A Multivariate GARCH-Jump Mixture Model

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Abstract

Jump models are useful in capturing skewed and/or leptokurtic financial returns. So far, research has focused on jumps in a single asset, including co-jumps between return and volatility. On the other hand, co-jumps among assets is also important especially in practices such as beta dynamics and portfolio allocation. This paper proposes a parsimonious yet flexible version of multivariate GARCH-jump mixture model (MGARCH-jump model) with multivariate jumps that allows both jump sizes and jump arrivals to be correlated. The model identifies co-jumps well and shows that both jump arrivals and jump sizes are highly correlated. The model also provides better prediction and better investment outcomes as opposed to the benchmark multivariate GARCH model with normal innovations (MGARCH-N model).

Key words: Multivariate, GARCH, Jump, Multinomial, Co-jump, beta dynamics, Value at Risk

1 Introduction

It is well-known that daily stock returns exhibit both continuous and occasional discontinued changes, also known as jumps. Popular volatility models include generalized autoregressive conditional heteroskedasticity (GARCH) (Bollerslev, 1986) and stochastic volatility (SV). Both models work well, especially when volatility is persistent. As regarding jumps, many efforts have been made to model a univariate stock price process since Press (1967), who appends a simple geometric Brownian motion with a compounded Poisson counting process. However, how jump arrivals and jump sizes affect each other among assets still remain unclear. This paper proposes a parsimonious and yet flexible model that allows both jump arrivals and jump sizes to be cross-sectionally correlated. We find that although jumps arrive infrequently in daily data, it’s more likely to have multiple assets jump together rather than independently. Moreover, whenever they jump together, they are very likely to jump in the same direction, and the magnitudes are clearly correlated.
Introducing jumps can affect conditional mean, conditional variance as well as higher-order unconditional moments such as skewness and kurtosis. This captures the empirical fact that unconditional distribution of stock returns is skewed and leptokurtic relative to a normal distribution (for example, Corrado and Su, 1996; Fama, 1965; Kon, 1984; Mandelbrot, 1963; Mills, 1995; Peiró, 1994, 1999; Praetz, 1972). Moreover, jumps are especially helpful in explaining large extreme return changes like market crushes.

Single-asset based jump models have been extensively investigated. The most commonly used model is the compounded Poisson process model introduced by Press (1967), where jump arrival is a Poisson counting process and jump size is normally independently distributed. Based on that, Ball and Torous (1983) provide a Bernoulli jump model through a discrete approximation. Jorion (1988) implements the Poisson jump model with an autoregressive conditional heteroskedasticity (ARCH) diffusion component on both foreign exchange and stock market. Vlaar and Palm (1993) test the Bernoulli jump model on the former European Monetary System for different drift (AR) and diffusion (ARCH/GARCH) representations, and Nieuwland et al. (1994) further test the Poisson model on the same topic. Bates (1996, 2000) and Pan (2002) find SV-jump (Poisson) model can best explain the price behaviour of foreign exchange and stock options than other available alternatives.

Aside from return jumps, Eraker (2004) and Caporin et al. (2014) confirm jumps also exist in volatility dynamics. Eraker et al. (2003) also study its impact on option pricing. Bandi and Renò (2016) and Jacod et al. (2017) take a further step and inspect the relation between return jumps and volatility jumps, and Chorro et al. (2017) further study how such return-volatility co-jumps affect density forecast. Another direction of extension from the basic jump-diffusion model is to consider time-varying jumps. Oldfield et al. (1977) expand the Poisson jump model into autoregressive jump sizes, while Chan and Maheu (2002) introduce an autoregressive conditional jump intensity (ARJI) model that allows jump intensities to be autocorrelated. Implementations of ARJI model include Maheu and McCurdy (2004), who found jumps usually arrive in cluster, Chan and Feng (2012) and Maheu et al. (2013), who inspect the relation of jumps and risk premiums.

As mentioned above, most of the research on jumps have been focused on modelling a univariate asset return, with covariation among different assets ignored. Bollerslev et al. (2008) is among the first who identify the existence of co-jumps and provide a test for co-jumps in multiple assets. Gilder et al. (2014) confirm the empirical findings in Bollerslev et al. and provide another test. Mancini and Gobbi (2012) suggest a nonparametric estimator based on realized covariation. Similarly, A¨ıt-Sahalia and Xiu (2016) decompose quadratic variation into continuous and discontinuous component to estimate co-jumps. Bibinger and Winkelmann (2015), Winkelmann et al. (2016) also concentrate on extracting co-jump from quadratic covariation and introduce a truncated estima-
tor. Caporin et al. (2017) further apply this estimator in a higher dimensional experiment. Other attempts are Gobbi and Mancini (2007) to derive a bivariate parametric co-jump estimator, and Novotný and Urga (2018) to introduce a new approach to test the existence of co-jumps.

All of the above co-jump estimators and tests rely on asymptotic assumptions and focus on identifying co-jumps but not the dynamics which are needed for prediction purpose. Moreover, exploiting co-jumps from quadratic covariation makes it impossible to study the cross-sectional relation of jump arrivals and jump sizes separately. This type of nonparametric estimators is designed to investigate the ex-post relation of jump sizes but ignores the information embedded in jump arrivals. Also, when extracting jumps from quadratic variation and bipower variation, one loses important information on the signs of jump sizes. One may argue that the nonparametric methods are superior as they don’t rely on any distributional assumption, but in order to forecast jumps, an additional econometric model is still required. And the proposed MGARCH-jump model can flexibly model jump size distributions and covariances, and all jump arrival combinations are also allowed.

For parametric models, Laurini and Mauad (2015) propose a bivariate SV model with built-in co-jumps, but idiosyncratic jumps are not allowed in the model; and Zhang et al. (2017) provide a goodness-of-fit test for this type of models. Chua and Tsiaplias (2017) introduce another model with correlated jump sizes but independent jump arrivals and autocorrelated jump intensities.

In this paper, I propose a new fully parametric model in which there’s an embedded component that allows the returns to jump separately or jump together with correlated jump sizes. This model overcomes the drawbacks mentioned above. The proposed MGARCH-jump model is well identified with reasonable GARCH parameters and high no-jump probabilities. The results also show that when jumps occur, it is more likely that several stocks jump together, with strong and positive jump size correlations. This is especially important when extreme returns occur, which is confirmed by predictive likelihoods. The MGARCH-jump model performs similarly to the benchmark MGARCH-N model in normal times, but outperforms it in high volatility times.

Some pure jump processes like Hawkes process (Hawkes, 1971) are also popular in modelling jumps. Examples of Hawkes process models include Bacry et al. (2013), Rambaldi et al. (2014), Aït-Sahalia et al. (2015), Aït-Sahalia and Hurd (2015), Bormetti et al. (2015), etc. However, these intriguing works view all price changes as pure jumps with self-exciting intensities, hence unrelated to the discussion of this paper.

In this paper, Section 2 formally describes the layout and some properties of the proposed multivariate GARCH-jump mixture model (MGARCH-jump model hereafter) in detail. Section 3
illustrates the data sources along with the cleaning and transformation methods. Section 4 discusses the estimated and forecast results from MGARCH-jump model for five trivariate examples of individual asset, corresponding industry and the market. Section 5 tests the model in higher dimension with five stocks all together. Section 6 shows several applications by the MGARCH-jump model, including the effect of jumps on beta dynamics and the predictive Value at Risk for an equally-weighted portfolio. And Section 7 is the conclusion.

2 Model

In this section, we present a discrete time MGARCH-jump model for financial returns. The model has a multinomial jump arrival and a multivariate normal jump size component. Let \( r_t = (r_{t,1}, r_{t,2}, \cdots, r_{t,N})' \) be a \( N \times 1 \) vector of returns of \( N \) assets at time \( t \). \( r_t \) is specified as

\[
    r_t = \mu + \epsilon_t, \\
    \epsilon_t = \epsilon_{1,t} + \epsilon_{2,t},
\]

where \( \mu = (\mu_1, \mu_2, \cdots, \mu_N)' \) is a \( N \times 1 \) vector of constant drift, \( \epsilon_{1,t} \) is a \( N \times 1 \) vector of return innovations with \( E(\epsilon_{1,t}|\mathcal{F}_{t-1}) = 0 \), where \( \mathcal{F}_{t-1} = \{r_1, r_2, \cdots, r_{t-1}\} \) is the information set at time \( t - 1 \). In particular,

\[
    \epsilon_{1,t} = H_{t}^{1/2} z_t, \\
    z_t \sim NID(0, I),
\]

where \( z_t \) is a \( N \times 1 \) vector of multivariate NID shocks, \( H_{t}^{1/2} \) is the Cholesky decomposition of a \( N \times N \) conditional covariance matrix regulated by a multivariate GARCH structure. \( \epsilon_{2,t} \) is a \( N \times 1 \) vector of jump innovations also with mean of zero.

\[
    \epsilon_{2,t} = J_t - E(J_t|\Theta, \mathcal{F}_{t-1}),
\]

where \( J_t = (J_{t,1}, J_{t,2}, \cdots, J_{t,N})' \) is a \( N \times 1 \) vector of jumps, \( \Theta \) is the union set of all parameters, and \( E(\epsilon_{2,t}|\mathcal{F}_{t-1}) = 0 \). Note that the conditional expectation of jumps is removed from the model, so \( E(r_t|\mathcal{F}_{t-1}) = \mu \) for all \( t \). This feature provides a constant drift without jump effects as in Merton (1976). \( \epsilon_{1,t} \) and \( \epsilon_{2,t} \) are contemporaneously independent with each other.

2.1 Vector-Diagonal GARCH (VD-GARCH)

There are many approaches to extend the traditional version of GARCH model (Bollerslev, 1986) from univariate to multivariate world. We use a slightly modified version of the vector diagonal
GARCH (VD-GARCH) model introduced by Ding and Engle (2001):

\[ H_t = CC' + \alpha \alpha' \odot \epsilon_{t-1} \epsilon_{t-1}' + \beta \beta' \odot H_{t-1}, \tag{6} \]

where \( \odot \) is the Hadamard product operator that performs element-by-element multiplication, \( C \) is a \( N \times N \) lower triangular matrix, and \( \alpha \) and \( \beta \) are all \( N \times 1 \) vector of parameters. \( \epsilon_{t-1} \) includes both continuous shocks \( \epsilon_{1,t-1} \) and jump shocks \( \epsilon_{2,t-1} \). It is also natural to consider \( \epsilon_{t-1} \) incorporating only continuous shocks with \( \epsilon_{t-1} = r_{t-1} - \mu - J_{t-1} + E (J_{t-1} | F_{t-2}) \), but then the past jump series are propagated into future \( H_t \) making the model likelihood path dependent and thus the sampling for jump components extremely difficult.

The VD-GARCH specification is a simplified version of the BEKK model (Engle & Kroner, 1995), and inherits the property that guarantees \( H_t \) to be positive definite if \( H_0 \) is positive definite. Specifically, for each element \( h_{t,ij} \) in matrix \( H_t \),

\[ h_{t,ij} = \omega_{ij} + \alpha_i \alpha_j \epsilon_{t-1,i} \epsilon_{t-1,j} + \beta_i \beta_j h_{t-1,ij}, \tag{7} \]

where \( \omega_{ij} = (CC')_{ij} \). No further restriction on parameters is required other than stationary conditions of \( \alpha_i^2 + \beta_i^2 < 1 \forall i \) and \( (C)_{ii} > 0 \) for better identification. Note that \( \omega_{ij} \) does not need to be restricted.

### 2.2 A Compound Multinomial Jump Structure

Most of the past univariate jump models parametrize jumps as a compound Poisson process follow Press (1967). Although a Poisson process fits well in univariate continuous-time models, it is not easily extended to higher dimension with sufficient flexibility and dependence. While empirically, all the observed data is discrete in time, so as suggested in Ball and Torous (1983), Bernoulli jump is a good discrete approximation of a Poisson process over a small time interval and also more intuitive. One nice feature of Bernoulli jump is that it’s much easier to generalize into the multivariate universe. A multinomially distribution with only one trial can perfectly fit all possible jump/co-jump combination patterns. Therefore, for vector \( J_t \),

\[ J_t = Y_t \odot B_t, \tag{8} \]

\[ Y_t \sim N (\mu_J, \Sigma_J), \tag{9} \]

where \( Y_t \) is a \( N \times 1 \) vector of jump sizes that are multivariate normally distributed with mean vector \( \mu_J \) and covariance matrix \( \Sigma_J \), and \( B_t = (B_{t,1}, B_{t,2}, \ldots, B_{t,N})' \) is a \( N \times 1 \) vector of jump indicators with each element \( B_{t,i} \) being the result of an Bernoulli trial, \( B_{t,i} \in \{0, 1\} \) for \( i = 1, \ldots, N \).
Let $L = 2^N$, 

$$B_t \sim \text{multinomial} \left(1, p_1, \ldots, p_L \right), \tag{10}$$

where $\sum_{j=1}^{L} p_j = 1$, $B_{t,i} = 1$ indicates there is a jump for asset $i$ at time $t$, and $B_{t,i} = 0$ otherwise. The parameter $p_j$ is the jump/co-jump probability. Unlike univariate models, where the jump intensity parameter represents the probability of jump arrivals, in this specification, the jump/co-jump probability $p_j$ is a separate probability assigned to each possible jump/co-jump outcome. Admittedly, it may look a little abstract and tedious, and the number of probabilities to be estimated increases exponentially as $N$ grows, but it’s necessary so to maintain the flexibility in jump arrivals, and it’s relatively easy to sample. To be more specific, define a $2^N \times N$ matrix $\Omega_B$ that contains all possible outcomes of $B_t$, with each row being one exclusive possible value of $B_t$, and $p = (p_1, p_2, \ldots, p_L)'$ is a vector of corresponding jump probabilities. In a trivariate case, there are $2^3 = 8$ possible outcomes of $B_t$: one trivariate co-jump $(1\ 1\ 1)$; three bivariate co-jumps $(1\ 1\ 0)$, $(1\ 0\ 1)$ and $(0\ 1\ 1)$; three idiosyncratic jumps $(1\ 0\ 0)$, $(0\ 1\ 0)$ and $(0\ 0\ 1)$; and one no jump outcome $(0\ 0\ 0)$. This covers all possible jump patterns including all-asset co-jumps and subset co-jumps. Each outcome is associated with one probability element in $p$. Note that $\Omega_B$ is neither a parameter or a latent variable. It’s a generated constant and solely depends on $N$. The order of combinations of $B_t$ in $\Omega_B$ (how rows are stacked in $\Omega_B$) does not matter, but it needs to be consistent throughout the sampling procedure to avoid any unnecessary order switching.

One merit of this specification is that one can easily verify whether the jumps are cross-sectionally independent through these probabilities. Our empirical results show that the jump arrivals are clearly correlated cross-sectionally, so the multinomial assumption offers an accurate model of jump dependencies.

Besides jump arrivals, the multivariate normal structure naturally connects jump sizes among assets through the covariance matrix $\Sigma_J$. As a result, in this model, one can easily extract correlation of jump arrivals and that of jump sizes separately, so question like “whether and when do they jump together” and “how do they jump together” can be answered explicitly.

### 2.3 Conditional Moments

Most of the past research regarding cross-sectional co-jumps focus on estimating the compound jump process $(J_t)$ directly, and study its effect on the conditional moments of return. Define the moments only conditional on the past information set as “ex-ante”, and the moments further conditional on jump arrivals $B_t$ as “ex-post”. The first two ex-ante conditional moments of jump
\( J_t \) are\(^1\)

\[
E(J_t|\Theta, \mathcal{F}_{t-1}) = \mu_J \odot \Omega_B' p = \mu_J \odot \left( \sum_{j=1}^{2N} \Omega_j p_j \right),
\]

(11)

and

\[
\text{Cov} (J_t|\Theta, \mathcal{F}_{t-1}) = (\Sigma_J + \mu_J \mu_J') \odot \left( \sum_{j=1}^{2N} p_j \Omega_j \Omega_j' \right) - \mu_J \mu_J' \odot \Omega_B' pp' \Omega_B.
\]

(12)

where \( \Theta = (\mu, \theta_H, p, \mu_J, \Sigma_J) \), and \( \Omega_j \) is the \( j \)th row of \( \Omega_B \). Similarly, the first two conditional moments of return are

\[
E (r_t|\Theta, \mathcal{F}_{t-1}) = \mu,
\]

(13)

\[
\text{Cov} (r_t|\Theta, \mathcal{F}_{t-1}) = H_t + \text{Cov} (J_t|\Theta, \mathcal{F}_{t-1}).
\]

(14)

The conditional mean of returns is simply \( \mu \) because both MGARCH volatility and jump innovation has mean of zero. The conditional covariance of returns is the aggregation of conditional MGARCH volatility and conditional covariance of jump.

Unique in this new model, the ex-post moments conditional on jump arrivals \( (B_t) \) show more interesting properties. The first two conditional moments are:

\[
E (r_t|B_t, \Theta, \mathcal{F}_{t-1}) = \mu + \mu_J \odot (B_t - \Omega_B' p)
\]

(15)

\[
\text{Cov} (r_t|B_t, \Theta, \mathcal{F}_{t-1}) = H_t + B_t B_t' \odot \Sigma_J.
\]

(16)

Because \( B_t B_t' \) is positive semi-definite, and both \( H_t \) and \( \Sigma_J \) are positive definite, the conditional covariance of \( r_t \) is positive definite. To be more specific,

\[
E (r_t|B_t, \Theta, \mathcal{F}_{t-1}) = \mu + \begin{pmatrix}
B_{t,1} \mu_{J,1} \\
B_{t,2} \mu_{J,2} \\
\vdots \\
B_{t,N} \mu_{J,N}
\end{pmatrix} - \mu_J \odot \Omega_B' p
\]

(17)

\[
\text{Cov} (r_t|B_t, \Theta, \mathcal{F}_{t-1}) = H_t + \begin{pmatrix}
B_{t,1}^2 \sigma_{J,1}^2 & B_{t,1} B_{t,2} \sigma_{J,12} & \cdots & B_{t,1} B_{t,N} \sigma_{J,1N} \\
B_{t,2} B_{t,1} \sigma_{J,21} & B_{t,2}^2 \sigma_{J,2}^2 & \cdots & B_{t,2} B_{t,N} \sigma_{J,2N} \\
\vdots & \vdots & \ddots & \vdots \\
B_{t,N} B_{t,1} \sigma_{J,N1} & B_{t,N} B_{t,2} \sigma_{J,N2} & \cdots & B_{t,N}^2 \sigma_{J,N}^2
\end{pmatrix}
\]

(18)

\(^1\)Proof can be found in Appendix A.
Clearly,

\[ B_{t,i}B_{t,j} = \begin{cases} 
1 & \text{if } B_{t,i} = B_{t,j} = 1 \\
0 & \text{otherwise.} 
\end{cases} \]  

This ensures which element(s) in \( \mu_j \) and \( \Sigma_j \) should be turned on and thus affect conditional means and covariances among asset returns. The corresponding element \( \mu_{J,i} \) and \( \sigma_{J,i}^2 \) will be turned on if and only if asset \( i \) jumps, and \( \sigma_{J,ij} \), where \( i \neq j \), will be turned on if and only if asset \( i \) and asset \( j \) both jump at the same time. This property helps to capture the co-jump behaviour among assets and reflect it directly to return covariances. If there’s no jump for all \( N \) assets, then \( B_t = (0, 0, \ldots, 0)' \), so \( \mathbb{E}(r_t|B_t, \Theta, \mathcal{F}_{t-1}) = \mu - \mu_j \odot \Omega_B'p \) and \( \text{Cov}(r_t|B_t, \Theta, \mathcal{F}_{t-1}) = H_t \), which reduces to the results from basic dynamic volatility models such as MGARCH. If all \( N \) assets jump, then \( B_t = (1, 1, \ldots, 1)' \), so \( \mathbb{E}(r_t|B_t, \Theta, \mathcal{F}_{t-1}) = \mu + \mu_j - \mu_j \odot \Omega_B'p \) and \( \text{Cov}(r_t|B_t, \Theta, \mathcal{F}_{t-1}) = H_t + \Sigma_j \). In other cases, only a sub-block of \( \Sigma_j \) is turned on. For example, in a trivariate case with a bivariate co-jump occurring, say \( B_t = (1, 1, 0)' \), two elements in \( \mu_j \) and four elements in \( \Sigma_j \) are turned on:

\[
\mathbb{E}(r_t|B_t, \Theta, \mathcal{F}_{t-1}) = \mu + \begin{pmatrix} 
\mu_{J,1} \\
\mu_{J,2} \\
0
\end{pmatrix} - \mu_j \odot \Omega_B'p
\]

\[
\text{Cov}(r_t|B_t, \Theta, \mathcal{F}_{t-1}) = H_t + \begin{pmatrix} 
\sigma_{J,1}^2 & \sigma_{J,12} & 0 \\
\sigma_{J,21} & \sigma_{J,2}^2 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

This is consistent with the intuition that conditional mean and variance can only be affected when the corresponding asset jumps, and conditional covariance can only affected when the two corresponding assets jump together. Obviously, this model supports all jump/co-jump possibilities and channels the jump/co-jump effects into conditional moments as desired. As for correlations, which is computed by \( \frac{h_{t,ij} + B_{t,i}B_{t,j} \sigma_{J,ij}}{\sqrt{(h_{t,i} + B_{t,i}^2 \sigma_{J,j}^2)(h_{t,j} + B_{t,j}^2 \sigma_{J,j}^2)}} \) for ex-post and \( \frac{h_{t,ij} + \sigma_{J,ij}}{\sqrt{(h_{t,i} + \sigma_{J,j}^2)(h_{t,j} + \sigma_{J,j}^2)}} \) if there’s a co-jump, the jump effect depends on the scale of the correlation computed by \( H_t \) (MGARCH correlation hereafter). Clearly, the co-jumps does not necessarily increase the overall return correlation as in the covariance case. For example, the jump impact on the overall ex-post correlation is determined by the comparison of the square of MGARCH correlation \( \rho^2_{MGARCH} = \frac{h_{t,ij}^2}{h_{t,i}h_{t,j}} \) and \( \rho^2 = \frac{\sigma_{J,ij}^2 + 2h_{t,i}h_{t,j} \sigma_{J,ij}}{\sigma_{J,i}^2 \sigma_{J,j}^2 + h_{t,i} \sigma_{J,j}^2 + h_{t,j} \sigma_{J,i}^2} \). If \( \rho^2_{MGARCH} \) is greater than \( \rho^2 \), then the overall return correlation will decrease, and vice versa. This behaviour clearly can affect diversification benefits in a portfolio.

### 2.4 Sampling Algorithm

This model consists two latent variables, \( Y_t \) and \( B_t \), so not easy to estimate by classical methods. Instead, we apply a typical Bayesian method, Markov chain Monte Carlo (MCMC). Bayesian
method is powerful since it allows one to treat latent variables as parameters to estimate, and to break complex problem into relatively simple pieces. In each piece, one can estimate only a subset of the parameters, and treat others as given. The estimation of this model also takes advantage of these properties. A full MCMC run contains $M_0 + M$ iterations, where the first $M_0 = 10000$ are burn-in samples, and the rest $M = 10000$ are posterior draws. Each MCMC iteration is as follow:

1. $\mu | r_{1:T}, H_{1:T}, \mu_J, \Sigma_J, B_{1:T}, p$.
2. $\theta_H | r_{1:T}, \mu, \mu_J, \Sigma_J, B_{1:T}, p$, where $\theta_H = (C, \alpha, \beta)'$.
3. $B_t | r_t, \mu, H_t, \mu_J, \Sigma_J, p$.
4. $p | r_{1:T}, \mu, H_{1:T}, \mu_J, \Sigma_J, B_{1:T}$.
5. $Y_t | r_t, \mu, H_t, \mu_J, \Sigma_J, B_t, p$.
6. $\mu_J | r_{1:T}, \mu, H_{1:T}, \Sigma_J, Y_{1:T}, B_{1:T}, p$.
7. $\Sigma_J | \mu_J, Y_{1:T}$.

Steps 1, 3, 5, 7 are simply Gibbs samplers, and steps 2, 4, 6 are Metropolis-Hastings (MH) due to unknown type of posterior distributions.² And the jump arrivals ($B_t$) and jump sizes ($Y_t$) can be estimated as:

$$E(B_t) \approx \frac{1}{M} \sum_{i=1}^{M} B^{(i)}_t$$  \hfill (20)

and

$$E(Y_t) \approx \frac{1}{M} \sum_{i=1}^{M} Y^{(i)}_t$$  \hfill (21)

We apply uninformative priors for all parameters and let data determine the posteriors. The prior choices are:

$$\mu \sim N(0, 100I)$$

$$\theta_H \sim N(0, 100I)$$

$$p \sim Dir(1, \ldots, 1)$$

$$\mu_J \sim N(0, 100I)$$

$$\Sigma_J \sim IW(N + 2, I)$$

²Details of each sampling step can be found in Appendix B.
2.5 Predictive Likelihood

Recall that \( \Theta = (\mu, \theta_H, p, \mu_J, \Sigma_J) \), and the predictive likelihood is computed by integrating out all parameters \( \Theta \). From equation (15) and (16), the conditional distribution of returns conditional on jump arrivals is simply a multivariate normal distribution. In order to compute the predictive likelihood for the whole model, one can first compute the likelihood of this conditional distribution \( p(r_{t+1}|r_{1:t}, \Theta, B_{t+1}) \) then integrate out jump arrivals:

\[
p(r_{t+1}|r_{1:t}) \approx \sum_{i=1}^{M} \sum_{j=1}^{L} p\left(r_{t+1}|r_{1:t}, \Theta^{(i)}, B_{t+1}^{(j)}\right) p_j^{(i)},
\]

where \( p(\Theta|r_{1:t}) \) is the posterior of all parameters conditional on sub-sample \( r_{1:t} \), \( \Theta^{(i)} \) and \( p_j^{(i)} \) are the parameters drawn in the \( i \)th iteration, and \( B_{t+1}^{(j)} \) is the \( j \)th possible outcome from \( \Omega_B \). The out-of-sample likelihood is product of the predictive likelihood evaluated in each period:

\[
p(r_{t+1:T}|r_{1:t}) = \prod_{t=t}^{T} p(r_{t+1}|r_{1:t}).
\]

In practice, one usually computes the log-predictive likelihood by summing the log-predictive likelihood evaluated in each out-of-sample period.

3 Data

The properties of this model enable easy solution to questions such as “What’s the jump/co-jump relation among different assets?” and “Does a particular asset more likely to jump idiosyncratically or with the corresponding industry or even with the market?” To answer these questions, we select General Electric (GE), Exxon (XOM), Wal-Mart (WMT), Microsoft (MSFT) and American Express (AXP), representing electrical equipment industry, petroleum and natural gas industry, retail industry, computer software industry and banking industry, respectively. All the return data is retrieved from Center for Research in Security Prices (CRSP) database, specifically daily holding-period return of each selected stock and the value-weighted market portfolio (MKT). Also, daily returns of risk-free rate and Fama-French 49 industry portfolio is obtained from Kenneth French’s website. In order to match each stock with its corresponding industry portfolio, SIC codes of the above stocks are also acquired from CRSP. All the returns are collected from January 1, 1990 to December 31, 2016, with 6805 observations in total after removing all non-number and missing...
values. Earning announcement dates are gathered from I/B/E/S database.

Table 1 illustrates descriptive statistics of daily continuously compounded returns for the selected stocks as well as the value-weighted market portfolio. The statistics are computed from log returns in percentage value after dropping all the missing values and non-number observations.

4 Individual Stocks, Corresponding Industry and the Market Co-Jumps

4.1 Estimation

The first example is to estimate the trivariate model each for GE, XOM, WMT, MSFT and AXP coupled with their corresponding industry and the market respectively. Table 2 reports the results for these trivariate estimates. All the posteriors are in reasonable regions proved by vast amount of previous researches, with small means ($\mu_i$ of 0.02 – 0.05), low shock parameters ($\alpha_i$ of 0.15 – 0.20), and high persistent parameter ($\beta_i$ of 0.97 – 0.98) from the MGARCH specification.

All trivariate models indicates that “no jump” is the most likely outcome. No-jump probabilities ($p_{STK,IND,MKT}$) ranges from 0.82 to 0.88. The jump size variances are considerably large, with jump size variance for individual stocks ($\sigma^2_{J,STK}$) ranging from 2.14 to 9.45, for industries ($\sigma^2_{J,IND}$) ranging from 1.61 to 3.66, for the market ($\sigma^2_{J,MKT}$) ranging from 1.00 to 1.52. Jump size covariances are all positive and also relatively large, with covariance for stocks and corresponding industry ($\sigma_{J,STK,IND}$) ranging from 1.83 to 4.25, for stocks and the market ($\sigma_{J,STK,MKT}$) ranging from 1.03 to 3.08, for industries and the market ($\sigma_{J,IND,MKT}$) ranging from 1.09 to 2.43. This confirms the fact that jumps are rare but extreme movements in stock returns.

From the basic probability rules, if jumps are cross-sectionally independent, a co-jump joint probability should be equal to the product of marginal jump probabilities for the corresponding assets. Panel A of Table 3 compares the co-jump joint probabilities with the product of its marginal probabilities. The co-jump probabilities range from 0.0595 to 0.0984, while the product of marginal probabilities ranges from 0.0007 to 0.0250. Clearly, jump arrivals are strongly correlated as the joint probabilities and product of marginal probabilities are very different from each other. The differences are even greater when the number of assets in a co-jump is greater. For example, the bivariate co-jump probabilities of GE and its industry, GE and the market, GE’s industry and the market are 0.0984, 0.0980, 0.0986 respectively, while the products of marginal jump probabilities are 0.0181, 0.0159, 0.0121, respectively. They are very different but still the same decimals. In contrast, the joint probability of a trivariate co-jump with GE, its industry and the market jump
all together is 0.0954, while the product of marginal jump probabilities is 0.0019, 50 times less than the corresponding co-jump probability.

Panel B further computes the co-jump probabilities conditional on different univariate jumps, which indicate the proportion of co-jumps in each univariate jumps. The results show that if the market jumps, each selected stock and its industry will most likely jump as well. Electrical equipment industry, retail industry and banking industry are more likely to jump along with the market when unusual condition occurs, more than half of jumps in petroleum and natural gas industry and software industry coincide the market jumps. For XOM, WMT and MSFT, all the firms in their industry are most likely to jump together with them, with probabilities of co-jump with their industries conditional on stock jumps being 0.9496, 0.9517 and 0.9783 respectively. When WMT jumps, the whole market is very likely to follow, with a probability of co-jump with the market conditional on stock jumps being 0.8102. GE and AXP also have strong influence on their industry when they jump, with co-jump probability conditional on stock jumps of 0.6392 and 0.5457 respectively.

Figure 1 plots the posteriors of jump arrival probabilities for each of the five stocks with their corresponding industry and the market. Most of the jump arrivals are aligned together, which confirms the results in panel B of Table 3. Figure 2 plots jump size realizations over time. The figure shows jump size realizations are relatively large (up to 10% and -10%) and infrequent. The results are more clear if we focus on a small time span. Take AXP from January 1, 2007 to December 31, 2009 as an example shown in Figure 3, jump probability is usually high around quarterly earnings announcement dates. Beyond that, progression of sub-prime mortgage crisis plays an important role on jump dynamics. For instance: on March 13, 2007, reacted to the potential risk of sub-prime mortgages, causing a -2.93% jump on AXP, a -2.73% jump on the banking industry and a -1.86% jump on the market. On November 1, 2007, after a previous interest rate cut, the Federal Reserve injected 41 billion dollars into money supply with a response of -3.14% AXP jump, -3.49% industrial jump and -2.12% market jump. On September 29, 2008, the House of Representatives rejected the bailout plan, accompanying with a -5.24% of AXP jump, a -3.85% of industrial jump and a -3.18% of market jump. All the above jumps have posterior jump probabilities greater than 0.9. Other major events can also be matched with these realizations. The proposed MGARCH-jump model correctly identifies the time and magnitude of a jump event.

Figure 4 scatters the jump probabilities and jump sizes across AXP, the industry and the market pairwise during the recent financial crisis. The top three graphs plot the jump probabilities with a 45-degree line. In most cases, the jump probabilities are low with points concentrating at the bottom left corner. Between AXP and the industry, it’s a lot more likely to jump only in AXP than only in the industry, as most of the points lie below the 45-degree line with only a few exceptions.
As for AXP and the market, nearly all the points are below the 45-degree line, indicating that any market jump is very likely an AXP-market co-jump but not conversely. In the top right graph, most of the points lie along the 45-degree line, so the financial industry and the market are more likely to jump together rather than jump separately. The bottom three graphs plot the jump sizes with the linear regression line of the vertical axis variable against the horizontal axis variable. In all three cases, the points spread quite well along the regression line, indicating the jump sizes are highly correlated.

Table 4 outlines the jump size correlations for the five selected stocks with their industry and the market. The first observation is all the co-jumps have positive correlation whenever it occurs. As for magnitude, all the five stocks are highly correlated with their corresponding industry, and each of five industry is also highly correlated with the market when co-jump arrives. GE, XOM and AXP strongly follows the market in jump sizes, while WMT and MSFT are just moderately correlated with the market. The relatively low jump size correlation between WMT and the market is probably because of the defensive nature of WMT in business cycle, while that between MSFT and the market is more likely due to the comparably lower stock-market co-jump probability. The high jump size correlations imply that when extreme events, for example crisis, occur, diversification benefits may be greatly affected as the overall correlation among asset returns could be significantly altered by jump effects. Details are further discussed in Section 5.

4.2 Prediction

This subsection compares the forecasts between the MGARCH-jump model and a benchmark MGARCH model (VD-GARCH) with normal renovations (MGARCH-N model) by computing their predictive likelihood respectively. These predictive likelihoods are computed by comparing each of the five stocks along with their corresponding industry and the market. The last 100 observations are used for out-of-sample density forecast evaluation, and prediction is implemented one period ahead recursive forecasting, following equation (22) to (25).

Log-Bayes factor is computed by subtracting the log-predictive likelihoods of MGARCH-N model from that of MGARCH-jump model. A rule of thumb of this measure is that if log-Bayes factor is greater than 5, then the evidence for MGARCH-jump is considered as “very strong”. Table 8 lists the log-predictive likelihoods and log-Bayes factors from different cases. The MGARCH-jump model overwhelms the benchmark MGARCH-N model in all six predictions, with log-Bayes factors from around 12.70 to 61.81, That is, the MGARCH-jump model is approximately \(3.2812 \times 10^5\) to \(6.9782 \times 10^{26}\) times better than the MGARCH-N model collectively in terms of predictive likelihood. This dominance is robust to larger sample and/or longer prediction horizons.
Figure 5 plots the predictive likelihoods at each period. During normal days, both model performs very similarly due to the same VD-GARCH component; while in days with drastic return change, the predictive likelihood is significantly greater for the MGARCH-jump model. This is especially important, because a risk-averse investor would like to be able to design some special strategies to diversify or hedge against these one-time, enormous risk events in advance. Comparing to the commonly used MGARCH-N models, adding a jump component is more suitable to design those strategies.

5 Jumps/Co-jumps among Individual Stocks

The second application is to estimate a 5-dimensional model with GE, XOM, WMT, MSFT and AXP all together. Table 5 lists posterior results for jump component in this model. Again, the posterior estimates are in reasonable region with low mean ($\mu$ of 0.03–0.06), low innovation parameter ($\alpha$ of 0.11–0.15) and high persistent parameter ($\beta$ greater than 0.98) as shown in Panel B of Table 5, and jump probabilities strongly favour “no jump” ($p_{\text{GE, XOM, WMT, MSFT, AXP}} = 0.7102$) in Panel C, and jump size variances are large (all greater than 4.6). This shows that the proposed model is correctly specified and well identified even with five stocks. Furthermore, the probability of only one stock jump while others don’t is higher than that of any co-jumps, as the former are all above 0.024 and the latter are generally below 0.01 with the only exception of a 5-asset mutual jump probability of 0.017.

Panel A of Table 6 compares the joint co-jump probability and the product of corresponding marginal univariate jump probabilities. The jump arrivals across assets are clearly correlated. For instance, the XOM, MSFT, AXP co-jump has the lowest joint probability among all of 0.0010, but the product of marginal jump probabilities is less than 0.00005. This pattern is consistent over all co-jump cases. The joint co-jump probability is at least 0.0010, while the product of marginal probabilities is at most 0.0001. Panel B exhibits that the majority of co-jumps among these five stocks are mutual co-jumps, which consist 20.41% of the GE-jump, 18.74% of the XOM-jump, 16.22% of the WMT-jump, 14.90% of the MSFT-jump and 19.41% of the AXP-jump. And partial co-jumps with one or more stocks not jumping are a lot less frequent, each type of which consists less than 10% of each asset jumps. This implies that for major stocks from different industries, they either all jump together, which is probably a market jump, or jump separately. One possible explanation is that when a jump arrives in some industry, it’s not very likely to spread into other not directly related industries unless it’s a market jump.

Table 7 lists jump size correlations among the five stocks. Most of the time, all five stocks jump to the same direction, and most of the stocks are also highly correlated in jump magnitude except for XOM in oil industry. As mentioned before, these jump size correlations could severely change
the overall return correlations. The final effect is rather complicated as shown in Figure 6, with plots the differences by subtracting the correlations of GARCH component from those of ex-ante and ex-post covariances separately. These differences are usually around zero (no jump or very low probability of jump), but they can also go up to 0.4 and down to -0.4 as a result of jumps. As shown in the figure, jumps increase the return correlations when the MGARCH correlation is relatively low and thus decrease the diversification benefits, and vice versa. This is consistent with the theoretical implications based on the model structure.

The out-of-sample forecast comparison results are robust in high dimension. In the last row of Table 8, the log-Bayes factor for the MGARCH-jump model relative to the MGARCH-N model is 67.43, equivalent to $1.9206 \times 10^{29}$ times better in terms of predictive likelihood. Similarly, in the bottom right plot in Figure 5, the MGARCH-jump model greatly outperforms the MGARCH-N model in a few particular periods, and performs about the same for the rest time.

6 Applications

6.1 Impact on beta Dynamics

Consider a bivariate volatility model for excess returns of some individual stock and the market, then beta of this stock can be computed simply, by definition, from covariance matrix. Therefore, dynamics of beta is equivalent to dynamics of the conditional covariance matrix. Compared to the traditional approach that treats beta as a regression slope, this method naturally allows for dynamic beta as long as the covariances changing over time.

Past researches defining beta dynamics varies considerably. Bali et al. (2016) emphasize the role of dynamic beta in investment practice; Engle (2016) derives an estimator based on dynamic conditional correlation (DCC) model for continuous beta; from an complete different perspective, Todorov and Bollerslev (2010) on the other hand disentangle jumps into systematic jumps and idiosyncratic jumps, and then estimate continuous beta and jump beta accordingly. As for MGARCH-jump model, we follow a similar method to Engle (2016) but with Bayesian techniques. Unlike Todorov and Bollerslev (2010), we do not specifically separate jump beta, but rather focus on how beta changes with or without jumps.

One can either predict an ex-ante beta before knowing the exact jump arrivals, or compute an ex-post beta after taking jump arrivals into account. Based on results from Section 2.3, if $\tilde{\mathbf{r}}_t = (\tilde{r}_{t,i}, \tilde{r}_{t,m})'$, where $\tilde{r}_{t,i}$ is the excess return of an arbitrary asset $i$, and $\tilde{r}_{t,m}$ is the excess return
of the market. Then,

\[
\text{Cov}(\tilde{r}_t | \mathbf{B}_t, \Theta, \mathcal{F}_{t-1}) = \begin{pmatrix}
    h_{t,ii} + B_{t,i}^2 \sigma_{J,i}^2 & h_{t,im} + B_{t,i}B_{t,m} \sigma_{J,im} \\
    h_{t,im} + B_{t,i}B_{t,m} \sigma_{J,im} & h_{t,mm} + B_{t,m}^2 \sigma_{J,m}^2
\end{pmatrix}
\] (26)

So an ex-post beta is

\[
\beta_{t,i} = \begin{cases} 
    \frac{h_{t,im} + \sigma_{J,im}}{h_{t,mm} + \sigma_{J,mm}} & \text{both jump} \\
    \frac{h_{t,im}}{h_{t,mm} + \sigma_{J,mm}} & \text{only market jumps} \\
    \frac{h_{t,im}}{h_{t,mm}} & \text{otherwise}
\end{cases}
\] (27)

This result nicely agrees with how beta relates to systematic risk: when the market doesn’t, there’s no change in systematic risk, so beta is not affected; if only the market jumps, then the stock’s relative exposure to the market decreases and so does beta; if there’s a co-jump, both market risk and stock risk increase, and the effect on beta depends on values in the jump size covariance matrix. Now systematic risk transfers through \( h_{im} \) and \( \sigma_{J,im} \) when co-jumps occur. Since a single stock is usually riskier than the market, ex-post beta is more likely to increase when co-jump occurs.

On the other hand, before knowing jump arrivals (integrate out \( B_t \)), conditional covariances of \( \tilde{r}_t \) is a dynamic volatility component (\( H_t \)) plus a constant correction of expected jump covariances. Given the rareness of jump events, this correction term is expected to be fairly small, and ex-ante beta should be relatively close to ex-post beta except for whenever market jumps occur. Additionally, because of the dominance of co-jumps among all jumps, ex-post beta should be mostly greater than ex-ante beta when jump arrives. Figure 7 shows the plots beta dynamics computed from bivariate models with excess returns of AXP and the market. These results confirms the above hypotheses.

Comparing to MGARCH-jump model, the benchmark MGARCH-N model tries to fit the jump extremes into smoothly changing volatilities. Thus the whole volatility dynamic is contaminated by jumps, and so as the beta dynamics. Empirically, MGARCH-N model tends to overestimate beta in general by not separating out jumps.

### 6.2 Impact on Value at Risk

Value at Risk (VaR) is an important measure that lends help in various investment decisions, especially in risk management. It’s defined by a VaR probability \( \alpha \), which means the loss an investor may potentially face when the worst \( \alpha \) circumstances happen. Consider an equally-weighted portfolio constructed by the five assets used in Section 5. The predictive VaR is computed
as following steps:

1. Simulate \( r_{t+1} | \mathcal{F}_t \) for \( M = 10000 \) times.
   
   (a) Propagate \( H_{t+1} \) and generate \( e_{1,t+1} \) from equation (3).
   
   (b) Generate \( B_{t+1} \) and \( Y_{t+1} \), and compute \( e_{2,t+1} \) from equation (5).
   
   (c) Compute \( r_{t+1} = \mu + e_{1,t+1} + e_{2,t+1} \).

2. Collect all the \( r_{t+1}^{(i)} | \mathcal{F}_t \) simulations, and compute the return of equally-weighted portfolio \( r_{EW,t+1}^{(i)} = w_{EW}' r_{t+1}^{(i)} \), where \( i = 1, \ldots, M \), and \( w_{EW} = (1/N, \ldots, 1/N)' \).

3. Find the \( M \alpha \)-th least value of \( r_{EW,t+1}^{(i)} \), where \( \alpha \) is the VaR probability.

Figure 8 plots the 10%, 5% and 1% predictive VaR respectively for both the MGARCH-jump model and the MGARCH-N model. In this 100 daily prediction periods, 10% predictive VaR’s are almost the same for both models, with the solid and dashed grey lines moving along with each other. The MGARCH-N model starts to slightly underestimate the 5% predictive VaR relatively than the MGARCH-jump model, with the red dashed line lies just above the red solid line for most of the periods. As for the 1% predictive VaR, the MGARCH-jump model provide clearly more conservative predictions, with the blue solid line always stay below the blue dashed line. The MGARCH-jump model predicts around 0.2% more potential daily loss agianst the MGARCH-N model when the worst 1% scenarios occur. It shows that the MGARCH-jump model provides a similar density prediction to the MGARCH-N model in general, and it’s more conservative only in the far left tail in the return distribution.

7 Conclusion

This paper proposes a multivariate GARCH-jump mixture model that is both parsimonious and flexible. The model consists two major components: a smooth shock in return due to volatility dynamics governed by a VD-GARCH component, and a drastic shock due to jumps governed by a compounded multinomial component. This jump component is a Hadamard product of a multivariate normal variable, indicating jump sizes, and a multinomial realization, a vector of jump arrival indicators, from all different possible jump arrival combinations. This structure allows for both jump sizes and arrivals to be correlated respectively, and the first two conditional moments exhibit desirable properties especially when further conditional on jump arrivals. The element-by-element jump effect on both conditional mean and conditional covariance will be activated/deactivated by the corresponding jump arrival indicators.
Results estimated by MCMC method show the model is well identified, and strong cross-sectional correlation in both jump sizes and jump arrivals. When modelling individual stocks with corresponding industry and the market, we found most of the market jumps are co-jumps, and a considerable proportion of industry/stock jumps are also co-jumps. This proportion depends on the degree of firm dominance in their industry and the sensitivity between industry and the market. The source of identified jumps are major unexpected events, including earning announcement surprises, bad public news, etc. As for jump sizes, individual stock, industry and the market always jump into the same direction; strong correlations are found both between stock and industry, and between industry and the market, while only moderate correlation between stock and the market. When modelling five stocks from different industries, most of the jumps are either mutual co-jumps or individual jumps. This shows low contagion effect across industries unless is market-wise. Again, jump size correlations are all positive, and mostly high except for XOM from oil industry.

Impact of jumps on return correlations is less trivial than that on return covariances, as it depends on the level of MGARCH correlations. When the MGARCH correlation is higher than the jump size correlation, jumps will decrease the overall correlation and increase the diversification benefits; and vice versa. This is especially of importance when an investor considers a portfolio diversification strategy.

In order to compare the proposed MGARCH-jump model with the benchmark MGARCH-N model, predictive likelihoods are computed for both model by rolling-forward out-of-sample forecast. In all six cases, prediction very strongly supports the MGARCH-jump model with log-Bayes factor over the MGARCH-N model. This dominance does not consistently exist over time. Both models predict similarly in normal days, but the MGARCH-jump model is able to predict abnormal events better as opposed to the MGARCH-N model. A risk-averse investor cares more about such events than regular days, and the MGARCH-jump model enables potentially better risk reduction strategies against these events in advance.

Beta dynamics can be extracted based on the MGARCH-jump model from conditional covariance matrices by definition. Beta can be computed either ex-post (conditional on jump arrivals) or ex-ante (only conditional on time). These two methods provide relatively similar estimates, while beta extracted from the MGARCH-N model is more different because of the excess smoothing when ignoring jumps.

The MGARCH-jump model produces very similar predictive Value at Risk as the MGARCH-N model for a five-asset equally-weighted portfolio, and more conservative loss predictions only in the far left tail. Both models provide almost identical predictive VaR at 10% level, and the
MGARCH-jump model starts to show slightly more conservativeness at 5% level. At 1% level, the MGARCH-jump model predicts clearly higher loss potential than the MGARCH-N model. This show that the MGARCH-jump model does not severly overestimate the potential losses at relatively normal periods, while gives more conservative guidelines for truly disastrous events.

References


Table 1: Descriptive Statistics for Daily Returns

<table>
<thead>
<tr>
<th></th>
<th>GE</th>
<th>XOM</th>
<th>WMT</th>
<th>MSFT</th>
<th>AXP</th>
<th>MKT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Means</td>
<td>0.0372</td>
<td>0.0409</td>
<td>0.0421</td>
<td>0.0737</td>
<td>0.0379</td>
<td>0.0353</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>1.7608</td>
<td>1.4729</td>
<td>1.6679</td>
<td>2.0451</td>
<td>2.2313</td>
<td>1.1099</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.0338</td>
<td>0.0657</td>
<td>0.1050</td>
<td>0.0217</td>
<td>0.0039</td>
<td>-0.3445</td>
</tr>
<tr>
<td>Ex. Kurtosis</td>
<td>8.4730</td>
<td>8.7859</td>
<td>3.9253</td>
<td>5.7624</td>
<td>7.8700</td>
<td>8.6008</td>
</tr>
<tr>
<td>Max</td>
<td>17.9844</td>
<td>15.8631</td>
<td>10.5018</td>
<td>17.8692</td>
<td>18.7711</td>
<td>10.8753</td>
</tr>
</tbody>
</table>

*Notes: From January 1, 1990 to December 31, 2016, 6805 observations.*
Table 2: Estimates of Selected Stocks, Corresponding Industry and the Market

\[ r_t = \mu + \epsilon_{1,t} + \epsilon_{2,t}, \quad \epsilon_{1,t} = H_t^{1/2} z_t, \quad z_t \sim NID(0, I), \quad \epsilon_{2,t} = J_t - \mu + \Omega_{B_t} ' \rho \]

\[ H_t = CC' + \alpha \alpha' \circ \epsilon_{t-1} \epsilon_{t-1}' + \beta \beta' \circ H_{t-1}, \quad \epsilon_{t-1} = r_{t-1} - \mu \]

\[ J_t = Y_t \circ B_t, \quad Y_t \sim N(\mu_J, \Sigma_J), \quad B_t \sim \text{multinomial}(1, p) \]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>GE</th>
<th>XOM</th>
<th>WMT</th>
<th>MSFT</th>
<th>AXP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>Mean, 0.95 DI</td>
<td>Mean, 0.95 DI</td>
<td>Mean, 0.95 DI</td>
<td>Mean, 0.95 DI</td>
<td>Mean, 0.95 DI</td>
</tr>
<tr>
<td>( \mu )</td>
<td>( 0.0604 \ (0.0307, 0.0873) )</td>
<td>( 0.0432 \ (0.0301, 0.0683) )</td>
<td>( 0.0261 \ (0.0068, 0.0440) )</td>
<td>( 0.0844 \ (0.0567, 0.1098) )</td>
<td>( 0.0654 \ (0.0446, 0.0877) )</td>
</tr>
<tr>
<td>( \sigma^2 )</td>
<td>( 3.7909 \ (3.0470, 4.6388) )</td>
<td>( 2.1381 \ (1.7064, 2.6699) )</td>
<td>( 4.3986 \ (3.5767, 5.3886) )</td>
<td>( 9.4538 \ (7.7795, 11.4719) )</td>
<td>( 5.9235 \ (4.9261, 7.0697) )</td>
</tr>
<tr>
<td>( \delta^2 )</td>
<td>( 2.5034 \ (2.0738, 3.1586) )</td>
<td>( 1.8704 \ (1.4839, 2.5497) )</td>
<td>( 1.6212 \ (1.2676, 2.0193) )</td>
<td>( 2.5496 \ (2.0470, 3.1482) )</td>
<td>( 3.6641 \ (2.8846, 4.6942) )</td>
</tr>
<tr>
<td>( \alpha^2 )</td>
<td>( 2.3086 \ (1.8405, 2.7742) )</td>
<td>( 1.2182 \ (0.9410, 1.5322) )</td>
<td>( 1.0306 \ (0.7271, 1.3770) )</td>
<td>( 1.7621 \ (1.2730, 2.3417) )</td>
<td>( 3.0789 \ (2.5331, 3.7240) )</td>
</tr>
<tr>
<td>( \beta^2 )</td>
<td>( 1.5230 \ (1.2214, 1.8870) )</td>
<td>( 1.0451 \ (0.8021, 1.3480) )</td>
<td>( 1.0099 \ (0.7873, 1.2793) )</td>
<td>( 1.2506 \ (0.9653, 1.5910) )</td>
<td>( 1.9048 \ (1.5145, 2.3804) )</td>
</tr>
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</table>
Table 3: Jump probabilities for GE, XOM, WMT, MSFT and AXP with corresponding industry and the market

### Panel A: marginal and joint probabilities

<table>
<thead>
<tr>
<th>Probabilities</th>
<th>GE</th>
<th>XOM</th>
<th>WMT</th>
<th>MSFT</th>
<th>AXP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Marginal</td>
<td>0.1539</td>
<td>0.1593</td>
<td>0.1196</td>
<td>0.0928</td>
<td>0.1560</td>
</tr>
<tr>
<td>Industry Marginal</td>
<td>0.1176</td>
<td>0.1569</td>
<td>0.1224</td>
<td>0.1102</td>
<td>0.0984</td>
</tr>
<tr>
<td>Market Marginal</td>
<td>0.1030</td>
<td>0.1010</td>
<td>0.1038</td>
<td>0.0672</td>
<td>0.0833</td>
</tr>
</tbody>
</table>

| Stock and Industry Joint | 0.0984 | 0.1513 | 0.1139 | 0.0908 | 0.0851 |
| (0.0181)                |        |        |        |        |        |
| Stock and Market Joint  | 0.0980 | 0.0953 | 0.0969 | 0.0602 | 0.0800 |
| (0.0159)                |        |        |        |        |        |
| Industry and Market Joint | 0.0986 | 0.0969 | 0.1014 | 0.0650 | 0.0810 |
| (0.0152)                |        |        |        |        |        |
| Stock, Industry and Market Joint | 0.0954 | 0.0943 | 0.0958 | 0.0595 | 0.0787 |
| (0.0019)                |        |        |        |        |        |

*Notes: Numbers in parentheses below the joint probabilities are the product of corresponding marginal probabilities.*

### Panel B: conditional probabilities

<table>
<thead>
<tr>
<th>Stock</th>
<th>Probabilities</th>
<th>Stk,Ind,Mkt</th>
<th>Stk,Ind</th>
<th>Stk,Mkt</th>
<th>Ind,Mkt</th>
</tr>
</thead>
<tbody>
<tr>
<td>GE</td>
<td>$p(co\text{-}jump</td>
<td>mkt\text{-}jump)$</td>
<td>0.9261</td>
<td>—</td>
<td>0.9516</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>ind\text{-}jump)$</td>
<td>0.8110</td>
<td>0.8363</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>stk\text{-}jump)$</td>
<td>0.6198</td>
<td>0.6392</td>
<td>0.6369</td>
</tr>
<tr>
<td>XOM</td>
<td>$p(co\text{-}jump</td>
<td>mkt\text{-}jump)$</td>
<td>0.9338</td>
<td>—</td>
<td>0.9432</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>ind\text{-}jump)$</td>
<td>0.6012</td>
<td>0.9647</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>stk\text{-}jump)$</td>
<td>0.5918</td>
<td>0.9496</td>
<td>0.5977</td>
</tr>
<tr>
<td>WMT</td>
<td>$p(co\text{-}jump</td>
<td>mkt\text{-}jump)$</td>
<td>0.9233</td>
<td>—</td>
<td>0.9337</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>ind\text{-}jump)$</td>
<td>0.7829</td>
<td>0.9300</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>stk\text{-}jump)$</td>
<td>0.8011</td>
<td>0.9517</td>
<td>0.8102</td>
</tr>
<tr>
<td>MSFT</td>
<td>$p(co\text{-}jump</td>
<td>mkt\text{-}jump)$</td>
<td>0.8855</td>
<td>—</td>
<td>0.8950</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>ind\text{-}jump)$</td>
<td>0.5402</td>
<td>0.8240</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>stk\text{-}jump)$</td>
<td>0.6414</td>
<td>0.9783</td>
<td>0.6483</td>
</tr>
<tr>
<td>AXP</td>
<td>$p(co\text{-}jump</td>
<td>mkt\text{-}jump)$</td>
<td>0.9452</td>
<td>—</td>
<td>0.9602</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>ind\text{-}jump)$</td>
<td>0.8004</td>
<td>0.8657</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$p(co\text{-}jump</td>
<td>stk\text{-}jump)$</td>
<td>0.5046</td>
<td>0.5457</td>
<td>0.5126</td>
</tr>
</tbody>
</table>

*Notes: Each column indicates a particular type of co-jumps. For example, column 3 shows conditional probabilities of stock-industry-market co-jumps for each stock.*
Table 4: Jump size correlations for GE, XOM, WMT, MSFT and AXP with corresponding industry and the market

<table>
<thead>
<tr>
<th></th>
<th>GE</th>
<th>IND</th>
<th>MKT</th>
<th>XOM</th>
<th>IND</th>
<th>MKT</th>
</tr>
</thead>
<tbody>
<tr>
<td>GE</td>
<td>1.0000</td>
<td>—</td>
<td>—</td>
<td>XOM</td>
<td>1.0000</td>
<td>—</td>
</tr>
<tr>
<td>IND</td>
<td>0.9070</td>
<td>1.0000</td>
<td>—</td>
<td>IND</td>
<td>0.9140</td>
<td>1.0000</td>
</tr>
<tr>
<td>MKT</td>
<td>0.9441</td>
<td>0.9534</td>
<td>1.0000</td>
<td>MKT</td>
<td>0.8106</td>
<td>0.8865</td>
</tr>
<tr>
<td>WMT</td>
<td>WMT</td>
<td>1.0000</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>IND</td>
<td>0.8029</td>
<td>1.0000</td>
<td>—</td>
<td>IND</td>
<td>0.8232</td>
<td>1.0000</td>
</tr>
<tr>
<td>MKT</td>
<td>0.4890</td>
<td>0.8557</td>
<td>1.0000</td>
<td>MKT</td>
<td>0.5125</td>
<td>0.8649</td>
</tr>
<tr>
<td>AXP</td>
<td>AXP</td>
<td>1.0000</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>IND</td>
<td>0.9127</td>
<td>1.0000</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>MKT</td>
<td>0.9166</td>
<td>0.9207</td>
<td>1.0000</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 5: Estimates among GE, XOM, WMT, MSFT and AXP

Panel A: drift and GARCH parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>GE</th>
<th>XOM</th>
<th>WMT</th>
<th>MSFT</th>
<th>AXP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>0.0313</td>
<td>0.0328</td>
<td>0.0364</td>
<td>0.0672</td>
<td>0.0372</td>
</tr>
<tr>
<td></td>
<td>( 0.0015, 0.0613)</td>
<td>( 0.0043, 0.0605)</td>
<td>( 0.0064, 0.0666)</td>
<td>( 0.0284, 0.1049)</td>
<td>(-0.0010, 0.0756)</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.1507</td>
<td>0.1516</td>
<td>0.1166</td>
<td>0.1363</td>
<td>0.1503</td>
</tr>
<tr>
<td></td>
<td>( 0.1381, 0.1644)</td>
<td>( 0.1389, 0.1654)</td>
<td>( 0.1054, 0.1502)</td>
<td>( 0.1244, 0.1493)</td>
<td>( 0.1391, 0.1627)</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.9832</td>
<td>0.9825</td>
<td>0.9898</td>
<td>0.9855</td>
<td>0.9839</td>
</tr>
<tr>
<td></td>
<td>( 0.9797, 0.9860)</td>
<td>( 0.9791, 0.9853)</td>
<td>( 0.9829, 0.9916)</td>
<td>( 0.9827, 0.9879)</td>
<td>( 0.9812, 0.9862)</td>
</tr>
<tr>
<td>( C )</td>
<td>1 0.0868</td>
<td>2 0.0717</td>
<td>3 0.0516</td>
<td>4 0.0196</td>
<td>5 0.0196</td>
</tr>
<tr>
<td></td>
<td>( 0.0668, 0.1049)</td>
<td>( 0.0371, 0.0737)</td>
<td>( 0.0515, 0.0898)</td>
<td>( 0.0207, 0.0573)</td>
<td>( 0.0366, 0.0713)</td>
</tr>
<tr>
<td></td>
<td>( 0.0207, 0.0573)</td>
<td>( 0.0223, 0.0225)</td>
<td>( 0.0029, 0.0347)</td>
<td>( 0.0019, 0.0593)</td>
<td>( 0.0019, 0.0481)</td>
</tr>
<tr>
<td></td>
<td>0.0521</td>
<td>0.0123</td>
<td>0.0334</td>
<td>0.0206</td>
<td>0.0249</td>
</tr>
<tr>
<td></td>
<td>( 0.0446, 0.0768)</td>
<td>( 0.0067, 0.0274)</td>
<td>( 0.0000, 0.0097)</td>
<td>( 0.0000, 0.0507)</td>
<td>( 0.0014, 0.0535)</td>
</tr>
</tbody>
</table>

Notes: Numbers in parentheses are 0.95 DI of the parameters above them.
Table 5: Estimates among GE, XOM, WMT, MSFT and AXP (Continued)

Panel B: jump/co-jump probabilities

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>0.95 DI</th>
<th>Parameter</th>
<th>Mean</th>
<th>0.95 DI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0170</td>
<td>(0.0076,0.0266)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0018</td>
<td>(0.0001,0.0051)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0022</td>
<td>(0.0001,0.0071)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0022</td>
<td>(0.0001,0.0074)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0042</td>
<td>(0.0003,0.0108)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0087</td>
<td>(0.0008,0.0231)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0024</td>
<td>(0.0001,0.0074)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0010</td>
<td>(0.0000,0.0036)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0021</td>
<td>(0.0001,0.0064)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0038</td>
<td>(0.0001,0.0126)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0030</td>
<td>(0.0001,0.0084)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0040</td>
<td>(0.0003,0.0105)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0031</td>
<td>(0.0001,0.0106)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0298</td>
<td>(0.0122,0.0521)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0020</td>
<td>(0.0001,0.0070)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0471</td>
<td>(0.0300,0.0660)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0021</td>
<td>(0.0001,0.0068)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0023</td>
<td>(0.0001,0.0076)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0020</td>
<td>(0.0001,0.0064)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0441</td>
<td>(0.0001,0.0143)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0030</td>
<td>(0.0001,0.0108)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0018</td>
<td>(0.0001,0.0059)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0037</td>
<td>(0.0001,0.0129)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0342</td>
<td>(0.0185,0.0496)</td>
</tr>
<tr>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.0241</td>
<td>(0.0115,0.0357)</td>
<td>$p_{GE,XOM,WMT,MSFT,AXP}$</td>
<td>0.7102</td>
<td>(0.6728,0.7454)</td>
</tr>
</tbody>
</table>

Panel C: jump size parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>GE</th>
<th>XOM</th>
<th>WMT</th>
<th>MSFT</th>
<th>AXP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_j$</td>
<td>0.0409</td>
<td>-0.2288</td>
<td>0.0713</td>
<td>0.4148</td>
<td>0.0279</td>
</tr>
<tr>
<td></td>
<td>(-0.1949,0.2692)</td>
<td>(-0.4219,-0.0489)</td>
<td>(-0.1186,0.2588)</td>
<td>(0.1857, 0.6384)</td>
<td>(-0.2911,0.3439)</td>
</tr>
<tr>
<td>GE</td>
<td>6.7958</td>
<td>3.0414</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XOM</td>
<td>(4.6207, 8.8016)</td>
<td>(0.1584, 5.4529)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WMT</td>
<td>5.9970</td>
<td>2.1946</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSFT</td>
<td>6.8466</td>
<td>6.3338</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AXP</td>
<td>10.1653</td>
<td>8.3923</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Notes: Numbers in parentheses are 0.95 DI of the parameters above them.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 6: Jump probabilities among GE, XOM, WMT, MSFT and AXP

Panel A: marginal and joint probabilities

<table>
<thead>
<tr>
<th>Stock</th>
<th>GE</th>
<th>XOM</th>
<th>WMT</th>
<th>MSFT</th>
<th>AXP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marginal probs</td>
<td>0.0832</td>
<td>0.0907</td>
<td>0.1047</td>
<td>0.1140</td>
<td>0.0875</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Co-jump</th>
<th>Joint Pr</th>
<th>Product</th>
<th>Co-jump</th>
<th>Joint Pr</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>All jump</td>
<td>0.0170</td>
<td>0.0000</td>
<td>GE,MSFT</td>
<td>0.0067</td>
<td>0.0001</td>
</tr>
<tr>
<td>GE,XOM,WMT,MSFT</td>
<td>0.0022</td>
<td>0.0000</td>
<td>GE,AXP</td>
<td>0.0037</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,XOM,WMT,AXP</td>
<td>0.0034</td>
<td>0.0000</td>
<td>XOM,WMT,MSFT,AXP</td>
<td>0.0018</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,XOM,WMT</td>
<td>0.0042</td>
<td>0.0000</td>
<td>XOM,WMT,MSFT</td>
<td>0.0018</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,XOM,MSFT,AXP</td>
<td>0.0024</td>
<td>0.0000</td>
<td>XOM,WMT,AXP</td>
<td>0.0022</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,XOM,WMT</td>
<td>0.0021</td>
<td>0.0000</td>
<td>XOM,MSFT</td>
<td>0.0087</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,XOM,AXP</td>
<td>0.0030</td>
<td>0.0000</td>
<td>XOM,MSFT,AXP</td>
<td>0.0010</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,XOM</td>
<td>0.0031</td>
<td>0.0000</td>
<td>XOM,MSFT</td>
<td>0.0038</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,WMT,MSFT,AXP</td>
<td>0.0020</td>
<td>0.0000</td>
<td>XOM,AXP</td>
<td>0.0040</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,WMT,MSFT</td>
<td>0.0021</td>
<td>0.0000</td>
<td>WMT,MSFT,AXP</td>
<td>0.0017</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,WMT,AXP</td>
<td>0.0020</td>
<td>0.0000</td>
<td>WMT,MSFT</td>
<td>0.0023</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,WMT</td>
<td>0.0020</td>
<td>0.0001</td>
<td>WMT,AXP</td>
<td>0.0041</td>
<td>0.0000</td>
</tr>
<tr>
<td>GE,MSFT,AXP</td>
<td>0.0030</td>
<td>0.0000</td>
<td>MSFT,AXP</td>
<td>0.0018</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Notes: Column “Product” is product of corresponding marginal probabilities.
Table 6: Jump probabilities among GE, XOM, WMT, MSFT and AXP (Continued)

Panel B: conditional probabilities

<table>
<thead>
<tr>
<th>Type of co-jumps</th>
<th>Conditional on</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GE-jump</td>
</tr>
<tr>
<td>All jump</td>
<td>0.2041</td>
</tr>
<tr>
<td>GE,XOM,WMT,MSFT</td>
<td>0.0259</td>
</tr>
<tr>
<td>GE,XOM,WMT,AXP</td>
<td>0.0409</td>
</tr>
<tr>
<td>GE,XOM,WMT</td>
<td>0.0507</td>
</tr>
<tr>
<td>GE,XOM,MSFT,AXP</td>
<td>0.0294</td>
</tr>
<tr>
<td>GE,XOM,MSFT</td>
<td>0.0255</td>
</tr>
<tr>
<td>GE,XOM,AXP</td>
<td>0.0365</td>
</tr>
<tr>
<td>GE,XOM</td>
<td>0.0378</td>
</tr>
<tr>
<td>GE,WMT,MSFT,AXP</td>
<td>0.0243</td>
</tr>
<tr>
<td>GE,WMT,MSFT</td>
<td>0.0257</td>
</tr>
<tr>
<td>GE,WMT,AXP</td>
<td>0.0240</td>
</tr>
<tr>
<td>GE,WMT</td>
<td>0.0243</td>
</tr>
<tr>
<td>GE,MSFT,AXP</td>
<td>0.0366</td>
</tr>
<tr>
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<td>0.0806</td>
</tr>
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<td>GE,AXP</td>
<td>0.0445</td>
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<tr>
<td>XOM,WMT,MSFT,AXP</td>
<td></td>
</tr>
<tr>
<td>XOM,WMT,MSFT</td>
<td></td>
</tr>
<tr>
<td>XOM,WMT,AXP</td>
<td></td>
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<td>XOM,MSFT</td>
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</tr>
<tr>
<td>WMT,MSFT,AXP</td>
<td></td>
</tr>
<tr>
<td>WMT,MSFT</td>
<td></td>
</tr>
<tr>
<td>WMT,AXP</td>
<td></td>
</tr>
<tr>
<td>MSFT,AXP</td>
<td></td>
</tr>
</tbody>
</table>

Notes: Each number above is a conditional probability of a particular co-jump type defined by row conditional on a particular univariate jump defined by column.

Table 7: Jump size correlations among GE, XOM, WMT, MSFT and AXP

<table>
<thead>
<tr>
<th>Stock</th>
<th>GE</th>
<th>XOM</th>
<th>WMT</th>
<th>MSFT</th>
<th>AXP</th>
</tr>
</thead>
<tbody>
<tr>
<td>GE</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XOM</td>
<td>0.5403</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WMT</td>
<td>0.8792</td>
<td>0.3884</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSFT</td>
<td>0.7986</td>
<td>0.6603</td>
<td>0.7592</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>AXP</td>
<td>0.8296</td>
<td>0.5874</td>
<td>0.7286</td>
<td>0.8509</td>
<td>1.0000</td>
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<td></td>
<td>MGARCH-jump</td>
<td>MGARCH-N</td>
<td>Log-Bayes factor</td>
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<tr>
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<tr>
<td>GE, XOM, WMT, MSFT, AXP</td>
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<td>-1653.5347</td>
<td>67.4276</td>
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Notes: A log-Bayes factor greater than 6 shows that prediction favours MGARCH-jump model.
Figure 1: Jump arrivals for GE, XOM, WMT, MSFT and AXP with corresponding industry and the market
Figure 2: Jump sizes for GE, XOM, WMT, MSFT and AXP with corresponding industry and the market
Figure 3: Jump probabilities and sizes for AXP over time from Jan 1, 2007 to Dec 31, 2009
Figure 4: Jump probabilities and sizes for AXP crossover from Jan 1, 2007 to Dec 31, 2009
Figure 5: Log-predictive likelihoods of rolling-forward forecasts
Figure 6: Impact of Jumps on Correlation Dynamics over GARCH Component
Figure 7: beta dynamics for AXP computed from MGARCH-jump model and MGARCH-N model

Figure 8: Predictive Value at Risk over time for a five-asset equally-weighted portfolio
A Proof of Conditional Moments of $J_t$

Proof. First, prove $E(J_t|\Theta, F_{t-1})$:

$$E(J_t|\Theta, F_{t-1}) = \mu_J \circ E(B_t|\Theta, F_{t-1})$$

$$E(B_t|\Theta, F_{t-1}) = \Omega_B'p = \sum_{j=1}^{2N} B_t^{(j)}p_j$$

Then, prove $Cov(J_t|\Theta, F_{t-1})$:

$$Cov(J_t|\Theta, F_{t-1}) = E[(Y_t \circ B_t)(Y_t \circ B_t)'|\Theta, F_{t-1}]$$

$$- E(Y_t \circ B_t|\Theta, F_{t-1}) E(Y_t \circ B_t'|\Theta, F_{t-1})'$$

$$= E(Y_tY_t'|B_tB_t'|\Theta, F_{t-1}) - (\mu_J \circ \Omega_B') (\mu_J \circ \Omega_B'p)'$$

$$= E(Y_tY_t'|B_tB_t'|\Theta, F_{t-1}) - \mu_J \mu_J' \circ \Omega_B'pp'\Omega_B$$

$$= E(Y_tY_t'|B_tB_t'|\Theta, F_{t-1}) = E\{E(Y_tY_t'|B_tB_t'|B_t, \Theta)|F_{t-1}\}$$

$$= E\{E(Cov(Y_t \circ B_t|B_t, \Theta) + E(Y_t \circ B_t|B_t, \Theta) E(Y_t \circ B_t|B_t, \Theta)'|F_{t-1}\}$$

$$= E[\Sigma_J \circ B_tB_t' + \mu_J \mu_J' \circ B_tB_t'|\Theta, F_{t-1}]$$

$$= (\Sigma_J + \mu_J \mu_J') \circ E(B_tB_t'|\Theta, F_{t-1})$$

$$= (\Sigma_J + \mu_J \mu_J') \circ \left(\sum_{j=1}^{2N} p_j \Omega_J j'\right) \qed$$

B Sampling Details

In each MCMC iteration,

1. $\mu|r_{1:T}, H_{1:T}, \mu_J, \Sigma_J, B_{1:T}, p$. With everything else given, it’s nothing more than a linear model with normal innovation, so the standard conjugate Gibbs result can be applied. Assuming $\mu$ has a normal prior $N(b_\mu, B_\mu)$, let $T_\mu = B_\mu^{-1}$, then

$$\mu|r_{1:T}, H_{1:T}, \mu_J, \Sigma_J, B_{1:T}, p \sim N(M_\mu, V_\mu)$$

$$M_\mu = V_\mu \left[ \sum_{t=1}^{T} (H_t + B_tB_t' \circ \Sigma_J)^{-1} r_t^* + T_\mu b_\mu \right]$$

$$V_\mu = \left[ \sum_{t=1}^{T} (H_t + B_tB_t' \circ \Sigma_J)^{-1} + T_\mu \right]^{-1}$$

where $r_t^* = r_t - \mu_J \circ (B_t - \Omega_B'p)$. 

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2. $\theta_H|r_{1:T}, \mu, \mu_J, \Sigma_J, B_{1:T}, p$, where $\theta_H = (C, \alpha, \beta)'$. The posterior is

$$p(\theta_H|r_{1:T}, \mu, \mu_J, \Sigma_J, B_{1:T}, p) \propto \prod_{t=1}^{T} p(r_t|\mu, H_t, \mu_J, \Sigma_J, B_t, p) p(\theta_H)$$

$$r_t|\mu, H_t, \mu_J, \Sigma_J, B_t, p \sim N(\mu + \mu_J \odot (B_t - \Omega_B'p), H_t + B_t B_t' \odot \Sigma_J)$$

where $H_t$ follows equation (6). Apply a standard random-walk Metropolis-Hastings (MH) algorithm.

3. $B_t|r_t, \mu, H_t, \mu_J, \Sigma_J, p$. There are $2^N$ different possible realizations of $B_t$, and the posterior is

$$p(B_t|p) = \prod_{i=1}^{2^N} p^{x_i}_{t,i}$$

where $x_i = \delta(B_t, \Omega_i)$. Here, $x_i$ indicates whether the $i$th row of $\Omega_B, \Omega_t$, is realized.

4. $p|r_{1:T}, \mu, H_{1:T}, \mu_J, \Sigma_J, B_{1:T}$. Assuming $p$ has a Dirichlet prior $\text{Dir}(a_1, \ldots, a_{2^N})$, the posterior is

$$p(p|r_{1:T}, \mu, H_{1:T}, \mu_J, \Sigma_J, B_{1:T}) \propto \prod_{t=1}^{T} p(r_t|\mu, H_t, \mu_J, \Sigma_J, B_t, p) p(B_{1:T}|p) p(p)$$

An asymmetric MH sampler instead of Gibbs need be applied. Since $B_t$'s are iid conditional on $p_t$, one asymmetric proposal density is the conjugate posterior of multinomial distribution:

$$p' \sim \text{Dir} \left( a_i + \sum_{t=1}^{T} x_{t,i} \right), \quad i \in \{1, \ldots, 2^N\}$$

and accept $p'$ with probability

$$\alpha(p^{(i)}, p') = \min \left\{ 1, \frac{\prod_{t=1}^{T} p(r_t|\mu, H_t, \mu_J, \Sigma_J, B_t', p')}{\prod_{t=1}^{T} p(r_t|\mu, H_t, \mu_J, \Sigma_J, B_t, p^{(i)})} \right\}$$

5. $Y_t|r_t, \mu, H_t, \mu_J, \Sigma_J, B_t, p$. After simple transformation, conjugate Gibbs result can be applied:

$$Y_t|r_t, \mu, H_t, \mu_J, \Sigma_J, B_t, p \sim N(M_{Y,t}, V_{Y,t})$$
where
\[
M_{Y,t} = V_{Y,t} \left[ B_t \odot H_t^{-1}(r_t - \mu + \mu_J \odot \Omega_E p) + \Sigma_J^{-1}\mu_J \right]
\]
\[
V_{Y,t} = (B_t B_t' \odot H_t^{-1} + \Sigma_J^{-1})^{-1}
\]

6. \( \mu_J | r_{1:T}, \mu, H_{1:T}, \Sigma_J, Y_{1:T}, B_{1:T}, p \). Assume a prior of \( \mu_J \sim N(b_{\mu_J}, B_{\mu_J}) \), then the posterior is
\[
p(\mu_J | r_{1:T}, \mu, H_{1:T}, \Sigma_J, Y_{1:T}, B_{1:T}, p)
\]
\[
\propto \prod_{t=1}^{T} p(r_t | \mu, H_t, \mu_J, \Sigma_J, B_t, p) p(Y_{1:T} | \mu_J, \Sigma_J) p(\mu_J)
\]

Similarly, a conjugate proposal density can be applied:
\[
\mu_J' \sim N(M_{\mu_J}, V_{\mu_J})
\]
\[
M_{\mu_J} = V_{\mu_J} \left( \Sigma_J^{-1} \sum_{t=1}^{T} Y_t + B_{\mu_J}^{-1}b_{\mu_J} \right)
\]
\[
V_{\mu_J} = (T\Sigma_J^{-1} + B_{\mu_J}^{-1})^{-1}
\]

accept \( \mu_J' \) with probability
\[
\alpha(\mu_J^{(i)}, \mu_J') = \min \left\{ 1, \frac{\prod_{t=1}^{T} p(r_t | \mu, H_t, \mu_J', \Sigma_J, B_t, p)}{\prod_{t=1}^{T} p(r_t | \mu, H_t, \mu_J^{(i)}, \Sigma_J, B_t, p)} \right\}
\]

7. \( \Sigma_J | \mu_J, Y_{1:T} \). Assume a prior of \( \Sigma_J \sim IW(\nu_p, V_p) \), then apply the standard conjugate Gibbs result
\[
\Sigma_J | \mu_J, Y_{1:T} \sim IW(\nu_J, V_J)
\]
\[
\nu_J = T + \nu_p
\]
\[
V_J = \sum_{t=1}^{T} (Y_t - \mu_J)(Y_t - \mu_J)' + V_p
\]