Asset Pricing Implications of Short-sale Constraints in Imperfectly Competitive Markets*

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We propose an equilibrium model to study the impact of short-sale constraints on market prices and liquidity in imperfectly competitive markets where market makers have significant market power and are averse to inventory risk. We show that short-sale constraints decrease bid because of the market power and increase ask because of the risk aversion. Our model can therefore help explain why short-sale constraints tend to increase bid-ask spread. In addition, short-sale constraints also decrease quote depths and trading volume, but increase volatility of bid-ask spreads. The adverse impact of short-sale constraints on market prices and liquidity is greater in more transparent markets. Our results are largely unaffected by endogenization of information acquisition.

*JEL Classification Codes: G11, G12, G14, D82.

Keywords: Short-sale Constraints, Bid-Ask Spread, Market Liquidity, Imperfect Competition.

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Abstract

We propose an equilibrium model to study the impact of short-sale constraints on market prices and liquidity in imperfectly competitive markets where market makers have significant market power and are averse to inventory risk. We show that short-sale constraints decrease bid because of the market power and increase ask because of the risk aversion. Our model can therefore help explain why short-sale constraints tend to increase bid-ask spread. In addition, short-sale constraints also decrease quote depths and trading volume, but increase volatility of bid-ask spreads. The adverse impact of short-sale constraints on market prices and liquidity is greater in more transparent markets. Our results are largely unaffected by endogenization of information acquisition.

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1. Introduction

Competitions in many financial markets are imperfect, even with the introduction of electronic trading platforms (e.g., Christie and Schultz (1994) and Biais, Bisi`ere and Spatt (2010)). Implicit and explicit short-sale constraints are prevalent in many financial markets, especially in imperfectly competitive ones such as over-the-counter markets. There is a vast literature on how short-sale constraints affect portfolio choice, market prices, and information efficiency in competitive markets.\textsuperscript{1} However, as far as we know, theoretical analysis on how short-sale constraints affect asset pricing and market liquidity, especially bid-ask spreads and depths, in imperfectly competitive markets is still absent. We attempt to fill this gap in this paper. Our analysis highlights that the impact of short-sale constraints on asset prices critically depends on buyers' market power and risk aversion toward inventory, and helps explain why short-sale constraints can significantly increase bid-ask spreads.\textsuperscript{2}

We solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed-form in a market with a monopolistic, risk averse market maker in the presence of short-sale constraints and asymmetric information. We find if the market maker has significant market power, short-sale constraints increase bid-ask spread and decrease bid, market depths and trading volume. In addition, as long as the imposition of short-sale constraints does not signal strongly negative information and the market maker is risk averse, short-sale constraints also increase ask. Furthermore, short-sale constraints increase volatility of the bid-ask spread. More public disclosure may increase the adverse impact of short-sale constraints on market liquidity. Our main results still hold for a large set of parameter values when the effect of short-sale constraints on information acquisition is taken into account.

Specifically, we consider an equilibrium model with two trading dates and three types of risk averse investors: informed investors, uninformed investors, and an uninformed market maker. On date 0, all investors optimally choose how to trade a


\textsuperscript{2}See, for example, Beber and Pagano (2013), Boehmer, Jones and Zhang (2013), Ang, Shtauber, and Tetlock (2013).
risk-free asset and a risky security (e.g., a stock in an imperfectly competitive market, a corporate bond, or a derivative security) to maximize their constant absolute risk aversion utility from the terminal wealth on date 1. Informed investors observe a private signal about the date 1 payoff of the security just before trading on date 0 and thus have trading demand motivated by private information. Informed investors also have non-information-based incentives to trade, which we term as a liquidity shock and model as a random endowment of a nontradable asset whose payoff is correlated with that of the risky security.\(^3\) The uninformed and the informed must trade through the market maker, possibly due to high search costs.

Because short-sale constraints restrict selling at the bid, one may expect that the equilibrium selling (bid) price increases, as found in the existing literature (e.g., Miller (1977), Wang (2014)), which is exactly the opposite to what we find. One key difference of our model from the existing literature is that competition is imperfect in our model. To pinpoint imperfect competition as the cause of the opposite result, we show in Theorems 2 and 3 in Appendix B that keeping everything else the same as in our model, if market makers did not have market power, then short-sale constraints would indeed increase equilibrium bid prices. The intuition for our opposite result is as follows. Suppose without short-sale constraints, the short-seller short-sells 10 shares (at the bid) in equilibrium and with the short-sale constraints, the short-seller can only short 5 shares. Because the number of shares the short-seller chooses to short decreases as the bid price decreases, a market maker with market power can lower the bid price to the level at which the constraints just start to bind (i.e., at this bid price, the short-seller shorts 5 shares when unconstrained). By doing this, the market maker pays a lower price for the shares without any adverse impact on the number of shares she can buy (still 5 shares). Therefore, because of the market power of the market maker, the equilibrium bid price is lower with the short-sale constraints. More generally, when some investors are restricted from selling more, if buyers do not have market power, then they will compete for the limited supply and thus drive up the prices.\(^3\)

\(^3\)These liquidity motivated trades can also be viewed as from other liquidity traders that have the same trading direction as the informed and are necessary for private information to be not fully revealed in equilibrium.
equilibrium trading price, as found in the existing literature with competitive markets. On the other hand, if buyers have significant market power, then the equilibrium price goes down, as found in this paper, because a lower price makes buyers better off. This shows how short-sale constraints affect the price at which short-sale occurs (bid price) critically depends on whether buyers have significant market power.

Because the market maker buys less from short-sellers as a result of the short-sale constraints, to achieve optimal inventory risk exposure, she prefers to sell less at the ask, and thus she posts a higher ask price. The simplest example to explain the intuition is when the market maker is infinitely risk averse. In this case, the market maker does not carry any inventory (and makes profit only from the spread). Therefore, when her purchase at bid decreases as a result of short-sale constraints, she also decreases the sale and charges a higher ask price. If the market maker is risk neutral, then the short-sale constraints do not affect ask price or ask depth, but still lower bid and bid depth and increase spread. This is because the change in the inventory risk due to the short-sale constraints is irrelevant for a risk neutral market maker and the same intuition for the determination of the bid price and bid depth still applies. This shows how short-sale constraints affect ask price and ask depth critically depends on a market maker’s risk aversion. Therefore, while the impact of short-sale constraints on the bid price is through the market maker’s market power, the impact on the ask price and the ask depth is through the market maker’s aversion to inventory risk. To the extent that for small stocks, markets are less competitive and inventories are riskier, our results are consistent with the empirical evidence that bid-ask spreads increase more for small-cap stocks as a result of short-sale bans (e.g., Beber and Pagno (2013)). Because short-sale constraints increase spread and decrease trading volume, they reduce market liquidity. In addition, with short-sale constraints, bid and ask prices become more sensitive to shocks in the economy because risk sharing is reduced by the constraints. Thus short-sale constraints also make the bid-ask spread more volatile.

More public disclosure about asset payoff reduces overall risk, increases investors’ trading demand, and thus makes short-sale constraints bind more often and have a greater effect. As a result, the adverse impact of short-sale constraints on prices and
market liquidity may be greater in more transparent markets. On the other hand, even with significant information asymmetry, our main results still hold.\textsuperscript{4}

Because the imposition of short-sale constraints may change the benefit of private information, we further study whether endogenizing information acquisition invalidates our main results. To this end, we assume that the cost for the private signal about the risky security payoff is an increasing and convex function of the signal’s precision. We find that for a large set of parameter values, our main results, such as the increase in the expected spread and spread volatility and the decrease in trading volume, remain valid.

As the existing literature on the impact of short-sale constraints, we do not take into account the possibility that imposition of short-sale constraints itself may convey negative information about the stock payoff. However, the effect of this negative signal is clear from our model: It decreases both bid and ask prices. Therefore, while the result that short-sale constraints increase ask price might be reversed if this negative information effect dominates, the result that short-sale constraints decrease bid price would be strengthened. In addition, if the negative information effect lowers bid and ask by a similar amount, the result that short-sale constraints increase spread would still hold.

To our knowledge, Diamond and Verrecchia (1987) (hereafter DV) is the only theoretical paper in the existing literature that considers the effect of short-sale constraints on bid-ask spreads. They show that when both the informed and the uninformed are prohibited from short selling, conditional on the same expected payoff, neither the bid nor the ask changes, and thus the bid-ask spread stays the same (see Corollary 2 in DV). Like most of the rational expectations models in market microstructure literature (e.g., Glosten and Milgrom (1985), Admati and Pfleiderer (1988)), DV consider a perfect competition market with risk neutral market makers, and thus impose a zero expected profit condition for each trade of a market maker. However, there are other

\textsuperscript{4}One of the main difference of our model from the existing literature on the impact of information asymmetry is that some agent (i.e., the market maker) can adjust trading prices to induce other discretionary investors to share the adverse selection problem from the agent’s trade with the informed. As a result, without short-sale constraints, the expected bid-ask spread can decrease in information asymmetry, as supported by some empirical studies (e.g., Brooks (1996), Huang and Stoll (1997), Acker, Stalker and Tonks (2002), Acharya and Johnson (2007)).
Table 1: Comparison of Predictions on Average Bid, Ask and Spread

<table>
<thead>
<tr>
<th>Cases</th>
<th>This Paper</th>
<th>Diamond and Verrecchia (1987)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Bid</td>
<td>Ask</td>
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<tr>
<td>Base case to Case 1</td>
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<td>Case 2 to Case 3</td>
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</tbody>
</table>

Base case: Unconstrained; Case 1: Short-sale prohibition for both the informed and the uninformed; Case 2: Only the informed can short; Case 3: Only the informed can short, conditional on shorting.

markets where competition among market makers are imperfect (e.g., Christie and Schultz (1994) and Biais, Bisière and Spatt (2010)), market makers are risk-averse to carrying inventory (e.g., Garman (1976), Lyons (1995)), and market makers can make offsetting trades to avoid significant inventory position (e.g., Sofianos (1993), Shachar (2012)). Our model focuses on these markets and generates several empirically testable implications that differ from the existing literature. Table 1 summarizes the difference that may help empirical tests differentiate our model and DV. For example, Table 1 implies that if before a short-sale ban it is mostly the informed who can short, then after imposing a short-sale ban DV implies that bid price goes up (because of a sell order contains less bad information), the ask stays the same, and thus the bid-ask spread decreases, while our model predicts that bid price goes down, ask price goes up and so does spread.

The remainder of the paper proceeds as follows. We first provide background for short-sale constraints in imperfectly competitive markets and additional literature review in the next section. In Section 3 we present the model. In Section 4 we derive the closed-form equilibrium results. In Section 5 we examine the effect of short-sale constraints on bid/ask prices, bid-ask spread and liquidity risk with and without endogenous information acquisition. We conclude in Section 6. All proofs are in Appendix A. In Appendix B, we provide related results for the competitive case to identify the source of difference of our results from those in the literature.
2. Short-sale constraints in imperfectly competitive markets and related literature

Competitions in many financial markets such as those for small stocks and OTC securities are imperfect. For example, Christie and Schultz (1994) suggest that Nasdaq dealers may implicitly collude to maintain wide spread. Biais, Bisière and Spatt (2010) analyze trades and order placement on Nasdaq and a competing electronic order book, Island. They conclude that competitions among market makers in these markets are still imperfect even after the introduction of electronic markets. The opaqueness and illiquidity of OTC markets make them even less competitive (e.g., Ang, Shtauber, and Tetlock (2013)).

Because there tend to be less liquidity and less trading volume in imperfectly competitive markets, implicit and explicit short-sale constraints are more prevalent in these markets. For example, short selling of small stocks and OTC stocks is difficult and rare, possibly due to low ownership by market makers and institutions (the main lenders of shares), which leads to high short-sale costs. Even though we model short-sale constraints in the form of explicit short position limit instead of short-sale costs, the qualitative results of these two alternative models are the same. This is because as the short-sale costs increase, the amount of short-sales goes down, and thus the same qualitative effect as short position limit arises. In addition, explicit short-sale constraints are also often imposed by market making firms in many imperfectly competitive markets (e.g., OTC markets). For example, Ang, Shtauber, and Tetlock (2013) collect short selling data for a sample of 50 OTC stocks and 50 similarly-sized (small) listed stocks in June 2012 and find “A retail customer of Fidelity could buy all 100 of these stocks, but the broker would allow short selling in only one of the OTC stocks and eight of the listed stocks.”

The Dodd-Frank Wall Street Reform and Consumer Protection Act will bring OTC derivatives markets, swap dealers and other market participants under the regulatory oversight of the Securities and Exchange Commission and the Commodity Futures Trading Commission (CFTC). In January 2011, the CFTC proposed a rule to establish both long and short position limits for metals, agricultural and energy derivatives, including contracts traded in the related OTC markets. As it will become clear, although our main model focuses on short-sale constraints (in the form of short position limit), the main results apply also to positive position limit. For example, both short position and long position limits drive down bid and trading volume, drive up ask and spread when the
Although explicit or implicit short-sale constraints are usually more stringent in imperfectly competitive markets, there is still significant outstanding short interest in these markets. For example, Asquith, Au, Covert, and Pathak (2013) find that between 2004 and 2007, trading activity averaged $17.3 billion per day with shorting represents 19.1% of all corporate bond trades. In addition, the stringency of short-sale constraints varies significantly across securities, as can be inferred from variations in explicit short-sale constraints and the cross sectional variation of security lending costs. This suggests the importance of understanding the impact of short-sale constraints in these markets and the feasibility of testing predictions of various theories in these markets.

There is a vast literature on the impact of short-sale constraints on asset prices such as Miller (1977), Harrison and Kreps (1978), Scheinkman and Xiong (2003), and Wang (2014). These models focus on competitive markets and find that short-sale constraints drive up trading prices. As our model, Hong and Stein (2003) and Bai et. al. (2006) show that short-sale constraints can cause trading prices to go down. However, the driving force in Hong and Stein (2003) and Bai et. al. (2006) is the assumption that short-sale constraints prevent some investors from revealing negative information. With less information revealed, uncertainty increases, demand for assets decreases, and thus prices may go down when this uncertainty effect dominates the effect of reduced sales, as shown in Bai et. al. (2006). When the negative information initially prevented from revealing is revealed later, prices decrease, as shown in Hong and Stein (2003). In contrast, the driving force behind our result that short-sale constraints can lower trading prices is buyers’ market power and therefore our result holds even when there is no information asymmetry. In addition, neither Hong and Stein (2003) nor Bai et. al. (2006) consider the impact of short-sale constraints on ask price or bid-ask spreads. Duffie, Gărleanu, and Pedersen (2002) consider the effect of lending fee on asset prices in a random matching model and find that the prospect of lending fee can push the initial price higher than even the most optimistic evaluation. Although Liu and Wang (2014) use a similar setting to this paper, short-sale constraints are absent and thus they are silent on the impact of constrained do not have market power.
short-sale constraints on market prices and liquidity. Nezafat, Schroder, and Wang (2014) consider a partial equilibrium model with endogenous information acquisition and short-sale constraints. In contrast to our model, they do not study the impact of short-sale constraints on equilibrium asset prices or bid-ask spreads, and there are no strategic traders in their model.

3. The model

We consider a one period setting with trading dates 0 and 1. There are a continuum of identical informed investors with mass $N_I$, a continuum of identical uninformed investors with mass $N_U$, and $N_M = 1$ designated market maker ($M$).\footnote{The results for the case with multiple market makers and oligopolistic competition are qualitatively the same and are available from the authors.} They can trade one risk-free asset and one risky security on date 0 and date 1 to maximize their constant absolute risk aversion (CARA) utility from the terminal wealth on date 1. There is a zero net supply of the risk-free asset, which also serves as the numeraire and thus the risk-free interest rate is normalized to 0. The total supply of the security is $N \times \bar{\theta} \geq 0$ shares where $N = N_I + N_U + N_M$ and the date 1 payoff of each share is $\tilde{V} = \tilde{v} + \tilde{u}$, where $\tilde{v} \sim N(\bar{v}, \sigma_v^2)$, $\tilde{u} \sim N(0, \sigma_u^2)$, $\tilde{v}$ and $\tilde{u}$ are independent, $\bar{v}$ is a constant, $\sigma_v > 0$, $\sigma_u > 0$ and $N(\cdot)$ denotes the normal distribution. The separate roles of $\tilde{v}$ and $\tilde{u}$ will become clear below. The aggregate risky asset endowment is $N_i\bar{\theta}$ shares for type $i \in \{I, U, M\}$ investors, but no investor is endowed with any risk-free asset.\footnote{Given the CARA preferences, having different cash endowment would not change any of the results.}

On date 0, informed investors observe a private signal

$$\hat{s} = \hat{v} - \tilde{v} + \tilde{\varepsilon} \quad (1)$$

about the payoff $\tilde{v}$, where $\tilde{\varepsilon}$ is independently normally distributed with mean zero and variance $\sigma_{\tilde{\varepsilon}}^2$.\footnote{Throughout this paper, “bar” variables are constants, “tilde” random variables are realized on date 1 and “hat” random variables are realized on date 0. Observing the private signal may also be reinterpreted as extracting more precise information from public news (e.g., Engelberg, Reed, and Ringgenberg (2012)).} Informed investors are also subject to a liquidity shock that is modeled
as a random endowment of \( \hat{X}_I \sim N(0, \sigma_X^2) \) units of a non-tradable risky asset on date 0, with \( \hat{X}_I \) realized on date 0 and only known to informed investors.\(^9\) The non-tradable asset has a per-unit payoff of \( \hat{N} \sim N(0, \sigma_N^2) \) that has a covariance of \( \sigma_{uN} \) with the second component of the payoff of the risky asset \( \hat{u} \) and is independent of the first component \( \hat{v} \). The payoff of the nontradable asset is realized and becomes public on date 1. The correlation between the non-tradable asset and the security results in a liquidity demand for the risky asset to hedge the non-tradable asset payoff. We assume that the payoff of the nontraded asset to be independent of the first component \( \hat{v} \) so that private information about the security payoff does not affect the hedging demand and thus information motivated trades are separated from hedging motivated trades. This way, the informed’s trades can also be viewed as pooled trades from pure information traders and pure liquidity traders.

In addition, we assume that there is a public signal

\[
\hat{S}_s = \hat{s} + \hat{\eta}
\]

about the informed’s private signal \( \hat{s} \) that all investors (i.e., the uninformed, the designated market maker, and the informed) can observe, where \( \hat{\eta} \) is independently normally distributed with mean zero and volatility \( \sigma_{\eta} > 0 \). This public signal represents public disclosure about the asset payoff determinants, such as macroeconomic conditions, cash flow news and regulation shocks, which is correlated with but less precise than the informed’s private signal. While this additional signal \( \hat{S}_s \) is not critical for our main results, it has three main benefits as explained in the subsection below.

All trades go through the designated market maker (dealer) whose market making cost is assumed to be 0 possibly because the counterparty search cost is high. Specifically, \( I \) and \( U \) investors sell to the designated market maker at the bid \( B \) or buy from her at the ask \( A \) or do not trade at all. Given that there is a continuum of informed and uninformed investors, we assume that they are price takers and there are no strategic interactions among them or with the designated market maker. We

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\(^9\)The random endowment can represent any shock in the demand for the security, such as a liquidity shock or a change in the needs for rebalancing an existing portfolio or a change in a highly illiquid asset.
assume that both $I$ and $U$ investors are subject to short-sale constraints, i.e., the signed order size $\theta_i \geq -\lambda_i \bar{\theta}$, $i = I, U$, where $\lambda_i \geq 1$ can be different for the informed and the uninformed, a smaller $\lambda_i$ means a more stringent short-sale constraint. If $\lambda_i = 1$, then type $i$ investors are prohibited from short selling. Heterogeneous short-sale constraint stringency for the informed and the uninformed captures the essence of possibly different short-sale costs across them (e.g., Kolasinski, Reed and Ringgenberg (2013)).

The market maker posts her price schedules first. Then informed and uninformed investors decide how much to trade. When deciding on what price schedules to post, the market maker takes into account the best response functions (i.e., the demand schedules) of the informed and the uninformed given the posted price schedules. For each type $i \in \{I, U, M\}$, investors of type $i$ are ex ante identical. Accordingly, we restrict our analysis to symmetric equilibria where all investors of the same type adopt the same trading strategy. Let $\mathcal{I}_i$ represent a type $i$ investor’s information set on date $0$ for $i \in \{I, U, M\}$.

For $i \in \{I, U\}$, a type $i$ investor’s problem is to choose the (signed) demand schedule $\theta_i(A, B)$ to solve
\[
\max E[-e^{-\delta \tilde{W}_i(I)}],
\]
where
\[
\tilde{W}_i = \theta_i^- B - \theta_i^+ A + (\tilde{\theta} + \theta_i)\tilde{V} + \tilde{X}_i \tilde{N},
\]
$\tilde{X}_U = 0$, $\delta > 0$ is the absolute risk-aversion parameter, $x^+ := \max(0, x)$, and $x^- := \max(0, -x)$. In addition, a type $i$ investor is subject to the short-sale constraint
\[
\theta_i \geq -\lambda_i \tilde{\theta}.
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Since $I$ and $U$ investors buy from the designated market maker at ask and sell to
them at bid, we can view these trades occur in two separate markets: the “ask” market
and the “bid” market. In the ask market, the market maker is the supplier, other
investors are demanders and the opposite is true in the bid market. The monopolistic
market maker chooses bid and ask prices, taking into account other investors’ demand
curve in the ask market and supply curve in the bid market.

Let the demand schedules of the informed and the uninformed be denoted as
$\theta^*_I(A, B)$ and $\theta^*_U(A, B)$ respectively. By market clearing conditions, the equilibrium
ask depth $\alpha$ must be equal to the total amount bought by other investors and the
equilibrium bid depth $\beta$ must be equal to the total amount sold by other investors,\(^\text{12}\)
i.e.,
\[
\alpha = \sum_{i=I, U} N_i \theta^*_i(A, B)^+, \quad \beta = \sum_{i=I, U} N_i \theta^*_i(A, B)^-.
\]
(6)

Note that if an investor decides to buy (sell), then only the ask (bid) price affects how
much he buys (sells), i.e., $\theta^*_i(A, B)^+$ only depends on $A$ and $\theta^*_i(A, B)^-$ only depends
on $B$. Therefore, the bid depth $\beta$ only depends on $B$, henceforth referred as $\beta(B)$
and the ask depth $\alpha$ only depends on $A$, henceforth referred as $\alpha(A)$.

Then the designated market maker’s problem is to choose ask price $A$ and bid
price $B$ to solve
\[
\max E \left[-e^{-\delta W_M} | I_M \right],
\]
subject to
\[
\tilde{W}_M = \alpha(A) A - \beta(B) B + (\bar{\theta} + \beta(B) - \alpha(A)) \tilde{V}.
\]
(8)

This leads to our definition of the Nash equilibrium.

**Definition 1** An equilibrium $(\theta^*_I(A, B), \theta^*_U(A, B), A^*, B^*, \alpha^*, \beta^*)$ is such that

1. given any $A$ and $B$, $\theta^*_i(A, B)$ solves a type $i$ investors’ Problem (3) for $i \in \{I, U\}$;

2. given $\theta^*_I(A, B)$ and $\theta^*_U(A, B)$, $A^*$ and $B^*$ solve the market maker’s Problem (7);

\(^{12}\)The risk-free asset market will be automatically cleared by the Walras’ law. To help remember,
Alpha denotes Ask depth and Beta denotes Bid depth.
3. market clearing condition (6) is satisfied by \((\theta^*_1(A, B), \theta^*_2(A, B), A^*, B^*, \alpha^*, \beta^*)\).

A. Discussions on the assumptions of the model

In this subsection, we discuss our main assumptions and discuss whether these assumptions are important for our main results.

The assumption that there is only one market maker is for expositional focus. A model with multiple market makers was solved in an earlier version of this paper, the results of which are available from authors. In this more general model with oligopolistic competition, we show that competition among market makers, while lowering spreads, does not change our main qualitative results (e.g., short-sale constraints increase expected bid-ask spread). In illiquid markets such as some OTC markets, it is costly for non-market-makers to find and directly trade with each other. Therefore, most trades are through a small number of market makers.

One important assumption is that the market maker can buy at the bid from some investors and sell at the ask to other investors at the same time. This assumption captures the fact that in many OTC markets, when a dealer receives an inquiry from a client, she often contacts other clients (or dealers) to see at which price and by how much she can unload the inquired trade before she trades with the initial client. In addition, even for markets where there is a delay between offsetting trades, using a dynamic model with sequential order arrival would unlikely yield qualitatively different results. For example, in such a dynamic model, short-sale constraints still decrease bid, increase ask, and decrease trading volume. This is because even when orders arrive sequentially and thus a market maker needs to wait a period of time for the offsetting trades, her market power would still result in a lower bid price, and her aversion to inventory risk would still make her raise the next sale price (ask) to achieve the optimal inventory position. In addition, as long as she has a good estimate of order flow, she would choose not only qualitatively but also quantitatively similar trading strategy.

To keep information from being fully revealed in equilibrium, we assume that informed investors have liquidity shocks in addition to private information. In a model with private information sources such as a private signal \(\hat{s}\) and a private liquidity shock
(e.g., Grossman and Stiglitz (1980), and Vayanos and Wang (2012)), assuming that all investors who are subject to liquidity shock also observe $\hat{s}$ is only for simplicity: even if they do not observe $\hat{s}$, they can infer it perfectly from the equilibrium price, because the equilibrium price is an invertible function of the weighted sum of $\hat{s}$ and the private liquidity shock (and they observe the liquidity shock). In this type of models asymmetric information can therefore exist only if some investors who do not have any liquidity shock do not observe $\hat{s}$ either and are thus uninformed, as in Vayanos and Wang (2012) for example. We assume that these investors are all uninformed for simplicity. The assumption that informed investors have liquidity shocks in addition to private information can be also interpreted as there are some pure liquidity traders who trade in the same direction as the informed. Alternatively, one can view an informed investor as a broker who combines information motivated trades and liquidity motivated trades. The assumption that all informed traders have the same information and the same liquidity shock is only for simplicity so that there are only two groups of non-market-makers in the model. Our main results still hold when they have different information and different liquidity shocks. Intuitively, no matter how many heterogenous investor groups there are, the equilibrium bid and ask prices would divide these groups into Buy group, Sell group, and No trade group. When short-sale constraints bind for the Sell group, bid price would be lower and ask price would be higher by the same intuition.

We also assume that the market maker posts price schedules first (after taking into account what would be the best responses of other investors), then other investors choose their optimal trading strategies taking the posted price schedules as given. As shown in Liu and Wang (2014), this assumption is equivalent to assuming that in a Nash Bargaining game between the market maker and other investors, the market maker has all the bargaining power. This is consistent with the common practice in OTC markets that a dealer making two-sided markets typically provides a take-it-or-leave-it pair of prices, a bid and an offer, to customers (e.g., Duffie (2012), Chapter 1).

Different from the existing models, we assume there is public signal that is correlated with the private signal of the informed. This additional signal is not critical
for our main results, but has three main benefits. First, it allows us to examine the impact of public disclosure about the asset payoff. Second, it can serve as a good measure of information asymmetry that does not affect aggregate information quality in the economy (measured by the precision of security payoff distribution conditional on all information in the economy).\textsuperscript{13} Third, its introduction also makes our model nest models with different degrees of information asymmetry in one unified setting.\textsuperscript{14}

4. The equilibrium

In this section, we solve the equilibrium bid and ask prices, bid and ask depth and trading volume in closed form.

Given $A$ and $B$, the optimal demand schedule for a type $i$ investor for $i \in \{I, U\}$ is

$$\theta^*_i(A, B) = \begin{cases} \frac{P^R_i - A}{\delta \text{Var}[\tilde{V} | I_i]} & A < P^R_i, \\ 0 & B \leq P^R_i \leq A, \\ \max \left[ -\lambda_i \tilde{\theta}, -\frac{B - P^R_i}{\delta \text{Var}[\tilde{V} | I_i]} \right] & B > P^R_i, \end{cases}$$ \hspace{1cm} (9)

where

$$P^R_i = E[\tilde{V} | I_i] - \delta \sigma_u X_i - \delta \text{Var}[\tilde{V} | I_i] \tilde{\theta}$$ \hspace{1cm} (10)

is the reservation price of a type $i$ investor (i.e., the critical price such that non-market-makers buy (sell, respectively) the security if and only if the ask price is lower (the bid price is higher, respectively) than this critical price).

Because the informed know exactly $\{\hat{s}, \hat{X}_I\}$ while equilibrium prices $A^*$ and $B^*$ and the public signal $\hat{S}_s$ are only noisy signals about $\{\hat{s}, \hat{X}_I\}$, the information set of the informed in equilibrium is

$$\mathcal{I}_I = \{\hat{s}, \hat{X}_I\},$$ \hspace{1cm} (11)

\textsuperscript{13}For example, the precision of a private signal about asset payoff would not be a good measure, because a change in the precision also changes the quality of aggregate information about the payoff and both information asymmetry and information quality can affect economic variables of interest (e.g., prices, liquidity).

\textsuperscript{14}For example, the case where $\sigma_\eta = 0$ implies that the uninformed and the market maker can perfectly observe $\hat{s}$ from the public signal and thus represents the symmetric information case. The case where $\sigma_\eta = \infty$, on the other hand, implies that the public signal is useless and thus corresponds to the asymmetric information case as modeled in the standard literature, i.e., there is no public signal about the private information.
which implies that
\[
E[\tilde{V}|I] = \bar{v} + \rho_I \hat{s}, \quad \Var[\tilde{V}|I] = \rho_I \sigma_e^2 + \sigma_u^2,
\] (12)

where
\[
\rho_I := \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2}.
\] (13)

Equation (10) then implies that
\[
P_I^R = \bar{v} + \hat{S} - \delta(\rho_I \sigma_e^2 + \sigma_u^2)\bar{\theta},
\] (14)

where \( \hat{S} := \rho_I \hat{s} + h \hat{X}_I \) and \( h = -\delta \sigma_u N \) represents the hedging premium per unit of liquidity shock.

While \( \hat{s} \) and \( \hat{X}_I \) both affect an informed investor’s demand and thus the equilibrium prices, other investors can only infer the value of \( \hat{S} \) from market prices because the joint impact of \( \hat{s} \) and \( \hat{X}_I \) on market prices is only in the form of \( \hat{S} \). In addition to \( \hat{S} \), other investors can also observe the public signal \( \hat{S}_s \) about the private signal \( \hat{s} \). Thus we conjecture that the equilibrium prices \( A^* \) and \( B^* \) depend on both \( \hat{S} \) and \( \hat{S}_s \).

Accordingly, the information sets for the uninformed investors and the market maker are\(^{15}\)
\[
\mathcal{I}_U = \mathcal{I}_M = \{\hat{S}, \hat{S}_s\}.
\] (15)

Then the conditional expectation and conditional variance of \( \tilde{V} \) for the uninformed and the market maker are respectively
\[
E[\tilde{V}|\mathcal{I}_U] = \bar{v} + \rho_U(1 - \rho_X)\hat{S} + \rho_U\rho_X\rho_I \hat{S}_s,
\] (16)
\[
\Var[\tilde{V}|\mathcal{I}_U] = \rho_U \rho_I (\sigma_e^2 + \rho_X \sigma_n^2) + \sigma_u^2,
\] (17)

where
\[
\rho_X := \frac{h^2 \sigma_X^2}{h^2 \sigma_X^2 + \rho_I^2 \sigma_n^2}, \quad \rho_U := \frac{\sigma_v^2}{\sigma_v^2 + \rho_X \rho_I \sigma_n^2}.
\] (18)

\(^{15}\)Note that uninformed only need to observe their own trading price, i.e., \( A^* \) or \( B^* \), not both \( A^* \) and \( B^* \). For OTC markets, investors may not be able to observe trading prices by others, although with improving transparency, this has also become possible in some markets (e.g., TRACE system in the bond market).
It follows that the reservation price for an uninformed investor and the market maker is

\[ P_{RU}^R = P_M^R = \bar{v} + \rho_U (1 - \rho_X) \hat{S} + \rho_U \rho_X p_I \hat{S}_s - \delta \rho_U p_I (\sigma_v^2 + \rho_X \sigma_n^2) \hat{\bar{\theta}} - \delta \sigma_n^2 \hat{\bar{\theta}}. \]  

(19)

Let \( \Delta \) denote the difference in the reservation prices of \( I \) and \( U \) investors, i.e.,

\[ \Delta := P_{RI}^R - P_{RU}^R = (1 - \rho_U) \left( \left( 1 + \frac{\sigma_v^2}{\rho_I \sigma_n^2} \right) \hat{S} - \frac{\sigma_v^2}{\sigma_n^2} \hat{S}_s + \delta \rho_I \sigma_n^2 \hat{\bar{\theta}} \right). \]  

(20)

Let

\[ \nu := \frac{\text{Var}[\tilde{V}|\mathcal{I}_U]}{\text{Var}[\tilde{V}|\mathcal{I}_I]} \geq 1 \]

be the ratio of the security payoff conditional variance of the uninformed to that of the informed, and

\[ \overline{N} := \nu N_I + N_U + 1 \geq N \]

be the information weighted total population. Define

\[ C_I := \frac{N_U + 2}{2 (\overline{N} + 1)}, \quad C_U := \frac{\nu N_I}{2 (\overline{N} + 1)}. \]

The following theorem provides the equilibrium bid and ask prices and equilibrium security demand in closed form.

**Theorem 1** 1. If \(-\frac{\text{Var}[\tilde{V}|\mathcal{I}_I] \lambda \bar{\theta}}{C_I} < \Delta < \frac{\text{Var}[\tilde{V}|\mathcal{I}_U] \lambda_U \bar{\theta}}{C_U}\), then short-sale constraints do not bind for any investors and

(a) the equilibrium bid and ask prices are

\[ A^* = P_{RU}^R + C_U \Delta + \frac{\Delta^+}{2}, \]  

(21)

\[ B^* = P_{RI}^R + C_U \Delta - \frac{\Delta^-}{2}, \]  

(22)

which implies that the bid-ask spread is

\[ A^* - B^* = \frac{|\Delta|}{2} = \frac{(1 - \rho_U) \left( \left( 1 + \frac{\sigma_v^2}{\rho_I \sigma_n^2} \right) \hat{S} - \frac{\sigma_v^2}{\sigma_n^2} \hat{S}_s + \delta \rho_I \sigma_n^2 \hat{\bar{\theta}} \right)}{2}. \]  

(23)
(b) the equilibrium quantities demanded are
\[ \theta_I^* = C_I \frac{\Delta}{\delta \text{Var}[\hat{V}|I]}; \quad \theta_U^* = -C_I \frac{\Delta}{\delta \text{Var}[\hat{V}|U]}; \quad \theta_M^* = 2\theta_U^*; \quad (24) \]

the equilibrium quote depths are
\[ \alpha^* = N_I (\theta_I^*)^+ + N_U (\theta_U^*)^+, \quad (25) \]
\[ \beta^* = N_I (\theta_I^*)^- + N_U (\theta_U^*)^- \quad (26) \]

2. If \( \Delta \leq -\frac{\delta \text{Var}[\hat{V}|I] \lambda_I \bar{\theta}}{C_I} \), then short-sale constraints bind for the informed and

(a) the equilibrium bid and ask prices are
\[ A_{c1}^* = P^R_I - \delta N_I \text{Var}[\hat{V}|I] \lambda_I \bar{\theta} \frac{N_I}{N_U + 2}, \quad (27) \]
\[ B_{c1}^* = P^R_I + \delta \text{Var}[\hat{V}|I] \lambda_I \bar{\theta}, \quad (28) \]

and bid-ask spread is
\[ A_{c1}^* - B_{c1}^* = -\Delta - \frac{\bar{N} + 1}{N_U} \delta \text{Var}[\hat{V}|I] \lambda_I \bar{\theta}; \quad (29) \]

(b) the equilibrium quantities demanded are
\[ \theta_{Ic1}^* = -\lambda_I \bar{\theta}; \quad \theta_{Uc1}^* = \frac{N_I \lambda_I \bar{\theta}}{N_U + 2}; \quad \theta_{Mc1}^* = \frac{2N_I \lambda_I \bar{\theta}}{N_U + 2}; \quad (30) \]

the equilibrium quote depths are
\[ \alpha_{c1}^* = \frac{N_I N_U \lambda_I \bar{\theta}}{N_U + 2}; \quad \beta_{c1}^* = N_I \lambda_I \bar{\theta}. \quad (31) \]

3. If \( \Delta \geq \frac{\delta \text{Var}[\hat{V}|U] \lambda_U \bar{\theta}}{C_U} \), then short-sale constraints bind for the uninformed and

(a) the equilibrium bid and ask prices are
\[ A_{c2}^* = P^R_I - \frac{\Delta + \delta N_U \text{Var}[\hat{V}|U] \lambda_U \bar{\theta}}{\nu N_I + 2}, \quad (32) \]
\[ B_{c2}^* = P^R_U + \delta \text{Var}[\hat{V}|U] \lambda_U \bar{\theta}; \quad (33) \]

and bid-ask spread is
\[ A_{c2}^* - B_{c2}^* = \frac{\nu N_I + 1}{\nu N_I + 2} \Delta - \frac{\bar{N} + 1}{\nu N_I + 2} \delta \text{Var}[\hat{V}|U] \lambda_U \bar{\theta}; \quad (34) \]
(b) the equilibrium quantities demanded are

\[ \theta^*_{c2} = \frac{\Delta + \delta N_U \text{Var}[\hat{V}|I_U] \lambda_U \bar{\theta}}{(\nu N_I + 2) \delta \text{Var}[V|I_I]} \], \quad \theta^*_{Uc2} = -\lambda_U \bar{\theta}, \quad (35) \]

\[ \theta^*_{Mc2} = \frac{-N_I \Delta + 2\delta N_U \text{Var}[\hat{V}|I_I] \lambda_U \bar{\theta}}{(\nu N_I + 2) \delta \text{Var}[V|I_I]} \], \quad (36) \]

the equilibrium quote depths are

\[ \alpha^*_{c2} = \frac{N_I \Delta + \delta N_I N_U \text{Var}[\hat{V}|I_U] \lambda_U \bar{\theta}}{(\nu N_I + 2) \delta \text{Var}[V|I_I]} \], \quad \beta^*_{c2} = N_U \lambda_U \bar{\theta}. \quad (37) \]

Theorem 1 shows that whether short-sale constraints bind depends on whether the magnitude of the reservation price difference is large. If the informed’s reservation price is close to that of the uninformed, then no one trades a large amount because the equilibrium prices are close to their reservation prices and thus short-sale constraints do not bind for any of the investors (Case 1). If the informed’s reservation price is much smaller than that of the uninformed, then the informed would like to sell a large amount because the equilibrium bid price in the no-constraint case is much higher than the reservation price of the informed and thus the short-sale constraint binds for the informed (Case 2). If the informed’s reservation price is much larger than that of the uninformed, on the other hand, then the uninformed would like to sell a large amount and thus the short-sale constraint binds for the uninformed (Case 3). The thresholds for the reservation price difference such that short-sale constraints bind are determined by equalizing the unconstrained equilibrium short-sale quantities (\(\theta^*_I\) or \(\theta^*_U\)) to the short-sale bounds (\(-\lambda_I \bar{\theta}\) or \(-\lambda_U \bar{\theta}\) respectively).

Part 1 of Theorem 1 implies that when short-sale constraints do not bind, in equilibrium both bid and ask prices are determined by the reservation price of the uninformed and the reservation price difference between the informed and the uninformed. In addition, given the public signal \(\hat{S}_s\), all investors can indeed infer \(\hat{S}\) from observing their trading prices as conjectured, because of the one-to-one mapping between the two.\(^{16}\) Furthermore, Part 1 shows that when short-sale constraints do not

\(^{16}\)Similar to the set-up of Glosten (1989), the market maker can infer how much informed investors are trading. However, she does not know how much of the informed’s trades is due to the private information on the security’s payoff or how much is due to the hedging demand.
bind, the equilibrium bid-ask spread is equal to the absolute value of the reservation price difference between the informed and the uninformed, divided by 2 (more generally by \( N_M + 1 \)).

When the short-sale constraints bind for the informed or the uninformed, the amount of purchase the market maker makes with the constrained investors is fixed, thus the market maker’s utility always decreases in the bid price in the region where the constraints bind. As explained in the next subsection, the market power of the market maker then implies that the optimal bid price when the short-sale constraints bind in equilibrium must be such that the short-sale constraints just start to bind. This relationship gives rise to the constrained equilibrium bid prices as in (33) and (28), and bid depths as in (37) and (31). Given these bid prices and depths, ask prices and depths are then determined optimally by the market maker facing the demand schedules of the buyers.

If the public signal \( \hat{S}_s \) is precise (i.e., \( \sigma_\eta = 0 \)), the information asymmetry disappears because other investors can exactly observe the informed’s private signal. Therefore, Theorem 1 applies to both the symmetric information case and the asymmetric information case. In addition, because as \( \sigma_\eta \) increases, the information asymmetry increases, but the aggregate information quality in the economy (i.e., the inverse of the conditional variance of the security payoff given all the information in the economy) does not change, we use \( \sigma_\eta \) as a measure of public disclosure and information asymmetry. Unlike some other measures such as the precision \( 1/\sigma_\varepsilon^2 \) of the informed’s private signal, our measure is not contaminated by information quality effect.

In our model, we assume that non-market-makers submit their demand schedules in all states and thus as in DV, the short-sale prohibition for both the informed and the uninformed does not change the information content of an order. One concern over this assumption may be that informed investors may choose not to submit demand schedules when they are constrained by the short-sale constraints. If so, then other investors may get less information than what we assume in the model. Our assumption is motivated by the following three observations. First, if information is symmetric, then clearly short-sale constraints do not change the information content of an order because there is no private information to reveal. Second, if these
informed investors initially own some shares or can short some shares, they submit
the schedules to sell some shares, which reveals the same amount of information even
if they sell less in equilibrium than they would without the constraints, because the
schedules specify the trade amount for each price. Third, consider the case where
these investors do not initially own any shares and cannot short sell at all, in which
case they cannot trade in equilibrium. In this case, they are indifferent between
submitting demand schedules or not, absent of submission costs. More importantly,
when the informed are constrained and cannot trade, what is important for our anal-
ysis is the information revealed by the demand schedules. As long as there are some
investors with the same or more precise information who have initial endowment or
can short sell submit their schedules to sell some shares, these schedules convey the
same or more precise information compared to the demand schedules we assume the
constrained to submit. Therefore, in an economy where there are both constrained
and unconstrained informed investors, the main results of our model still hold by the
same driving forces: the market maker’s market power and risk aversion. In addition,
compared to those who do not have any initial endowment, investors with large initial
endowment may as likely sell but less likely short sell and have greater incentives to
obtain more precise information because they have more at risk and the short-sale
constraints limit less the potential benefit of more precise information (because the
constraints bind less often). Thus not only they are less likely constrained by the
short-sale constraints but also the information revealed by their demand schedules
gets greater weight. Accordingly, short-sale constraints restrict negative information
revelation only when all long position holders have less incentives to acquire informa-
tion than those who do not hold any shares (short position holders are few for most
securities with positive supply). Therefore, short-sale constraints unlikely affect the
overall information revelation in markets with large long position holders or essen-
tially unconstrained informed investors. In this paper, we focus on informed investors
with positive initial positions or can short sell some shares and thus without much
loss of economics, we assume that the informed submit their demand schedules even
when they are constrained.\footnote{One minor modification of our model is to have two
groups of informed investors with one}
5. The effect of short-sale constraints

In this section, we analyze the effect of short-sale constraints on bid prices, ask prices, bid-ask spreads, liquidity risk with and without endogenous information acquisition.

A. Bid/ask prices, bid-ask spread, and trading volume

Theorem 1 implies that

Proposition 1

1. As short-sale constraints become more stringent, equilibrium bid price decreases, equilibrium ask price increases, and so does equilibrium bid-ask spread.

2. As short-sale constraints become more stringent, equilibrium bid depth, ask depth and trading volume decrease.

Proposition 1 implies that prohibition of short-sales for both the informed and the uninformed decreases bid and increases ask. This is different from the conclusion of DV who show that short-sale prohibition imposed on both the informed and the uninformed does not change either the bid or the ask price and thus does not affect the bid-ask spread, as long as the prohibition does not change the expected payoff (see Corollary 2 in DV). The intuition in DV is that since short-sale prohibition restricts both the uninformed and the informed symmetrically, conditional on a sell order, the percentage of the informed trading does not change and thus the conditional expected payoff stays the same. Because for a risk neutral, competitive market maker, the bid price is equal to the conditional expected payoff, the equilibrium bid price also remains the same. In addition, since ask price is equal to the expected payoff conditional on a buy order and short-sale prohibition does not affect an investor’s purchasing decision in their model, ask price also remains the same. In contrast, in our model equilibrium bid, ask, and spread all change. As we show below, the main driving forces are the market power and the risk aversion of the market maker.

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Because short-sale constraints restrict sales at the bid, one might expect that short-sale constraints increase the equilibrium bid price. In contrast, Proposition 1 states the opposite. We next provide the essential intuition for this seemingly counter-intuitive result and other implications of Proposition 1 through graphical illustrations. Suppose $P_I^R < P_U^R$ and thus the informed sell and the uninformed buy. The market clearing condition (6) implies that the inverse demand and supply functions faced by the market maker are respectively

$$A = P_U^R - k_U \alpha, \quad B = P_I^R + k_I \beta,$$

where

$$k_i = \frac{\delta \text{Var}[\tilde{V} | I_i]}{N_i}, \quad i = I, U.$$

To make the intuition as simple as possible, we first plot the above inverse demand and supply functions and equilibrium spreads in Figure 1(a) for the extreme case where the market maker has infinite risk aversion and no initial endowment of the risky security. Then we illustrate in Figure 1(b) the case where the market maker has the same risk aversion and initial endowment as other investors. Figure 1(a) shows that as the market maker increases ask (decreases bid) other investors buy (sell) less. Facing the inverse demand and supply functions, a monopolistic market maker optimally trades off the prices and quantities. Similar to the results of monopolistic competition models, the bid and ask spread is equal to the absolute value of the reservation price difference $|\Delta|$, divided by 2 (by $N_M + 1$ with multiple market makers engaging in Cournot competition). Because the market maker has infinite risk aversion and no initial endowment, the market maker buys and sells the same amount so that there is zero inventory carried to date 1. With short-sale constraints, a market maker can only buy from the informed up to $N_I \lambda_I \theta$. Because the market maker has market power and obtains a greater utility with a lower bid price when the amount of purchase is fixed, the market maker chooses a bid price such that the short-sale constraint never strictly binds. Therefore, if the unconstrained equilibrium sale amount from the informed is

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18 Even though in the main model, we assume that the market maker has the same risk aversion as other investors, this extreme case can be easily solved to yield the results shown in this figure, because the ask depth is always equal to the bid depth and the market maker maximizes only the profit from the spread and carries no inventory.
Figure 1: Inverse Demand/Supply Functions and Bid/Ask Prices with and without Short-sale Constraints.
larger than the upper bound $N_I \lambda I \bar{\theta}$, the market maker lowers the bid price such that in the constrained equilibrium, the informed sell less and the short-sale constraint just starts to bind. Because the market maker buys less from the informed in equilibrium, the market maker also sells less to the uninformed at the ask than the unconstrained case to avoid inventory risk. This implies the equilibrium ask price is higher and the ask depth is also smaller, because the utility of the market maker increases in the ask price. When the market maker has positive but finite risk aversion, the same motive of reducing inventory risk also drives up the ask price and drives down the ask depth, although the market maker may choose to carry some inventory. On the other hand, if the market maker is risk neutral, then short-sale constraints only reduce bid price and bid depth, but do not affect ask price or ask depth because inventory risk is irrelevant for a risk neutral market maker. The above intuition suggests that position limit on long position would have the same qualitative impact: increasing ask, decreasing bid, increasing bid-ask spread, and decreasing depths and trading volume.

To further identify the driving force behind the reduction of bid price due to short-sale constraints, in Theorem 3 in Appendix B we report the equilibrium results for an alternative model where the market maker is a price taker in the “bid” market as in most of the existing literature, but a monopolist in the “ask” market as in the main model. Theorem 3 shows the same qualitative results for the impact of short-sale constraints on ask price, depths, and trading volume. However, in contrast to the main model, Theorem 3 implies that short-sale constraints increase equilibrium bid price. Because this alternative model differs from our main model only in that the market maker is a price taker in the “bid” market, this shows that the driving force behind our result that short-sale constraints decrease bid price is indeed the market maker’s market power. If buyers do not have market power (i.e., are price takers), then they compete for the reduced supply and thus the constrained equilibrium price becomes higher. In this case, the constrained equilibrium price is determined by the unconstrained’s (i.e., buyers’) demand. On the other hand, if buyers have all

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19 When the market maker is risk neutral, she maximizes $A \alpha(A) - B \beta(B) + (\bar{\theta} + \beta(B) - \alpha(A))E[\tilde{V} | \mathcal{I}_M]$ and therefore choices of $B$ and $A$ are independent.

20 The results on the effect of position limits (on both long and short positions) on prices and depths are available from authors.
the market power, then because their utility increases as the bid price decreases, they dictate a lower bid price. In this case, the constrained equilibrium price is determined by the constrained’s (i.e., short-sellers’) reduced supply. More generally, if both the constrained short-sellers and the buyers have market power, then the bid price can be higher or lower depending on whether the buyers’ market power is smaller or greater respectively than the short-sellers. As far as we know, this paper is the first to show that when buyers have market power, equilibrium selling price can go down when sellers are constrained.

Our model predicts that in markets where market makers have significant market power and are risk averse, imposing short-sale constraints will cause bid prices to go down and the ask prices to go up. There is a caveat for this result: we do not model the information content of the imposition itself. The imposition of short-sale constraints by regulators may signal some negative information about the stocks being regulated. If this negative information content were taken into account (e.g., in terms of a lower unconditional expected payoff upon the imposition, i.e., smaller $\bar{v}$), then the joint impact of this negative signal and the short-sale constraints would lower the bid price further, but might also lower the ask price in the net, because negative information drives both bid price and ask price down, as implied by Theorem 1. On the other hand, because negative information drives both bid price and ask price down, the information content of the imposition of the constraints affects less the result that short-sale constraints increase bid-ask spread, as long as the magnitude of the impact is similar on bid and ask.\footnote{If one models the impact of the information content of the short-sale constraints imposition as having a lower unconditional expected payoff $\bar{v}$ in the case with short-sale constraints than without, then our model implies that equilibrium bid and ask are lowered by the same amount and thus spread is unaffected.}

To illustrate the magnitude of the impact, we plot the percentage changes in expected bid, expected ask and expected spread against the information asymmetry measure $\sigma_\eta$ in Figure 2.\footnote{The key random variable for computing the expected changes in prices, depths, spread and trading volume is the reservation price difference $\Delta$. It can be shown that $\Delta$ is normally distributed with mean $\mu_\Delta$ and variance $\sigma^2_\Delta$ where

\[ \mu_\Delta = \delta \rho_U (1 - \rho_U) \sigma^2_\theta, \quad \sigma^2_\Delta = h^2 \sigma^2_X - \rho_U (1 - \rho_U) \sigma^2_\eta. \]}

\[ (38) \]
Figure 2: The percentage changes in expected bid price (dashed), expected ask price (solid), expected mid-price (dot-dashed), and expected bid-ask spread with short-sale constraints against $\sigma_\eta$. The parameter values are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_N = 0.9$, $\rho_{uN} = 0.9$, $\bar{v} = 3$, $\sigma_X = 0.8$, $\sigma_v = 0.9$, $N_I = 10$, $N_M = 1$, $N_U = 100$, $\bar{\theta} = 1/(N_I + N_U + N_M)$, $\lambda_I = \lambda_U = 1$.

Figure 3: The percentage changes in expected bid depth (dashed), ask depth (solid), and expected trading volume (dot-dashed) with short-sale constraints against $\sigma_\eta$ and $\sigma_X$. The parameter values are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_N = 0.9$, $\rho_{uN} = 0.9$, $\bar{v} = 3$, $\sigma_X = 0.8$, $\sigma_v = 0.9$, $N_I = 10$, $N_M = 1$, $N_U = 100$, $\bar{\theta} = 1/(N_I + N_U + N_M)$, $\lambda_I = \lambda_U = 1$. 
Figure 2 shows that short-sale prohibition can have significant impact on bid and ask prices. For example, the bid price can go down by more than 1% and the ask can go up by almost 1%, even though on average there is no liquidity shock and the private signal is 0 for the informed. In addition, the increase in the expected spread can be much greater. For example, with symmetric information ($\sigma_\eta = 0$), there is a more than 75% increase in the expected spread due to the short-sale prohibition. The much larger effect on the spread in percentage term is because the expected spread is much smaller than the expected bid/ask prices for the parameter values used for the figure. As public disclosure increases, both the increase in the expected ask and the decrease in the expected bid go up, and so does the increase in the expected spread. Intuitively, as public disclosure improves (i.e., $\sigma_\eta$ decreases), information asymmetry and uncertainty decrease, investors increase the amount of short-sales and thus the probability of the constraint binding is greater, which in turn increases the impact of the short-sale constraints. In Figure 3, we plot the expected percentage change in trading volume against the information asymmetry measure $\sigma_\eta$ and the volatility of the liquidity shock $\sigma_X$. Figure 3 shows that the impact of short-sale prohibition on the trading volume can be quite significant, as high as more than 80%. As public disclosure increases, the effect of the ban increases for the same reason as above. These findings imply that more public disclosure, which reduces information asymmetry, can magnify the adverse impact of short-sale constraints on market prices and liquidity. Accordingly, one testable empirical prediction is that the effect of short-sale constraints on bid-ask spread is greater in more transparent markets. On the other hand, Figures 2 and 3 show that even with significant information asymmetry, our main results still hold.

While information asymmetry tends to decrease the impact of short-sale constraints, Figure 3 suggests that as the volatility of the liquidity shock $\sigma_X$ increases, the impact of the short-sale ban increases significantly. This is because a greater volatility implies a higher probability of the ban becoming binding.

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23 All the figures in the paper are for illustrations of qualitative results only and we do not attempt to calibrate to some OTC markets because of the complexity and opaqueness of OTC markets. We normalize the total supply of the security to 1 share.
To facilitate future empirical analysis, next we compare the predictions of our model with those of DV for different scenarios. For this purpose, we restrict to four main cases: Base case: No short-sale constraints; Case 1: Short-sale prohibition for both the informed and the uninformed; Case 2: Only the informed can short; Case 3: Only the informed can short, conditional on shorting. We consider the impact of changing from one case to another. Cases 2 and 3 are motivated by empirical evidence that short-sale costs are smaller for relatively informed investors (e.g., for some institutional traders) than for relatively uninformed investors (e.g., most retail investors). In our model, whenever additional investors, whether informed or uninformed, become effectively subject to short-sale constraints, expected bid price and depths go down, while expected ask price and spread go up. In addition, conditional on a short-sale by the informed, the news is on average negative. Therefore, our model predicts that conditional on a short-sale by informed investors, on average both bid and ask are lower, but the spread can increase or decrease. In contrast, in DV, whether the informed or the uninformed become constrained is critical for the prediction. First, consider a change from Base case to Case 2. In DV, because less uninformed investors submit sell order, the expected payoff conditional on a sell order goes down and thus bid price goes down, which is consistent with our model’s prediction, although for different reasons. A key differentiation in this case is that our model predicts that ask price goes up, but DV predict that it stays the same. Second, consider a change from Case 2 to Case 1. Because the ban prohibits the informed from shorting and thus a sell order becomes less likely from the informed, DV model implies that in these markets the ban increases the expected bid price. As explained above, in DV model, short-sale constraints do not have any impact on ask price because market makers are risk neutral and short-sale constraints do not change the information content of a buy order. This implies that DV model predicts that the expected spread goes down after a short-sale ban. Third, consider a change from Case 2 to Case 3. Because conditional on a short-sale by the informed, the news is on average negative, DV predict that while on average bid price is lower, the ask price does not change. The main differences are summarized in Table 1 in the introduction. One can use these differences in predictions to test which theory applies better in
which markets.

B. Liquidity risk

Next, we study how short-sale constraints affect liquidity risk measured by the volatility of bid-ask spreads. We have

**Proposition 2** Short-sales constraints increase stock’s liquidity risk measured by the volatility of bid-ask spread, i.e., \( \text{Vol}(A^* - B^*) \geq \text{Vol}(A^* - B^*) \).

![Bid-Ask Spreads](image)

Figure 4: Spread as a function of reservation price difference \( \Delta \). The parameters are: \( \delta = 1, \sigma_u = 0.4, \sigma_N = 0.9, \rho_{uN} = 0.9, \bar{v} = 3, \sigma_X = 0.8, \sigma_v = 0.9, N_I = 10, N_M = 1, N_U = 100, \bar{\theta} = 1/(N_I + N_U + N_M), \lambda_I = \lambda_U = 1 \).

The main intuition for Proposition 2 is as follows. When short-sale constraints bind, there is less risk sharing among investors and thus bid and ask prices change more in response to a random shock. For example, keeping everything else constant, we plot the bid-ask spreads as a function of reservation price difference \( \Delta \) for the case without constraints (dashed lines) and the case with short-sale constraints (solid lines) in Figure 4. This figure shows that indeed when the constraints bind, for the same change in the reservation price difference, spread changes more (i.e., steeper lines), which in turn implies that the volatility of spread goes up.

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24Since the spread is zero at time 1, this volatility is the same as time series volatility in our model.
To examine the impact of information asymmetry on the volatility of spread, we plot bid-ask spread volatilities against information asymmetry $\sigma_\eta$ for the case where both investors are subject to short-sale constraints (solid red), the case without short-sale constraints (dashed purple), and the case when only uninformed are subject to short-sale constraints (dotted blue). As illustrated in Figure 5, spread volatility is the greatest when both the informed and the uninformed are subject to short-sale constraints, which is significantly greater than (almost double) the lowest volatility when no one is subject to short-sale constraints. Figure 5 suggests that the impact of short-sale constraints on volatility of spreads can be significant. As information asymmetry increases, the volatility decreases because with higher uncertainty investors trade less and the short-sale constraints bind less often. Thus more public disclosure also increases the impact of short-sale constraints on the spread volatility.

C. Robustness with information acquisition

So far, we have assumed that the informed’s information quality is not affected by the imposition of short-sale constraints. As argued before, if some investors have a large long position possibly because of previous positive information and/or lower risk aversion, then these investors have greater incentive to acquire more precise...
information. Because they are less likely to be constrained by short-sale constraints and their sale orders would reveal more precise information than the orders of short sellers, short-sale constraints unlikely change much aggregate information quality in most markets. Still, it is worthwhile to examine whether our results can still hold when aggregate information quality is indeed affected by short-sale constraints.

To this extent, we assume that, on date 0, there is only one type of $I$-investors as in our model and they can acquire a costly signal $\hat{s}$ as defined in (1) with precision of $\rho_{\epsilon} = \frac{1}{\sigma_{\epsilon}^2}$ at a cost of $c(\rho_{\epsilon}) := k\rho_{\epsilon}^2$, where $k$ is a positive constant.

If the feasible precision is bounded above, then when the cost of information acquisition is low enough, the optimal precision will be the same with and without short-sale constraints and therefore endogenous information acquisition will not have any impact on our results in this case. If in the other extreme, the information cost is large enough, then no one would acquire any private information with and without short-sale constraints and therefore endogenizing information precision would not have impact either. Next, we show that even when the precision can change with constraints our results still hold for a large set of parameter values.

As expected, for moderate information acquisition cost, there exists an optimal level of precision for informed investors who trade off the benefit from trading with more precise private information and the cost of acquiring more precise private information. We find that the optimal precision of private information for informed investors in the presence of short-sale constraints tend to be lower than that in the absence of short-sale constraints. Intuitively, the presence of short-sale constraints may reduce the incentive of investors to produce more precise information because investors are restricted from short selling after observing a bad signal, which implies a smaller benefit of more precise information.

Most importantly, Figure 6 shows that short-sale constraints may still increase the expected ask price and spread volatility, and decrease the expected bid price even after the impact of short-sale constraints on endogenous information acquisition is taken into account. For a large set of parameter values, we obtain similar patterns to those in Figure 6. This suggests in particular that the main result that short-sale constraints increase bid-ask spread is largely unaffected by endogenization of
Figure 6: The percentage changes of expected bid and ask prices, and the optimal precisions of information with and without short-sale constraints. The default parameters are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_N = 0.9$, $\rho_{uN} = 0.9$, $\bar{v} = 3$, $\sigma_X = 0.8$, $\sigma_v = 0.9$, $N_I = 1$, $N_M = 1$, $N_U = 10$, $\bar{\theta} = 0.01$, $\lambda_I = \lambda_U = 1$, $k = 0.001$. 
information acquisition. In addition, Figure 6 illustrates that more disclosure (i.e., smaller $\sigma_{\eta}$) might actually increase the incentive of informed investors to acquire more precise private information. This is because public disclosure reduces information asymmetry and the loss of the informed from the adverse selection problem. Figure 6 also suggests that the optimal precision increases with liquidity shock. Intuitively, high liquidity shock volatility tends to increase the informed’s trading volume and thus makes them benefit more from more precise information.

6. Conclusions

In this paper, we develop an equilibrium model to help explain the empirical evidence that short-sale constraints tend to increase bid-ask spread. In contrast to the existing literature, our analysis suggests that if market makers have significant market power and are risk averse, then short-sale constraints drive bid price down and ask price up, which implies a greater equilibrium bid-ask spread. In addition, short-sale constraints decrease market trading volume and increase liquidity risk measured by the volatility of bid-ask spreads. More public disclosure that reduces information asymmetry can further magnify the adverse impact of short-sale constraints on asset prices and market liquidity. Furthermore, the main result that short-sale constraints increase bid-ask spread is largely unaffected by endogenization of information acquisition.

Our model provides some new empirically testable implications. For example,

1. in markets where market makers have significant market power, short-sale constraints decrease average bid and average trading volume, but increase average spread and spread volatility;

2. the impact of short-sale constraints is greater in more transparent markets.
References


Appendix A

Proof of Theorem 1:

We prove the case when $\Delta < 0$. In this case, we conjecture that $I$ investors sell and $U$ investors buy. First, suppose no investors are constrained. Given bid price $B$ and ask price $A$, the optimal demand of $I$ and $U$ are respectively:

$$\theta^*_I = \frac{P^R_I - B}{\delta \text{Var}[\tilde{V}|I_I]} \quad \text{and} \quad \theta^*_U = \frac{P^R_U - A}{\delta \text{Var}[\tilde{V}|I_U]}.$$  \hfill (39)

Substituting (39) into the market clearing condition (6), we get that the market clearing ask and bid depths are respectively:

$$\beta = -N_I \theta^*_I = N_I \frac{B - P^R_I}{\delta \text{Var}[\tilde{V}|I_I]}, \quad \alpha = N_U \theta^*_U = N_U \frac{P^R_U - A}{\delta \text{Var}[\tilde{V}|I_U]}.$$ \hfill (40)

Because of the CARA utility and normal distribution of the date 1 wealth, the market maker’s problem is equivalent to:

$$\max_{A,B} \alpha A - \beta B + (\bar{\theta} + \beta - \alpha)E[\tilde{V}|I_M] - \frac{1}{2} \delta \text{Var}[\tilde{V}|I_M](\bar{\theta} + \beta - \alpha)^2,$$ \hfill (41)

subject to (40). The F.O.C with respect to $B$ (noting that $\beta$ is a function of $B$) gives us:

$$-\beta - B \frac{N_I}{\delta \text{Var}[\tilde{V}|I_I]} + E[\tilde{V}|I_M] \frac{N_I}{\delta \text{Var}[\tilde{V}|I_I]} - \delta \text{Var}[\tilde{V}|I_M](\bar{\theta} + \beta - \alpha) \frac{N_I}{\delta \text{Var}[\tilde{V}|I_I]} = 0,$$

which can be reduced to

$$(\nu N_I + 2)\beta - \nu N_I \alpha = -\frac{N_I \Delta}{\delta \text{Var}[\tilde{V}|I_I]},$$ \hfill (42)

by (10), (19), and expressing $B$ in terms of $\beta$ using (40).

Similarly using the F.O.C with respect to $A$, we get:

$$\alpha + A \left( -\frac{N_U}{\delta \text{Var}[\tilde{V}|I_U]} \right) - E[\tilde{V}|I_M] \left( -\frac{N_U}{\delta \text{Var}[\tilde{V}|I_U]} \right) + \delta \text{Var}[\tilde{V}|I_M](\bar{\theta} + \beta - \alpha) \left( -\frac{N_U}{\delta \text{Var}[\tilde{V}|I_U]} \right) = 0,$$ \hfill (43)

which can be reduced to

$$(N_U + 2)\alpha - N_U \beta = 0,$$ \hfill (44)
by using (10), expressing $A$ in terms of $\alpha$ using (40), and noting that $I_M = I_U$.

Solving (44) and (42), we can get the equilibrium ask depth and bid depth $\alpha^*$ and $\beta^*$ as in (25) and (26). Substituting $\alpha^*$ and $\beta^*$ into (40), we can get the equilibrium ask and bid prices $A^*$ and $B^*$ as in (21) and (22). In addition, by the market clearing condition, we have $\theta^*_U = \alpha^*/N_U, \theta^*_I = -\beta^*/N_I, \theta^*_M = \beta^* - \alpha^*$, which can be simplified into equation (24).

The short-sale constraint binds for the informed if and only if $\theta^*_I \leq -\lambda_I \tilde{\theta}$, equivalently, if and only if $\Delta < -\frac{\delta \text{Var}[\tilde{V}|I]}{C_I} \lambda_I \tilde{\theta}$. When the short-sale constraint binds for the informed, we have $\theta^*_{Ic1} = -\lambda_I \tilde{\theta}$ and $\beta^*_{c1} = N_I \lambda_I \tilde{\theta}$. Because the first order condition (44) with respect to $\alpha$ remains the same, we have:

$$\alpha^*_{c1} = \frac{N_I N_U}{N_U + 2 \lambda_I \tilde{\theta}}.$$

Then from (40), we get the equilibrium bid price $B^*_{c1}$ and ask price $A^*_{c1}$ when short-sale constraints bind for the informed. Other quantities can then be derived. Similarly, we can prove Theorem 1 for the other case where $I$ investors buy and $U$ investors sell. 

Proof of Proposition 1: We prove this proposition for the case when $I$ investors sell, the proof of the other case is very similar and we thus skip it here. Conditional on the constraint binding for the informed, it is clear from Theorem 1 that $A^*_{c1}$ decreases in $\lambda_I$, $B^*_{c1}$ increases in $\lambda_I$ and $(A^*_{c1} - B^*_{c1})$ decreases in $\lambda_I$. We next show that compared to the case without short-sale constraints, bid is lower and ask is higher with the constraints. By Theorem 1, we have

$$B^*_{c1} - B^* = C_I \Delta + \delta \text{Var}[\tilde{V}|I] \lambda_I \tilde{\theta}, \quad \text{(45)}$$

and

$$A^*_{c1} - A^* = -C_U \Delta - \frac{N_I}{N_U + 2} \delta \text{Var}[\tilde{V}|U] \lambda_I \tilde{\theta}. \quad \text{(46)}$$

The condition $\Delta < -\frac{\delta \text{Var}[\tilde{V}|I] \lambda_I \tilde{\theta}}{C_I}$ implies that $B^*_{c1} \leq B^*$ and $A^*_{c1} \geq A^*$, which leads to $A^*_{c1} - B^*_{c1} \geq A^* - B^*$. Similarly, the results on depths and trading volume can be shown. 

The following lemma is used to prove Proposition 2.
Lemma 1 Let \( f(x) := |x| \) and

\[
g(x) := \begin{cases} 
    c_1(k_1x - h_1) & x > \frac{h_1}{k_1} \\
    c_2(-k_2x - h_2) & x < \frac{-h_2}{k_2} \\
    0 & \text{otherwise,}
\end{cases}
\]

where \( k_1, k_2, h_1, h_2, c_1 \) and \( c_2 \) are positive constants and \( x \) is randomly distributed in \(( -\infty, +\infty )\) with probability density function \( p(x) \) which is an even function. Then we have \( \text{Cov}(f(x), g(x)) > 0 \).

Proof:

\[
\text{Cov}(f(x), g(x)) = E(f(x)g(x)) - E(f(x))E(g(x))
\]

\[
= \int_{-\infty}^{+\infty} f(x)g(x)p(x)dx - \int_{-\infty}^{+\infty} f(x)p(x)dx \int_{-\infty}^{+\infty} g(x)p(x)dx
\]

\[
= \int_{-\infty}^{+\infty} p(y)dy \int_{-\infty}^{+\infty} f(x)g(x)p(x)dx - \int_{-\infty}^{+\infty} f(y)p(y)dy \int_{-\infty}^{+\infty} g(x)p(x)dx
\]

\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x)g(x) - f(y)g(x))p(x)p(y)dxdy
\]

\[
= \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))(g(x) - g(y))p(x)p(y)dxdy. \tag{47}
\]

Since \( p(-x) = p(x) \) and \( p(-y) = p(y) \), we have

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(-x) - f(-y))(g(-x) - g(-y))p(x)p(y)dxdy
\]

\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))(g(x) - g(y))p(x)p(y)dxdy. \tag{48}
\]

From (47) and (48), we have \( \text{Cov}(f(x), g(x)) = \)

\[
\frac{1}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))(g(x) + g(-x) - g(y) - g(-y))p(x)p(y)dxdy. \tag{49}
\]

(1) If \( x \) and \( y \) have the same sign, the term inside the integral can be written as \((f(x) - f(y))(g(x) - g(y)) + (f(-x) - f(-y))(g(-x) - g(-y))\), which is non-negative.

(2) If \( x < 0 \) and \( y > 0 \), the term inside the integral can be written as \((f(-x) - f(y))(g(-x) - g(y)) + (f(x) - f(-y))(g(x) - g(-y))\), which is non-negative.

(3) If \( x > 0 \) and \( y < 0 \), the term inside the integral can be written as
\[(f(x) - f(-y))(g(x) - g(-y)) + (f(-x) - f(y))(g(-x) - g(y)),\] which is non-negative. In addition, at least for some \(x\) and \(y\), the term inside the integral is non-zero. Therefore, \(\text{Cov}(f(x), g(x)) > 0\). \(\text{Q.E.D.}\)

**Proof of Proposition 2:** The spread with short-sale constraints \(A_c^* - B_c^*\) can be written as \(f(\Delta) + g(\Delta)\), where

\[f(\Delta) = A^* - B^* = \frac{|\Delta|}{2}\]

and

\[g(\Delta) = \begin{cases} \frac{\nu N_I M}{2(\nu N_I + 2)} \Delta - \frac{\bar{N} + 1}{\nu N_I + 2} \delta \text{Var}[\tilde{V}|I]\lambda U \bar{\theta} & \Delta \geq \frac{\delta \text{Var}[\tilde{V}|I]\lambda U \bar{\theta}}{C_I} \\ -\frac{1}{2} \Delta - \frac{\bar{N} + 1}{\nu N_I + 2} \delta \text{Var}[\tilde{V}|I]\lambda I \bar{\theta} & \Delta \leq -\frac{\delta \text{Var}[\tilde{V}|I]\lambda I \bar{\theta}}{C_I} \\ 0 & \text{otherwise} \end{cases}\]

By Lemma 1, \(f(\Delta)\) and \(g(\Delta)\) are positively correlated. Then it follows that \(\text{Var}(A_c^* - B_c^*) > \text{Var}(A^* - B^*)\). \(\text{Q.E.D.}\)

### Appendix B

**Equilibrium with a price taking market maker**

To show the impact of the market maker’s market power on equilibrium results, in this Appendix, we consider the case where the market maker is also a price taker, which yields the same results as in a perfect competition market. When the market maker is a price taker, the equilibrium bid and ask prices must be equal, which can also be easily seen from the first order conditions. Let \(P\) denote the stock price. Given \(P\), the optimal demand schedule for a type \(i\) investor for \(i \in \{I, U\}\) is

\[\theta_i^*(P) = \max \left[-\lambda_i \tilde{\theta}, -\frac{P - P_i^R}{\delta \text{Var}[V|I_i]}\right].\] (50)

Solving for the equilibrium, we have

**Theorem 2** 1. If \(-\frac{\bar{N} \delta \text{Var}[\tilde{V}|I_i]\lambda_i \tilde{\theta}}{\nu N_I + 1} < \Delta < \frac{\bar{N} \delta \text{Var}[\tilde{V}|I_i]\lambda_i \tilde{\theta}}{\nu N_I}\), then no one is constrained and the equilibrium price is

\[P^* = \frac{\nu N_I}{N} P_I^R + \frac{N_U}{N} P_U^R + \frac{1}{N} P_M^R\] (51)
and the investors’ optimal stock demand are given by
\[ \theta^*_I = \frac{N_U + 1}{N} \frac{\Delta}{\delta \text{Var}[V|I]}, \quad \theta^*_U = \theta^*_M = -\frac{\nu N_I}{N} \frac{\Delta}{\delta \text{Var}[V|I]}, \]
\[ \text{(52)} \]

2. If \( \Delta \geq \frac{\delta \text{Var}[\tilde{V}|I_U] \lambda_U \theta}{\nu N_I} \), then the short-sale constraint binds for the uninformed and the equilibrium price is
\[ P^*_{c1} = P^R_I - \frac{\Delta + N_U \delta \text{Var}[\tilde{V}|I_U] \lambda_U \theta}{\nu N_I + 1}, \]
\[ \text{(53)} \]

and the investors’ optimal stock demand are given by
\[ \theta^*_{Ic1} = \frac{\Delta + N_U \delta \text{Var}[\tilde{V}|I_U] \lambda_U \theta}{(\nu N_I + 1) \delta \text{Var}[\tilde{V}|I]}, \quad \theta^*_{Uc1} = -\lambda_U \theta, \]
\[ \theta^*_M_{c1} = -\frac{-\nu N_I \Delta + N_U \delta \text{Var}[\tilde{V}|I_U] \lambda_U \theta}{(\nu N_I + 1) \delta \text{Var}[\tilde{V}|I_U]}, \]
\[ \text{(54)} \]
\[ \text{(55)} \]

3. If \( \Delta \leq \frac{\delta \text{Var}[\tilde{V}|I_U] \lambda_U \theta}{N_U + 1} \), then the short-sale constraint binds for the informed and the equilibrium price is
\[ P^*_{c2} = P^R_U - \frac{N_I \delta \text{Var}[\tilde{V}|I_U] \lambda_I \theta}{N_U + 1}, \]
\[ \text{(56)} \]

and the investors’ optimal stock demand are given by
\[ \theta^*_{Ic2} = -\lambda_I \theta, \quad \theta^*_{Uc2} = \theta^*_M_{c2} = \frac{N_I \lambda_I \theta}{N_U + 1}. \]
\[ \text{(57)} \]

As shown by Theorem 2, the equilibrium price is a weighted average of the reservation prices of the investors in the economy. In addition, it is easy to show when the short-sale constraint binds, the equilibrium selling price goes up (i.e., \( P^*_{c1} > P^* \) and \( P^*_{c2} > P^* \)) and trading volume decreases, as in the existing literature. Theorem 2 assumes that the market maker is a price taker in both “bid” and “ask” markets. Next, to isolate the impact of the market power on how short-sale constraints affect bid price, we assume that the market maker is a monopolist in the “ask” market, as in our main model, but is a price taker in the “bid” market. Under this assumption, we have,
Theorem 3  

1. If $0 < \Delta < \frac{\nu N_I + 2(N_U + 1)\delta \text{Var}[\hat{V} | I_U] \lambda_U \bar{\theta}}{\nu N_I}$, then no one is constrained and the equilibrium prices are

$$A^*_1 = P^R_U + \frac{N}{2(N_U + 1) + \nu N_I} \Delta, \quad B^*_1 = P^R_U + \frac{\nu N_I}{2(N_U + 1) + \nu N_I} \Delta,$$

which implies the bid-ask spread is

$$A^*_1 - B^*_1 = \frac{N_U + 1}{2(N_U + 1) + \nu N_I} \Delta,$$

the equilibrium depths are

$$\alpha^*_1 = \frac{\nu N_I(N_U + 1)}{2(N_U + 1) + \nu N_I} \frac{\Delta}{\delta \text{Var}[\hat{V} | I_U]}, \quad \beta^*_1 = \frac{N_U}{N_U + 1} \alpha^*_1.$$

the investors’ optimal stock demand are given by

$$\theta^*_I_1 = \frac{\alpha^*_1}{N_I}, \quad \theta^*_U_1 = -\frac{\beta^*_1}{N_U}, \quad \theta^*_M_1 = -\frac{\alpha^*_1}{N_U + 1}.$$

2. If $-\frac{2(\nu N_I + 1 + N_U)}{N_U + 2} \delta \text{Var}[\hat{V} | I_U] \lambda_I \bar{\theta} < \Delta < 0$, then no one is constrained and the equilibrium prices are

$$A^*_2 = P^R_U + \frac{\nu N_I}{2\nu N_I + N_U + 2} \Delta, \quad B^*_2 = P^R_U + \frac{2\nu N_I}{2\nu N_I + N_U + 2} \Delta$$

which implies the bid-ask spread is

$$A^*_2 - B^*_2 = -\frac{\nu N_I}{2\nu N_I + N_U + 2} \Delta$$

the equilibrium depths are

$$\alpha^*_2 = -\frac{\nu N_I N_U}{N_U + 2 + 2\nu N_I} \frac{\Delta}{\delta \text{Var}[\hat{V} | I_U]}, \quad \beta^*_2 = \frac{N_U + 2}{N_U} \alpha^*_2,$$

the investors’ optimal stock demand are given by

$$\theta^*_I_2 = -\frac{\beta^*_2}{N_I}, \quad \theta^*_U_2 = \frac{\alpha^*_2}{N_U}, \quad \theta^*_M_2 = \frac{2}{N_U + 2} \beta^*_2.$$

3. If $\Delta \geq \frac{\nu N_I + 2(N_U + 1)}{\nu N_I} \delta \text{Var}[\hat{V} | I_U] \lambda_U \bar{\theta}$, then the short-sale constraint binds for the uninformed and the equilibrium prices are

$$A^*_{c1} = P^R_U + \frac{(\nu N_I + 1)\Delta - N_U \delta \text{Var}[\hat{V} | I_U] \lambda_U \bar{\theta}}{\nu N_I + 2},$$

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\[ B_{c1}^* = P_U^R + \frac{\nu N_I \Delta - 2 N_U \delta \text{Var}[\tilde{V} | I_U] \lambda_U \bar{\theta}}{\nu N_I + 2}, \] 

which implies the bid-ask spread is

\[ A_{c1}^* - B_{c1}^* = \frac{\Delta + N_U \delta \text{Var}[\tilde{V} | I_U] \lambda_U \bar{\theta}}{\nu N_I + 2}, \]

the equilibrium depths are

\[ \alpha_{c1}^* = \frac{\nu N_I \Delta}{\delta \text{Var}[\tilde{V} | I_U]} + \frac{\nu N_I N_U \lambda_U \bar{\theta}}{\nu N_I + 2}, \quad \beta_{c1}^* = N_U \lambda_U \bar{\theta}. \]

the investors’ optimal stock demand are given by

\[ \theta_{Ic1}^* = \frac{\alpha_{c1}^*}{N_I}, \quad \theta_{Uc1}^* = -\lambda_U \bar{\theta}, \quad \theta_{Mc1}^* = N_U \lambda_U \bar{\theta} - \alpha_{c1}^*. \] 

4. If \( \Delta < -\frac{2(\nu N_I + 1)}{N_U + 2} \delta \text{Var}[\tilde{V} | I_U] \lambda_I \bar{\theta} \), then the short-sale constraint binds for the informed and the equilibrium prices are

\[ A_{c2}^* = P_U^R - \frac{N_I}{N_U + 2} \delta \text{Var}[\tilde{V} | I_U] \lambda_I \bar{\theta}, \quad B_{c2}^* = P_U^R - \frac{2 N_I}{N_U + 2} \delta \text{Var}[\tilde{V} | I_U] \lambda_I \bar{\theta}, \] 

which implies the bid-ask spread is

\[ A_{c2}^* - B_{c2}^* = \frac{N_I}{N_U + 2} \delta \text{Var}[\tilde{V} | I_U] \lambda_I \bar{\theta}, \]

the equilibrium depths are

\[ \alpha_{c2}^* = \frac{N_U}{N_U + 2} N_I \lambda_I \bar{\theta}, \quad \beta_{c2}^* = N_I \lambda_I \bar{\theta}. \]

the investors’ optimal stock demand are given by

\[ \theta_{Ic2}^* = -\lambda_I \bar{\theta}, \quad \theta_{Uc2}^* = \frac{1}{N_U + 2} N_I \lambda_I \bar{\theta}, \quad \theta_{Mc2}^* = \frac{2}{N_U + 2} N_I \lambda_I \bar{\theta}. \] 

Given the results stated in Theorem 3, it is easy to show that as in our main model, short-sale constraints increases equilibrium ask price, decreases bid/ask depths (and thus trading volume). In contrast to our main model, Theorem 3 implies short-sale constraints increase equilibrium bid price (i.e., \( B_{c1}^* > B_1^* \) and \( B_{c2}^* > B_2^* \)).