Over-the-Counter Markets: Market Making with Asymmetric Information, Inventory Risk and Imperfect Competition

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Abstract

We develop an equilibrium market making model to study how information asymmetry, inventory risk, and imperfect competition among market makers jointly affect bid and ask prices, market liquidity, and trading volume in over-the-counter markets. We solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed-form. We also develop a new measure of information asymmetry. Our model can help explain some empirical puzzles such as why bid-ask spreads can be lower with asymmetric information. Moreover, we find that information asymmetry may reduce the welfare loss from market power.

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*Keywords:* Illiquidity, Bid-Ask Spread, Asymmetric Information, Imperfect Competition, Welfare, Over-the-counter Market.
Microstructure theories can be broadly divided into inventory-based models (e.g., Garman (1976), Stoll (1978), and Ho and Stoll (1981)) and information-based ones (e.g., Kyle (1985), Glosten and Milgrom (1985)). As O’Hara (1997) points out, these two types of theories have been largely kept separate. For example, inventory-based models abstract away the impact of information asymmetry, while most information-based models ignore the effect of inventory risk and the ability of market makers to make profit from bid-ask spreads. However, both information and inventory are important determinants of market prices and market liquidity for many financial markets such as over-the-counter (OTC) markets, which are generally characterized by great information asymmetry, high inventory risk, and significant market power and risk aversion of market makers (see, e.g., Garman (1976), Lyons (1995), Green, Hollifield, and Schrhoff (2007), Gravelle (2010), Ang, Shtauber, and Tetlock (2011)). In this paper, we develop an equilibrium model to study how information asymmetry, inventory risk, and market power jointly affect market prices, market liquidity and trading volume in OTC markets.

Although this model incorporates many important features in these markets, such as asymmetric information, inventory risk, imperfect competition, and risk aversion, and allows both bid/ask prices and depths as well as all demand schedules to be endogenous, the model is still tractable. Indeed, we solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed-form even when investors have different risk aversion, different inventory levels, different liquidity shocks, different resale values of the risky asset and heterogeneous private information. These explicit solutions make it possible to conduct reliable comparative statics and to generate empirically testable implications. Another contribution of this paper is the development of a new measure of information asymmetry that is not contaminated by any information quality effect. In addition to the methodology contribution, our model can also help explain some puzzling empirical findings in the literature, such as average bid-ask spreads can be lower with asymmetric information (e.g., Brooks (1996), Huang and Stoll (1997), Chordia, Roll, and Subrahmanyam (2001), Acker, Stalker and Tonks (2002)) and the bid-ask spread can be positively correlated with trading volume.
Specifically, we consider a one-period economy with three types of risk averse investors: informed investors, uninformed investors, and market makers who are also uninformed. On date 0, all investors optimally choose how to trade a risk-free asset and a risky security (e.g., OTC stocks, corporate bonds, and derivatives) to maximize their expected constant absolute risk averse (CARA) utility from the terminal wealth on date 1 and all may be endowed with some shares of the risky security but not the risk-free asset. On date 0 informed investors can observe a private signal about the date 1 payoff of the security before trading and thus they have trading demand motivated by the private information. They are also subject to a liquidity shock that is realized on date 0 before trading. We model the liquidity shock as a random endowment of a nontradable asset whose payoff is correlated with that of the security. Accordingly, informed investors also have trading demand motivated by the liquidity needs for hedging. Different from the existing literature, we assume that there is a public signal about the realized liquidity shock that everyone can observe on date 0 before trading (e.g., macroeconomic news that is related with liquidity shocks). There is a continuum of the informed and the uninformed and thus neither the informed nor the uninformed trade strategically. As in most OTC markets, informed and uninformed investors must trade through market makers.

Following Kyle (1989), we assume that informed and uninformed investors submit their demand schedules before trading. Different from Kyle (1989), however, the demand schedules are dependent on bid and ask prices (rather than a single price) and are submitted not to an auctioneer but to market makers who then determine how to trade at the bid and the ask. To model the market power of market makers in OTC markets, we allow the competition among market makers to be imperfect. In contrast to the standard literature that implicitly assumes Bertrand competition among market makers, we model the competition among market makers as a Cournot competition: after they observe the demand schedules, they simultaneously choose how much to buy at the bid

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1While some of these findings are for exchange traded stocks, the sample period lies where our model would also apply well to these stock markets (e.g., NASDAQ) for which traditional market makers were critical.
and how much to sell at the ask (instead of what prices to post), taking into account the price impact of their trades.\footnote{Most of the existing rational expectations microstructure models assume market makers engage in Bertrand competition, have unlimited capital, and are risk neutral (e.g., Copeland and Galai (1983), Kyle (1985), Glosten and Milgrom (1985)). As is well-known, it takes only two Bertrand competitors to reach the perfect competition equilibrium prices and one implicit assumption of Bertrand competition is that competitors have unlimited capacity to serve the entire market so that undercutting is a credible threat. However, market prices can be far from the perfect competition ones and market makers have limited inventory risk bearing capacity, especially in OTC markets (e.g., Christie and Schultz (1994), Chen and Ritter (2000), and Biais, Bisière and Spatt (2003)). In addition, the capital of market makers is likely finite and market makers can be risk-averse (e.g., Garman (1976), Lyons (1995)).} The equilibrium bid and ask prices are then determined by the market clearing conditions at the bid and at the ask, i.e., the total amount market makers buy (sell) at the bid (ask) is equal to the total amount other investors sell (buy). In equilibrium, both the risky security market and the risk-free asset market clear.

We show that in equilibrium, the bid-ask spread, the amount traded by the informed and the uninformed, and market trading volume all increase with the absolute value of the reservation price difference between the informed and the uninformed, which can be viewed as a measure of the size of the pie that is shared among all investors. Intuitively, the greater the reservation price difference, the greater the total gain to trades, the more the informed and the uninformed trade, and the more market makers can extract by indirectly charging a higher spread. This implies that in contrast to the literature on portfolio selection with transaction costs (e.g., Davis and Norman (1990), Liu (2004)), bid-ask spreads can be positively correlated with trading volume.

Why can expected bid-ask spreads be smaller with asymmetric information? This is because asymmetric information can reduce the reservation price difference between the informed and the uninformed. Unlike “noise traders” in most rational expectations models, the uninformed in our model rationally revise their reservation price upon observing market prices. With asymmetric information, the higher estimation risk premium required by the uninformed and the uninformed’s overestimation or underestimation of the expected security payoff relative to the informed may reduce the reservation price difference and hence may decrease the spread. This result is not driven by market makers’ inventory risk because the reservation price difference between the informed and the uninformed is independent of market makers’ risk aversion. Consistent with the finding
that expected spreads may decrease with information asymmetry, we show that greater information asymmetry can also reduce the social welfare loss from market power.

While information asymmetry has been a focal point of most of the existing information-based models, there is still no good measure of information asymmetry as far as we know. Changes in a good measure of information asymmetry should change information asymmetry, but not the aggregate information quality (measured by the precision of security payoff distribution conditional on all the information in the economy), because both information asymmetry and information quality can affect economic variables of interest (e.g., prices, liquidity). For example, a candidate information asymmetry measure may be the precision of the private signal about the security payoff. However, as the private signal becomes more precise, the aggregate information quality also increases. In contrast, we use the variance of the public signal about the liquidity shock as a measure of information asymmetry. Because informed investors observe their liquidity shock perfectly, this variance does not change the aggregate information quality. On the other hand, because market prices reveal the combined demand from information and liquidity shock, as this variance decreases, the uninformed can better estimate the security payoff and thus information asymmetry decreases. For example, when this variance decreases to 0, the uninformed can perfectly observe the liquidity shock, can then fully back out the private signal of the informed and thus the model becomes one with symmetric information. In the other extreme, as this variance increases to infinity, it becomes equivalent to the standard asymmetric information case where there is no public signal about the liquidity shock and thus achieves the maximum information asymmetry, \textit{ceteris paribus}.\footnote{This implies one additional benefit from introducing the public signal: our model nests both the symmetric information case and the standard asymmetric information case.}

We also solve in closed-form a generalized model where investors can have different risk aversion, different inventory levels, different liquidity shocks, different resale values of the security and heterogeneous private information. There are eight types of equilibria characterized by the trading directions of investors, e.g., some investors may choose not to trade in equilibrium and both the
informed and the uninformed can trade in the same directions. Although our model abstracts away sequential order arrivals and allow market makers to trade both at the bid and at the ask on the same date, market makers may choose to trade only on one side on a trading date in the generalized model, similar to the trading behavior in a sequential order arrival model. Some empirically testable implications from the generalized model include: average bid-ask spread is more sensitive to market makers’ inventory level in less active markets; as market makers’ inventory increases, average bid and ask prices decrease, average ask depth increases and average bid depth decreases. The generalized model can indirectly capture some features such as search, pre- and post-trade opacity that might be important for some OTC markets such as bond and pink sheets markets. For example, when search cost is high, search takes a long time, and the resale value of the security is with large uncertainty, one can model this as a low mean and high volatility distribution for the resale value of the security acquired by market makers. This is clearly just a reduced form, but likely indirectly captures the first order effect of these features. For example, when search cost is high and the uncertainty about the resale value of the security is large, market makers charge a higher premium for the security on date 0 and the bid-ask spread increases in a search model (e.g., Duffie, Gârleanu, and Pedersen (2005, 2007)). With a lower mean and higher volatility for the resale value, our model can generate the same qualitative result.

Different from inventory-based models, our model takes into account the impact of information asymmetry on bid and ask prices and inventory levels, which is especially important for OTC markets. In addition, we also allow market makers to vary bid and ask depths to better manage inventory risk. In contrast to information-based (rational expectations) models, our model recognizes that market makers can profit from bid-ask spreads and inventory risk can be significant for market makers. Most of the information-based models assume a zero expected profit condition for each trade of a market maker, implying a zero expected profit from acquired inventory. This is inconsistent with the liquidity provision mandate for most designated market makers. In addition,

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4Madhavan and Sofianos (1998) find that market makers mainly adjust quote depths to manage inventories.
as shown by the existing empirical literature, market makers tend to offset their trades to reduce inventory risk and make their profit mostly from the bid-ask spread (e.g., Sofianos (1993)). In contrast to most information-based models, market makers in our model may be willing to lose money from a particular trade in expectation (at bid or at ask) in equilibrium especially when they have high initial inventory and they can make significant profit from the bid-ask spread. On the other hand, when market makers do not have any initial inventory they make positive expected profit from inventory because of the required inventory risk premium, consistent with the findings of Hendershott, Moulton, and Seasholes (2007).

There is a large literature on dealership markets. For example, Amihud and Mendelson (1980) consider the problem of a price-setting monopolistic market-maker in a dealership market where the stochastic demand and supply follow independent Poisson processes. Darrell, Gärleanu, and Pedersen (2005) analyze market making in over-the-counter markets using a search model. Neither of these papers examines the effect of asymmetric information on market prices and market liquidity. Our model is also related to Diamond and Verrecchia (1981), Subrahmanyam (1991), Diamond and Verrecchia (1991), Dennert (1993), Naik, Neuberger, and Viswanathan (1999), Back and Baruch (2004), Vayanos and Wang (2011), and Rosu (2010). Diamond and Verrecchia (1981) analyze a rational expectations equilibrium model of a competitive security market in which traders possess independent pieces of information about the return of a risky asset. Subrahmanyam (1991) finds that increasing the precision of private information intensifies competition between risk averse informed investors and thus can increase market liquidity. Diamond and Verrecchia (1991) show that reducing information asymmetry can increase liquidity and security prices may be nonmonotonic in information asymmetry because of the potential exit of market makers. In all these three papers, market makers post a single price, the trading needs of some of the uninformed investors (i.e., “noise investors”) are exogenous and thus do not respond to price changes. Therefore if market makers

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5In the special case where market makers are extremely risk averse and thus do not hold any inventories across trading periods, the potential date 1 payoff of the stock is irrelevant for market makers’ pricing or trading decision. This special case bears some similarity to high frequency market makers who only carry any significant inventory for at most very short period of time (typically less than 1 hour) and private information about the fundamentals of the security is thus less relevant.
were allowed to post bid and ask prices, then in contrast to our predictions, the bid-ask spread would always be increasing in information asymmetry. Dennert (1993) examines how price competition among risk neutral market makers affects market quality, assuming market makers must post bid and ask prices before observing order flows and informed traders are risk neutral and have unlimited capacity. He shows that an increase in the number of market makers induces more insider trading and may decrease market quality. Naik, Neuberger, and Viswanathan (1999) examine whether full and prompt disclosure of public-trade details improves the welfare of a risk-averse investor in a two-stage dealership market. Similar to Diamond and Verrecchia (1991), market makers post a single price which is the conditional expected payoff of the security. Back and Baruch (2004) solve a version of the Glosten-Milgrom model with a single informed investor, in which the informed investor chooses his trading times optimally. As in the original Glosten-Milgrom model, they find that the bid-ask spread is greater with asymmetric information. Vayanos and Wang (2011) examine how liquidity and asset prices are affected by market imperfections such as asymmetric information, leverage constraints, and transaction costs. They find that imperfections may decrease expected returns, and can have opposite impact on market illiquidity. In contrast to our model, Vayanos and Wang (2011) assume a zero bid-ask spread and transaction costs are exogenous. Rosu (2010) finds that bid-ask spread can decrease with the fraction of informed investors because the competition among them intensifies (similar to the effect of signal precision on competition in Subrahmanyam (1991)) and they can trade with impatient investors who are assumed to always submit market orders. Another strand of literature studies the effect of illiquidity on portfolio choice and asset pricing (e.g., Constantinides (1986), Vayanos (1998), Liu and Loewenstein (2002), Lo, Mamaysky and Wang (2004), Liu (2004), Acharya and Pedersen (2005)). In this literature, illiquidity is generally modeled as exogenous transaction costs and therefore the fundamental question of what affects illiquidity is largely unanswered.

The remainder of the paper proceeds as follows. In Section 1 we present the model. In Section 2 we derive the equilibrium. In Section 3 we provide some comparative statics on asset prices,
illiquidity, and welfare. In Section 4, we present, solve and discuss a generalized model. We conclude in Section 5. All proofs are in the Appendix.

1. The model

We consider a one period setting with trading dates 0 and 1. There are a continuum of identical informed investors with mass \( N_I \), a continuum of identical uninformed investors with mass \( N_U \), and \( N_M \) identical designated market makers (\( M \)) who are also uninformed. They can trade one risk-free asset and one risky security on date 0 and date 1 to maximize their expected constant absolute risk aversion (CARA) utility from their wealth on date 1. There is a zero net supply of the risk-free asset, which also serves as the numeraire and thus the risk-free interest rate is normalized to 0. The total supply of the security is \( N \times \bar{\theta} \geq 0 \) shares where \( N = N_I + N_U + N_M \) and the date 1 payoff of each share \( \bar{V} \sim N(\bar{V}, \sigma^2_V) \) becomes public on date 1, where \( \bar{V} \) is a constant, \( \sigma_V > 0 \), and \( N \) denotes the normal distribution. The aggregate risky asset endowment is \( N_i \bar{\theta} \) shares for type \( i \in \{I, U, M\} \) investors. No investor is endowed with any risk-free asset.

On date 0, informed investors observe a private signal

\[
\hat{s} = \bar{V} - \tilde{V} + \tilde{\varepsilon}
\]  

(1)

about the payoff \( \bar{V} \), where \( \tilde{\varepsilon} \) is independently normally distributed with mean zero and variance \( \sigma^2_\varepsilon \). In addition to the security, every informed investor is also subject to a liquidity shock that is modeled as a random endowment of \( \hat{X}_I \sim N(0, \sigma^2_X) \) units of a non-tradable risky asset on date 0, with \( \hat{X}_I \) realized on date 0 and only directly known to informed investors.\(^7\) The non-tradable asset has a per-unit payoff of \( \bar{N} \sim N(0, \sigma^2_N) \) that has a covariance of \( \sigma_{VN} \) with \( \bar{V} \) and is realized and becomes public on date 1. The correlation between the non-tradable asset and the security results

\(^6\)Throughout this paper, “bar” variables are constants, “tilde” random variables are realized on date 1 and “hat” random variables are realized on date 0.

\(^7\)The random endowment can represent any shock in the demand for the security, such as a liquidity shock or a change in the needs for rebalancing an existing portfolio or a change in a highly illiquid asset.
in a liquidity demand for the risky asset to hedge the non-tradable asset payoff. In a model with private information sources such as a private signal \( \hat{s} \) and a private liquidity shock (e.g., Grossman and Stiglitz (1980), Wang (1994), O’Hara (2003), and Vayanos and Wang (2011)), assuming that all investors who are subject to liquidity shock also observe \( \hat{s} \) is only for simplicity: even if they do not observe \( \hat{s} \), they can infer it perfectly from the equilibrium price, because the equilibrium price is an invertible function of the weighted sum of \( \hat{s} \) and the private liquidity shock. In this type of models asymmetric information can therefore exist only if some investors who do not have any liquidity shock do not observe \( \hat{s} \) either and are thus uninformed, as in Vayanos and Wang (2011) for example. We assume that these investors are all uninformed for simplicity.\(^8\)

In contrast to the existing literature that assume uninformed investors cannot observe any signal about liquidity demands (e.g., those from noise traders) while the informed can perfectly infer the liquidity demands from their private information about the security and the equilibrium asset price, we assume that there is a public signal

\[
\hat{S}_x = \hat{X}_I + \hat{\eta}
\]  

about the liquidity shock \( \hat{X}_I \) that all investors (i.e., the uninformed, market makers, and the informed) can observe, where \( \hat{\eta} \) is independently normally distributed with mean zero and variance \( \sigma^2_\eta \).\(^9\) This public signal represents public news about liquidity demand determinants, such as fund flows, macroeconomic conditions and regulation shocks.\(^10\) While this additional signal \( \hat{S}_x \) is not critical for our main results (e.g., spread can be smaller with asymmetric information), it allows us to model different degrees of information asymmetry in one unified setting. For example, the

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\(^8\) Alternatively, one can view an informed investor as a broker who combines information motivated trades and liquidity motivated trades. The assumption that all informed traders have the same information is only for simplicity. Our main results still hold when they have differential information (see Section 4).

\(^9\) While this public signal can also be observed by the informed, it is useless to them because they can already perfectly observe it privately.

\(^10\) As another example, in a companion paper where we consider two independent securities that are both correlated with liquidity demands, the equilibrium price of the other security can serve as a public signal about the liquidity demand.
case where \( \sigma_q^2 = 0 \) implies that the uninformed and market makers can perfectly observe \( \hat{X}_I \) from the public signal and thus can in turn perfectly infer the private signal \( \hat{s} \) from the equilibrium security price, as will be shown later. Therefore, the case where \( \sigma_q^2 = 0 \) represents the symmetric information case. The case where \( \sigma_q^2 = \infty \), on the other hand, implies that the signal \( \hat{S}_x \) is useless and thus corresponds to the asymmetric information case as modeled in the standard literature, i.e., there is no public news about liquidity demand. In addition, as shown later, as the variance \( \sigma_q^2 \) increases, the difference between the security payoff conditional variances of the informed and the uninformed increases while other important factors such as the aggregate information quality (to be defined) about the security payoff remain the same. Thus, the variance \( \sigma_q^2 \) can serve as a measure of the degree of information asymmetry.

All trades must go through the designated market makers (dealers) whose market making cost is assumed to be 0.\(^{11}\) Specifically, \( I \) and \( U \) investors sell to market makers at the bid \( B \) or buy from them at the ask \( A \) or do not trade at all. Given that there is a continuum of informed and uninformed investors, we assume that they are price takers and there are no strategic interactions among them or with market makers.\(^{12}\)

Following Kyle (1989), we assume that informed and uninformed investors submit their demand schedules before trading. Different from Kyle (1989), however, the demand schedules are submitted not to an auctioneer but to designated market makers who then determine how to trade, and the schedules depend on both the bid and ask prices rather than a single trading price. This assumption is consistent with a dealership market microstructure where designated market makers observe order flows before determining bid and ask depths and prices.

For each \( i \in \{I, U, M\} \), investors of type \( i \) are identical both before and after realizations of signals on date 0 and thus adopt the same trading strategy. Let \( I_i \) represent a type \( i \) investor’s information set on date 0 for \( i \in \{I, U\} \). For \( i \in \{I, U\} \), a type \( i \) investor’s problem is to choose

\(^{11}\)Assuming zero market making cost is only for better focus and expositional simplicity. Market making cost is considered in an earlier version, where a potential market maker must pay a fixed market-making utility cost on date 0 to become a market maker. We show that no results in this paper are altered by this fixed cost.

\(^{12}\)Strategic informed traders have been studied in the existing literature (e.g., Kyle (1989)). We abstract away this strategic interaction to make what drives our main results clearer.
the (signed) demand schedule \( \theta_i(A, B) \) to

\[
\max E[-e^{-\delta \tilde{W}_i} | I_i],
\]

where

\[
\tilde{W}_i = \theta_i^- B - \theta_i^+ A + (\tilde{\theta} + \theta_i) \tilde{V} + \tilde{X} \tilde{N},
\]

\( \tilde{X} U = 0, \delta > 0 \) is the absolute risk-aversion parameter, \( x^+ := \max(0, x) \), and \( x^- := \max(0, -x) \).

Since \( I \) and \( U \) investors buy from market makers at ask and sell to them at bid, we can view these trades occur in two separate markets: the “ask” market and the “bid” market. In the “ask” market, market makers are suppliers and other investors are demanders and the opposite is true in the “bid” market. As market makers supply (sell) more in the “ask” market, the ask price goes down and as market makers demand (buy) more in the “bid” market, the bid price goes up. Accordingly, in contrast to the standard microstructure literature where market makers directly choose market prices, we assume market makers directly choose how much to buy at bid given the inverse supply function (a function of the market makers’ purchasing quantity) of all other participants and how much to sell at ask given the inverse demand function (a function of the market makers’ selling quantity) of all other participants. The posted bid and ask prices are the required prices to achieve the optimal amount market makers choose to trade. Since all trades must go through market makers, market makers can have market powers especially when the number of market makers is small. To model the oligopolistic competition among the market makers, we use the notion of the Cournot competition that is well studied and understood. Specifically, we assume that market makers simultaneously choose the optimal numbers of shares to sell at ask and to buy at bid, taking into account the price impact of their trades.

Let \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_{NM})^\top \) and \( \beta = (\beta_1, \beta_2, ..., \beta_{NM})^\top \) be the vector of the number of shares market makers sell at ask (i.e., ask depth) and buy at bid (i.e., bid depth) respectively.\(^{13}\) Given

\(^{13}\)To help remember, Alpha denotes Ask depth and Beta denotes Bid depth.
the demand schedules of the informed and the uninformed \((\theta^*_I(A, B) \text{ and } \theta^*_U(A, B))\), the bid price \(B(\beta)\) (i.e., the inverse supply function in the bid market) and the ask price \(A(\alpha)\) (i.e., the inverse demand function in the ask market) can be determined by the following security market clearing conditions at the bid and ask prices.\(^{14}\)

\[
\sum_{j=1}^{N_M} \alpha_j = \sum_{i=I, U} N_i \theta^*_I(A, B)^+, \quad \sum_{j=1}^{N_M} \beta_j = \sum_{i=I, U} N_i \theta^*_U(A, B)^-, \quad (5)
\]

where the left-hand sides represent the total sales and purchases by market makers respectively and the right-hand sides represent the total purchases and sales by other investors respectively.

Then for \(j = 1, 2, ..., N_M\), the designated market maker \(M_j\)'s problem is to choose ask depth \(\alpha_j \geq 0\) and bid depth \(\beta_j \geq 0\) to

\[
\max E \left[ -e^{-\delta \tilde{W}_{M_j} |I_M|} \right], \quad (6)
\]

where

\[
\tilde{W}_{M_j} = \alpha_j A(\alpha) - \beta_j B(\beta) + (\bar{\theta} + \beta_j - \alpha_j) \bar{V}. \quad (7)
\]

Note that different from other investors, a market maker takes into account the price impact of her own trades, i.e., recognizing that both \(A\) and \(B\) will be affected by her trades.

This leads to our definition of the Nash equilibrium of the Cournot competition.\(^{15}\)

**Definition 1** An equilibrium \((\theta^*_I(A, B), \theta^*_U(A, B), A^*, B^*, \alpha^*, \beta^*)\) is such that

1. given any \(A\) and \(B\), \(\theta^*_i(A, B)\) solves a type \(i\) investor’s Problem (3) for \(i \in \{I, U\}\);

2. given \(\theta^*_I(A, B)\) and \(\theta^*_U(A, B)\), \(\alpha_j^*\) and \(\beta_j^*\) solve market maker \(M_j\)'s Problem (6), for \(j = 1, 2, ..., N_M\);

\(^{14}\)The risk-free asset market will be automatically cleared by the Walras’ law. A buyer’s (seller’s) trade only depends on ask \(A\) (bid \(B\)). So \(A\) only depends on \(\alpha\) and \(B\) only depends on \(\beta\).

\(^{15}\)Deviations by undercutting prices can be prevented by matching prices by other market makers in subsequent periods in a repeated-game setting. As in standard Cournot competition models, varying prices is not in the strategy space.
3. \( A^* := A(\alpha^*) \) and \( B^* := B(\beta^*) \) clear both the risky security and the risk-free asset markets, where \( A(\alpha) \) and \( B(\beta) \) solve Equation (5).

2. The equilibrium

In this section, we solve the equilibrium bid and ask prices, bid and ask depth and trading volume in closed form.

Given \( A \) and \( B \), the optimal demand schedule for a type \( i \) investor for \( i \in \{I, U\} \) is

\[
\theta^*_i(A, B) = \begin{cases} 
\frac{P^R - A}{\delta \text{Var}[V|\tilde{L}_i]} & A < P^R_i, \\
0 & B \leq P^R_i \leq A, \\
\frac{B - P^R_i}{\delta \text{Var}[V|\tilde{L}_i]} & B > P^R_i, 
\end{cases}
\]

(8)

where

\[
P^R_i = E[\tilde{V}|\tilde{L}_i] - \delta \text{Cov}[\tilde{V}, \tilde{N}|\tilde{L}_i]\tilde{X}_i - \delta \text{Var}[\tilde{V}|\tilde{L}_i]\tilde{\theta}
\]

(9)

is the reservation price of a type \( i \) investor (i.e., the critical price such that non-market-makers buy (sell, respectively) the security if and only if the ask price is lower (the bid price is higher, respectively) than this critical price). Equations (9) also shows that if there is no aggregate risk (\( \tilde{\theta} = 0 \), e.g., for derivatives whose aggregate supply is zero), the reservation prices only depend on the expected payoff and hedging premium (\(-\delta \text{Cov}[\tilde{V}, \tilde{N}|\tilde{L}_i]\tilde{X}_i\)), but not on payoff conditional variance.

Because the informed know exactly \( \{\tilde{s}, \tilde{X}_I\} \) while equilibrium prices \( A^* \) and \( B^* \) and the public signal \( \tilde{S}_x \) are only noisy signals about \( \{\tilde{s}, \tilde{X}_I\} \), the information set of the informed in equilibrium is

\[
\mathcal{I}_I = \{\tilde{s}, \tilde{X}_I, \tilde{S}_x, A^*, B^*\} = \{\tilde{s}, \tilde{X}_I\},
\]

(10)
which implies that

\[
E[\tilde{V} | I] = \tilde{V} + \rho_I \hat{s}, \quad \text{Var}[\tilde{V} | I] = \rho_I \sigma_\tilde{V}^2, \quad \text{Cov}[\tilde{V}, \tilde{N} | I] = (1 - \rho_I) \sigma_{V,N},
\]

(11)

where

\[
\rho_I := \frac{\sigma_{\tilde{V}}}{\sigma_{\tilde{V}}^2 + \sigma_\tilde{x}^2}.
\]

(12)

Equation (9) then implies that

\[
P_I^R = \tilde{V} + \tilde{S} - \delta \rho_I \sigma_\tilde{V}^2 \tilde{\theta},
\]

(13)

where \( \tilde{S} := \rho_I \hat{s} - \delta (1 - \rho_I) \sigma_{V,N} \hat{X}_I \).

While \( \hat{s} \) and \( \hat{X}_I \) both affect the informed investor’s demand, other investors can only infer the value of \( \hat{S} \) from an informed investor’s order because the joint impact of \( \hat{s} \) and \( \hat{X}_I \) on the informed’s order is only in the form of \( \hat{S} \). In addition to \( \hat{S} \), other investors can also observe the public signal \( \hat{S}_x \) about the liquidity shock \( \hat{X}_I \). Thus we conjecture that the equilibrium prices \( A^* \) and \( B^* \) depend on both \( \hat{S} \) and \( \hat{S}_x \). Accordingly, the information sets for the uninformed investors and market makers are

\[ I_U = I_M = \{ A^*, B^*, \hat{S}_x \} = \{ \hat{S}, \hat{S}_x \}. \]

(14)

Then the conditional expectation and variance of \( \tilde{V} \) for the uninformed and market makers are respectively

\[
E[\tilde{V} | I_U] = \tilde{V} + \rho_U \hat{S} + \delta (1 - \rho_I) \sigma_{V,N} \rho_U \rho_X \hat{S}_x,
\]

(15)

\[
\text{Var}[\tilde{V} | I_U] = \rho_U \rho_I \sigma_\tilde{V}^2 + \frac{\rho_U}{\rho_I} \delta^2 (1 - \rho_I)^2 \sigma_{V,N}^2 \rho_X^2 \sigma_\eta^2,
\]

(16)

where

\[
\rho_X := \frac{\sigma_X^2}{\sigma_X^2 + \sigma_\eta^2}, \quad \rho_U := \frac{\rho_I \sigma_\tilde{V}^2}{\rho_I \sigma_\tilde{V}^2 + \delta^2 (1 - \rho_I)^2 \sigma_{V,N}^2 \rho_X \sigma_\eta^2} \leq 1.
\]

(17)
It follows that the reservation price for a $U$ investor and an $M$ investor is

$$P^{R}_U = P^{R}_M = \bar{V} + \rho_U \left( \hat{S} + \delta (1 - \rho_I) \sigma_{VN} \rho_X \hat{S}_x \right) - \frac{\rho_U}{\rho_I} \left( \rho_I^2 \sigma_{\hat{Z}}^2 + \delta^2 (1 - \rho_I)^2 \sigma_{V_N}^2 \rho_X \sigma_{\eta}^2 \right) \bar{\theta}.$$ \hspace{1cm} (18)

Let $\Delta RP$ denote the difference in the reservation prices of $I$ and $U$ investors. We then have

$$\Delta RP := P^{R}_I - P^{R}_U = (1 - \rho_U) \left( \hat{S} - \frac{\sigma_{\hat{V}}^4}{\delta \sigma_{\hat{Z}}^2 \sigma_{V_N} \sigma_{\eta}^2} \hat{S}_x + \delta \rho_I \sigma_{\hat{V}}^2 \bar{\theta} \right).$$ \hspace{1cm} (19)

Define

$$C_I := \frac{N_M (N_U + N_M + 1)}{(N_M + 1) (N + 1)}, \quad C_U := \frac{\nu N_M N_I}{(N_M + 1) (N + 1)},$$ \hspace{1cm} (20)

where

$$\bar{N} := \nu N_I + N_M + N_U \geq N$$

and

$$\nu := \frac{\text{Var}[\hat{V} | I_U]}{\text{Var}[\hat{V} | I_I]} = \rho_U + \frac{\rho_U \rho_X \delta^2 \sigma_{V_N}^2 \sigma_{\hat{Z}}^2 \sigma_{\eta}^2}{\sigma_{\hat{V}}^2} \geq 1$$

is the ratio of the security payoff conditional variance of the uninformed to that of the informed. The following theorem provides the equilibrium bid and ask prices and equilibrium security demand. \hspace{1cm} (16)

**Theorem 1**

1. The equilibrium bid and ask prices are

$$A^* := A(\alpha^*) = P^{R}_U + C_U \Delta RP + \frac{\Delta RP^+}{N_M + 1},$$

$$B^* := B(\beta^*) = P^{R}_U + C_U \Delta RP - \frac{\Delta RP^-}{N_M + 1},$$

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\(16\) Since market makers are identical, we use notations without the subscript \(j\) to save notation. Because all utility functions are strictly concave, all budget constraints are linear in the amount invested in the security and both the informed and the uninformed are price takers, there is a unique solution to the problem of each informed and each uninformed given the bid and ask prices. Because the inverse demand and supply functions implied by the market clearing conditions are linear in market depths and when a market maker trades a strictly positive amount with both the informed and the uninformed, there is a unique solution to her utility maximization problem (which already takes into account the market clearing conditions). This implies that there is a unique equilibrium when all investors trade in equilibrium. When some investors do not trade in equilibrium, there are multiple equilibria because either bid or ask would not be unique (see Theorem 2).
and we have $A^* > P^* > B^*$, where

$$P^* = \frac{\nu N_I}{N} P^R_I + \frac{N_U}{N} P^R_U + \frac{N_M}{N} P^R_M$$

(21)

is the equilibrium price of a perfect competition equilibrium where market makers are also price takers. The bid-ask spread is

$$A^* - B^* = \frac{|\Delta RP|}{N_M + 1} = \frac{(1 - \rho_U) \hat{S} - \frac{\sigma_t^2}{\delta \sigma_v \sigma_1} \hat{S}_x + \delta \rho_1 \sigma_v^2 \hat{\theta}}{N_M + 1}.$$  

(22)

2. The equilibrium quantities demanded are

$$\theta_I^* = C_I \frac{\Delta RP}{\delta \text{Var}[V|\hat{I}_I]}, \quad \theta_U^* = -C_U \frac{\Delta RP}{\delta \text{Var}[V|\hat{I}_U]}, \quad \theta_M^* = \frac{N_M + 1}{N_M} \theta_U^*;$$

(23)

the equilibrium quote depths are

$$\alpha^* = \frac{N_I}{N_M} (\theta_I^*)^+ + \frac{N_U}{N_M} (\theta_U^*)^+, \quad \beta^* = \frac{N_I}{N_M} (\theta_I^*)^- + \frac{N_U}{N_M} (\theta_U^*)^-;$$

(24)

(25)

which implies that the equilibrium trading volume is

$$N_M (\alpha^* + \beta^*) = \frac{N_M N_I (N_M + 2N_U + 1)}{(N_M + 1)(N + 1)} \left( \frac{|\Delta RP|}{\delta \text{Var}[V|\hat{I}_I]} \right).$$

(26)

Part 1 of Theorem 1 implies that in equilibrium both bid and ask prices are determined by the reservation price of the uninformed and the reservation price difference between the informed and the uninformed. In addition, given the public signal $\hat{S}_x$, all investors can indeed infer $\hat{S}$ from observing the equilibrium prices as conjectured, because of the one-to-one mapping between
As illustrated in Figure 1, Part 1 also implies that holding constant the aggregate population of noninformed investors \((N_M + N_U)\), as \(N_M\) increases, the bid (ask) price monotonically increases (decreases) to the competitive market equilibrium price \(P^*\), which is simply equal to the weighted average of reservation prices across all investors. Furthermore, Part 1 shows that the equilibrium bid-ask spread is equal to the absolute value of the reservation price difference between the informed and the uninformed, divided by the number of market makers plus one. Therefore, the spread increases with the absolute value of the reservation price difference and decreases with competition.

Equation (23) implies that \(I\) investors buy and \(U\) investors sell if and only if \(I\) investors have a higher reservation price than \(U\) investors. Because market makers have the same reservation price as the \(U\) investors, in the net they trade in the same direction as \(U\) investors. However, because market makers can make profit from the spread, they trade more in the net than \(U\) investors.

In the standard literature on portfolio choice with transaction costs (e.g., Liu and Loewenstein (2002), Liu (2004)), it is well established that as the bid-ask spread increases, investors reduce

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\(^{17}\)In our model, market makers observe order flow and can infer how much informed investors are trading. However, they do not know how much of the informed investor’s order is due to information on the security’s payoff or how much is due to the hedging demand. This is similar to the set-up of Glosten (1989) and Vayanos and Wang (2011).
trading volume to save on transaction costs and thus trading volume and bid-ask spread move in the opposite directions in these models. In contrast, Theorem 1 implies that bid-ask spreads and trading volume can move in the same direction, because both trading volume and bid-ask spread increase with $|\Delta RP|$. Lin, Sanger and Booth (1995) find that trading volume and effective spreads are positively correlated at the beginning and the end of the day. Chordia, Roll, and Subrahmanyam (2001) find that the effective bid-ask spread is positively correlated with trading volume. Our model suggests that these positive correlations may be caused by changes in the valuation difference of investors.

The net order size is the magnitude of the difference between the total buy order size and total sell order size and is equal to $N_M|\alpha^* - \beta^*|$ by market clearing condition (5). Theorem 1 implies that net order size and bid-ask spread are positively correlated even in the symmetric information case ($\sigma^2 \rightarrow 0$, as shown in Proposition 1), because both increase with $|\Delta RP|$. A typical justification of this positive correlation (e.g., Easley and O’Hara (1987, 1992)) is that as the net order size increases, the adverse selection effect of information asymmetry increases and thus the bid-ask spread increases. We offer an alternative explanation: it may be that changes in both the net order size and the spread are driven by changes in the reservation price difference. Intuitively, as the reservation price difference increases, the spread increases, market makers are willing to sell or buy more in the net and thus the net order size also increases.

Next we provide the essential intuition for the results in Theorem 1 through graphical illustrations. Suppose $P^R_I > P^R_U$ and thus $I$ investors buy and $U$ investors sell. The market clearing condition (5) implies that the inverse demand and supply functions faced by the market makers are respectively

$$A = P^R_I - k_I\alpha, \quad B = P^R_U + k_U\beta,$$

where

$$k_i = \frac{N_M\delta \text{Var}[^{\hat{V}}|Z_i]}{N_i}, \quad i = I, U.$$
\( N_M = 1 \)

(a) The Informed Buy and the Uninformed Sell

(b) The Uninformed Buy and the Informed Sell

Figure 2: Inverse Demand/Supply Functions and Bid/Ask Spreads.
We plot the above inverse demand and supply functions and equilibrium spreads in Figures 2(a) with $N_M = 1$. Similarly, in Figure 2(b), we plot the case where the informed sell, the uninformed buy and $N_M = 3$. Figure 2 shows that as market makers buy (sell) more at the bid (ask), the bid (ask) price goes up (down). Facing the inverse demand and supply functions, market makers optimally trade off the prices and quantities, as in a standard Cournot competition. Similar to the results of classical Cournot competition models of multiple firms who compete through choosing the amount of output of a homogeneous product, the bid and ask spread is equal to the absolute value of the reservation price difference $|ΔRP|$, divided by the number of market makers plus one. In addition, as implied by Theorem 1, Figure 2 illustrates that the difference between $P_{R}^{I}$ ($P_{R}^{U}$) and the ask (bid) price is also proportional to the reservation price difference $|ΔRP|$. Therefore the trading amount of both $I$ and $U$ investors and thus the aggregate trading volume all increase with $|ΔRP|$. The shaded areas represent the profits (min($α^∗, β^∗$)$($A^∗ − B^∗$)) made from the bid-ask spread at time 0.

In contrast to the standard rational expectations literature which assumes zero expected profit for each trade (e.g., Glosten and Milgrom (1985)), Theorem 1 implies that a market maker may lose money in expectation on a particular trade. For example, suppose $ΔRP > 0$ (which implies that the informed buy at the ask and the uninformed sell at the bid), the per share expected profit of the market maker from the trade at the bid (not including the profit from the spread) is equal to

$$E[\tilde{V}|\mathcal{I}_M] − B^∗ = δ\text{Var}[\tilde{V}|\mathcal{I}_M]\tilde{θ} − C_U ΔRP,$$

(27)

which can be negative if $ΔRP$ is large, in which case the market maker on average loses to the uninformed and makes money from the informed. The market maker is willing to buy from the uninformed in anticipation of a loss from this trade because she can sell the purchased shares at a higher price (i.e., ask). Because of the hedging benefit, the informed may be willing to buy from the market maker in anticipation of a loss. This same intuition applies to a dynamic setting where orders arrive sequentially. For example, seeing an order to sell at the bid, if the market maker
expects that she will be able to unwind part of her purchase later at a higher price, she would be willing to accommodate the sell order even in anticipation of a loss. This suggests that using a dynamic model does not change these qualitative results, while making the analysis intractable.

Theorem 1 and Equation (27) implies when $\Delta RP < 0$, market makers buy in the net and they make positive expected profit from inventory carried over if they do not have any initial inventory (i.e., $\theta = 0$), because of the required inventory risk premium. This is consistent with the findings of Hendershott, Moulton, and Seasholes (2007).

In the following proposition, we show that the symmetric information case where all investors observe the same signal $\hat{s}$ as the informed and the standard asymmetric information case where there is no public signal about the hedging demand are all special cases of the asymmetric information case we consider above.\textsuperscript{18}

**Proposition 1**

1. As $\sigma_n^2 \downarrow 0$, we have

(a) $\text{Var}[\hat{V}|I_U] \to \text{Var}[\hat{V}|I_I]$ and $E[\hat{V}|I_U] \to E[\hat{V}|I_I]$, therefore the asymmetric information converges to the symmetric information case where all investors observe the signal $\hat{s}$;

(b) In addition, $\nu \downarrow 1$, $\mathcal{N}_{\downarrow} \to N$, $\rho_X \uparrow 1$, $\rho_U \uparrow 1$, $\hat{S}_x \to \hat{X}_I$, $\Delta RP \to -\delta(1 - \rho_I)\sigma_V \hat{X}_I$;

2. As $\sigma_n^2 \uparrow \infty$, we have

(a) $\text{Var}[\hat{V}|I_U]$ converges to $\text{Var}[\hat{V}|\hat{S}] = \rho_U \rho_I (\sigma_z^2 + \delta^2(1 - \rho_I)^2 \sigma_V \sigma_X / \rho_I^2)$ and $E[\hat{V}|I_U]$ converges to $E[\hat{V}|\hat{S}] = \hat{V} + \rho_U \hat{S}$, therefore the asymmetric information case where there is a public signal about $\hat{X}_I$ converges to the standard asymmetric information case where there is no such public signal.

(b) In addition, $\rho_X \downarrow 0$, $\rho_X \sigma_n^2 \to \sigma_X^2$,

$$\Delta RP \to (1 - \rho_U) \left( \hat{S} + \delta \rho_I \sigma_V \theta \right),$$

\textsuperscript{18}The proof is straightforward from direct computation and thus omitted.
\[
\rho_U \to \frac{\rho_1 \sigma_V^2}{\rho_1 \sigma_V^2 + \delta^2 (1 - \rho_1)^2 \sigma_{V_N}^2 \sigma_X^2};
\]

3. As \( \sigma_e^2 \uparrow \infty \), we have both \( \text{Var}[\tilde{V} | I_I] \) and \( \text{Var}[\tilde{V} | I_U] \) converge to the unconditional variance \( \sigma_V^2 \), both \( E[\tilde{V} | I_I] \) and \( E[\tilde{V} | I_U] \) converge to \( \bar{V} \), and therefore the asymmetric information case converges to the symmetric information case without the signal \( \hat{s} \).

Using Parts 1 and 2 of Proposition 1, one can directly obtain the equilibria expressions for the symmetric information case and the standard asymmetric information case (i.e., without the public signal).\(^{19}\)

3. **Comparative statics**

In this section, we provide some comparative statics on asset prices, market illiquidity, and welfare, focusing on the impact of information asymmetry, liquidity shock volatility and market power.

3.1. **A measure of information asymmetry**

While there is a vast literature on the impact of information asymmetry on asset pricing and market liquidity, to our knowledge, there is still not a good measure of information asymmetry (i.e., a change of which does not affect other relevant economic variables such as the quality of information about the security payoff). For example, the precision of a private signal about asset payoff would not be a good measure, because a change in the precision also changes the information quality about the payoff and both information asymmetry and information quality can affect economic variables of interest (e.g., prices, liquidity). Even a comparison between the cases with and without asymmetric information cannot attribute the difference to the impact of information asymmetry alone, as long as the information quality is different across these two cases. We next propose a measure of information asymmetry.

\(^{19}\)As in Vayanos and Wang (2011), one can further subdivide the symmetric information case into two subcases: “full-information,” where all investors observe \( \hat{s} \), and “no-information,” where no one observes \( \hat{s} \), or equivalently \( \sigma_e = \infty \).
One of the fundamental manifestations of asymmetric information is that the security payoff conditional variance for the uninformed is greater than that for the informed, i.e.,

$$\text{Var}(\tilde{V}|\mathcal{I}_U) - \text{Var}(\tilde{V}|\mathcal{I}_I) = \left(\frac{(\sigma_z^2 + \sigma_{\tilde{V}}^2)(\sigma_{\tilde{S}}^2 + \sigma_N^2)}{\delta^2 \sigma_N^2 \sigma_{\tilde{V}}^2 \sigma_{\tilde{S}}^2} + \frac{\sigma_{\tilde{S}}^2 + \sigma_{\tilde{V}}^2}{\sigma_{\tilde{V}}^2}\right)^{-1} \geq 0. \quad (28)$$

The greater this conditional variance difference, the greater the information asymmetry. This difference is monotonically increasing in $\sigma_N^2$, $\sigma_{\tilde{V}}^2$, and $\sigma_{\tilde{S}}^2$, but nonmonotonic in $\sigma_z^2$ and $\sigma_{\tilde{V}}^2$. A change in $\sigma_N^2$ would change the correlation between the nontraded asset and the risky security while a change in $\sigma_{\tilde{S}}^2$ would change the unconditional hedging demand uncertainty. In addition to the undesirable nonmonotonicity, a change in $\sigma_z^2$ or $\sigma_{\tilde{V}}^2$ would change the aggregate information quality about the security payoff. In contrast, a change in $\sigma_N^2$ only changes the information asymmetry but not the aggregate information quality or the unconditional hedging demand uncertainty or the correlation between the nontraded asset and the risky security. Accordingly, to isolate the impact of information asymmetry in the subsequent analysis, we use $\sigma_N^2$ as the measure of information asymmetry. From Proposition 1, as $\sigma_N^2 \to 0$, the information asymmetry goes to 0 and the asymmetric information case converges to the symmetric information case, while as $\sigma_N^2 \to \infty$, the information asymmetry tends to the level in the standard asymmetric information case without the public signal $\tilde{S}_x$.

The above analysis shows that in this model, the noisiness of the public signal $(\sigma_N^2)$ about the liquidity shock can serve as a measure of information asymmetry. Similar idea extends to other models with information asymmetry. For example, in models with noise trades where the informed’s

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20 The nonmonotonicity follows because as $\sigma_z^2$ decreases or $\sigma_{\tilde{V}}^2$ increases, the conditional covariance magnitude $|(1 - \rho_I)\sigma_{\tilde{V}}^2|$ decreases, thus the noise from the hedging demand decreases and hence the conditional security payoff variance of the uninformed may get closer to that of the informed.

21 The aggregate information quality about the security payoff is measured by the inverse of the security payoff variance conditional on all the information in the economy, i.e.,

$$\left(\text{Var}(\tilde{V}|\mathcal{I}_I \cup \mathcal{I}_U \cup \mathcal{I}_M)\right)^{-1} = \left(\text{Var}(\tilde{V}|\mathcal{I}_I)\right)^{-1} = \frac{\sigma_{\tilde{V}}^2 + \sigma_z^2}{\sigma_{\tilde{V}}^2 \sigma_z^2}, \quad (29)$$

where the first equality follows from the fact that the informed have better information than the rest and the second from (11).
orders can depend on market prices (e.g., Kyle (1989)), the noisiness of a public signal about the noise trades can also serve as a measure of information asymmetry.\footnote{For example, because the informed can always back out the noise trades from market prices in these models, observing the public signal does not change aggregate information quality.}

3.2. Bid-ask spread, market depths, and trading volume

The following proposition shows that in contrast to most of the existing literature (e.g., Glosten and Milgrom (1985)), the expected bid-ask spread (before the realization of \(\hat{s}, \hat{X}_I,\) and \(\hat{S}_x\)) can decrease as information asymmetry increases.

**Proposition 2** 1. The expected bid-ask spread is equal to:

\[
E[(A^* - B^*)] = \frac{\rho^I(1 - \rho^U)\delta\sigma^2_V}{NM + 1} \left( \frac{2n(\bar{z}\theta)}{z} + (2N(z\bar{\theta}) - 1)\bar{\theta} \right),
\] (30)

where \(n\) and \(N\) are respectively the pdf and cdf of the standard normal distribution and

\[
z = \frac{\delta\sigma^2_V}{\sqrt{\sigma^2_V + \sigma^2_v + \left( \frac{1}{\rho^I} + \frac{\sigma^2_v}{\delta^2(1 - \rho^I)} \right)^2 \frac{2}{\delta^2} (1 - \rho^I)^2 \sigma^2_{V,N}\sigma^2_{X} + \frac{\sigma^2_{V,N}}{\delta^2(1 - \rho^I)^2 \sigma^2_{V,N}}} \right].
\] (31)

2. Suppose \(\bar{\theta} > 0.\footnote{If \(\theta = 0\), then it can be shown that the expected bid-ask spread always decreases with information asymmetry.} Let \(z\) be as defined in (31), \(\bar{z} > 0\) and \(\bar{z} > 0\) be the unique solution to \(f(x) = 0\) and \(g(x) = 0\) respectively, where

\[
f(x) := -\frac{1}{\delta^2|\sigma_{V,N}|(1 - \rho^I)\sigma_X}n(\bar{\theta}x) + (2N(\bar{\theta}x) - 1)\bar{\theta}
\] (32)

and

\[
g(x) = -\frac{\sigma^2_V}{\delta^2(1 - \rho^I)^2 \sigma^2_{V,N} \sigma^2_X} \times \frac{n(\bar{\theta}x)}{x} + (2N(\bar{\theta}x) - 1)\bar{\theta}.
\] (33)

Then the expected bid-ask spread decreases (increases, resp.) with information asymmetry \(\sigma^2_{\theta}\)

if \(z < \bar{z}\) (\(z > \bar{z}\), resp.). This implies if \(\sigma^2_{\theta}\) is small enough, then the expected bid-ask spread
with asymmetric information is smaller than with symmetric information.

3. If

\[
0 < \sigma_X < \frac{\sigma_Y^2}{\delta \sigma_{VN}(1-\rho I) \left( \sqrt{\sigma_Y^2 + \sigma_Z^2 + \delta \sigma_Y^2 \theta \sqrt{\pi}} \right)},
\]

then \( E[(A^* - B^*)_{\sigma_{eN}}] < E[(A^* - B^*)_{\sigma_{e0}}] = \frac{2\delta \sigma_{VN}(1-\rho I)\sigma_X}{(N_M+1)\sqrt{2\pi}} \).

4. The expected bid-ask spread increases with both the liquidity shock volatility \( \sigma_X \) and the covariance magnitude \( |\sigma_{VN}| \), but decreases with competition (measured by \( N_M \)).

Part 2 of Proposition 2 implies that for small enough \( \sigma_{\eta}^2 \) or large enough \( \sigma_e^2 \), the expected spread decreases with information asymmetry \( \sigma_{\eta}^2 \). In particular, this implies that when \( \sigma_{\eta}^2 \) is small enough, the expected bid-ask spread with asymmetric information is smaller than with symmetric information. In addition, Part 3 shows if the liquidity shock volatility (\( \sigma_X \)) is small, then the expected bid-ask spread with significant information asymmetry (\( \sigma_{\eta} = \infty \)) is also smaller than with symmetric information. Consistent with these results, Figure 3 shows that when \( \sigma_{\eta}^2 \) or \( \sigma_X \) is small, the expected spread decreases with information asymmetry. One empirically testable implication of these results is that for stocks with small information asymmetry, average spreads

\[24\] This is because as \( \sigma_{\eta}^2 \to 0 \) or \( \sigma_e^2 \to \infty \), we have \( z \to 0 \). In an earlier version, we show that the relative spread \( E \left[ \frac{A^* - B^*}{(A^* + B^*)^{1/2}} \right] \) can also decrease with information asymmetry.
before major news announcement are smaller than after.

To illustrate the intuition behind these seemingly counterintuitive results, we first show that for some realizations of \( \hat{s} \), \( \hat{X}_I \), and \( \hat{S}_x \) (i.e., ex post), bid-ask spread can be smaller with asymmetric information, then explain why even the expected spread (i.e., averaging across all possible realizations) can also be smaller with asymmetric information. For this purpose, we can rewrite the reservation price difference (19) as

\[
\Delta RP = -\delta(1 - \rho_I)\sigma_{VN}\hat{X}_I + \left( E[\hat{V}|I_I] - E[\hat{V}|I_U] \right) + \left( \delta \text{Var} [\hat{V}|I_U] \hat{\theta} - \delta \text{Var} [\hat{V}|I_I] \hat{\theta} \right)
\]

\[
= -\delta(1 - \rho_I)\sigma_{VN}\hat{X}_I + \rho_U(1 - \rho_U) \left( \hat{s} + \frac{\sigma_V^2}{\delta(1 - \rho_I)\sigma_{VN}} \left( \hat{X}_I - \hat{S}_x \right) \right) + \rho_U(1 - \rho_U)\delta \sigma_V^2 \hat{\theta} ,
\]

where the first term is from the difference in the hedging demand between the informed and the uninformed (“hedging effect”), the second term is the difference in the estimation of the expected security payoff (“estimation error effect”), and the third term is the difference in the risk premium required for the estimation risk (“estimation risk effect”).

Either the estimation risk effect or the estimation error effect can cause spreads with asymmetric information to be smaller than with symmetric information. As a simple example, suppose the hedging effect is \(-1\) and the uninformed have the same estimate of the expected payoff as the informed, i.e., the estimation error effect is zero. Since the private signal observed by the informed is more informative than the security price (i.e., \( \nu \geq 1 \)), the uninformed require a higher risk premium, and thus the estimation risk effect is always positive. Suppose the estimation risk effect is equal to 0.5. Then the spread is equal to \(0.5/(N_M + 1)\) with asymmetric information, while the spread is equal to \(1/(N_M + 1)\) with symmetric information, because only the hedging effect is present with symmetric information. In this case, it is the estimation risk effect in the presence of
asymmetric information that drives down the spread.

Alternatively, the estimation error effect can also reduce the spread. For example, suppose \( \Delta RP > 0 \) and thus \( I \) buy and \( U \) sell. When the magnitude of the realized private signal \( |\hat{s}| \) is relatively small for a given positive hedging demand, the uninformed can over-attributing the demand of the informed to the private information and thus overestimate the expected payoff. In this case, the estimation error effect is negative and the net of the estimation error effect and the estimation risk effect can cancel out some of the positive hedging effect. Therefore, the reservation price difference and hence the bid-ask spread with asymmetric information can be lower than with symmetric information. As an extreme case, by (19), if

\[
\hat{s} = \hat{s}^* := \frac{\sigma_x^2}{\sigma_{\hat{v}}} \delta \sigma_{V_N} \hat{X}_I + \frac{\sigma_{\hat{v}}^2}{2 \sigma_x^2 \sigma_{V_N} \sigma_{\eta}} \hat{S}_x - \delta \sigma_{\hat{v}} \hat{\theta},
\]

then the reservation prices are the same for \( I \) and \( U \) investors, because the net of the estimation error effect and the estimation risk effect exactly cancels out the hedging effect. Therefore, the equilibrium bid-ask spread must be zero, which is clearly smaller than the spread in the symmetric information case. By continuity, if \( \hat{s} \) is close to \( \hat{s}^* \), then the bid-ask spread is also smaller than with symmetric information.

The intuition behind the results on expected spreads in Proposition 2 then follows closely from the above intuition for ex post spreads: (1) \( E[\Delta RP] > 0 \) because of the estimation risk premium required by the uninformed and so the informed buy on average; (2) When \( \sigma_x^2 \) is small or \( \sigma_{\eta}^2 \) is small or \( \sigma_{\hat{v}}^2 \) is large, the uncertainty about the security payoff is much greater than that about the hedging demand, the uninformed can thus significantly overestimate the expected security payoff with a greater probability; (3) As information asymmetry increases, the uncertainty about the payoff further increases, the overestimation can thus further increase when the uncertainty about the security payoff is greater than that about the hedging demand, and the estimation error effect can offset even more of the hedging effect and the estimation risk effect, making the expected spread
even smaller.

Proposition 2 also implies that as liquidity shocks become more volatile or the payoffs of the security and the nontraded asset covary more, the expected bid-ask spread increases. Intuitively, as $\sigma_X^2$ or $|\sigma_{VN}|$ increases, the fluctuation of the hedging demand increases, and thus the volatility of the reservation price difference increases, which drives up the expected spread.

Because market makers face both information asymmetry and inventory risk, it would be helpful to separate the roles of information asymmetry and inventory risk in affecting equilibrium asset prices and liquidity. It seems impossible to completely separate the effects of information asymmetry and inventory risks in every single case for every economic variable of interest because in general these two effects interact with each other and cannot be perfectly disentangled. However, we can separate them for some important economic variables in some important cases. First, clearly, in the symmetric information case, there is no information asymmetry effect. Second, the effect of inventory risk is through market makers’ risk aversion. For example, if market makers are risk neutral, then market makers’ inventory risk has no effect on asset prices. Because the spread is determined by the reservation price difference between the informed and the uninformed and this difference is independent of market makers’ risk aversion, the spread is not affected by market makers’ inventory risk. Therefore our results in Proposition 2 and Figure 3 on how information asymmetry affects expected spread are free of the inventory risk effect.

Next we examine how expected market depths and trading volume change with information asymmetry, liquidity shock volatility and competition.\footnote{As can be verified from Theorem 2 in the generalized model, when market makers have zero initial inventory and are extremely risk averse (and thus do not have date 1 inventory either), results in Proposition 3 and Figure 4 still hold.}

**Proposition 3**

1. If $\bar{\theta} > 0$ and $N_U$ is large enough and $z > \bar{z}$, then the expected trading volume increases with information asymmetry, i.e., $\frac{\partial E[N_M(\alpha^*+\beta^*)]}{\partial \sigma^2} > 0$, where $z$ and $\bar{z}$ are as defined in Proposition 2.

2. As the liquidity shock volatility $\sigma_X$ or the covariance magnitude $|\sigma_{VN}|$ increases, the expected...
As many studies of asymmetric information show (e.g., Akerlof (1970)), information asymmetry decreases trading volume because of the well known “lemon” problem. In contrast, as shown in Part 1 of Proposition 3 and Figure 4, the average trading volume can increase with information asymmetry when the population of noninformed investors is relatively large. This is because trading volume increases with the magnitude of the reservation price difference, the expected magnitude of the reservation price difference can increase with information asymmetry, and the marginal impact of adverse effect of information asymmetry on each noninformed investor is small when their population size is large. In addition, because as the liquidity shock volatility or the covariance magnitude $|\sigma_{VN}|$ increases, the expected magnitude of the reservation price difference increases as implied by Part 4 of Proposition 2, so does the expected trading volume. Part 3 of Proposition 3 shows that although competition decreases the expected quote depths of each market maker, it increases the expected market trading volume when the number of market makers is small, as also shown in Figure 4. Intuitively, when $N_M$ is small, with one additional market maker, the decrease
in the total depths from competition is small and dominated by the increase of the trading volume from the additional market maker.

3.3. Utility loss due to market power

In this subsection, we analyze the welfare loss due to market power. Not surprisingly, it can be shown that market makers’ market power makes themselves better off but non-market-makers worse off. Also as expected, because of market friction, the social welfare loss (measured by the total certainty equivalent wealth loss) due to market power is positive and decreases with competition. This implies that there exists a Pareto improvement wealth transfer and market regulation mechanism that limit market bid-ask spreads and depths and make all investors (including market makers) strictly better off. It also suggests the importance of promoting competition among market makers on improving market liquidity and social welfare.

Next, we examine how information asymmetry affects the welfare loss from market power. Let $U_i$ and $\bar{U}_i$ denote the expected utility of type $i$ ($i = I, U, M$) investors with imperfect and perfect competition (where market makers are also price takers as others) respectively given realizations of signals on date 0 and $f_i$ and $\bar{f}_i$ be the corresponding certainty equivalent wealth, i.e., $U_i = -\exp(-\delta f_i)$, and $\bar{U}_i = -\exp(-\delta \bar{f}_i)$.

**Definition 2** The certainty equivalent wealth loss of a type $i$ investor ($i = I, U, M$) due to market power is $\bar{f}_i - f_i$ and the total certainty equivalent wealth loss $WL$ is $N_U(\bar{f}_U - f_U) + N_I(\bar{f}_I - f_I) + N_M(\bar{f}_M - f_M)$.

The next result shows how the expected total certainty equivalent wealth loss due to market power (before date 0 signal realizations) changes with information asymmetry.

**Proposition 4** If the difference in conditional variances between the uninformed and the informed (i.e., $\text{Var}(\bar{V}|I_U) - \text{Var}(\bar{V}|I_I)$) is small, then the expected total certainty equivalent wealth loss due to market power decreases with information asymmetry (i.e., $\frac{\partial E[WL]}{\partial \sigma^2} < 0$).
Proposition 4 implies that when information asymmetry is small, information asymmetry may decrease the investors’ expected total certainty equivalent wealth loss due to market power, as also shown in Figure 5. Intuitively, this is because the expected bid-ask spread can decrease with information asymmetry and the increase in investors’ utility from the smaller expected spread can offset the cost of information asymmetry.

3.4. Price impact

Another measure of illiquidity is the price impact of an investor’s additional trade, similar to Kyle’s lambda. To study this illiquidity measure, we assume that a (non-market-maker) buyer buys a small additional amount $\varepsilon_b \geq 0$ at ask and a seller sells a small additional amount $\varepsilon_s \geq 0$ at bid. For example, if $\Delta RP > 0$, then an $I$ investor is a buyer and submits an order

$$\hat{\vartheta}_I^* = \theta_I^* + \varepsilon_b,$$

while a $U$ investor is a seller and submits an order

$$\hat{\vartheta}_U^* = \theta_U^* - \varepsilon_s.$$
We then examine the marginal impact of these additional trades on equilibrium prices.

Solving for the equilibrium prices with the additional trades, we have the following result.

**Proposition 5**

1. If \( \Delta R P > 0 \), then at \( \varepsilon_s = \varepsilon_b = 0 \), we have

\[
\frac{\partial A^*}{\partial \varepsilon_b} = \frac{(\nu N_I N_M + \bar{N} + 1)\delta \text{Var}[\hat{V}|I_I]}{(N_M + 1)(\bar{N} + 1)}, \quad \frac{\partial B^*}{\partial \varepsilon_b} = \frac{\nu N_I N_M \delta \text{Var}[\hat{V}|I_I]}{(N_M + 1)(\bar{N} + 1)}. \tag{36}
\]

2. The magnitude of each of the partial derivatives in (36)-(38) increases in the information asymmetry \( \sigma^2_\eta \), the liquidity shock volatility \( \sigma_X \) and the covariance magnitude \( |\sigma_{VN}| \).

3. The corresponding result for \( \Delta R P < 0 \) is obtained by switching \( A^* \) and \( B^* \), and replacing \( \varepsilon_b \) with \(-\varepsilon_s\), and \( \varepsilon_s \) with \(-\varepsilon_b\) in (36)-(38).

Proposition 5 implies that both the bid and the ask prices decrease in extra sales and increase in extra purchases. In addition, both extra sales and extra purchases drive up the bid-ask spread, because as the trading demand increases, market makers need a greater spread to compensate them for meeting the additional demand. Moreover, as the information asymmetry measure \( \sigma^2_\eta \) or the liquidity shock volatility \( \sigma_X \) or the covariance magnitude \( |\sigma_{VN}| \) increases, by (28) the conditional variance difference between the uninformed and the informed (which represents the total information asymmetry) increases, and thus the price impact increases. These properties of the price impact measure are similar to those of the expected spread measure of illiquidity. Different from the expected spread measure, the price impact measure always increases with information asymmetry, because of the increase in the adverse selection effect of asymmetric information.
4. A generalized model

To simplify exposition, in the main model studied above we assume that all investors have the same risk aversion, the same initial inventory, the same date 1 resale value of the security, and only the informed have private information and liquidity shocks. In this section, we show that our model can be generalized to allow investors to have different risk aversion, different inventory levels, different liquidity shocks, different resale values of the security on date 1, and heterogeneous private information. Still, this generalized model is tractable and solved in closed-form. This generalized model can be used to conduct many interesting analyses such as the effect of market makers’ inventory (e.g., Garman (1976)), private information (Van der Wel et. al. (2009)), and liquidity shocks (e.g., Acharya and Pedersen (2005)). Let $\bar{\theta}_i$, $\delta_i$, $\bar{X}_i$, $\tilde{V}_i$ and $I_i$ denote respectively the initial inventory, risk aversion coefficient, liquidity shock, date 1 resale value of the security and information set for a type $i$ investor for $i \in \{I, U, M\}$. Then by the same argument as before, a type $i$ investor’s reservation price can be written as

$$P^R_i = E[\tilde{V}_i | I_i] - \delta_i \text{Cov}[\tilde{V}_i, N[I_i]]\bar{X}_i - \delta_i \text{Var}[\tilde{V}_i | I_i] \bar{\theta}_i, \quad i \in \{I, U, M\}. \tag{39}$$

Let $\Delta RP_{ij} := P^R_i - P^R_j$ denote the reservation price difference between type $i$ and type $j$ investors for $i, j \in \{I, U, M\}$. In this generalized model, there are eight cases corresponding to eight different trading direction combinations of the informed and the uninformed, as illustrated in Figure 6.\footnote{The case where both informed and uninformed do not trade is a measure zero event that occurs only when the reservation prices of all investors are exactly the same, i.e., at the origin of the figure.}

Figure 6 shows that the trading directions are determined by the ratio of the reservation price difference between the informed and the uninformed ($\Delta RP_{IU}$) to the reservation price difference between the uninformed and market makers ($\Delta RP_{UM}$). When the magnitude of this ratio is large enough (Cases (1) and (5)), the informed and the uninformed trade in opposite directions. If it is small enough (Cases (3) and (7)), on the other hand, they trade in the same direction. In between, either the informed or the uninformed do not trade.
Figure 6: Eight cases of equilibria characterized by the trading directions of the informed and the uninformed, where $b_1, b_2, b_3$ and $b_4$ are defined in (40), (41) and (71).

To save space, we only present the equilibrium results for Cases (1), (2), and (5) in this section, where Cases (1) and (5) are a direct generalization of the main model studied above and Case (2) illustrates what happens if some investors do not trade. The rest are similar and are provided in the Appendix. Define

$$b_1 = \frac{\delta_U(N_M + 1)\nu_1}{\delta_M \nu_2 N_U + \delta_U \nu_1(N_M + 1)}, \quad b_2 = \frac{\delta_I(N_M + 1)}{\delta_M \nu_2 N_I},$$

(40)

$$b_3 = \frac{\delta_I N_M}{\delta_I + \delta_M \nu_2 N_I} \leq b_2,$$

(41)

and

$$\hat{C}_U := \frac{\nu_2 N_M N_I \delta_M}{(N_M + 1)\delta_I \left(\hat{N} + 1\right)},$$

(42)

where

$$\nu_1 = \frac{\text{Var}[\hat{V}_U|\mathcal{I}_U]}{\text{Var}[\hat{V}_I|\mathcal{I}_I]}, \quad \nu_2 = \frac{\text{Var}[\hat{V}_M|\mathcal{I}_M]}{\text{Var}[\hat{V}_I|\mathcal{I}_I]}, \quad \hat{N} := \frac{\delta_M}{\delta_I} \nu_2 N_I + N_M + \frac{\delta_M \nu_2}{\delta_U \nu_1} N_U.$$
Theorem 2: For the generalized model, we have:

1. The informed buy and the uninformed sell (Case (1)) if and only if

\[ \Delta RPU_{IU} > \max \{-b_1 \Delta RPU_M, b_2 \Delta RPU_M\}. \]  \hspace{1cm} (43)

The informed sell and the uninformed buy (Case (5)) if and only if

\[ \Delta RPU_{IU} < \min \{-b_1 \Delta RPU_M, b_2 \Delta RPU_M\}. \]

For Cases (1) and (5), the equilibrium bid and ask prices are

\[ A^* = P^R_U + \hat{C}_U \Delta RPU_{IU} - \frac{N_M \Delta RPU_M}{N + 1} + \frac{\Delta RPU^+_{IU}}{N + 1}, \]

\[ B^* = P^R_U + \hat{C}_U \Delta RPU_{IU} - \frac{N_M \Delta RPU_M}{N + 1} - \frac{\Delta RPU^-_{IU}}{N + 1}, \]

and the bid-ask spread is

\[ A^* - B^* = \frac{|\Delta RPU_{IU}|}{N + 1}; \]  \hspace{1cm} (44)

the equilibrium security quantities demanded are

\[ \theta^*_I = \frac{N_M((\delta_M v_2 N_U + \delta_U v_1 (N_M + 1)) \Delta RPU_{IU} + \delta_U v_1 (N_M + 1) \Delta RPU_M)}{(1 + N_M)(N + 1) \delta_U \delta_I v_1 \text{Var}[V_I|Z_I]}. \]  \hspace{1cm} (45)

\[ \theta^*_U = \frac{N_M(-\delta_M v_2 N_I \Delta RPU_{IU} + \delta_I (N_M + 1) \Delta RPU_M)}{(1 + N_M)(N + 1) \delta_U \delta_I v_1 \text{Var}[V_I|Z_I]}, \]  \hspace{1cm} (46)

\[ \theta^*_M = -\left( \frac{N_I}{N_M} \theta^*_I + \frac{N_U}{N_M} \theta^*_U \right); \]  \hspace{1cm} (47)
the equilibrium quote depths are

\[ \alpha^* = \frac{N_I}{N_M} (\theta_1^I)^+ + \frac{N_U}{N_M} (\theta_U^*)^+, \]  

\[ \beta^* = \frac{N_I}{N_M} (\theta_1^I)^- + \frac{N_U}{N_M} (\theta_U^*)^- . \]  

(48)  

(49)

2. The informed buy and the uninformed do not trade (Case (2)) if and only if

\[ b_3 \Delta RP_{UM} \leq \Delta RP_{IU} \leq b_2 \Delta RP_{UM}. \]  

(50)

For Case (2), the equilibrium bid and ask prices are

\[ A^* = P_I^R - \frac{N_M \Delta RP_{IM}}{N_M + 1 + N_I \nu_2 \delta_M / \delta_I}, \quad B^* \leq P_U^R; \]  

(51)

the equilibrium security quantities demanded are

\[ \theta_1^I = \frac{N_M \Delta RP_{IM}}{(\delta_I (N_M + 1) + N_I \nu_2 \delta_M) \text{Var}[V_I | Z_I]}, \quad \theta_U^* = 0, \quad \theta_M^* = -\frac{N_I}{N_M} \theta_1^I; \]  

(52)

the equilibrium quote depths are

\[ \alpha^* = \frac{N_I \Delta RP_{IM}}{(\delta_I (N_M + 1) + N_I \nu_2 \delta_M) \text{Var}[V_I | Z_I]}, \quad \beta^* = 0. \]  

(53)

Theorem 1, a special case of Part 1 of Theorem 2 (Cases (1) and (5)), implies when market makers and the uninformed have the same reservation price, all investors trade in equilibrium and market makers trade at both the bid and the ask as long as the reservation price of the informed is different. In contrast, Part 2 of Theorem 2 shows when market makers and the uninformed have different reservation prices, there may exist equilibria where some investors do not trade and market

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makers only trade on one side. For example, in Case (2), the reservation price of the uninformed is lower than that of the informed but higher than that of the market maker, the market maker chooses not to trade with the uninformed to avoid buying from the uninformed at a price that is significantly higher than her reservation price because in this case the profit from the spread and the benefit from reducing net trade are relatively small. Other examples include Cases (3), (4), (6), (7), and (8) presented in the Appendix. This shows that while market makers can trade both at the bid and at the ask on date 0, they may choose to trade only on one side, as in all the cases except (1) and (5). These equilibria where market makers trade only on one side at a time imply similar trading behaviors to those implied by a sequential trading model. Cases (1) and (5) are more applicable to more active markets such as OTCQX and OTCQB stock markets where search cost is low, trading frequency is relatively high and thus market makers have a better estimate of the order flow on the other side, while the rest is more representative of less active markets where search cost is high and time between trades is relatively long (e.g., bond markets and pink sheets markets).27

As Theorem 1, Theorem 2 reveals that conditional on the uninformed and the informed trading in the opposite directions (i.e., Cases (1) and (5)), the equilibrium spread only depends on the reservation price difference between the informed and the uninformed and the competition among market makers, but not on the initial inventory, or the risk aversion, or the private valuation of a market maker. Intuitively, the initial inventory, the risk aversion, and the private valuation of a market maker only affect the certainty equivalent wealth from the net inventory and a market maker can (indirectly) change the spread without changing inventory by varying the bid depth and the ask depth by the same amount. For Case (2), however, the spread in general depends on the characteristics of the market maker as implied by (51) with $B^*$ set to $P_U^R$, this is because the market maker is not making offsetting trades at the bid and thus any trade at the ask changes inventory. This result suggests whether the initial inventory, the risk aversion, or the private valuation of a market maker

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27OTCQX and OTCQB are top tier OTC markets for equity securities (more than 3,700 stocks) with a combined market capitalization of more than $1 trillion and more than 2 billion daily share trading volume.
market maker is important for the spread depends on whether market makers can frequently make offsetting trades. One empirically testable implication of this result is that in relatively less active markets, the average spread is more sensitive to the inventory level and the private information of market makers.

Although inventory risk does not affect the spread in Cases (1) and (5), it always affects active depths and prices (i.e., at which trades occur). For example, for Cases (1) and (5), Theorem 2 implies when the initial inventory is large and market makers’ risk aversion is high, market makers reduce the inventory by increasing the ask depth and decreasing the bid depth, which results in a decrease in both the ask and the bid prices that encourages purchases and discourages sales by other investors.\(^{28}\) Accordingly, another empirically testable implication is that average ask depth increases, but average bid depth decreases with market makers’ initial inventory.

A limitation of the model in Section 2 is the absence of some features such as search, pre- and post-trade opacity that might be important for some OTC markets such as bond markets and pink sheets markets. The generalized version can serve as a reduced form model that can indirectly capture some of these features. The date 1 resale value \(\tilde{V}_M\) of the security represents what price a market maker can sell the security for on date 1. In the model in Section 2, for expositional simplicity, we assume that the true value of the security is publicly announced on date 1 and thus the resale value on date 1 is the same across all investors and does not vary with market features like search costs. In the generalized model, the date 1 utility function can represent the continuation value function in a multi-period setting and one can adapt the distribution of \(\tilde{V}_M\) to model indirectly market conditions such as searching costs and opacity. For example, when search cost is high, search takes a long time, and the resale value of the security is with large uncertainty, one can approximate this situation by using a low mean and high volatility distribution for \(\tilde{V}_M\). This is clearly just a reduced form, but likely indirectly captures the first order effect of these features. For example, when search cost is high and the uncertainty about the resale value of the

\(^{28}\)This is because the reservation price of a market maker decreases with the initial inventory and risk aversion, and thus \(\Delta R_P U_M\) increases with it.
security is high, market makers charge a higher premium for the security on date 0 and the bid-ask spread increases in a search model (e.g., Duffie, Gârleanu, and Pedersen (2005, 2007)). With a lower mean and higher volatility for $\tilde{V}_M$, it can be shown that our model can generate the same result. Intuitively, an increase in the volatility or a decrease in the mean of the resale value on date 1 reduces the value of the security on date 0. In addition, when market makers buy from one type of investors and the other type do not trade in equilibrium (Cases (6) and (8) in the Appendix), the bid price goes down and ask price does not change, and thus the spread goes up.

5. Concluding remarks

Microstructure theories have been largely kept separate between inventory-based models and information-based ones. However, both information and inventory risk can be important in many financial markets. In this paper, we develop an equilibrium model to study how information asymmetry, inventory risk, and imperfect competition among market makers jointly affect market prices and market liquidity in these markets. We solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed-form even when investors have different risk aversion, different inventory levels, different liquidity shocks, and heterogeneous private information. Our model can generate a rich set of equilibria, such as those where some investors do not trade, where the informed and the uninformed trade in opposite directions, and where they trade in the same direction. Market makers may trade both at bid and at ask or trade only on one side. The trading behavior in these equilibria is largely consistent with those observed in a wide range of financial markets. In addition, we develop a measure of information asymmetry that can separate information asymmetry effect from other effects (e.g., information quality effect). Moreover, our model can also help explain some puzzling empirical findings such as bid-ask spreads can be lower with asymmetric information.
Appendix

Proof of Theorem 1: We prove the case when $\Delta RP < 0$. In this case, we conjecture that $I$ investors sell at the bid and $U$ investors buy at the ask. Given bid price $B$ and ask price $A$, the optimal demand of $I$ and $U$ are:

$$
\theta_I^* = \frac{E[\hat{V}|I] - \delta(1 - \rho_I)\sigma_V \hat{X}_I - B}{\delta \text{Var}[\hat{V}|I]} - \bar{\theta} \quad \text{and} \quad \theta_U^* = \frac{E[\hat{V}|U] - A}{\delta \text{Var}[\hat{V}|U]} - \bar{\theta}.
$$

(54)

Substituting (54) into the market clearing conditions (5), we get that the market clearing bid and ask prices are:

$$
A = E[\hat{V}|U] - \delta \text{Var}[\hat{V}|U]\bar{\theta} - \frac{\delta \text{Var}[\hat{V}|U]}{N_U} \sum_{j=1}^{N_M} \alpha_j,
$$

$$
B = E[\hat{V}|I] - \delta(1 - \rho_I)\sigma_V \hat{X}_I - \delta \text{Var}[\hat{V}|I]\bar{\theta} + \frac{\delta \text{Var}[\hat{V}|I]}{N_I} \sum_{j=1}^{N_M} \beta_j,
$$

(55)

where $\beta_j$ and $\alpha_j$ are the optimal shares of security $M_j$ choose to buy from $I$ investors and sell to $U$ investors respectively. Market maker $M_j$’s problem is:

$$
\min_{\alpha_j, \beta_j} -\delta(\alpha_j A - \beta_j B) - \delta(\bar{\theta} + \beta_j - \alpha_j)E[\hat{V}|U] + \frac{1}{2} \delta^2 \text{Var}[\hat{V}|U](\bar{\theta} + \beta_j - \alpha_j)^2,
$$

(56)

where $A$ and $B$ are the market clearing prices given in (55). F.O.C with respect to $\beta_j$ gives us:

$$
E[\hat{V}|I] - \delta(1 - \rho_I)\sigma_V \hat{X}_I = E[\hat{V}|U] + \delta \left( \text{Var}[\hat{V}|U] - \text{Var}[\hat{V}|I] \right) \bar{\theta} + \frac{\delta \text{Var}[\hat{V}|I]}{N_I} \sum_{j=1}^{N_M} \beta_j + \left( \frac{\text{Var}[\hat{V}|I]}{N_I} + \text{Var}[\hat{V}|U] \right) \delta \beta_j - \delta \text{Var}[\hat{V}|U]\alpha_j = 0.
$$

(57)
Summing all, we get:

\[
N_M \left( E[\hat{V}|I_I] - \delta(1 - \rho_I)\sigma_{VN} \hat{X}_I - E[\hat{V}|I_U] + \delta \left( \text{Var}[\hat{V}|I_U] - \text{Var}[\hat{V}|I_I] \right) \hat{\theta} \right)
\]

\[+
\frac{\delta \text{Var}[\hat{V}|I_I]}{N_I} N_M \sum_{j=1}^{N_M} \beta_j + \left( \frac{\text{Var}[\hat{V}|I_I]}{N_I} + \text{Var}[\hat{V}|I_U] \right) \delta \sum_{j=1}^{N_M} \beta_j - \delta \text{Var}[\hat{V}|I_U] \sum_{j=1}^{N_M} \alpha_j = 0. \tag{58}
\]

Using the F.O.C with respect to \( \alpha_j \), we get:

\[
\frac{\delta}{N_U} \sum_{j=1}^{N_M} \alpha_j - \delta(\beta_j - \alpha_j) + \frac{\delta}{N_U} \alpha_j = 0. \tag{59}
\]

Summing all, we get:

\[
\sum_{j=1}^{N_M} \beta_j = \frac{N_U + N_M + 1}{N_U} \sum_{j=1}^{N_M} \alpha_j. \tag{60}
\]

Substituting (60) into (58), we get:

\[
\sum_{j=1}^{N_M} \alpha_j = - \frac{N_M N_I N_U}{(N_M + 1) (N + 1)} \frac{\Delta RP}{\delta \text{Var}[\hat{V}|I_I]}. \tag{61}
\]

Substituting (61) into (55), we can get the equilibrium ask and bid price \( A^* \) and \( B^* \). And then substituting \( A^* \) and \( B^* \) into (54), we can get the optimal security holdings of \( I \) and \( U \) investors as stated in Theorem 1. In addition, \( A^* < P_U^R \) and \( B^* > P_I^R \) are equivalent to \( \Delta RP < 0 \) which is exactly the condition we conjecture for \( I \) investors to sell and \( U \) investors to buy. Similarly, we can prove the other case of this Theorem when \( I \) investors buy and \( U \) investors sell. \( Q.E.D. \)

**Proof of Proposition 2:** Part 1: Letting \( \hat{S}_w = \frac{s}{\rho_I} - \frac{\sigma_{V_N}^2}{\delta(1 - \rho_I)\sigma_{VN}^2} \hat{S}_z \), then

\[
\hat{S}_w = s - \left( \frac{1}{\rho_I} + \frac{\sigma_{V_N}^2}{\delta^2(1 - \rho_I)^2\sigma_{V_N}^2} \right) \delta(1 - \rho_I)\sigma_{V_N} \hat{X}_I - \frac{\sigma_{V}^2}{\delta(1 - \rho_I)\sigma_{V_N} \hat{\eta}},
\]

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which implies that $\hat{S}_w$ is normally distributed with mean 0 and variance
\[
\sigma_V^2 + \sigma_\varepsilon^2 + \left( \frac{1}{\rho_I} + \frac{\sigma_V^2}{\delta^2(1 - \rho_I)^2\sigma_N^2} \right)^2 \delta^2(1 - \rho_I)^2\sigma_N^2 \sigma_X^2 + \frac{\sigma_V^4}{\delta^2(1 - \rho_I)^2\sigma_N^2 \sigma_X^2}.
\]

Direct computation then yields (30).

Part 2:
\[
\frac{\partial E[A^* - B^*]}{\partial \sigma_N^2} = \frac{\rho_I^2 \rho_X^2 \delta^3(1 - \rho_I)^2\sigma_N^2 \sigma_Y^4}{(N_M + 1) \left( \rho_X \delta^2(1 - \rho_I)^2\sigma_N^2 \sigma_Y^2 + \rho_I \sigma_Y^2 \right)^2} \times C,
\]
where
\[
C = -\frac{\sigma_Y^4 \sigma_N^2}{\sigma_X^2 ((\sigma_\varepsilon^2 + \sigma_Y^2) \delta^2(1 - \rho_I)^2 \sigma_N^2 \sigma_Y^2 + \sigma_Y^2)} \times \frac{n(\hat{\theta}z)}{z} + (2N(\hat{\theta}z) - 1)\hat{\theta} \leq f(z),
\]
where the inequality follows from $z \leq \frac{\delta^2 |\sigma_{VN}|(1 - \rho_I)\sigma_N^2 \sigma_Y^2}{\sigma_X^2 ((\sigma_\varepsilon^2 + \sigma_Y^2) \delta^2(1 - \rho_I)^2 \sigma_N^2 \sigma_Y^2 + \sigma_Y^2)}$ with equality if $\sigma_N^2 = 0$. Because $f(0) < 0$, $f(\infty) > 0$, and $f'(x) > 0$, there is a unique positive solution to $f(x) = 0$. Define this unique positive solution to be $\bar{z}$, which is only dependent on $\delta^2 (1 - \rho_I) |\sigma_{VN}| \sigma_X$ and $\bar{\theta}$. We have if $z < \bar{z}$, then $C < 0$. In addition, because as $\sigma_N^2 \to 0$ or $\sigma_\varepsilon^2 \to \infty$, $z \to 0$ (and thus satisfies $z < \bar{z}$), the spread with asymmetric information is smaller than with symmetric information when $\sigma_N^2$ is small.

Moreover, because $-\frac{\sigma_Y^4 \sigma_N^2}{\sigma_X^2 ((\sigma_\varepsilon^2 + \sigma_Y^2) \delta^2(1 - \rho_I)^2 \sigma_N^2 \sigma_Y^2 + \sigma_Y^2)} > -\frac{\sigma_Y^2}{\delta^2 (1 - \rho_I) \sigma_{VN} \sigma_X^2}$, we have $C > g(z)$. Since $g(0) < 0$, $g(\infty) > 0$, and $g'(x) > 0$, we have that $C > 0$ if $z > \bar{z}$.

Part 3: By (30), we have the expected spread with symmetric information is
\[
E[(A^* - B^*)_{\sigma_\varepsilon^2 \to 0}] = \frac{2\delta |\sigma_{VN}| (1 - \rho_I) \sigma_X}{(N_M + 1) \sqrt{2\pi}}.
\]
In addition, we have

\[
E[(A^* - B^*)_{\sigma^2_{\theta} \to \infty}]
\leq \frac{\delta^2(1 - \rho_I)^2 \sigma^2_{V_N} \sigma^2_X E \left( \rho_I |\bar{V} - \bar{V} + \bar{\varepsilon} | + |\delta(1 - \rho_I)\sigma_{V_N} \bar{X}_I| + \delta \rho_I \sigma^2_{V} \bar{\theta} \right)}{(N_M + 1)(\rho_I \sigma^2_{V} + \delta^2(1 - \rho_I)^2 \sigma^2_{V_N} \sigma^2_X)}
\]

\[
= \frac{\delta^2(1 - \rho_I)^2 \sigma^2_{V_N} \sigma^2_X \left( 2\rho_I \sqrt{\sigma^2_{V} + \sigma^2_{\theta}} + \frac{2\delta \sigma_{V_N}(1 - \rho_I)}{\sqrt{2\pi}} + \delta \rho_I \sigma^2_{V} \bar{\theta} \right)}{(N_M + 1)(\rho_I \sigma^2_{V} + \delta^2(1 - \rho_I)^2 \sigma^2_{V_N} \sigma^2_X)}. 
\]

(63)

Then (34) follows from comparing (62) with (63).

Part 4: It can be shown the first term in (30) increases in \( \sigma^2_X \). In addition,

\[
\frac{\partial}{\partial \sigma^2_X} \left( 2n(\bar{z}) + (2N(\bar{\theta}z) - 1)\bar{\theta} \right) = -2n(\bar{\theta}z) \times \frac{\partial z}{\partial \sigma^2_X} > 0, 
\]

(64)
i.e., the second term in (30) also increases in \( \sigma^2_X \) and thus \( \frac{\partial E[A^* - B^*]}{\partial \sigma^2_X} > 0 \). Similarly, it follows from straightforward (but tedious) computation that \( \frac{\partial E[A^* - B^*]}{\partial |\sigma_{V_N}|} > 0 \).

Q.E.D.

**Proof of Proposition 3:** Part 1: From the expression of \( N_M(\alpha^* + \beta^*) \) in Theorem 1, we have

\[
Sign \left( \frac{\partial E[N_M(\alpha^* + \beta^*)]}{\partial \sigma^2_{\theta}} \right) = Sign \left( \frac{\partial E[RP]}{\partial \sigma^2_{\theta}} - \frac{E[RP]}{N + 1} \frac{\partial \bar{N}}{\partial \sigma^2_{\theta}} \right). 
\]

(65)

Because \( \frac{\partial \bar{N}}{\partial \sigma^2_{\theta}} = A_1 \bar{N}_I \), where \( A_1 := \frac{\delta^2(1 - \rho_I)^2 \sigma^2_{V_N} \sigma^2_X (\sigma^2 + \sigma^2_{\varepsilon})}{\sigma^2(\sigma^2 + \delta^2(1 - \rho_I)^2 \sigma^2_{V_N} \sigma^2_X (\sigma^2 + \sigma^2_{\varepsilon}) + (\sigma^2_{\theta} + \sigma^2_{\theta}))} \). Therefore, if \( N_U \) is large enough, we have \( \frac{\partial E[N_M(\alpha^* + \beta^*)]}{\partial \sigma^2_{\theta}} > 0 \) if \( \frac{\partial E[RP]}{\partial \sigma^2_{\theta}} > 0 \), which is in turn implied by \( z > \bar{z} \), as shown in the proof of Proposition 2. We have that when \( N_U \) and \( z \) are large enough, \( \frac{\partial E[N_M(\alpha^* + \beta^*)]}{\partial \sigma^2_{\theta}} > 0 \).

Part 2:

\[
E[\Delta RP] = \frac{\delta \rho_I \sigma^2_{V}(1 - \rho_U)}{\rho_I \sigma^2_{V} + \rho_X \delta^2(1 - \rho_I)^2 \sigma^2_{V_N} \sigma^2_{\theta}} \left( 2n(\bar{\theta}z) \frac{\bar{\theta}}{z} + (2N(\bar{\theta}z) - 1)\bar{\theta} \right),
\]

43
where \( z \) is as defined in (31). It can be shown that 
\[
\frac{\partial}{\partial \sigma^2_X \sigma^2_Y} \left( \frac{1 - \rho_Y}{\sigma^2_X + \rho_Y \sigma^2_Y (1 - \rho_Y)^2 \sigma^2_X \sigma^2_Y} \right) > 0.
\]
From (26) and (64), we have \( \frac{\partial E[N_M(\alpha^* + \beta^*)]}{\partial \sigma^2_X} > 0 \). Similarly, straightforward computation yields \( \frac{\partial E[N_M(\alpha^* + \beta^*)]}{\partial \sigma^2_Y} > 0 \).

For Part 3, fixing \( N_U + N_M \), we have
\[
\frac{\partial E[\alpha^*]}{\partial N_M} = \frac{\partial E[\beta^*]}{\partial N_M} = -\frac{N_I(1 + N_U + N_M)E[\Delta RP]}{(N_M + 1)^2(N + 1)\delta \text{Var}[\tilde{V}|I]} < 0,
\]
\[
\frac{\partial \sum_{n=1}^{N_M} E[\alpha^* + \beta^*]}{\partial N_M} = \frac{N_I(2N_U + 1 - N^2_M)E[\Delta RP]}{(N_M + 1)^2(N + 1)\delta \text{Var}[\tilde{V}|I]},
\]
which is positive when \( N_M \) is small. \( \text{Q.E.D.} \)

**Proof of Proposition 4:** The total equivalent wealth loss with asymmetric information is:
\[
WL = \frac{(\Delta RP)^2}{\delta \text{Var}[\tilde{V}|I]} D,
\]
where \( D \) is defined as follows.
\[
D = \frac{N_I(N_U(N + 1)^2 + N_M(N + 1)N_U(N + 1) + N_M(N_M + 1)(N_U + N_M + 1))}{2(N_M + 1)^2N(N + 1)^2}.
\]
(66)

Since \( N \) increases in \( \sigma^2 \), \( D \) decreases with information asymmetry \( \sigma^2 \). In addition, because
\[
\frac{\partial E[\Delta RP^2]}{\partial \sigma^2} = \frac{-\delta^2(1 - \rho I)^2\sigma^2_Y \sigma^4_V \sigma^4_Y (\sigma^2_X \delta^2(1 - \rho I)^2\sigma^2_Y \sigma^2_Y \sigma^2_Y \sigma^2_Y + \sigma^2_Y - 2\delta^2 \sigma^2_Y \bar{\theta}^2) + (\sigma^2_X + \sigma^2_Y) \sigma^4_Y}{(\sigma^2_X \delta^2(1 - \rho I)^2\sigma^2_Y \sigma^2_Y \sigma^2_Y \sigma^2_Y + \sigma^2_Y + \sigma^4_Y) ^3},
\]
(67)
we have, \( \frac{\partial E[\Delta RP^2]}{\partial \sigma^2} < 0 \), if and only if
\[
\text{Var}(\tilde{V}|I_U) - \text{Var}(\tilde{V}|I_I) = \left( \frac{(\sigma^2_Z + \sigma^2_Y) \delta^2 \sigma^2_Y \sigma^2_Y \sigma^2_Y \sigma^2_Y + \sigma^2_Y + \sigma^2_Y}{\sigma^4_Y} \right)^{-1} < (2\delta^2 \bar{\theta}^2)^{-1}.
\]
(68)

Therefore, \( E[WL] \) decreases with information asymmetry \( \sigma^2 \) if \( \text{Var}(\tilde{V}|I_U) - \text{Var}(\tilde{V}|I_I) \) is small.
Proof of Proposition 5: The proof of Part 1 is similar to the proof of Theorem 1. Suppose $\Delta R P > 0$, i.e., $I$ investors buy and $U$ investors sell. Substituting (54) into the market clearing conditions

$$
\sum_{j=1}^{N_M} \beta_j^* = N_U \hat{\theta}_U^*(A, B)^-, \quad \sum_{j=1}^{N_M} \alpha_j^* = N_I \hat{\theta}_I^*(A, B)^+,
$$

we get that the market clearing bid and ask prices are:

$$
B = E[\tilde{V}|I_U] - \delta \text{Var}[\tilde{V}|I_U] \tilde{\theta} + \frac{\delta \text{Var}[\tilde{V}|I_U]}{N_U} \left( \sum_{j=1}^{N_M} \beta_j - N_U \tilde{\epsilon}_s \right),
$$

$$
A = E[\tilde{V}|I_I] - (1 - \mu_I) \sigma_{\tilde{X}} \hat{X}_I - \delta \text{Var}[\tilde{V}|I_I] \tilde{\theta} - \frac{\delta \text{Var}[\tilde{V}|I_I]}{N_I} \left( \sum_{j=1}^{N_M} \alpha_j - N_I \tilde{\epsilon}_b \right),
$$

where $\beta_j$ and $\alpha_j$ are the optimal shares of stock $M_j$ choose to buy from $U$ investors and sell to $I$ investors respectively. Taking the first order conditions with respect to $\beta_j$ and $\alpha_j$ in the market maker $M_j$’s problem, we can get $\alpha_j^*$ and $\beta_j^*$. Substituting $\alpha_j^*$ and $\beta_j^*$ into (70), we can get $A^*$ and $B^*$ and then we can get $\frac{\partial A^*}{\partial \epsilon_s}$, $\frac{\partial B^*}{\partial \epsilon_b}$, $\frac{\partial B^*}{\partial \epsilon_s}$ and $\frac{\partial B^*}{\partial \epsilon_b}$ as stated in Proposition 5. The proof for $\Delta R P < 0$ is similar. Part 2 follows from that both $\nu$ and $\frac{\nu}{N+1}$ increase in $\sigma^2_{\tilde{X}}, \sigma_X, \text{ and } |\sigma_{\tilde{V}}|$. Q.E.D.

Next, we provide the results for the rest of cases for the generalized model. Define

$$
b_4 = \frac{N_M(N_U \delta_I + \nu_I N_I \delta_U)}{\delta_I N_U (N + 1) + \nu_I N_I N_M \delta_U} < b_1.
$$

Theorem 3 1. Both the informed and uninformed buy (Case (3)) if and only if

$$
-b_4 \Delta R P_{UM} < \Delta R P_{IU} < b_3 \Delta R P_{UM}.
$$
For Case (3), the equilibrium prices are

\[ A^* = \frac{N_I \nu_1 \delta_U P^R_I + N_U \delta_I P^R_U}{N_I \nu_1 \delta_U + N_U \delta_I} - \frac{N_M (N_I \nu_1 \delta_U \Delta R P_{IM} + N_U \delta_I \Delta R P_{UM})}{(N + 1)(N_I \nu_1 \delta_U + N_U \delta_I)}, \quad B^* \leq A^*; \quad (73) \]

the equilibrium security quantities demanded are

\[ \theta_I^* = \frac{N_M \Delta R P_{IM}}{(N + 1) \delta_I \text{Var}[\tilde{V}_I|\tilde{I}_I]}, \quad \theta_U^* = \frac{N_M \Delta R P_{UM}}{(N + 1) \delta_U \text{Var}[\tilde{V}_U|\tilde{I}_U]}, \quad \theta_M^* = -\frac{N_I}{N_M} \theta_I^* - \frac{N_U}{N_M} \theta_U^*; \quad (74) \]

and the equilibrium depths are

\[ \alpha^* = \frac{N_I}{N_M} \theta_I^* + \frac{N_U}{N_M} \theta_U^*, \quad \beta^* = 0. \quad (75) \]

2. The informed do not trade and uninformed buy (Case (4)) if and only if

\[ -b_1 \Delta R P_{UM} \leq \Delta R P_{IU} \leq -b_4 \Delta R P_{UM}; \quad (76) \]

For Case (4), the equilibrium prices are

\[ A^* = P^R_U - \frac{N_M \Delta R P_{UM}}{N_M + 1 + N_U \nu_2 \delta_M / (\nu_1 \delta_U)}, \quad B \leq P^R_U; \quad (77) \]

the equilibrium security quantities demanded are

\[ \theta_I^* = 0, \quad \theta_U^* = \frac{N_M \Delta R P_{UM}}{((N + 1) + N_U \nu_2 \delta_M / (\nu_1 \delta_U)) \delta_U \text{Var}[\tilde{V}_U|\tilde{I}_U]}, \quad \theta_M^* = -\frac{N_U}{N_M} \theta_U^*; \quad (78) \]

and the equilibrium depths are

\[ \alpha^* = \frac{N_U}{N_M} \theta_U^*, \quad \beta^* = 0. \quad (79) \]
3. The informed sell and the uninformed do not trade (Case (6)) if and only if

\[ b_2 \Delta R_{PIU} \leq \Delta R_{PIU} \leq b_3 \Delta R_{PIU} \]. \hfill (80)

For Case (6), the equilibrium prices are

\[ B^* = P^R_I - \frac{N_M \Delta R_{PIU}}{N_M + 1 + N_1 \nu_2 \delta_M \delta_I}, \quad A^* \geq P^R_U. \hfill (81) \]

the equilibrium security quantities demanded are

\[ \theta_I^* = \frac{N_M \Delta R_{PIU}}{(N_M + 1 + N_1 \nu_2 \delta_M \delta_I) \text{Var}[\bar{V}_I | \bar{Z}_I]}, \quad \theta_U^* = 0, \quad \theta_M^* = -\frac{N_I}{N_M} \theta_I^*; \hfill (82) \]

and the equilibrium depths are

\[ \alpha^* = 0, \quad \beta^* = -\frac{N_I}{N_M} \theta_I^*. \hfill (83) \]

4. Both the informed and uninformed sell (Case (7)) if and only if

\[ b_3 \Delta R_{PIU} < \Delta R_{PIU} < -b_4 \Delta R_{PIU}, \hfill (84) \]

For Case (7), the equilibrium prices are

\[ B^* = \frac{N_1 \nu_1 \delta_U P^R_I + N_U \delta_I P^R_U}{N_1 \nu_1 \delta_U + N_U \delta_I} - \frac{N_M (N_1 \nu_1 \delta_U \Delta R_{PIU} + N_U \delta_I \Delta R_{PIU})}{(N + 1)(N_1 \nu_1 \delta_U + N_U \delta_I)}, \hfill (85) \]

and \( A^* \geq B^* \); the equilibrium security quantities demanded are

\[ \theta_I^* = \frac{N_M \Delta R_{PIU}}{(N + 1) \delta_I \text{Var}[\bar{V}_I | \bar{Z}_I]}, \quad \theta_U^* = \frac{N_M \Delta R_{PIU}}{(N + 1) \delta_U \text{Var}[\bar{V}_U | \bar{Z}_U]}, \hfill (86) \]
and the equilibrium depths are

\[ \alpha^* = 0, \quad \beta^* = -\frac{N_1 I^*_I}{N_M} - \frac{N_U}{N_M} \theta^*_U. \]  (87)

5. The informed do not trade and the uninformed sell (Case (8)) if and only if

\[ -b_4 \Delta R_{P_{UM}} \leq \Delta R_{P_{IU}} \leq -b_1 \Delta R_{P_{UM}}. \]  (88)

For Case (8), the equilibrium prices are

\[ B^* = P_R^U - \frac{N_M \Delta R_{P_{UM}}}{N_M + 1 + N_U \nu_2 \delta_M/(\nu_1 \delta_U)}, \quad A^* \geq P_I^R. \]  (89)

the equilibrium security quantities demanded are

\[ \theta^*_I = 0, \quad \theta^*_U = \frac{N_M \Delta R_{P_{UM}}}{(N_M + 1 + N_U \nu_2 \delta_M/(\nu_1 \delta_U)) \delta_U \text{Var}[\tilde{V}_U|U]} \quad \theta^*_M = -\frac{N_U}{N_M} \theta^*_U; \]  (90)

and the equilibrium depths are

\[ \alpha^* = 0, \quad \beta^* = -\frac{N_U}{N_M} \theta^*_U. \]  (91)

**Proof of Theorems 2 and 3:** This is similar to the proof of Theorem 1. We only sketch the main steps. First, for each case, conditional on the trading directions (or no trade), we derive the equilibrium depths, prices, and trading quantities, similar to the proof of Theorem 1. Then we verify that under the specified conditions it is indeed optimal for the assumed trading directions.

Q.E.D.
References


