Inventory Rationing for Multiple Class Demand under Continuous Review

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We consider how a firm should ration inventory to multiple classes in a stochastic demand environment with partial, class-dependent backlogging where the firm incurs a fixed setup cost when ordering from its supplier. We present an infinite-horizon, average cost criterion Markov decision process formulation for the case with zero lead times. We provide an algorithm that determines the optimal rationing policy, and show how to find the optimal base stock reorder policy. Numerical studies indicate that the optimal policy is similar to that given by the equivalent deterministic problem and relies on tracking both the current inventory and the rate that backorder costs are accumulating. Our study of the case of non-zero lead time shows that a heuristic combining the optimal, zero lead time policy with an allocation policy based on a single-period profit management problem is effective.

1. Introduction

A common problem in distribution is the determination of which customers to serve when there is a limited supply of inventory. Often customers are characterized by the level of service they expect, either through explicit contracts or implicit in their business relationships. For example, by contract, suppliers may distinguish customers by the price they pay or the distribution channel they use. Alternatively, suppliers may characterize customers by the volume of business they engage in on an annual basis, preferring to provide higher levels of service to their best customers in expectation of furthering the relationship. A similar problem also occurs in production environments where the marginal value of a unit depends on its use. For example Dekker et al. (1998) discuss a case study on inventory control for spare parts in a large petrochemical plant, where parts are installed in equipment of different criticality. Other applications are given in the survey paper by Kleijn and Dekker (1999). The problem faced by the firm is to determine an inventory ordering policy from its supplier as well as a policy for allocating the capacity or inventory to different classes of customers.
In this paper we consider a single item, multiple class, continuous time inventory problem with class-dependent partial backlogging and setup costs. A firm purchases inventory from a supplier, incurring a fixed setup cost for each order it places. We consider Poisson unit demand (extending to compound Poisson). For each arrival, the firm must determine whether to serve the demand from stock, offer to backorder it and fill it from the next supply order, or place an order from the supplier that, under a zero lead time assumption, will fill the demand. Each demand class is characterized by a demand rate and a contribution margin per unit filled. Further, customer classes are distinguished by the likelihood they will wait for demand fulfillment, if they are offered to be backordered, and the cost per unit time of waiting for fulfillment. That is, there is partial, class-dependent backordering where the backorder cost is proportional to the time until fulfillment. We determine the optimal rationing and ordering policy to maximize the expected average profit per unit time.

This paper contributes to the literature by providing an efficient algorithm to solve the continuous-time inventory rationing model with setup costs. Previous work has considered \((Q,R)\) policies or single rationing thresholds based on inventory position and has only considered tracking the total number of units backlogged. This makes sense in a single-class problem (or two-class problem where only one class is backordered). However, in a multiple-class problem, the cost to a firm of backlogging one customer versus another is likely to differ. For example, consider the cost of backlogging an infrequent customer with low service expectations versus a high value customer that a firm wishes to retain. Thus, we formulate the problem with the vector of backorders from each class as a state variable. Further, our objective function generalizes the full backlogging case to allow probabilistic customer acceptance of backlogging. However we show that we can collapse the state space so that we only need to track the total backorder cost rate in order to determine the optimal policy. We solve a dynamic program to maximize the single period profit when an auxiliary operating cost per unit time is incurred and establish the structure of the optimal policy. We then show that this solution, for an appropriately chosen operating cost, maximizes the average profit.

In particular we show that a \((\theta_i, I_0)\) base-stock policy with class-dependent rationing is optimal for the partial backlogging, infinite horizon case with an average profit criterion. That is, there is a class-dependent threshold \(\theta_i\) that determines the maximum backorder cost rate the firm will incur prior to reordering and the firm orders to \(I_0\). As such it combines the previous work on multiple-class rationing without setup costs with the approach to the price sensitive, continuous time model of Feng and Chen (2003) and Chen and Simchi-Levi (2004a).

In addition, we show that the dynamic rationing policy provides insights into the problem. First, classes are denied service in the order of the marginal benefit accrued if they wait for delayed service, multiplied by the odds of their agreeing for such a delay. Thus inventory is protected
for classes with a greater marginal benefit for immediate service. Further, we show that as the inventory or total backorder cost rate increases, more classes are served from stock. Also, reorders are not placed until inventory is depleted and then one must consider the class designation of the arriving customer. A customer that is likely to wait for fulfillment and whose delay cost is low, does not trigger a reorder. By delaying the reorder the firm can spread the fixed costs over a greater number of customers, increasing the average profit. We show that the dynamic rationing policy performs significantly better than a static policy when there are relatively few high valued customers or when such customers are willing to wait for service.

In the next section we review the related literature. In Section 3 we present our model. We then discuss its solution and provide proofs of the optimal policy in Section 4. In Section 5 we compare the optimal policy to heuristics based on the deterministic policy to gain insight into how the rationing policy performs. We discuss extensions to the model in Section 6. In particular, throughout we assume that the firm incurs a class-dependent cost per unit time for all backordered items. We note in our extensions that a class-dependent fixed cost per unit backordered can also be accommodated by the model, with only minor changes to the results. We also discuss allowing non-stationary probability of acceptance of the backorder and non-stationary backorder charges, and compound Poisson demand. We address the assumption of a zero lead time for delivery from the supplier in Section 7 by proposing a heuristic solution allowing a single outstanding order with a positive lead time. The fundamental difficulty in including a positive lead time is that the decisions on ordering and rationing take place on different time scales. Allowing for a positive lead time would disrupt the renewal process we use to prove our results in that it would confound the operational decisions of rationing with the periodic decision of reordering. This difficulty is consistent with a similar one found in previous work in the \( (s, S, p) \) literature such as in Chen and Simchi-Levi (2004a). We discuss our results in Section 8.

2. Literature Review

The paper links research on optimal rationing of inventory to multiple classes of customers with work considering joint inventory and pricing policies. Both of these areas may be considered related to the broader topic of revenue management (see Talluri and van Ryzin (2004) for a comprehensive treatment of this literature). Topkis (1968) considers the rationing of inventory to multiple demand classes in a periodic review model where a period is subdivided into several intervals. He shows a class-dependent threshold, base-stock policy is optimal. Frank et al. (2003) extend the periodic review problem to a multiple period problem with a deterministic priority class and a stochastic secondary class. Ayvaz and Huh (2010) study an application of allocating hospital beds to two classes of patients, one lost-sale and one backlogging. The capacity of each period is identical
on a finite horizon. Perhaps more relevant to the current work are several papers that consider continuous time models. Nahmias and Demmy (1981) analyze a rationing policy with two-customer classes and Poisson demand assuming a positive lead time and backorders. They study a \((Q,R)\) policy with rationing and a single outstanding order to evaluate fill rates for the classes. Moon and Kang (1998) extend this model to a compound Poisson process demand. Melchiors et al. (2000) consider a similar problem in a lost sales context. Deshpande et al. (2003) analyze a similar problem to Nahmias and Demmy (1981) but allow for multiple outstanding orders. They consider optimizing the \((Q,R)\) inventory parameters and a backorder fulfillment policy to ration inventory. Arslan et al. (2007) consider a model with multiple demand classes with service-level requirements. The paper provides an efficient heuristic in a \((Q,R)\) continuous review framework with rationing to minimize the expected average on-hand inventory level on an infinite horizon.

Additional work has been conducted in production and inventory rationing contexts. Work such as Ha (1997) considers inventory rationing to multiple classes for a make-to-stock production system. Benjaafar and Elhafsi (2006) consider an assemble-to-order system with multiple customer classes. Benjaafar et al. (2010) has recently studied a make-to-order system with partial backlogging.

Other relevant research has considered the single-item price sensitive demand with setup costs. Rather than rationing capacity to several classes, the demand rate is varied by changing the price. Zheng (1991) provides an alternate proof of the optimality of a \((s,S)\) policy as shown by Scarf (1960), using the renewal reward argument we employ in this paper. Adopting this argument, Chen and Simchi-Levi (2004c) show the optimality of a \((s,S,p)\) policy where demand is controlled by pricing in a periodic review model for a finite horizon with additive demand and backlogging. Chen and Simchi-Levi (2004b) prove a similar result for the infinite-horizon case for both discounted-profit and average profit. Feng and Chen (2003) and Chen and Simchi-Levi (2004a) show the optimality of a \((s,S,p)\) policy for the average profit case in a continuous review environment. Recently, Huh and Janakiraman (2008) have developed an alternate approach for proving \((s,S)\) policies with demand control. This approach has been used in a context of multiplicative demand and lost sales by Song et al. (2009).

The paper is also related to Ding et al. (2006) and Ding et al. (2007) which allow for discount-based, partial backordering. Ding et al. (2006) consider a \textit{periodic review} problem similar to the problem studied by Topkis (1968). The authors assume the probability that a customer waits for delayed demand fulfillment is based on a discount offered. As in Topkis (1968), the period is divided into multiple stages, allowing updating of demand information and making allocation decisions. They develop an algorithmic approach to determine the optimal allocation and discounts. In comparison, Ding et al. (2007) consider a \textit{deterministic}, multi-class inventory allocation model
with partial backordering in an infinite horizon, EOQ-like setting. The paper provides closed-form solutions to that case and insights into the drivers of firm policy. Among these it shows that both high and low value customers can enjoy higher service rates and lower average delays through the optimal policy compared with a first-come/first-served policy that offers discounts to retain customers.

3. Model

We consider the problem faced by a firm that must determine when to purchase inventory from a supplier and how to allocate the inventory to multiple customer classes in a continuous-review environment. Customers are alternately served from stock or their demand is backlogged with same probability. Each class is defined by its expected demand rate, contribution margin, likelihood to accept a backlog, and backlogging cost.

Let $N$ be the number of customer classes and let subscript $i = 1, \ldots, N$ designate the class. We assume customers from class-$i$ arrive according to a Poisson process with rate $\lambda_i$ and let $\lambda = \sum_i \lambda_i$. Let the probability an arrival is from class $i$ be $\alpha_i = \lambda_i/\lambda$. Let $r_i$ be the revenue per unit, $b_i$ be the backorder cost per unit per unit time, and $\gamma_i$ be the probability a customer accepts a backlog. That is, $\gamma_i b_i$ expresses the expected cost per unit time of delaying a customer (the unit cost times the probability a customer accepts a backlog), and $\gamma_i r_i$ expresses the expected revenue from serving a class-$i$ customer from backlog. The constants $\gamma_i$ and $b_i$ represent the outcome of a policy used to encourage customers to accept delayed order fulfillment. We assume $b_i \in \mathbb{N}$ (the set of non-negative integers).

Inventory held costs $h$ per unit time. We assume that there is a fixed order setup cost for the firm, $K$ (typically relates to fixed costs of order generation and transportation), and that there is no lead
time for order fulfillment. The state of the system, \( s \), is defined by the on-hand inventory, \( I \), and the vector of backlogged demands \( \vec{B} = \{B_1, \ldots, B_N\} \) where \( B_i \) is the number of backlogged customers from class \( i \). That is, let \( s = \{I, \vec{B}\} \in S = \mathbb{N}^{N+1} \) be the state of the system. Let \( B = \sum_{i \in N} b_i B_i \) be the total backorder cost rate in state \( s \). Table 1 provides a summary of the notation; additional notation will be defined as needed.

When a customer arrives, the firm must determine whether to serve the customer from stock, deny the customer and perhaps serve the customer from backlog (delay the customer), or place an order and serve the customer immediately from the order (reorder). At an arrival epoch of a class-\( i \) customer at time \( t \), let \( a_i(s) = \{y_i(s), z_i(s)\} \) be the action in state \( s = \{I, \vec{B}\} \) where

\[
y_i(s) = \begin{cases} 
1 & \text{if a class-}i\text{ customer is served in state } s \\
0 & \text{if a class-}i\text{ customer is delayed in state } s \\
-1 & \text{if a reorder is placed upon the arrival of a class-}i\text{ customer in state } s 
\end{cases}
\]

and \( z_i(s) \) is the on-hand inventory after a purchase is made in state \( s \). That is, if \( y_i(s) = -1 \), the resulting state is \( s' = \{I', \vec{B}'\} \) where \( I' = z_i(s) \) and \( 0 \leq B_j' \leq B_j \) for all \( j \in N \). (We show below \( B_j' = 0 \) for all \( j \) (all backorders are filled), and so for simplicity of notation write the action as \( a_i(s) = \{y_i(s), z_i(s)\} \) as opposed to clarifying the decision on how to treat the backorders.)

The expected revenue received in state \( s \) for decision \( y_i(s) \) is

\[
r(s, a_i(s)) = \begin{cases} 
r_i & y_i(s) = 1 \\
\gamma_i r_i & y_i(s) = 0 \\
r_i & y_i(s) = -1 
\end{cases}
\]

and the expected cost rate incurred by the firm from the arrival to state \( S \) under decision \( a_i(s) \) is

\[
c(s, a_i(s)) = \begin{cases} 
h(I - 1) + B & y_i(s) = 1 \\
hI + B + \gamma_i b_i & y_i(s) = 0 \\
hz(s) + B' & y_i(s) = -1 
\end{cases}
\]

Let \( \pi \) be some policy \( \{y^\pi(s), z^\pi(s)\} \). Let \( \{X_i\}_{t>0} \) be the Poisson arrival process for class-\( i \) and let \( \{X\}_{t>0} \) be the vector of arrival processes for all classes. Let \( x \) be realization of \( \{X\}_{t>0} \). Let \( \sigma = \{I_0, B_0\} \) be the state of the system at time \( t = 0 \). Let \( \Sigma_n(x) \) be the time of the \( n^{th} \) arrival from all classes and let \( \nu_t(x) \) be the number of arrivals from all classes by time \( t \). Define \( \nu_0 = 0 \) (we assume no arrivals at time \( t = 0 \)). Let \( s^\pi_n(x) \) be the state of the system at the time of the \( n^{th} \) arrival, given policy \( \pi \), initial state \( \sigma \) and arrival process realization \( x \in X \).

Then suppressing the dependence of \( s^\pi_n(x) \) on \( x \), we can define the expected revenue received from the \( n^{th} \) arrival as

\[
r_n^\pi(x) = r(s_n^\pi, a(s_n^\pi))
\]

and the inventory and backorder cost incurred for \( t, \Sigma_n(x) \leq t < \Sigma_{n+1}(x) \) as

\[
c_t^\pi(x) = c(s_{\nu_t}^\pi, a(s_{\nu_t}^\pi))
\]
We define a cycle as the time between placing orders, that is, if for some arrival time $\Sigma_n$, $y(s_n^{\pi,\sigma}) = -1$, and for some integer $m > n$, $y(s_m^{\pi,\sigma}) = -1$, and $y(s_k^{\pi,\sigma}) \neq -1$ for all $k, n < k < m$, then a cycle starts at $\Sigma_n$ and finishes at $\Sigma_m$. We define the first cycle to start at $t = 0$. Let $T^{\pi,\sigma}_M(x) \equiv \Sigma_n(x)$ be the time to complete $M$ cycles, where the $n^{th}$ arrival completes the $M^{th}$ cycle.

Let $u^{\pi,\sigma}_M(x) = \nu_{T^{\pi,\sigma}_M}$ be the number of arrivals at the completion of the $M^{th}$ cycle. Note that $T^{\pi,\sigma}_M(x)$ is a stopping time so that we can take expectations over all $X$ until time $T^{\pi,\sigma}_M(x)$ and similarly until the $u^{\pi,\sigma}_M(x)^{th}$ arrival. Then the expected profit received under policy $\pi$ over $M$ cycles with initial state $\sigma$ is defined as

$$ v^{\pi,\sigma}_M \equiv E_X \left[ \sum_{n=1}^{u^{\pi,\sigma}_M(x)} r^{\pi,\sigma}_n(x) - \int_0^{T^{\pi,\sigma}_M(x)} c^{\pi,\sigma}_t(x) dt - MK \right]. $$

(4)

The profit expresses the total revenue received over $M$ cycles less the inventory and backorder costs incurred less the total setup costs over the $M$ cycles. The average reward for policy $\pi$ given initial state $\sigma$ is defined as

$$ h^{\pi,\sigma} \equiv \lim_{M \to \infty} \frac{v^{\pi,\sigma}_M}{E[T^{\pi,\sigma}_M(x)]} $$

(5)

We solve for the policy $\pi^*$ that maximizes $h^{\pi,\sigma}$.

### 3.1. Discussion of Assumptions

In the formulation we have made the implicit assumption that reorders do not take place between arrivals. This is without loss of generality. First, because of the Poisson assumption, reorders would only take place either when a customer arrives, or immediately after that customer is served or delayed. Observe that if a customer is served, the firm could instead have placed an order and served the customer from the reorder. Thus serving a customer from stock and then reordering immediately is equivalent to just reordering. On the other hand, if a customer is delayed, the firm receives expected revenue $\gamma_i r_i$ which is less than $r_i$, the revenue received from reordering. So delaying the customer, and then reordering immediately if they choose to stay, has lower expected value than just reordering.

Similar to other backlogging models, we assume that backlogged customers orders are filled through a reorder and not from inventory. While there may be instances where a firm would prefer to fill a backlogged customer earlier than from the next reorder, the benefit in doing so is to lower the backlog cost, and not increase revenue as the revenue of a backlogged customer has already been recognized. Thus, a firm would do so only if the backlogging cost, $b_i$, was high. Observe, if $b_i$ were high, the firm would backlog the customer only if there were a small chance of acceptance, $\gamma_i$. 
Further, high $b_i$ would tend to be associated with high revenue customers, $r_i$, further supporting the assumption. We note that in the case of high $r_i$, it is unlikely that a customer would have been backlogged at all. To summarize, it is possible for a firm to prefer to fill a backlog from inventory, but it would seem unlikely to find itself in such a situation.

It is also possible that the firm may offer a backorder to a customer, hoping that they reject it. A consequence of the preceding assumption is that if the customer chooses to be backordered, then it might be beneficial for the firm to subsequently place a new order to fill the backordered customers, even while holding inventory. To prove the optimality of our policy, and in particular the non-optimality of ordering while holding inventory, we assume $b_i < \lambda E[(1 - \gamma_j) r_j] + h$ for all customer classes $i$. That is, the backlogging cost rate for all customers is less than the expected lost sales cost rate of future demand plus the holding cost. Similar to the preceding argument, the firm is unlikely to backorder customers if this inequality does not hold, i.e., if $b_i$ is high, $\gamma_i$ is close to 1, and $r_i$ is high. Thus the assumption is not overly restrictive. However, if it does not hold, the policy given below would be approximately correct, filling orders from stock instead of by reorder, ensuring the backorder cost did not increase.

As noted we address several extensions allowing various additional costs, state-dependent parameters, and compound Poisson demand in Section 6. We address the assumption of zero lead time in Section 7.

4. Solution Approach

We approach the problem by first considering a single period version of the problem of maximizing the expected profit where an additional exogenous operating cost, $\phi$, is charged per unit time. Assuming an initial inventory we find the optimal allocation policy and initial backorder values for the single period problem by first considering the problem on a reduced state space tracking only the total backorder cost rate. We then show the resulting policy also solves the single period problem on the full state space. Next we find the optimal initial inventory level for the single period problem. We then show that any optimal policy for the infinite period, average profit maximizing problem, defines a renewal process. Finally we establish that the policy that maximizes (5) is given by a policy that maximizes the optimal single period profit for an appropriately chosen value of $\phi$.

4.1. Single Period Problem

Consider the following single period problem. As above, customers arrive from $N$ classes according to independent Poisson processes with associated parameters $\lambda_i, r_i, b_i$ and $\gamma_i$. Similar to above, for each arrival the firm must choose whether to serve, delay, or, in this case of a single period, close the firm. As above let the state $s = \{I, B\}$ and let $y_i(s)$ be the action taken in state $s$ as defined in (1) though now we redefine $y_i(s) = -1$ to indicate the firm is closed in state $s$ upon the arrival
of a class-\(i\) customer. For consistency with the multiple period model, the revenue from all delayed customers orders (who accept the delay) is received when they are delayed and their orders are filled after the firm closes. Also, the firm receives \(r_i\) from the final customer (the customer for which \(y_i(s) = -1\)).

Let \(\phi\) be an exogenously-given additional operating cost of the firm, a constant cost incurred per unit time while the firm is in operation.

Let \(\{I_0, \vec{B}_0\}\) be the initial inventory and backorder vector. Let \(\Pi^{I_0, \vec{B}_0}(Y)\) be the firm’s expected net contribution (revenue less holding and backorder costs) for the single period under a given policy \(Y\) and \(\{I_0, \vec{B}_0\}\) and let \(T^{I_0, \vec{B}_0}(Y)\) be the firm’s expected operating time. Then we solve the following single period problem:

\[
\max_{Y, \{I_0, \vec{B}_0\}} \Pi^{I_0, \vec{B}_0}(Y) - \phi T^{I_0, \vec{B}_0}(Y). \tag{6}
\]

That is, we solve for the initial state \(\{I_0, \vec{B}_0\}\) and policy \(Y\) that maximizes the revenue less operating time cost, valuing time at the rate \(\phi\) dollars per unit time. Let \(Y^{I_0, \vec{B}_0}(\phi)\) be the maximizing solution to (6), if it exists, and let \(\Pi^{I_0, \vec{B}_0}(\phi)\equiv\Pi^{I_0, \vec{B}_0}(Y^{\phi})\) and \(T^{I_0, \vec{B}_0}(\phi)\equiv T^{I_0, \vec{B}_0}(Y^{\phi})\) be the implied contribution and operating time, respectively. We use backward induction to solve (6) to determine \(\Pi^{I_0, \vec{B}_0}(\phi)\) and \(T^{I_0, \vec{B}_0}(\phi)\).

4.1.1. Reduced State Space We first consider the problem on a reduced state space where rather than tracking the number of backorders from each class, \(\vec{B}\), we only track the total backorder cost rate given by \(B = \sum_j b_j B_j\). That is, we consider states \(\tilde{s} = (I, B)\in \tilde{S} = \{\mathbb{N}^2\}\), where \(\tilde{S}\) is the reduced state space. We solve

\[
\max_{\tilde{Y}, \{I_0, \vec{B}_0\}} \Pi^{I_0, \vec{B}_0}(\tilde{Y}) - \phi T^{I_0, \vec{B}_0}(\tilde{Y}). \tag{7}
\]

where \(\tilde{Y}\) is determined with respect to the state space \(\tilde{S}\).

Let \(\tilde{g}(I, B)\) be the expected margin-to-go in state \((I, B)\) under the optimal policy. For a type-\(i\) arrival in state \(\tilde{s} = (I, B)\), let \(\tilde{T}_i^{I, B}\) be the expected time remaining before the firm closes in this auxiliary problem and let \(\tilde{\Pi}_i^{I, B}\) be the expected profit generated, given the optimal (\(\phi\)-dependent) policy, \(\tilde{Y}(\phi)\) is followed. Then \(\tilde{V}_i^{I, B} = \tilde{\Pi}_i^{I, B} - \phi \tilde{T}_i^{I, B}\) is the expected margin-to-go (profit less cost of time where time is evaluated at rate \(\phi\) dollars per unit time) under the optimal policy given a type-\(i\) arrival in state \((I, B)\). Let \(\tilde{V}_i^{I, B}\) be the value given decision \(\tilde{y}\) is made in state \((I, B)\).

We inductively solve for \(\tilde{Y}_i^{I, B}\), the optimal decision in state \((I, B)\) with a type-\(i\) arrival, and by implication \(\tilde{\Pi}_i^{I, B}, \tilde{T}_i^{I, B},\) and \(\tilde{V}_i^{I, B}\) assuming that \(\tilde{g}(I, B)\) is given for future states. We show that the optimal initial backorder \(B_0 = 0\) (Lemma 1) and the resulting policy, \(\tilde{Y}^{I_0, 0}\), is the optimal solution
to (7) for a given $I_0$. We then in Section 4.1.2 show that the resulting policy $\hat{Y}(\phi) = \{\hat{y}_{i}^{I,B}\}$ is optimal also for the full state space $S$, i.e., for all $s \in S$, the optimal $y_{i}^{I,B} = \hat{y}_{i}^{I,B}$ where $B = \sum_{j} b_{j}$.

Relying on the Poisson assumption, we can uniformize the problem so that each arrival occurs at a rate $\lambda$ and the expected revenue and costs per arrival are then accounted for at this rate. As shown below, the solution to (7) is given by $\tilde{g}(I_0,0)$ found through the dynamic program

$$\tilde{g}(I,B) = - \frac{hI + B + \phi}{\lambda} + E[\tilde{V}^{I,B}]$$

where

$$\tilde{V}_{i}^{I,B} = \max \left[ \begin{array}{c} r_{i} + \tilde{g}(I-1,B), \\
\gamma_{i}r_{i} + \gamma_{i}\tilde{g}(I,B+b_{i}) + (1-\gamma_{i})\tilde{g}(I,B), \\
r_{i} \end{array} \right]$$

for $I \in \{0,1,\ldots,I_0\}$ and $B < \bar{B}$ where $\bar{B} = \sum_i \lambda_i r_i - \phi$. The boundary values for the DP are given by $E[\tilde{V}^{0,\bar{B}}] = \sum_{j} \alpha_{j} r_{j}$ for $\bar{B} \leq B \leq \bar{B} + \max_{i} b_{i}$.

As $\sum \lambda_i r_i$ is the maximum expected revenue rate and $\phi$ is the operating cost rate, $\bar{B} = \sum \lambda_i r_i - \phi$ is the maximum backorder rate that the firm can incur and have a non-negative profit rate. We claim that the firm will close in the single period problem if the backorder cost rate exceeds $\bar{B}$ (see Lemma 2 below). We initialize the induction in state $(I,B) = (0,\bar{B})$ (no inventory, maximum backorder cost). Then for this initial state we set $\tilde{T}_{i}^{0,\bar{B}} = 0$ for all $i$, indicating the firm closes and set $\tilde{\Pi}_{i}^{0,\bar{B}} = r_{i}$, indicating the final customer is served (this is assumed to maintain consistency with the multiple period formulation given above – see (2)). Thus $E[\tilde{V}^{0,\bar{B}}] = \sum_{j} \alpha_{j} r_{j}$.

The rows in the maximization, (8), correspond to $y = 1, 0, \text{ and } -1$, respectively. We show that if $I > 0$, then $\tilde{y}_{i}^{I,B} \in \{0,1\}$ and if $I = 0$, $\tilde{y}_{i}^{I,B} \in \{0,-1\}$. Thus we choose $\tilde{y}_{i}^{I,B}$ as follows:

If $I > 0$, for each $i \in N$

$$\tilde{y}_{i}^{I,B} = \begin{cases} 
1 & \text{if } \tilde{V}_{i,y=1}^{I,B} \geq \tilde{V}_{i,y=0}^{I,B}, \\
0 & \text{otherwise.}
\end{cases}$$

(9a)

If $I = 0$, for each $i \in N$

$$\tilde{y}_{i}^{I,B} = \begin{cases} 
0 & \text{if } \tilde{V}_{i,y=0}^{I,B} \geq \tilde{V}_{i,y=-1}^{I,B}, \\
-1 & \text{otherwise.}
\end{cases}$$

(9b)

Depending on the decision, we then recognize the expected revenue received and account for the expected holding and backorder costs that are incurred until the next arrival, unless the firm is closed.

We next present the Rationing Algorithm for the restricted state space that determines $\tilde{Y}, \tilde{\Pi}(\phi)$, and $\tilde{T}(\phi)$ through backward induction. We then prove it provides the optimal solution to (7) given $I_0$.

**Rationing Algorithm given initial inventory $I_0$ and cost rate $\phi$.**
Step 0. Initialize the boundary. Let $\bar{B} = \sum_{i} \lambda_{i} r_{i} - \phi$. For $i \in N$, $I = 0, \ldots, I_{0}$, and $B = B + 1, \ldots, B + \max_{j \in N} b_{j}$, let $\tilde{V}_{i}^{B} = r_{i}$. Also let $\Pi_{i}^{B} = r_{i}$ and $T_{i}^{B} = 0$ Let $\tilde{V}_{i}^{0,\bar{B}}, \Pi_{i}^{0,\bar{B}} = r_{i}$ and $T_{i}^{0,\bar{B}} = 0$. Let $I = 0$ and $B = \bar{B} - 1$.

Step 1. Determine $J$ the set of classes to delay.

For $i = 1, \ldots, N$ find:

for $I = 0$, let

$$\beta_{i}^{0,B} = -r_{i} + \sum_{j} \alpha_{j} r_{j} - \frac{(B + \gamma_{i} b_{i} + \phi)}{\lambda (1 - \gamma_{i})} + \frac{\gamma_{i}}{1 - \gamma_{i}} \sum_{j} \alpha_{j} \tilde{V}_{j}^{0,B+b_{i}};$$

and for $I > 0$, let

$$\beta_{i}^{I,B} = -r_{i} + \sum_{j} \alpha_{j} r_{j} - \frac{(h + \gamma_{i} b_{i})}{\lambda (1 - \gamma_{i})} - \frac{h(I - 1) + B + \phi}{\lambda} + \frac{\gamma_{i}}{1 - \gamma_{i}} \sum_{j} \alpha_{j} (\tilde{V}_{j}^{I,B+b_{i}} - \tilde{V}_{j}^{I-1,B-1});$$

Order the $\beta_{i}^{I,B}$ so that $\beta_{i(1)}^{I,B} > \beta_{i(2)}^{I,B} > \ldots > \beta_{i(N)}^{I,B}$ where $i(1)$ is the index of the largest $\beta_{i}^{I,B}$, etc. If $\beta_{i(1)}^{I,B} < 0$, let $J = \emptyset$. Otherwise, let $k < N$ be the smallest index such that

$$-\sum_{j=1}^{k} \beta_{i(j)}^{I,B} \alpha_{i(j)} (1 - \gamma_{i(j)}) > \beta_{i(k+1)}^{I,B} \left(1 - \sum_{j=1}^{k} \alpha_{i(j)} (1 - \gamma_{i(j)})\right),$$

or, if none exists, $k = N$. Let $J = \{i(1), \ldots, i(k)\}$.

Step 2. Fix decision $\tilde{y}_{i}^{I,B}$.

If $I > 0$, for each $i \in N$, let

$$\tilde{y}_{i}^{I,B} = \begin{cases} 1 & \text{if } i \notin J, \\ 0 & \text{if } i \in J; \end{cases}$$

If $I = 0$, for each $i \in N$, let

$$\tilde{y}_{i}^{I,B} = \begin{cases} -1 & \text{if } i \notin J, \\ 0 & \text{if } i \in J. \end{cases}$$

Step 3. Assign values. Assign the optimal profit- and time-to-go for each $i \in N$:

Let

$$Serve_{i}^{I,B} = \begin{cases} r_{i} - (h(I - 1) + B + \phi)/\lambda + \sum_{j} \alpha_{j} \tilde{V}_{j}^{I-1,B} & \text{if } I > 0; \\ r_{i} & \text{if } I = 0. \end{cases}$$

and $x_{i} = (1 - \gamma_{i}) \left(\beta_{i}^{I,B} + \frac{\sum_{j \in J} \beta_{j}^{I,B} \alpha_{j} (1 - \gamma_{j})}{\gamma_{i} - \sum_{j \in J} \alpha_{j} (1 - \gamma_{j})}\right)$ for $i \in J$ and $x_{i} = 0$ for $i \notin J$. Then $\tilde{V}_{i}^{I,B} = Serve_{i}^{I,B} + x_{i}$.

Note (shown in Lemma 6) this is equivalent to letting $\tilde{V}_{i}^{I,B} = \tilde{\Pi}_{i}^{I,B} - \phi T_{i}^{I,B}$ where

if $\tilde{y}_{i}^{I,B} = 1$:

$$\tilde{\Pi}_{i}^{I,B} = r_{i} - (h(I - 1) + B)/\lambda + \sum_{j} \alpha_{j} \tilde{\Pi}_{j}^{I-1,B},$$
\[ \tilde{T}^I_B = 1/\lambda + \sum_j \alpha_j \tilde{T}^{I-1}_j; \]

if \( \tilde{y}^I_B = -1 \):

\[ \tilde{\Pi}^I_B = r_i, \quad \tilde{T}^I_B = 0; \]

and for all \( i \in J \), \( \tilde{\Pi}^I_B \) and \( \tilde{T}^I_B \) solve the linear equations:

\[ \tilde{\Pi}^I_B = \gamma_i r_i - (hI + B + \gamma_i b_i) / \lambda + \gamma_i \sum_j \alpha_j \tilde{\Pi}^{I+B}_j + (1 - \gamma_i) \sum_j \alpha_j \tilde{\Pi}^{I-B}_j, \]

\[ \tilde{T}^I_B = 1/\lambda + \gamma_i \sum_j \alpha_j \tilde{T}^{I+B}_j + (1 - \gamma_i) \sum_j \alpha_j \tilde{T}^{I-B}_j. \]

**Step 4. Iterate.** If \( (I, B) = (I_0, 0) \), Stop. Otherwise, revise the state:

If \( B \geq 1 \), set \( B := B - 1 \). Otherwise set \( I := I + 1 \) and \( B = \bar{B} \). Go to Step 1.

Briefly, Step 0 sets the boundary values and initializes the recursion. In Step 1 we find the set of classes that should be delayed in state \( \{I, B\} \). The resulting decisions are set in Step 2 and the value-to-go, \( \tilde{V}^I_B \) is set in Step 3. We also set the auxiliary variables \( \tilde{\Pi}^I_B \) and \( \tilde{T}^I_B \), the profit- and time-to-go in the period where the profit does not include the cost \( \phi \). We observe if \( y^I_B = 1 \), the profit is received and the future values are determined with one less unit of inventory; if \( y^I_B = -1 \), the current customer is served and the firm is closed; and for all \( i \) such that \( y^I_B = 0 \) (\( i \in J \)), we solve for the simultaneous profit- and time-to-go. In Step 4 we reset \( I \) and \( B \), first decreasing \( B \) iteratively to 0, and subsequently increasing \( I \) to \( I_0 \), stopping in state \( \{I_0, 0\} \).

We now show that the Rationing Algorithm determines the optimal policy \( \{\hat{Y}^*\} \) for the reduced state space. First we show that the margin decreases in the backlogging cost rate.

**Lemma 1.** \( \tilde{V}^I_B \) is non-increasing in \( B \).

(All proofs appear in Appendix A.)

The lemma implies that there is no value in starting the period with backordered units. Thus, \( B_0 = 0 \) in (7) and the Rationing Algorithm is correct in concluding in Step 4 when \( B = 0 \).

**Lemma 2.** If \( B \geq \bar{B} \), then \( y^I_B = -1 \) and \( \tilde{V}^I_B = r_i \) for all \( i \).

The lemma implies that when the backorder cost rate exceeds the maximum expected cost rate, the firm should close as \( \tilde{V}^I_B = r_i \) only in this case. All backorders are filled at that time (potentially from inventory) and the final customer is served.

Next we determine the optimal policy depending on the total backorder rate and inventory. The following three lemmas establish that the firm cannot close when there is positive inventory. This establishes that the policy given in (9a) and (9b) is correct.
First, we show there is a lower bound on $B$, above which the firm will not delay a customer.

**Lemma 3.** For the state $(I, B)$ with $I \geq 1$ and $B \geq \bar{B} - hI$, the optimal policy for (7) is either to serve an arrival from stock or close.

The next lemma establishes an upper bound on $B$ below which closing is not optimal. Note that this differs from the preceding threshold by $h$.

**Lemma 4.** For the state $(I, B)$ with $I \geq 1$ and $B \leq \bar{B} - h(I - 1)$, the optimal policy for (7) is either to serve an arrival from stock or delay.

The next lemma indicates that a customer should not be delayed if by doing so, $B$ would then exceed $\bar{B} - h(I - 1)$. If this did not hold, then it would be possible for the firm to close with $I > 0$, and, in particular if $I = 1$, by Lemma 2, it would certainly close upon the next arrival.

**Lemma 5.** If $I \geq 1$, $B \leq \bar{B} - h(I - 1)$, and $B + b_j > \bar{B} - h(I - 1)$, then $y_{j}^{I,B} = 1$.

Next we show that the optimal value-to-go, $\tilde{V}_{i}^{I,B}$ is given by the algorithm. For each class $i$ and state $(I, B)$ assign the following variables:

$$
\text{Serve}_{i}^{I,B} = \begin{cases} 
 r_i - (h(I - 1) + B + \phi)/\lambda + \sum_j \alpha_j \tilde{V}_{j}^{I-1,B} & \text{if } I > 0; \\
 r_i & \text{if } I = 0.
\end{cases} \tag{10}
$$

$$
\text{Delay}_{i}^{I,B} = \gamma_i r_i - (hI + B + \gamma_i b_i + \phi)/\lambda + \gamma_i \sum_j \alpha_j \tilde{V}_{j}^{I,B+b_i} + (1 - \gamma_i) \sum_j \alpha_j \tilde{V}_{j}^{I,B} \tag{11}
$$

Observe, from (8), that $\text{Serve}_{i}^{I,B}$ is the expected value of $\tilde{V}_{i}^{I,B}$ if the customer is served for $I > 0$, or if the firm is closed for $I = 0$. $\text{Delay}_{i}^{I,B}$ is its expected value if the customer is delayed. Note that $\tilde{V}_{j}^{I,B}$ is not evaluated in the assignment for $\text{Delay}_{i}^{I,B}$. Then setting $\tilde{V}_{i}^{I,B}$ to the maximum of these implies $\tilde{V}^{I,B}$ solves the following linear program:

$$
\min \sum_j \tilde{V}_{j}^{I,B} \tag{12a}
$$

s.t.

$$
\tilde{V}_{i}^{I,B} \geq \text{Delay}_{i}^{I,B} \text{ for all } i \in N \tag{12b}
$$

$$
\tilde{V}_{i}^{I,B} \geq \text{Serve}_{i}^{I,B} \text{ for all } i \in N \tag{12c}
$$

The following lemma shows that in Step 2 we efficiently solve the linear program (12a)–(12c) to determine the classes to serve and to delay. Furthermore, the solution provides the value of $\tilde{V}_{i}^{I,B}$.

**Lemma 6.** For $I = 0$, let

$$
\beta_{i}^{0,B} = -r_i + \sum_j \alpha_j r_j - \frac{(B + \gamma_i b_i + \phi)}{\lambda(1-\gamma_i)} + \frac{\gamma_i}{1-\gamma_i} \sum_j \alpha_j \tilde{V}_{j}^{0,B+b_i}, \tag{13}
$$
and for $I > 0$, let

$$\beta_i^{I,B} = -r_i + \sum_j \alpha_j r_j - \frac{(h + \gamma_i b_i)}{\lambda(1 - \gamma_i)} - \frac{h(I - 1) + B + \phi}{\lambda} + \frac{\gamma_i}{1 - \gamma_i} \sum_j \alpha_j \left( \tilde{V}_j^{I,B} + b_i - \tilde{V}_j^{I-1,B} \right).$$  \hspace{1cm} (14)$$

Assume w.l.o.g. $\beta_1^{I,B} > \beta_2^{I,B} > \ldots > \beta_N^{I,B}$. If $\beta_1^{I,B} < 0$, $x_i = 0$ for all $i \in N$. Otherwise, let $k < N$ be the smallest index such that

$$- \sum_{j=1}^k \beta_j^{I,B} \alpha_j (1 - \gamma_j) > \beta_k^{I,B} \left( 1 - \sum_{j=1}^k \alpha_j (1 - \gamma_j) \right),$$  \hspace{1cm} (15)$$
or, if none exists, $k = N$. Let $J = \{1, \ldots, k\}$ and $x_i = (1 - \gamma_i) \left( \beta_i^{I,B} + \frac{\sum_{j \in J} \beta_j^{I,B} \alpha_j (1 - \gamma_j)}{1 - \sum_{j \in J} \alpha_j (1 - \gamma_j)} \right)$ for $i \in J$ and $x_j = 0$ for $i \notin J$. Then $\tilde{V}_i^{I,B} = \text{Serve}_i^{I,B} + x_i$.

Lemma 6 provides the set of classes, $J$, to be delayed and the set of classes $N \setminus J$ to be served inventory. The lemma establishes the option value of delaying a customer, $x_i$, that is the difference between $\text{Delay}_i$ and $\text{Serve}_i$ as given in Step 3. We note that in Step 3, we also define the auxiliary variables $\tilde{\Pi}_i^{I,B}$ and $\tilde{T}_i^{I,B}$. Observe that as defined both

$$\text{Serve}_i^{I,B} = \tilde{\Pi}_i^{I,B} - \phi \tilde{T}_i^{I,B} \text{ for } i \notin J \quad \text{and} \quad \text{Delay}_i^{I,B} = \tilde{\Pi}_i^{I,B} - \phi \tilde{T}_i^{I,B} \text{ for } i \in J.$$

Therefore, $\tilde{\Pi}_i^{I,B}$ and $\tilde{T}_i^{I,B}$ correctly determine the profit-to-go (not including cost $\phi$) and the time-to-go in the single period, respectively.

Observe $J$ is determined by ordering the classes by their $\beta_i^{I,B}$’s. Substituting in the definition of $\text{Serve}_i^{I,B}$ and $\text{Delay}_i^{I,B}$, we can show

$$\beta_i^{I,B} = \frac{1}{1 - \gamma_i} \left( \text{Delay}_i^{I,B} - \text{Serve}_i^{I,B} \right) - (1 - \gamma_i) \sum_j \alpha_j \left( \tilde{V}_j^{I,B} - \text{Serve}_j^{I,B} \right).$$  \hspace{1cm} (16)$$

Let $\delta_i^{I,B}$ be the marginal value of an accepted delay over serving the customer. Then

$$\text{Delay}_i^{I,B} - \text{Serve}_i^{I,B} = \gamma_i \delta_i^{I,B} + (1 - \gamma_i) \sum_j \alpha_j \left( \tilde{V}_j^{I,B} - \text{Serve}_j^{I,B} \right).$$

Comparing with (16) implies $\beta_i^{I,B}$ equals the marginal value of an accepted delay times the odds of the customer agreeing, $\gamma_i / (1 - \gamma_i)$. Therefore the $\beta_i^{I,B}$’s give the natural order for the customer classes to delay and (15) provides a test to determine the first class whose marginal benefit of delaying does not justify doing so. Note that the test can result in a class $i$ being delayed even if $\beta_i^{I,B}$ is negative, i.e., the marginal value of serving customer exceeds the marginal value of delaying him. This occurs when the difference is small and the odds of the customer agreeing to wait are
small. By delaying such a customer, the firm effectively ignores the customer, neither incurring his backorder cost, nor expending inventory on a low value customer.

Further, observe the $\beta^{I,B}_i$’s provide an ordering for the rationing of inventory. If $\beta^{I,B}_i$ is decreasing in $I$ and $B$ for all $i$, the set $J$ in the lemma provides a switching curve for each class $i$ in $(I,B)$-space such that for sufficiently high (low) inventory or backorder rate, class-$i$ customers are served (delayed). From Lemma 1, we observe for $I = 0$, $\beta^{0,B}_i$ is decreasing in $B$. Similarly, for $I > 0$, the fourth term in (14) defining $\beta^{I,B}_i$ is decreasing in $I$ and $B$. Thus, considering the fifth term, for some small $\Delta B > 0$, we can show $\beta^{I,B}_i$ is decreasing in $B$ and $I$, respectively, if

$$E[\tilde{V}^{I,B} - \Delta B] - E[\tilde{V}^{I,B}] \geq E[\tilde{V}^{I+1,B+b_i - \Delta B}] - E[\tilde{V}^{I+1,B+b_i}]$$

and

$$E[\tilde{V}^{I+1,B}] - E[\tilde{V}^{I,B}] \geq E[\tilde{V}^{I+2,B+b_i}] - E[\tilde{V}^{I+1,B+b_i}]$$

hold. The equations imply the marginal value of reducing the backorder or holding an extra unit of inventory is higher at $(I,B)$ rather than at $(I + 1, B + b_i)$. We claim both are true. Considering (18), the marginal benefit of the additional unit of inventory is reduced on the right-hand-side by the increased duration in which the additional backorder cost is incurred. Similarly, the marginal benefit of a reduction in the backorder cost is tempered on the right-hand-side of (17) by the additional holding cost incurred for the extra unit of inventory.

Next we establish a bound on $B$ below which delaying dominates closing.

**Lemma 7.** If $B \leq \bar{B} - \gamma_i b_i - (1 - \gamma_i) r_i \lambda - h I$, the optimal policy for (7) is either to serve an arrival from stock or delay.

Together the preceding lemmas establish that the optimal closing policy of the firm.

**Proposition 1.** If $0 < I_0 \leq \lceil \bar{B}/h \rceil$ and $B_0 = 0$, then there exists a threshold $\theta_i$ such that in an optimal policy to (7) the firm is closed upon the arrival of a class-$i$ customer if and only if $I = 0$ and $B \geq \theta_i$. Further,

$$\theta_i \geq \bar{B} - \gamma_i b_i - (1 - \gamma_i) r_i \lambda.$$

Observe that the bound on $\theta_i$ in Proposition 1 implies that the firm closes when $B + \gamma_i b_i + (1 - \gamma_i) r_i \lambda \geq \bar{B}$. That is, the firm closes when the maximum backorder cost, $\bar{B}$, is exceeded by the current backorder plus the expected additional backorder cost for delaying a class-$i$ customer plus the expected lost revenue until the next arrival. Noting that in practice the value of $\theta_i$ is close to its lower bound, this emphasizes the need to consider the class of customer, their revenue rate, delay costs and likelihood of accepting a delay in determining when the firm should close, and as will be shown, when to reorder.

Based on the preceding results, we establish

**Proposition 2.** Given $\phi$ and $I_0$, the Rationing Algorithm provides the policy $\{\tilde{Y}(\phi)\}$ that solves (7).
4.1.2. Full State Space  We now show that the policy solution \( \{ \tilde{y}_i^I, B \} \) for \( I \in \{0, \ldots, I_0\} \) and \( B \in \{0, \ldots, B \} \) can be transformed to solve (6), the single period problem on the full state space for a given \( I_0 \). To do so we rewrite the dynamic program (8) on the state space where a state is defined as 
\[
s = \{ I, \vec{B} \}. \]
The inventory state space for the dynamic program is naturally given as \( I \in \{0, \ldots, I_0\} \). To define the state space applicable to \( \vec{B} \), recall \( \vec{B} = \sum_i \lambda_i r_i - \phi \) and let \( \vec{B}_i = \lceil \vec{B} / b_i \rceil \). That is, \( \vec{B}_i \) is the minimum number of backorders from class \( i \) such that the backorder cost rate attributable to that class exceeds \( \vec{B} \). As \( \vec{B} \) is the maximum profitable backorder rate the firm can incur, \( \vec{B}_i - 1 \) is the maximum number of backorders from class \( i \) that the firm can allow and be profitable. We then define the space for the backorders \( \vec{B} \in \mathbf{B} = \{ \{0, \ldots, \vec{B}_1\} \times \{0, \ldots, \vec{B}_2\} \times \ldots \times \{0, \ldots, \vec{B}_N\} \} \). Also define \( \bar{\mathbf{B}} = \{ \vec{B} \in \mathbf{B} : \sum_i b_i \vec{B}_i \geq \vec{B} \} \). Let \( \mathbf{0} \) be the \( N \)-vector of 0’s and let \( \mathbf{e}_i \) be the \( i^{th} \) unit vector of length \( N \), i.e., \( \mathbf{e}_i = \{0, \ldots, 0, 1, 0, \ldots, 0\} \) where the 1 is in the \( i^{th} \) position.

We claim the optimal value for (6) is given by \( g(I_0, \mathbf{0}) \) found through the dynamic program:

\[
g(I, \vec{B}) = -\frac{h I + \sum_i b_i \vec{B}_i + \phi}{\lambda} + E[V_I, \vec{B}] \tag{19}\]

where
\[
V_i[I, \vec{B}] = \max \left[ \begin{array}{c} r_i + g(I - 1, \vec{B}), \\
\sum_j \gamma_j \gamma_i \sum_j \gamma_j \gamma_i g(I, \vec{B} + \mathbf{e}_i) + (1 - \gamma_i) g(I, \vec{B}), \end{array} \right]
\]
for \( I \in \{0, 1, \ldots, I_0\} \) and \( \vec{B} \in \mathbf{B} \). The boundary values for the DP are given by \( E[V^0, \vec{B}] = \sum_j \alpha_j r_j \) for \( \vec{B} \in \bar{\mathbf{B}} \).

**Proposition 3.** Let \( \tilde{y}_i[I, \vec{B}] = \tilde{y}_i[I, \sum_i b_i B_i] \) where \( \tilde{y}_i[I, B_i] \) is given through the Rationing Algorithm. Then given \( \phi \) and \( I_0 \), the policy \( \{ Y(\phi) \}^{I, \vec{B}}_i \) for \( I \in \{0, \ldots, I\} \) and \( \vec{B} \in \mathbf{B} \) solves (6).

4.1.3. Optimal Initial Inventory  Next we find the initial inventory \( I_0 \) that maximizes \( \tilde{g}(I_0, 0) \). We have the following:

**Proposition 4.** \( \tilde{g}(I, 0) \) is maximized by either\( I_0(\phi) = \lceil \bar{B} / h \rceil \) or \( I_0(\phi) = \lfloor \bar{B} / h \rfloor \).

Propositions 2, 3 and 4 imply \( Y(\phi) \) and \( \{ I_0, \mathbf{0} \} \) solve (6).

4.2. Multiple Period Model  We now show that the solution to the single period problem for an appropriately chosen value of \( \phi \) solves for the maximum average profit given in (5). First we show that the optimal policy defines a renewal process on the state space \( (I, \vec{B}) \). We then find the policy that maximizes the average profit per renewal. Recall \( \pi = \{ Y, Z \} \) is a policy defining the rationing of inventory by \( Y \) and the ordering policy \( Z \).

**Lemma 8.** There exists \( I_0^* > 0 \) such that under the policy \( \pi^* = \{ Y^*, Z^* \} \) that maximizes (5), if \( y^{\pi^*}(s) = -1 \), then after ordering the resulting state is \( \{ I, \vec{B} \} = \{ I_0^*, \mathbf{0} \} \).
Let } \pi(\phi) = \{Y(\phi), Z(\phi)\} \text{ be the } \phi\text{-dependent policy for the multiple period model where } Y(\phi) \text{ is the } \phi\text{-dependent optimal rationing policy given by the Rationing Algorithm and } Z(\phi) \text{ is the policy that orders up to the } (\phi\text{-dependent) } I_0(\phi) \text{ given in Proposition 4, and clears all backorders so that } \bar{B} = 0 \text{ after ordering. In Step 3 of the Rationing Algorithm, we track for } \{I, B\}, \tilde{\Pi}^{I,B}, \text{ the profit-to-go without the } \phi \text{ terms, and } \tilde{T}^{I,B}, \text{ the time-to-go, given a class-}i \text{ arrival. Then following policy } \pi(\phi) \text{ in the multiple period model, starting in state } \hat{\sigma} = \{I_0(\phi), 0\}, \text{ the expected profit- and time-to-go until another order is placed are }

\begin{align*}
\Pi(\phi) &= E[\Pi_i^{I_0(\phi), 0}] - hI_0(\phi)/\lambda \quad \text{and} \\
T(\phi) &= E[T_i^{I_0(\phi), 0}] + 1/\lambda
\end{align*}

where the expectations are taken over the class of the first arrival. For the single period model the expected reward less operating cost until closing is given by } g(I_0(\phi), 0) = \Pi(\phi) - \phi T(\phi). \text{ In the multiple period model, we have the following: }

**Lemma 9.** The expected reward in each renewal cycle under policy } \pi(\phi) \text{ starting in state } \hat{\sigma} = \{I_0(\phi), 0\} \text{ is given by }

\[ v^{\pi(\phi), \hat{\sigma}}_1 \equiv E \left[ \sum_{n=1}^{\nu(x)} r^{\pi(\phi), \hat{\sigma}}_n(x) - \int_0^{T^{\pi(\phi), \hat{\sigma}}_1(x)} c^{\hat{\sigma}}_t(x) \, dt - K \right] = \Pi(\phi) - K. \]

Together Lemmas 8 and 9 imply

**Proposition 5.** The average reward under policy } \pi(\phi) \text{ is }

\[ h^{\pi(\phi)} = \frac{\Pi(\phi) - K}{T(\phi)}. \]

Assuming there is a value of } \phi \text{ inducing a policy } \pi(\phi) \text{ such that } \Pi(\phi) > K, \text{ we can find the optimal policy. That is, we observe }

**Proposition 6.** If there exists } \phi > 0 \text{ such that } \Pi(\phi) > K, \text{ then there exists a unique } \phi^* \text{ such that } \pi(\phi^*) \equiv \{Y^*, I_0^*\} = \{Y(\phi^*), I_0(\phi^*)\} \text{ maximizes the average policy reward given an initial state (as formally defined in (5)) and } \phi^* = h^{\pi(\phi^*)} \text{ is the maximum average profit. }

Let } \bar{\phi} = \sum_i \lambda_i r_i. \text{ An } \epsilon \text{-optimal policy can be found by bisection search on } \phi \in [0, \bar{\phi}], \text{ solving the Rationing Algorithm for } f(\phi) = h^{\pi(\phi)} = (\Pi(\phi) - K)/T(\phi) \text{ and adjusting } \phi \text{ until } |f(\phi) - \phi| < \epsilon \text{ for some small } \epsilon > 0. \text{ Because } I_0 \text{ and } \bar{B} \text{ are both } O(\bar{\phi}), \text{ there are } O(\bar{\phi}^2) \text{ states } (I, B). \text{ From Step 1 of the Algorithm, for each state } (I, B), \text{ we need to determine and sort the } \beta_i, \text{ for } i \in N \text{ taking }
O(N log N) time. The remaining steps take O(N) time. So the total time required for each value of \( \phi \) is \( O(\bar{\phi}^2 N \log N) \). Thus determining the optimal policy requires \( O(\bar{\phi}^2 N \log (\bar{\phi}/\epsilon)) \) time. Note that the state space for the problem tracking the number of backorders from all classes is \( O(\bar{\phi} N + 1) \).

We can make several comments regarding the optimal policy. By allowing the firm to choose which customer classes to delay (as opposed to a first-come/first-served policy), the average profit per unit time, \( \phi^* \), is greater than what it would be without rationing. We then observe this has the effect of decreasing the purchase quantity, \( I_0^* \approx (\bar{\phi} - \phi^*)/h \). Similarly, this also has the effect of lowering the bound for the point where a new order is placed because \( \theta_i^* \approx \bar{\phi} - \phi^* - \gamma_i b_i - (1 - \gamma_i) r_i \lambda \). Thus, the flexibility provided by the ability to ration inventory allows the firm to order less inventory with fewer backlogs.

5. Comparison to Static and Single Class Policies

In this section we consider two questions: First, in what instances is it valuable to allocate inventory through a dynamic policy rather than a static one? Second, what is the value of considering the customer class of an arrival when determining if the firm should place a reorder? These questions are at the heart of the current paper as they address the need to track the current inventory, backorder cost rate, and class of each arriving customer as is done in the Rationing Algorithm. To address these questions we compare the optimal policy and its profit rate to two static heuristics. These are based on the results of the problem set in a deterministic EOQ-like environment presented in Ding et al. (2007). We emphasize that we use these heuristics to gain insight into the properties of the optimal solution and not as a solution technique in and of themselves.

Ding et al. (2007) determine an optimal policy defined by a base stock \( S^{\text{Det}} \), a reorder time \( T^{\text{Det}} \), and run-out times \( t_i^{\text{Det}} \) for each class \( i \), prior to which demand from class \( i \) is served and after which it is either backordered or lost, depending on \( \gamma_i \). Let \( y_{it}^{\text{Det}} = 1 \) if \( t < t_i^{\text{Det}} \) and 0 otherwise, and let \( \phi^{\text{Det}} \) be the average profit rate given in the deterministic setting. Then by definition, \( S^{\text{Det}} = \sum_j \lambda_j t_j^{\text{Det}} \).

The heuristics we consider transform the deterministic policy into a feasible policy for problem (7). Without loss of generality assume \( t_1^{\text{Det}} > t_2^{\text{Det}} > \cdots > t_N^{\text{Det}} \), i.e., class-1 is the highest class. Suppose we serve class-\( i \) customers from stock as long as the inventory on hand exceeds \( S_{i-1} \). That is, let \( S_i \) be the protected stock for classes \( i \) and higher. Then \( S_i = \sum_{j=1}^i \lambda_j (t_j^{\text{Det}} - t_{i+1}^{\text{Det}}) \) noting \( S_N = S^{\text{Det}} \) by letting \( t_{N+1}^{\text{Det}} = 0 \). Also define \( S_0 = 0 \).

In the two deterministic heuristics, a reorder is placed when the total effective backorder cost rate exceeds some deterministic threshold. The total effective backorder rate, \( B_i \), is the sum of the backorder cost rates, \( b_i \), for all delayed customers arriving by time \( t \) who agree to wait to be served
from a reorder. The heuristics differ only on the definition of the threshold. From Proposition 1, we know the optimal threshold

$$\theta_i \geq \bar{B} - \gamma_i b_i - (1 - \gamma_i) r_i \lambda = \sum_i (\lambda_i r_i - \phi) - \gamma_i b_i - (1 - \gamma_i) r_i \lambda.$$  

Suppose we substitute in $\phi^{Det}$ for $\phi$ and define

$$\theta_i^{Det} \equiv \sum_i (\lambda_i r_i - \phi^{Det}) - \gamma_i b_i - (1 - \gamma_i) r_i \lambda.$$  

The Class-Dependent Heuristic (CDH) serves class-$i$ customers from stock as long as the inventory on hand exceeds $S_{i-1}$ and places a reorder to bring the inventory up to $S^{Det}$ when a class-$i$ customer arrives and the total effective backorder cost rate, $B_t \geq \theta_i^{Det}$. The policy is static in that the cut-off values $S_i$ are not determined dynamically and the reorder threshold, $\theta_i^{Det}$, is defined by the static profit rate.

To address our second question, we consider a Class-Independent Heuristic (CIH). Under this policy, a reorder is placed when inventory is depleted and the backorder cost rate is sufficiently high, exceeding $\theta_i^{Det} \equiv \sum_i \lambda_i \int_0^{T_i^{Det}} (1 - y_{it}^{Det}) (\gamma_i b_i) dt$. For the CIH we place a reorder to bring the inventory up to $S^{Det}$ when no inventory remains and the actual total effective backorder cost, $B_t \geq \theta_i^{Det}$. Thus, the reorder decision is made independent of the arriving customer class.

We illustrate the difference between the optimal policy and those given by the heuristics through a two-customer class example. We let $r_1 = 15$, $r_2 = 10$, $b_1 = 1.5$, $b_2 = 1.0$, $\gamma_2 = 1$, $h = 0.5$, and $K = 1000$. For Test Case-1 we let $\lambda_1 = \lambda_2 = 10$, and vary the percentage of class-1 customers willing to wait, $\gamma_1$, from 0 to 100%. In Test Case-2, we let $\gamma_1 = 0$ and, while maintaining $\lambda_1 + \lambda_2 = 20$, vary $\lambda_1$ from 0 to 20, illustrating the effect of the customer mix. For each test case we simulated 100,000 time periods.

In Figure 1(a) we present the percentage of the optimal average profit achieved by each heuristic as a function of the percentage of class-1 customers waiting as given by Test Case-1. We observe that the two heuristics perform adequately, though on differing segments (the optimality gap of the CDH is about 0.5% on average; the CIH, 1.0%). The performance can be traced to how each heuristic treats reorders, in particular, those triggered or not by the arrival of a high-value class-1 customer. In Figure 1(b) we present the reorder thresholds as the percentage of class-1 customers waiting varies for both CIH and CDH heuristics and the optimal policy. We observe the CIH reorder threshold lies between the optimal ones for classes-1 and -2. When class-1 customers are unlikely to wait, the reorder point for the CIH is significantly higher than that of the optimal class-1 threshold. Thus, the CIH does not place an order for a class-1 customer, as it does not differentiate between classes. In doing so it delays orders and incurs lost revenue from class-1. When class-1 customers
are likely to wait, the CDH places a reorder where the optimal policy and CIH imply a delay. The CDH does not take fully into account the expected value of the additional class-1 customers who would arrive and be served through a backorder during a lengthened cycle time.

In Figure 2(a) and (b) we present the percentage of the optimal profit and the reorder thresholds, respectively, for Test Case-2. The CDH performs well over the range (except at very low class-1 demand) with an average optimality gap of less than 0.3%. The CIH performs increasingly poorly as the percentage of class-1 customers increases with a maximum optimality gap of 11%. The contrast between the results illustrates neatly the importance of tracking not just the backorder in determining the need to reorder, but also the class of the arriving customer when considering reordering. In Figure 2(b) we observe for low class-1 demand, the CDH reorders at a lower backorder
cost than the optimal policy, reducing profitability. As the class-1 demand increases, the CIH approaches the class-1 optimal policy, but differs greatly from the class-2 optimal policy. This incurs high setup costs when class-2 customers arrive and trigger setups rather than being placed on backorder.

Taken together, the two test cases show that the optimal policy demonstrates three qualities held by the static heuristic: (1) It will cut-off provision of supply to the second customer class prior to cutting off supply to the first; (2) it will reorder sufficient inventory to fill the demand of any waiting customers plus enough to last through the next cycle; and (3) it will place an order approximately when the backorder cost rate plus the cost of losing the current customer exceeds the average operating cost rate. Further, because the ordering policy of the CDH is based on the expected profit rate, $\phi^{Det}$, the optimal profit rate may be close to the profit rate from the deterministic case and thus $\phi^{Det}$ may provide a very good starting point for the search for the optimal $\phi^*$. Numerical testing (not presented) confirms this.

On the other hand, we observe that the static CDH can lead to significant losses in two cases: when class-1 customers are more likely to wait for a delayed service and when class-1 customers are relatively rare. In both cases, the firm should trade-off the lost revenue from customers that do not wait with the savings in setup costs and the revenue gains from those customers (class-1 and -2) that do wait for delayed service. This illustrates the need to dynamically take the future value of customers arriving during an order cycle into account when making costly reorder decisions.

6. Model Extensions

In our model formulation we let $r_i$ represent the revenue per unit. With no change in the model, $r_i$ can also represent the margin per unit (the revenue less cost per unit). In addition, if there is a per unit lost sale penalty, then $r_i$ can represent the margin per unit plus the lost sale penalty. These all follow from standard inventory theory noting $r_i$ represents the relative benefit of serving a customer over delaying or losing the customer. Also in the model we assume that the firm incurs a cost per unit time, $b_i$, for backlogging a class-$i$ customer. We can extend the model to also include a fixed cost per customer backlogged, independent of time. That is, letting $d_i$ be the fixed cost of delaying a customer (assuming they accept the delay), the expected costs of delaying a customer is $\gamma_i(r_i + d_i) + (hI + B + \gamma_i b_i + \phi)/\lambda + \gamma_i g(I, B + b_i) + (1 - \gamma)g(I, B)$. All of the lemmas and propositions hold for this extension, the only change being the lower bound given in Proposition 1 decreases by $\gamma_i d_i \lambda$, i.e., $\theta_i \geq B - \gamma_i (b_i + d_i \lambda) - (1 - \gamma_i) r_i \lambda$.

We have assumed $b_i$ and $\gamma_i$ to depend only on the class of the customer. However, we can allow these parameter values to depend also on the inventory $I$ and the total backorder rate $B$. That is, let $b_i^I, B$ represent the discount offered class-$i$ to accept delayed delivery in state $(I, B)$ and $\gamma_i^I, B$
represent the likelihood the offer is accepted. Then all of the theoretical results developed in the model can be shown to hold for this case as well. Observe that the firm could announce the expected time until a delayed customer is served, $T^{I,B}_i$, so that $\gamma^{I,B}_i$ could depend on both the discount and the expected time until a delayed customer is served. Alternatively, the firm could use the values of $T^{I,B'}_j$ for future states $\{I',B'\}$ and the dynamic program solution to estimate a time until service that occurs with some high likelihood. Thus customers can make their decision to accept or balk based on the discount $b_i$ and information regarding delay. (See Armony et al. (2009) who discuss such announcements and customer balking in call center contexts.)

Further, we can allow the firm to optimize over $b^{I,B}_i$. From Lemma 6 we observe that the expected total value-to-go for all classes in state $(I,B)$ in the objective function in (12a) is given by

$$\text{Total Value}^{I,B} \equiv \sum_{i \in J} (1 - \gamma^{I,B}_i) \left( \beta^{I,B}_i + \sum_{j \in J} \beta^{I,B}_j \alpha_j \left( \gamma^{I,B}_j \right) \right) + \sum_{i \in N} \text{Serve}_i$$

where $\beta^{I,B}_i$ and $J$ are given in the statement of Lemma 6. Then if $\gamma^{I,B}_i$ were a function of $b^{I,B}_i$, the firm could solve the nonlinear program $\max_{b^{I,B}_i} \text{Total Value}^{I,B}$ for each state $(I,B)$ where $b^{I,B} = \{b^{I,B}_1, \ldots, b^{I,B}_N\}$. A difficulty in solving this nonlinear program is that the set $J$ depends on $\beta^{I,B}_i$. However, we can address the problem using an approximate dynamic programming approach presented in Appendix B.

We can extend the model to compound Poisson as follows. Define an $(i,q)$-customer as a class-$i$ customer with demand for $q$ units. Let $f_i(q)$ be the probability distribution of $q$ defined on $(1, \ldots, Q_i)$, $Q_i \in \mathbb{N}$. Upon arrival of an $(i,q)$-customer, the firm offers to fulfill $p$ out of the $q$ units immediately and delay fulfillment of the remaining $(q - p)$. Considering the restricted state space tracking only the inventory and total backorder cost, $B$, let $y^{I,B}_{i,q}$ be the number of units offered in the optimal policy. Let $P_q$ be the set of potential values of $p$. For example, if $P_q = \{0, q\}$, the firm either serves the entire order or delays it. Alternatively $P_q = \{0, \ldots, q\}$ implies the firm can partially fulfill an order. Let $\gamma^{I,B}_{i,q}$ be the likelihood that the offer is accepted by the customer. By definition $\gamma^{I,B}_{i,q} = 1$. We assume that if an offer is not accepted, the customer departs with none of the order fulfilled. (Of course, alternate models of customer behavior are possible, but we restrict ourselves to this case.) Let $W^{I,B}_{i,q}(p)$ be the expected value-to-go for an $(i,q)$-customer in state $(I,B)$ offered fulfillment on $p$ units of demand. Then

$$W^{I,B}_{i,q}(p) = \gamma^{I,B}_{i,q} qr_i - (h(I - \gamma^{I,B}_{i,q} p) + B + \gamma^{I,B}_{i,q} (q - p) b_i + \phi) / \lambda$$

$$+ \gamma^{I,B}_{i,q} \sum_j \alpha_j V^{I-p,B+(q-p)b_i} + (1 - \gamma^{I,B}_{i,q}) \sum_j \alpha_j V^{I,B}_j$$

for $p \in P_q$ and $p \leq I$;

$$W^{I,B}_{i,q}(q) = qr_i \text{ for } q > I.$$
Then we can replace the linear program (12a)–(12c) with

\[
\begin{align*}
\text{min} & \quad \sum_{i,q} V_{i,q}^{IB} \\
\text{s.t.} & \quad V_{i,q}^{IB} \geq W_{i,q}^{IB}(p) \text{ for all } p \in P_q, q \in Q_i, i \in N \\
& \quad V_i^{IB} = \sum_{q=1}^Q f_i(q) V_{i,q}^{IB} \text{ for all } i \in N
\end{align*}
\]  

(22a) \quad (22b) \quad (22c)

The solution to (22a)–(22c) provides the optimal values of \( y_{i,q}^{IB} \). Step 2 of the Rationing Algorithm must be slightly altered so \( y_{i,q}^{IB} = p \) if \( V_{i,q}^{IB} = W_{i,q}^{IB}(p) \), and \( y_{i,q}^{IB} = -1 \) (a reorder is placed) if \( V_{i,q}^{IB} = W_{i,q}^{IB}(q) \) for \( q > I \). Step 3 of the algorithm must be altered so that the values of \( \Pi_{i,q}^{IB} \) and \( T_{i,q}^{IB} \) for serving \( p \) out of \( q \) units solve the simultaneous linear equations:

\[
\begin{align*}
\Pi_{i,q}^{IB} &= r_i - (\gamma^{p}_{i,q} \Pi_{i,q}^{IB}) + B + \gamma^{p}_{i,q} (q-p) b_i) / \lambda + \gamma^{p}_{i,q} \sum_{j} \alpha_j \Pi_{j}^{IB} + (q-p) b_i) / \lambda + \gamma^{p}_{i,q} \sum_{j} \alpha_j \Pi_{j}^{IB} \\
T_{i,q}^{IB} &= 1 / \lambda + \gamma^{p}_{i,q} \sum_{j} \alpha_j T_{j}^{IB} + (q-p) b_i) / \lambda + \gamma^{p}_{i,q} \sum_{j} \alpha_j T_{j}^{IB} + (q-p) b_i) / \lambda + \gamma^{p}_{i,q} \sum_{j} \alpha_j T_{j}^{IB} \\
\Pi_{i,q}^{IB} &= r_i; \quad T_{i,q}^{IB} = 0 \text{ for } y_{i,q}^{IB} = -1.
\end{align*}
\]

We note the solution given in Lemma 6 still holds. Observe that \( W_{i,q}^{IB}(p) \) depends on \( V_{i,q}^{IB} \) similar to how \( \text{Delay}_{i}^{IB} \) does. In the compound Poisson case, \( V_{i,q}^{IB} \) is given by the linking constraint (22c). The structure of (22a)–(22c) is the same as (12a)–(12c). This is shown by substituting (22c) into (22b) and then noting the terms \( W_{i,q}^{IB}(q) \) give the value-to-go of serving an order, or reordering if \( q > I \). By defining \( \beta_{i,q}^{IB} \) analogous to that in Lemma 6 and ordering them, we can efficiently determine \( V_{i,q}^{IB} \). Letting \( Q \) be the maximum value of \( Q_i \) over all \( i \in N \), we can solve (22a)–(22c) in \( O(N\bar{Q} \log N\bar{Q}) \) steps.

7. The Case of a Positive Lead Time with One Outstanding Order

Next we consider the case of a positive lead time for delivery of an order. We assume that only one outstanding order is allowed. In this case, after an order is placed, the firm faces the problem of allocating the remaining inventory over the lead time to the various classes. We approach the problem through dynamic programming using the standard multiple class approach, e.g., Talluri and van Ryzin (2004). We discretize the time remaining until the order is fulfilled into a small amounts \( \Delta t \) so that the probability of a class-\( i \) arrival in an interval is \( \lambda_i \Delta t \) and the probability more than one customer arriving in any interval \( \Delta t \) is small.

Let \( V_i(x) \) be the expected revenue with \( t \) time units to go until the order is fulfilled with \( x \) units of inventory remaining. If a class-\( i \) customer arrives with time \( t \) to go, by serving the customer
from stock the firm receives $r_i$ and incurs the holding cost for the remaining $x - 1$ units for a time period, $(x - 1)h\Delta t$. If the firm does not serve the customer, it receives the expected revenue for a back ordered customer, $\gamma_i r_i$, less the expected backorder charge, $\gamma_i b_i t$, and the holding cost on $x$ units, $x h \Delta t$. Letting $y = 1$ if a customer is served and $y = 0$ otherwise, the reward on the arrival of a class-$i$ customer with $x$ units in inventory and $t$ time units to go, $R_{it}(x, y)$, is

$$R_{it}(x, y) = \begin{cases} r_i - (x - 1)h\Delta t & \text{if } y = 1, \\ \gamma_i r_i - \gamma_i b_i t - x h \Delta t & \text{if } y = 0. \end{cases}$$

Then,

$$V_t(x) = \sum_i \lambda_i \Delta t \left( \max_{y \in \{0, 1\}} R_{it}(x, y) + V_{t-\Delta t}(x - y) \right) + \left( 1 - \sum_i \lambda_i \Delta t \right) V_{t-\Delta t}(x). \tag{23}$$

For convenience, define $y_{it}(x)$ as the optimal decision in state $(x, t)$ for class $i$ and note $y_{it}(0) = 0$ (customers are only served from stock if $x > 0$).

We require two boundary conditions. First, $V_t(0) = 0$ for $t > 0$ as $V_t$ expresses the marginal benefit from serving a customer from stock. Second, we require the boundary condition for $V_0(x)$. This is not as simple because inventory held when the next order arrives may negatively affect the average profit rate in the future. In theory one would like the inventory upon delivery, after clearing backorders, to equal $I_0^*$. Based on the case with zero lead time, we can approximate the cost of an additional unit in the next period as

$$(\Pi_i^{I_0^*-1,0} - \phi^* T_i^{I_0^*-1,0}) \approx (\Pi_i^{I_0^*-1,0} - \phi^* T_i^{I_0^*-1,0}) - (\Pi_i^{I_0^*-1,0} - \phi^* T_i^{I_0^*-1,0}) = 0.$$

Then an approximate boundary condition is given by

$$V_0(x) = \left( (\Pi_i^{I_0^*-1,0} - \phi^* T_i^{I_0^*-1,0}) - (\Pi_i^{I_0^*-1,0} - \phi^* T_i^{I_0^*-1,0}) \right) x.$$

The problem can now be solved through standard backward induction to determine the values of $y_{it}(x)$ for $0 \leq t \leq L$ and $0 \leq x \leq I_0^*$.

Let $D_t(x)$ be the expected demand backordered after an order has been placed with $t$ time units to go until delivery given $x$ units of inventory. We can find $D_t(x)$ for $t \leq L$ and $0 \leq x \leq I_0^*$ through the following recursion:

$$D_t(x) = \sum_{i=1}^N \lambda_i \Delta t (\gamma_i (1 - y_{it}) + D_{t-\Delta t}(x - y_{it})) + (1 - \sum_{i=1}^N \lambda_i \Delta t) D_{t-\Delta t}(x)$$

where $D_0(x) = 0$ for $x \geq 0$.

Let $W^{1,B}$ be the demand backordered in state $\{I, B\}$. This is just the accounting variable tracking the decisions of customers denied service from stock and who agree to being served by backorder
prior to placing the order with $L$ time units to go. Then at the time an order is placed, the order quantity required to bring the expected order up to $I_0^*$ is

$$Q^{I,B} = I_0^* + W^{I,B} + D_L(I).$$  \(24\)

Together these ideas imply the following dynamic heuristic for the case of non-zero lead time.

**Dynamic Heuristic:**

**Step 0. Initialize.** Solve the zero lead time case for $\phi^*, \{Y^*, I_0^*\}$. Solve (23) for $x = 0, \ldots, I_0^*$ to determine $y_{it}$. Place an order for $I_0^*$ and observe the demand process. Given an arrival of class $i$:

**Step 1. Given an arrival of class $i$**

- If $T_{i,B} > L$ and there are no outstanding orders, the service policy is given by $y_{iL}$.
- If $T_{i,B} \leq L$ and there are no outstanding orders, place an order from the supplier for $Q^{I,B}$ given in (24). The service policy is given by $y_{it(x)}$.
- If there is an outstanding order with $t$ time units to go before its arrival, the service policy is given by $y_{it(x)}$.

**Step 2. Order Arrival** When an order arrives, fill the backlogged demand as much as possible.

The dynamic heuristic divides the period in two. It follows the optimal zero lead time policy prior to ordering. An order is placed when the expected remaining cycle time is less than the lead time. For the remainder of the period, the policy follows a dynamic allocation policy. As the positive lead time case policy is feasible for the case of zero lead time, the average profit for the zero lead time problem is an upper bound to the positive lead time case.

In Table 2, we compare the performance of the dynamic heuristic as a percentage of the upper bound. (Here we present the results for the case considered previously with $\gamma_1 = 0$ and $\gamma_2 = 1$.) We vary the class-1 demand ratio from 10% to 90% and the lead time from 1 to 10. Note that the optimal cycle time varies from approximately 35 time units to 15 time units as the ratio of class-1 customer demand increases. We observe that for even significant lead times (up to approximately 2/3 of the order cycle time) the heuristic performs well, within 3% of its upper bound.

8. Discussion

In this paper we have introduced a multiple-class, continuous time inventory problem with partial, class-dependent backlogging and fixed setup costs. The problem is solved by defining a suitable dynamic program that uses an initial profit rate to find an allocation and ordering policy. The resulting policy then defines an expected profit rate. By comparing the initial and resulting rates, the algorithm adjusts its search and approaches the optimal, feasible expected profit rate. We prove that a $(\theta, I_0)$ basestock policy with class-dependent rationing and reordering is optimal, with reorders placed when the ongoing backorder cost plus lost revenue exceeds the average operating
cost rate, and sufficient inventory purchased to cover expected demand plus backorders. Further, we show the inventory allocation policy is given by ordering customers according to the marginal benefit provided by delayed customers that wait. This benefit decreases as the inventory or backorder cost rate increases. Thus, the paper links previous work on rationing to multiple classes differentiated by price and other factors with no setup costs, with work on inventory and pricing \((s,S,p)\) policies for continuous time models with setup costs and a single, price-dependent class.

Table 2: Performance of the heuristic as a percentage of the upper bound vs. lead time for alternate Class-1 demand percentages.

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In comparing the optimal policy to heuristics based on the deterministic fluid model, we show that in cases with greater amount of impatient, high-value demand, good solutions require class-dependent reorder-triggering thresholds, similar to those provided by the static solution. However, when there are relatively fewer high-value customers that are more likely to wait, policies based on the static solution are inadequate. The optimal solution in this case considers the dynamic trade-off between additional setup costs and the expected sales that may be received during an extended ordering cycle.

We extend the model to include lost sales costs, fixed and variable costs for delaying customers, state-dependent backorder costs (representing incentives for customers to wait for delayed service), and likelihoods of acceptance of a delay. Further, we show that we can optimize over the incentives, doing so readily through an approximate dynamic programming formulation. We also can extend the model to allow for compound Poisson demand. The extension to the case with positive lead times shows that merging the Rationing Algorithm with a single-period, revenue management allocation heuristic performs well.

We have made several assumptions in this paper. We have restricted our policies to the class where revision of previous decisions is not allowed. Allowing the firm to fill previously delayed demand prior to the receipt of an order should be considered in future research. In solving for the solution in the positive lead time case, we made the assumption that a single order may be outstanding at any given time. Such an assumption is defensible when setup costs are high. We
note that the heuristic approach is still very good even as the lead time approaches the cycle time. Extending the problem to allowing multiple outstanding orders would be useful.

References


Appendix A - Proofs

Proof of Lemma 1 Let \( \omega_i^{I,B} \in \Omega \) represent the optimal policy followed from an arrival epoch from class \( i \) in state \((I,B)\) onward and let \( \tilde{\Pi}_i^{I,B-1}(\omega_i^{I,B}) \) and \( \tilde{T}_i^{I,B-1}(\omega_i^{I,B}) \) be the profit to go and time to order, respectively, if policy \( \omega_i^{I,B} \) is employed in state \((I,B-1)\) for class \( i \). Observe \( \tilde{T}_i^{I,B-1}(\omega_i^{I,B}) = \tilde{T}_i^{I,B}(\omega_i^{I,B}) = \tilde{\Pi}_i^{I,B} \) as the same policy is followed. Also, \( \tilde{\Pi}_i^{I,B-1}(\omega_i^{I,B}) = \tilde{\Pi}_i^{I,B} + \tilde{T}_i^{I,B} \) as the additional backlogging unit is carried for \( \tilde{T}_i^{I,B} \) time units. Therefore

\[
\tilde{\Pi}_i^{I,B-1} - \tilde{\Pi}_i^{I,B} = (\tilde{\Pi}_i^{I,B-1}(\omega_i^{I,B}) - \phi \tilde{T}_i^{I,B-1}(\omega_i^{I,B})) - (\tilde{\Pi}_i^{I,B} - \phi \tilde{T}_i^{I,B})
\]

\[
= (\tilde{\Pi}_i^{I,B} + \tilde{T}_i^{I,B} - \phi \tilde{T}_i^{I,B}) - (\tilde{\Pi}_i^{I,B} - \phi \tilde{T}_i^{I,B})
\]

\[
= \tilde{T}_i^{I,B} \geq 0.
\]

Proof of Lemma 2 Consider the base case with no inventory, \( I = 0 \).

By definition \( \tilde{V}_{i,0}^{0,B} = \max[r_i, \tilde{V}_{i,y=0}^{0,B}] \). Now,

\[
\tilde{V}_{i,y=0}^{0,B} = \gamma_i r_i - 1/\lambda \left( B + \gamma_i b_i + \phi - \left( \gamma_i \sum_j \lambda_j (\tilde{V}_j^{0,B+b_i}) + (1-\gamma_i) \sum_j \lambda_j (\tilde{V}_j^{0,B}) \right) \right)
\]

\[
\leq r_i - 1/\lambda \left( \sum_j \lambda_j r_j + \gamma_i b_i - \left( \gamma_i \sum_j \lambda_j (\tilde{V}_j^{0,B+b_i}) + (1-\gamma_i) \sum_j \lambda_j (\tilde{V}_j^{0,B}) \right) \right)
\]

\[
\leq r_i - \gamma_i b_i/\lambda + 1/\lambda \sum_j \lambda_j (\tilde{V}_j^{0,B} - r_j)).
\]

The first inequality holds because, by the Lemma’s assumption, \( B > \bar{B} \), \( \gamma_i < 1 \), and by definition \( \sum_j \lambda_j r_j = \bar{B} + \phi \). The second holds because \( \tilde{V}_j^{0,B+b_i} \leq \tilde{V}_j^{0,B} \) as any policy optimizing the case with backorder cost \( B+b_i \) is feasible and at lower cost when the backorder cost is \( B \). Thus we have

\[
\tilde{V}_{i,y=0}^{0,B} - r_i \leq -\gamma_i b_i/\lambda + 1/\lambda \left( \sum_j \lambda_j (\tilde{V}_j^{0,B} - r_j) \right).
\]

Suppose for some set \( \Psi \subseteq N, y_j^{0,B} = 0 \) for \( j \in \Psi \), and \( y_j^{0,B} = -1 \) for \( j \notin \Psi \). Then

\[
\sum_{j \in \Psi} \lambda_j (\tilde{V}_j^{0,B} - r_j) \leq -\sum_{j \in \Psi} \lambda_j \gamma_j b_j/\lambda + \sum_{j \notin \Psi} \lambda_j \sum_j \lambda_j (\tilde{V}_j^{0,B} - r_j).
\]

Observe that

\[
\sum_j \lambda_j (\tilde{V}_j^{0,B} - r_j) = \sum_{j \in \Psi} \lambda_j (\tilde{V}_j^{0,B} - r_j) + \sum_{j \notin \Psi} \lambda_j (\tilde{V}_j^{0,B} - r_j)
\]

\[
= \sum_{j \in \Psi} \lambda_j (\tilde{V}_j^{0,B} - r_j) \quad (26)
\]
because for \( j \notin \Psi \), \( \Pi_{j,y=-1}^{0,B} = r_j \) and \( T_{j,y=-1}^{0,B} = 0 \). Therefore substituting (26) into (25) and rearranging terms we have
\[
\sum_{j \notin \Psi} \frac{\lambda_j}{\lambda} \sum_j \lambda_j (\tilde{V}_j^{0,B} - r_j) \leq -\sum_{j \in \Psi} \frac{\lambda_j \gamma_j b_j}{\lambda}.
\]
Since the l.h.s. of the inequality is non-negative we have a contradiction. Therefore \( y_i^{0,B} = -1 \) for all \( i \).

**Induction:** Now assume for some \( I > 0 \), \( \tilde{V}_{i,I-1,B} = r_i \) for all \( i \) and \( I' \leq I - 1 \). Consider an arrival from class \( i \) in state \((I,B)\). Observe
\[
\tilde{V}_{i,y=1}^{I,B} - \tilde{V}_{i,y=-1}^{I,B} = -\frac{1}{\lambda} \left( \phi + h(I - 1) + B - \sum_j \lambda_j \tilde{V}_j^{I-1,B} \right)
\]
\[
= -\frac{1}{\lambda} \left( \phi + h(I - 1) + B - \sum_j \lambda_j r_j \right)
\]
\[
\leq -\frac{1}{\lambda} \left( \phi + \sum_j \lambda_j r_j - \phi - \sum_j \lambda_j r_j \right)
\]
\[
= 0,
\]
where the second equality holds by the induction assumption and the inequality holds as \( B \geq \bar{B} \equiv \sum_j \lambda_j r_j - \phi \). Therefore serving the demand from inventory is suboptimal for all \( I \).

Let \( \Psi = \{ i \text{ such that } y_i^{I',B} = 0 \} \) and assume w.l.o.g. \( \Psi = 1, \ldots, k \) for some \( k \leq N \). Then for \( i \in \Psi \), \( \tilde{V}_{i,y=0} - r_i > 0 \). Observe \( i \in \Psi \),
\[
\tilde{V}_{i,y=0}^{I,B} - r_i = (\gamma_i - 1)r_i - (hI + B + \gamma_i b_i + \phi)/\lambda + \gamma_i \sum_j \alpha_j \tilde{V}_j^{I,B} + (1 - \gamma_i) \sum_j \alpha_j \tilde{V}_j^{I,B}
\]
\[
\leq (\gamma_i - 1)r_i - (\bar{B} + \phi)/\lambda + \sum_j \alpha_j \tilde{V}_j^{I,B}
\]
\[
= (\gamma_i - 1)r_i - \sum_j \alpha_j r_j + \sum_{j \in \Psi} \alpha_j \tilde{V}_{j,y=0}^{I,B} + \sum_{j \notin \Psi} \alpha_j r_j
\]
\[
= (\gamma_i - 1)r_i + \sum_{j \in \Psi} \alpha_j (\tilde{V}_{j,y=0}^{I,B} - r_j),
\]
where the inequality follows from Lemma 1. Thus we have
\[
(1 - \alpha_i) \left( \tilde{V}_{i,y=0}^{I,B} - r_i \right) \leq (\gamma_i - 1)r_i + \sum_{j \in \Psi, j \neq i} \alpha_j (\tilde{V}_{j,y=0}^{I,B} - r_j)
\] (27)
for all \( i \in \Psi \).

Let
\[
A = \begin{bmatrix}
(1 - \alpha_1) & -\alpha_2 & \ldots & -\alpha_k \\
-\alpha_1 & (1 - \alpha_2) & \ldots & -\alpha_k \\
\vdots & \ddots & \ddots & \vdots \\
-\alpha_1 & -\alpha_2 & \ldots & (1 - \alpha_k)
\end{bmatrix}.
\]
Also, let \( \beta_i = (\gamma_i - 1)r_i \), and \( \beta = [\beta_1, \beta_2, \ldots, \beta_k]' \). Then (27) implies \( Ax \leq \beta \) where \( x_i = \tilde{V}_{i,y=0}^I - r_i \).

Observe

\[
A^{-1} = \frac{1}{1 - \sum_{i \in \Psi} \alpha_i} \begin{pmatrix}
(1 - \sum_{j \in \Psi, j \neq 1} \alpha_j) & \alpha_2 & \cdots & \alpha_k \\
\alpha_1 & (1 - \sum_{j \in \Psi, j \neq 2} \alpha_j) & \cdots & \alpha_k \\
\vdots & \alpha_1 & \alpha_2 & \cdots & (1 - \sum_{j \in \Psi, j \neq k} \alpha_j)
\end{pmatrix}
\]

\( A^{-1} \) has all positive entries while \( \beta < 0 \). Therefore \( x = A^{-1}\beta < 0 \). Contradiction. Therefore \( \Psi = \emptyset \) and \( y_i^I,B = -1 \) for all \( i \).

**Proof of Lemma 3** Observe

\[
g(I,B) = -\left(\frac{hI + B + \phi}{\lambda}\right) + E \left( \max \left\{ \gamma_ir_i + \gamma_i g(I,B+b_i) + (1-\gamma_i)g(I,B), \frac{r_i + g(I-1,B)}{r_i} \right\} \right)
\]

\[
\leq \sum_i \alpha_i \left( r_i - \frac{hI + B + \phi}{\lambda} \right) + \sum_i \alpha_i \max \left\{ \frac{g(I-1,B)}{r_i}, -\left(1-\gamma_i\right)r_i + g(I,B) \right\}
\]

by Lemma 1. So

\[
0 \leq \sum_i \alpha_i \left( r_i - \frac{hI + B + \phi}{\lambda} \right) + \sum_i \alpha_i \max \left\{ \frac{g(I-1,B) - g(I,B)}{r_i}, -\left(1-\gamma_i\right)r_i, -g(I,B) \right\}
\]

(28)

By the assumption \( B > \bar{B} - hI, \sum_i \alpha_i \left( r_i - \frac{hI + B + \phi}{\lambda} \right) \leq 0 \), so that \( g(I-1,B) \geq g(I,B) \). Thus

\[
\tilde{V}_{i,y=0}^{I,B} = \gamma_ir_i + \gamma_ig(I,B+b_i) + (1-\gamma_i)g(I,B)
\]

\[
\leq r_i + g(I,B)
\]

\[
\leq r_i + g(I-1,B)
\]

\[
= \tilde{V}_{i,y=1}^{I,B}
\]

**Proof of Lemma 4**

\[
V_{i,y=1}^{I,B} - V_{i,y=-1}^{I,B} = g(I-1,B)
\]

\[
\geq \sum_i \alpha_ir_i - \left(\frac{h(I-1) + B + \phi}{\lambda}\right)
\]

\[
\geq 0.
\]

The first inequality holds by definition of \( g(I-1,B) \) and the second by the assumption of the lemma.
Proof of Lemma 5 If $b_j \leq h$, then $B > \bar{B} - h(I - 1) - b_j \geq \bar{B} - hI$. Since $B \leq \bar{B} - h(I - 1)$, by Lemmas 3 and 4, $y_i^{I,B} = 1$.

If $b_j > h$, from (28), $g(I - 1, B) \geq g(I, B)$ if
\[
\sum_i \alpha_i \left( r_i - \frac{hI + B + \phi}{\lambda} \right) + \sum_i \alpha_i \left( -(1 - \gamma_i)r_i \right) = E[\gamma_i r_i] - \frac{hI + B + \phi}{\lambda} < 0.
\]
But $B + b_j > \bar{B} - h(I - 1)$ implies $E[\gamma_i r_i] - (hI + B + \phi)/\lambda \leq (b_j - h)/\lambda - E[(1 - \gamma_i)r_i] < 0$ by assumption.

So
\[
\tilde{V}_{j,y=0}^{I,B} = \gamma_j r_j + \gamma_j g(I, B + b_j) + (1 - \gamma_j)g(I, B) \\
\leq r_j + g(I, B) \\
\leq r_j + g(I - 1, B) \\
= \tilde{V}_{j,y=1}^{I,B}.
\]

Proof of Lemma 6 Let $\beta_i^{I,B} = \gamma_i r_i - (hI + B + \gamma_i b_i + \phi)/\lambda + \gamma_i \sum_j \alpha_j \tilde{V}_{j,y=1}^{I,B+b_i}$. In the following we suppress the superscripts (I,B). Linear program (12a)–(12c) may be written as
\[
\begin{align*}
\min & \sum_j \tilde{V}_j \\
\text{s.t.} & \tilde{V}_i \geq \beta_i + (1 - \gamma_i) \sum_j \alpha_j \tilde{V}_j \text{ for all } i \in N \\\n& \tilde{V}_i \geq \text{Serve}_i \text{ for all } i \in N.
\end{align*}
\]

Then letting $x_i = \tilde{V}_i - \text{Serve}_i$ and $\beta_i'' = \beta_i - \text{Serve}_i + (1 - \gamma_i) \sum_j \alpha_j \text{Serve}_j$ the LP is equivalent to
\[
\begin{align*}
\min & \sum_j x_j \\
\text{s.t.} & x_i \geq \beta_i'' + (1 - \gamma_i) \sum_j \alpha_j x_j \text{ for all } i \in N \\
& x_i \geq 0 \text{ for all } i \in N.
\end{align*}
\]

Observe, if $\gamma_i = 1$ then in the solution to the LP $x_i^* = \max[0, \beta_i'']$. Then substituting in $x_i^*$, we can eliminate the variable $x_i$ and the rows corresponding to $i$. Thus, we proceed assuming without loss of generality that $\gamma_i < 1$. Letting
\[
A = \begin{bmatrix}
\frac{1}{1-\gamma_1} - \alpha_1 & -\alpha_2 & \ldots & -\alpha_N \\
-\alpha_1 & \frac{1}{1-\gamma_2} - \alpha_2 & \ldots & -\alpha_N \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_1 & -\alpha_2 & \ldots & \frac{1}{1-\gamma_N} - \alpha_N
\end{bmatrix}
\]
and \( \beta_i = \beta''_i / (1 - \gamma_i) \), we can write the LP as

\[
\min \sum_j x_j \quad \text{s.t.} \quad Ax \geq \beta; \; x \geq 0.
\] (29)

Substituting in the definition of \( \text{Serve}_i^{1:B} \) from (10) and simplifying gives (13) and (14). Observe because \( 0 < \alpha_i < 1 \) and \( 0 \leq \gamma_i < 1 \), the diagonal of \( A \) is positive and the remaining entries in each row are negative. Without loss of generality, we assume the \( \beta_i \) are ordered as \( \beta_1 > \beta_2 > \ldots > \beta_N \).

That \( x = 0 \) is optimal if \( \beta_1 < 0 \) is immediate. For \( \beta_1 > 0 \), let \( u_i \) be the dual variable associated with row \( i \) in the constraint set in (29). Letting \( U = \sum_i u_i \), the dual of (29) is

\[
\max \sum_i u_i \beta_i \\
\text{s.t.} \\
(D) \quad \frac{1}{1 - \gamma_i} u_i - \alpha_i U \leq 1 \text{ for all } i \\
\sum_i u_i - U = 0 \\
u_i \geq 0 \text{ for all } i
\] (D)

Dualizing the constraint \( \sum_i u_i - U = 0 \), the Lagrangian dual of (D) is

\[
LD = \min_\eta \left[ \max_{u_i,U} \sum_j (\beta_j - \eta)u_j + \eta U \right] \\
\text{s.t.} \\
u_i \leq (1 + \alpha_i U)(1 - \gamma_i) \\
u_i \geq 0
\] (30)

Observe \( \beta_i - \eta < 0 \) implies \( u_i = 0 \) and \( \beta_i - \eta \geq 0 \) implies \( u_i = (1 + \alpha_i U)(1 - \gamma_i) \). Letting \( J = \{ j : \beta_j - \eta \geq 0 \} \) the dual is equivalent to

\[
\min_\eta \left[ \max_{U \geq 0} \sum_{j \in J} (\beta_j - \eta)(1 + \alpha_j U)(1 - \gamma_j) + \eta U \right].
\] (31)

Because the term inside the brackets is linear in \( U \), in order that the solution be non-trivial, it must be that the coefficient of \( U \) is 0. That is,

\[
\sum_{j \in J} (\alpha_j (\beta_j - \eta)(1 - \gamma_j)) + \eta = 0,
\] or

\[
\eta = -\frac{\sum_{j \in J} \beta_j \alpha_j (1 - \gamma_j)}{1 - \sum_{j \in J} \alpha_j (1 - \gamma_j)}.
\]
Because the elements of $J$ depend on $\eta$ and change at the points where $\eta = \beta_j$, we need to find the first $k < N$ where

$$\beta_{k+1}(1 - \sum_{j=1}^{k} \alpha_j(1 - \gamma_j)) < -\sum_{j=1}^{k} \beta_j \alpha_j(1 - \gamma_j).$$

If none exists then $k = N$. Noting the independence of the function value on $U$, (30) implies

$$LD = \sum_{i \in J} (1 - \gamma_i)(\beta_i - \eta) = \sum_{i \in J} (1 - \gamma_i) \left( \beta_i + \frac{\sum_{j \in J} \beta_j \alpha_j(1 - \gamma_j)}{1 - \sum_{j \in J} \alpha_j(1 - \gamma_j)} \right)$$

Let $x_i = (1 - \gamma_i) \left( \beta_i + \frac{\sum_{j \in J} \beta_j \alpha_j(1 - \gamma_j)}{1 - \sum_{j \in J} \alpha_j(1 - \gamma_j)} \right)$ for $j \in J$ and $x_j = 0$ for $j \notin J$. Then we find $LD = \sum_i x_i = \sum_{j \in J} x_j$ and $Ax \geq \beta$ so $x$ is the optimal solution by the Fundamental Duality Theorem for linear programs.

**Proof of Lemma 7**

$$\tilde{V}_{i,y=0}^{I,B} - \tilde{V}_{i,y=-1}^{I,B} = -(1 - \gamma_i)r_i + \gamma_i \tilde{g}(I,B) + (1 - \gamma_i)\tilde{g}(I,B) = -(1 - \gamma_i)r_i - \left( \frac{hI + B + \gamma_i b_i + \phi}{\lambda} \right) + \gamma_i \sum_{j} \alpha_j \tilde{V}_{j,B}^{I,B} + (1 - \gamma_i) \sum_{j} \alpha_j \tilde{V}_{j,B}^{I,B} \geq \left( \frac{(1 - \gamma_i)r_i - hI - B - \gamma_i b_i - \phi}{\lambda} \right) + \sum_{j} \alpha_j r_j \geq 0.$$  

The first inequality holds by definition of $\tilde{V}_{j,B}^{I,B}$ and the second by the assumption of the lemma.

**Proof of Proposition 1** $I_0 \leq \lfloor \bar{B}/h \rfloor$ implies $\bar{B} - h(I_0 - 1) \geq \bar{B} - h(\lfloor \bar{B}/h \rfloor - 1) > 0$. Therefore, $B_0 < B - h(I_0 - 1)$ and the first customer is either served or delayed. Then considering an arrival to some state $(I,B)$, $I \geq 1$ and $B \leq \bar{B} - h(I - 1)$, by Lemma 5, if $I \geq 1$ then $B \leq \bar{B} - h(I - 1)$ and therefore, by Lemma 4, $y_{i,B} \in \{0,1\}$. Thus by induction, the firm only closes if $I = 0$. Finally, by Lemma 7, if $I = 0$, the firm only closes if $B \geq \theta_i \geq \bar{B} - \gamma_i b_i - (1 - \gamma_i)r_i\lambda$.

**Proof of Proposition 2** From Lemma 2, we only need to consider $B \in [0,\bar{B}]$. In Step 0, the initial case $(I,B) = (0,\bar{B})$ establishes that the firm is closed with received revenue $r_i$ and the cycle time remaining is 0. This boundary is also established for $\bar{B} < B \leq \bar{B} + \max_i b_i$ on $I \in [0,I_0]$ to facilitate the dynamic program. Observe in Step 1 for $I > 0$, $\text{Serve}_{i,B}^{I,B} = \tilde{V}_{i,y=1}^{I,B}$ and for $I = 0$, $\text{Serve}_{i,B}^{I,B} = \tilde{V}_{i,y=-1}^{I,B} = r_i$; and $\text{Delay}_{i,B}^{I,B} = \tilde{V}_{i,y=0}^{I,B}$. Then if $\tilde{V}_{i,B}^{I,B}$ is the solution vector to (12a), then $\tilde{V}_{i,B}^{I,B} = \text{Serve}_{i,B}^{I,B}$ if $\text{Serve}_{i,B}^{I,B} \geq \text{Delay}_{i,B}^{I,B}$ and $\tilde{V}_{i,B}^{I,B} = \text{Delay}_{i,B}^{I,B}$ otherwise. Thus (9a) and (9b) are satisfied. By Lemma 6 this is the unique solution. Noting $\text{Serve}_{i,B}^{I,B}$ is given as the value for a
continuing operation if $I > 0$ and the terminal value if $I = 0$, then by Proposition 1, Step 2 specifies the optimal policy for $Y(\phi)$. Step 3 determines the auxiliary values $\hat{\Pi}_i^I, B$ and $\tilde{T}_i^I, B$. Note $\tilde{V}_i^I, B = \hat{\Pi}_i^I, B - \phi \tilde{T}_i^I, B$. In Step 4 the algorithm iterates, first over $B$ and subsequently over $I$. Because the arrivals are independent, the policy given for each state $(i, I, B)$ is optimal, and by Lemma 1 the optimal policy for (7) is given by $\hat{Y}$ starting with $I = I_0$ and $B = 0$.

**Proof of Proposition 3** Base: For all $\tilde{B} \in \tilde{B}$, $E[V_{\tilde{I}, \tilde{B}}^0] = \sum_j \alpha_j r_j = E[\tilde{V}^0, B]$ where $B = \sum_j b_j B_j$.

Induction: Assume $E[V_{I', \tilde{B}}^l] = E[\tilde{V}_{I, \tilde{B}}^l]$ for $I \leq I'$ and all $\tilde{B}$ such that $\sum_j b_j B_j \geq B'$ for some $I'$ and $B'$ such that $0 \leq I' \leq I_0$ and $0 \leq B' \leq \tilde{B}$, $\{I', B'\} \neq \{I_0, 0\}$. There are two cases. Case 1: Let $I = I'$ and $B = B' - 1$ (for $B' > 0$). Consider some $\tilde{B}$ such that $\sum_j b_j B_j = B$. Then letting $y_i^I, \tilde{B} = y_i^I, B$ for all $i \in N$, we observe

a. If $y_i^I, B = 1$, $V_i^{I, \tilde{B}} = r_i + g(I - 1, \tilde{B})$

   $= r_i - \frac{h(I-1) + \sum_j b_j B_j + \phi}{\lambda} + E[V_{I-1, \tilde{B}}]

   = \tilde{V}_{i}^{I, B};$

b. If $y_i^I, B = -1$, $V_i^{I, \tilde{B}} = r_i = \tilde{V}_{i}^{I, B};$

c. If $y_i^I, B = 0$, $V_i^{I, \tilde{B}} = \gamma_i r_i + \gamma_i g(I, \tilde{B} + e_j) + (1 - \gamma_i) g(I, \tilde{B})$

   $= \gamma_i r_i + \gamma_i \left( - \frac{h(I' + \sum_j b_j B_j + \phi)}{\lambda} + g(I', \tilde{B} + e_j) \right) + (1 - \gamma_i) \left( - \frac{h(I' + \sum_j b_j B_j + \phi)}{\lambda} + g(I', \tilde{B}) \right)$

   $\geq \tilde{V}_{i}^{I, B}. $

The solution to the linear equations (32) for all $i$ such that $y_i^I, \tilde{B} = 0$ must equal the solution to (8) given in the Rationing Algorithm as all terms are identical to those solving the cases where $y_i^I, B = 0$ (solved through (12b)). Therefore $V_i^{I, \tilde{B}} = \tilde{V}_{i}^{I, B}$ for all $\tilde{B}$ such that $\sum_j b_j B_j = B$.

Case 2: Let $I = I' + 1$ and $B = B'$ and consider some $\tilde{B}$ such that $\sum_j b_j B_j = B$. Then the same argument as in (a)–(c) holds. By induction, on $B'$ and $I'$, $Y = \{y_i^I, B\}$ maximizes $V_{I, B}^{1, 0}$. ■

**Proof of Proposition 4** Let $I^* = \tilde{B}/s$ so $\tilde{I} = [I^*]$.

For $I \leq \tilde{I} - 1$,

$$\tilde{g}(I, B) \geq - \left( \frac{hI + B + \phi}{\lambda} \right) + \sum_j \alpha_j r_j + \tilde{g}(I - 1, B)$$

$$\geq \tilde{g}(I - 1, B)$$

for $B \leq \tilde{B}$. Then by Lemma 1, $g(\tilde{I}, 0) \geq g(I, B)$ for $0 \leq I \leq \tilde{I} - 1$ and $0 \leq B \leq \tilde{B}$.

We next show $\tilde{g}(\tilde{I}, 0) \geq \tilde{g}(I, B)$ for $I \geq \tilde{I}$ and $0 \leq B \leq \tilde{B}$. To do so we show by induction $E[V_{I, 0}^l] \geq E[V_{I, B}^l]$ for $I \geq \tilde{I}$ and $0 \leq B \leq \tilde{B}$. First, from Lemma 2, for $B \geq \tilde{B}$, $E[V_{I+1, B}^l] = \sum_i \alpha_i r_i \leq E[V_{I, B}^l]$. ■
Now assume \( E[V^{I+1,B}] = \sum_i \alpha_i r_i \leq E[V^{I,0}] \) for \( B = B' + 1, B' + 2, \ldots \) for some \( B' \), \( 0 \leq B' < \bar{B} \). Then

\[
E[V^{I+1,B'}] = \sum_i \alpha_i \left( r_i - \frac{hI^* + \phi}{\lambda} \right)
+ \sum_i \alpha_i \max \left( \frac{\hat{g}(I,B') + \frac{hI^* + \phi}{\lambda}}{hI^* + \phi}, \frac{(\gamma_i - 1)r_i + \gamma_i \hat{g}(I + 1, B' + b_i) + (1 - \gamma_i)\hat{g}(I + 1, B') + \frac{hI^* + \phi}{\lambda}}{hI^* + \phi} \right)
\leq \sum_i \alpha_i \max \left( \frac{E[V^{I,B'}]}{\lambda}, \frac{E[V^{I+1,B'}] + (1 - \gamma_i)E[V^{I+1,B'}]}{\lambda} \right)
\leq E[V^{I,0}] + \sum_i \alpha_i \max \left( 0, (1 - \gamma_i)(E[V^{I+1,B'}] - E[V^{I,0}]) \right)
\]  

(33)

where the first inequality follows from removing the negative terms and noting \( \sum_i \alpha_i r_i - (h + \phi)/\lambda = 0 \). The second inequality follows from the induction assumption and Lemma 1. But (33) implies \( E[V^{I+1,B'}] - E[V^{I,0}] \leq E[1 - \gamma_i] \max(0, \gamma_i - 1) \) \( \leq E[V^{I+1,B'}] - E[V^{I,0}] \) or \( E[V^{I+1,B'}] \leq E[V^{I,0}] \). Therefore, by induction, \( E[V^{I+1,0}] \leq E[V^{I,0}] \). Then repeating these arguments, for \( I = \hat{I} + 2, \hat{I} + 3, \ldots \) establishes \( E[V^{I+1,B}] \leq E[V^{I,0}] \) for \( I \geq \hat{I} \) and \( 0 \leq B \leq \bar{B} \). Finally observe this implies

\[
\hat{g}(I,0) = -\left( \frac{hI + \phi}{\lambda} \right) + E[V^{I,0}]
\leq -\left( \frac{hI + \phi}{\lambda} \right) + E[V^{I,B}]
\leq -\left( \frac{hI + B + \phi}{\lambda} \right) + E[V^{I,B}]
= \hat{g}(I,B)
\]

for \( I \geq \hat{I} \) and \( 0 \leq B \leq \bar{B} \).}

**Proof of Lemma 8** First observe that the state space to consider for any optimal policy is finite: By definition \( I \geq 0 \) and \( B \geq 0 \). For the expected revenue rate to exceed the holding cost, \( hI \leq \lfloor \sum_i \lambda r_i \rfloor \). For the expected revenue rate to be greater than the backorder cost \( B \leq \lfloor \sum_i \lambda r_i \rfloor \). Therefore there must be at least one set of recurrent states.

For any state \( s = \{I, B\} \), let \( I(s) = I \) and \( B(s) = B \). Consider a stationary policy \( \pi^A \). For each sample path \( x \), let \( \bar{s}_m(x), m = 1, \ldots \), be the state immediately after the \( m^{th} \) reorder. Assume \( \bar{B}(\bar{s}_m(x)) \neq 0 \) for some \( m \). That is, if \( y^{\pi^A}(x_m) = -1 \) (an order is placed for the \( n^{th} \) arrival under \( \pi^A \) and this is the \( m^{th} \) order, then the state at the time of the \( n+1^{st} \) arrival is \( \bar{s}_m(x) \). Now consider the
path-dependent set of decisions (labeled \( \pi^B \)) that makes the same set of decisions as \( \pi^A \) for each arrival, but does so for each order cycle starting with no backorders, but with the same inventory as \( s_m(x), m = 1, \ldots \). Therefore, for a given \( x \), \( \pi^B \) follows \( \pi^A \) until the first order where \( \pi^B \) places an order to state \( s_1 = \{ I(\bar{s}_1(x)), 0 \} \) instead of \( \{ I(\bar{s}_1(x)), \bar{B}(\bar{s}_m) \} \). Then until the next order we define the decision as \( y_{i,1}^B(s) = y_i^A(\{ I(s), \bar{B}(s) + \bar{B}(\bar{s}_1) \}) \). Similarly, for the \( m^{th} \) order, \( \pi^B \) places an order so the state is \( s_m = \{ I(\bar{s}_m(x)), 0 \} \) and we let \( y_{i,m}^B(s) = y_i^A(\{ I(s), \bar{B}(s) + \bar{B}(\bar{s}_m) \}) \) until the \( m + 1^{st} \) order is placed.

We show that \( h_{x|\sigma} \), the average profit rate under \( \pi^B \) starting in state \( \sigma \), is greater than \( h_{x|\sigma}^A \). First observe for any \( M > 0 \), \( E[T_{x|\sigma}^M] = E[T_{x|\sigma}^M] \) as the \( m^{th} \) order is placed for the \( u_{x|\sigma}^A(m) \) arrival for both \( \pi^A \) and \( \pi^B \). Let \( T_0^A, \sigma = 0 \) and \( \bar{B}(\bar{s}_0) = 0 \). Because the backorder cost using \( \pi^B \), \( \sum b_i \bar{B}_i(\bar{s}_m) \), is less than that of \( \pi^A \) in the \( m^{th} \) order cycle, the total profit until the \( M^{th} \) order for \( \pi^B \) is

\[
v_{x|\sigma}^B \leq E_x \left[ \sum_{n=1}^M t_{n-1}^B(\bar{s}_m(x)) \right] - E_x \left[ \sum_{n=1}^M t_{n-1}^B(\bar{s}_m(x)) \right] \leq v_{x|\sigma}^A
\]

where the inequality is strict if \( \bar{B}(\bar{s}_m(x)) \neq 0 \) for some \( m \leq M \). Therefore,

\[
h_{x|\sigma}^B = \lim_{M \to \infty} E \left[ T_{x|\sigma}^M \right] > \lim_{M \to \infty} E \left[ T_{x|\sigma}^M \right] = h_{x|\sigma}^A.
\]

Thus, \( \pi^B \) is a (possibly path-dependent) policy ordering to \( \bar{s}_m(x) = \{ I_m, 0 \}, m = 1, \ldots \) and has a higher profit rate than that of \( \pi^A \). Let \( \bar{I} = \{ I_m \}_{x \in X, m = 1, \ldots} \), be the set of \( I_m \) for all paths \( x \) and all \( m \) for policy \( \pi^B \), and let \( \bar{K} = |\bar{I}_m| \). Let \( \bar{I} \) be the \( k^{th} \) element of \( \bar{I} \). Then let \( \pi^k \) be the average cost maximizing policy that always orders up to \( \bar{I}_k \), filling all backorders. Because \( \pi^B \) always orders up to some \( I \in \bar{I} \), \( h_{x|\sigma}^B = \sum w_k h_{x|\sigma}^k \) for some weights \( w_k \geq 0, \sum w_k = 1 \). But for some \( k' \), \( h_{x|\sigma}^{k'} = \max_k [h_{x|\sigma}^k] \geq \sum w_k h_{x|\sigma}^k \geq h_{x|\sigma}^A \) and so for any policy \( \pi^A \), there is a dominating policy \( \pi^{k'} \).

Because \( \pi^A \) was chosen arbitrarily, there must exist \( I_0 \).

**Proof of Lemma 9** Consider a sample path \( x \). Recall revenue \( r_{n|\bar{B}(\bar{s}_m(x))}^\phi(x) \) is received for customers \( n = 1 \) to \( n = u_{x|\bar{B}(\bar{s}_m(x))}^\phi(x) \) customers. By following \( Y(\phi) \), for the \( k^{th} \) customer in a sample path \( x \) where \( 1 \leq k \leq u_{x|\bar{B}(\bar{s}_m(x))}^\phi(x) \), if the state upon arrival is \( \{ I, \bar{B} \} \), \( r_{n|\bar{B}(\bar{s}_m(x))}^\phi(x) = r_i \) if \( y_{i,B} = 1 \) or \( y_{i,B} = -1 \), or \( r_{n|\bar{B}(\bar{s}_m(x))}^\phi(x) = \gamma_i r_i \) if \( y_{i,B} = 0 \). Letting the initial state be \( \hat{\sigma} = \{ I_0(\phi), 0 \} \) and taking expectations
over all sample paths, the expected revenue defined by (2) equals the revenue terms in $\Pi(\phi)$ as defined by Step 3 of the rationing algorithm and (20).

Similarly, $c_i^\pi\sigma(x)$, the operating cost (the holding plus backorder cost) depends only on the state at each time $t$ between 0 and $T_i\pi\sigma(x)$, the time of the first order. If under $Y(\phi)$, the system is in state $\{I, \bar{B}\}$ from time $t_1$ to $t_2$, $0 \leq t_1 < t_2 \leq T_i\pi\sigma(x)$, then the cost incurred is $(t_2 - t_1)(hI + \sum x_iB_i)$. Noting that the expected holding cost until the first order is received, $hI_0(\phi)/\lambda$, is added to $E[\Pi_i^{\pi(\phi),0}]$, and again taking expectations over all sample paths, the operating cost defined in (3) equals the operating costs in $\Pi(\phi)$ as defined by Step 3 of the rationing algorithm and (20). Subtracting the ordering cost, $K$, gives the result. 

**Proof of Proposition 5** Lemma 8 implies that under policy $\pi(\phi)$, there is a renewal in state $\{I_0(\phi), 0\}$ and by Lemma 9, the reward between renewals following $\pi(\phi)$ is given by $\Pi(\phi) - K$. The expected time between renewals, $T(\phi)$, is defined by Step 3 and (21). By the Renewal Reward Theorem (see, e.g., Ross (1983), 3.6.1), the average reward is independent of the initial state so that for any initial state $\sigma$,

$$h^{\pi(\phi)} = \lim_{M \to \infty} \frac{\nu_M^{\pi\sigma}}{E[T_M^{\pi\sigma}(x)]} = \frac{\Pi(\phi) - K}{T(\phi)}.$$ 

**Proof of Proposition 6** Let $\bar{\phi} = \sum \lambda_i r_i$. Let $K(\phi) = \Pi(\phi) - \phi T(\phi)$. Observe for $0 < \phi_1 < \phi_2 < \bar{\phi}$,

$$K(\phi_1) = \Pi(\phi_1) - \phi_1 T(\phi_1) \geq \Pi(\phi_2) - \phi_1 T(\phi_2) \geq \Pi(\phi_2) - \phi_2 T(\phi_2) = K(\phi_2),$$

so that $K(\phi)$ is decreasing. Recall $\Pi(\phi)$ is the total profit in a cycle when an exogenous cost $\phi$ is charged per unit time. As $\phi$ increases, $T(\phi)$, the cycle length decreases. Therefore, $\Pi(\phi)$ is decreasing in $\phi$. (Formally, from (13), (14), and (15), we observe the number of classes to delay, $|J|$, is decreasing in $\phi$. Therefore, by Lemma 6 and the assignment of $\Pi_i^{I,B}$ in Step 3 of the Rationing Algorithm, both $T(\phi)$ and $\Pi(\phi)$ are decreasing.)

Observe $K(0) = \Pi(0) > K$ since there exists $\phi$ such that $\Pi(\phi) > K$. Also, $K(\bar{\phi}) \leq 0$ since $\Pi(\phi)/T(\phi) \leq \bar{\phi}$ for all $0 < \phi < \bar{\phi}$. Therefore there exists a unique $\phi^*$ such that $K(\phi^*) = K$. Then letting $f(\phi) = (\Pi(\phi) - K)/T(\phi)$,

$$\phi^* = \frac{\Pi(\phi^*) - K(\phi^*)}{T(\phi^*)} = \frac{\Pi(\phi^*) - K}{T(\phi^*)} = f(\phi^*).$$

so $\phi^*$ is the unique fixed point of $f(\phi)$.

Then for any policy $\{Y', Z'\}$, $\Pi(Y', Z') - \phi^* T(Y', Z') \leq \Pi(\phi^*) - \phi^* T(\phi^*) = K$ or

$$\frac{\Pi(Y', Z') - K}{T(Y', Z')} \leq \phi^*.$$ 

Thus the policy $\pi^* = \{Y(\phi^*), Z(\phi^*)\}$ maximizes the average profit rate.
Appendix B - Approximate Dynamic Program

A difficulty in solving for the rationing and ordering policy of the firm is the partial backlogging of customers where customers can choose whether to accept an offer of delay or not. This leads to the linear program (12a)-(12c) in which we simultaneously solve for $V_{I,B}^j$ for all customer classes $j$ simultaneously for a given $I$ and $B$. While we are able to solve the LP through Lemma 6, a simplifying approximation that effectively decomposes the problem by customer class may prove useful for the extension of the problem where the firm optimizes the discounts offered to the customers.

Recall that $V_{I,B}^i$ is defined on the positive integer set $B \in [0, \ldots, \bar{B}]$ and under the Rationing Algorithm, upon reaching a state $(I,B)$, we know $V_{I,B'}^j$ for all $B < B' \leq \bar{B}$. Let $\hat{V}_{I,x}^i$ be an approximating function of $V_{I,B}^i$ defined for $x$ in the interval $[B, \bar{B}]$. For example, if $\hat{V}_{I,x}^i$ is the linear interpolating approximation, then

$$\hat{V}_{I,x}^i = \left\lfloor \frac{x}{|x|} \right\rfloor V_{I,\lfloor x \rfloor}^i + \frac{x - \lfloor x \rfloor}{|x| - \lfloor x \rfloor} V_{I,\lceil x \rceil}^i.$$ 

Then we can change Steps 1 and 2 of the Rationing Algorithm by redefining the value-to-go if a customer is delayed to be

$$\text{Delay}_{I,B}^i = \gamma_i r_i - (hI + B + \gamma_i b_i + \phi) / \lambda + \sum_j \alpha_j \hat{V}_{I,B}^j + \gamma_i b_i$$

while $\text{Serve}_{I,B}^i$ is as defined in 10. Thus $\text{Delay}_{I,B}^i$ now expresses the approximate value of the expected resulting state of delaying the customer, rather than the expected value for the states where the customer accepts or rejects the offer. Then rather than solving for the $\beta_i$’s and ordering them to find the set $J$ of classes to delay in Step 1, we can simply determine $y_{I,B}^i$ as follows:

For $I > 0$, for each $i \in N$,

$$y_{I,B}^i = \begin{cases} 
1 & \text{if } \text{Serve}_{I,B}^i \geq \text{Delay}_{I,B}^i, \\
0 & \text{otherwise}.
\end{cases}$$

For $I = 0$, for each $i \in N$,

$$y_{I,B}^i = \begin{cases} 
-1 & \text{if } \text{Serve}_{I,B}^i \geq \text{Delay}_{I,B}^i, \\
0 & \text{otherwise}.
\end{cases}$$

The remainder of the algorithm is unchanged.

Consider now the case where the cost per unit time of backordering a class-$i$ customer in state $(I,B)$ is state dependent and given by $b_{I,B}^i + x$ where $x$ is an incentive given to the customer to accept a delay. Then let $\gamma_{I,B}^i(x)$ be the probability that a delay is accepted with incentive $x$. Then the approximate value of delaying a customer can be optimized over $x$, i.e.,

$$\text{Delay}_{I,B}^i = \max_x \gamma_{I,B}^i(x) r_i - (hI + B + (b_{I,B}^i + x) \gamma_{I,B}^i(x) + \phi) / \lambda + \sum_j \alpha_j \hat{V}_{I,B}^j + (b_{I,B}^i + x) \gamma_{I,B}^i(x).$$
If \( \hat{V}_{j}^{I,x} \) is given by a continuous function in \( x \), then this is a non-linear optimization problem in one variable. For example, if \( \gamma_{i}^{I,B}(x) = \zeta + \xi x \) for some \( \zeta, \xi > 0 \) and the \( \hat{V}_{i}^{I,B} \) is the linear interpolating approximation, then the maximizing value of \( x \) can be found by solving for the maximizer of a quadratic equation, while checking the end points for each interval where values of \( x \) are such that

\[
(b_{i}^{I,B} + x)\gamma_{i}^{I,B}(x) = B'
\]

for \( B' \in \{B, B + 1, \ldots, \bar{B}\} \).