Dynamic Financial Hedging Strategies for a Storable Commodity with Demand Uncertainty

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We consider a firm purchasing and processing a storable commodity in a volatile commodity price market. The firm has access to both a commodity spot market and an associated financial derivatives market. The purchased commodity serves as a raw material which is then processed into an end product with uncertain demand. The objective of the firm is to coordinate the replenishment and financial hedging decisions to maximize the mean-variance utility of its terminal wealth over a finite horizon. We employ a dynamic programming approach to characterize the structure of optimal time-consistent policies for inventory and financial hedging decisions of the firm. Assuming unmet demand is lost, we show that under forward hedges the optimal inventory policy can be characterized by a myopic state-dependent base-stock level. The optimal hedging policy can be obtained by minimizing the variance of the hedging portfolio, the value of excess inventory and the profit-to-go as a function of future price. In the presence of a continuum of option strikes, we demonstrate how to construct custom exotic derivatives using forwards and options of all strikes to replicate the profit-to-go function. The financial hedging decisions are derived using the expected profit function evaluated under the optimal inventory policy. These results shed new light into the corporate risk and operations management strategies: inventory replenishment decisions can be separated from the financial hedging decisions as long as forwards are in place, and the dynamic inventory decision problem reduces to a sequence of myopic optimization problems. In contrast to previous results, our work implies that financial hedges do affect optimal operational policies, and inventory and financial hedges can be substitutes. Finally, we extend our analysis to the cases with backorders, price-sensitive demand, and variable transaction costs.

Key words: Finance-Operations Interface, commodity price risk, inventory management, financial hedging.

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1. Introduction

Commodity price volatility has a significant impact on raw material purchasing cost and inventory management in many industries. For example, in chemical industry, manufacturers’ (e.g., BASF) cost structures largely depend on the price of crude oil which experienced tremendous fluctuations in the past several years, ranging from around $30 in 2003 up to a July 2008 high of $147.30 but then down to a December 2008 low of $32 (Berling and Martínez-de-Albéniz 2011). Managing commodity

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material price risk has become one of the biggest challenges for supply chain management. More and more firms have realized the strategic importance of integrating the financial risk management and operational management in their supply chains. For example, in December 2001, Ford announced a loss of $1 billion on its precious-metals inventory and forward-contract agreements. On the contrary, HP was able to cut its purchasing price risk for flash memory using its procurement risk management strategy based on a contract portfolio approach (Nagali et al. 2008). The survey of Bodnar et al. (1998) shows that 50% of U.S. non-financial firms were using financial derivatives as risk management instruments.

There is a growing research interest in studying the financial risk management in supply chain and operational contexts. We refer to Kleindorfer (2008a,b) for excellent overviews of the recent developments and the research challenges to use financial instruments (e.g., forwards, futures and options) in production and supply chain management. Our paper adds to this literature by addressing a joint dynamic financial hedging and inventory control problem for a risk-averse firm with mean-variance utility of the terminal wealth. We attempt to answer the following questions: (1) How to find a time-consistent integrated inventory and financial hedging policy under the mean-variance framework? (2) What is the structure of the optimal joint replenishment and financial hedging policy? (3) What is the distinguishing role of inventory and financial hedges in managing commodity price and demand risks? and (4) Are inventory and financial hedges complements or substitutes?

More specifically, we consider a firm purchasing a storable commodity from a volatile (physical) spot market with the access to associated financial derivatives. This commodity serves as a raw material which is then processed into an end product with fixed selling prices. The demand of the end product is random and depends on spot price. Unmet demand is treated as lost sales. The spot price evolves as a Markov process. The physical spot market and the related financial derivatives market (with the financial instruments written on the spot price) are assumed to be liquid in that no transaction cost is incurred. Moreover, we assume that there is no arbitrage opportunity in both the physical spot market and financial market. The objective of the firm is to maximize the mean-variance utility of terminal wealth.

We formulate this problem as a stochastic dynamic program. Note that the mean-variance criterion is not time consistent which will induce a decision maker to deviate from the previous decision at a later time. This is because the variability of terminal wealth that the decision maker perceives at any point of time is higher than that of a later date. Inspired by Basak and Chabakauri (2010), we apply the law of total variance (Weiss 2005) to develop the recursive equation by decomposing the conditional mean-variance utility at any date into two parts. The first part is the expected future mean-variance utility and the second the conditional variance of expected future cash flow.
The second part serves as an additional risk adjustment due to the greater perceived variability compared to the variability that will be perceived in next period, which provides an incentive to deviate from the decisions made in previous periods. Note that although the mean-variance criterion is not time-consistent, the derived policy in our model is time-consistent, namely, the optimal decisions in the future are also optimal now.

We analyze two types of financial hedging strategies. We first assume that only the vanilla derivative contracts such as forwards and simple options (e.g., European puts and calls) are used. We show that, in the presence of a forward contract maturing at next period, the joint replenishment and hedging optimization problem can be decomposed into two sub-problems: One is a single-period inventory decision problem and the other involves the financial hedging decisions. Therefore, myopic state-dependent base-stock policies are optimal. On the other hand, the dynamic financial hedging decisions can be obtained by minimizing the variance associated with the expected future cash flow conditioned on the next-period spot price. We then show how to construct exotic derivative contracts with European calls and puts of all strikes to fully replicate the expected future cash flow conditioned on the next period’s price. Note that exotic derivatives refer to all those non-standard and tailor-made derivative contracts which normally cannot be replicated by a finite number of standard exchange-traded vanilla derivatives, and mostly traded in over-the-counter (OTC) markets (Briys et al. 1998). Thus the continuum of option strikes also belongs to the exotic derivatives.

In order to examine whether the insight applies to more general settings, we study several important extensions: (1) demand backordering; (2) price-sensitive demand; and (3) transaction costs. When demand is backlogged, we show that the replenishment decisions are still separable from the financial hedging decisions, while the hedging decision must take into account the optimal pricing and inventory policies when predicting future cash flows. When demand is price-sensitive, we show that the joint pricing and replenishment decisions can still be made myopically and separated from financial hedging decisions. When there exist variable transaction costs, the myopic policies are no longer optimal, but we can still use the exotic contracts to replicate the profit-to-go, which implies that the replenishment decisions can still be separated from the financial hedging decisions.

Our contributions to the literature are two-fold. First, we develop a dynamic programming framework to derive the time-consistent optimal inventory and financial hedging policies under an inter-period mean-variance utility. Second, our analysis sheds new light into effective commodity risk management practices: Inventory is used to hedge demand (quantity) risk while financial derivatives are used to hedge the price risk. The inventory replenishment decisions can be separated from financial hedging decisions if the portfolio of hedging derivatives is structured properly.
Financial hedging does affect inventory policies and often simplifies them. However, optimal financial hedging strategies are heavily dependent on inventory policies as they determine future cash flows. In other words, inventory decisions can be separated from financial hedging decisions, but not the other way around.

This paper is organized as follows. Section 2 briefly reviews the related literature. Section 3 presents the model and the dynamic programming formulation. Section 4 characterizes the optimal joint replenishment and hedging policies under different hedging strategies. Sections 5, 6 and 7 discuss the relationship between financial hedges and inventory hedges, elaborate on how financial hedges create value, and illustrate numerically some of the obtained insights. Section 9 concludes the paper. All the proofs are in the Appendix.

2. Related Literature
This section reviews the related literature. We first briefly review the most relevant financial literature and then focus on the operations management literature.

Starting from the seminar paper of Markowitz (1952), the mean-variance analysis has long been seen as the cornerstone of modern portfolio theory and has been widely adopted in both academia and industry. There is an extensive literature about the mean-variance analysis of portfolio optimization. We refer to Basak and Chabakauri (2010) for a comprehensive review of its recent developments. In a complete market, it has been shown that a (static) pre-commitment portfolio strategy chosen at an initial date also solves the one with a quadratic objective for some specific parameters (e.g., Li and Ng 2000). However, Basak and Chabakauri (2010) point out that an investor may subsequently find it optimal to deviate from the initial policy if market conditions change in the future. They solve a continuous-time dynamic asset allocation problem of a mean-variance optimizer in a complete market setting and provide a simple dynamic portfolio policy. Their work appears to be the first to obtain in an incomplete market environment a closed-form solution of the dynamically optimal policy, from which the investor has no incentive to deviate, namely, time-consistent policies. In a similar setting, Basak and Chabakauri (2011) study how to hedge a non-tradable asset with a tradable asset. They obtain time-consistent hedges by dynamic programming and show that their dynamically optimal hedges typically outperform their static and myopic counterparts. Inspired by the developments of Basak and Chabakauri (2010, 2011), we develop a discrete time dynamic programming framework to address the joint replenishment and financial hedging problem. Our model differs from their models mainly in two aspects: (1) We explicitly incorporate and address demand uncertainty. (2) We consider the integration of dynamically optimal operational (inventory) hedges and financial hedges, while there are no operational decisions in their model. (3) We consider a discrete time setting as opposed to their continuous
time model, which allows us to decompose the inventory and financial hedging decisions and better explore the structural insights.

Our work also relates to the optimal positioning problem in derivative securities. Carr and Madan (2001) show that a twice-differentiable payoff function can be replicated by a risk-less asset (the bond), a single risky asset (the stock) and European options of all strikes (derivatives). Oum et al. (2006) employ Carr and Madan’s (2001) approach to develop the optimal static hedging strategy in a competitive electricity wholesale market when both demand and price are uncertain. Our model further extends Carr and Madan’s (2001) analysis to construct the dynamic replication strategies in an inventory system.

Our paper is in line with the literature on the joint operational and financial risk management; see Kleindorfer (2008a,b), Zhu and Kapuscinski (2011) and Kouvelis et al. (2013) for reviews of recent developments. Compared to the financial literature, there is relatively less attention that has been paid to the financial risk management in operational environments. Although some of the tools in finance, such as risk averse decision criterion (e.g., mean-variance analysis of Sobel 1994, expected utility framework of Bonakiz and Sobel 1992, Eeckhoudt et al. 1995 and Agrawal and Seshadri 2000), portfolio optimization (e.g., Martínez-de-Albéniz and Simchi-Levi 2005) and financial option theory (e.g., Ding et al. 2007), have been introduced into operations research, they do not take into account the unique aspects of commodity risk management in uncertain demand environments. Next, we review the related literature in the single-period (newsvendor) and multi-period inventory setting, respectively.

Recent developments on the interfaces of operational and financial risk management mainly focus on the single-period (newsvendor) setting. Gaur and Seshadri (2005) address a joint inventory ordering and financial hedging problem for a retailer with demand being correlated to the price of a financial asset. They first show how to construct single-period hedging strategies in both the mean-variance and the more general utility-maximization frameworks. They then investigate the impact of financial hedges on the expected utility and the inventory levels. They show that financial hedging increases the optimal inventory level for the risk averse newsvendor, and brings it closer to the risk-neutral profit-maximization decision. With weather sensitive demand, Gao et al. (2011) study a joint inventory ordering and weather derivative hedging decision problem under a CVaR framework and obtain similar insights. Ding et al. (2007) consider the integration of operational and financial hedging strategies for a global firm facing exchange rate risk. Chod et al. (2010) examine the relationship between the operational flexibility and financial hedging in capacity investment decisions. They show that product flexibility and financial hedging tend to be complements (substitutes) when product demands are positively (negatively) correlated, whereas postponement flexibility is a substitute to financial hedging. This literature has also been extended
to supply chain settings (e.g., Caldentey and Haugh 2009). Caldentey and Haugh (2009) address the impact of financial hedging on the supply chain performance and contract design, and show that in a typical two-stage supply chain the supplier always prefers the flexible contract with financial hedging to the flexible contract without financial hedging whereas the buyer might or might not prefer the flexible contract with hedging.

There are only a few papers addressing the joint financial hedging and operational decisions in multi-period settings. With a quadratic utility function, Caldentey and Haugh (2006) provide a modeling framework that allows continuous trading in the financial market to hedge against a cash flow driven by an operational policy. With an additive exponential utility function, Smith and Nau (1995) address a project valuation problem and show that the operational and financial hedging decisions can be separated from each other. Applying the same framework of Smith and Nau (1995), Chen et al. (2007) propose a multi-period consumption model that integrates pricing, inventory control, consumption and financial hedging decisions. Assuming that financial hedging decisions are made after observing the realized demand, they show that the inventory and pricing decisions can be separated from the consumption and financial hedging decisions. Their results indicate that the financial hedging does not change the inventory and pricing decisions when compared to the system without financial hedges. In a similar setting, Geman and Ohana (2008) address the problem of managing a storable commodity portfolio that includes physical assets and hedging positions in spot and forward markets. Their discussion focuses on the time-consistency of the utility functions. Note that the utility functions used by Chen et al. (2007) and Geman and Ohana (2008) are both intra-period (see Kouvelis et al. 2013 for a detailed discussion), as opposed to our inter-period utility.

Kouvelis et al. (2013) is the first paper to employ an inter-period utility for the joint inventory control and financial hedging problem for a storable commodity in a multi-period setting. More specifically, they consider a risk-averse buyer procuring a storable commodity from a spot market and a fixed-price long-term contractual channel, and having access to the financial derivatives. In each period, the buyer maximizes the mean-variance utility of the net present value (NPV) of the total cash flow over the remaining periods, taking into account the future decisions under the same criterion. However, it is notable that since the mean-variance utility is not time-consistent, the future decisions under the same criterion may not be optimal for the objective function of current period.

Our paper addresses the same issue of Kouvelis et al. (2013), but without the presence of long-term contract, and with the objective of maximizing the mean-variance utility of the terminal wealth. We employ the dynamic programming approach to obtain a time-consistent optimal policy, explicitly accounting for the fact that the mean-variance utility itself is not time-consistent. Our
analysis shows that the joint replenishment and hedging policy can be decomposed as follows: (1) the optimal inventory policy is derived from solving a sequence of myopic optimization problems, and thus myopic base-stock policies are optimal; and (2) the financial hedging portfolio is to offset the price risk associated with the inventory carried over to the next period and the profit-to-go as a function of future price. Our result confirms that the operational (inventory) decisions can be separated from financial decisions. Differing from results in Chen et al. (2007), we show that financial hedges do affect optimal inventory policies (myopic policies are no longer optimal without financial (forward) hedges). In the last period, financial hedges increase inventory order-up-to level, whereas in previous periods financial hedges can reduce inventory order-up-to level in comparison to the unhedged solution. We further extend our analysis to account for backlogs, price-sensitive demand, and transaction costs. In particular, in the presence of transaction costs, the myopic inventory policy is no longer optimal, but the inventory decision is still separable from financial hedging decisions when exotic hedges with forwards and options of all strikes may be used.

3. The Model
3.1. Problem Description
Consider a risk averse firm purchasing a storable commodity from a volatile spot market. This commodity serves as a raw material which can be processed into an end product with uncertain demand and fixed selling prices. Assume that there exists a spot market and an associated financial market for financial derivative contracts written on the spot price. The inventory is reviewed periodically and the periods are indexed by \( t = 0, 1, 2, \ldots, T + 1 \). Period 0 is the beginning period and period \( T + 1 \) is the ending period. The firm serves the demand occurring in period \( t = 1, \ldots, T \), denoted by \( d_t \), with a fixed unit selling price \( r_t \). Unmet demand is lost. The excess inventory is carried over to the next period which incurs a holding cost \( h_t \) for each unit of inventory.

We assume that this commodity product can be sold or bought from spot market with current spot price at the beginning of each period. The replenishment leadtime is zero. Assume that the commodity spot market is liquid in the sense that there is no transaction cost and the bid-ask spread is zero. Let \( S_t \) be the risk-adjusted spot price at the beginning of period \( t \). Denote by \( s \) the realization of the spot price. The spot price process \( \{ S_t, t = 0, 1, \ldots, T \} \) is a Markov process.

Demand in each period may depend on the spot price. Without loss of generality, we assume that the support of demand is \( \mathbb{R}_+ \). Let \( \Phi_t(\cdot | s) \) and \( \phi_t(\cdot | s) \) be the distribution and density functions of demand given a spot price \( s \). Assume that \( \Phi_t(\xi, s) \) is twice differentiable in \( (\xi, s) \). As Gaur and Seshadri (2005), one may specify the demand as the form

\[
d_t = a + bs + \epsilon_t,
\]
where $\epsilon_t$ is an error term independent of $s$ such that $E_t[\epsilon_t] = 0$ and $E[\epsilon_t^2] < \infty$.

We denote by $\mathcal{F}_t$ the filtration generated by $(S_t, d_t)$. Throughout the paper, we make the no arbitrage assumption and assume a risk-neutral probability measure $Q$ (see, e.g., Geman and Ohana 2008). For convenience, let $E_t[\cdot] = E[\cdot | \mathcal{F}_t] = E[\cdot | S_t = s]$ denote the conditional expectation operator under $Q$.

According to the competitive storage theory, the commodity price satisfies the following no-arbitrage arguments (see, e.g., Working 1948, Williams and Wright 1991, Routledge et al. 2000).

**Assumption 1.** The spot price and holding cost satisfy the no-arbitrage condition, i.e.,

$$s + h_t - \alpha E[S_{t+1} | S(t) = s] \geq 0,$$

where $\alpha$ is the risk-free discount factor.

The financial market is assumed to be complete. All the financial contracts (e.g., forwards/futures and options) are written on spot price and fairly priced. For example, the forward prices are determined by the expected spot price under a risk neutral probability measure at maturity of the contract. Let $f_{t,\tau}(s)$ be the forward price quoted in period $t$ and for delivery in period $\tau \geq t$ given the spot price at $t$, $s$. In particular, $f_{t,\tau}(s) = E_t[S_\tau]$ and $f_{t,t} = S_t$. The no arbitrage assumption implies that the forward prices are martingale under $\mathbb{P}$, i.e., $E_t[f_{t,\tau}] = f_{t,t} = S_t$. Let $H_t(S_{t+1})$ be the net profit (or marked-to-market payoff) of a portfolio under the hedging strategy in period $t$ with a random payoff depending on the realization of the spot price $S_{t+1}$ in period $t + 1$. Without loss of generality, we let function $H_t$ represent a hedging strategy. Let $\mathcal{H}_t$ be the set of feasible hedging strategies. We impose the following assumption on the payoff function of the hedging portfolio.

**Assumption 2.** For all $t$ and for any $H_t \in \mathcal{H}_t$, the expected profit from the hedging strategy is zero, i.e., $E_t[H_t(S_{t+1})] = 0$.

This assumption implies that the financial derivatives are fairly priced and thus the expected profit earned from the hedging portfolio is zero. This is a common assumption in finance literature (see, e.g., Brown and Toft 2002).

### 3.2. Dynamic Programming Formulation

Note that in period 0 only hedging decision is made. From period 1 to $T$, a joint inventory and hedging decision is made. Let $x_t$ be the inventory level at the beginning of period $t$ and $y_t$ be the order-up-to level in period $t$ such that $y_t \in \mathbb{R}_+$. The sequence of events in each period $t > 0$ is as following: (1) observe the spot price $S_t = s$ and obtain the payoff of last period’s financial hedge $H_{t-1}(S_t)$; (2) place an order $y_t - x_t$ (bid if $y_t > x_t$ or ask if $y_t < x_t$) from spot market, and choose
a hedging portfolio $H_t$ which matures at the next period; (3) demand $d_t$ occurs; and (4) holding costs are incurred for ending inventory of the level $(y_t - d_t)^+$. Let $\mathcal{U}$ be the set of admissible joint replenishment and financial hedging policies. Under any policy $u \in \mathcal{U}$, the dynamics of inventory level are

$$x_{t+1} = (y_t - d_t)^+, t = 1, \cdots, T.$$  

The operational profit function of each period is denoted by

$$\tilde{\pi}_t(x_t, y_t, S_t) = r_t \min(y_t, d_t) - S_t(y_t - x_t) - h_t(y_t - d_t)^+, t = 1, 2, \cdots, T,$$

$$\tilde{\pi}_{T+1}(x_{T+1}, S_{T+1}) = x_{T+1}S_{T+1}.$$  

Note that $\tilde{\pi}_0 = 0$. For simplicity, we assume that $x_1 = x_0 = 0$. Let $\tilde{W}_t$ be the initial wealth at the beginning of period $t$. Under a joint inventory and hedging strategy, the dynamics of wealth are represented by

$$\tilde{W}_{t+1} = \alpha^{-1} [\tilde{W}_t + \tilde{\pi}_t(x_t, y_t, S_t)] + H_t(S_{t+1}), t = 1, \cdots, T$$

and the terminal wealth is expressed as

$$\tilde{W}_{T+1} = \alpha^{-1} [\tilde{W}_T + \tilde{\pi}_T(x_T, y_T, S_T)] + S_{T+1}x_{T+1} + H_T(S_{T+1})$$

$$= \alpha^{-(T+1)} \tilde{W}_0 + \sum_{t=1}^{T} \alpha^{-(T+1-t)} [\tilde{\pi}_t(x_t, y_t, S_t) + H_{t-1}(S_t)] + S_{T+1}x_{T+1} + H_T(S_{T+1}),$$

where $W_0$ is the initial wealth level which is an exogenously given positive constant. For convenience, we redefine the operational profit associated with period $t$’s operation as

$$\pi_t(y_t, S_t, S_{t+1}) = r_t \min(y_t, d_t) - S_t(y_t - x_t) - h_t(y_t - d_t)^+ + \alpha S_{t+1}(y_t - d_t)^+,$$

$$= (r_t - S_t)y_t - (r_t + h_t - \alpha S_{t+1})(y_t - d_t)^+,$$

with $\pi_0 = 0$ and $\pi_{T+1} = 0$. This profit function counts the total profit from the sales revenue less the holding cost and plus the clear value of the ending inventory at the beginning of the next period. Since the market is liquid and there is no backlog, the excess inventory can be sold back to the market at the spot price of the next period without any transaction cost, and thus the inventory replenishment decision of the next period does not depend on how much inventory is on hand. This treatment effectively allows us to separate the cash flow related to the current period’s inventory decision from the cash flow related to the next period’s inventory decision.

Redefine the dynamics of wealth process as

$$W_{t+1} = \alpha^{-1} [W_t + \pi_t(y_t, S_t, S_{t+1}) + \alpha H_t(S_{t+1})], t = 0, 1, \cdots, T. (1)$$
The terminal wealth can be expressed as

\[ W_{T+1} = \sum_{t=0}^{T} \alpha^{-(T+1-t)}[\pi_t(y_t, S_t, S_{t+1}) + \alpha H_t(S_{t+1})]. \]

Notice that \( W_{T+1} = \tilde{W}_{T+1} \).

The firm’s objective is to maximize the mean-variance utility of the terminal wealth at the end of the planning horizon. In particular, the dynamic joint replenishment and financial hedging problem of the firm is given by

\[
\max_{u \in U} \{ U_t := E_t[W_{T+1}] - \lambda \text{Var}_t[W_{T+1}] \},
\]

subject to the wealth dynamics (1).

**Remark 1 (Time Consistency).** The variance risk measure is time-inconsistent while the expectation operation is time-consistent. To see this, we apply the law-of-total-variance (e.g., Weiss 2005) such that

\[ \text{Var}_t(W_{T+1}) = E_t[\text{Var}_{t+1}(W_{T+1})] + \text{Var}_t[E_{t+1}[W_{T+1}]], t < T + 1. \]

Clearly, the period-\((t + 1)\) variance is larger than the expected variance at period-\((t + 1)\). Thus, the hedging strategy in period \(t\) should account not only for the expected period-\((t + 1)\) variance of the terminal wealth, but also the variance of period-\((t + 1)\) expected terminal wealth. Then, given any admissible policy \(u\), the mean-variance utility \(U_t\) satisfies the recursive representation

\[ U_t = E_t[U_{t+1}] - \lambda \text{Var}_t[E_{t+1}[W_{T+1}]]. \]

That is, the expected mean-variance utility in period \(t\) consists of the expected future utility and an adjustment. This adjustment enables us to determine the optimal policy by backward induction. This derived optimal policy is time-consistent in the sense that the firm optimally chooses the policy taking into account that he or she will act optimally in the future. We refer to Basak and Chabakauri (2010) for a more detailed discussion about the dynamic mean-variance analysis framework and the time consistence of the derived optimal policy.

Let \( \tilde{J}_t(W_t, s) \) as the mean-variance utility under the optimal policy \( u^* \) which solves (2) given the wealth level and spot price at the beginning of period \(t\). The following proposition presents the dynamic programming formulation for \( \tilde{J}_t \).

**Proposition 1.** \( \tilde{J}_0(W_0, s) \) satisfies the following optimality equation:

\[
\tilde{J}_0(W_0, s) = E_0[\tilde{J}_1(W_1, S_1)] - \lambda \min_{H_0 \in \mathcal{H}_0} \text{Var}_0 \left[ \alpha^{-T} H_0(S_1) + v_1(S_1) \right].
\]
For \( t = 1, \ldots, T \), \( \tilde{J}_{t}(W_{t}, s) \) satisfies the following optimality equation:

\[
\tilde{J}_{t}(W_{t}, s) = \max_{y_{t} \geq 0, H_{t} \in \mathcal{H}_{t}} \left\{ E_{t}[\tilde{J}_{t+1}(W_{t+1}, S_{t+1})] - \lambda \text{Var}_{t} [\alpha^{-\left( T+1-t \right)} \pi_{t} + \alpha H_{t}(S_{t+1})] + v_{t+1}(S_{t+1}) \right\}
\]

where \( v_{t+1}(s) \) is the expected profit-to-go function (accounted to period \( T+1 \)) under the optimal policy and satisfies the following recursive equation:

\[
v_{t}(s) = E_{t}[\alpha^{-\left( T+1-t \right)} \pi_{t}(y_{t}^{*}(s), S_{t}, S_{t+1})] + E_{t}[v_{t+1}(S_{t+1})],
\]

with the terminal value \( v_{T+1}(s) = 0 \). Here, \( y_{t}^{*}(s) \) is the optimal order-up-to level in period \( t \) when the spot price is \( s \).

Equation (3) shows that in period 0 only the hedging decision is made. For \( t = 1, \ldots, T \), the optimality equation (4) states that the period-\( t \) mean-variance of terminal wealth under the optimal policy equals the maximum of expected period-(\( t+1 \)) mean-variance of terminal wealth less the product of \( \lambda \) and the variance of period-(\( t+1 \)) expected terminal wealth. The second term serves as a time-consistent adjustment term which enables us to determine the optimal policy by backward induction. The derived optimal policy is \textit{time-consistent} in that the firm optimally chooses the decisions taking into account that she will act optimally in the future periods.

The following proposition shows that the formulation (4) for the problem (2) can be transformed into a simpler equivalent problem by separating the wealth at the beginning of each period from the objective function.

\textbf{Proposition 2.} The optimal mean-variance utility function \( \tilde{J}_{t}(W_{t}, s) \) can be decomposed into two parts as \( \tilde{J}_{t}(W_{t}, s) = J_{t}(s) + \alpha^{-\left( T+1-t \right)} W_{t} \), where \( J_{t}(s) \) satisfies the following optimality equations:

\[
J_{0}(s) = E_{0}[J_{1}(S_{1})] - \lambda \alpha^{-2T} \min_{H_{0} \in \mathcal{H}_{0}} \text{Var}_{0}[\alpha^{T} v_{1}(S_{1}) + H_{0}(S_{1})],
\]

\[
J_{t}(s) = \max_{y_{t} \geq 0} \{ \alpha^{-\left( T+1-t \right)} g_{t}(y, s) \} + E_{t}[J_{t+1}(S_{t+1})], t = 1, \ldots, T,
\]

where \( J_{T+1}(s) = 0 \) and

\[
g_{t}(y, s) = E_{t}[\pi_{t}] - \lambda \alpha^{-\left( T+1-t \right)} \left[ A_{t}(s) \text{Var}_{t}[(y - d_{t})^{+}] + \alpha^{2} \psi_{t}(y, s) \right],
\]

\[
\psi_{t}(y, s) = \min_{H_{t} \in \mathcal{H}_{t}} \left\{ \text{Var}_{t}[S_{t+1}E_{t}[(y - d_{t})^{+}] + \alpha^{T-t} v_{t+1}(S_{t+1}) + H_{t}(S_{t+1})] \right\}.
\]

where \( A_{t}(s) = E_{t}[(r_{t} + h_{t} - \alpha S_{t})^{2}] = (r_{t} + h_{t} - \alpha f_{t}(s))^{2} + \alpha^{2} \text{Var}_{t}[S_{t+1}] \).

The equations (6) and (7) are equivalent to (3) and (4). Note that \( J_{t}(s) \) is the period-\( t \) expected mean-variance of the terminal wealth with subtraction of the initial wealth in period \( t \). The separation of the initial wealth in the objective function allows us to separate \( J_{t+1} \) from the optimization
problem in each period and concentrate on the expected single period profit \( E_t[\pi_t] \) and the time-consistent adjustment term.

The optimality equation (7) implies that in each period \( t \) the optimal policy is independent of the initial wealth of this period. The inventory and financial hedging decisions depend only on the initial price and thus the current inventory decision will not affect next period’s decisions. Moreover, the time-consistent term is divided into two parts: the first one is related to the variance of overage inventory which is carried over to the next period, and the second one is about the variance of the cash flows related to the spot price: the value of expected overage inventory carried to next period, the period-(\(t+1\)) expected profit-go-to, and the financial hedging term. This decomposition implies that the financial hedges are used to mitigate the price risk (by minimizing the variance of the cash flows related to the spot price of the next period), while the risk due to demand uncertainty (i.e., the variance of overage inventory) cannot be financially hedged. In other words, demand risk is hedged with physical inventory and price risk can be hedged by financial instruments. In general, the demand risk cannot be perfectly hedged.

The optimal policy can be solved by backward induction with (7). The optimality equation (7) also indicates that the joint inventory and financial hedging decision can be solved by two sequential steps. First, given any inventory decision \( y \), the optimal hedging strategy is derived by solving the problem (9) from which we can see that the financial hedges should be constructed to offset the cash flow at the beginning of the next period, \( S_{t+1}E_t[(y - d_t)^+] \), and the discounted profit-to-go \( \alpha^{T-t}v_{t+1}(S_{t+1}) \). Second, the optimal inventory decision is obtained by solving the problem in (7), given the optimal inventory level dependent hedging strategy.

Before closing this section, we introduce the evolution of the variance. Let \( V_t(s) \) be the conditional variance of the terminal wealth in period \( t \), \( t = 0, 1, ..., T \). The preceding analysis implies that \( V_t(s) \) satisfies the following recursive equations:

\[
V_0(s) = E_0[V_1(S_1)] + \alpha^{-2T}Var_0[\alpha^{-T}v_1(S_1) + H_0(S_1)],
\]

\[
V_t(s) = E_t[V_{t+1}(S_{t+1})] + \alpha^{-2(T+1-t)}[A_t(s)Var[(y_t^* - d_t)^+] + \alpha^2\psi_t(y_t^*, s)], t = 1, ..., T.
\]

4. Optimal Policies

This section first addresses the optimal hedging and inventory control policy with forwards and then develop the replication strategy when there exist options of all strikes.

4.1. Hedging with Vanilla Derivatives

Vanilla derivatives refer to the standard derivative contracts such as futures, forward and simple options. Futures and forward contracts are the most commonly traded derivative contracts in commodity markets. The trading mechanisms of the futures and forwards are quite similar except
the fact that futures are traded on exchanges and forwards are traded in over-the-counter (OTC) market. For the sake of simplicity, we ignore the difference in trading details between the futures and forwards.

The following theorem characterizes the structure of the integrated inventory replenishment and hedging policies with vanilla derivatives.

**THEOREM 1 (Forward Hedge).** Assume that \( H_t \) contains at least a forward contract maturing at next period. For all \( t \),

(a) a myopic base-stock policy is optimal and the optimal base-stock level is obtained by solving the following sub-problem:

\[
\max_{y \geq 0} \left\{ E_t[\pi_t] - \lambda \alpha^{-(T+1-t)} A_t(s) \text{Var}[(y - d_t)^+] \right\}, t = 1, \ldots, T. \tag{10}
\]

In particular, if \( r_t \leq s \), the least optimal inventory level \( y_t^*(s) = 0 \); otherwise, \( y_t^*(s) \) is the unique solution of the following equation:

\[
[r_t + h_t - \alpha f_t(s) - 2\alpha^{-(T+1-t)} \lambda E[(y - d_t)^+] A_t(s)] \tilde{\Phi}_t(y|s) - (s + h_t - \alpha f_t(s)) = 0. \tag{11}
\]

(b) the optimal hedging portfolio consists of a position of shorting \( E[(y_t^*(s) - d_t)^+] \) units of forwards maturing at next period and a portfolio \( \bar{H}_t \) which is the solution of the following sub-problem:

\[
\min_{\bar{H}_t \in H_t} \text{Var}_t[\bar{H}_t(S_{t+1}) + \alpha^{T-t} v_{t+1}(S_{t+1})] \tag{12}
\]

(c) in particular, if only forward contracts maturing at next period are available, then

\[
\bar{H}_t = -\alpha^{T-t} \frac{\text{Cov}_t[S_{t+1}, v_{t+1}(S_{t+1})]}{\text{Var}_t[S_{t+1}]} [S_{t+1} - f_t(s)],
\]

and

\[
\psi_t(y, s) = \alpha^{2(T-t)} \left[ \text{Var}_t[v_{t+1}(S_{t+1})] - \left( \frac{\text{Cov}_t[S_{t+1}, v_{t+1}(S_{t+1})]}{\text{Var}_t[S_{t+1}]} \right)^2 \right].
\]

Theorem 1 characterizes the optimal policy in the presence of forward contracts. Specifically, part (a) shows that a myopic base-stock policy is optimal. Part (b) shows that the optimal hedging position of the forward contracts consists of two parts: Short \( E[(y_t^*(s) - d_t)^+] \) units of forward and long \( -\alpha^{T-t} \frac{\text{Cov}_t[S_{t+1}, v_{t+1}(S_{t+1})]}{\text{Var}_t[S_{t+1}]} \) (short if it is negative) units of forwards that mature next period.

The first part offsets the variation due to the excess inventory carried over to next period and the second part is to hedge against the risk associated with the future cash flow, \( v_{t+1}(S_{t+1}) \). The final hedging position is the combination of the two parts:

\[
H_t^* = -\left( E[(y_t^*(s) - d_t)^+] + \alpha^{T-t} \frac{\text{Cov}_t[S_{t+1}, v_{t+1}(S_{t+1})]}{\text{Var}_t[S_{t+1}]} \right) [S_{t+1} - f_t(s)].
\]
It is a long (short) position if $E[(y_t^*(s) - d_t)^+] + \alpha^{T-t} \frac{\text{Cov}_{t+1} \text{Var}_{t+1}}{\text{Var}_{t+1}}$ is negative (positive).

This theorem implies that, in the presence of the forward contract maturing at the next period, one could make the inventory decision independently, knowing that the risk associated with the expected excess inventory will be eliminated by a forward hedge. Then $J_t$ can be rewritten as

$$J_t(s) = \alpha^{-(T+1-t)} \tilde{g}_t(s) - \lambda \alpha^{-(T-t)} \tilde{\psi}_t(s) + E_t[J_{t+1}(S_{t+1})],$$

$$\tilde{g}_t(s) = \max_{y \geq 0} \left\{ E_t[\pi_t] - \lambda \alpha^{-(T+1-t)} A_t(s) \text{Var}_t[(y - d_t)^+] \right\},$$

$$\tilde{\psi}_t(s) = \min_{\tilde{H}_t \in \mathcal{H}_t} \text{Var}_t \left[ \alpha^{T-t} v_{t+1}(S_{t+1}) + \tilde{H}_t(S_{t+1}) \right].$$

That is, the joint optimization problem (7)-(9) can be decomposed into two sub-problems: One only involves inventory decisions and the other only involves hedging decisions. This decomposition allows us to compute the original dynamic program in four steps: (1) Derive the optimal myopic base-stock level for each period by solving the inventory-related sub-problem (13); (2) Compute the expected profit-to-go function $v_t$ for all the periods; (3) Compute the optimal portfolio $\tilde{H}_t^*$ of (14) using the $v_t$; and (4) Obtain the optimal hedging portfolio by combing the short position related to the expected excess inventory and $\tilde{H}_t^*$.

Note that although we restrict our attention to the forward contracts, it is straightforward to observe from (14) that this decomposition applies to the general setting where the hedging portfolio contains not just forwards but also other derivatives contracts (e.g., options).

**Remark 2.** The optimality of the myopic base-stock policy relies on the existence of the forwards (maturing in next period). Without the forward contract maturing at next period, the risk associated with overage inventory cannot be perfectly replicated and therefore the determinant of the optimal base-stock level $y_t^*(s)$ involves the profit-to-go $v_{t+1}$, which implies that the optimal base-stock policy may not be myopic any more and the joint optimization problem may not be decomposed into two separate sub-problems.

This decomposition has an important practical implication. In practice, inventory decisions are made by operations managers while the financial hedging decisions are made by financial managers. Our result suggests that the management can allow the operations manager to manage the inventory decision separately and myopically. But, when making the hedging decisions, the risk manager must take into account the optimal inventory policy to forecast the resulting future cash flow. Therefore, the financial hedges simplify the operations manager’s task and help clarify the relationship between operations management and financial risk management within an organization.

The following corollary is a direct application of Theorem 1.
Corollary 1. Suppose the demand is perfectly correlated to the spot price and has the specification \( d_t = a_t + b_t s \) for constants \( a_t, b_t \) such that \( d_t \geq 0 \). Then, for \( t = 1, \ldots, T \),

(a) \( y_t^*(s) = a + bs \) if \( r_t > s \) and \( y_t^*(s) = 0 \) if \( r_t \leq s \).

(b) \( v_t(s) = E_t[\sum_{i=t}^{T} \alpha_t^{-i}(r_t - S_i)^+ (a_t + b_t S_i)] \).

(c) If only the forwards contracts maturing at next period are available, then the optimal hedging portfolio is

\[
H_t^* = -\frac{\text{Cov}_t[S_{t+1}, \alpha_t^{T-t}v_{t+1}(S_{t+1})]}{\text{Var}_t[S_{t+1}]}[S_{t+1} - f_t(s)].
\]

(d) If options contracts maturing at any future periods are available and \( b = 0 \), i.e., \( d_t = a \), then the optimal hedging strategies is to short \( a_t \) units of put options with strike price \( r_t \) maturing at future periods \( i = t + 1, \ldots, T \).

Corollary 1 shows that when the demand is perfectly correlated to the spot price, i.e., only price risk exists, the ordering quantity of each period is simply to order the demanded quantity. But \( v_t(s) \) is still nonlinear in \( s \) (due to the term \( (r_t - S_i)^+ \)), which implies that the price risk cannot be perfectly hedged by forward contracts. For deterministic demand, the price risk can be perfectly hedged by entering the short positions in put options.

We next examine the impact of risk-aversion on the inventory policy. Let \( y_t^*(s; \lambda) \) be the optimal inventory decision with respect to \( \lambda \). The following corollary shows the effect of risk aversion on the optimal inventory decision.

Corollary 2. For all \( t \), \( y_t^*(s; \lambda) \) is decreasing in \( \lambda \).

This result shows that the more risk averse the decision maker is the less inventory the firm orders, and thus the fewer forwards to short. In particular, as \( \lambda = 0 \), the \( y_t^*(s; \lambda) \) reduces to a newsvendor solution (if \( r_t > s \)):

\[
y_t^* = \Phi_t^{-1}((s + h_t - \alpha f_t(s))/(r_t + h_t - \alpha f_t(s))).
\]

Let \( \pi_t^*, U_t^*, E_t[W_{T+1}^*] \) and \( \text{Var}_t[W_{T+1}^*] \) be the period-\( t \) expected operational profit, mean-variance, the expected terminal wealth and variance of the terminal wealth under the optimal policy. We have the following intuitive result.

Corollary 3. For all \( t \), \( E[\pi_t^*], U_t^*, E_t[W_{T+1}^*] \) and \( \text{Var}_t[W_{T+1}^*] \) are all decreasing in \( \lambda \).
4.2. Hedging with Continuum of Option Strikes

In financial markets, since option contracts traded in exchanges normally have limited number of strikes, exotic derivatives are structured to develop more sophisticated hedging or trading strategies. In financial literature, it is common to assume that there exists a continuum of options of all strikes as a proxy to a frictionless financial market. As argued by Carr and Madan (2001), the assumption of a continuum of strikes is essentially the counterpart of the standard assumption of continuous trading, which serves as a reasonable approximation to a market environment where there is a large but finite number of European options strikes (e.g., options for S&P 500). In a single-period hedging model, they show that a twice-continuously differentiable payoff function can be replicated by a portfolio that consists of a spectrum of European put and call options with all strike prices, forwards and risk-less bonds. This type of derivative contracts belong to the exotic derivative contracts. In the following analysis, we will follow Car and Madan’s (2001) approach to construct the optimal exotic hedging strategies.

Note that Car and Madan’s (2001) approach requires the payoff function to be twice differentiable. To employ their approach, we assume that the spot price satisfies the following condition.

**Assumption 3.** For all \( t \) and any real function \( v(\cdot) \) which is twice differentiable almost everywhere, the conditional expectation \( E[v(S_{t+1})|S_t = s] \) is twice differentiable in \( s \) almost everywhere.

For example, for the geometric Ornstein-Uhlenbeck process (also called Schwartz’s One-Factor model, Schwartz 1997):

\[
d\ln S = \kappa(\alpha^* - \ln S)dt + \sigma dz^*(t)
\]

(15)

where \( \kappa \) is the mean-reverting rate, \( \alpha^* = \eta - \sigma^2/2 - \gamma \) is the risk adjusted long-run mean (\( \eta \) is the long-run mean and \( \gamma \) is the market price of risk) and \( z^* \) is a Brownian motion under the risk neutral measure. Assume the length of each period is equal to \( \Delta \). It follows from Schwartz (1997) that the discrete-time dynamics of \( S_{t+1} \) are of the form

\[
S_{t+1} = \exp \left( \alpha^* \left( 1 - e^{-\kappa\Delta} \right) + e^{-\kappa\Delta} \ln S_t + \epsilon_{t+1} \right),
\]

(16)

where \( \epsilon_t \sim N \left( 0, \sqrt{\frac{1-e^{-2\kappa\Delta}}{2\kappa} \sigma^2} \right) \).

Apparently, for any twice differentiable function \( v \) with finite \( E[v(S)] \) and \( E[v''(S)] \), \( v(S_{t+1}) \) is twice differentiable in \( S_t \) and then the conditional expectation \( E[v(S_{t+1})|S_t = s] = E \left[ v \left( s^{e^{-\kappa\Delta}} \exp (\alpha^* (1 - e^{-\kappa\Delta}) + e^{-\kappa\Delta} + \epsilon_{t+1}) \right) \right] \) is twice differentiable in \( s \).

We now proceed to present our key result of this section.

**Theorem 2 (Exotic Hedge).** Suppose that there exist forwards and European calls and puts of all strikes, and Assumption 3 holds. For all \( t \),
(a) the myopic base-stock policy is optimal. If \( r_t \leq s \), the least optimal base-stock level \( y_t^*(s) = 0 \); otherwise, \( y_t^*(s) \) is the unique solution of the following FOC (first-order-condition):

\[
[r_t + h_t - \alpha f_t(s) - 2\alpha^{-(T+1-t)}\lambda E[(y - d_t)^+]]A_t(s) + \Phi_t(y|s) - (s + h_t - \alpha f_t(s)) = 0. \tag{17}
\]

(b) the optimal hedging portfolio is to short \( E_t[(y_t^*(s) - d_t)^+] + \alpha^{T-t}v_{t+1}'(s) \) units of forwards, \( \alpha^{T-t}v_{t+1}''(K)dK \) puts for all strikes \( K < s \) and \( \alpha^{T-t}v_{t+1}''(K)dK \) calls for all strikes \( K > s \) maturing at next period. That is,

\[
H_t^*(S_t+1) = -(E_t[(y_t^*(s) - d_t)^+] + \alpha^{T-t}v_{t+1}'(s))[S_{t+1} - f_t(s)]
+ \int_0^s \alpha^{T-t}v_{t+1}''(K)[(K - S_{t+1})^+ - P_t(K)]dK + \int_s^\infty \alpha^{T-t}v_{t+1}''(K)[(S_{t+1} - K)^+ - C_t(K)]dK,
\]

where \( P_t(K) = E_t(K - S_{t+1})^+ \) and \( C_t(K) = E_t(S_{t+1} - K)^+ \).

Theorem 2 characterizes the structure of the optimal joint inventory control and hedging strategies with complete term structure. Part (a) shows that the optimal ordering policy is described by a myopic state-dependent base-stock level, \( y_t^*(s) \). In any period \( t = 1, \cdots, T \), if \( r_t > s \), then order up/sell down to \( y_t^*(s) \); otherwise, no order is placed. The ordering policy is myopic since the optimal base stock level in a period does not depend on the decisions to make in the future. This significantly simplifies the computation of the optimal solution.

Part (b) describes the structure of the optimal hedging strategy. The hedging portfolio consists of two parts that are associated with the inventory decision of current period and the profit-to-go function of the next period, respectively. First, the optimal hedging strategy requires shorting \( E_t[(y_t^*(s) - d_t)^+] \) units of forwards that mature next period. The intuition behind this is as follows.

In the first part, given the order-up-to inventory level, \( y_t^*(s) \), the expected inventory carried over to next period is \( E_t[(y_t^*(s) - d_t)^+] \). Thus, the associated risky payoff is \( E_t[(y_t^*(s) - d_t)^+]\alpha S_{t+1} \) with the variance being \( (E_t[(y_t^*(s) - d_t)^+])^2Var_t(S_{t+1}) \). Shorting \( E_t[(y_t^*(s) - d_t)^+] \) units of forwards offsets the variation caused by the excess inventory carried over to next period.

The second part of the hedging strategy is related to \( v_{t+1}(S_{t+1}) \), the profit-to-go function which represents the expected total profit from next period. We follow Carr and Madan (2001) to decompose the profit-to-go function into three components which can be interpreted by the portfolio of the forwards and options. Specifically, the optimal hedging decision requires to short \( \alpha^{T-t}v_{t+1}'(s) \) units of forward (in addition to those related to inventory decision), \( \alpha^{T-t}v_{t+1}''(K)dK \) puts for all strikes \( K < s \) and \( \alpha^{T-t}v_{t+1}''(K)dK \) calls for all strikes \( K > s \) maturing at next period. As a result, under the optimal hedging strategy, the next-period expected profit \( v_{t+1}(S_{t+1}) \) is completely offset by the proposed portfolio.
Now, under the optimal hedging strategies, the equation (8) is equal to

$$g_t(y, s) = E_t[\pi_t] - \lambda \alpha^{-(T+1-t)} A_t(s) \text{Var}[(y - d_t)^+],$$

where the second term involves the variance associated with $d_t$. This implies that even with complete term structure of the financial instruments, we are not able to completely hedge all the risk. In fact, we can only mitigate risks related to market spot price (i.e., $S_{t+1} E_t[(y_t^*(s) - d_t)^+]$ and $v_{t+1}(S_{t+1})$) with the financial instruments. The risk associated with the demand is hedged with physical inventory. This demonstrates the different roles of inventory and financial hedge in managing the demand and market price risks.

5. Inventory Hedges versus Financial Hedges

This section discusses the setting without the financial hedges, namely inventory hedge. Without the financial hedges, for $t = 1, \ldots, T$, the optimality equation becomes

$$J_t(s) = \max_{y \geq 0, z \geq 0} \{\alpha^{-(T+1-t)} \hat{g}_t(y, s) \} + E_t[J_{t+1}(S_{t+1})],$$

where

$$\hat{g}_t(y, s) = E_t[\pi_t] - \lambda \alpha^{-(T+1-t)} \left[A_t(s) \text{Var}[(y - d_t)^+] + \alpha^2 \psi_t(y, s)\right],$$

$$\psi_t(y, s) = \text{Var}_t[S_{t+1} E_t[(y - d_t)^+] + \alpha^{T-t} v_{t+1}(S_{t+1})].$$

Let $\hat{y}_t^*$ be the optimal inventory decision without the financial hedges.

We first analyze the last period’s problem.

**Proposition 3.** In period $T$, without financial hedges,

(a) if $r_t \leq s$, the optimal inventory decision $\hat{y}_T^*(s) = 0$; otherwise, the optimal inventory decision $\hat{y}_T^*(s)$ satisfies the following first-order condition:

$$0 = \left[r_t + h_T - \alpha f_T(s) - 2\alpha^{-1} \lambda E[(y - d_T)^+] A_T(s) \right] \Phi_T(y, s) - (s + h_T - \alpha f_T(s))$$

$$-2\alpha \text{Var}_T[S_{T+1}] \Phi_T(y, s) E_T[(y - d_T)^+].$$

(b) in particular, when $r_t > s$, $\hat{y}_T^*(s) < y_T^*(s)$, i.e., the unhedged inventory level is lower than hedged inventory decision.

Part (a) characterizes the optimal inventory solution. Part (b) states that the financial hedges lead to a higher inventory level compared to the system without financial hedges and therefore provides a better service level. This result is consistent with the numerical findings in Kouvelis et al. (2013) that the financial hedges increase customer service level by increasing the inventory level. In other
words, the inventory hedges are complementary to the financial hedges. Similar insights are also found in newsvendor models (see, e.g., Gaur and Seshadri 2005 and Gao et al. 2011).

Although the relationship between inventory and financial hedges in the last period are essentially the same as that in the newsvendor model, it may be different in the previous periods. The following proposition gives a condition that the hedged inventory decision is less than the unhedged.

**Proposition 4.** For any period \( t = 1, \cdots, T - 1 \), suppose that \( y^*_t(s) > 0 \). Then, \( \hat{y}^*_t(s) > y^*_t(s) \) if and only if
\[
E[(y^*_t(s) - d_t)^+] < -\alpha^{t-t} \frac{\text{Cov}_t(S_{t+1}, v_{t+1}(S_{t+1}))}{\text{Var}_t(S_{t+1})}.
\]

This proposition shows that the optimal unhedged inventory decision may not necessarily be less than the optimal inventory decision under financial hedges when the spot price is negatively correlated to the expected next-period profit-to-go (as a function of the next period’s price). Intuitively, the expected profit is a decreasing function of the purchasing price and thus the profit-to-go is likely to be negatively correlated to the spot price. To justify this intuition, we consider the following special case.

**Proposition 5.** If the demand is perfectly negatively correlated with the spot price with the form \( d_t = a - bs, a > 0, b > 0 \), then \( \hat{y}^*_t > d_t \) if and only if \( \lambda < -\frac{a + b - \alpha f_t}{2a\text{Cov}(S_{t+1}, v_{t+1}(S_{t+1}))} \).

Note that \( v_t(s) = E_t[\sum_{i=t}^T \alpha^{t-i}(r_i - S_i)^+(a_i - b_i S_i)] \) which is a strictly decreasing function of \( S_t = s \). Then it is clear that \( S_{t+1} \) and \( v_{t+1}(S_{t+1}) \) are negatively correlated, i.e., \( \text{Cov}_t(S_{t+1}, v_{t+1}(S_{t+1})) < 0 \).

This proposition shows that, when the demand is perfectly negatively correlated with spot price and the risk-averse degree is sufficiently high, the unhedged strategy orders more inventory in period \( t \) to hedge the future inventory risk. Recall that, in the presence of financial hedges, the optimal inventory decision is to order \( d_t \).

The discussion above sheds new light into the relationship between physical inventory and financial hedges: Financial hedges can be a substitute of physical inventory in a multi-period inventory system, whereas it is a complement to the inventory in a single-period inventory system where financial hedges lead to a higher inventory and thus a higher service level. This is the key difference between the storable commodities (e.g., oil and gas) and perishable commodities (e.g., electricity). The implication gained from the newsvendor model may no longer apply to the dynamic model of storable commodity inventories though in conventional inventory models many results found in newsvendor model generally hold in the corresponding dynamic models. This justifies the importance of explicitly addressing the dynamic hedging and inventory control problem.
6. How Financial Hedges Create Value

A conventional thought is that hedging under no arbitrage condition does not increase or decrease profit. Recall that for a fixed $\lambda$, exotic hedges have the same expected terminal wealth and inventory level as that of forward hedges, though the variance of exotic hedges is smaller.

For convenience, we use superscripts $e$, $f$ and $u$ to indicate these strategies (or market profiles): hedging with forwards and options with all strikes (exotic hedges), hedging with forwards only (forward hedges) and unhedged inventory policy. The following lemma compares the mean-variance utilities, expected terminal wealth and variances under difference hedging strategies for the same $\lambda$. The proof is straightforward and is therefore omitted.

**Lemma 1.** For any given $\lambda$, wealth and price levels at time $t$, $U^e_t \geq U^f_t \geq U^u_t$, but $E_t[W^e_{T+1}] = E_t[W^f_{T+1}]$ and $\text{Var}_t[W^e_{T+1}] \leq \text{Var}_t[W^f_{T+1}]$.

Note that the risk-averse parameter $\lambda$ reflects the risk tolerance level of a decision maker. A higher risk tolerance implies a smaller $\lambda$. In practice, instead of giving $\lambda$ directly, one normally specifies the maximum variance level that he/she can tolerate while maximizing the expected return, i.e.,

$$\max_{\omega \in \mathcal{U}} E_0 W^\omega_{T+1},$$

subject to

$$\text{Var}_0 [W^\omega_{T+1}] \leq \delta$$

where $\omega$ is a policy and $\mathcal{U}$ is the set of feasible policies, and $\delta > 0$ is a given variance level. Let $\lambda$ be the Lagrange multiplier. Then the problem becomes

$$\min_{\lambda} \max_{\omega} \{E_0 W^\omega_{T+1} - \lambda [\text{Var}_0[W^\omega_{T+1}] - \delta]\}. \quad (22)$$

For any $i \in \{e, f, u\}$, let $\lambda^{*i}$ be the corresponding optimal Lagrange multiplier derived from the Lagrange problem (22).

Let $\delta_0$ be the variance of the terminal wealth under the risk-neutral solution and $W^0_{T+1}$ the expected risk-neutral terminal wealth. The following theorem characterizes the efficient frontier of the three hedging strategies.

**Theorem 3.** Given the same wealth level and spot price at the beginning of the planning horizon, and a tolerance level $\delta < \delta_0$, the following assertions hold.

(a) Exotic hedges dominate forward hedges which then dominate unhedged inventory policies, i.e.,

$$U^e_0 \geq U^f_0 \geq U^u_0.$$

(b) For any $i \in \{e, f, u\}$, $\lambda^{*i} > 0$ and $\text{Var}_0[W^i_{T+1}] = \delta$ but

$$E[W^e_{T+1}] \geq E[W^f_{T+1}] \geq E[W^u_{T+1}].$$
(c) $\lambda^i$ is decreasing in $\delta$ and then $E[W^i_{T+1}]$ is increasing in $\delta$. In particular, as $\delta \to \delta_0$, $E[W^i_{T+1}]$ converges to risk-neutral expected terminal wealth $W^0_{T+1}$.

(d) $\lambda^{*e} \leq \lambda^{*f}$ and thus $y^{*e}_t \leq y^{*f}_t$.

Theorem 3 shows that the efficient frontiers are increasing and converge to the expected risk-neutral terminal wealth level (see parts (a) and (c)). The efficient frontier under exotic hedge dominates that under forward hedge and the efficient frontier under forward hedge dominates that of unhedged inventory policies. Part (b) and (d) imply that for any given tolerance level, the exotic hedges provide a higher expected terminal wealth level than forward hedges do; and the forward hedges in turn provide a higher expected terminal wealth level than unhedged inventory policies do. This indicates that financial hedging is profitable. Finally, part (d) shows that the complete hedges yield a higher inventory level (service level) than the forward hedge.

These results have important practical implications. First, the risk-aversion parameter $\lambda$ is rarely specified directly in practice. Instead, it is natural to identify the risk-tolerance level $\delta$, which then determines the selection of $\lambda$ as a Lagrange multiplier to penalize the variance. From this perspective, one can argue that the risk-aversion parameter can be endogenously selected.

Second, for any given risk-tolerance level, exotic hedges dominate the vanilla (forward or simple option) hedges and the optimal selection of the Lagrange multiplier binds the variance constraint, which implies that the expected terminal wealth level under exotic hedges is higher than that under a less complete hedge. Third, exotic hedges yield a higher inventory level than forward hedges do, which also explains why hedging can be profitable. Fourth, exotic hedges have a lower risk-aversion parameter $\lambda$ than forward hedges do, which implies that compared to forward hedges, exotic hedges allow one to be less sensitive to the risk and therefore to be able to gain more profit.

7. Numerical Example

This section presents a numerical example to demonstrate and compare the structures of the optimal integrated financial hedging and replenishment policies and their mean and variance performances under different hedging strategies: (1) forward hedges, (2) exotic hedges, and (3) no (financial) hedge.

Specifically, we consider a two-period model, i.e., $T = 2$, with the length of each period being $\Delta = 1$. Assume that the risk-adjusted spot price process follows the Geometric Mean-Reverting process (16). We use the calibration of Schwartz (1997) for crude oil price to specify the parameters of the price process:

$$\kappa = 0.428, \mu = 2.991, \sigma = 0.257, \gamma = 0.002.$$ 

The demand is expressed as $d_t = \mu_d + \epsilon_t$ where $\mu_d$ is the mean of the demand and the error terms $\epsilon_t$ are i.i.d. normally distributed with mean 0 and variance $\sigma_d^2$. This implies that the demand is
normally distributed with mean $\mu_d$ and variance $\sigma_d^2$. Set $\mu_d = 10$ and $\sigma_d = 5$. The initial wealth level is assumed to be zero. The other cost and model parameters are $r_t = 50, h_t = 5, \alpha = 0.9$ and $\lambda = 0.01$. The optimal policy is computed by applying the value iteration approach on the corresponding optimality equations. The results are demonstrated in figures (1)-(3).

Figures 1 demonstrates the optimal order-up-to levels with or without financial hedges. Recall that as long as the forward contracts are used, the risk associated with the inventory carried over to the next period is perfectly hedged and the firm solves the same optimization problem regardless what other derivative contracts are used. Thus, the inventory decisions under forward hedges and exotic hedges are identical. When comparing the hedged and unhedged inventory decisions, one can see the relationship between the hedged and unhedged inventory order-up-to levels changes when moving from the last period to the previous periods: In the last period $t = 2$, the hedged order-up-to level is greater than the unhedged order-up-to level whereas in period 1 the hedged is smaller than the unhedged. In other words, financial hedge increases the order-up-to level in the last period but decreases the order-up-to level in the first period. Recall that newsvendor models predict that financial hedges induce the firm to order more and thus financial hedges and inventory are complements to each other (see, e.g., Gaur and Seshadri 2005). However, in a dynamic system like ours, such a prediction is no longer true. In fact the opposite relationship is more likely to hold in periods before the last period. This can be explained by the relationship between the profit-to-go function $v_t(s)$ and the spot price $s$. As shown in Figure 3, the profit-to-go functions are decreasing in the spot price, which implies that in each period $t < T$ the next period’s price $S_{t+1}$ and the profit-to-go $v_{t+1}(S_{t+1})$ tend to be negatively correlated. Since the negative correlation implies that to offset the variation of the expected future cash flow it is optimal to enter a long position in forward (or to buy forward contracts) if forwards are used, when there is no financial hedge, the inventory carried over to the next period can play the role of forward hedges. That is, the financial hedges and inventory are substitutes for each other.

Figure 2 demonstrates the hedging solutions. Note that the positive (negative) sign indicates a long (short) position. We first discuss the solution under the forward hedge. In the last period, the optimal hedging policy should offset the risk associated with the excess inventory, which implies a short position. Thus the sign is negative. But in the previous period, the hedging positions are all positive. This echoes previous discussion that the negative correlation between the spot price and profit-to-go suggests that the forward hedger should enter a long position. Moreover, the forward hedges will buy (long) more in period 0 than in period 1. This can be explained by the fact that the downward slop of $v_1$ is greater than $v_2$ (the change of $v_1$ is greater the price increases from 1 to 40), i.e., the $S_{t+1}$ and $V_{t+1}$ are more negatively correlated in period 0. Under the exotic hedge, we have the similar observations. Comparing the forward positions under the forward hedge and
exotic hedge at the same period, one can observe that the forward positions under the forward hedge are higher than that under the exotic hedge. This is can be explained by the fact that some of the risks are hedged by the options.

Recall that only the period 0 and period 1 need the options. Different from the corresponding forward positions, the period 0 has a lower position than period 1, implying that it is more likely to enter a short position in period 0 than period 1. Note that the short position is due to the convexity of the profit-to-go function (the second order derivative is positive). In general, compared to the forward position, the number of options to long or short is relatively smaller, which can be explained by the fact the profit-to-go functions are decreasing almost linearly.
Finally, we discuss the mean and variance performances under hedging or unhedged strategy. As mentioned above, the left panels of Figure 3 show that the profit function is decreasing in the spot price. The profits under the forward hedges and exotic hedges are identical and they are very close to the profit without hedges. In the last period, the hedged profit is slightly higher than the unhedged profit but the opposite is observed in the previous periods. This can be explained as follows: The order-up-to level under financial hedges in the last period is higher (and thus the profit is closer to the risk neutral one). But in the previous period the order-up-to levels under financial hedges is lower, which leads to lower profits. Nevertheless, in this example, the difference in profit is very small.

The right panels of Figure 3 show that in periods 0 and 1 the unhedged variances are greater than the variances under forward hedges and the later are greater than that under exotic hedges. In the last period, the forward hedge and exotic hedge have the same variance whereas the unhedged
variance is smaller than the hedged variance. This is due to the different optimal inventory decisions under hedged and unhedged strategies.

In summary, this example demonstrates and compares the structural relationships between inventory control and financial hedging under different hedging strategies. The dynamic system does behave differently from the static models such as the newsvendor model and the difference between them sheds new light into the commodity inventory management.

8. Extensions

This section extends the analysis for the basic lost-sales case to several important extensions: backorder, price-sensitive demand, and transaction costs. In these extensions, we focus on examining the relationship between the inventory control and financial hedges.

8.1. Backorders

In many B2B businesses, unmet demand is often backlogged due to the contractual obligation to meet all the demand, which may incur backlog costs. Let $b_t$ be the unit backlog cost of period $t$.

Assume that each backorder occurs in a period must be met in the next period, which is subject to the contractual arrangement. That is, if there is backlog at the beginning of a period, the firm will order at least as much as the backlog to meet the backlog even when the spot price is high. Then the inventory order-up-to level of each period can be any nonnegative value, which implies that the profit-to-go function $v_{t+1}$ only depends on spot price. The backordering model yields the same Bellman equation as (7) except that the operational profit of period $t$, $\pi_t$, is replaced by

$$\pi_t(y_t, S_t, S_{t+1}) = r_t d_t - S_t y_t - h_t(y_t - d_t)^+ - b_t(d_t - y)^+ + \alpha S_{t+1}(y_t - d_t),$$  \hspace{0.5cm} (23)$$

Hence, the analysis of the lost-sales model applies directly to the backordering case and similar insights can be obtained.

8.2. Price-Sensitive Demand

In the preceding analysis, the demand is exogenously given. However, in many industries, especially B2C businesses, customers may have certain degree of price sensitivity though the underlying products are commodities or made of commodity materials (e.g., retail gasoline and heating oils, food, etc.) . This section aims to extend the preceding analysis to the case of price-sensitive demand.

Specifically, we assume that the demand $d_t$ is of the additive form:

$$d_t = \mu_t(r, s) + \epsilon_t,$$

where $\mu_t(r, s)$ is a deterministic and strictly decreasing function of the price $r$ and dependent on the spot price, and $\epsilon_t$ is a random variable with zero mean. The distribution function of $\epsilon_t$ can be
dependent of the spot price of period $t$, $S_t$. The price level is chosen from an interval $[r, r]$. Then the operational profit in period $t$ is

$$\pi_t(r, y, S_t, S_{t+1}) = (r - S_t)y - (r + h_t - \alpha S_{t+1})(y - d_t)^+. $$

The profit-to-go function $v_t$ depends only on the spot price.

Following the analysis of Section 3.2, we can obtain the Bellman equation: For $t = 1, ..., T$,

$$J_t(s) = \max_{y \geq 0, r \in [r, r]} \{\alpha^{-(T+1-t)} g_t(r, y, s) + E_t[J_{t+1}(S_{t+1})]\},
$$

where

$$g_t(r, y, s) = E_t[\pi_t] - \lambda \alpha^{-(T+1-t)} \left[ A_t(r, s) \text{Var}_t((y - d_t)^+) + \alpha^2 \psi_t(r, y, s) \right],
$$

$$\psi_t(r, y, s) = \min_{H_t} \left\{ \text{Var}_t[S_{t+1} E_t[(y - d_t)^+] + \alpha^{T-t} v_{t+1}(S_{t+1}) + H_t(S_{t+1})] \right\}. $$

Clearly, for any fixed $r$, the objective function is unimodal in $y$ and thus the optimal ordering solution satisfies the first order condition. The optimal price can be searched linearly.

Using the optimal pricing and inventory decisions, the hedger can estimate the expected future cash flows and obtain the profit-to-go function $v_t$. Finally, the the optimal hedging decision can be obtained by solving the problem (25).

Thus, the insight obtained in Section 4 can be strengthened by incorporating the pricing decisions: In the presence of financial hedges (in particular, the forward contract maturing at next period), both the pricing and inventory decisions can be obtained myopically, which implies that the marketing (pricing) and operational (inventory) decisions can be made independently. But the financial hedging decision must take into account of the inventory and pricing decisions which influence the prediction of future cash flows.
8.3. Transaction Costs

Trading in the spot market often incurs transaction costs for both buying and selling activities. The transaction costs include the market cost (bid/ask spread) and operational cost (e.g., transportation cost). Our analysis can be extended to include all relevant variable transaction costs.

Let $k_b$ and $k_s$ be the unit transaction costs for purchases or sales in the spot market. In the presence of the transaction costs, it is no longer costless to sell all the on-hand inventory to the market and then buy up to the optimal inventory level. Then, at the beginning of each period, the on-hand inventory does influence the optimal inventory decision. But, for consistency, we still represent the operational profit of period $t$ in a similar way of Section 3.2:

$$
\pi_t(x_t, y_t, S_t, S_{t+1}) = (r_t - y_t)(y_t - (r_t + h_t - \alpha S_{t+1})(y_t - d_t)^+ - k_b(y_t - x_t)^+ - k_s(x_t - y_t)^+).
$$

Then the Bellman equation is

$$
J_t(x, s) = \max_{y \geq 0} \left\{ E_t[J_{t+1}(y - d_t, S_{t+1})] + \alpha^{-(T+1-t)} E_t[\pi_t] - \alpha^{-2(T+1-t)} \lambda E_t[Var_{d_t}[\pi_t + \alpha^{T+1-t}v_{t+1}((y - d_t)^+, S_{t+1})]|S_{t+1}] \right\} 
$$

(26)

and the profit-to-go, $v_t(x, s)$, has the following dynamics

$$
v_t(x, s) = E_t[\alpha^{-2(T+1-t)}] + E_t[v_{t+1}(y^*_t - d_t, S_{t+1})]
$$

where the the terminal profit function is

$$
v_{T+1}(x, s) = x(s - k_s)^+ - xs = -x \min(s, k_s).
$$

In the objective of the optimal hedging decision problem, since the profit-to-go is also a function of $y$, the risk associated with the excessive inventory $E_{d_t}[(y - d_t)^+]$ cannot be perfected hedged by simply entering a short position in forwards. To isolate the inventory decision problem from the joint optimization, we need sophisticated derivative portfolio. Under similar conditions to subsection 4.2, we can use forwards and options of all strikes (exotic derivatives) to replicate the expected profit-to-go $E_{d_t}[v_{t+1}((y - d_t)^+, S_{t+1})]$ as a function of $y$ and $S_{t+1}$. Note that the expectation is taken on the demand $d_t$. Then, the optimal inventory decision can be obtained by solving the following Bellman equation:

$$
J_t(x, s) = \max_{y \geq 0} \left\{ E_t[J_{t+1}(y - d_t, S_{t+1})] + \alpha^{-(T+1-t)} E_t[\pi_t] - \alpha^{-2(T+1-t)} \lambda E_t[Var_{d_t}[\pi_t + \alpha^{T+1-t}v_{t+1}((y - d_t)^+, S_{t+1})]|S_{t+1}] \right\}.
$$
Thus, although the myopic inventory policy is no longer optimal when there are transaction costs, the inventory decision can still be isolated from the financial hedging decision. With the optimal inventory policies derived from the above dynamic program, one can then evaluate and replicate the profit-to-go as a function of price and the current inventory decision.

The above discussion implies that if there are sufficient derivative instruments in the market then the operations manager can make inventory decisions separately whereas the risk manager must use the inventory policy to predict the future cash flows. This concurs with insight previously obtained.

9. Concluding Remarks

This paper addresses the joint inventory and financial hedging decision problem for a commodity inventory system with lost sales. We provide a novel dynamic programming formulation which allows us to derive a time-consistent optimal policy. We characterize the structure of the optimal policies under both forward hedges and exotic hedges. We show that as long as the forward contract is used the inventory decisions can be separated from the financial hedging decisions and myopic base-stock policies are optimal. However, to derive the optimal hedging decision, one needs to evaluate the profit-to-go based on the optimal (state-dependent) base-stock levels in the future. When there exist forwards and options of all strikes, a hedging portfolio can be constructed to replicate the profit-to-go, which allows us to separate the inventory and financial decision in a very general context. We also show that different from those newsvendor models in literature, financial hedges may lead to lower order-up-to level. We further show that these results hold in rather general frameworks (e.g., backlog, price-dependent demand and transaction cost). These results provide new insights into the coordinated risk and operational management strategies.

Firstly, operational (inventory) decisions and financial hedging decisions interplay with each other. On the one hand, financial hedges allow us to decompose a dynamic inventory decision problem into a sequence of myopic decision problems, which significantly simplifies the inventory decision process. On the other hand, financial hedging decisions rely on inventory strategies, since the future cash flow depends on the inventory policies. Although some of these points have been shown by Kouvelis et al. (2013), our results show that they hold in more general frameworks even when myopic inventory policies may no longer be optimal (e.g., when there are transaction costs). In real business world, the operational decisions and financial decisions are often made separately, partially because of the difficulty in coordinating the decisions between operational and financial managers. Our model suggests that the operational managers can still enjoy their independence (of the financial decisions) while financial managers must be able to understand the cash flow implications of operational strategies.
Secondly, financial hedges could be substitutes to inventory. In the single-period newsvendor model, or the last period of the dynamic model, the financial hedges and inventory levels are complementary. However, in the previous periods of the dynamic model, financial hedges may lead to lower inventory levels, which implies that the financial hedges could be a substitute for inventory. In practice, a firm tends to order more inventory in anticipation of a higher price in the future. Our result suggests that financial derivatives are better instruments to hedge the future price risk, while the inventory decisions focus on the current demand (quantity) risk which cannot be hedged by financial instruments.

Thirdly, financial hedging does create values. We show that given a risk constraint, financial hedges lead to a higher terminal wealth. This is also quite intuitive: Financial hedging reduces the risk, which allows the firm to focus more on its value maximization decisions.

In summary, integrating operational and financial risk management strategies is the best way to address commodity risk in an uncertainty demand environment. Our analysis framework provides us a useful conceptual tool to explore these insights.

References


**Appendix. Proofs of Statements**

**Proof of Proposition 1**

By induction, we assume that the optimality equation (4) holds for \( t + 1, \ldots, T \). Let \( \pi^*_i \) denote the profit function in period \( i \) under the optimal policy, \( i = t + 1, \ldots, T \). Taking expectation on \( W_{T+1} \) in period \( t \) yields

\[
E_t[W_{T+1}] = \alpha^{-(T+1-t)}W_t + v_t(s) \\
= E_t[\alpha^{-(T+1-t)}(W_t + \pi_t(y_t,S_t,S_{t+1}) + \alpha H_t(S_{t+1})) + v_{t+1}(S_{t+1})] \\
= E_t[E_{t+1}[\alpha^{-(T-t)}W_{t+1} + v_{t+1}(S_{t+1})]] \\
= E_t[E_{t+1}[W_{T+1}]],
\]

where

\[ v_{t+1}(S_{t+1}) = E_t\left[ \sum_{i=t+1}^{T} \alpha^{-(T+1-i)}[\pi^*_i(y^*_i(S_i),S_i,S_{i+1})] \right], t = 0,1,\ldots,T. \]
It is not hard to see that $v_t(s)$ satisfies the recursive equation (5). Therefore, given the initial wealth level in period $t$, the expected terminal wealth level depends only on current spot price $s$ (as the inventory level information has been included in $W_t$).

By the law-of-total-variance (e.g., Weiss 2005), the variance of the terminal wealth is given by

\[ Var_t \left[ W_t + \alpha^{-\left(T+1-t\right)} \left[ \pi_t + \alpha H_t(S_{t+1}) \right] + \sum_{i=t+1}^{T+1} \alpha^{-\left(T+1-i\right)} \left[ \pi_i^* + \alpha H_i(S_{i+1}) \right] \right] \]

\[ = E_t \left[ Var_{t+1} \left[ \alpha^{-\left(T+1-t\right)} \left[ \pi_t + \alpha H_t(S_{t+1}) \right] + \sum_{i=t+1}^{T} \alpha^{-\left(T+1-i\right)} \left[ \pi_i^* + \alpha H_i(S_{i+1}) \right] \right] \right] \]

\[ + Var_t \left[ E_{t+1} \left[ \alpha^{-\left(T+1-t\right)} \left[ \pi_t + \alpha H_t(S_{t+1}) \right] + \sum_{i=t+1}^{T} \alpha^{-\left(T+1-i\right)} \left[ \pi_i^* + \alpha H_i(S_{i+1}) \right] \right] \right] \]

\[ = E_t \left[ Var_{t+1} \left[ \sum_{i=t+1}^{T} \alpha^{-\left(T+1-i\right)} \left[ \pi_i^* + \alpha H_i(S_{i+1}) \right] \right] \right] \]

\[ + Var_t \left[ \alpha^{-\left(T+1-t\right)} \left[ \pi_t + \alpha H_t(S_{t+1}) \right] + E_{t+1} \left[ \sum_{i=t+1}^{T} \alpha^{-\left(T+1-i\right)} \pi_i^* \right] \right] \]

\[ = E_t \left[ Var_{t+1} \left[ \sum_{i=t+1}^{T} \alpha^{-\left(T+1-i\right)} \left[ \pi_i^* + \alpha H_i(S_{i+1}) \right] \right] + Var_t \left[ \alpha^{-\left(T+1-t\right)} \left[ \pi_t + \alpha H_t(S_{t+1}) \right] + v_{t+1}(S_{t+1}) \right] \right] \]

\[ = E_t \left[ Var_{t+1}[W_{T+1}] \right] + Var_t \left[ \alpha^{-\left(T+1-t\right)} \left[ \pi_t + \alpha H_t(S_{t+1}) \right] + v_{t+1}(S_{t+1}) \right] . \]

Notice that $\tilde{J}_{t+1}(W_{t+1}, S_{t+1}) = E_{t+1}[W_{T+1}] - \lambda Var_{t+1}[W_{T+1}]$. Then the equation (4) holds.

**Proof of Proposition 2**

Observe that the variance of the terminal wealth conditional on period $t$ does not depend on the initial wealth $W_t$ in period $t$ and the expected terminal wealth is summation of $\alpha^{-\left(T+1-t\right)}W_t$ and $v_t(s)$. Replacing $\tilde{J}_{t}(W_{t}, s)$ by $J_{t}(s)$ and $\alpha^{-\left(T+1-t\right)}W_t$ in (4) yields

\[ J_t(s) + \alpha^{-\left(T+1-t\right)}W_t = \max_{s' \geq 0, H_t \in \mathcal{H}_t} \left\{ \alpha^{-\left(T+1-t\right)}W_t + \alpha^{-\left(T+1-t\right)}E_t[\pi_t] + E_t[J_{t+1}(S_{t+1})] \right\} \]

\[ - \lambda Var_t \left[ \alpha^{-\left(T+1-t\right)} \left[ \pi_t + \alpha H_t(S_{t+1}) \right] + v_{t+1}(S_{t+1}) \right] . \]

Subtracting $\alpha^{-\left(T+1-t\right)}W_t$ from both sides of above equation leads to

\[ J_t(s) = \max_{s' \geq 0, H_t \in \mathcal{H}_t} \left\{ E_t[J_{t+1}(S_{t+1})] \right\} \]

\[ + \alpha^{-\left(T+1-t\right)} \left[ E_t[\pi_t] - \alpha^{-\left(T+1-t\right)} \lambda \min_{H_t \in \mathcal{H}_t} \left\{ Var_t \left[ \pi_t + \alpha H_t(S_{t+1}) + \alpha^{-T+1-t}v_{t+1}(S_{t+1}) \right] \right\} \right] , \]

\[ = \max_{s' \geq 0} \left\{ E_t[J_{t+1}(S_{t+1})] \right\} \]

\[ + \alpha^{-\left(T+1-t\right)} \left[ E_t[\pi_t] - \alpha^{-\left(T+1-t\right)} \lambda \min_{H_t \in \mathcal{H}_t} \left\{ Var_t \left[ \pi_t + \alpha H_t(S_{t+1}) + \alpha^{-T+1-t}v_{t+1}(S_{t+1}) \right] \right\} \right] . \]

(28)

Note that $d_t$ and $S_{t+1}$ are independent for any given $S_t = s$. Applying the law of total variance yields

\[ Var_t[\pi_t + \alpha H_t(S_{t+1}) + \alpha^{-T+1-t}v_{t+1}(S_{t+1})] \]

\[ = E_t[Var_t[\pi_t + \alpha H_t(S_{t+1}) + \alpha^{-T+1-t}v_{t+1}(S_{t+1})] | S_{t+1}] + Var_t[E_t[\pi_t + \alpha H_t(S_{t+1}) + \alpha^{-T+1-t}v_{t+1}(S_{t+1})] | S_{t+1}] \]
= E_i[Var_i[\pi_i|S_{t+1}]] + Var_i[(r_t - s)y - (r_t + h_t - \alpha S_{t+1})E_i[(y - d_i)^+] + \alpha H_i(S_{t+1}) + \alpha^T + 1 - t_v + 1(S_{t+1})]
= E_i[(r_t + h_t - \alpha S_{t+1})^2Var_i[(y - d_i)^+]] + Var_i[\alpha S_{t+1}E_i[(y - d_i)^+] + H_i(S_{t+1}) + \alpha^T + 1 - t_v + 1(S_{t+1})]
= E_i[(r_t + h_t - \alpha S_{t+1})^2]Var_i[(y - d_i)^+] + Var_i[\alpha S_{t+1}E_i[(y - d_i)^+] + H_i(S_{t+1}) + \alpha^T + 1 - t_v + 1(S_{t+1})].

(29)

Here the first equality applies the law of total variance directly, the second is due to the fact that the payoff of the hedge, \alpha H_i(S_{t+1}), becomes a constant when \alpha S_{t+1} is given (conditioned on), and the third equality is by the fact that

\[ Var_i[\pi_i|S_{t+1}] = Var_i[(r_t - s)y - (r_t + h_t - S_{t+1})(y - d_i)^+|S_{t+1}] = (r_t + h_t - S_{t+1})^2Var_i[(y - d_i)^+]. \]

Replacing the variance term in the RHS of (28) by the RHS of (29) yields (7).

**Proof of Corollary 2**

Note that

\[ \frac{\partial^2 g_i(y,s)}{\partial y \partial \lambda} = -2\alpha^{-(T + 1 - t)}E[(y - d_i)^+]A_i(s)\Phi_i(y,s) \leq 0, \]

which implies that \( g_i(y,s) \) is submodular in \((y, \lambda)\). By Theorem 2.8.1 of Topkis (1998), \( y_i^*(s; \lambda) \) is decreasing in \( \lambda \).

**Proof of Corollary 3**

First, we know that \( E[\pi_i(y,s,S_{t+1})] \) is concave in \( y \), which implies that it is increasing on the left hand side of the risk-neutral inventory level \( y_i^*(s;0) \). For any \( \lambda > 0 \), it follows from Corollary 2 that \( y_i^*(s; \lambda) \) is decreasing in \( \lambda \), which implies that \( y_i^*(s; \lambda) \leq y_i^*(s;0) \). Then, combining the concavity of \( E[\pi_i(y,s,S_{t+1})] \) and monotonicity of \( y_i^*(s; \lambda) \), we know that \( E[\pi_i^*] \) is decreasing in \( \lambda \), which in turn implies that \( E_i[W_{T+1}^*] \) is also decreasing in \( \lambda \).

It is obvious that \( U_i \) is decreasing in \( \lambda \) for any given policy. Then, \( U_i^* \) must be decreasing in \( \lambda \). We argue that \( Var_i[W_{T+1}^*] \) must be decreasing in \( \lambda \). Suppose, otherwise, there exist \( \lambda_1 > \lambda_2 \) such that the corresponding variances, denoted by \( V_{1}^* \) and \( V_{2}^* \) respectively, satisfy \( V_{1}^* > V_{2}^* \). Let \( u_1 \) and \( u_2 \) be the corresponding optimal policies and \( W_{T+1}^{u_1} \) and \( W_{T+1}^{u_2} \) the terminal wealth levels under policies \( u_1 \) and \( u_2 \). Note that \( W_{T+1}^{u_1} \leq W_{T+1}^{u_2} \). Then, we have

\[ U_i^{u_1} = E_i[W_{T+1}^{u_1}] - \lambda_1 Var_i[W_{T+1}^{u_1}] < E_i[W_{T+1}^{u_2}] - \lambda_1 Var_i[W_{T+1}^{u_2}] = U_i^{u_2}, \]

that is policy \( u_2 \) is strictly better than the optimal policy \( u_1 \) with respect to \( \lambda_1 \), which leads to a contradiction. The desired result holds.

**A Technical Lemma**

The Fundamental Theorem of Calculus states that:

A function \( f : [a,b] \rightarrow \mathbb{R} \) is absolutely continuous if and only if it is differentiable almost everywhere, its derivative \( f' \in L^1[a,b] \) and for any \( t \in [a,b] \),

\[ f(t) = f(a) + \int_a^t f'(s)ds. \]
LEMMA 2. Suppose that function $f : \mathbb{R}_{+} \to \mathbb{R}$ is first-order differentiable and $f'$ is continuous in $(0, \infty)$, and differentiable in $(0, r)$, $|f''(r-)| < \infty$ and $f' \equiv 0$ in $(r, \infty)$. Then, $f'$ is differentiable almost everywhere, $f' \in L^{1}[0, \infty)$ and, for any $S > s$,

$$f'(S) = f'(s) + \int_{s}^{S} f''(u)du.$$

That is, $f'$ is absolutely continuous.

**Proof.** Note that $f''(s) = 0$ for $s > r$ and $f'$ is differentiable in $(0, r)$. Then, $f'$ is differentiable almost everywhere. Moreover, $|f''(r-)| < \infty$ and $f''(r+) = 0$, which implies that $f'' \in L^{1}[0, \infty)$. For any $S > s \geq 0$, if $S \leq r$, by the Fundamental Theorem of Calculus, we have

$$f'(S) = f'(s) + \int_{s}^{S} f''(u)du.$$

If $S > r > s$,

$$f'(S) = f'(r) = f'(s) + \int_{s}^{r} f''(u)du + f'(r) + \int_{r}^{S} f''(u)du = f'(s) + \int_{s}^{S} f''(u)du.$$

If $S > s > r$,

$$f'(S) = f'(s) = f'(r) + \int_{s}^{S} f''(u)du.$$

The desired result holds. Q.E.D.

**Proof of Theorem 2**

The proof is by induction. Note that $v_{t+1}(s) = 0$. For any $t$, suppose that $v_{t+1}(\cdot)$ is twice differentiable almost everywhere. Then, in a similar way to Carr and Madan (2001), applying Lemma 2 and the Fundamental Theorem of Calculus, $v_{t+1}(S_{t+1})$ can be decomposed as

$$v_{t+1}(S_{t+1}) = [v_{t+1}(s) - v'_{t+1}(s)s] + v'_{t+1}(s)S_{t+1} + \int_{0}^{s} v''_{t+1}(K)(K - S_{t+1})^{+} dK + \int_{S_{t+1}}^{\infty} v''_{t+1}(K)(S_{t+1} - K)^{+} dK.$$

That is, the $v_{t+1}(S_{t+1})$ can be perfectly replicated by bonds, forwards, and options. For any $y$, with complete term and strike structure, the optimal hedging decision is to offset $S_{t+1}E_{t}[(y - d_{i})^{+}]$ and $\alpha^{T-t}v'_{t+1}(S_{t+1})$ so that $\psi_{t}(y, s) = 0$. That is,

$$H_{t}(S_{t+1}) = -\left[ E_{t}[(y - d_{i})^{+}] + \alpha^{T-t}v'_{t+1}(s)\right] + \int_{0}^{\infty} \alpha^{T-t}v''_{t+1}(K)[(K - S_{t+1})^{+} - P_{t}(K)]dK + \int_{S_{t+1}}^{\infty} \alpha^{T-t}v''_{t+1}(K)(S_{t+1} - K)^{+} - C_{t}(K)dK.$$

That is, given any inventory level $y$, the optimal hedging strategy is to short $E_{t}[(y - d_{i})^{+}] + \alpha^{T-t}v'_{t+1}(s)$ units of forwards, $\alpha^{T-t}v''_{t+1}(K)dK$ puts for all strikes $K < s$ and $\alpha^{T-t}v''_{t+1}(K)dK$ calls for all strikes $K > s$ maturing at next period. Then,

$$g_{t}(y, s) = E_{t}[\pi_{t}] - \lambda \alpha^{-(T+1-t)}A_{t}(s)Var[(y - d_{i})^{+}].$$

Taking derivative with respect to $y$ yields

$$\frac{\partial g_{t}(y, s)}{\partial y} = \left[ r_{t} - (r_{t} + h_{t} - \alpha f_{t}(s))\Phi_{t}(y|s) - 2\lambda \alpha^{-(T+1-t)}E[(y - d_{i})^{+}]A_{t}(s)\Phi_{t}(y|s)\right] \Phi_{t}(y|s) - (s + h_{t} - \alpha f_{t}(s)).$$

(30)
Note that both factors of of the first term in the RHS of the second equality are decreasing in \( y \). By Assumption 1, \( s + h_t - \alpha f_t(s) \geq 0 \). If the first term is negative at some point \( y^0 \), then \( g_t(y,s) \) is decreasing in \( y \geq y^0 \). This implies that the optimal point \( y^*_t \leq y^0 \). When the first term is nonnegative, it is clear that \( \frac{\partial g_t(y,s)}{\partial y} \) is decreasing in \( y \), i.e., \( g_t(y,s) \) is concave in \( y \). More precisely, it is strictly concave in \( y \). Then, \( g_t(y,s) \) is unimodal in \( y \) for any \( s \). Note that \( \lim_{s \to 0^+} \frac{\partial g_t(y,s)}{\partial y} = r_t - s \). Then, if \( r_t > s \), \( y^*_t(s) \) is the unique maximizer of \( g_t(\cdot, s) \) which satisfies (17). If \( r_t \leq s \), the unimodularity of \( g_t \) implies that \( g_t \) is decreasing in \( y \), which implies that the least maximizer \( y^*_t = 0 \).

Notice that all the terms of the RHS of (30) are differentiable in \( s \), which implies that \( \frac{\partial g_t(y,s)}{\partial y} \) is also differentiable in \( s \).

Taking derivatives on both sides of (30) with respect to \( s \) and \( y \), respectively, and letting \( y = y^*_t(s) \), we have

\[
\frac{\partial^2 g_t(y^*_t(s), s)}{\partial y \partial s} = -1 + \alpha f_t(s) \Phi_t(y^*_t(s), s) - (r_t + h_t - \alpha f_t(s)) \phi_t(y^*_t(s), s)
\]

\[
-2\lambda_\alpha \Phi_t(y^*_t(s), s) + A_t(s) \phi_t(y^*_t(s), s),
\]

where the second equality is by the first-order condition.

When \( s < r_t \), we know that \( \frac{\partial^2 g_t(y^*_t(s), s)}{\partial y^2} < 0 \). This implies that \( y^*_t(s) \) is differentiable in \( s \) and satisfies

\[
\frac{dy^*_t(s)}{ds} = -\frac{\frac{\partial^2 g_t(y^*_t(s), s)}{\partial y^2}}{\frac{\partial^2 g_t(y^*_t(s), s)}{\partial y \partial s}}.
\]

It is clear that \( \frac{\partial \Phi_t(y^*_t(s), s)}{\partial y} \) is differentiable in \( s \) since \( \Phi_t \) is twice differentiable and \( y^*_t(s) \) is differentiable. Then, all the terms of the RHS of (31) are differentiable. Similarly, all the terms of the RHS of (32) are differentiable. Then \( \frac{dy^*_t(s)}{ds} \) is differentiable in \( s < r_t \), i.e., \( y^*_t(s) \) is twice differentiable in \( s < r_t \).

When \( s > r_t \), we have shown that \( y^*_t(s) = 0 \), which implies that the first- and second-order derivatives of \( y^*_t(s) \) are both zero as \( s > r_t \). However, as \( r_t = s = \alpha f_t(s) - h_t \), \( \frac{\partial^2 g_t(y^*_t(s), s)}{\partial y^2} = 0 \) which implies that \( y^*_t(s) \) may not be differentiable at the point \( s = r_t \). Thus, \( y^*_t(s) \) is twice differentiable in \( s \) almost everywhere.

Note that

\[
E_t[\pi_t] = (r_t - s)y - (r_t + h_t - \alpha f_t(s)) \int_0^y \Phi_t(\xi, s)d\xi
\]

is twice differentiable in \( s \) by Assumption 3 since \( f_t(s) = E[S_{t+1}|S_t = s] \) and \( \Phi_t(\xi, s) \) are twice differentiable in \( s \). Let \( \psi_t(s) = E_t[\pi_t(y^*_t(s), s, S_{t+1})] \). We next show that \( \psi_t(s) \) is twice differentiable in \( s \) almost everywhere. First, for \( s < r_t \)

\[
\frac{\partial \psi_t(s)}{\partial s} = -y^*_t(s) + \alpha f_t(s) \int_0^{y^*_t(s)} \Phi_t(\xi, s)d\xi - (r_t + h_t - \alpha f_t(s)) \int_0^{y^*_t(s)} \phi_t(\xi, s)d\xi
\]

\[+ [ (r_t - s) - (r_t + h_t - \alpha f_t(s)) \phi_t(y^*_t(s), s)] \frac{\partial y^*_t(s)}{\partial s}. \]

Since \( y^*_t(s) \) is twice differentiable in \( s < r_t \), then \( E_t[\pi_t(y^*_t(s), s, S_{t+1})] \) is twice differentiable in \( s < r_t \).
When \( s > r_t \), we know that \( y'(s) = 0 \), which infers that \( \psi_t(s) = E_t[\pi_t(y_t'(s), s, S_{t+1})] = 0 \), and therefore, \( \psi_t(s) \) is twice differentiable in \( s \) for \( s > r_t \). It is obvious that \( \lim_{s \to r_t^+} \frac{\partial \psi_t(s)}{\partial s} = 0 \). Then, \( \psi_t(s) \) is differentiable in \( s \) for all \( s > 0 \) and \( \psi_t(s) \) is continuous in \( s \) with \( \psi_t(r_t) = 0 \).

Thus, \( \psi_t(s) = E_t[\pi_t(y_t'(s), s, S_{t+1})] \) is twice differentiable in \( s \) almost everywhere.

By induction assumption that \( \nu_{t+1}(s) \) is twice differentiable in \( s \) almost everywhere and Assumption (3), we know that \( E_t[\nu_{t+1}(S_{t+1})] \) is twice differentiable in \( s \) almost everywhere as well. By recursive equation (5), we know that \( \nu_t(s) \) is twice differentiable in \( s \) almost everywhere as well. This completes the induction.

**Proof of Theorem 1**

For any \( t \), let \( H_t(S_{t+1}) = -E_t[(y - d_t)^+](S_{t+1} - f_t(s)) + \tilde{H}_t(S_{t+1}) \). Then, we have

\[
\psi(y, s) = \max_{\bar{\alpha} \in \bar{\alpha}_t} \{ Var_t[\alpha T^{-1}v_{t+1}(S_{t+1}) + \tilde{H}_t(S_{t+1})] \},
\]

which is independent of \( y \). Then, we have

\[
\frac{\partial \psi_t(y, s)}{\partial y} = \left[ r_t + h_t - \alpha f_t(s) - 2\alpha^{-1}(T-1-t) \lambda E_t((y - d_t)^+ A_t(s)) \right] \Phi_t(y|s) - (s + h_t - \alpha f_t(s)) + \nonumber
\]

In a similar way of the Proof of Theorem 2, we can show Parts (a) and (b).

If only forward contracts maturing at next period are available, letting \( \bar{H}_t = q[S_{t+1} - f_t(s)] \) and solving

\[
\min \ Var_t[q[S_{t+1} - f_t(s)] + \alpha T^{-1}v_{t+1}(S_{t+1})]
\]

yields the optimal solution

\[
q^*_t = \frac{-Cov_t[S_{t+1}, \alpha T^{-1}v_{t+1}(S_{t+1})]}{Var_t[S_{t+1}]}.
\]

**Proof of Proposition 3**

Note that \( v_{t+1}(s) = 0 \). Taking first order derivative for \( \hat{g}_t(y, s) \) yields

\[
\frac{\partial \hat{g}_t(y, s)}{\partial y} = \left[ r_t + h_t - \alpha f_t(s) - 2\alpha^{-1}(T-1-t) \lambda E_t((y - d_t)^+ A_t(s)) \right] \Phi_T(y|s) - (s + h_t - \alpha f_t(s))
\]

Observe that the first and the third terms of the RHS of above equation are decreasing in \( y \). Similar to the proof of Theorem 2, we can show that \( \hat{g}_t(y, s) \) is unimodal in \( y \).

When \( r_t \leq s \), \( \hat{g}_t(y, s) \) is decreasing in \( y \) and thus \( \hat{g}_t^*(s) = 0 \). Otherwise, \( \hat{g}_t(y, s) \) is unimodal in \( y \) and the optimal solution satisfies (21). By Theorem 2 and Theorem 1, the hedged optimal inventory decision \( y_T^*(s) \) satisfies

\[
[r_t + h_T - \alpha f_T(s) - 2\alpha^{-1}(T-1-t) \lambda E_t((y - d_T)^+ A_T(s)) \] \( F_T(y, s) - (s + h_T - \alpha f_T(s)) = 0.
\]

This implies that

\[
\frac{\partial \hat{g}_t(y_T^*, s)}{\partial y} = -2\alpha Var_t[S_{t+1}] F_T(y_T^*(s), s) E_T[y_T^*(s) - d_T]^+ < 0.
\]

The unimodularity of \( \hat{g}_t(y, s) \) in \( y \) implies that \( \hat{g}_T^*(s) < y_T^*(s) \) as \( r_t > s \).
Proof of Proposition 4

Differentiating $\hat{g}_t$ with respect to $y$ yields
\[
\frac{\partial}{\partial y} \hat{g}_t(y, s) = \left[ r_t + h_t - \alpha f_t(s) - 2\alpha^{-(T-1-t)}\lambda E[(y - d_t)^+]A_t(s) \right] \Phi_t(y|s) - \left[ s + h_t - \alpha f_t(s) \right] - 2\alpha^{-(T-t-1)}\Phi_t(y|s) Var_t(S_{t+1}) \left[ E[(y - d_t)^+] + \alpha \frac{\tau - Cov_t(S_{t+1}, v_{t+1}(S_{t+1}))}{Var_t(S_{t+1})} \right].
\] (33)

Not that the sum of the first two terms of RHS of (33) is strictly positive if $0 < y < y^*_t$ while the third term is decreasing in $y$. If $\hat{E}[(y^*_t(s) - d_t)^+] < -\alpha^{T-t} \frac{Cov_t(S_{t+1}, v_{t+1}(S_{t+1}))}{Var_t(S_{t+1})}$, then the third term is positive, which implies that $\frac{\partial}{\partial y} \hat{g}_t(y, s) > 0$ as $y \leq y^*_t$ and therefore the optimal solution $\hat{y}^*_t(s)$ is increasing in $y$. If, on the contrary, $\hat{E}[(y^*_t(s) - d_t)^+] \geq -\alpha^{T-t} \frac{Cov_t(S_{t+1}, v_{t+1}(S_{t+1}))}{Var_t(S_{t+1})}$, then the third term is negative and decreasing in $y$, which implies that $\frac{\partial}{\partial y} \hat{g}_t(y, s) \leq 0$ as $y \geq y^*_t$ and therefore the optimal solution $\hat{y}^*_t(s) \leq y^*_t(s)$.

Proof of Proposition 5

Let $z = y - d_t$. Then,
\[
g_t(d_t + z, s) = (r_t - s)(d_t + z) - (r_t + h_t - \alpha f_t)z^+ - \lambda \alpha^{-(T-1-t)}Var_t[S_{t+1}z^+ + \alpha^{T-1}v_{t+1}(S_{t+1})].
\]

When $z > 0$, we have
\[
\frac{\partial}{\partial z} g_t(z, s) = -(s + h_t - \alpha f_t) - 2\alpha^{-(T-1-t)}Var_t(S_{t+1})[z + \alpha^{T-t}Cov_t(S_{t+1}, v_{t+1}(S_{t+1}))/Var_t(S_{t+1})],
\]
which is decreasing in $z$. This implies that $g_t(d_t + z, s)$ is concave in $z$ as $z > 0$.

It is clear that for any $s$, $\frac{\partial}{\partial z} g_t(z, s) \to -\infty$ as $z \to \infty$. When $z = 0$,
\[
\frac{\partial}{\partial z} g_t(d_t, s) = -(s + h_t - \alpha f_t) - 2\alpha Cov_t(S_{t+1}, v_{t+1}(S_{t+1})).
\]

If $Cov_t(S_{t+1}, v_{t+1}(S_{t+1})) < -\frac{s + h_t - \alpha f_t}{2\alpha}$, then $\frac{\partial}{\partial z} g_t(d_t, s)$ and the optimal inventory decision must be greater than $d_t$; otherwise, the optimal inventory decision must be less than $d_t$.

Proof of Theorem 3

First, the inequalities of part (a) follow directly from Lemma 1 since the inequalities can be preserved under the minimization operations. Part (a) holds.

Note that the optimal lagrange multipliers must be nonnegative. We argue that they must be positive as $\delta < \delta_0$. If, otherwise, $\lambda^* = 0$, the resulted variance must be equal to $\delta_0$, which violates the constraint. By KKT theorem, a positive lagrange multiplier implies that the constraint is binding. Then, by part (a), we know that the inequalities of part (b) hold.

Note that the objective function of the lagrange problem is supermodular in $(\lambda, \delta)$. Then, under the minimization operation with respect to $\lambda$, the supermodularity implies that $\lambda^*$ must be decreasing in $\delta$. Since $E[W_{t+1}^\pi]$ is decreasing in $\delta$, we know that $E[W_{t+1}^\pi]$ is increasing in $\delta$. It is clear that as $\delta \to \delta_0$, $E[W_{t+1}^\pi]$ converges to risk-neutral expected terminal wealth. Part (c) holds.

Recall that for any $\lambda$, $E_0W_{t+1}^\pi = E_0W_{t+1}$ but $Var_0[W_{t+1}^\pi] \leq Var_0[W_{t+1}^\pi]$ (by Lemma 1). Then, when $\lambda = \lambda^*$, $E_0W_{t+1}^\pi = E_0W_{t+1}$ but $Var_0[W_{t+1}^\pi] \leq Var_0[W_{t+1}^\pi] = \delta$. By Corollary 3, we know that to bind the variance constraint, one must lower the value $\lambda$, i.e., $\lambda^* \leq \lambda^*$. Applying Corollary 2, we know that $y^*_t \leq y^*_t$. Part (d) holds.