This paper studies a firm’s joint decisions of \textit{ex ante} inventory procurement and capacity investment, and \textit{ex post} production and pricing. The firm invests in multiple capacities and input inventories in the face of end-product market uncertainty; the input inventory can be transformed into end-products by the capacities through certain production technologies at a later time when the market sizes are observed. The end-product demands are endogenously determined by the realized market sizes and prices set by the firm. By modeling the stochastic decision process as a responsive pricing newsvendor network, we are able to show the existence and uniqueness of the optimal inventory procurement and capacity investment levels under the general assumption of concave revenue function. We present a guided approach to the \textit{ex post} analysis on the production and pricing decisions through careful partitioning of the demand space. We also clearly state the optimality equations for the \textit{ex ante} decisions. We study in detail the additive linear demand model, obtain optimal solutions, and explore the comparative statics of the optimal value function with respect to the market size uncertainty and demand substitution/complementarity parameters. Our paper advances the study of the newsvendor network problem from exogenous demand to endogenous demand by providing a general framework and a guided approach to analysis and solutions of the newsvendor network problem with responsive pricing.

\textit{Key words}: newsvendor network, responsive pricing, inventory procurement, capacity investment

1. \textbf{Introduction}

Consider a firm facing two potential markets. For each market, one product is produced from certain input (inventory) by some capacity of production technology and is brought to the market to fulfill demand, where the demand is endogenously determined by the market size and the price set by the firm. Both of the inventory and capacity are invested when the market size is unknown, while production and pricing is postponed until the market size is observed. If the inventories and
capacities are dedicated to the production of one product, the firm shall have to invest in inventory and capacity for each product, and bring the end-product to the specified market separately. However, in many cases the firm owns the flexibility of transforming one input into multiple products, and utilizes one capacity to accomplish two or more such transformations. The combination of production flexibility and the pricing power creates values for firms producing and selling multiple products whose demands are affected by not only their own prices but also by prices of other products. On the supply side, the firm enjoys the benefit of risk-pooling, as both common input and flexible capacity reduce the risk of over-investing and under-investing; on the demand side, with pricing power the firm is able to influence the demand realizations in the two markets and to choose the actual demand level to meet in each market, mitigating the mismatch of supply and demand to maximize the expected firm value.

Nevertheless, coming along with the desirable flexibility on both supply and demand sides is the complication of the decision analysis. It is necessary to employ the tool of newsvendor network modeling, with the additional consideration of responsive pricing. The concept of newsvendor network is introduced by van Mieghem and Rudi (2002). Specifically, a newsvendor network “is defined by a linear production technology, which describes how input (supply) are transformed into outputs to fill end-product demand, a linear financial structure, and a probability distribution of end-product demand” (van Mieghem and Rudi 2002, p.318). Newsvendor network models capture two salient characteristics that are common in many production/storage/distribution systems, namely, multidimensionality (i.e., multiple inputs are transformed into multiple outputs) and discretionary activities (i.e., more than one way to use inventory and/or resource capacities). Responsive pricing newsvendor networks retain these two characteristics, and add a third characteristic of \textit{ex post} pricing flexibility to allow prices to become part of the decisions taking place after uncertainties in the output market resolve. The demand data set for responsive pricing newsvendor networks consists of the probability distributions of the market sizes and the demand functions that establish the relationship of the actual end-product demands, the realized market sizes and prices set by the firm. There can be many modeling choices for demand data set, including the form of the demand
function, the interdependence between products (substitutes or complements), the type of random shocks to demand function (additive or multiplicative), and correlations of the random shocks. It is worthwhile to investigate whether insights gained from newsvendor network models remain valid for the responsive pricing newsvendor networks, and to explore new questions that are important and relevant in an environment where firms pricing can influence the demand for its products.

A few special cases of responsive pricing newsvendor networks have been studied in the operations literature, with simplification on certain dimensions of the problem. For example, a major stream of the research on this topic considers a firm investing in a single flexible capacity only, and hence the capacity variable is one-dimensional, which simplifies the \textit{ex ante} capacity/inventory decision significantly. In another stream of the research, both dedicated and flexible capacities are allowed, but the demand functions are assumed independent, i.e., products are neither substitutes nor complements, which simplifies the \textit{ex post} production and pricing decisions accordingly. (Detailed review of the existing literature will be given in the next section)

The absence of a unified framework studying the general responsive pricing newsvendor network problems motivated our research. In this paper, we strive to conduct a comprehensive analysis of the problem, characterizing the optimal \textit{ex ante} and \textit{ex post} policies, and exploring comparative statics to the extent possible. We adopt the original definition of newsvendor network in van Mieghem and Rudi (2002) with a modification of the demand data set to reflect a price-sensitive demand environment, and use the same example as they carry through their discussion, for the purpose of illustration. Because the model is already heavily complicated by the incorporation of responsive pricing, we focus only on the single period decision and leave the dynamic optimality of inventory and capacity policies for future research.

The objective of this paper is three-fold. First, we present a general framework for the modeling of a responsive pricing newsvendor network problem, which unifies and generalizes existing results in the literature. With a mild assumption on the revenue function, we show that the optimal inventory procurement and capacity investment levels exist and are unique, and inherit the classic “critical fractile” format. Second, going beyond result unification, we provide a detailed, guided approach
for the *ex post* analysis, which is valid for all demand functions satisfying the revenue concavity assumption. This guided approach is clearly illustrated in a classic example used in the newsvendor network literature. In a comparison with the *ex post* analysis for the classic newsvendor network, we show how demand substitution and complementarity can affect the marginal value of capacities and inventories, highlighting the modeling flexibility of the responsive pricing newsvendor network. Finally, we use the comparative statics of the optimal value of the firm to demonstrate a wide range of questions that can be potentially explored by the adoption of the responsive newsvendor network regarding the nature of the price-dependent demand (functional form of demand curve), the demand curve interdependence (substitutes vs. complements), and the type of demand shock (additive vs. multiplicative).

The remainder of this paper is organized as follows: In the next section, we provide a review of the existing literature. In Section 3, we propose the modeling framework for responsive pricing newsvendor network problems. In Section 4, using an illustrative example we demonstrate the formulation of a responsive pricing newsvendor network problem and further analyze the optimal solutions, while in Sections 5 we investigate comparative statics of the illustrative example with additive linear demand model. We extend our model to some other demand models and conclude in Section 6. Technical proofs are collected in the Appendix.

2. Literature Review

van Mieghem and Rudi (2002) (vMR in short hereafter) choose the term “newsvendor network” in order to stress the link of a class of “processing-storage network” models to the classic newsvendor model. Porteus (1990) gives an excellent review of the classic newsvendor problem, while Khouja (1999) provides an extensive survey of its variations and extensions. One of those extensions is from exogenous demand to endogenous demand by assuming that the newsvendor has the pricing power that affects the demand for the product. Petruzzi and Dada (1999) examine the literature on such extension and develop some additional results. By allowing the quantity (production) and price decisions to be made at different stages and including the additional capacity investment
decision, van Mieghem and Dada (1999) present a comparative analysis of possible postponement strategies in a two-stage decision model.

Another recent and significant extension of newsvendor research is from one dimensional to multidimensional, by Harrison and van Mieghem (1999) and van Mieghem (1998). vMR further extends the multidimensional newsvendor model, which considers the *ex ante* capacity investment decisions and the *ex post* production decisions only, to incorporate the *ex ante* input inventory procurement decisions, and coins the term “the newsvendor network model.” With both inventory and capacity as multidimensional decision variables, the newsvendor network framework subsumes not only the multidimensional newsvendor model, but also several other multi-item models with inventories/capacities features such as component commonality, inventory substitution, transshipment, etc. Interested readers are referred to vMR for more discussion on the relationship between such models and the newsvendor network model, as well as the related literature. Due to space restriction, we focus our survey of the literature on newsvendor networks with responsive pricing.

A stream of papers extend a classic newsvendor network model of van Mieghem (1998) to the situation where demand is sensitive to price and the firm sets the price in response to the realized market condition. Specifically, van Mieghem (1998) studies the optimal capacity investment decision of a two-product firm with the option to invest in two dedicated (product-specified) capacities as well as a flexible capacity that can produce both products. Bish and Wang (2004) focus on characterizing a price-setting firm’s optimal capacity portfolio, i.e., which capacity(s) shall be invested and which shall not, and the threshold policy of the flexible capacity investment decision. Whereas Bish and Wang (2004) assumes no cross-price effects in two demand curves, Ceryan et al. (2013) extend the Bish and Wang (2004) model to the dynamic setting, with the consideration of cross-price effects, and their focus is on the structure of the optimal production and pricing decisions for given capacity levels.

One variation of van Mieghem (1998) model is to let the firm invest in either two dedicated capacities or one flexible capacity but not both of them. Goyal and Netessine (2007) consider two
such firms competing in two product markets, and the two products can be substitutes or complements. The two firms simultaneously make decisions on technology choices (product-flexible or product-dedicated) before making the corresponding capacity investment and production quantity decisions. Under the assumption that each firm is always forced to produce both products, and for each product, to produce at capacity level, they show that asymmetric equilibrium is possible even when firms are perfectly symmetric. Goyal and Netessine (2011) investigate the interplay between product flexibility and volume flexibility for a monopoly firm.

Another variation of van Mieghem (1998) model is to let firm invest only in a flexible capacity that can produce two products (no dedicated capacities are available). Under the assumption of the additive demand uncertainty and normally distributed demand intercepts, Chod and Rudi (2005) focuses on the effects of two key drivers of flexibility: demand variability and demand correlation. They find that: 1) the optimal flexible capacity level is always increasing in both demand variability and demand correlation; 2) the optimal expected profit is increasing in demand variability; and 3) the optimal expected profit is decreasing in demand correlation in the case of no cross-price effect. All but the last result contrast with the scenarios of no responsive pricing, which highlights the key role that responsive pricing plays in the management of flexible capacities. Bish et al. (2010) revisit the Chod and Rudi (2005) problem and show that such results hold for the exact expressions of the optimal expected profit and optimal capacity under any arbitrary continuous distribution of demand intercepts. They also point out that the additive demand uncertainty is a critical assumption for the results to hold. In a parallel work, Bish et al. (2009) study the flexible-capacity-only problem in a wider extension. They explore the impact of market size and risk on the optimal capacity decision under both additive and multiplicative demand shocks. In particular, they find that under the additive demand shock, the firm benefits from an increase in either market size or market risk by investing more in the flexible capacity, while under the multiplicative demand shock the impact of market size is the same as in the additive model but the impact of market risk is different for different demand correlations and is no longer monotone in some cases. In another parallel paper, Bish and Suwandechochaib (2010) focus on the impact
of two new drivers: the degree of substitution between the products and the level of operational postponement. Their primary finding is that operational postponement and capacity can be either strategic substitutes or strategic complements, a result that differs from the well known result in a single-product setting, which states that operational postponement and capacity are strategic complements (see van Mieghem and Dada 1999).

Other newsvendor network structures have been explored in the context of responsive pricing. For example, Dong et al. (2010) study the global facility network design problem for a global firm who sells to both the domestic market and a foreign market, with the consideration of transshipment and responsive pricing. Chod et al. (2010) considers a firm that produces \( n \) differentiated product by using \( m \) different components and has the option to trade components in a secondary market ex post. They show how the value of production flexibility (the difference in expected profits between the make-to-order scenario and the make-to-stock scenario) can be affected by the degree of component commonalities and demand correlations.

For comparative statics analysis of responsive pricing newsvendor networks problems Bish et al. (2012) provide a framework relying solely on properties of the objective functions and the corresponding dual variables. Theoretically, their approach applies to any network and distribution-free demand so long as the capacity dual vector is well behaved, i.e., being continuous, linear and joint convex in the demand intercept vector. However, for analytical tractability they have to restrict their study to networks in which the production route of each product is fixed, which excludes any networks with alternative routes (i.e., any networks with discretionary common input and/or flexible capacity such as the one of our illustrative example). Also, an additive linear demand is assumed and is also crucial to the validity of their results.

In this paper we aim to provide a general framework for the responsive pricing newsvendor network problems following a parallel path to that of van Mieghem and Rudi (2002) for the standard newsvendor network problems. Thus, we pay close attention to general modeling issues, offering specific guidelines on demand (market size) space partitioning, and providing the existence and uniqueness of optimal solutions. We do offer some properties of our obtained optimal solutions
via comparative statics, but such results are by nature more limited as they are usually case-dependent. The illustrative example we use to demonstrate the market size space partition scheme features two inputs (inventories) and two capacities, while one inventory is a discretionary common input for the production of both products and one capacity has the flexibility of producing both products (ref. to Figure 1, also Figure 1 in vMR). This example, by removing the availability of the non-common input and relaxing the capacity constraints, includes the flexible-capacity-only situation as a special case. In summary, our paper appropriately links the related literature, thus offering a unifying treatment. Moreover, it goes beyond unification to extending current results for a comprehensive newsvendor network and any demand function fitting the revenue concavity assumption. Furthermore, we offer a general analysis and solution approach to such problems. More importantly, the responsive pricing newsvendor network allows us to extend the study to a rich set of demand models with different demand function forms, different nature of random shocks, market correlations, and cross-price effect in demand curves.

3. Modeling Framework

A responsive pricing newsvendor network is a stochastic decision model with recourse and pricing. It is defined by three data sets: network data, financial data, and demand data. The network data set consists of the parameters of the production technology, which describes how inputs are transformed into outputs to fill end-product demand. The financial data set includes the revenue and cost parameters. Both the production technology and the cost structure are linear, as in a standard newsvendor network model (van Mieghem and Rudi (2002)), except that the prices are endogenous rather than exogenous. The demand data, however, is different from that in a standard newsvendor network model. In a responsive pricing environment, the end-product demands depend on the realized market sizes and prices set by the firm. So, the demand data set contains: (i) the probability distributions of the market sizes, and (ii) the demand functions that establish the relationship of the actual end-product demands, the realized market sizes, and the prices. The three data sets are detailed as in below.
(1) Network data: input matrix $R_S$, capacity consumption matrix $A$, and output matrix $R_D$.
There are $l$ different capacities that consume $m$ distinct inputs (inventories) to produce $n$ distinct outputs by means of $p$ different processing activities. So, the input matrix $R_S$ is $m \times p$, the capacity consumption matrix $A$ is $l \times p$, and the output matrix $R_D$ is $n \times p$.

(2) Financial data: inventory procurement cost $c_S$, capacity investment cost $c_K$, and processing cost $c$. $c_S \in \mathbb{R}_+^m$ is the vector of the per unit order cost for input materials, $c_K \in \mathbb{R}_+^l$ is the vector of the per unit capacity investment cost, while $c \in \mathbb{R}_+^p$ is the vector of the per unit processing cost associated with the various processing activities.

(3) Demand data: market condition $\xi$, market size realization $\epsilon$, inverse demand function $P(Q, \epsilon)$. $\xi$ is a $n$-dimensional random variables with a continuous probability distribution and positive support; $\epsilon \in \mathbb{R}_+^n$ is the vector of a realization of $\xi$; $Q \in \mathbb{R}_+^n$ is the vector of selling quantities, and $P(Q, \epsilon) \in \mathbb{R}_+^n$ is the price vector for given $Q$ and $\epsilon$.

We assume that the revenue function satisfies the following assumption:

(A1). The total revenue $R(Q, \epsilon) = Q'P(Q, \epsilon) = \sum_{i=1}^{n} Q_i P_i(Q, \epsilon)$ is joint concave in $Q$.

In the first stage, the firm makes the capacity and inventory decisions when the market condition is still uncertain. Specifically, by setting the capacity level $K \in \mathbb{R}_+^l$ and inventory level $S \in \mathbb{R}_+^m$, the firm maximizes the expected firm value defined by

$$V(K, S) \equiv E_{\xi} \pi(K, S, \xi) - c'_S S - c'_K K,$$

where $\pi(K, S, \xi)$ is the maximal operating profit for given $K$ and $S$.

In the second stage, after the uncertainty of the market sizes is resolved, the firm maximizes the operating profit by choosing the processing activity vector $x \in \mathbb{R}_+^p$. As an optimality property of responsive pricing models, so long as the end-products are optimally priced, the output vector $R_D x$ is always equal to the vector of the selling quantities $Q$. So, we have $Q = R_D x$ and $R(x, \epsilon) = (R_D x)' P(R_D x, \epsilon)$. The operating profit is defined by

$$\tau(x, \epsilon) \equiv R(x, \epsilon) - c' x,$$
subject to the inventory and capacity constraints. Then we can formally define the two stage stochastic model as:

\[ \textbf{ex post:} \quad \pi(K, S, \epsilon) = \max_{x \geq 0} \{ \tau(x, \epsilon) = R(x, \epsilon) - c'x \} \] (1)

\[ \text{s.t.} \quad R_S x \leq S, \quad Ax \leq K \] (2)

\[ \textbf{ex ante:} \quad \max_{K, S \geq 0} \{ V(K, S) = E_\xi \pi(K, S, \xi) - c'_S S - c'_K K \} . \] (3)

The following proposition states that the optimal capacity vector \( K^* \) and inventory vector \( S^* \) are unique and can be obtained from the first order conditions. In general, \( S^* \) is dependent on \( K^* \).

**Proposition 1.** Both \( \pi(K, S, \epsilon) \) and \( V(K, S) \) are joint concave. There exists a market size space partition \( \Omega \) in which the gradients of \( V \) are given by

\[ \nabla_K V(K, S) = E \nabla_K \pi(K, S, \xi) - c_K, \] (4)

\[ \nabla_S V(K, S) = E \nabla_S \pi(K, S, \xi) - c_S, \] (5)

where

\[ E \nabla_K \pi(K, S, \xi) = \sum_j \int_{\Omega_j(K, S, \epsilon)} \lambda_{K,j}(K, S, \epsilon) f(\epsilon) d\epsilon, \] (6)

\[ E \nabla_S \pi(K, S, \xi) = \sum_j \int_{\Omega_j(K, S, \epsilon)} \lambda_{S,j}(K, S, \epsilon) f(\epsilon) d\epsilon. \] (7)

where \( \lambda_{K,j}(K, S, \epsilon) \) and \( \lambda_{S,j}(K, S, \epsilon) \) are the Lagrange multipliers of the capacity and inventory constraints, respectively, in \( \Omega_j(K, S, D) \), and \( f(\cdot) \) is the probability density function of \( \xi \).

Table 1 provides the comparison of the responsive pricing newsvendor networks and the standard newsvendor networks. Comparing the two models, we can see that the key difference lies in the operating profit functions. In the standard model, the revenue function is linear in \( x \) and so is the operating profit function, and consequently, an optimal solution to the ex post problem is always a corner solution. The marginal values of input vector \( S \) and capacity vector \( K \), although vary by regions \( \Omega_j(K, S, D) \), are constant in each region. In the responsive pricing model, however, the revenue function is concave in \( x \) by Assumption 1, and so is the operating profit function (as \(-c'x \)
is also a concave function), and thus an optimal solution may be a corner solution, or an interior solution that is a function of $\epsilon$. Because of the existence of interior solutions and their dependence on $\epsilon$, the marginal values of input vector $S$ and capacity vector $K$ are all $\epsilon$-dependent, and hence, the critical fractile solutions are much more involved than those of the standard model. Meanwhile, the market size space partition is largely complicated, too, as will be shown in the next section.

**Table 1 Comparing Responsive Pricing and Standard Newsvendor Networks Models**

<table>
<thead>
<tr>
<th></th>
<th>Responsive Pricing Newsvendor Networks Model</th>
<th>Standard Newsvendor Networks Model</th>
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<tbody>
<tr>
<td><strong>Ex post Problem</strong></td>
<td>$\pi(K,S,\epsilon) = \max_{x \geq 0} R(x,\epsilon) - c'x$ s.t. $R_Sx \leq S$, $Ax \leq K$</td>
<td>$\pi(K,S,D) = \max_{x \geq 0} [r - c + R'_Dc_P + R'_Dc_H]'x$ s.t. $R_Dx \leq D$, $R_Sx \leq S$, $Ax \leq K$</td>
</tr>
<tr>
<td><strong>Ex ante Problem</strong></td>
<td>$\max_{K,S \geq 0} E_\xi[\pi(K,S,\xi)] - c_S'S - c'_K K$</td>
<td>$\max_{K,S \geq 0} E_D[\pi(K,S,D)] - (c_S + c_H)'S - c'_K K$</td>
</tr>
<tr>
<td><strong>“Critical Fractile” Solution</strong></td>
<td>$\sum_j \int_{\Omega_j(K,S,\epsilon)} \lambda_{K,j}(K,S,\epsilon)f(\epsilon) = c_K$ $\sum_j \lambda_{K,j}P_j = c_K$</td>
<td>$\sum_j \lambda_{S,j}P_j = c_S + c_H$</td>
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We shall note that in the standard newsvendor network model, the demand vector is denoted by $D$, the output matrix by $R_D$, and for the purpose of dynamic extension, both penalty cost and holding cost are explicitly accounted and are denoted by $c_P$ and $c_H$, respectively. However, in the responsive pricing version there is no shortage penalty cost in the *ex post* problem, as the products are appropriately priced so that the actual demand in each market exactly matches the quantity produced. Also, we do not explicitly account for holding cost and salvage value of any leftover input inventory in the *ex ante* problem, as for a single period problem they can be easily incorporated into the processing cost and the inventory procurement cost. Finally, we note that in the standard model $P_j$ is shorthand for $P(\Omega_j(K,S,D))$.

### 4. Problem Formulation and Solutions

In this section we use an illustrative example to demonstrate a guided approach on how a specific responsive pricing newsvendor network problem is formulated, how the *ex post* analysis is performed via the demand space partitioning, and finally how the optimal solutions are obtained. The
benefit of this exercise is to highlight the subtle, yet important difference between the two types of newsvendor network. To facilitate the comparison between the responsive pricing newsvendor network model and the standard newsvendor network model, we employ the classic example depicted in vMR (vMR, p.314, Figure 1), which they carry throughout their discussions. The example features a discretionary common input 1 and a flexible processing capacity 2, as shown in Figure 1.

![Diagram](image)

**Figure 1** An example with a discretionary common input 1 and a flexible processing capacity 2.

Both the production technology and the financial structure are assumed linear. In particular, one unit of inventory (input) \( i \) is transformed into one unit of product (output) \( i \), consuming one unit of capacity \( i \), \( i = 1, 2 \); in addition, one unit of inventory 1 can be transformed into one unit of product 2, consuming \( \alpha^{-1} \) \((0 \leq \alpha \leq 1)\) unit of capacity 2. Thus, the inventory and capacity constraints are:

\[
\begin{align*}
  x_1 + x_3 & \leq S_1, \\
  x_1 & \leq K_1, \\
  x_2 & \leq S_2, \\
  x_2 + \alpha^{-1} x_3 & \leq K_2.
\end{align*}
\]

Recall that the optimal *ex post* prices should match the production quantity with the demand exactly, i.e., selling quantities \( Q_1 = x_1 \) and \( Q_2 = x_2 + x_3 \). That is, the optimal prices \( P^* (Q, \epsilon) \) are implied by the optimal activity levels \( x_i^* \)’s. In matrix form, the network data are:
\[ R_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad R_S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha^{-1} \end{pmatrix}. \]

In vMR’s terms, the \( i + i \to i \) transformation (input \( i \), by capacity \( i \), generates output \( i \)) is called “normal” or “basic”, while the \( 1 + 2 \to 2 \) transformation is called “alternative” or “non-basic”. In general, a non-basic activity incurs higher processing cost and consumes more capacity than a basic activity, and so it is assumed that both \( c_3 > c_2 \), and \( \alpha^{-1} \geq 1 \), i.e., \( 0 \leq \alpha \leq 1 \), hold. Moreover, as input 1 has the flexibility of producing both outputs 1 (through Activity 1) and 2 (through Activity 2), and the input consumption rates are assumed the same, we assume that \( c_{S1} > c_{S2} \) holds. To sum up, we make the following assumptions on the production technology and cost structure.

(A2). The processing costs satisfy \( c_3 > c_2 \);

(A3). The capacity consumption rate of Activity 3 satisfies \( 0 \leq \alpha \leq 1 \);

(A4). The inventory procurement costs satisfy \( c_{S1} > c_{S2} \).

As demonstrated in vMR (page 322-323, Examples (1)-(7)), by appropriately adjusting the matrices \( R_D, R_S, \) and \( A \), the network structure in Figure 1 can represent various newsvendor-type systems such as assembly systems, systems with component commonality or transshipment.

Relating this example to the existing responsive pricing newsvendor literature, if we set \( \alpha = 1 \), \( K_1 = K_2 = \infty \), and remove activity \( x_2 \), then we obtain a system with a single flexible capacity; if we remove \( x_3 \), then we obtain a system with two dedicated capacities.

4.1. Problem Formulation

With the above assumptions we can formulate the problem. Let \( MR_i(x, \epsilon) \equiv \frac{\partial R(x, \epsilon)}{\partial x_i} \) denote the marginal revenue of activity \( x_i \). First of all, we shall note that although capacity 2 is capable of processing both \( x_2 \) and \( x_3 \), the priority is given to Activity 2 for two reasons: (1) its marginal profit is higher than that of Activity 3, i.e., \( \frac{\partial \tau}{\partial x_2}(x, \epsilon) > \frac{\partial \tau}{\partial x_3}(x, \epsilon) \), as the marginal revenues are equal (\( MR_2(x, \epsilon) = MR_3(x, \epsilon) \)) but the marginal costs satisfy \( c_2 < c_3 \) by Assumption 2; and (2) its capacity consumption rate is lower than that of Activity 3 by Assumption 3. Based on this observation together with Assumption 4, we have the following property on the optimal solutions to the \( ex \ ante \) problem.
Property 1 The optimal inventory and capacity levels satisfy: (1) $S_1^* = K_1^*$; and (2) $S_2^* \leq K_2^* \leq S_2^* + \alpha^{-1} S_1^*$.

Property 1 helps us reduce the dimension of the decision space and the number of constraints, as illustrated below. Therefore, without loss of optimality we can consider only $K_1 = S_1$. Then, $x_1 \leq K_1$ is equivalent to $x_1 \leq S_1$. However, as $x_3 \geq 0$, $x_1 + x_3 - S_1 \leq 0$ implies $x_1 \leq S_1$. So, the first capacity constraint in (9) is redundant. Meanwhile, as the priority is given to Activity 2, inventory 1 is not consumed by Activity 3 unless inventory 2 is exhausted, and so $x_3 > 0$ holds only for $x_2 = S_2$. Thus, $x_2 + \alpha^{-1} x_3 \leq K_2$ is equivalent to $x_3 \leq \alpha(K_2 - S_2)$ for $x_3 > 0$, and is simply $x_2 \leq K_2$ for $x_3 = 0$. As $S_2^* \leq K_2^*$ by Property 1, $x_2 \leq K_2$ is implied by $x_2 \leq S_2$. Noting that $K_2 - S_2$ is the effective capacity 2 that can be used by Activity 3 and $\alpha(K_2 - S_2)$ serves as the actual capacity constraint for $x_3$, we define

$$S_3 \equiv \alpha(K_2 - S_2).$$

Then, the second capacity constraint in (9) can be replaced by a new constraint of $x_3 \leq S_3$. Applying $S_3$ to the two-stage models, we reduce the 4-dimensional decision variables to the augmented vector $S \equiv (S_1, S_2, S_3)$. The capacity levels become dependent variables given by

$$K_1 = S_1, \quad K_2 = S_2 + \alpha^{-1} S_3.$$

Now the ex post problem can be rewritten as

$$\pi(S, \epsilon) = \max \left\{ \tau(x, \epsilon) = R(x, \epsilon) - \sum_{i=1}^{3} c_i x_i \right\}$$

s.t. $x_1 + x_3 - S_1 \leq 0,$

$$x_2 - S_2 \leq 0,$$

$$x_3 - S_3 \leq 0,$$

$$x_i \geq 0, \; i = 1, 2, 3.$$

The ex ante problem is to maximize the expected firm value redefined as

$$V(S) = E_\xi [\pi(S, \xi) - (c_{S1} + c_{K1}) S_1 - (c_{S2} + c_{K2}) S_2 - \alpha^{-1} c_{K2} S_3].$$
4.2. Market Size Space Partition

For any realization of market sizes, the profit-maximizing selling quantity (equivalently, price) should equalize the marginal revenue and the marginal production cost of that product. For given input inventory levels \( S_1, S_2, \) and \( S_3, \) we partition the market size space \((\epsilon_1, \epsilon_2)\) into 15 regions, each corresponding to a unique scenario of production activities. For ease of exposition, we arrange the regions in a \(5 \times 3\) matrix and the corresponding scenarios in another matrix in expression (17), and provide a graphical illustration in Figure 2.

The regions are numbered in the current way with subregions such as \( \Omega_{0i}, i = 1, \ldots, 4, \) rather than from 1 to 15, because we want to emphasize the similarity among these subregions. For example, in all the regions \( \Omega_{0i} \) none of the three activities is constrained, while in all the other regions at least one of the three activities is constrained. For simplicity sometimes we write \( \Omega_j \) to denote the joint of all subregions \( \Omega_{ji}, \) for example, \( \Omega_4 = \Omega_{41} \cup \Omega_{42}. \)

<table>
<thead>
<tr>
<th>( \epsilon_2 )</th>
<th>( \Omega_{61} : (0, S_2, S_3) )</th>
<th>( \Omega_{62} : (x_1^*, S_2, S_3) )</th>
<th>( \Omega_7 : (S_1 - S_3, S_2, S_3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_1 )</td>
<td>( \Omega_{41} : (0, S_2, x_3^*) )</td>
<td>( \Omega_{42} : (x_1^<em>, S_2, x_3^</em>) )</td>
<td>( \Omega_5 : (x_1^<em>, S_2, x_3^</em>) )</td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>0</td>
<td>( \Omega_{21} : (0, S_2, 0) )</td>
<td>( \Omega_{22} : (x_1^*, S_2, 0) )</td>
<td>( \Omega_3 : (S_1, S_2, 0) )</td>
</tr>
<tr>
<td></td>
<td>( \Omega_{03} : (0, x_2^*, 0) )</td>
<td>( \Omega_{04} : (x_1^<em>, x_2^</em>, 0) )</td>
<td>( \Omega_{12} : (S_1, x_2^*, 0) )</td>
</tr>
<tr>
<td></td>
<td>( \Omega_{01} : (0, 0, 0) )</td>
<td>( \Omega_{02} : (x_1^*, 0, 0) )</td>
<td>( \Omega_{11} : (S_1, 0, 0) )</td>
</tr>
</tbody>
</table>

Figure 2    Market size space partition, general demand.
The logic of the partition is detailed here. Recall that Activity 3 is discretionary because it is more expensive and less efficient than Activity 2 (i.e., $c_3 > c_2$ and $\alpha \leq 1$). Thus, it is optimal to use Activity 3 only when market 2 size is high enough that using input inventory 1 through Activity 3 is profitable even after input 2 inventory is exhausted. When market 2 demand is low, it is optimal to set $x_3 = 0$. For any scenario with $x_3^* = 0$, the constraint $x_1 + x_3 \leq S_1$ becomes $x_1 \leq S_1$. So, we have two possible boundary solutions $x_i^* = \{0, S_i\}$ and one interior solution $x_i^* \in (0, S_i)$ for both $i = 1$ and 2, and thus, there are totally $3 \times 3 = 9$ scenarios, corresponding to the 9 regions with $x_3^* = 0$ (the three bottom rows in the matrix (17)). For any scenario with $x_3^* > 0$ we have $x_2^* = S_2$. For scenarios with $x_3^* < S_3$, $x_1^*$ may be the boundary solution $x_1^* = 0$, or an interior solution satisfying $x_1^* + x_3^* < S_1$, or the boundary solution defined by $x_1^* = S_1 - x_3^*$, and the corresponding scenarios are the 3 regions in the second row, i.e., $\Omega_{41}$, $\Omega_{42}$ and $\Omega_5$, respectively; for scenarios with $x_3^* = S_3$, $x_1^*$ may be one of the two boundary solutions $x_1^* = \{0, S_1 - S_3\}$, or an interior solutions satisfying $0 < x_1^* < S_1 - S_3$, and the corresponding scenarios are the 3 regions in the first row.

The inventory allocation decision for a standard newsvendor network is different in two ways, both are driven by the fact that the allocation priority is solely determined by the exogenous financial data. First, in vMR it is assumed that $v_1 > v_2 \geq v_3 > 0$ holds, where $v_i$ is the net value (i.e., marginal profit) of Activity $i$. The all positive net value assumption means that the entire demand space $\mathbb{R}_+^2$ is profitable, and hence there exists no non-production region, i.e., all regions in the first column and/or the bottom row do not appear in the newsvendor network. Second, the given ranking of net values decides that Activity 3 has the lowest priority, taking place only if input 1 has inventory left after serving product 1 demand. In the responsive pricing network, however, Activity 3 has higher priority than Activity 1 when the marginal profit of market 2 is higher than that of market 1. This priority is clearly reflected in Region $\Omega_7$ in Figure 2, where
input 1 is used for producing product 2 via Activity 3 up to the process capacity of Resource 2, the leftover inventory is used for producing product 1. Those distinctions will play important roles in the sensitivity of the firm’s value to demand characteristics.

4.3. Optimal Solutions

With the demand space appropriately partitioned, it remains to determine the \( x_i^* \)'s in Figure 2, and then solve the first stage problem. As Proposition 2 below shows, for regions with only one interior solution, the optimal solutions are obtained by solving one single first order condition

\[
\frac{\partial \text{MR}_i}{\partial x_i}(x, \epsilon) = 0, \quad \text{i.e., } \text{MR}_i(x, \epsilon) = c_i, \quad \text{for } x_i^* \text{ in } \Omega_{02}, \Omega_{22}, \Omega_{62}, x_2^* \text{ in } \Omega_{03} \text{ and } \Omega_{12}, x_3^* \text{ in } \Omega_{41};
\]

for regions with two interior solutions \((x_i^*, x_j^*)\), the optimal solutions are obtained by jointly solving

\[
\text{MR}_i(x, \epsilon) = c_i \quad \text{and} \quad \text{MR}_j(x, \epsilon) = c_j, \quad \text{i.e., } (x_1^*, x_2^*) \text{ in } \Omega_{04} \text{ and } (x_1^*, x_3^*) \text{ in } \Omega_{42}; \text{ in region } \Omega_5, \text{ inventory } S_1 \text{ is used up jointly by Activities 1 and 3, the optimal solutions are obtained by jointly solving } x_1 + x_3 = S_1 \text{ and } \text{MR}_1(x, \epsilon) - c_1 = \text{MR}_2(x, \epsilon) - c_2.
\]

**Proposition 2.** Assume a demand function satisfying the revenue concavity assumption (A1). For all regions with interior solution(s), the optimal solutions are provided below.

1. For \( \Omega_{02}, \Omega_{22}, \Omega_{62}, x_1^* \) is the solution of \( \text{MR}_1(x, \epsilon) = c_1 \);
2. for \( \Omega_{03}, \Omega_{12}, x_2^* \) is the solution of \( \text{MR}_2(x, \epsilon) = c_2 \);
3. for \( \Omega_{41}, x_3^* \) is the solution of \( \text{MR}_3((0, S_2, x_3), \epsilon) = c_3 \);
4. for \( \Omega_{04}, (x_1^*, x_2^*) \) is the solution of \( \begin{cases} \text{MR}_1((x_1, x_2, 0), \epsilon) = c_1, \\ \text{MR}_2((x_1, x_2, 0), \epsilon) = c_2; \end{cases} \)
5. for \( \Omega_{42}, (x_1^*, x_3^*) \) is the solution of \( \begin{cases} \text{MR}_1((x_1, S_2, x_3), \epsilon) = c_1, \\ \text{MR}_2((x_1, S_2, x_3), \epsilon) = c_3. \end{cases} \)
6. for \( \Omega_5, (x_1^*, x_3^*) \) is the solution of \( \begin{cases} x_1 + x_3 = S_1, \\ \text{MR}_1((x_1, S_2, x_3), \epsilon) - c_1 = \text{MR}_2((x_1, S_2, x_3), \epsilon) - c_2. \end{cases} \)

With all the \( x_i^* \)'s determined, we obtain the marginal values of the vector \( S \) in all regions, as shown in Table 2. Hereinafter we drop \( K \) from the parenthesis and \( S \) from the subscript in \( \lambda_{S,j}(K, S, \Omega_i) \), since neither is necessary after the re-definition of the first-stage decision variables. We shall note that \( \lambda_j(S, \Omega_i) = 0, j = 1, 2, 3 \), holds for all unlisted regions.

With the marginal values of \( S \) in each region we solve the first stage problem by the first order conditions, which are necessary and sufficient according to Proposition 1. We note that in the
concrete lines to separate the regions. In the next section we characterize market size partition
of the boundaries depend on the demand functions. For this reason we use dashed lines rather than
2, which are summarized in Tables 4 and 5 in the Appendix. We note that the shape and position

\( \lambda_1(S, \Omega_i) = \sum_{j=1}^{N_1} \lambda_j(S^*, \Omega_i) f(\epsilon_1, \epsilon_2) \)

\( \Omega_i \lambda_2(S, \Omega_i) = \Omega_21 MR_2((S_1, S_2, S_3), \epsilon) - c_1 \)

\( MR_3((0, S_1, S_2), \epsilon) - c_1 \)

\( MR_2((S_1, S_2, S_3), \epsilon) - c_3 \)

\( MR_2((x^*_1, x^*_2, S_3), \epsilon) - c_3 \)

\( MR_2((S_1 - S_3, S_2, S_3), \epsilon) - c_3 \)

\( -MR_1((S_1 - S_3, S_2, S_3), \epsilon) + c_1 \)

following proposition the equations are just the first order conditions of \( V(S) \) with respect to \( S \),
where \( V(S) \) is defined by (16), and thus the results are presented without proof.

**Proposition 3.** A vector \( S^* \in \mathbb{R}^3_+ \) is optimal if and only if there exists a non-negative vector
\( \mu \in \mathbb{R}^3_+ \) that satisfies \( \mu S^* = 0 \) and the following conditions:

\[
\sum_{i \in N_1} \int \int \lambda_1(S^*, \Omega_i) f(\epsilon_1, \epsilon_2) \, d\epsilon_1 \, d\epsilon_2 = c_{S1} + c_{K1} + \mu_1,
\]

\[
\sum_{i \in N_2} \int \int \lambda_2(S^*, \Omega_i) f(\epsilon_1, \epsilon_2) \, d\epsilon_1 \, d\epsilon_2 = c_{S2} + c_{K2} + \mu_2,
\]

\[
\sum_{i \in N_3} \int \int \lambda_3(S^*, \Omega_i) f(\epsilon_1, \epsilon_2) \, d\epsilon_1 \, d\epsilon_2 = \alpha^{-1} c_{K2} + \mu_3,
\]

where \( N_1 = \{11, 12, 3, 5, 7\}, N_2 = \{21, 22, 3, 41, 42, 5, 61, 62, 7\}, N_3 = \{61, 62, 7\}, \) and \( \lambda_j(S^*, \Omega_i), \ i \in \{N_1, N_2, N_3\}, j = 1, 2, 3, \) are given by Table 2.

We complete this section with a brief discussion of the boundaries between the regions in Figure
2, which are summarized in Tables 4 and 5 in the Appendix. We note that the shape and position
of the boundaries depend on the demand functions. For this reason we use dashed lines rather than
concrete lines to separate the regions. In the next section we characterize market size partition
schemes with exact boundaries, for the additive linear demand functions.

5. The Additive Linear Demand Model

This section makes further comparison between the responsive pricing newsvendor network and
the standard newsvendor network, from the *ex post* analysis of the demand/market-size space
partition to the comparative statics of the firm’s value function, with the objective of deepening our
understanding of the similarity and difference of these types of models. Such comparisons require
us to specify the demand functions and explicitly solve the problem. We will first define a two
dimensional demand function, and then apply it to the illustrative example introduced in Section
4.

Let $\mathbb{D}$ be a $2 \times 2$ matrix defined by

$$
\mathbb{D} = \begin{pmatrix}
a_1 & b \\
b & a_2
\end{pmatrix},
$$

where $a_i > 0$, $i = 1, 2$, measures the individual self-price effect and $b$, $|b| < \min\{a_1, a_2\}$, measures
the cross-price effect. The inverse demand function is defined by

$$
P(Q, \epsilon) = \mathbb{D}(\epsilon - Q) \iff P_i(\epsilon - Q) = a_i(\epsilon_i - Q_i) + b(\epsilon_j - Q_j) \text{ for } i = 1, 2 \text{ and } j = 2, 1. \quad (18)
$$

The corresponding demand function is additive linear:

$$
Q = \epsilon - \mathbb{D}^{-1}P \iff Q_i(\epsilon) = \epsilon_i - \alpha_iP_i + \beta P_j \text{ for } i = 1, 2 \text{ and } j = 2, 1 \quad (19)
$$

where

$$
\beta = \frac{b}{a_1a_2 - b^2}, \quad \alpha_i = \frac{a_j}{a_1a_2 - b^2}, \quad i = 1, 2, \quad j = 2, 1.
$$

Note that the substitutability/complementarity between the two products can be captured by
both of the parameters $b$ and $\beta$: a positive (resp. negative) $b$ or $\beta$ means that the products are
substitutes (resp. complements), while $b = 0$ or $\beta = 0$ implies that the demands of the two products
are independent. The $b$-measure has been well accepted by the economics community, and is intro-
duced to the operations management community by recent papers such as Goyal and Netessine
(2007), Lus and Muriel (2009), and Goyal and Netessine (2011). Following these works, we adopt
the $b$-measure and base our analysis on the demand functions defined by (18)-(19), which defines
the revenue function:

$$
R(Q, \epsilon) = Q'P(Q, \epsilon) = Q'\mathbb{D}(\epsilon - Q).
$$

1 As pointed out by Lus and Muriel (2009), $b$ is better than $\beta$ in measuring the substitutability/complementarity, as
the latter can lead to unnatural comparative static results.
Then, the marginal revenue is

\[ MR(Q, \epsilon) = \frac{\partial R(Q, \epsilon)}{\partial Q} = \mathbb{D}(\epsilon - 2Q), \]

and the negative Hessian of \( R(Q, \epsilon) \) is \(-H_R = 4\mathbb{D}\). It is straightforward to verify that Assumption (A1) of \( R(Q, \epsilon) \) being strictly joint concave in \( Q \) holds.

Recall \( x_1 = Q_1 \) and \( x_2 + x_3 = Q_2 \). Then the \( \text{ex post} \) problem becomes

\[
\pi(S, \epsilon) = \max \{ \tau(x, \epsilon) = (a_1 \epsilon_1 + b \epsilon_2)x_1 + (a_2 \epsilon_2 + b \epsilon_1)(x_2 + x_3) - a_1 x_1^2 - 2b x_1 x_3 - a_2 (x_2 + x_3)^2 - \sum_{i=1}^{3} c_i x_i \}, \tag{20}
\]

subject to the constraints (12)-(15).

5.1. Optimal Policy Characterization

We define some thresholds for the realized market size:

\[
\xi_1 = \frac{a_2 c_1 - bc_2}{a_1 a_2 - b^2}, \quad \xi_2 = \frac{a_1 c_2 - bc_1}{a_1 a_2 - b^2}, \quad \bar{\epsilon}_1 = \frac{a_2 c_1 - bc_3}{a_1 a_2 - b^2}, \quad \bar{\epsilon}_2 = \frac{a_3 c_1 - bc_1}{a_1 a_2 - b^2}.
\]

Note that for \( b = 0 \), we have \( \xi_1 = \bar{\epsilon}_1 = \xi_1^0 \equiv \frac{c_1}{a_1}, \quad \xi_2 = \xi_2^0 \equiv \frac{c_2}{a_2} \) and \( \bar{\epsilon}_2 = \bar{\epsilon}_2^0 \equiv \frac{c_3}{a_2} \).

The marginal revenues are:

\[
MR_1(x, \epsilon) = a_1(\epsilon_1 - 2x_1) + b[\epsilon_2 - 2(x_2 + x_3)], \tag{21}
\]

\[
MR_i(x, \epsilon) = a_2[\epsilon_2 - 2(x_2 + x_3)] + b(\epsilon_1 - 2x_1), \quad i = 2, 3. \tag{22}
\]

Substituting the above marginal revenues into the equations in Proposition 2 and solving them give the optimal solutions, as shown in Corollary 1.

**Corollary 1.** Assume a linear demand function \( P(Q, \epsilon) = \mathbb{D}(\epsilon - Q) \). For all regions with interior solutions, the optimal solutions are provided below.

1. For \( \Omega_{02} \), \( x_1^* = \frac{1}{2}(\epsilon_1 - \xi_1^0) + \frac{b}{2a_1} \epsilon_2 \), for \( \Omega_{22} \), \( x_1^* = \frac{1}{2}(\epsilon_1 - \xi_1^0) + \frac{b}{2a_1}(\epsilon_2 - 2S_2) \), for \( \Omega_{02} \), \( x_1^* = \frac{1}{2}(\epsilon_1 - \xi_1^0) + \frac{b}{2a_1}(\epsilon_2 - 2(S_2 + S_3)) \);

2. For \( \Omega_{03} \), \( x_2^* = \frac{1}{2}(\epsilon_2 - \xi_2^0) + \frac{b}{2a_2} \epsilon_1 \), for \( \Omega_{12} \), \( x_2^* = \frac{1}{2}(\epsilon_2 - \xi_2^0) + \frac{b}{2a_2}(\epsilon_1 - 2S_1) \);

3. For \( \Omega_{11} \), \( x_3^* = \frac{1}{2}(\epsilon_2 - \xi_2^0 - 2S_2) + \frac{b}{2a_2} \epsilon_1 \).
(4) for $\Omega_{04}$, $x_i^* = \frac{1}{2}(\epsilon_i - \bar{\epsilon}_i)$, $i = 1, 2$;

(5) for $\Omega_{42}$, $x_1^* = \frac{1}{2}(\epsilon_1 - \bar{\epsilon}_1)$, $x_3^* = \frac{1}{2}(\epsilon_2 - \bar{\epsilon}_2 - 2S_2)$.

(6) for $\Omega_5$, $x_1^* = \frac{1}{2(a_1 + a_2 - 2b)} \left[ (a_1 - b)\epsilon_1 - (a_2 - b)(\epsilon_2 - 2(S_1 + S_2)) - c_1 + c_3 \right]$, $x_3^* = S_1 - x_1^*$.

Substituting (21) and (22) and the optimal solutions obtained from Corollary 1 into Tables 4-5 (in Appendix) gives the boundary lines for partitioning the market size space (see Tables 6-7 in the Appendix). The partition scheme is shown in Figures 3-5, for cases $b > 0$, $b = 0$, and $b < 0$, respectively. We first provide some observations for the case of $b = 0$, the case in which there is no cross-price effect between the two products, and then compare the three cases to highlight the impact of demand substitution/complementarity on the partition of the market size space. For $b = 0$, the boundary lines between regions are strictly horizontal lines and vertical lines except those between Region $\Omega_5$ and its neighboring regions, Regions $\Omega_3$, $\Omega_{42}$, and $\Omega_7$. This is an intuitive observation. When the two products do not share input inventory, or share inventory and inventory is ample, i.e., the regions toward the bottom and left sides in Figure 4, as the market $i$ size increases, production of product $i$ increases from 0 to positive $x_i^*$, and then to $S_i$, $i = 1, 2$, regardless of the market size of the other product. Region $\Omega_5$ represents the scenario in which market sizes for both products are large and it is profitable to use input 1 to produce product 2 (via Activity 3), and input 1 inventory is fully allocated between $x_1$ and $x_2$. In Region $\Omega_5$, the inventory allocation decision is based on comparing the relative profitability of the two markets so as to balance the marginal profit from the two markets, and thus the boundaries lines between regions are dependent on both $\epsilon_1$ and $\epsilon_2$. Specifically, as market 2 size decreases, it becomes less profitable to allocate input 1 to product 2 and $x_3^*$ decreases, the boundary between $\Omega_5$ and $\Omega_3$ is where the marginal profit of $x_1$ at $x_1 = S_1$ is equal to the marginal profit of $x_3$ at $x_3 = 0$, and the boundary between $\Omega_5$ and $\Omega_{42}$ is where the marginal profit of $x_1$ at $x_1 = x_1^*$ is equal to the marginal profit of $x_3$ at $x_3 = S_1 - x_1^*$. On the other hand, as market 2 size increases, it becomes more profitable to allocate input 1 to product 2 and $x_3^*$ increases, the boundary between $\Omega_5$ and $\Omega_7$ is where the marginal profit of $x_1$ at $x_1 = S_1 - S_3$ is equal to the marginal profit of $x_3 = S_3$. 
The most salient observation from comparing Figures 3-5 is that some of the horizontal boundary lines in Figure 4 rotate clockwise in Figure 3 (case of $b > 0$) and rotate counter-clockwise in Figure 5 (case of $b < 0$), and some of the vertical boundary lines in Figure 4 rotate counter-clockwise in Figure 3 and rotate clockwise in Figure 5. This observation reflects the fact that selling decisions of the two products are interdependent due to the cross-price effect. Moreover, in comparison to the case of zero cross-price effect ($b = 0$), when products are substitutes ($b > 0$)/complements ($b < 0$), the regions where the total inventory is not fully utilized move towards the lower/larger market size realizations. In other words, inventory is more/less likely to be fully utilized when products are substitutes/complements than when products are independent. This is because the presence of partially substitutable/complementary products gives rise to an incentive to lower/raise the selling prices so as to stimulate/suppress demand to counterbalance the potential sales cannibalization/competition for scarce inventory, which in turn gives higher/lower utilization of inventory, as evident from the left-ward/right-ward shift of the bottom-row regions, and from the downward/upward shift of the first-column regions in Figure 3/Figure 5. This observation drives the comparative statics of the firm’s value with respect to the cross-price parameter $b$.

It is worth mentioning that Corollary 1 and Figures 3-5 subsume the optimal solution characterization of the special case of the responsive newsvendor network, and therefore are consistent with results developed in the existing literature for those cases.
Finally, substituting the optimal solutions obtained from Corollary 1 into (21) and (22) we have the updated marginal values in Table 3. Substituting these new data into Proposition 3, we can further analyze the first-stage investment strategy. As mentioned earlier, such analysis becomes quickly involved and fewer analytical results can be expected.

Table 3  The marginal values of $S$ in market size domain, general $b$

<table>
<thead>
<tr>
<th>$\Omega_i$</th>
<th>$\lambda_1(S, \Omega_i)$</th>
<th>$\Omega_i$</th>
<th>$\lambda_2(S, \Omega_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_{11}$</td>
<td>$a_1(e_1 - 2S_1) + b\epsilon_2 - c_1$</td>
<td>$\Omega_{21}$</td>
<td>$a_2(e_2 - 2S_2) + b\epsilon_1 - c_2$</td>
</tr>
<tr>
<td>$\Omega_{12}$</td>
<td>$\frac{a_1a_2-b^2}{a_2}(e_1 - 2S_1) - c_1 + \frac{b}{a_2}c_2$</td>
<td>$\Omega_{22}$</td>
<td>$\frac{a_1a_2-b^2}{a_1}(e_2 - 2S_2) - c_2 + \frac{a}{a_1}c_1$</td>
</tr>
<tr>
<td>$\Omega_3$</td>
<td>$a_1(e_1 - 2S_1) + b(e_2 - 2S_2) - c_1$</td>
<td>$\Omega_3$</td>
<td>$a_2(e_2 - 2S_2) + b(e_1 - 2S_1) - c_2$</td>
</tr>
<tr>
<td>$\Omega_4$</td>
<td>$\lambda_3(S, \Omega_4)$</td>
<td>$\Omega_4$</td>
<td>$c_3 - c_2$</td>
</tr>
<tr>
<td>$\Omega_{61}$</td>
<td>$a_2[e_2 - 2(S_2 + S_3)] + b\epsilon_1 - c_3$</td>
<td>$\Omega_{61}$</td>
<td>$a_2[e_2 - 2(S_2 + S_3)] + b\epsilon_1 - c_2$</td>
</tr>
<tr>
<td>$\Omega_{62}$</td>
<td>$\frac{a_1a_2-b^2}{a_1}(e_2 - 2(S_2 + S_3)) - c_3 + \frac{b}{a_1}c_1$</td>
<td>$\Omega_{62}$</td>
<td>$\frac{a_1a_2-b^2}{a_1}[e_2 - 2(S_2 + S_3)] - c_2 + \frac{b}{a_1}c_1$</td>
</tr>
</tbody>
</table>

5.2. Comparative Static Analyses

In this subsection we focus our study on the substitution/complementarity effects and the impacts of the stochastic characteristics of the market sizes on the expected firm value. We do not examine the impacts on the optimal inventory and capacity levels for two reasons: tractability and limited applicability. vMR indicated in their study of newsvendor networks that comparative static analyses of the optimal inventory or capacity levels become involved quickly and one often has to turn to numerical approach. Such difficulty persist in the responsive pricing newsvendor networks. Moreover, in general the optimal inventory and capacity levels depend critically on the network data, e.g., the availability and flexibility of the inventories and capacities, and hence, the significance of such analyses is limited.

We start by clarifying the impact of the substitutability/complementarity between the two products, which is measured by the parameter $b$, as shown in the following proposition.

PROPOSITION 4. Assume a linear demand function $P(Q, \epsilon) = \mathbb{D}(\epsilon - Q)$. The maximal expected firm value $V^*(S)$ is increasing in the substitutability/complementarity parameter $b$. 

As we highlighted in the comparison of Figures 3-5, when the products become more substitutable or less complementary, responsive pricing allows the inventory to be more fully utilized, and the firm benefits from higher utilization of inventory.

We now turn our attention to the impact of stochastic demand on the firm’s value function. We begin by introducing some definitions of stochastic comparison, and then apply them to the study of the impacts of various characteristics of the stochastic market sizes.

Definition 1. (Müller and Stoyan 2002, p.90, 98, 113) Let $X$ and $Y$ be $n$-dimensional random vectors with finite expectations. Then,

1. $X$ is smaller than $Y$ in the usual stochastic order, written $X \leq_{st} Y$, if $E[f(X)] \leq E[f(Y)]$ for all bounded increasing functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the expectations exist;

2. $X$ is smaller than $Y$ in the convex order, written $X \leq_{cx} Y$, if $E[f(X)] \leq E[f(Y)]$ for all convex functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the expectations exist;

3. $X$ is smaller than $Y$ in the supermodular order, written $X \leq_{sm} Y$, if $E[f(X)] \leq E[f(Y)]$ for all supermodular functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the expectations exist.

Both the usual stochastic order and convex order have clear, intuitive implications for one-dimensional random variables, i.e., $X \leq_{st} Y$ implies $EX \leq EY$, and $X \leq_{cx} Y$ implies $Var(X) \leq Var(Y)$. But the implications of these stochastic orders are not as intuitive for general multi-dimensional random variables. For tractability, we choose the multivariate normal distribution (a most frequently used distribution in the operations literature) to help us demonstrate the stochastic comparison results. Specifically, consider two 2-dimensional random variables $X$ and $Y$, both following bivariate normal (BVN) distribution, $X \sim \mathcal{N}(\mu_X, \Sigma_X)$ and $Y \sim \mathcal{N}(\mu_Y, \Sigma_Y)$, with variance-covariance matrices defined by

$$
\Sigma_X = \begin{pmatrix}
\sigma_{X1}^2 & \sigma_{X1} \sigma_{X2} \\
\sigma_{X1} \sigma_{X2} & \sigma_{X2}^2
\end{pmatrix}
$$

and

$$
\Sigma_Y = \begin{pmatrix}
\sigma_{Y1}^2 & \sigma_{Y1} \sigma_{Y2} \\
\sigma_{Y1} \sigma_{Y2} & \sigma_{Y2}^2
\end{pmatrix}
$$

and correlation coefficient $\rho_X = \frac{\sigma_{X1} \sigma_{X2}}{\sigma_{X1} \sigma_{X2}}$ and $\rho_Y = \frac{\sigma_{Y1} \sigma_{Y2}}{\sigma_{Y1} \sigma_{Y2}}$. The following properties hold (Müller and Stoyan 2002, Theorem 3.3.13, Theorem 3.4.7, and Theorem 3.13.5):

1. if $\Sigma_X = \Sigma_Y$, then $\mu_X \leq \mu_Y$ is equivalent to $X \leq_{st} Y$;
(2) if \( \mu_X = \mu_Y \), then \( \Sigma_Y - \Sigma_X \) is non-negative definite is equivalent to \( X \leq_{ca} Y \). Moreover, if \( \mu_X = \mu_Y = \mu \), \( \sigma_{X1} = \sigma_{X2} = \sigma_X \), and \( \sigma_{Y1} = \sigma_{Y2} = \sigma_Y \), then \( \sigma_X \leq \sigma_Y \) is equivalent to \( X \leq_{ca} Y \); 

(3) if \( \mu_X = \mu_Y = \mu \), \( \sigma_{X1} = \sigma_{X2} = \sigma_X \), and \( \sigma_{Y1} = \sigma_{Y2} = \sigma_Y \), then \( \sigma_X \leq \sigma_Y \) is equivalent to \( X \leq_{sm} Y \).

In establishing following comparative statics for the firm’s value function, we show that the firm’s value function \( \pi(S, \epsilon) \) is an increasing, jointly convex function of \( \epsilon \), and \(-\pi(S, \epsilon)\) is supermodular in \( \epsilon \). Recall that \( \epsilon \) is a realization of \( \xi \).

**Proposition 5.** Assume a linear demand function \( P(Q, \epsilon) = D(\epsilon - Q) \). The maximal expected firm value \( V^*(S) \)

(1) increases if \( \xi \) increases in usual stochastic order;

(2) increases if \( \xi \) increases in convex order;

(3) decreases if \( \xi \) increases in supermodular order and \( b \leq 0 \).

If \( \xi \) follows a BVN distribution, Proposition 5 states that: (1) for fixed variance-covariance matrix \( \Sigma \), \( V^*(S) \) increases in the expectation \( \mu \); (2) for fixed \( \mu \), \( V^*(S) \) increases in the common standard deviation \( \sigma \) \( (\sigma = \sigma_1 = \sigma_2) \); (3) for fixed \( \mu \) and \( \sigma \), \( V^*(S) \) increases in the correlation coefficient \( \rho \).

Now we take the BVN distributed \( \xi \) as an instance to explore the managerial implications of the monotonic results in Proposition 5. The monotone increasing property of \( V^*(S) \) in \( \mu \) is intuitive, as any increase in the expected market sizes is always beneficial to the firm. The monotone increasing property of \( V^*(S) \) in \( \sigma \) is interesting and is consistent with the findings for the special case of this example (e.g., the single flexible capacity only case studied by Chod and Rudi 2005), and is the natural outcome of pricing commitment being postponed to after the uncertainty is resolved. Specifically, while under/over-investment is inevitable in the presence of demand variability, responsive pricing enable the firm to take advantage of the large market size by selling at a high price and mitigate the impact of small market size by charging a low price; the more volatile is the market size, the higher value responsive pricing provides to the firm.

The monotone decreasing property of \( V^*(S) \) in \( \rho \) in the case of \( b \leq 0 \) can be explained by the fact that the product complementarity and increasing market size correlation reinforcing the effect
of each other, and prices would have to be further increased to reduce the demand for inventory, and the firm’s value suffers. Product substitution, on the other hand, counterbalances the effect of increasing correlation of market size realizations, and hence, the firm’s value may increase in \( \rho \) in some cases (as we observed in numerical investigations, but not included in this paper). Proposition 5(3) highlights the difference between the stochastic correlation of market sizes (\( \rho \)) and the cross-price effect (\( b \)) in demand curves, which could not be modeled or differentiated in a standard newsvendor network model.

6. Extension and Conclusion

As an extension, we consider special cases of a multiplicative demand model. Let us define the demand functions as

\[ Q_i(P, \epsilon) = \epsilon_i (\bar{Q}_i - \alpha_i P_i + \beta P_j), \quad i = 1, 2, \quad j = 2, 1, \tag{23} \]

where \( \alpha_i > 0, |\beta| < \min\{\alpha_1, \alpha_2\} \), and \( \bar{Q}_i \) is a constant, \( i = 1, 2 \). Define

\[ b = \frac{\beta}{\alpha_1 \alpha_2 - \beta^2}, \quad a_i = \frac{\alpha_i}{\alpha_1 \alpha_2 - \beta^2}, \quad \bar{P}_i = a_i \bar{Q}_i + b \bar{Q}_j, \quad i = 1, 2, \quad j = 2, 1. \]

Then the inverse demand functions are

\[ P_i = \bar{P}_i - a_i \bar{Q}_i \epsilon_i^{-1} - b Q_j \epsilon_j^{-1}, \quad i = 1, 2, \quad j = 2, 1, \]

where \( \bar{P}_i \) is the market price of product \( i \) with \( Q = 0 \), which is also the upper bound of \( P_i, i = 1, 2 \); \( b \) satisfies \(|b| < \min\{a_1, a_2\}\), which is equivalent to \(|\beta| < \min\{\alpha_1, \alpha_2\}\).

The revenue function now is

\[ R(Q, \epsilon) = \sum_{i=1}^{2} P_i Q_i = \bar{P}_1 Q_1 + \bar{P}_2 Q_2 - a_1 Q_1^2 \epsilon_1^{-1} - a_2 Q_2^2 \epsilon_2^{-1} - b Q_1 Q_2 (\epsilon_1^{-1} + \epsilon_2^{-1}). \]

The negative Hessian of \( R(Q, \epsilon) \) is

\[ -H_R = \begin{vmatrix} 2a_1 \epsilon_1^{-1} & b (\epsilon_1^{-1} + \epsilon_2^{-1}) \\ b (\epsilon_1^{-1} + \epsilon_2^{-1}) & 2a_2 \epsilon_2^{-1} \end{vmatrix}. \]
which is not necessarily positive-definite for any realization of \( \epsilon \). Nevertheless, in the special case of \( b = 0 \), i.e., the no demand substitutability/complementarity case, we have \( \text{Det}(-H_R) = 4a_1a_2(\epsilon_1\epsilon_2)^{-1} > 0 \). (Recall that \( \epsilon \in \mathbb{R}^2_+ \).) Hence, the revenue concavity assumption is satisfied, and so our approach is applicable. We relegate the derivation of the optimal activity level \( x^* \) and the optimal inventory level \( S^* \) to the Appendix, which follows the procedure described in Section 4.

Following the same logic of the proof of Proposition 5, we have the following comparative statics.

**Proposition 6.** Assume a multiplicative demand function \( P_i = \bar{P}_i - a_i Q_i \epsilon_i^{-1} \), \( i = 1, 2 \). The maximal expected firm value \( V^*(S) \)

1. increases if \( \xi \) increases in usual stochastic order;
2. decreases if \( \xi \) increases in convex order;
3. decreases if \( \xi \) increases in supermodular order.

Comparing the above monotonic results with those in Proposition 5 we see that the maximal expected firm value \( V^*(S) \) behaves differently if \( \epsilon \) increases in convex order. Take the BVN distributed \( \xi \) as an instance. In the additive linear demand model the monotone increasing property of \( V^*(S) \) in \( \sigma \) is attributed to the power of responsive pricing, since the firm is able to take advantage of large market size by selling at a high price and mitigate the impact of small market size by charging a low price. In the multiplicative demand model pricing still plays a key role in response to demand variability but is less flexible in responding to different market size scenarios. To see this, let us compare the impact of \( \epsilon \) on pricing decisions for additive linear demand with \( b = 0 \), \( Q_i = \epsilon_i - \alpha_i P_i \) and multiplicative linear demand with \( b = 0 \), \( Q_i = \epsilon_i(\bar{Q}_i - \alpha_i P_i) \). For the additive linear demand, as \( \epsilon_i \) increases, we can always increase \( P_i \) proportionally so that \( Q_i \) at least does not decrease, and then the maximal expected firm value \( V^*(S) \) increases for sure. For multiplicative linear demand, however, because \( P_i < \bar{P}_i = a_i \bar{Q}_i \) is required to guarantee a positive demand (Recall

\[ k = \frac{\epsilon_1}{\epsilon_2} \]

To see this, let \( k = \epsilon_1/\epsilon_2 \) be the ratio of the two demand realizations. For \( k = 1 \) we have \( \text{Det}(-H_R) = 4(a_1a_2 - b^2)\epsilon_1^{-2} > 0 \), as \( |b| < \min\{a_1, a_2\} \) implies \( a_1a_2 - b^2 > 0 \). However, for \( k = 2 \), we have \( \text{Det}(-H_R) = (8a_1a_2 - 9b^2)\epsilon_1^{-2} \), which is negative for \( b > 2\sqrt{2a_1a_2}/3 \). In general, the sign of \( \text{Det}(-H_R) \) depends on the ratio \( k \) as well as the parameters \( a_i \) and \( b \).
\[ \alpha_i = 1/a_i \text{ for } b = 0, \]  
\[ P_i \text{ is bounded by } \bar{P} \text{ from the above. So, the power of responsive pricing vanishes as } P_i \text{ approaches to } \bar{P}, \text{ and hence the benefit that the firm can capture from the increase of market size diminishes. Consequently, the maximal expected firm value } V^*(S) \text{ decreases as demand variability increases.} \]

Besides the additive linear demand models and the special case of the multiplicative demand models with \( b = 0 \), our framework applies to other demand models, too, as long as the revenue function \( R(Q, \epsilon) \) is concave in \( Q \). For example, consider the isoelastic demand model with \( P_i(Q, \epsilon) = \theta_i Q_i^{-a_i} \epsilon_i, \theta_i \in \mathbb{R}^+, a_i \in (0, 1], i = 1, 2 \). As \( -\frac{\partial^2 R}{\partial Q_i^2} = a_i(1-a_i)\theta_i Q_i^{-1-a_i} \epsilon_i > 0, i = 1, 2 \), and \( \text{Det}(-H_R) = a_1a_2(1-a_1)(1-a_2)\theta_1 \theta_2 Q_1^{-1-a_1}Q_2^{-1-a_2} \epsilon_1 \epsilon_2 > 0 \), the revenue concavity assumption is satisfied. For more general demand models, such as the isoelastic demand with cross-price effect, more conditions may be needed to guarantee the revenue concavity.

In conclusion, we extend vMR’s newsvendor network model to environments where demands are sensitive to prices and firms have the flexibility of pricing responsively to the realized market condition. We show that the problem is well behaved under a rather generous assumption on revenue functions. We provide a guided approach to the optimal solutions, and demonstrate the similarity and difference between the two types of newsvendor network models in both the \textit{ex post} quantity allocation and the comparative statics of the firm’s value function. We believe that responsive pricing newsvendor networks, by enriching the modeling of the demand side, can serve as a building block for future research that incorporates intricate interaction between supply chain and markets.

\textbf{References}


7. Appendix

Remark 1. As $\epsilon$ is continuous in the market size space $\mathbb{R}_+^2$, it is easily seen that $\pi(S, \epsilon)$ and $\tau(x, \epsilon)$ are continuously differentiable in $\epsilon$, and thus satisfy the Leibniz’s rule (see Harrison and van Mieghem 1999). So, differentiation and integration can be interchanged in the following proofs.

Proof of Proposition 1. Since $P(Q, \epsilon)Q$ is joint concave in $Q$ and $R_Dx = Q$ holds, $P'(x, \epsilon)R_Dx$ is joint concave in $x$. Then one can verify that Assumptions 1 and 2 in page 123 of Birge and Louveaux (1997) are satisfied. Thus, according to Theorem 32 and Corollary 34 following the assumptions, $\pi(K, S, \epsilon)$ is joint concave in $(K, S)$ for any realization of $\xi$, and $V(K, S)$ is joint concave in $(K, S)$. Then the gradients follow. $
$

Proof of Property 1. The idea of our proof is similar to that of Results 1 and 2 of vMR. Since capacity 1 is dedicated to Activity 1 which depletes inventory 1 only, it is never optimal to set $K_1 > S_1$. Similarly, inventory 2 is depleted by Activity 2 through capacity 2 only, and so it is suboptimal to set $S_2 > K_2$. Thus, the optimal solutions must satisfy $S_1^* \geq K_1^*$ and $S_2^* \leq K_2^*$. Then, it remains to show that $S_1^* \geq K_1^*$ is binding and $K_2^*$ is upper-bounded by $S_2^* + \alpha^{-1}S_1^*$.

Now suppose $S_1^* > K_1^*$. On the one hand, although inventory 1 can be depleted by both Activities 1 and 3, the amount of $\Delta S_1 = S_1^* - K_1^* > 0$ is solely reserved for Activity 3, as Activity 1 is constrained by capacity $K_1^*$. On the other hand, although capacity 2 can be consumed by both Activities 2 and 3, the priority is given to Activity 2, and so Activity 3 consumes only the excessive capacity $K_2^* - S_2^*$. Considering the greater than 1 consumption rate of Activity 3, we must have $K_2^* - S_2^* > \Delta S_1$, i.e., $K_2^* > S_2^* + \Delta S_1$. So, it is feasible to reduce $S_1^*$ by $\Delta S_1$ and increase $S_2^*$ by the same amount, as $S_2^* \leq K_2^*$ still holds for the increased $S_2^*$. After the adjustment of the inventory levels, the expected operating profit increases, as the marginal profit of Activity 2 is higher than that of Activity 3, while the inventory procurement cost decreases, as $c_{S1} > c_{S2}$ by Assumption 4. Thus, the adjustment is beneficial and so it is suboptimal to set $S_1^* > K_1^*$.

Finally, as $x_2 \leq S_2^*$ and $x_3 \leq S_1^* \geq S_1^*$ ($x_1 + x_3 \leq S_1^*$ and $x_1 \geq 0$), the total of capacity 2 consumed by Activities 2 and 3 is no more than $S_2^* + \alpha^{-1}S_1^*$, and hence $K_2^*$ is upper-bounded by $S_2^* + \alpha^{-1}S_1^*$. $

Proof of Proposition 2. As all the interior solutions are derived from the FOC(s), it remains to prove the result of $\Omega_5$. We note that in $\Omega_5$ the constraint $x_1 + x_3 \leq S_1$ is binding, and so $(x^*_1, x^*_3)$ is not an interior solution. To solve it, a new pair of equations rather than the joint FOC’s is needed. The first equation is the binding constraint $x_1 + x_3 = S_1$. The second equation is derived from the fact that in $\Omega_5$, Activities 1 and 3 compete for inventory 1. Recall that $MR_2(x, \epsilon) = MR_3(x, \epsilon)$ holds for any $\epsilon$. So, under the reasonable assumption of $c_2 < c_3$ (Assumption 2), we have $\frac{\partial \tau}{\partial x_2}(x, \epsilon) > \frac{\partial \tau}{\partial x_3}(x, \epsilon)$, and hence, whenever Activities 2 and 3 compete for capacity 2, the priority is given to Activity 2. However, as there is no dominance by either $MR_1(x, \epsilon)$ or $MR_3(x, \epsilon)$, it is suboptimal to assign priority to either of Activities 1 and 3 for the entire market size space. In fact, for some $\epsilon$, it is optimal to set $x^*_i > 0$ for both $i = 1$ and 3. Such $\epsilon$’s define $\Omega_5$. Moreover, in equilibrium the optimality requires $\frac{\partial \tau}{\partial x_1}(x, \epsilon) = \frac{\partial \tau}{\partial x_3}(x, \epsilon)$, i.e., $MR_1((x_1, S_2, x_3), \epsilon) - c_1 = MR_2((x_1, S_2, x_3), \epsilon) - c_3$. □

Remark 2. The following results will be used in the proof of Propositions 4-5.

We argue that for any parameters $y$, the following equation holds in every region.

$$\frac{d\pi}{dy}(S, \epsilon) = \frac{\partial \tau}{\partial y}(x^*, \epsilon). \quad (24)$$

To be concise, we omit the domains of $\pi(S, \epsilon)$ and $\tau(x^*, \epsilon)$ whenever there is no ambiguity. So, it suffices to show that the second term in the following equation is equal to zero in each region.

$$\frac{d\pi}{dy} = \frac{\partial \tau}{\partial y}(x^*, \epsilon) + \sum_{i=1}^{3} \frac{\partial \tau}{\partial x_i^*} \frac{\partial x_i^*}{\partial y}.$$

According to the analysis in Section 4.3, all the optimal solutions $x^*_i$ (excluding $x^*_1$ and $x^*_3$ of $\Omega_5$, as they are not interior solutions) are derived from the first order condition(s) $\frac{\partial \tau}{\partial x_i^*} = 0$. Meanwhile, for all the boundary solutions $x^*_i = 0$ and $x^*_i = S_i$, including $x^*_1 = S_1 - S_3$ in region $\Omega_7$, $\frac{\partial x_i^*}{\partial y} = 0$ holds. Therefore, it remains to show that the second term in the above equation is equal to zero in $\Omega_5$. Recall that for $\Omega_5$ we have $x^*_2 = S_2$, $x^*_1 + x^*_3 = S_1$, and $\frac{\partial \tau}{\partial x_1^*} = \frac{\partial \tau}{\partial x_3^*}$. As $\frac{\partial x_i^*}{\partial y} = 0$ holds for $x^*_2 = S_2$ and $x^*_1 + x^*_3 = S_1$ implies $\frac{\partial x^*_1}{\partial y} = -\frac{\partial x^*_3}{\partial y}$, we have $\sum_{i=1}^{3} \frac{\partial \tau}{\partial x_i^*} \frac{\partial x_i^*}{\partial y} = \frac{\partial x^*_1}{\partial y} (\frac{\partial \tau}{\partial x_1^*} - \frac{\partial \tau}{\partial x_3^*}) = 0$.

Remark 3. The following FOC results will be used in the proof of Proposition 4.
Recall the *ex post* problem (20) and the constraints (12)-(15):

$$\pi(S, \epsilon) = \max \{ \tau(x, \epsilon) = (a_1 \epsilon_1 + b \epsilon_2)x_1 + (a_2 \epsilon_2 + b \epsilon_1)(x_2 + x_3) - a_1 x_1^2 - 2b x_1(x_2 + x_3) - a_2(x_2 + x_3)^2 - \sum_{i=1}^3 c_i x_i \}, \quad (25)$$

s.t. $x_1 + x_3 - S_1 \leq 0, \ (\eta_1)$

$x_2 - S_2 \leq 0, \ (\eta_2)$

$x_3 - S_3 \leq 0, \ (\eta_3)$

$x_i \geq 0, \ i = 1, 2, 3. \ (\zeta_i)$

where $\eta_i \geq 0$ and $\zeta_i \leq 0, \ i = 1, 2, 3, \eta$ are the Lagrange multipliers associated with the constraints.

The first order conditions are:

$$\frac{\partial \tau}{\partial x_1} = a_1(\epsilon_1 - 2x_1^*) + b[\epsilon_2 - 2(x_2^* + x_3^*)] - c_1 = \xi_1 + \zeta_1, \quad (30)$$

$$\frac{\partial \tau}{\partial x_2} = b(\epsilon_1 - 2x_1^*) + a_2[\epsilon_2 - 2(x_2^* + x_3^*)] - c_2 = \xi_2 + \zeta_2, \quad (31)$$

$$\frac{\partial \tau}{\partial x_3^*} = b(\epsilon_1 - 2x_1^*) + a_2[\epsilon_2 - 2(x_2^* + x_3^*)] - c_3 = \xi_1 + \xi_3 + \zeta_3, \quad (32)$$

where $\xi_i \geq 0$ and $\zeta_i \leq 0, \ i = 1, 2, 3, \xi$ satisfy the complementary slackness:

$$\xi_1(x_1^* + x_3^* - S_1) = 0,$$

$$\xi_i(x_i^* - S_i) = 0, \ i = 2, 3,$$

$$\zeta_i x_i^* = 0, \ i = 1, 2, 3.$$

**Proof of Proposition 4.** We show that for any constant $S, \frac{dV}{db}(S) \geq 0$ holds. It suffices to show that $\frac{dV}{db}(S, \epsilon) \geq 0$ holds everywhere in the market size space. By (24) we need to show $\frac{\partial \tau}{\partial b}(x^*, \epsilon) \geq 0$.

By (25),

$$\frac{\partial \tau}{\partial b}(x^*, \epsilon) = \epsilon_2 x_1^* + \epsilon_1(x_2^* + x_3^*) - 2x_1^*(x_2^* + x_3^*)$$

$$= \epsilon_2 x_1^* + (x_2^* + x_3^*)(\epsilon_1 - 2x_1^*)$$

$$= \epsilon_1(x_2^* + x_3^*) + x_1^*[\epsilon_2 - 2(x_2^* + x_3^*)].$$
As $\epsilon_i \geq 0$, $i = 1, 2$, and $x_j^* \geq 0$, $j = 1, 2, 3$, it is sufficient to show either $x_1^* \leq \epsilon_1/2$ or $x_2^* + x_3^* \leq \epsilon_2/2$ hold in every region in the market size space partition. Suppose neither of the two inequalities holds in some region. That is to say, both $x_1^* > \epsilon_1/2$ and $x_2^* + x_3^* > \epsilon_2/2$ holds. As $x_2^* = 0$ implies $x_3^* = 0$, $x_2^* + x_3^* > \epsilon_2/2$ implies $x_2^* > 0$. Also, $x_1^* > \epsilon_1/2$ means $x_1^* > 0$. So, we have $\zeta_i = 0$, $i = 1, 2$. Adding (30) and (31) gives

$$(a_1 + b)(\epsilon_1 - 2x_1^*) + (a_2 + b)(\epsilon_2 - 2(x_2^* + x_3^*)) - c_1 - c_2 = \eta_1 + \eta_2.$$ 

The LHS of the above equation is negative, as $x_1^* > \epsilon_1/2$ implies $\epsilon_2 - 2x_1^* < 0$, $x_2^* + x_3^* > \epsilon_2/2$ implies $\epsilon_2 - 2(x_2^* + x_3^*) < 0$, and $|b| < \min\{a_1, a_2\}$ implies $a_i + b > 0$, $i = 1, 2$. But the RHS is nonnegative, as $\eta_i \geq 0$ holds for $i = 1, 2$. So, $x_1^* > \epsilon_1/2$ and $x_2^* + x_3^* > \epsilon_2/2$ cannot hold in the same region, and thus for any region we have either $x_1^* \leq \epsilon_1/2$ or $x_2^* + x_3^* \leq \epsilon_2/2$. ■

**Proof of Proposition 5.**

1. By Definition 1, it suffices to show that $\pi(S, \xi)$ is a bounded increasing function. By (11)-(15), $\pi(S, \epsilon)$ is bounded for any $\epsilon$ (the realization of $\xi$). So, it remains to show that $\frac{\partial \pi}{\partial \epsilon}(S, \epsilon) \geq 0$ holds for all $\epsilon \in \mathbb{R}_+^2$. By (24) we need to show $\frac{\partial \pi}{\partial \epsilon}(x^*, \epsilon) \geq 0$ in each region. As $b \geq 0$, $a_i > 0$, $i = 1, 2$, and $x_j^* \geq 0$, $j = 1, 2, 3$, from (20) we have

$$\frac{\partial \tau}{\partial \epsilon_1}(x^*, \epsilon) = a_1 x_1^* + b(x_2^* + x_3^*) \geq 0,$$

$$\frac{\partial \tau}{\partial \epsilon_2}(x^*, \epsilon) = bx_1^* + a_2(x_2^* + x_3^*) \geq 0,$$

in each of the 15 regions in the market size space $\epsilon \in \mathbb{R}_+^2$. Moreover, by the definition of the boundary lines (where $x_i^*$ changes from a boundary solution to an interior one, and vice versa), both $\frac{\partial \tau}{\partial \epsilon_1}(x^*, \epsilon)$ and $\frac{\partial \tau}{\partial \epsilon_2}(x^*, \epsilon)$ are continuous at the boundaries. Thus, $\frac{\partial \tau}{\partial \epsilon}(x^*, \epsilon) \geq 0$ holds for all $\epsilon \in \mathbb{R}_+^2$, and so does $\frac{\partial \pi}{\partial \epsilon}(S, \epsilon) \geq 0$.

2. By Definition 1, it suffices to show that $\pi(S, \epsilon)$ is a convex function in the market size space $\mathbb{R}_+^2$. As $a_i > 0$, $i = 1, 2$, and $|b| < \min\{a_1, a_2\}$ implies $a_1 + a_2 - 2b > 0$, from Table 9 we see that $\tau(x, \epsilon)$ is convex in each of the 15 regions. Moreover, both $\frac{\partial \tau}{\partial \epsilon_1}(x^*, \epsilon)$ and $\frac{\partial \tau}{\partial \epsilon_2}(x^*, \epsilon)$ are continuous at the boundaries. So, for fixed $\epsilon_i$, at any boundary line $\frac{\partial^2 \tau}{\partial \epsilon_j^2}$ may be equal to the second order
derivative of the region in either side, \( i = 1, 2 \) and \( j = 2, 1 \). As all the second order derivatives are non-negative (see Table 9), \( \tau(x, \epsilon) \) is convex for all \( \epsilon \in \mathbb{R}_+^2 \), and so is \( \pi(S, \epsilon) \).

(3) By Definition 1, it suffices to show that \( -\pi(S, \epsilon) \) is a supermodular function in the market size space \( \mathbb{R}_+^2 \). As \( \frac{\partial^2 \tau}{\partial \epsilon_1 \partial \epsilon_2} \leq 0 \) for \( b \leq 0 \) in each of the 15 regions (see Table 9), according to Theorem 3.9.3 of Müller and Stoyan (2002), \( -\tau(x, \epsilon) \) is a supermodular function in each region, and so is \( -\pi(S, \epsilon) \). Then, as \( \epsilon \) increases in supermodular order, \( -V^*(S) \) increases and \( V^*(S) \) decreases.

Results for multiplicative linear demand

Recall that \( x_1 = Q_1 \) and \( x_2 + x_3 = Q_2 \). With \( b = 0 \) the marginal values can be calculated.

\[
MR_1(x, \epsilon) = \bar{P}_1 - 2a_1x_1\epsilon_1^{-1},
\]
\[
MR_i(x, \epsilon) = \bar{P}_2 - 2a_2(x_2 + x_3)\epsilon_2^{-1}, \ i = 2, 3.
\]

The first order conditions \( MR_i(x^*, \epsilon) = c_i, \ i = 1, 2, \) and \( MR_3(x^*, \epsilon) = c_3 \) with \( x_2^* = S_2 \) give the following property.

**Proposition 7.** Assume a multiplicative demand function \( P_i = \bar{P}_i - a_iQ_i\epsilon_i^{-1}, \ i = 1, 2 \). For all regions with interior solution(s), the optimal solutions are defined by

\[
x_1^* = \frac{1}{2a_1}(\bar{P}_1 - c_1)\epsilon_1, \ x_2^* = \frac{1}{2a_2}(\bar{P}_2 - c_2)\epsilon_2, \ x_3^* = \frac{1}{2a_2}(\bar{P}_2 - c_3)\epsilon_2 - S_2.
\]

For \( \Omega_5 \), \( x_1^* = \frac{c_1}{2(a_2\epsilon_1 + a_2\epsilon_2)}[2a_2(S_1 + S_2) + M\epsilon_2], \ x_2^* = \frac{1}{2(a_2\epsilon_1 + a_2\epsilon_2)}[2(a_1\epsilon_2S_1 - a_2\epsilon_1S_2) - M\epsilon_1\epsilon_2], \) where

\( M = \bar{P}_1 - \bar{P}_2 + c_3 - c_1. \)

As \( \tau(x, \epsilon) = R(x, \epsilon) - c'x \), we have

\[
\frac{\partial \tau}{\partial \epsilon_1}(x^*, \epsilon) = \frac{\partial R}{\partial \epsilon_1}(x^*, \epsilon) = a_1(x_1^*\epsilon_1^{-1})^2 \geq 0,
\]
\[
\frac{\partial \tau}{\partial \epsilon_2}(x^*, \epsilon) = \frac{\partial R}{\partial \epsilon_2}(x^*, \epsilon) = a_2[(x_2^* + x_3^*)\epsilon_2^{-1}]^2 \geq 0.
\]

The first and second order derivatives and the Hessians are given in Tables 10.
Table 4  The boundaries between the regions in each column, general demand

<table>
<thead>
<tr>
<th></th>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows 1 &amp; 2</td>
<td>$MR_2((0, S_2, S_3), \epsilon) = c_3$</td>
<td>$x_1^*(\Omega_{42}) = S_3$</td>
<td>$\Delta MR((S_1 - S_3, S_2, S_3), \epsilon) = c_3 - c_1$</td>
</tr>
<tr>
<td>Rows 2 &amp; 3</td>
<td>$MR_2((0, S_2, 0), \epsilon) = c_3$</td>
<td>$x_1^*(\Omega_{42}) = 0$</td>
<td>$\Delta MR((S_1, S_2, 0), \epsilon) = c_3 - c_1$</td>
</tr>
<tr>
<td>Rows 3 &amp; 4</td>
<td>$MR_2((0, S_2, 0), \epsilon) = c_2$</td>
<td>$x_2^*(\Omega_{44}) = S_2$</td>
<td>$MR_2((S_1, S_2, 0), \epsilon) = c_2$</td>
</tr>
<tr>
<td>Rows 4 &amp; 5</td>
<td>$MR_2((0, 0, 0), \epsilon) = c_2$</td>
<td>$x_2^*(\Omega_{44}) = 0$</td>
<td>$MR_2((S_1, 0, 0), \epsilon) = c_2$</td>
</tr>
</tbody>
</table>

†: $\Delta MR(x, \epsilon) = MR_2(x, \epsilon) - MR_1(x, \epsilon)$

Table 5  The boundaries between the columns, general demand

<table>
<thead>
<tr>
<th></th>
<th>Columns 1 &amp; 2</th>
<th>Columns 2 &amp; 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>$MR_1((0, S_2, S_3), \epsilon) = c_1$</td>
<td>$MR_1((S_1 - S_3, S_2, S_3), \epsilon) = c_1$</td>
</tr>
<tr>
<td>Row 2</td>
<td>$x_1^*(\Omega_{42}) = 0$</td>
<td>$x_1^<em>(\Omega_{42}) + x_1^</em>(\Omega_{42}) = S_1$</td>
</tr>
<tr>
<td>Row 3</td>
<td>$MR_1((0, S_2, 0), \epsilon) = c_1$</td>
<td>$MR_1((S_1, S_2, 0), \epsilon) = c_1$</td>
</tr>
<tr>
<td>Row 4</td>
<td>$x_1^*(\Omega_{44}) = 0$</td>
<td>$x_1^*(\Omega_{44}) = S_1$</td>
</tr>
<tr>
<td>Row 5</td>
<td>$MR_1((0, 0, 0), \epsilon) = c_1$</td>
<td>$MR_1((S_1, 0, 0), \epsilon) = c_1$</td>
</tr>
</tbody>
</table>

Table 6  The lines separating the regions in each column, general $b$

<table>
<thead>
<tr>
<th></th>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows 1 &amp; 2</td>
<td>$\epsilon_2 + \frac{b}{a_2}\epsilon_1 = \epsilon_1^0 + 2(S_2 + S_3)$</td>
<td>$\epsilon_2 = \epsilon_2 + 2(S_2 + S_3)$</td>
<td>†</td>
</tr>
<tr>
<td>Rows 2 &amp; 3</td>
<td>$\epsilon_2 + \frac{b}{a_2}\epsilon_1 = \epsilon_1^0 + 2S_2$</td>
<td>$\epsilon_2 = \epsilon_2 + 2S_2$</td>
<td>†</td>
</tr>
<tr>
<td>Rows 3 &amp; 4</td>
<td>$\epsilon_2 + \frac{b}{a_2}\epsilon_1 = \epsilon_1^0 + 2S_2$</td>
<td>$\epsilon_2 = \epsilon_2 + 2S_2$</td>
<td>$\epsilon_2 + \frac{b}{a_2}\epsilon_1 = \epsilon_1^0 + 2(S_2 + \frac{1}{a_2}S_1)$</td>
</tr>
<tr>
<td>Rows 4 &amp; 5</td>
<td>$\epsilon_2 + \frac{b}{a_2}\epsilon_1 = \epsilon_1^0$</td>
<td>$\epsilon_2 = \epsilon_2$</td>
<td>$\epsilon_2 + \frac{b}{a_2}\epsilon_1 = \epsilon_1^0 + 2S_1$</td>
</tr>
</tbody>
</table>

†: $(a_2 - b)[\epsilon_2 - 2(S_2 + S_3)] - (a_1 - b)[\epsilon_1 - 2(1 - S_3)] = c_3 - c_1$
†: $(a_2 - b)(\epsilon_2 - 2S_2) - (a_1 - b)(\epsilon_1 - 2S_1) = c_3 - c_1$

Table 7  The lines separating the columns, general $b$

<table>
<thead>
<tr>
<th></th>
<th>Column 1</th>
<th>Column 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows 1</td>
<td>$\epsilon_1 + \frac{b}{a_1}\epsilon_2 = \epsilon_1^0 + \frac{2b}{a_1}(S_2 + S_3)$</td>
<td>$\epsilon_1 + \frac{b}{a_1}\epsilon_2 = \epsilon_1^0 + 2(S_1 - S_3) + \frac{2b}{a_1}(S_2 + S_3)$</td>
</tr>
<tr>
<td>Rows 2</td>
<td>$\epsilon_1 = \epsilon_1^0$</td>
<td>$\epsilon_1 + \epsilon_2 = \epsilon_1 + \epsilon_2 + 2(S_1 + S_2)$</td>
</tr>
<tr>
<td>Rows 3</td>
<td>$\epsilon_1 + \frac{b}{a_1}\epsilon_2 = \epsilon_1^0 + \frac{2b}{a_1}S_2$</td>
<td>$\epsilon_1 + \frac{b}{a_1}\epsilon_2 = \epsilon_1^0 + 2(S_1 + \frac{1}{a_1}S_2)$</td>
</tr>
<tr>
<td>Rows 4</td>
<td>$\epsilon_1 = \epsilon_1^0$</td>
<td>$\epsilon_1 = \epsilon_1^0 + 2S_1$</td>
</tr>
<tr>
<td>Rows 5</td>
<td>$\epsilon_1 + \frac{b}{a_1}\epsilon_2 = \epsilon_1^0$</td>
<td>$\epsilon_1 + \frac{b}{a_1}\epsilon_2 = \epsilon_1^0 + 2S_1$</td>
</tr>
</tbody>
</table>
Table 8  First Order Derivatives of $\tau$ in $\epsilon$, additive demand

<table>
<thead>
<tr>
<th>Region</th>
<th>$\frac{\partial \tau}{\partial \epsilon_1}$</th>
<th>$\frac{\partial \tau}{\partial \epsilon_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_{01}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_{02}$</td>
<td>$\frac{1}{2}(a_1 \epsilon_1 + b \epsilon_2 - c_1)$</td>
<td>$\frac{b}{2a_1}(a_1 \epsilon_1 + b \epsilon_2 - c_1)$</td>
</tr>
<tr>
<td>$\Omega_{03}$</td>
<td>$\frac{b}{2a_2}(a_2 \epsilon_2 + b \epsilon_1 - c_2)$</td>
<td>$\frac{1}{2}(a_2 \epsilon_2 + b \epsilon_1 - c_2)$</td>
</tr>
<tr>
<td>$\Omega_{04}$</td>
<td>$\frac{1}{2}(a_1 \epsilon_1 + b \epsilon_2 - c_1)$</td>
<td>$\frac{1}{2}(a_2 \epsilon_2 + b \epsilon_1 - c_2)$</td>
</tr>
<tr>
<td>$\Omega_{11}$</td>
<td>$a_1 S_1$</td>
<td>$b S_1$</td>
</tr>
<tr>
<td>$\Omega_{12}$</td>
<td>$\frac{b}{2a_2}(a_2 \epsilon_2 + b \epsilon_1 - c_2) + \frac{a_1 a_2 - b^2}{a_2} S_1$</td>
<td>$\frac{1}{2}(a_2 \epsilon_2 + b \epsilon_1 - c_2)$</td>
</tr>
<tr>
<td>$\Omega_{21}$</td>
<td>$b S_2$</td>
<td>$a_2 S_2$</td>
</tr>
<tr>
<td>$\Omega_{22}$</td>
<td>$\frac{1}{2}(a_1 \epsilon_1 + b \epsilon_2 - c_1)$</td>
<td>$\frac{b}{2a_1}(a_1 \epsilon_1 + b \epsilon_2 - c_1) + \frac{a_1 a_2 - b^2}{a_2} S_2$</td>
</tr>
<tr>
<td>$\Omega_{3}$</td>
<td>$a_1 S_1 + b S_2$</td>
<td>$b S_1 + a_2 S_2$</td>
</tr>
<tr>
<td>$\Omega_{41}$</td>
<td>$\frac{b}{2a_2}(a_2 \epsilon_2 + b \epsilon_1 - a_2 \epsilon_2)$</td>
<td>$\frac{1}{2}(a_2 \epsilon_2 + b \epsilon_1 - a_2 \epsilon_2)$</td>
</tr>
<tr>
<td>$\Omega_{42}$</td>
<td>$\frac{1}{2}(a_1 \epsilon_1 + b \epsilon_2 - c_1)$</td>
<td>$\frac{1}{2}(a_2 \epsilon_2 + b \epsilon_1 - c_3)$</td>
</tr>
<tr>
<td>$\Omega_{61}$</td>
<td>$b(S_2 + S_3)$</td>
<td>$a_2 (S_2 + S_3)$</td>
</tr>
<tr>
<td>$\Omega_{62}$</td>
<td>$\frac{1}{2}(a_1 \epsilon_1 + b \epsilon_2 - c_1)$</td>
<td>$\frac{b}{2a_1}(a_1 \epsilon_1 + b \epsilon_2 - c_1) + \frac{a_1 a_2 - b^2}{a_2} (S_2 + S_3)$</td>
</tr>
<tr>
<td>$\Omega_{7}$</td>
<td>$a_1 (S_1 - S_3) + b(S_2 + S_3)$</td>
<td>$b(S_1 - S_3) + a_2(S_2 + S_3)$</td>
</tr>
</tbody>
</table>

For $\Omega_5$:
\[
\frac{\partial \tau}{\partial \epsilon_1} = \frac{a_1 - b}{2(a_1 + a_2 - 2b)} [(a_1 - b) \epsilon_1 - (a_2 - b) \epsilon_2 c_1 + c_3] + \frac{a_1 a_2 - b^2}{2(a_1 + a_2 - 2b)} (S_1 + S_2)
\]
\[
\frac{\partial \tau}{\partial \epsilon_2} = \frac{a_2 - b}{2(a_1 + a_2 - 2b)} [(a_2 - b) \epsilon_2 - (a_1 - b) \epsilon_1 c_1 - c_3] + \frac{a_1 a_2 - b^2}{2(a_1 + a_2 - 2b)} (S_1 + S_2)
\]

Table 9  Second Order Derivatives of $\tau$ in $\epsilon$, additive demand

| $\Omega_i$ | $\frac{\partial^2 \tau}{\partial \epsilon_1^2}$ | $\frac{\partial^2 \tau}{\partial \epsilon_2^2}$ | $\frac{\partial^2 \tau}{\partial \epsilon_1 \partial \epsilon_2}$ | $||H||$ |
|------------|---------------------------------|---------------------------------|-------------------|----|
| $01, 11, 21, 3, 61, 7$ | 0 | 0 | 0 | 0 |
| $02$ | $\frac{a_1^2}{2}$ | $\frac{b^2}{2}$ | $\frac{1}{2}(a_1 a_2 - b^2)$ | 0 |
| $03$ | $\frac{b^2}{2}$ | $\frac{a_2^2}{2}$ | $\frac{1}{2}(a_1 a_2 - b^2)$ | 0 |
| $04$ | $\frac{b^2}{2}$ | $\frac{a_2^2}{2}$ | $\frac{1}{2}(a_1 a_2 - b^2)$ | 0 |
| $12$ | $\frac{a_1^2}{2}$ | $\frac{b^2}{2}$ | $\frac{1}{2}(a_1 a_2 - b^2)$ | 0 |
| $22$ | $\frac{b^2}{2}$ | $\frac{a_2^2}{2}$ | $\frac{1}{2}(a_1 a_2 - b^2)$ | 0 |
| $41$ | $\frac{b^2}{2}$ | $\frac{a_2^2}{2}$ | $\frac{1}{2}(a_1 a_2 - b^2)$ | 0 |
| $42$ | $\frac{a_1^2}{2}$ | $\frac{b^2}{2}$ | $\frac{1}{2}(a_1 a_2 - b^2)$ | 0 |
| $62$ | $\frac{a_1^2}{2}$ | $\frac{b^2}{2}$ | $\frac{1}{2}(a_1 a_2 - b^2)$ | 0 |
| $5$ | $\frac{(a_1 - b)^2}{2(a_1 + a_2 - 2b)}$ | $\frac{(a_2 - b)^2}{2(a_1 + a_2 - 2b)}$ | $\frac{(a_1 - b)(a_2 - b)}{2(a_1 + a_2 - 2b)}$ | 0 |
### Table 10 Derivatives of $\tau$ in $\epsilon$, multiplicative demand with $b = 0$

<table>
<thead>
<tr>
<th>Region</th>
<th>$\frac{\partial \tau}{\partial \epsilon_1}$</th>
<th>$\frac{\partial \tau}{\partial \epsilon_2}$</th>
<th>$\frac{\partial^2 \tau}{\partial \epsilon_1^2}$</th>
<th>$\frac{\partial^2 \tau}{\partial \epsilon_2^2}$</th>
<th>$\frac{\partial^2 \tau}{\partial \epsilon_1 \partial \epsilon_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_{01}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_{02}$</td>
<td>$\frac{1}{4a_1} (\bar{P}_1 - c_1)^2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_{03}$</td>
<td>0</td>
<td>$\frac{1}{4a_2} (\bar{P}_2 - c_2)^2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_{04}$</td>
<td>$\frac{1}{4a_1} (\bar{P}_1 - c_1)^2$</td>
<td>$\frac{1}{4a_2} (\bar{P}_2 - c_2)^2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_{05}$</td>
<td>$a_1 S_1^2 \epsilon_1^{-2}$</td>
<td>0</td>
<td>$-2a_1 S_1^2 \epsilon_1^{-3}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_{06}$</td>
<td>0</td>
<td>$a_2 S_2^2 \epsilon_2^{-2}$</td>
<td>0</td>
<td>$-2a_2 S_2^2 \epsilon_2^{-3}$</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_{07}$</td>
<td>$a_1 (S_1 - S_3)^2 \epsilon_1^{-2}$</td>
<td>$a_2 (S_2 + S_3)^2 \epsilon_2^{-2}$</td>
<td>$-2a_1 (S_1 - S_3)^2 \epsilon_1^{-3}$</td>
<td>$-2a_2 (S_2 + S_3)^2 \epsilon_2^{-3}$</td>
<td>0</td>
</tr>
</tbody>
</table>

For $\Omega_5$:

$$
\frac{\partial \tau}{\partial \epsilon_1} = \frac{a_1}{4(a_2 \epsilon_1 + a_1 \epsilon_2)^2} \left[ 2a_2 (S_1 + S_2) + M \epsilon_2 \right]^2,
$$

$$
\frac{\partial \tau}{\partial \epsilon_2} = \frac{a_2}{4(a_2 \epsilon_1 + a_1 \epsilon_2)^2} \left[ 2a_1 (S_1 + S_2) - M \epsilon_1 \right]^2,
$$

$$
\frac{\partial^2 \tau}{\partial \epsilon_1^2} = -\frac{a_1 a_2}{2(a_2 \epsilon_1 + a_1 \epsilon_2)^3} \left[ 2a_2 (S_1 + S_2) + M \epsilon_2 \right]^2,
$$

$$
\frac{\partial^2 \tau}{\partial \epsilon_2^2} = -\frac{a_1 a_2}{2(a_2 \epsilon_1 + a_1 \epsilon_2)^3} \left[ 2a_1 (S_1 + S_2) - M \epsilon_1 \right]^2,
$$

$$
\frac{\partial^2 \tau}{\partial \epsilon_1 \partial \epsilon_2} = -\frac{a_1 a_2}{2(a_2 \epsilon_1 + a_1 \epsilon_2)^3} \left[ 2a_2 (S_1 + S_2) + M \epsilon_2 \right] \left[ 2a_1 (S_1 + S_2) - M \epsilon_1 \right]^{\dagger},
$$

where $M = \bar{P}_1 - \bar{P}_2 + c_3 - c_1$.

For $\Omega_3$: $|\mathbb{H}| = 4a_1 a_2 (S_1 S_2)^2 (\epsilon_1 \epsilon_2)^{-3}$.

For $\Omega_7$: $|\mathbb{H}| = 4a_1 a_2 (S_1 - S_3)^2 (S_2 + S_3)^2 (\epsilon_1 \epsilon_2)^{-3}$.

For all other regions: $|\mathbb{H}| = 0$.

†: The cross-derivative can be rewritten as $\frac{\partial^2 \tau}{\partial \epsilon_1 \partial \epsilon_2} = -\frac{2a_1 a_2 S_1 (S_2 + x_1)}{x_1 \epsilon_2 (a_2 \epsilon_1 + a_1 \epsilon_2)} < 0$. 