Two-Wholesale-Price Contracts: Push, Pull, and Advance-Purchase Discount Contracts

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The allocation of inventory ownership affects the inventory availability in a supply chain, which in turn determines the supply chain performance. In this paper, we consider a supplier-retailer supply chain in which the supplier starts production well in advance of the selling season, and the retailer is offered two ordering opportunities at different points in time. An early order is allowed before the supplier’s production decision, and a late order is allowed after the completion of production and after observing the demand. When the two wholesale prices change, we illustrate how the inventory decision rights and ownership are shifted and/or shared between the two firms, resulting in push, pull, or advance-purchase discount contracts. We then characterize the complete set of Pareto-dominant contracts for any given two-wholesale-price contract. We find that Pareto improvement can be achieved when inventory ownership is shifted from individual to shared and sometimes vice versa. In the latter case, push contracts not only are more likely to offer Pareto improvement but also can achieve higher supply chain efficiency than pull contracts. We also identify conditions that enable Pareto improvement by introducing a new ordering opportunity to firms who were bound by a single ordering opportunity without renegotiating the existing wholesale price, and we demonstrate through a numerical study that the adoption of the new ordering opportunity can significantly improve supply chain efficiency. We show that such Pareto improvement is more likely to happen when demand is more volatile.

1 Introduction

Both the inventory available in the supply chain and the inventory allocation among supply chain partners greatly affect supply chain performance. In a study of wholesale price contracts in a supplier-retailer supply chain, Cachon (2004) shows that the retailer and the supplier can significantly improve supply chain efficiency (the ratio of the supply chain profit to its maximum profit) if they shift and/or adjust inventory responsibility between them; the allocation of inventory responsibility is in turn determined by the supply contracts adopted in the supply chain.

Consider a supplier who has to decide the production quantity well in advance of the selling season (e.g., due to a long production leadtime). The retailer might place an early order (before the
supplier’s production decision) and/or a late order (shortly before or during the selling season after observing demand). If only one ordering opportunity is available to the retailer, then the inventory responsibility is borne either by the retailer alone or by the supplier alone, depending on the timing of whichever ordering opportunity is available. Specifically, if only the early order is available, then the corresponding contract assumes the nature of push, meaning that the system inventory is completely pushed to the retailer when demand is uncertain and that the supplier merely responds to the retailer’s early order (i.e., the supplier sells to a newsvendor retailer). If only the late order is available, then the corresponding contract takes on the nature of pull, meaning that the supplier assumes the full inventory responsibility in the supply chain and that the retailer merely pulls inventory from the supplier as demand occurs, though he is constrained by the supplier’s inventory availability (i.e., the retailer buys from a newsvendor supplier). It is also possible for both firms to share inventory responsibility between themselves when both ordering opportunities are available, often with some discounts applied to the early order. The resulting contract is then referred to as an advance-purchase discount contract. While the single-purchase practice is widespread for products with a long production leadtime and a short selling season, a double/multiple-purchase practice is not uncommon in fashion industries, as exemplified by Sport Obermeyer and other apparel manufacturers (Fisher et al. 1994 and Simon 2000).

The pricing of the ordering opportunities is vitally important since it determines whether the supply chain partners will exploit or ignore an ordering opportunity and whether such an opportunity offers Pareto improvement to the supply chain. This paper studies push, pull, and advance-purchase discount contracts in a unified two-wholesale-price contract framework. Instead of viewing the push and pull contracts as resulting from the availability of the early or late ordering opportunity, we find it convenient to view them as consequences of various wholesale price settings for the two ordering opportunities (Cachon 2004 is among the first who took this view in studying push and pull contracts). This treatment allows us to understand the well-studied push and pull systems in a more general setting and to see how inventory decision rights, ownership, and risk are transferred from one firm to the other when wholesale prices change. Such understanding holds the key to addressing the following questions, which are the focus of this paper: (1) How can firms achieve Pareto improvement by adjusting wholesale prices in their two-wholesale-price contracts? (2) What Pareto improvement opportunities associated with a new ordering opportunity are open to firms who were bound by a single ordering opportunity at a given wholesale price?

Our work is primarily related to research that studies the effect of the retailer’s purchasing timing in a system where the supplier’s production occurs once, well in advance of the selling
season, when the demand is uncertain. Much of this literature focuses on comparing two distinct alternatives: the retailer commits to inventory either early or late. Ferguson (2003) and Ferguson et al. (2005) study how firms’ preferences for one ordering time over the other are affected by the power structure of the supply chain and the demand uncertainty reduction achieved between the two points in time. Perakis and Roels (2006) establish bounds on the efficiency loss of push and pull contracts in Stackelberg game settings relative to the centralized supply chain. The timing of the retailer’s order placement not only affects the inventory ownership but also influences the retailer’s effort both to promote demand and to acquire and share demand information. Taylor (2006), from the perspective of a powerful supplier, examines several facets of the timing problem, including the implications of the retailer’s sales effort and his private demand information in a price-sensitive retail environment.

Within works that assume the availability of both the early and the late ordering opportunities, Deng and Yano (2002) study the impact of a late ordering opportunity on a push system, assuming that the price of the late order is random. Özer et al. (2005) show that the supply chain can achieve Pareto improvement if an early ordering opportunity is added to a push system at the time when the retailer knows even less about the demand. Erhun et al. (2000) focus on a capacitated supply chain in which the supplier has the pricing power and the system faces a two-state uncertainty in a linear demand function. Cachon and Lariviere (2001) study credible demand forecast sharing through a contract consisting of firm commitments and options, when the downstream firm has private demand forecast and dominating power. Özer and Wei (2006) examine the use of advance-purchase contracts for the same purpose, but assume that the upstream firm has dominating power; they show that advance purchase is a strategic quantity commitment that enables the downstream firm to reveal its private forecast information and illustrate Pareto improvement of the advance-purchase contract with the pull contract as one of the benchmarks.

Our work differs from the above research in that we adopt a unified two-wholesale-price contract framework that treats both push and pull contracts as special cases to study Pareto improvement opportunities available to systems that use any of the three contracts: push, pull, and advance-purchase discount. More importantly, we identify the effect of wholesale pricing on the inventory decision rights and ownership in the absence of the “usual” reasons for advance-purchase discounts such as information asymmetry and promotion/information acquisition effort – hence the fundamental relationship between inventory ownership and profit allocation under wholesale price contracts.

Cachon (2004) is the closest to our work in that he considers push and pull contracts as special
cases of the two-wholesale-price contract, and he introduces and studies the Pareto contract set in this context (a contract is Pareto efficient if there does not exist an alternative contract such that no firm is worse off and one firm is strictly better off). He shows that the single-wholesale-price Pareto set consists of both the push and pull contracts that bring the system inventory above a certain threshold. He also demonstrates that advance-purchase discount contracts can achieve full supply chain efficiency and allow arbitrary allocation of profits. However, the implementation of Pareto-efficient contracts sometimes requires firms to drastically change existing wholesale prices, which might not be feasible since price negotiations are often quite sensitive, involving multiple parties in both firms. Thus, it is important to study the behavior of the two-wholesale-price system and provide insights into the achievement of Pareto improvement through contracts of various price settings. Our work complements Cachon (2004) in that (1) we offer a more complete analytical treatment of two-wholesale-price contracts, allowing us to characterize analytically (in Section 2) some of the interesting insights that he finds numerically; (2) we characterize the complete set of Pareto improvement contracts for any given wholesale price contract, a result that allows firms with practical pricing constraints to identify Pareto improvement opportunities and also leads to the identification of the Pareto set. The latter distinction enables us to show further interesting results: we provide conditions under which Pareto improvement can be achieved by introducing a new ordering opportunity without changing the existing wholesale price, and we demonstrate through a numerical study that supply chain efficiency can be improved significantly. We also find that when demand becomes more volatile, Pareto improvement is more likely to be achieved through introducing the new ordering opportunity.

The outline of this paper is as follows. In Section 2, we set up the two-wholesale-price contract model, establish the relationship between wholesale prices and three possible operating regimes, conduct sensitivity analysis over wholesale prices, and characterize Pareto improvement opportunities for any given two-wholesale-price contract. In Section 3, we illustrate the magnitude of supply chain efficiency improvement associated with a new ordering opportunity. Finally, we give concluding remarks in Section 4.

2 A Model of Two-Wholesale-Price Contracts

We consider a single-retailer-single-supplier system – the same as the one studied by Cachon (2004). The retailer (he) buys a product from the supplier (she) and sells it over a single selling season. The demand $\xi$ in the selling season is uncertain and is assumed to follow a probability distribution with density function $f$ and cumulative distribution function $F$. We further assume that the demand
distribution has an increasing general failure rate (IGFR) with mean \( \mu \), i.e., \( \frac{\xi f(\xi)}{1 - F(\xi)} \) increases in \( \xi \) (e.g., Lariviere and Porteus 2001).

The event sequence is as follows. Well in advance of the selling season, the retailer prebooks \( q_r1 \) and pays \( w_1q_r1 \) for the prebook quantity. Upon receiving the prebook order from the retailer, the supplier produces a quantity of \( q_s \) at unit production cost \( c \). We assume that, due to the long production leadtime, the supplier has only one production opportunity before the selling season and has to fully satisfy the retailer’s prebook quantity, i.e., \( q_s \geq q_r1 \). When the selling season starts, the retailer observes the demand \( \xi \), decides the at-once quantity \( q_r2 \) to order from the supplier, and pays \( w_2q_r2 \). The supplier is not obligated to fulfill the retailer’s at-once order; thus the retailer’s at-once order is limited by the supplier’s production quantity \( q_s \), i.e., \( q_r2 \leq q_s - q_r1 \).

Wholesale prices \( \{w_1, w_2\} \) are set before orders and production take place. The setting of the wholesale prices can be either the result of industry competition or the outcome of negotiation between the supplier and the retailer. In this study, we abstract from the modeling of price formation, which allows us to focus on demonstrating Pareto improvement opportunities for various wholesale price levels. During the selling season, the retailer receives \( p \) for each unit sold. Without loss of generality, we assume zero loss of goodwill cost and salvage value. The results can be extended easily to the case of a positive loss of goodwill cost and to the case of the same salvage value for the supplier and the retailer. We also assume a symmetric information environment where demand and financial information are public knowledge.

The integrated supply chain maximizes the system’s total profit

\[
\Pi(q) = pS(q) - cq,
\]

where \( S(q) \equiv E[\min\{\xi, q\}] \) gives the expected sales with \( q \) units in stock. The optimal system profit is defined as \( \Pi^* \), which is achieved at \( q^* = F^{-1}\left(1 - \frac{c}{p}\right) \).

In a decentralized system, for a given wholesale price pair \( \{w_1, w_2\} \), the retailer and the supplier will choose \( q_r1 \) and \( q_s \), respectively, to maximize their expected profits. The inventory commitment of the retailer and the supplier \( \{q_r1, q_s\} \) falls into one of three general regimes:

1. **Pull**: \( q_r1 = 0 \). The retailer takes no inventory responsibility and pulls inventory from the supplier after he observes the demand realization.

2. **Partial advance-booking (PAB)**: \( 0 < q_r1 < q_s \). The retailer commits to a positive amount of inventory, but the supplier has “left-over” inventory \( q_s - q_r1 \) available for the at-once order.

3. **Push**: \( q_r1 = q_s \). The retailer takes full inventory responsibility, leaving the supplier with no “left-over” inventory available for the at-once order.
We first establish the mapping between the wholesale price space \( \{w_1, w_2\} \) and the inventory allocation space \( \{q_{r1}, q_s\} \). For ease of exposition, we will use push/pull contracts in the context of single-wholesale-price contracts, and push/pull/PAB-regime contracts in the context of two-wholesale-price contracts. For instance, by a push contract, we mean a single-wholesale-price contract that allows the early ordering opportunity, and by a push-regime contract, we mean a two-wholesale-price contract that runs in the push regime. To avoid trivial cases, we will focus on the wholesale price region \( \{w_1, w_2\} \in [c, p] \times [c, p] \)\(^1\).

We start our analysis by solving for the supplier’s production decision \( q_s \). Given the wholesale price pair \( \{w_1, w_2\} \) and the retailer’s prebook quantity \( q_{r1} \), the supplier’s decision problem is

\[
\max_{q_s \geq q_{r1}} \{w_1 q_{r1} - c q_s + w_2 [S(q_s) - S(q_{r1})]\},
\]

where \( S(q_s) - S(q_{r1}) \) is the expected at-once order placed by the retailer. By offering two ordering opportunities, the supplier is effectively a newsvendor with the obligation to satisfy the early order in full. We denote the supplier’s optimal production quantity as \( q_s^* \).

Under the assumption of symmetric information, the retailer is aware of the supplier’s production decision \( q_s^* \) and thus decides upon the prebook quantity \( q_{r1} \) to maximize his expected profit:

\[
\max_{q_{r1} \geq 0} \{p S(q_s^*) - w_1 q_{r1} - w_2 [S(q_s^*) - S(q_{r1})]\}.
\]

**Lemma 1**

1. The supplier’s expected profit is concave in the production quantity \( q_s \) and the optimal production quantity is \( q_s^* = \max \{q_{r1}, K_s\} \), where \( K_s = F^{-1} \left( 1 - \frac{c}{w_2} \right) \).

2. The retailer’s expected profit is locally concave in \( q_{r1} \) on two intervals, \([0, K_s]\) and \([K_s, \infty)\), respectively, but is not globally concave in \( q_{r1} \). The local optima are achieved at \( K_a \in [0, K_s) \), and at \( K_r \in [K_s, \infty) \) if and only if (iff) \( w_1 w_2 \leq pc \), where \( K_a = F^{-1} \left( 1 - \frac{w_1}{w_2} \right) \) and \( K_r = F^{-1} \left( 1 - \frac{w_1}{p} \right) \).

Lemma 1 describes the concavity property of the supplier’s expected profit and the non-global concavity of the retailer’s expected profit, and also provides several important quantity thresholds that are helpful to our understanding of the inventory decision-making. Specifically, the at-once ordering opportunity offers the supplier an opportunity to determine the inventory level for the supply chain by taking on the inventory risk at unit overage cost \( c \) and unit underage cost \( w_2 - c \). Thus, \( K_s \) is the supplier’s unconstrained optimal production quantity, i.e., the inventory responsibility that the supplier is willing to take without the retailer’s prebook, and is the system total

\(^1\)The retailer will not accept wholesale prices \( \{w_1, w_2\} > (p, p) \), and the supplier will reject \( w_1 < c \) regardless of the value of \( w_2 \). The wholesale prices \( w_1 > p \) and \( w_2 < c \) will also be rejected since the retailer will choose zero commitment and the supplier will choose not to produce at all. When \( w_1 \in (c, p) \) and \( w_2 < c \), or when \( w_1 \in [c, p] \) and \( w_2 > p \), the push regime will rule. It is also obvious that when \( w_1 > p \) and \( w_2 \in [c, p] \), the pull regime will rule.
inventory if the supply chain runs in the pull or PAB regime. Hereafter, we refer to $K_s$ as the supplier’s newsvendor quantity. The prebook ordering opportunity offers the retailer an opportunity to take the inventory responsibility at unit overage cost $w_1$. If the retailer chooses to prebook less than the supplier’s newsvendor quantity, i.e., $q_r \in [0, K_s)$, then his unit underage cost is $w_2 - w_1$; if he prebooks more than the supplier’s newsvendor quantity, i.e., $q_r \in [K_s, \infty)$, then he takes the full inventory responsibility with a unit underage cost $p - w_1$. Quantity $K_a$ ($K_r$) is the optimal partial (full) inventory responsibility that the retailer is willing to take. Thus, a comparison between taking partial and full inventory responsibility is necessary for the retailer when $0 \leq K_a < K_s \leq K_r$, i.e., when $w_1 w_2 \leq pc$, with $w_1 w_2 = pc$ representing the wholesale settings that yield $K_r = F^{-1} \left( 1 - \frac{w_1}{p} \right) = F^{-1} \left( 1 - \frac{c}{w_2} \right) = K_s$ (the retailer’s taking full inventory responsibility leads to the same system inventory level as what the supplier offers if the retailer takes partial inventory responsibility). Note that, unless $w_1 = c$ or $w_2 = p$, the supply chain suffers from double marginalization and is understocked (i.e., stocks less than the system optimal $q^\ast$). Define

$$
\pi_{r1} (w_1, w_2) \equiv pS(K_s) - w_1 K_a - w_2 [S(K_s) - S(K_a)]
$$

(1a)

$$
= (p - w_1) \mu - [w_1 E(K_a - \xi)^+ + (w_2 - w_1) E(\xi - K_a)^+ + (p - w_2) E(\xi - K_s)^+],
$$

(1b)

$$
\pi_{r2} (w_1, w_2) \equiv pS(K_r) - w_1 K_r
$$

(1c)

$$
= (p - w_1) \mu - [w_1 E(K_r - \xi)^+ + (p - w_1) E(\xi - K_r)^+],
$$

(1d)

$$
g (w_1, w_2) \equiv \pi_{r2} (w_1, w_2) - \pi_{r1} (w_1, w_2).
$$

Effectively, $\pi_{r1}$ (resp. $\pi_{r2}$) represents the retailer’s optimal expected profit by taking partial (resp. full) inventory responsibility, i.e., $q_r = K_a < K_s$ (resp. $q_r = K_r \geq K_s$), and $g$ represents the retailer’s expected profit advantage of making a full inventory commitment instead of a partial inventory commitment. Note that we provide alternative representations of $\pi_{r1}$ and $\pi_{r2}$ in (1b) and (1d) respectively, which interpret the retailer’s profit as the profit under perfect demand information (PDI), i.e., no demand uncertainty, minus the cost due to demand uncertainty (CU) consisting of inventory overage and underage costs.

**Lemma 2** For a given $w_2 \in (c, p)$,

1. $g (w_1, w_2)$ decreases in $w_1$ for $w_1 \in \left[ c, \frac{pc}{w_2} \right]$;
2. there exists a unique $\bar{w}_1 (w_2) \in \left( c, \frac{pc}{w_2} \right)$ such that $\bar{w}_1 (w_2)$ is the solution to $g (\cdot, w_2) = 0$, thus $g (w_1, w_2) > 0$ for $w_1 \in [c, \bar{w}_1 (w_2))$ and $g (w_1, w_2) < 0$ for $w_1 \in \left( \bar{w}_1 (w_2), \frac{pc}{w_2} \right]$.

Lemma 2 states that for a given at-once wholesale price $w_2$, the retailer’s profit advantage from full inventory commitment decreases as the prebook wholesale price $w_1$ increases, reaching zero
at \( \tilde{w}_1(w_2) \) and becoming negative when \( w_1 \) exceeds that threshold (Deng and Yano 2002 prove the same result in a study of the impact of the late ordering opportunity). The threshold \( \tilde{w}_1(w_2) \) represents a dividing line in the wholesale price space \( \{w_1, w_2\} \) between the push regime and the pull\( \cup \)PAB regime. The fact that \( \tilde{w}_1(w_2) < \frac{pw_2}{w_2} \) implies that the retailer prefers the pull or PAB regime to the push regime when \( w_1 \in (\tilde{w}_1(w_2), \frac{pw_2}{w_2}] \), i.e., even when the push regime can offer higher system inventory \( (K_r > K_s) \) and higher supply chain efficiency. This is because the unit overage cost \( w_1 \in (\tilde{w}_1(w_2), \frac{pw_2}{w_2}] \) is still too high for the retailer to make full inventory commitment, and the availability of the at-once ordering opportunity enables the retailer to make partial inventory commitment and gain profit at the expense of supply chain efficiency and the supplier’s profit.

Theorem 1 summarizes both the retailer’s and the supplier’s behavior under various combinations of wholesale prices, as illustrated in Figure 1.

**Theorem 1** The system runs

1. in the push regime \( (q^*_r = q^*_s = K_r) \) when \( w_1 \in [c, \tilde{w}_1(w_2)] \) and \( w_2 \in [c, p] \);
2. in the PAB regime \( (q^*_r > 0 \text{ and } q^*_s = K_s) \) when \( w_1 \in (\tilde{w}_1(w_2), w_2] \) and \( w_2 \in [c, p] \);
3. in the pull regime \( (q^*_r = 0, q^*_s = K_s) \) when \( w_1 \in \max \{\tilde{w}_1(w_2), w_2\}, p \} \) and \( w_2 \in [c, p] \).

In Figure 1, the dark area represents the push regime, the white area represents the pull regime, and the gray area represents the PAB regime. The curve of \( \tilde{w}_1(w_2) \) is represented by the right-side boundary of the dark area, which is bounded above by curve \( w_1w_2 = pc \). Figure 1 is representative of the operating-regime structure under two-wholesale-price contracts, in which the \( \tilde{w}_1(w_2) \) curve shifts as parameters of the demand distribution change (more explanations are provided in Corollary 3 and illustrations are given subsequently in Section 3). It is clear that \( \tilde{w}_1(w_2) \) is not monotone in \( w_2 \), which means that this boundary cannot be described as a function of \( w_1 \). Alternatively, for a given \( w_1 \), we define sets \( F(w_1) \equiv \{w_2 : g(w_1, w_2) \geq 0\} \) and \( P(w_1) \equiv \{w_2 : g(w_1, w_2) < 0\} \) such that the system runs in the push regime and the pull\( \cup \)PAB regime at \( \{w_1, w_2\} \) for \( w_2 \in F(w_1) \) and \( w_2 \in P(w_1) \), respectively. Lemma 3 further characterizes \( \tilde{w}_1(w_2) \) in terms of \( w_2 \) and \( F(w_1) \) in terms of \( w_1 \).

**Lemma 3** (1) There exists a unique \( w_2^0 \) such that \( \tilde{w}_1(w_2^0) = w_2^0 \).

2. \( F(w_1) \supseteq F(w'_1) \) for \( w_1 < w'_1 \) and \( w_1, w'_1 \in [c, p] \).

The boundary line \( \tilde{w}_1(w_2) \) crosses the line \( w_1 = w_2 \) exactly once at \( w_2 = w_2^0 \). Figure 1 shows that when \( w_2 \leq w_2^0 \), the retailer should be willing to take full inventory responsibility even with the prebook wholesale price no less than the at-once wholesale price, i.e., with \( w_1 \in [w_2, \tilde{w}_1(w_2)] \). This is
Figure 1: The effects of wholesale prices on the supply chain operating regimes.
Note. Demand follows a standard Gamma distribution with shape = (0.3)^{-2}, \( p = 10, c = 7.5 \).

because the supplier’s newsvendor quantity at a low at-once wholesale price is so low for the retailer
that he would rather take full inventory responsibility by prebooking more than the supplier chooses
to offer than take partial inventory responsibility and be constrained by the imposed inventory
limitation on the at-once order. Cachon (2004) uses “the push challenge to the pull contract” to refer
to the possibility that the retailer may prebook inventory even if \( w_1 = w_2 \). Lemma 3(1) identifies the
at-once wholesale price range \([c, w_2^\circ] \) within which the push challenge to the pull contract succeeds.
Moreover, it shows that the retailer will consider taking partial inventory responsibility only when
the at-once wholesale price \( w_2 \) exceeds \( w_2^\circ \), i.e., when the supplier’s underage cost \( w_2 - c \) drives her
to produce a large enough quantity so that at a reasonable prebook wholesale price \( w_1 \) the retailer
is better off taking partial rather than full or zero inventory responsibility. Lemma 3(2) states that
\( F(w_1) \) expands as \( w_1 \) decreases, i.e., the lower the prebook wholesale price, the more attractive
the full inventory commitment is to the retailer.

Figure 1 offers us an interesting perspective from which to view contracts. Well-studied single-
order contracts such as push and pull are special cases of the two-wholesale-price contract. In
particular, a push contract with wholesale price \( w_1 \in [c, p] \) can be viewed as a two-wholesale-price contract
with \( \{w_1, c\} \) (represented by the bottom borderline of Figure 1); that is, with an at-once
ordering opportunity offered at price \( c \), the supplier will not produce beyond the amount that the
retailer prebooks. Similarly, a pull contract with wholesale price \( w_2 \in [c, p] \) can be viewed as a
two-wholesale-price contract with \( \{p, w_2\} \) (represented by the right borderline of Figure 1); that
is, when the prebook wholesale price is offered at the full retail price, the retailer is not willing
to prebook. Although the push system and the pull system appear to be very different, yielding opposite inventory ownerships in the supply chain, we see now that the supply chain can migrate from one inventory ownership to the other by properly adjusting the two wholesale prices.

Cachon (2004) studies the Pareto set within the push and pull contracts, which is equivalent to considering contracts corresponding to \( \{w_1, c\} \) and \( \{p, w_2\} \) with \( w_1, w_2 \in [c, p] \). As we discussed previously, the \( w_1w_2 = pc \) curve represents wholesale price settings at which the push contract with wholesale price \( w_1 \) and the pull contract with wholesale price \( w_2 \) achieve the same level of system inventory and supply chain efficiency. Cachon shows that the push and pull contracts with system inventory level above a threshold \( q^P \) belong to the Pareto set of single-wholesale-price contracts, which requires \( w_1 \leq w_1^P \) for push and \( w_2 \geq w_2^P \) for pull, where \( F^{-1} \left( 1 - \frac{w_1^P}{p} \right) = F^{-1} \left( 1 - \frac{c}{w_2^P} \right) = q^P \) (see Figure 1). Hereafter, we refer to the supply chain efficiency achieved by \( q^P \) as the lower-bound-efficiency for the single-wholesale-price Pareto set. When considering the Pareto set among push, pull, and advance-purchase discount contracts, he finds that advance-purchase discount contracts with \( \{w_1, p\} \) (which correspond to the top borderline of Figure 1) can achieve 100% supply chain efficiency. Thus, the implementation of the contracts in the Pareto set may require significant adjustment of the wholesale prices to achieve the desired system inventory level and profit allocation.

Our work complements Cachon (2004) by characterizing the system behavior at any wholesale price settings and the corresponding set of Pareto-dominant contracts (as will be shown below), which can help firms identify profit improvement opportunities, especially when two negotiating firms face practical constraints in the adjustment of wholesale prices (Iyer and Bergen 1997).

For any given contract to achieve Pareto improvement, the system profit has to increase and neither party can be worse off. As a step towards identifying Pareto improvement opportunities, we now study how the system inventory level, the profit of the retailer, the profit of the supplier, and the system profit change as wholesale prices \( w_1 \) and \( w_2 \) change respectively. The retailer’s profit function is summarized below:

\[
\pi_r (w_1, w_2) = \begin{cases} 
(p - w_1) \mu - \left[ w_1 E (K_r - \xi)^+ + (p - w_1) E (\xi - K_r)^+ \right] & w_1 \leq \bar{w}_1 (w_2) \ [push] \\
(p - w_1) \mu - \left[ w_1 E (K_a - \xi)^+ + (w_2 - w_1) E (\xi - K_a)^+ \right] + (p - w_2) E (\xi - K_s)^+ & \bar{w}_1 (w_2) < w_1 \leq w_2 \ [PAB] \\
(p - w_2) \mu - (p - w_2) E (\xi - K_s)^+ & w_1 > \max \{\bar{w}_1 (w_2), w_2\} \ [pull]
\end{cases}
\]

The supplier’s profit is \( \pi_s = \Pi (K_r) - \pi_r \) in the push regime and \( \pi_s = \Pi (K_s) - \pi_r \) in the PAB and
pull regimes, that is,

\[
\pi_s(w_1, w_2) = \begin{cases} 
(w_1 - c) \mu - (w_1 - c) E (\xi - K_s)^+ - (w_1 - c) E (K_r - \xi)^+ & w_1 \leq \bar{w}_1(w_2) \quad \text{[push]} \\
(w_1 - c) \mu - (w_2 - c) E (\xi - K_s)^+ + cE (K_s - \xi)^+ & \bar{w}_1(w_2) < w_1 \leq w_2 \quad \text{[PAB]} \\
-w_1E (K_a - \xi)^+ - (w_2 - w_1) E (\xi - K_a)^+ & w_1 > \max \{\bar{w}_1(w_2), w_2\} \quad \text{[pull]} 
\end{cases}
\]  

(3)

Lemma 4 characterizes the effect of the prebook wholesale price \( w_1 \) (as illustrated in Figure 2).

**Lemma 4** For a given \( w_2 \in [c, p] \), we have

1. \( K_s \) is independent of \( w_1; K_a \) and \( K_r \) decrease in \( w_1 \).
2. \( \pi_r + \pi_s \) decreases in \( w_1 \) for \( w_1 \in [c, \bar{w}_1(w_2)] \) and is constant for \( w_1 \in (\bar{w}_1(w_2), p] \).
3. \( \pi_s \) is unimodal in \( w_1 \) for \( w_1 \in [c, \bar{w}_1(w_2)] \), increases for \( w_1 \in (\bar{w}_1(w_2), w_2] \), and is constant for \( w_1 \in (\max \{\bar{w}_1(w_2), w_2\}, p] \). Moreover, for any \( w_2 \in (c, p] \), \( \pi_s(\bar{w}_1(w_2), w_2) = \lim_{\varepsilon \to 0^+} \pi_s(\bar{w}_1(w_2) + \varepsilon, w_2) \).
4. \( \pi_r \) decreases in \( w_1 \) for \( w_1 \in [c, \max \{\bar{w}_1(w_2), w_2\}] \) and is constant for \( w_1 \in (\max \{\bar{w}_1(w_2), w_2\}, p] \).

![Figure 2](image-url)  

Figure 2: The effect of \( w_1 \) given \( w_2 = 8.3 \) (\( \bar{w}_1(8.3) = 8.15 \)). (a) Production and order quantities as functions of \( w_1 \), with the system inventory level highlighted. (b) Profits as functions of \( w_1 \). Note. Demand follows a standard Gamma distribution with shape= \( (0.3)^{-2} \), \( p = 10 \), \( c = 7.5 \).
PAB regime, the retailer’s profit continues to decrease in \( w_1 \) while the supplier’s profit increases as the retailer buys more through the at-once order. In the pull regime, both firms’ profits are no longer affected by \( w_1 \). Overall, the changes of the prebook wholesale price \( w_1 \) adjust the inventory and profit allocation between the supplier and the retailer without altering the system inventory, except when it is small enough to induce the push regime and the decreasing of its value increases the system inventory.

Interestingly, Lemma 4(3) shows that when the inventory decision rights shift from the supplier to the retailer at \( \{\tilde{w}_1(w_2), w_2\} \) (i.e., the operating regime changes from PAB or pull to push), the retailer sets the system inventory level significantly higher than the supplier’s newsvendor quantity (i.e., \( K_r > K_s \)), and the supplier’s profit (and hence the system profit) increases discontinuously. That is, when the retailer’s overage cost \( w_1 \) decreases to \( \tilde{w}_1(w_2) \), his interest is aligned with that of the system. He takes full inventory responsibility and brings the system inventory level from \( K_s \) to \( K_r \), which recovers the system efficiency lost in the PAB regime and allocates it to the supplier. Thus, the supply chain achieves Pareto improvement when the inventory decision rights shift from the supplier to the retailer. As an interesting contrast, when \( w_1 \) increases to \( w_2 \), the retailer’s profit decreases and the supplier’s profit increases smoothly to the constant pull-regime profit levels and no Pareto improvement occurs, suggesting that a pull-regime contract is the limit of the PAB-regime contract (in terms of both inventory levels and firms’ profits) as the retailer’s inventory commitment decreases to zero. This contrast is important to understand our later finding that preferences should be given to push contracts when firms seek for push- or pull-regime Pareto-dominant contracts for PAB contracts.

Lemma 5 characterizes the effect of the at-once wholesale price \( w_2 \) (as illustrated in Figure 3).

**Lemma 5** For a given \( w_1 \in [c, p] \), we have

1. \( K_s, K_a, \) and \( K_a/K_s \) all increase in \( w_2 \); \( K_r \) is independent of \( w_2 \).
2. Both \( \pi_s \) and \( \pi_s + \pi_r \) are independent of \( w_2 \) for \( w_2 \in \mathcal{F}(w_1) \) and increase for \( w_2 \in \mathcal{P}(w_1) \).
3. \( \pi_r \) is independent of \( w_2 \) for \( w_2 \in \mathcal{F}(w_1) \) and is unimodal for \( w_2 \in \mathcal{P}(w_1) \cap \{w_2 : w_2 < w_1\} \).

When the prebook wholesale price \( w_1 \) is fixed, the increase of the at-once wholesale price \( w_2 \) creates incentives for the supplier to produce more and for the retailer to prebook more, and the retailer’s relative inventory responsibility in the PAB regime, \( K_a/K_s \), increases as \( w_2 \) increases.

As \( w_2 \) increases, the operating regime of the supply chain can follow different transition paths depending on the value of \( w_1 \): push-PAB, push-pull-push-PAB, or push-pull-PAB, as illustrated in Figure 1. In the push regime, neither the retailer’s nor the supplier’s profits are affected by
In the PAB or pull regime, the system and the supplier’s profits increase in $w_2$ because the system total inventory, $K_s$, increases. Effectively, the retailer’s profit corresponds to the difference between two increasing functions and thus can be of any shape. In particular, in the pull regime, the retailer’s profit is unimodal, which is a result of the trade-off between decreasing profit margin and increasing expected sales. In the PAB regime, the prebook adjustment, $-w_1K_a + w_2S(K_a)$ (in (1a)), increases as $w_2$ increases and can produce multiple modes in the retailer’s profit when combined with the unimodal pull profit. Overall, the increase of the at-once wholesale price increases the system total inventory except in the push regime, where it cedes the control of inventory to the prebook wholesale price.

![Graph showing production and order quantities](image1)
![Graph showing profits](image2)

Figure 3: The effect of $w_2$ given $w_1 = 8.15$. (a) Production and order quantities as functions of $w_2$, with the system inventory level highlighted. (b) Profits as functions of $w_2$. Note. Demand follows a standard Gamma distribution with shape $\theta = 0.3^{-2}$, $p = 10$, $c = 7.5$.

Define $w_1^*$ as the supplier’s optimal prebook price in the push regime and $w_2^*$ as the retailer’s optimal at-once price in the pull regime, i.e., $w_1^* \equiv \arg \max_{w_1} \{ \pi_s(w_1, c) \}$ and $w_2^* \equiv \arg \max_{w_2} \{ \pi_r(p, w_2) \}$. Theorem 2 characterizes the complete set of Pareto-dominant contracts for any given two-wholesale-price contract. Figure 4(a) illustrates the Pareto-dominant sets for nine two-wholesale-price contracts with “+” indicating the location of the given contract $\{\hat{w}_1, \hat{w}_2\}$ in the wholesale price space and with the dark area representing its Pareto-dominant set.

**Theorem 2** Assume that a two-wholesale-price contract $\{\hat{w}_1, \hat{w}_2\}$ is used by a retailer and his supplier with resulting profit $\hat{\pi}_r$ and $\hat{\pi}_s$, respectively. If $\hat{w}_1 = c$ or $\hat{w}_2 = p$, then the contract is Pareto efficient; otherwise, the set of Pareto-dominant contracts is a union of three sets located in the three regimes: $S_{push}$, $S_{pull}$, and $S_{PAB}$, where

$S_{push} = \{ (w_1, w_2) : w_1 \in (w_1^L, w_1^R), w_2 \in F(w_1) \}$, where $w_1^L$ is the solution to $\pi_s(\cdot, c) = \hat{\pi}_s$ within...
[c, w_1^*], and w_1^R is the solution to \( \pi_r (\cdot, c) = \hat{\pi}_r \);

\[ S_{\text{pull}} = \{ \{w_1, w_2\} : w_2 \in (w_2^L, w_2^R), w_1 \in (\max \{w_2, \bar{w}_1 (w_2)\}, p) \} \],

where \( w_2^R \) is the solution to \( \pi_r (p, \cdot) = \hat{\pi}_r \) within \((w_2^L, p)\), and \( w_2^L \) is the solution to \( \pi_s (p, \cdot) = \hat{\pi}_s \) if \( \{\hat{w}_1, \hat{w}_2\} \in \text{push regime} \) and otherwise is the solution to \( \pi_r (p, \cdot) = \hat{\pi}_r \) within \([c, w_2^R]\); and

\[ S_{\text{PAB}} = \{ \{w_1, w_2\} : w_2 \in (w_2, p), w_1 \in (\max \{W_1^L(w_2), \bar{w}_1 (w_2)\}, \min \{W_1^R(w_2), w_2\}) \} \],

where \( w_2 = \max \left\{ w_2^L, \frac{w_2}{\alpha} \right\} \) if \( \{\hat{w}_1, \hat{w}_2\} \in \text{push regime} \) and otherwise \( w_2 = \hat{w}_2 \), and \( W_1^L(w_2) \) and \( W_1^R(w_2) \) are the solutions to \( \pi_s (\cdot, w_2) = \hat{\pi}_s \) and \( \pi_r (\cdot, w_2) = \hat{\pi}_r \) for any given \( w_2 \), respectively.

Figure 4: (a) Illustration of the Pareto dominant set for push-regime contracts (i)-(iii), for pull-regime contracts (iv)-(vi), and for PAB-regime contracts (vii)-(ix). (b) Partition of the wholesale price space demonstrating that types of Pareto dominant contracts can vary for different contracts. Note. Demand follows a standard Gamma distribution with shape = 2, \( p = 10, c = 4 \).

The characterization of the set of Pareto-dominant contracts allows firms with various pricing constraints to fully evaluate their Pareto improvement opportunities and price negotiation possibilities. Strict Pareto dominance has two requirements: the system profit increases, and neither party is worse off. For an understocked supply chain, the system inventory has to increase to improve
the system profit. That is, double marginalization has to be mitigated by setting wholesale prices closer to the system optimal (i.e., \( w_1 \) closer to \( c \), and/or \( w_2 \) closer to \( p \)). In fact, Theorem 2 states that Pareto efficiency can be achieved only at \( w_1 = c \) (the left borderline of \( [c,p] \times [c,p] \)), where the retailer captures the optimal system profit completely, or at \( w_2 = p \) (the top borderline of \( [c,p] \times [c,p] \)), where any division of the optimal system profit can be achieved through PAB-regime contracts by changing \( w_1 \). This is an extension to Theorem 7 of Cachon (2004). By the continuity argument, firms with Pareto-inferior contracts (a contract is Pareto inferior if some other contract exists that makes one firm better off and no firm worse off) can always find a set of PAB-regime contracts that are strictly Pareto dominant. Because the retailer’s profit decreases in \( w_1 \) and the supplier’s profit increases in \( w_1 \) in the PAB regime, the boundary of the PAB-regime Pareto-dominant set is specified by \( W^L_1(w_2) \) and \( W^R_1(w_2) \), with the retailer preferring a \( w_1 \) lower than \( W^R_1(w_2) \) and the supplier preferring a \( w_1 \) higher than \( W^L_1(w_2) \).

Push- and pull-regime contracts can also offer Pareto improvement but in different, and limited, ways. From Lemmas 4 and 5 (Figures 2 and 3) we see that a push (pull)-regime contract increases the system profit through decreasing the prebook (increasing the at-once) wholesale price. Such price adjustments are biased towards the retailer (supplier) in the sense that they can increase the retailer’s (supplier’s) profit all the way to the system optimal, and may first increase but eventually decrease the supplier’s (retailer’s) profit to zero.

If firms look for push-regime contracts for Pareto improvement, then it is sufficient to focus on the bottom boundary line \( \{ w_1, c \} \) with \( w_1 \in [c,p] \). The lower bound \( w^L_1 \) of the push-regime Pareto-dominant set \( S_{push} \) represents the lowest \( w_1 \) the supplier can accept, and the upper bound \( w^R_1 \) represents the highest \( w_1 \) the retailer and the system can accept. If firms look for pull-regime contracts for Pareto improvement, then it is sufficient to focus on the right boundary line \( \{ p, w_2 \} \) with \( w_2 \in [c,p] \). The lower bound \( w^L_2 \) of the pull-regime Pareto-dominant set \( S_{pull} \) represents the lowest \( w_2 \) the supplier (retailer) can accept if the current contract is in the push (pull or PAB) regime, and \( w^R_2 \) represents the highest \( w_2 \) the retailer can accept.

Not all contracts can be Pareto improved by push- or pull-regime contracts: because of its bias towards the retailer (supplier), the key factor for a push(pull)-regime contract to offer Pareto improvement is whether it can improve the supplier’s (retailer’s) profit. Theorem 3 states necessary and sufficient conditions under which the set of Pareto-dominant contracts in push (and pull) regimes is not empty. Figure 4(b) shows the partition of the wholesale price space with “-PUSH” (“-PULL”) representing contracts whose Pareto-dominant sets do not include push-regime (pull-regime) contracts.
Theorem 3 Notations \( \{ \hat{w}_1, \hat{w}_2 \} \), \( S_{\text{push}} \), and \( S_{\text{pull}} \) are as defined in Theorem 2.

(1) If \( \{ \hat{w}_1, \hat{w}_2 \} \in \text{push regime} \), \( S_{\text{push}} \neq \emptyset \) iff \( \hat{w}_1 \in (w_1^*, p) \), and \( S_{\text{pull}} \neq \emptyset \) iff \( \hat{w}_1 \in (w_1^P, p) \);

(2) if \( \{ \hat{w}_1, \hat{w}_2 \} \in \text{pull regime} \), \( S_{\text{push}} \neq \emptyset \) iff \( \hat{w}_2 \in [c, w_2^P] \), and \( S_{\text{pull}} \neq \emptyset \) iff \( \hat{w}_2 \in [c, w_2^*] \); and

(3) if \( \{ \hat{w}_1, \hat{w}_2 \} \in \text{PAB regime} \), \( S_{\text{push}} \neq \emptyset \) iff \( \hat{w}_2 \in (w_2^P, p) \) and \( \hat{w}_1 \in \left( \bar{w}_1(\hat{w}_2), \min \left\{ w_1^{\text{push}}(\hat{w}_2), \hat{w}_2 \right\} \right) \), where \( w_1^{\text{push}}(\hat{w}_2) \) is the solution to \( \pi_r (\cdot, \hat{w}_2) = \pi_r \left( \frac{p-c}{\hat{w}_2}, c \right) \), and \( S_{\text{pull}} \neq \emptyset \) iff \( \hat{w}_2 \in (w_2^P, w_2^*) \) and \( \hat{w}_1 \in \left( \max \left\{ w_1^{\text{pull}}(\hat{w}_2), \bar{w}_1(\hat{w}_2) \right\}, \hat{w}_2 \right) \), where \( w_1^{\text{pull}}(\hat{w}_2) \) is the solution to \( \pi_r (\cdot, \hat{w}_2) = \pi_r (p, w_2^*) \).

The conditions specified in Theorem 3 give firms clear directions on the types of contracts they can negotiate to achieve Pareto improvement. Theorem 3(1) states that a supply chain under a push-regime contract can achieve Pareto improvement by adopting another push-regime (a pull-regime) contract only when the current prebook wholesale price \( \hat{w}_1 \) is higher than \( w_1^* \) (higher than \( w_1^P \)), in which case the supplier’s (retailer’s) profit is low enough to be improved when the retailer’s (supplier’s) profit is improved by decreasing \( w_1 \) (increasing \( w_2 \)). A similar explanation applies to Theorem 3(2), the case of a supply chain under a pull-regime contract.

Those conditions (as illustrated in Figure 4(b)) also explain a systematic pattern shown in Figure 4(a) (i)-(iii) and (iv)-(vi): as the wholesale price of a push (pull) contract increases (decreases), the system profit decreases, and more contracts, as well as types of contracts, should become available to offer Pareto improvement.

Theorem 3(3) provides necessary and sufficient conditions for a most surprising result: PAB-regime contracts can be Pareto dominated by push- or pull-regime contracts, meaning that switching from shared to individual inventory ownership can improve the standing of both parties. Intuitively, this can happen to the PAB-regime contracts with low total system profit (i.e., with a low at-once wholesale price \( \hat{w}_2 \)). Moreover, push-regime contracts can provide Pareto improvement to supply chains with a high system profit but with the supplier capturing only a small portion of it (i.e., with high \( \hat{w}_2 \) but low \( \hat{w}_1 \)). In an extreme case, for any PAB contracts with \( \hat{w}_1 \) set close enough to \( \hat{w}_1 \), as Lemma 4(3) indicates, a switch to the push regime with a prebook wholesale price slightly lower than \( \hat{w}_1 \) will increase the supplier’s profit discontinuously. On the other hand, pull-regime contracts cannot Pareto improve a PAB-regime contract under which the total system profit is high but the retailer’s profit is low. This is because, as we notice in the discussion of Lemma 4, the retailer’s profit decreases to his pull-regime profit as the prebook wholesale price increases and the PAB regime converges to the pull regime. If a PAB-regime contract already has the at-once wholesale price higher than \( w_2^* \), a pull-regime contract that raises \( w_2 \) to increase the system profit can only decrease the retailer’s profit further. This explains the interesting observation in Figure
4(b) that the set of PAB contracts that can be Pareto dominated by push contracts corresponds to a significant portion of the PAB regime, which includes those that can be Pareto dominated by pull contracts.

Corollary 1 further demonstrates the supply chain efficiency advantage of push contracts over pull contracts in Pareto improvement for PAB contracts.

**Corollary 1** Assume that the demand has an increasing failure rate (IFR), \( \{ \hat{w}_1, \hat{w}_2 \} \in \text{PAB regime} \), and \( w_1^R \) and \( w_2^R \) are as defined in Theorem 2. Then, \( \pi_r (w_1^R, c) + \pi_s (w_1^R, c) > \pi_r (p, w_2^R) + \pi_s (p, w_2^R) \) and \( w_1^R < w_1^P \).

We note that the demand having an IFR (i.e., \( f(\xi) / (1 - F(\xi)) \) increases in \( \xi \)) is a stronger assumption than IGFR in the sense that the demand has relatively fat tails. This is not a restrictive assumption because many well-known distributions such as uniform and normal, as well as Gamma and Weibull distributions with shape parameters greater than 1, have IFR. Under this assumption, Corollary 1 states that when a PAB contract can be Pareto improved by both push and pull contracts, the lowest efficiency achieved by push contracts is higher than the highest efficiency achieved by pull contracts, and is higher than the lower-bound-efficiency of the single-wholesale-price Pareto set (achieved by \( q^P \)). This result suggests that individual inventory ownership should often be granted to the retailer, which is an even more appealing scenario because push contracts typically involve a single transportation arrangement instead of the possible multiple shipments under pull contacts. The rationale of this result is as follows. Recall that the PAB regime is possible only when the supplier’s newsvendor quantity is high enough that the retailer is willing to take only partial inventory responsibility at a reasonable prebook wholesale price. When the demand has IFR, the possibility of understock is relatively high. The PAB regime has to ensure a relatively high inventory availability, which leads to a relatively high retailer’s profit in the PAB regime. Because the pull regime is limited in improving the retailer’s profit and the push regime can increase the retailer’s profit up to the system optimal, a high retailer PAB-profit will restrict the pull Pareto-dominant contracts from reaching high supply chain efficiency, but will lead to the push Pareto-dominant contracts reaching high supply chain efficiency.

For firms that are currently using a single-wholesale-price contract and are considering introducing a new ordering opportunity, it is valuable to know when Pareto improvement is feasible without renegotiating the existing price, which can simplify the adoption of the new ordering opportunity.

**Proposition 1** Assume that a single-wholesale-price contract is used by a retailer and his supplier. (1) If it is a push contract with wholesale price \( \hat{w}_1 \), then an at-once ordering opportunity brings
strict Pareto improvement iff the at-once wholesale price \( w_2 \in (\bar{w}_2, p] \), where \( \bar{w}_2 > pc/\hat{w}_1 \) is the unique solution to \( \pi_s(\hat{w}_1, \cdot) = \pi_s(\hat{w}_1, c) \) for \( w_2 \in P(\hat{w}_1) \). Moreover, \( \bar{w}_2 < \hat{w}_1 \) iff \( \hat{w}_1 > W_1 \), where \( W_1 \) solves \( \pi_s(W_1, W_1) = \pi_s(W_1, c) \).

(2) If it is a pull contract with wholesale price \( \bar{w}_2 \), then there exists a \( W_2 > w_{\circ}^2 \) such that a prebook ordering opportunity brings strict Pareto improvement iff \( \bar{w}_2 \leq W_2 \) and the prebook wholesale price \( w_1 \in [\hat{w}_1, \bar{w}_1(\bar{w}_2)] \), where \( \bar{w}_1 \) satisfies \( \pi_s(\bar{w}_1, c) = \pi_s(p, \bar{w}_2) \) and \( \bar{w}_1 < \min\{w_{\ast}^1, \bar{w}_2\} \).

A new ordering opportunity offers a possibility for one firm to take over the system inventory decision from the other and to mitigate double marginalization by setting the corresponding new wholesale price closer to the system optimal. For the retailer, a new ordering opportunity, either early or late, is an option that gives him the right but not the obligation to place an order, and thus it will never make him worse off as long as the price for the existing ordering opportunity remains the same. For the supplier, the early order is an obligation to fulfill, and the late order is an option to decide on an inventory level for the system, so the effects of the two ordering opportunities are different. Proposition 1 shows that adding an at-once ordering opportunity with a properly chosen at-once wholesale price always brings strict Pareto improvement to a push system, but adding a prebook ordering opportunity does not always bring Pareto improvement to a pull system. We will illustrate the magnitude of supply chain efficiency improvement in Section 3.

Proposition 1(1) shows that for a push system, the at-once opportunity benefits the supplier only when the corresponding price is set to exceed \( \bar{w}_2 \) so that the supplier decides to produce more than the retailer’s push quantity to improve the system profit and to change the operating regime to pull or PAB. The threshold \( \bar{w}_2 \) represents the at-once wholesale price threshold above which the supplier is better off with the resulting non-push-regime contract. Moreover, pull-regime Pareto-dominant contracts exist only when \( \hat{w}_1 > W_1 \), i.e., when the retailer’s push profit is low enough to be improved by pull contracts.

For a pull system, a prebook ordering opportunity allows the supplier to off-load some of her inventory risk onto the retailer at the cost of a prebook discount, but will not bring Pareto improvement unless the retailer reacts by prebooking more than the supplier offers in the pull system so as to improve the system profit and to change the operating regime to push. Proposition 1(2) shows that a prebook ordering opportunity can offer Pareto improvement iff the existing at-once wholesale price is lower than threshold \( W_2 \), below which the supplier’s pull profit is low enough to be improved by push contracts.

Proposition 1 has interesting implications for firms’ contract preferences when only single-
wholesale-price contracts are considered, as shown in Corollary 2.

**Corollary 2** For firms considering the adoption of a single-wholesale-price contract with a given wholesale price \( w \), we have

1. if \( w < w_2^0 \), both the retailer and the supplier prefer a push contract;
2. if \( w \in [w_2^0, W_1] \), the retailer prefers a pull contract and the supplier prefers a push contract;
3. if \( w \in [W_1, p] \), both the retailer and the supplier prefer a pull contract.

Single-wholesale-price contracts, due to their simplicity, can appeal to firms. Corollary 2 shows that when the wholesale price is low (high), both parties will prefer push (pull) contracts, which enables high system inventory levels and high supply chain efficiency; when the wholesale price is moderate, i.e., between \( w_2^0 \) and \( W_1 \), each party prefers that the other party take on the inventory responsibility. Cachon (2004) shows numerically that the firms can be better off switching from the supplier’s optimal push contract with \( w_1^* \) to a pull contract, or from the retailer’s optimal pull contract with \( w_2^* \) to a push contract, keeping the wholesale price the same. Corollary 2 analytically characterizes this transition for a broader set of contracts.

**Corollary 3** Assume that the demand follows a normal distribution \( N(\mu_N, \sigma_N) \). Then

1. \( \tilde{w}_2 \) is nonincreasing in \( \sigma_N \) for any given \( w_1 \);
2. \( \tilde{w}_1(w_2) \) is nondecreasing and \( \tilde{w}_1 \) is nonincreasing in \( \sigma_N \) for any given \( w_2 \).

Because the Pareto-dominant set for a given push contract with prebook wholesale price \( \hat{w}_1 \) is characterized by \( w_2 \in (\tilde{w}_2(\hat{w}_1), p] \), and because the Pareto-dominant set for a given pull contract with at-once wholesale price \( \tilde{w}_2 \) is characterized by \( w_1 \in [\tilde{w}_1(\tilde{w}_2), \tilde{w}_1(\hat{w}_2)] \), Corollary 3 states that under a normal demand distribution, these Pareto-dominant sets will not contract as the demand uncertainty increases. In other words, letting the other party take over or share inventory responsibility might yield more Pareto-dominant contracts. Let us consider the Pareto-dominant set \( (\tilde{w}_2(\hat{w}_1), p] \) for push contracts, where \( \tilde{w}_2(\hat{w}_1) \) is the at-once wholesale price threshold above which the supplier’s pull or PAB profit is greater than her push profit (recall that the retailer is never worse off when the prebook wholesale price remains the same). Note that expression (3) suggests that at \( \{\hat{w}_1, \tilde{w}_2(\hat{w}_1)\} \), the supplier’s PDI profit under push is no less than it is under pull or PAB, so the reason that contract \( \{\hat{w}_1, \tilde{w}_2(\hat{w}_1)\} \) offers Pareto improvement must be that the supplier’s CU under a pull or PAB contract is less than it is under a push contract. Since CU is proportionate to the standard deviation of demand, the increase of demand uncertainty does not change her PDI profit and can only strengthen the advantage of low CU under pull or PAB regime.
Hence, contract \{\hat{w}_1, \bar{w}_2(\hat{w}_1)\} remains a Pareto-dominant contract, i.e., the boundary of the pull or PAB Pareto-dominant set can only expand if not staying the same.

We now discuss the behavior of \(\hat{w}_1\) when demand uncertainty increases. When prebook wholesale price \(w_1 \in [c, \bar{w}_1]\), the retailer’s push profit is higher than his PAB or pull profit. Expression (2) suggests that under contract \{\hat{w}_1(\bar{w}_2), \bar{w}_2\}, the retailer’s PDI profit under push is always lower than or equal to that under PAB or pull. Therefore, the reason that his push profit equals his PAB or pull profit at \{\hat{w}_1(\bar{w}_2), \bar{w}_2\} is that his CU under push is lower than that under PAB or pull. As demand uncertainty increases, the advantage of low CU associated with push is strengthened while the corresponding disadvantage of low PDI profit does not change; thus, the interval \([c, \bar{w}_1]\) will not contract. A similar rationale applies to the nonincreasing property of threshold \(\bar{w}_1(\hat{w}_2)\) in demand uncertainty. Therefore the Pareto-dominant set \((\hat{w}_1(\bar{w}_2), \hat{w}_1(\bar{w}_2))\) does not contract.

3  Numerical Illustration

In this section, we will illustrate through numerical examples the Pareto-dominant sets for push and pull contracts holding the existing wholesale price constant. We will also demonstrate the magnitude of the Pareto improvement and the effect of demand uncertainty. Throughout the numerical study, we assume a standard Gamma demand distribution, and vary the coefficient of variation (c.v.) by changing the shape parameter (shape= (c.v.))\(^{-2}\); we set retail price \(p = 10\), and vary the cost structure by changing the supplier’s unit production cost \(c\).

Figure 5 shows that the push regime (5a), the Pareto-dominant set for push contracts (5b), and the Pareto-dominant set for pull contracts (5c) all expand as the demand becomes more uncertain; this finding is consistent with Corollary 3’s results for the normal distribution.

Figure 5(b) illustrates the at-once wholesale price range \(w_2 \in (\bar{w}_2(\hat{w}_1), p]\) that yields pull- or PAB-regime Pareto-dominant contracts \(\{w_1, w_2\}\) for a push contract with \(w_1\). For a supply chain that already practices a Pareto push contract, i.e., the push contract with \(w_1 \leq w_1^*\), fewer pull contracts can provide Pareto improvement, as \(w_1^*\) decreases in the demand uncertainty.

Table 1 illustrates the magnitude of the Pareto improvement to the Pareto push contracts with \(w_1 \leq w_1^*\). We consider two cases: the supplier-preferred at-once wholesale price and the retailer-preferred at-once wholesale price. Within \((\hat{w}_2, p]\), the supplier prefers the at-once wholesale price \(w_2\) to be the full retail price \(p\) so that the system achieves 100% efficiency and the supplier appropriates all of the incremental profits, leaving the retailer with the same profit as in the push contract. The retailer, on the other hand, prefers the at-once wholesale price \(w_2 \in (\hat{w}_2, p]\) that maximizes his profit, which, of course, leads to less than 100% supply chain efficiency. Since the
Figure 5: (a) \( \tilde{w}_1(w_2) \) curves. (b) \( \bar{w}_2(w_1) \) curves and lines \( w_1 = w_1^* \). (c) Curves on the left side in each graph represent \( \tilde{w}_1(w_2) \); curves on right side are part of \( \bar{w}_1(w_2) \) curves; and lines \( w_2 = w_2^* \).

Note. \( (p-c)/p = 0.75, 0.5, 0.25 \) and c.v. = 0.3, 0.75, 1.5.
Table 1: Pareto Improvement Opportunities for Push Contracts

<table>
<thead>
<tr>
<th>c.v.</th>
<th>( \frac{p-c}{p} )</th>
<th>( w_1 )</th>
<th>Ret’s profit share ( \tau_r(c) )</th>
<th>Sup’s profit share ( \tau_s(c) )</th>
<th>Efficiency</th>
<th>Ret’s profit share ( \tau_r(p) )</th>
<th>Sup’s profit share ( \tau_s(p) )</th>
<th>Efficiency</th>
</tr>
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<td>0.3</td>
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<td>8.57</td>
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<td>96.7%</td>
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<td>53.9%</td>
</tr>
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<td>9.64</td>
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<td>80.4%</td>
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<td>73.9%</td>
<td>69.6%</td>
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<td>48.9%</td>
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</table>

\( \dagger \) \text{: } w_{2r} \text{ is the retailer-preferred at-once wholesale price.} \ \dagger \text{: to save space, we omit the prebook price in the profit functions. Two } w_1 \text{ values are selected for each combination of c.v. and } (p-c)/p \text{ using } \begin{align*} w_1 &= c + i(w^*_1 - c)/2, \quad i = 1, 2. \end{align*}

contract negotiation will lead to an at-once wholesale price that falls between that preferred by the retailer and that preferred by the supplier, the supply chain efficiency associated with the retailer-preferred two-wholesale-price contract represents the minimum supply chain efficiency achievable by adding an at-once ordering opportunity. Table 1 shows that even the minimum achievable efficiency, 92.2% – 99.5%, is significantly higher than that achieved by the push contracts, 71.6% – 96.7%.

Figure 5(c) illustrates the prebook wholesale price range \( w_1 \in (\bar{w}_1, \bar{w}_1] \) that yields push-regime Pareto-dominant contracts with \( \{w_1, w_2\} \) for a pull contract with \( w_2 \). As Proposition 1(2) shows, no contract can deliver Pareto improvement over the pull contract when the at-once wholesale price \( w_2 \) is high. For the supply chain that uses a Pareto pull contract (with \( w_2 \geq w^*_2 \)), fewer Pareto pull contracts can be Pareto improved by push contracts, as \( w^*_2 \) increases in the demand uncertainty.

Table 2 illustrates the magnitude of the Pareto improvement to the Pareto pull contracts with \( w_2 \geq w^*_2 \). We also consider two cases: the retailer-preferred prebook wholesale price and the

22
Table 2: Pareto Improvement Opportunities for Pull Contracts

<table>
<thead>
<tr>
<th>c.v.</th>
<th>( \frac{p-c}{p} )</th>
<th>( w_2 )</th>
<th>PULL {p, w_2}</th>
<th>Supplier preferred push leaving retailer no worse-off {w_{1s}, w_2} \textsuperscript{†}</th>
<th>Retailer preferred push leaving supplier no worse-off {w_{1r}, w_2} \textsuperscript{‡}</th>
</tr>
</thead>
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<td>( \pi_{s}(p) )</td>
<td>( \pi_{s}(w_{1s}) )</td>
<td>( \pi_{r}(p) )</td>
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</tbody>
</table>

\( w_{1s} \) and \( w_{1r} \) represent the supplier- and the retailer-preferred prebook prices, respectively. \( \textsuperscript{†} \): to save space, we omit the at-once price in the profit functions. Two \( w_2 \) values are selected for each combination of c.v. and \( (p-c)/p \) using \( w_2 = w_2^* + i(W_2 - w_2^*)/2, i = 1, 2 \), where \( W_2 \) is as defined in Proposition 3.

supplier-preferred prebook wholesale price. Because the system profit increases as the prebook wholesale price \( w_1 \) decreases and the retailer always prefers a lower \( w_1 \), the retailer-preferred two-wholesale-price contract always offers higher supply chain efficiency than the supplier-preferred contract. Thus, the former contract provides the maximum achievable efficiency, and the latter provides the minimum achievable efficiency. Table 2 indicates that the minimum achievable efficiency represents significant improvement over the pull contracts, from 74.7% – 91.1% to 92.4% – 99.1%.

4 Concluding Remarks

This paper studies the implication of the two-wholesale-price contract on the inventory and profit allocation in a supply chain consisting of a single supplier and a single retailer. The characterization of the mapping between the wholesale price space and the operating-regime space illustrates the relationship among three commonly practiced contracts: push, pull, and advance-purchase discount.
contracts. The sensitivity analysis on the wholesale prices provides insights into possible Pareto improvement opportunities for any given two-wholesale-price contract. Pareto improvement is achieved when prices are adjusted to mitigate the double marginalization and newly gained system profits are allocated between two parties. We find that firms can always achieve Pareto improvement through advance-purchase discount contracts. Interestingly, some advance-purchase discount contracts can also be Pareto improved by pull and/or push contracts, and push contracts offer higher supply chain efficiency than pull contracts. We also characterize Pareto improvement opportunities for firms who are considering only negotiating the price associated with a new ordering opportunity, and show numerically that supply chain efficiency can be improved significantly.

Firms consider adding additional ordering opportunities at different points in time for various reasons: responding to the demand forecast updates, giving incentives for revealing private information and/or acquiring information and/or influencing demand through a channel effort. In this paper, we study the simplest case of a perfect demand forecast update between the early and the late orders and symmetric information without information acquisition or channel effort. This approach allows us to focus on the inventory risk allocation issue, which is the most fundamental issue in such a supply chain and will permeate in various more complex models that consider issues such as information acquisition, promotion effort, and asymmetric information. Those more complex models raise a number of interesting questions: How do the decisions regarding promotion/information acquisition efforts interact with the inventory decision and profit allocation? How do changes in wholesale prices affect a supply chain partner’s information acquisition effort as well as the inventory decision? Comparing the mapping of the price space and operating-regime space and the set of Pareto-dominant contracts in the simple setting with those in the more complex model will aid the understanding of the exact effect of a particular factor.

Acknowledgments:

The authors thank Gérard Cachon, the associate editor, and three anonymous referees for their invaluable comments and suggestions, which have helped to improve the paper significantly. The authors also thank Tava Olsen for insightful discussions. The second author was supported in part by Hong Kong Research Grants Council under grant HKUST6152/04E.

Appendix

Throughout the Appendix, we use the notation \( \bar{F} \equiv 1 - F \) and \( L(q) = \mu - S(q) = \int_{\xi=q}^{\infty} (\xi - q) f(\xi) d\xi \). Much of the analysis is based on the following profit functions, which are equivalent to expressions...
Expressions (4) and (5) indicate that the retailer’s expected profit and the constraint $w$ if $w$ is minimized at $q^*_r = \max \{q^*_r, K_s\}$, where $K_s = F^{-1} \left( 1 - \frac{q^*_r}{w_2} \right)$.

(2) Given that the retailer is aware of the supplier’s decision rule $q^*_s = \max \{q^*_r, K_s\}$, the retailer’s profit can be written as \( \max_{q^*_r \geq 0} \{ pS (\max \{q^*_r, K_s\}) - w_1q^*_r - w_2 [S (\max \{q^*_r, K_s\}) - S (q^*_r)] \} \).

The first derivative of the retailer’s expected profit with respect to $q^*_r$ is

\[-w_1 + w_2 (1 - F (q^*_r)) + (p - w_2) (1 - F (q^*_r))^1_{(q^*_r \geq K_s)} \]

and the second derivative is

\[-w_2 f (q^*_r) - (p - w_2) f (q^*_r) \] \(1_{(q^*_r \geq K_s)} \).

Expressions (4) and (5) indicate that the retailer’s expected profit is locally concave in two intervals, \([0, K_s]\) and \([K_s, \infty)\), but is not globally concave since its first derivative has an upward jump at $K_s$. Straightforward algebra shows that in the interval \([0, K_s]\), a local optimum is achieved at $K_a = F^{-1} \left( 1 - \frac{w_1}{w_2} \right)$. In the interval \([K_s, \infty)\), a local optimum $K_r = F^{-1} \left( 1 - \frac{w_1}{p} \right)$ is achieved only if $w_1w_2 \leq pc$. Otherwise, due to $K_r < K_s$, the global optimum falls in \([0, K_s]\) and is $K_a$.

**Proof of Lemma 2.** (1) Note that $g (w_1, w_2)$ is decreasing in $w_1$ since

\[ \frac{\partial g (w_1, w_2)}{\partial w_1} = K_a - K_r = F^{-1} \left( 1 - \frac{w_1}{w_2} \right) - F^{-1} \left( 1 - \frac{w_1}{p} \right) < 0. \]

(2) For $w_1 = c$, we have $K_a = K_s$, and we can show that

\[ g (c, w_2) = w_1K_s + pL (K_s) - w_1K_r - pL (K_r) > 0, \]

where the inequality comes from the fact that $w_1 K + pL (K)$ is minimized at $K_r = F^{-1} \left( 1 - \frac{w_1}{p} \right)$.

For $w_1 = \frac{pc}{w_2}$, we have $K_r = K_s$, and we can show that

\[ g \left( \frac{pc}{w_2}, w_2 \right) = w_1K_a + w_2L (K_a) - w_1K_s - w_2L (K_s) < 0, \]
where the inequality comes from the fact that \( w_1 K + w_2 L(K) \) is minimized at \( K_a = F^{-1} \left( 1 - \frac{w_1}{w_2} \right) \).

Together with the result of part (1), the above analysis indicates the existence of a unique \( \bar{w}_1 (w_2) \in \left( c, \frac{pc}{w_2} \right) \) such that \( \bar{w}_1 (w_2) \) is the solution to \( g (\cdot, w_2) = 0. \)

**Proof of Theorem 1.** The result is straightforward from Lemmas 1 and 2.

**Proof of Lemma 3.** We first show the existence of \( w_2^0 \). Since \( g(c,p) = 0 \) and \( g(p,c) = 0 \), it follows that \( \bar{w}_1 (c) = p \) and \( \bar{w}_1 (p) = c \). Treating \( \bar{w}_1 (w) - w \) as a function of \( w \), we have \( \bar{w}_1 (c) - c > 0 \) and \( \bar{w}_1 (p) - p < 0 \). Since this function is continuous, by the Intermediate Value Theorem, the set \( \{ w : \bar{w}_1 (w) - w = 0 \} \) is not empty.

We then prove the uniqueness. Because \( \bar{w}_1 (w) = w \) implies \( \pi_r (w, c) - \pi_r (p, w) = 0 \), we show equivalently that there exists a unique \( w \in (c, p) \) such that \( \pi_r (w, c) - \pi_r (p, w) = 0 \). Note \( \pi_r (c, c) - \pi_r (p, c) > 0 \) and \( \pi_r (p, c) - \pi_r (p, p) = 0 \); it is sufficient to prove that \( \pi_r (w, c) - \pi_r (p, w) \) is unimodal in \( w \). Since \( \frac{\partial \pi_r (w, c)}{\partial w} = -K_r \) and \( \frac{\partial \pi_r (p, w)}{\partial w} = -S(K_s) + \frac{(p-w) \left[ \bar{F}(K_s) \right]^2}{f(K_s)} \), we obtain

\[
\frac{d [\pi_r (w, c) - \pi_r (p, w)]}{dw} = S(K_s) \left( 1 - \frac{K_r}{S(K_s)} - \frac{(p-w) \left[ \bar{F}(K_s) \right]^2}{f(K_s) S(K_s)} \right),
\]

where \( \frac{K_r}{S(K_s)} \) is decreasing in \( w \) and so is \( \frac{\left[ \bar{F}(K_s) \right]^2}{f(K_s) S(K_s)} \) (due to the IGFR property of the demand distribution; see Lemma 1 in Cachon (2004)). Therefore, there is at most one \( w \) satisfying \( \frac{d \pi_r (w,c)}{dw} = \frac{d \pi_r (p,w)}{dw} = 0 \), which implies that \( \pi_r (w, c) - \pi_r (p, w) \) is unimodal.

(2) It is sufficient to show that if \( w_2 \in \mathcal{F}(w'_1) \) then \( w_2 \in \mathcal{F}(w_1) \) for \( w_1 < w'_1 \), which is equivalent to showing that if \( g(w'_1, w_2) > 0 \) then \( g(w_1, w_2) > 0 \). This is true because the proof of Lemma 2 (1) has shown that \( g \) decreases in \( w_1 \).

**Proof of Lemma 4.** (1) & (4) This can be easily shown by taking a derivative with respect to \( w_1 \).

(2) & (3) For \( w_1 \leq \bar{w}_1 (w_2) \), the system runs in the push regime, i.e., the supplier sells to a newsvendor-type retailer. By Lariviere and Porteus (2001), the supplier’s profit is unimodal in \( w_1 \). The system profit is \( \pi_r + \pi_s = pS(K_r) - cK_r \). Since \( \frac{\partial [\pi_r + \pi_s]}{\partial w_1} = \frac{\partial \pi_r}{\partial K_r} \frac{\partial K_r}{\partial w_1} + \left( p\bar{F}(K_r) - c \right) \frac{\partial K_r}{\partial w_1} = (w_1 - c) \frac{\partial K_r}{\partial w_1} \), and \( K_r \) decreases in \( w_1 \), \( \pi_r + \pi_s \) decreases in \( w_1 \). For \( w_1 > \bar{w}_1 (w_2) \), the system runs in the PAB regime. Since \( K_s \) is independent of \( w_1 \), \( \frac{\partial \pi_s}{\partial w_1} = \frac{\partial [w_1 K_a + w_2 L(K_s)]}{\partial w_1} = K_a \geq 0 \) with equality holding for \( w_1 = w_2 \). For the system profit, \( \pi_r + \pi_s = pS(K_s) - cK_s \) is independent of \( w_1 \).

Finally, to show \( \pi_s (\bar{w}_1 (w_2), w_2) > \lim_{\varepsilon \to 0^+} \pi_s (\bar{w}_1 (w_2) + \varepsilon, w_2) \), we note \( K_a < K_s < K_r \) for
\[ c < w_1 < \frac{cp}{w_2}. \] Then, we evaluate the difference

\[
\pi_s|_{(q_1, q_s)=(K_r, K_r)} - \pi_s|_{(q_1, q_s)=(K_a, K_a)} \\
= (w_1 - c) (K_r - K_a) + c (K_s - K_a) - w_2 (S(K_s) - S(K_a)) \\
> (w_1 - c) (K_s - K_a) + c (K_s - K_a) - w_2 (S(K_s) - S(K_a)) \\
= w_1 (K_s - K_a) - w_2 (K_s - K_a) \hat{F}(K),
\]

where the last equality is due to the Mean Value Theorem and \( K \in (K_a, K_s). \) Since \( \hat{F}(K_s) < \hat{F}(K) < \hat{F}(K_a), \) i.e., \( \frac{c}{w_2} < \hat{F}(K) < \frac{w_1}{w_2}, \) we obtain \( \pi_s(w_1, w_2)|_{K_r, K_s} - \pi_s(w_1, w_2)|_{K_a, K_s} > 0. \]

**Proof of Lemma 5.** (1) We can show

\[
\frac{\partial (K_a/K_s)}{\partial w_2} = \frac{w_1}{w_2 f(K_s)} F^{-1} \left( 1 - \frac{c}{w_2} \right) - F^{-1} \left( 1 - \frac{w_1}{w_2} \right) \frac{c}{w_2 f(K_s)} \\
= \frac{K_a}{w_2 K_s} \left( \frac{\hat{F}(K_a)}{K_a f(K_a)} - \frac{\hat{F}(K_s)}{K_s f(K_s)} \right) > 0,
\]

where the inequality comes from \( K_a < K_s \) and the IGFR property of the demand distribution.

(2) For \( w_2 \in \mathcal{F}(w_1), \) since \( K_r \) is independent of \( w_2, \pi_s, \pi_r, \) and thus \( \pi_s + \pi_r \) are all independent of \( w_2. \) For \( w_2 \in \mathcal{P}(w_1), \) from \( \pi_s + \pi_r = pS(K_s) - cK_s, \) we obtain \( \frac{\partial (\pi_s + \pi_r)}{\partial w_2} = \frac{c(p-w_2)\hat{F}(K_s)}{w_2 f(K_s)} > 0. \) In the PAB regime, from \( \pi_s = w_1K_a - cK_s = w_2 (S(K_s) - S(K_a)), \) we obtain \( \frac{\partial \pi_s}{\partial w_2} = S(K_s) - S(K_a) > 0, \) while in the pull regime, due to \( \pi_s = w_2S(K_s) - cK_s, \) we obtain \( \frac{\partial \pi_s}{\partial w_2} = S(K_s) > 0. \)

(3) It is straightforward that \( \pi_r \) is independent of \( w_2 \) for \( w_2 \in \mathcal{F}(w_1). \) For \( w_2 \in \mathcal{P}(w_1) \cap \{w_2 : w_2 < w_1\}, \) the system runs in the pull regime, and \( \pi_r = (p - w_2)S(K_s). \) In this case, we obtain \( \frac{\partial \pi_r}{\partial w_2} = -S(K_s) + \frac{(p-w_2)c\hat{F}(K_s)}{w_2 f(K_s)} = S(K_s) \left[ \frac{(p-w_2)\hat{F}(K_s)}{w_2 f(K_s) S(K_s)} - 1 \right]. \) So we conclude that \( \pi_r \) is unimodal in \( w_2 \) because \( K_s \) increases in \( w_2 \) and \( \frac{\hat{F}(K_s)^2}{f(K_s) S(K_s)} \) decreases in \( K_s \) due to the IGFR property. ■

**Proof of Theorem 2 & 3.** For ease of presentation, we put the proofs of Theorem 2 and 3 together. Consider a contract \( \{\hat{w}_1, \hat{w}_2\} \) that yields \( \hat{\pi}_s \) and \( \hat{\pi}_r. \) When \( \hat{w}_1 = c \) or \( \hat{w}_2 = p, \) the system inventory is exactly \( q^o, \) implying that the contract is Pareto efficient. Otherwise, we discuss its Pareto-dominant contracts in the three operating regimes as follows:

1) First, we analyze \( S_{push}. \) Obviously, \( w_1^R \) is determined from solving \( \pi_s(\cdot,c) = \hat{\pi}_s \) within \( [c, w_1^R] \) due to the unimodality of \( \pi_s(w_1, c). \) As to \( w_1^R, \) we have

\[ w_1^R = \min \{w_1^{R1}, w_1^{R2}\}, \]

where \( w_1^{R1} \) is the solution to \( \pi_s(\cdot,c) = \hat{\pi}_s \) within \( (w_1^*, p] \) and \( w_1^{R2} \) is the solution to \( \pi_r(\cdot,c) = \hat{\pi}_r. \)

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To determine $w_1^R$, we need to discuss the specific type of the contract $\{\hat{w}_1, \hat{w}_2\}$. Prior to doing so, we present a claim that will be used in the derivation. We claim $w_1^L < w_1^{R2} < w_1^{R1}$ and hence $w_1^L < w_1^R = \min\{w_1^{R1}, w_1^{R2}\} = w_1^{R2}$ if $\{\hat{w}_1, \hat{w}_2\} \in \text{pull}\cup\text{PAB regime}$ and $\hat{\pi}_s = \pi_s\left(\frac{pc}{w_2}, c\right)$, or equivalently, $\hat{\pi}_r > \pi_r\left(\frac{pc}{w_2}, c\right)$. We prove the claim as follows: Given the conditions in the claim, we obtain that $w_1^L$ and $w_1^{R1}$, as the two solutions to $\pi_s (\cdot, c) = \hat{\pi}_s$, are located on the two sides of $\frac{pc}{w_2}$, i.e., $w_1^L < \frac{pc}{w_2} < w_1^{R1}$. On the other hand, $w_1^{R2}$, as the solution to $\pi_r (\cdot, c) = \hat{\pi}_r$, satisfies $w_1^{R2} < \frac{pc}{w_2}$. Therefore, we obtain $w_1^{R2} < w_1^{R1}$. To prove $w_1^L < w_1^{R2}$, we use contradiction. Assume $w_1^{R2} \leq w_1^L$. This assumption, together with the fact that $\pi_r(w_1, c)$ is decreasing in $w_1$, implies $\pi_r(w_1^{R2}, c) \geq \pi_r(w_1^L, c)$. Thus, we obtain $\hat{\pi}_r \geq \pi_r(w_1^L, c)$ due to $\pi_r(w_1^{R2}, c) = \hat{\pi}_r$. Because $\pi_s (w_1^L, c) = \hat{\pi}_s$, we in turn have $\pi_s (w_1^L, c) + \pi_r(w_1^L, c) \leq \hat{\pi}_s + \hat{\pi}_r = \pi_s \left(\frac{pc}{w_2}, c\right) + \pi_r \left(\frac{pc}{w_2}, c\right)$, which contradicts the fact $w_1^L < \frac{pc}{w_2}$.

When $\{\hat{w}_1, \hat{w}_2\} \in \text{push regime}$, it is straightforward that $S_{\text{push}}$ is non-empty if and only if (iff) $\hat{w}_1 \in (w_1^*, p]$. In this case, $\hat{w}_1$ solves both $\pi_s (\cdot, c) = \hat{\pi}_s$ within $(w_1^*, p]$ and $\pi_r (\cdot, c) = \hat{\pi}_r$. Therefore, $w_1^{R1} = w_1^{R2} = \hat{w}_1$ and hence $w_1^R = \min\{w_1^{R1}, w_1^{R2}\} = \hat{w}_1$.

When $\{\hat{w}_1, \hat{w}_2\} \in \text{pull regime}$, $S_{\text{push}}$ is non-empty iff $\hat{w}_2 \in [c, w_2^P)$. In this case, consider the push contract with $\frac{pc}{w_2} > w_2^P$. Lemma 4 of Cachon (2004) implies $\pi_s \left(\frac{pc}{w_2}, c\right) > \pi_s (p, \hat{w}_2) = \hat{\pi}_s$, which, together with the above claim, yields $w_1^R = \min\{w_1^{R1}, w_1^{R2}\} = w_1^{R2}$.

When $\{\hat{w}_1, \hat{w}_2\} \in \text{PAB regime}$, we can limit our analysis to $w_1 < \frac{pc}{w_2}$ because $K_r > K_s$ holds, and hence the supply chain efficiency becomes larger only in this case. We note that the definition of $w_1^{\text{push}} (w_2)$ is valid iff $w_2 \geq w_2^P$, the argument of which is the following: When $w_2 \geq w_2^P$, we have $\pi_r(\hat{w}_1(w_2), w_2) \geq \pi_r \left(\frac{pc}{w_2}, c\right) \geq \pi_r (p, w_2)$; otherwise, $\pi_r(w_1, w_2) \geq \pi_r (p, w_2) > \pi_r \left(\frac{pc}{w_2}, c\right)$ for any $w_1$ causing $(w_1, w_2) \in \text{PAB regime}$. It is also straightforward to show $w_1^{\text{push}} (w_2^P) = w_2^P$, $w_1^{\text{push}} (p) = c$, and $\hat{w}_1(w_2) < \frac{pc}{w_2} < w_1^{\text{push}} (w_2) < w_2$ for $w_2 \in (w_2^P, p)$. Because the system profit at $(w_1^{\text{push}} (w_2), w_2)$ is the same as that at $(\frac{pc}{w_2}, c)$, we obtain $\pi_s (w_1^{\text{push}} (w_2), w_2) = \pi_s \left(\frac{pc}{w_2}, c\right)$ as well.

We discuss the following cases:

Case 1: $\hat{w}_2 \geq w_2^P$ and $\hat{w}_1(\hat{w}_2) < \hat{w}_1 < w_1^{\text{push}} (\hat{w}_2)$. Note that $\hat{w}_1 < w_1^{\text{push}} (\hat{w}_2)$ implies $\hat{\pi}_s < \pi_s (w_1^{\text{push}} (\hat{w}_2), \hat{w}_2)$ and $\hat{\pi}_r > \pi_r (w_1^{\text{push}} (\hat{w}_2), \hat{w}_2)$ in the PAB regime. Due to $\pi_s (w_1^{\text{push}} (\hat{w}_2), \hat{w}_2) = \pi_s \left(\frac{pc}{w_2}, c\right)$ and $\pi_r (w_1^{\text{push}} (\hat{w}_2), \hat{w}_2) = \pi_r \left(\frac{pc}{w_2}, c\right)$, we obtain $\hat{\pi}_s < \pi_s (w_1^{\text{push}} (\hat{w}_2), \hat{w}_2)$ and $\hat{\pi}_r < \pi_r (\frac{pc}{w_2}, c)$. Thus, applying the above claim yields $w_1^R = w_1^{R2}$.

Case 2: $\hat{w}_2 \geq w_2^P$ and $\hat{w}_1 \geq w_1^{\text{push}} (\hat{w}_2)$. Note that $\hat{w}_2 \geq w_2^P$ implies $\frac{pc}{w_2} \leq w_1^P < w_1^*$. As mentioned above, we can limit our analysis to $w_1 < \frac{pc}{w_2}$, in which case $\pi_s (w_1, c)$ is increasing in $w_1$ due to $\frac{pc}{w_2} < w_1^*$. Therefore, any push contract with such $w_1$ cannot Pareto dominate $\{\hat{w}_1, \hat{w}_2\}$ due to $\pi_s (w_1, c) < \pi_s (\frac{pc}{w_2}, c) = \pi_s (w_1^{\text{push}} (\hat{w}_2), \hat{w}_2) < \pi_s (\hat{w}_1, \hat{w}_2)$. 

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Case 3: $\hat{w}_2 < w^*_2$. In this case, we have $\pi_s(p, \hat{w}_2) < \pi_s(p, \frac{pc}{w_2}, c)$ and $\pi_r(p, \hat{w}_2) > \pi_r(p, \frac{pc}{w_2}, c)$. In addition, we obtain from Lemma 4 $\hat{\pi}_s < \pi_s(p, \hat{w}_2)$ and $\hat{\pi}_r > \pi_r(p, \hat{w}_2)$. Therefore, we conclude $\hat{\pi}_s < \pi_s(p, \frac{pc}{w_2}, c)$ and $\hat{\pi}_r(\hat{w}_1, \hat{w}_2) > \pi_r(p, \frac{pc}{w_2}, c)$. Thus, applying the above claim yields $w^R_1 = w^{R2}_1$.

The discussion of the above three cases indicates that the contract $\{\hat{w}_1, \hat{w}_2\}$ has Pareto-dominant contracts in the push regime iff $\tilde{w}_1(\hat{w}_2) < \tilde{w}_1 < \min \left\{w^{push}_1, \hat{w}_2 \right\}$. This gives the first result presented in Theorem 3 (3).

In short, in determining $w^R_1$, we always have $w^{R2}_1 \leq w^{R1}_1$, which allows us to write $w^R_1 = w^{R2}_1$ and gives the characterization of $S_{push}$ in Theorem 2.

2) Then, we analyze $S_{pull}$. Obviously, $w^R_2$ is determined from solving $\pi_r(p, \cdot) = \hat{\pi}_r$ within $(w^*_2, p]$ due to the unimodality of $\pi_r(p, w_2)$. As to $w^L_2$, we have

$$w^L_2 = \max \left\{w^{L1}_2, w^{L2}_2 \right\},$$

where $w^{L1}_2$ is the solution to $\pi_r(p, \cdot) = \hat{\pi}_r$ within $[c, w^*_2]$ and $w^{L2}_2$ is the solution to $\pi_s(p, \cdot) = \hat{\pi}_s$.

When $\{\tilde{w}_1, \tilde{w}_2\} \in$ pull regime, the derivation of $S_{pull}$ follows the same logic as that of $S_{push}$ for $\{\hat{w}_1, \hat{w}_2\} \in$ push regime. Therefore, we conclude $w^{L1}_2 = w^{L2}_2 = \hat{w}_2$ and hence $w^L_2 = \max \left\{w^{L1}_2, w^{L2}_2 \right\} = \hat{w}_2$.

When $\{\tilde{w}_1, \tilde{w}_2\} \in$ push regime, the derivation of $S_{pull}$ follows the same logic as that of $S_{push}$ for $\{\tilde{w}_1, \tilde{w}_2\} \in$ pull regime. Therefore, we conclude $w^{L1}_2 < w^{L2}_2$ and hence $w^L_2 = \max \left\{w^{L1}_2, w^{L2}_2 \right\} = w^{L2}_2$.

When $\{\tilde{w}_1, \tilde{w}_2\} \in$ PAB regime, to derive $S_{pull}$, we note that for the supplier’s profit, $\pi_s(w_1, w_2)$ is increasing in $w_1$ in the PAB regime, is independent of $w_1$ in the pull regime, and is increasing in $w_2$ in the pull regime. Thus, we obtain that $w^{L2}_2$, as the solution to $\pi_s(p, \cdot) = \hat{\pi}_s$, satisfies $w^{L2}_2 < \hat{w}_2$. On the other hand, for the retailer’s profit, $\pi_r(w_1, w_2)$ is decreasing in $w_1$ in the PAB regime, keeps constant over $w_1$ in the pull regime, and is unimodal in $w_2$ in the pull regime. Therefore, if $\hat{w}_2 \geq w^*_2$, no pull contract with $w_2 > \hat{w}_2$ can make the retailer better off. If $\hat{w}_2 < w^*_2$, there exist pull contracts that make the retailer better off iff $\hat{\pi}_r < \pi_r(p, w^*_2)$. So using the fact that $\pi_r(w_1, w_2)$ is decreasing in $w_1$ in the PAB regime, we can define $w^{pull}_1(w_2)$, as stated in Theorem 3, for $w_2 < w^*_2$ such that iff $\hat{w}_2 \in (\tilde{w}_2, w^*_2)$ and $\max \left\{w^{pull}_1(\tilde{w}_2), \tilde{w}_1(\tilde{w}_2) \right\} < \tilde{w}_1 < \hat{w}_2$, $\hat{\pi}_r < \pi_r(p, w_2)$ holds, i.e., $S_{pull}$ is non-empty. This gives the second result presented in Theorem 3 (3). In this case, $w^{L1}_2$ and $w^R_2$, as the solutions to $\pi_r(p, \cdot) = \hat{\pi}_r$, satisfy $\hat{w}_2 < w^{L1}_2 < w^*_2 < w^R_2$. Due to $w^{L2}_2 < \hat{w}_2$, we conclude $w^{L1}_2 > w^{L2}_2$ and hence $w^L_2 = \max \left\{w^{L1}_2, w^{L2}_2 \right\} = w^{L1}_2$.

In short, in determining $w^L_2$, we have $w^{L1}_2 < w^{L2}_2$ if $\{\tilde{w}_1, \tilde{w}_2\} \in$ push regime and otherwise $w^{L1}_2 \geq w^{L2}_2$, which gives the characterization of $S_{pull}$ in Theorem 2.
3) We finally analyze $S_{PAB}$. Notes that $S_{PAB}$ is always non-empty because at $w_2 = p$ (the top boundary of $[c, p] \times [c, p]$), any division of the optimal system profit can be achieved through PAB-regime contracts by changing $w_1$. Besides, $\pi_s(w_1, w_2)$ is increasing in $w_1$ and $\pi_r(w_1, w_2)$ is decreasing in the PAB regime. Therefore, the defined $W^L_1(w_2)$ and $W^R_1(w_2)$ give bounds on $w_1$ such that a PAB contract $\{w_1, w_2\}$ can Pareto dominate the contract $\{\hat{w}_1, \hat{w}_2\}$. To derive $\hat{w}_2$, we need to discuss the type of contract $\{\hat{w}_1, \hat{w}_2\}$.

When $\{\hat{w}_1, \hat{w}_2\} \in$ push regime, we consider whether $S_{pull}$ is empty. If $S_{pull} \neq \emptyset$, i.e., $\hat{w}_1 > w_1^P$, then $w_1 = W^L_1(w_2)$ intersects with $w_1 = w_2$ at $w_2 = w_2^L$, while $w_1 = W^R_1(w_2)$ intersects with $w_1 = w_2$ at $w_2 = w_2^R$. Thus, we obtain $w_2 = w_2^L$. In addition, $w_2^L > \frac{pc}{w_1}$ holds due to the Pareto dominance of the pull contract with $w_2^L$ over the push contract with $\hat{w}_1$. On the other hand, if $S_{pull} = \emptyset$, i.e., $\hat{w}_1 \leq w_1^P$, we note that a PAB contract with $w_2 = \frac{pc}{w_1}$ yields the same supply chain efficiency as that by $\{\hat{w}_1, \hat{w}_2\}$ and furthermore, the contract $\{w_1^{push} \left( \frac{pc}{w_1} \right), \frac{pc}{w_1} \}$ results in the same profit division between the two parties. Therefore, $w_1 = W^L_1(w_2)$ and $w_1 = W^R_1(w_2)$ intersect at $w_2 = \frac{pc}{w_1}$, which indicates $w_2 = \frac{pc}{w_1}$. In this case, $\frac{pc}{w_1} \geq w_2^L$ holds due to $\hat{w}_1 \leq w_1^P$ and thus $\frac{pc}{w_1} \geq w_2^P$.

In short, we have $w_2 = \max \left\{ w_2^L, \frac{pc}{w_1} \right\}$.

When $\{\hat{w}_1, \hat{w}_2\} \in$ pull regime, we consider whether $S_{pull}$ is empty. If $S_{pull} \neq \emptyset$, i.e., $\hat{w}_2 < w_2^P$, then $w_1 = W^L_1(w_2)$ intersects with $w_1 = w_2$ at $w_2 = w_2^L = \hat{w}_2$, while $w_1 = W^R_1(w_2)$ intersects with $w_1 = w_2$ at $w_2 = w_2^R$. If $S_{pull} = \emptyset$, i.e., $\hat{w}_2 \geq w_2^P$, then $w_1 = W^L_1(w_2)$ intersects with $w_1 = w_2$ at $w_2 = \hat{w}_2$, so does $w_1 = W^R_1(w_2)$. In short, we have $w_2 = \hat{w}_2$.

When $\{\hat{w}_1, \hat{w}_2\} \in$ PAB regime, $W^L_1(w_2)$ and $W^R_1(w_2)$ intersect at $w_2 = \hat{w}_2$. Hence, we have $\bar{w}_2 = \hat{w}_2$.

**Proof of Corollary 1.** We claim $w_1^P < \sqrt{pc} < w_2^P$, which we prove as follows. Consider the function $\pi_s(w_1, c) - \pi_s \left( p, \frac{pc}{w_1} \right)$, which is positive iff $w_1 > w_1^P$. When $w_1 = \sqrt{pc}$, we have

$$\pi_s(w_1, c) - \pi_s \left( p, \frac{pc}{w_1} \right) > (\sqrt{pc} - c) F^{-1} \left( 1 - \sqrt{\frac{c}{p}} \right) - (\sqrt{pc} - c) F^{-1} \left( 1 - \sqrt{\frac{c}{p}} \right) = 0.$$

Thus, we have $\sqrt{pc} > w_1^P$, which leads to $w_2^P = \frac{pc}{w_1^P} > \sqrt{pc}$.

We now prove $w_1^P > w_2^P$ when the demand has an IFR. Because $w_2^P \in (c, \sqrt{pc})$ and $w_1^P \in (c, \sqrt{pc})$ are solutions to $\pi_r(w, c) - \pi_r(p, w) = 0$ and $\pi_r(w, c) - \pi_r(p, \frac{pc}{w}) = 0$, respectively, and $\pi_r(w, c)$ is decreasing in $w$, it is sufficient to show $\pi_r(p, w) > \pi_r(p, \frac{pc}{w})$ for any $w \in (c, \sqrt{pc})$. We compute $\pi_r(p, w) = (p - w) S(K_s) = pF(K_r) S(K_s)$ and $\pi_r(p, \frac{pc}{w}) = (p - \frac{pc}{w}) S(K_r) = pF(K_s) S(K_r)$, where $K_s = F^{-1}(1 - \frac{c}{w})$, $K_r = F^{-1}(1 - \frac{w}{p})$, and $K_s < K_r$ due to $w \in \left( c, \sqrt{pc} \right)$. As $\pi_r(p, w) > \pi_r(p, \frac{pc}{w})$ is equivalent to $\frac{F(K_s)}{S(K_s)} > \frac{F(K_r)}{S(K_r)}$, we take the derivative of $\frac{F(q)}{S(q)}$ and obtain $\frac{d}{dq} \left[ \frac{F(q)}{S(q)} \right] = \frac{f(q)S(q) - F(q)F(q)}{[S(q)]^2}$. We claim $h(q) \equiv f(q)S(q) - F(q)F(q) \geq 0$ for any $q \geq 0$. We prove the claim
by contradiction. Note $h(0) = 0$ and assume $h(q) < 0$ for some $q > 0$. Thus, some $\hat{q} \in [0, q)$ must exist such that $h(\hat{q}) = 0$ and $h'(\hat{q}) \leq 0$. But $h'(\hat{q}) = \frac{d}{dq} \left[ \frac{f(q)}{F(q)}S(q)\bar{F}(q) - F(q)\bar{F}(q) \right] = \frac{d}{dq} \left[ \frac{f(q)}{F(q)}S(q)\bar{F}(q) - F(q)\bar{F}(q) \right] = \frac{d}{dq} \left[ \frac{f(q)}{F(q)}S(q)\bar{F}(q) > 0. \right.$

For a PAB contract with $(\hat{w}_1, \hat{w}_2)$ for $\hat{w}_2 \in [w_2^0, w_2^R]$ being Pareto dominated by pull contracts, we have $\pi_r(w_1^R, c) = \pi_r(\hat{w}_1, \hat{w}_2) > \pi_r(p, \hat{w}_2) \geq \pi_r(w_1^R, c) > \pi_r(w_1^P, c)$, which implies $w_1^R < w_1^P$ and thus $\pi_r(w_1^R, c) + \pi_s(w_1^R, c) > \pi_r(w_1^P, c) + \pi_s(w_1^P, c)$. Similarly, we can show $\pi_r(p, w_2^R) = \pi_r(\hat{w}_1, \hat{w}_2) > \pi_r(p, w_2^R) = \pi_r(p, w_2^P)$, which implies $w_2^R < w_2^P$ and thus $\pi_r(p, w_2^R) + \pi_s(p, w_2^R) < \pi_r(p, w_2^P) + \pi_s(p, w_2^P)$. Since $\pi_r(w_1^P, c) + \pi_s(w_1^P, c) = \pi_r(p, w_2^P) + \pi_s(p, w_2^P)$, we can obtain the result of this corollary.

**Proof of Proposition 1.** (1) Pareto improvement can be achieved only if $w_2 \in \mathcal{P}(\hat{w}_1)$ because $\pi_s + \pi_r$ remains constant when $w_2 \in \mathcal{F}(\hat{w}_1)$, as indicated by Lemma 5 (2). For any $w_2 \in \mathcal{P}(\hat{w}_1)$, we obtain $\pi_r(\hat{w}_1, \hat{w}_2) > \pi_r(\hat{w}_1, c)$ because otherwise the retailer would order more than the supplier’s newsvendor quantity and hence the system would run in the push regime, which contradicts $w_2 \in \mathcal{P}(\hat{w}_1)$. We let $\hat{w}_2$ be the unique solution to $\pi_s(\hat{w}_1, \cdot) = \pi_s(\hat{w}_1, c)$ for $w_2 \in \mathcal{P}(\hat{w}_1)$. Two conditions underwrite the existence of such a solution. First, when $w_2 = p$, we have

$$\pi_s(\hat{w}_1, p) > \pi_s(\hat{w}_1, c),$$

which comes from the straightforward result that $\pi_r(\hat{w}_1, p) = \pi_r(\hat{w}_1, c)$ and the fact that the system achieves the first-best solution at $(\hat{w}_1, p)$ so that $\pi_r(\hat{w}_1, p) + \pi_s(\hat{w}_1, p) > \pi_r(\hat{w}_1, c) + \pi_s(\hat{w}_1, c)$.

Second, when $w_2 = \frac{pc}{\hat{w}_1}$, we have $\pi_r(\hat{w}_1, \frac{pc}{\hat{w}_1}) + \pi_s(\hat{w}_1, \frac{pc}{\hat{w}_1}) = \pi_r(\hat{w}_1, c) + \pi_s(\hat{w}_1, c)$, which, together with $\pi_r(\hat{w}_1, \hat{w}_2) > \pi_r(\hat{w}_1, c)$ for any $w_2 \in \mathcal{P}(\hat{w}_1)$, leads to

$$\pi_s\left(\hat{w}_1, \frac{pc}{\hat{w}_1}\right) < \pi_s(\hat{w}_1, c).$$

So by the the Intermediate Value Theorem, a solution exists and satisfies $\hat{w}_2 > \frac{pc}{\hat{w}_1}$, which indicates that $(\hat{w}_1, \hat{w}_2)$ lies above the line $w_2 = \frac{pc}{\hat{w}_1}$ and thus to the right of the line $w_1 = \hat{w}_1(\hat{w}_2)$. The uniqueness of the solution comes from the fact that, as indicated by Lemma 5 (2), $\pi_s$ increases in $w_2$ when $w_2 \in \mathcal{P}(\hat{w}_1)$.

We next claim that $w_2 = w_1$ and $w_2 = \hat{w}_2(w_1)$ have exactly one intersection, which in turn gives $W_1$. Because such an intersection means that the supplier’s push profit equals her pull profit given the same wholesale price, we show equivalently that there is exactly one solution to $\pi_s(w, c) - \pi_s(p, w) = 0$ for $w \in (c, p)$. We compute

$$\frac{\partial [\pi_s(w, c) - \pi_s(p, w)]}{\partial w} = \bar{F}(K_r) \left[ \frac{K_r f(K_r)}{F(K_r)} \left( 1 - \frac{S(K_s)}{K_r} \right) - 1 + \frac{c}{pF(K_r)} \right].$$
in which \( \frac{K_{r}f(K_{r})}{\bar{F}(K_{r})} \) and \( \frac{c}{pF(K_{r})} \) are decreasing in \( w \) but \( K_{s} \) and thus \( S(K_{s}) \) are increasing in \( w \). We discuss two cases:

**Case 1:** \( S(K_{s}) < K_{r} \). We have \( \frac{K_{r}f(K_{r})}{\bar{F}(K_{r})} \cdot \left(1 - \frac{S(K_{s})}{K_{r}}\right) \) and \( \frac{c}{pF(K_{r})} \) all being positive and decreasing in \( w \). So \( \frac{\partial [\pi_{s}(w,c) - \pi_{s}(p,w)]}{\partial w} \) changes from positive to negative as \( w \) increases.

**Case 2:** \( S(K_{s}) \geq K_{r} \). We have \( \frac{\partial [\pi_{s}(w,c) - \pi_{s}(p,w)]}{\partial w} \) < 0 because \( 1 - \frac{S(K_{s})}{K_{r}} \leq 0 \) and \( -1 + \frac{c}{pF(K_{r})} < 0 \) due to \( \bar{F}(K_{r}) = 1 - F(K_{r}) > \frac{c}{p} \).

In short, we have \( \frac{\partial [\pi_{s}(w,c) - \pi_{s}(p,w)]}{\partial w} \) changes from positive to negative as \( w \) increases, which in turn implies that \( \pi_{s}(w,c) - \pi_{s}(p,w) \) is unimodal in \( w \). Since \( \pi_{s}(c,c) - \pi_{s}(p,c) = 0 \) and \( \pi_{s}(p,c) - \pi_{s}(p,p) < 0 \), we conclude exactly one solution exists to \( \pi_{s}(w,c) - \pi_{s}(p,w) = 0 \) for \( w \in (c,p) \), which completes the proof of this part.

(2) By Lemma 4, both \( \pi_{r} \) and \( \pi_{s} \) are constant in \( w_{1} \) for \( w_{1} > \max \{ \bar{w}_{1}(\bar{w}_{2}), \bar{w}_{2} \} \), and \( \pi_{s} \) increases in \( w_{1} \) for \( \bar{w}_{1}(\bar{w}_{2}) < w_{1} \leq \bar{w}_{2} \). Therefore, Pareto improvement can be achieved only if \( w_{1} \leq \bar{w}_{1}(\bar{w}_{2}) \), i.e., in the push regime.

We know from Lemma 4 that the retailer’s profit in the push regime is greater than it is in the pull regime for any given \( \bar{w}_{2} \). Thus, we only need to derive the condition under which the supplier’s profit is greater as well. Note that the supplier’s push profit equaling her pull profit, i.e., \( \pi_{s}(w_{1},c) = \pi_{s}(p,w_{2}) \), defines \( w_{2} \) as a unimodal function of \( w_{1} \in [c,p] \), which increases for \( w_{1} \in [c,w_{1}^{*}] \) and then decreases. It can be easily shown that this function intersects with \( w_{2} = \frac{pc}{w_{1}} \) exactly once at \( w_{1}^{P} \) for \( w_{1} \in (c,p) \). Due to \( w_{1}^{P} < w_{1}^{*} \), the monotonicity of this function for \( w_{1} \in [c,w_{1}^{P}] \) allows us to define an inverse function \( \bar{w}_{1} \) for \( w_{2} \in [c,w_{2}^{P}] \). Note that \( \bar{w}_{1} \leq w_{1}^{P} < w_{1}^{*} \) and \( \bar{w}_{1}(w_{2}) \leq \frac{pc}{w_{2}} \) imply \( \pi_{s}(w_{1},w_{2}) > \pi_{s}(p,w) \) for \( \bar{w}_{1} < w_{1} \leq \bar{w}_{1}(w_{2}) \). In the following, we analyze any intersection between \( w_{1} = \bar{w}_{1}(w_{2}) \) and \( w_{1} = \bar{w}_{1}(w_{2}) \). Because at least one intersection exists, we let \( W_{2} < w_{2}^{P} \) be the largest at-once wholesale price associated with the intersections. We claim that on \( w_{1} = \bar{w}_{1}(w_{2}) \), \( g(\bar{w}_{1},w_{2}) \) is decreasing for \( w_{2} \in [c,W_{2}] \) and prove the claim by the chain rule as follows:

\[
\frac{dg(\bar{w}_{1},w_{2})}{dw_{2}} = \frac{\partial g(\bar{w}_{1},w_{2})}{\partial w_{2}} + \frac{\partial g(\bar{w}_{1},w_{2})}{\partial \bar{w}_{1}} \frac{\partial \bar{w}_{1}}{\partial w_{2}}
\]
\[
= \left[ S(K_{s}) - S(K_{a}) - \frac{(p - w_{2})c}{w_{2}} \frac{\partial K_{s}}{\partial w_{2}} \right] + (K_{a} - K_{r}) \frac{S(K_{s})}{K_{r} + (\bar{w}_{1} - c) \frac{\partial K_{r}}{\partial \bar{w}_{1}}} \]
\[
= \left[ K_{a} + (\bar{w}_{1} - c) \frac{\partial K_{r}}{\partial \bar{w}_{1}} - \frac{S(K_{a})}{S(K_{s})} \left( K_{r} + (\bar{w}_{1} - c) \frac{\partial K_{r}}{\partial \bar{w}_{1}} \right) \right] \frac{S(K_{s})}{K_{r} + (\bar{w}_{1} - c) \frac{\partial K_{r}}{\partial \bar{w}_{1}}} \]
\[
- \frac{(p - w_{2})c}{w_{2}} \frac{\partial K_{s}}{\partial w_{2}}
\]

Note that \( \frac{\partial K_{r}}{\partial \bar{w}_{1}} < 0 \), \( \frac{\partial K_{r}}{\partial \bar{w}_{2}} > 0 \), and, due to \( \bar{w}_{1} \leq w_{1}^{P} < w_{1}^{*} \), \( \frac{\partial \pi_{s}(\bar{w}_{1},c)}{\partial \bar{w}_{1}} = K_{r} + (\bar{w}_{1} - c) \frac{\partial K_{r}}{\partial \bar{w}_{1}} > 0 \). In
addition, from a sample path approach, we obtain \( \frac{S(K_s)}{S(K_r)} \geq \frac{K_r}{K_s} \), which, together with \( K_r > K_s \) due to \( \bar{w}_1 < w_1^P \) and \( w_2 < w_2^P \), leads to \( \frac{S(K_s)}{S(K_r)} \geq \frac{K_r}{K_s} \). Thus, we have

\[
\frac{dg(\bar{w}_1, w_2)}{dw_1} \leq \left[ \left( 1 - \frac{K_a}{K_r} \right) (\bar{w}_1 - c) \frac{\partial K_r}{\partial w_1} \right] \frac{S(K_s)}{K_r} + (\bar{w}_1 - c) \frac{\partial K_r}{\partial w_1} - \frac{(p - w_2) c}{w_2} \frac{\partial K_r}{\partial w_2} < 0.
\]

The above inequality indicates that \( g(\bar{w}_1, w_2) > 0 \) for \( w_2 \in [c, W_2] \), and hence \( \bar{w}_1 < \bar{w}_1(w_2) \) holds. Therefore, \((\bar{w}_1(W_2), W_2)\) actually is the only intersection of \( w_1 = \bar{w}_1(w_2) \) and \( w_1 = \bar{w}_1(w_2) \), which completes the proof of this part. ■

**Proof of Corollary 2.** In the proof of Lemma 3, we have shown that the retailer’s profit difference between push and pull contracts given the same wholesale price \( w, \pi_r(w, c) - \pi_r(p, w) \), is unimodal and equals 0 iff \( w = w_3^0 \). On the other hand, in the proof of Proposition 1, we have shown that the supplier’s profit difference, \( \pi_s(w, c) - \pi_s(p, w) \), is also unimodal and equals 0 iff \( w = W_1 \). So we obtain the result of this corollary. ■

**Proof of Corollary 3.** Let \( \phi \) and \( \Phi \) be the density function and cumulative distribution function for the standard Normal distribution. We have \( K_a = \mu_N + \sigma_N z_a, K_s = \mu_N + \sigma_N z_s, K_r = \mu_N + \sigma_N z_r \), where \( z_a = \Phi^{-1}(1 - \frac{w_1}{w_2}), z_s = \Phi^{-1}(1 - \frac{c}{w_2}), z_r = \Phi^{-1}(1 - \frac{w_1}{p}) \), and \( \Phi^{-1} \) is the inverse of \( \Phi \). We also have \( L(q) = \sigma_N L_N(z) \) for \( q = \mu_N + \sigma_N z \), where \( L_N(z) = \int_z^\infty (\xi - z)^+ \phi(\xi) d\xi \).

By definition, for given \( w_2, \bar{w}_1 \) is the solution to

\[
g(\bar{w}_1, w_2) = \bar{w}_1 K_a + (p - w_2) L(K_a) + w_2 L(K_s) - \bar{w}_1 K_r - p L(K_r)
\]

\[
= \begin{cases} 
\sigma_N [\bar{w}_1 z_a + (p - w_2) L_N(z_a) + w_2 L_N(z_a) - \bar{w}_1 z_r - p L_N(z_r)] & \text{for } K_a > 0, \text{i.e., } \bar{w}_1 < w_2 \\
(w_2 - \bar{w}_1) \mu_N + \sigma_N [(p - w_2) L_N(z_a) - \bar{w}_1 z_r - p L_N(z_r)] & \text{for } K_a = 0, \text{i.e., } \bar{w}_1 \geq w_2
\end{cases}
\]

Clearly for the case of \( K_a > 0 \), i.e., \( \bar{w}_1 < w_2 \), \( g(\bar{w}_1, w_2) = 0 \) is independent of the value of \( \sigma_N \), and \( \bar{w}_1(w_2) \) is not affected by \( \sigma_N \). For the case of \( K_a = 0 \), i.e., \( \bar{w}_1 \geq w_2 \), we obtain \( (p - w_2) L_N(z_a) - \bar{w}_1 z_r - p L_N(z_r) \geq 0 \) from \( g(\bar{w}_1, w_2) = 0 \). Applying the Implicit Function Theorem yields \( \frac{\partial \bar{w}_1}{\partial \sigma_N} = \frac{(p - w_2) L_N(z_a) - \bar{w}_1 z_r - p L_N(z_r)}{K_r} \geq 0 \). Thus, \( \bar{w}_1(w_2) \) is nondecreasing in \( \sigma_N \).

\( \bar{w}_1 \) comes from the solution to \( \pi_s(\cdot, c) = \pi_s(p, w_2) \) for \( w_2 \in [c, w_2^P] \). As proved, there is exactly one solution to \( \pi_s(w, c) - \pi_s(p, w) = 0 \). Due to \( \bar{w}_1 < w_1^P \), we obtain \( \bar{w}_1 < w_2 \) for any \( w_2 \in [c, w_2^P] \).

Now taking into account the assumption of normal distribution, we can write \( \pi_s(\bar{w}_1, c) - \pi_s(p, w_2) = 0 \) as

\[
(\bar{w}_1 - w_2) \mu_N + \sigma_N [\bar{w}_1 z_r - cz_r + w_2 L_N(z_a) + cz_a] = 0.
\]

Note \( \bar{w}_1 z_r - cz_r + w_2 L_N(z_a) + cz_a \geq 0 \) due to \( \bar{w}_1 < w_2 \). Applying the Implicit Function Theorem yields \( \frac{\partial \bar{w}_1}{\partial \sigma_N} = -\frac{\bar{w}_1 z_r - cz_r + w_2 L_N(z_a) + cz_a}{S(K_r)} \leq 0 \). Therefore, \( \bar{w}_1 \) is nonincreasing in \( \sigma_N \).
For \( \bar{w}_2 \), it comes from the solution to \( \pi_s(w_1, \cdot) = \pi_s(w_1, c) \) for \( w_2 \in \mathcal{P}(w_1) \). We analyze two cases. First, we examine the case in which some \( \bar{w}_2 > w_1 \) makes the supplier indifferent between the push regime and the PAB regime. That is, at this \( \bar{w}_2 \), the difference between the supplier’s profits in the two regimes is

\[
\sigma_N [(w_1 - c) z_r - w_1 z_s + cz_s - \bar{w}_2 (L_N(z_a) - L_N(z_s))] = 0.
\]

It is clear that a change in \( \sigma_N \) will not affect the equality, thus \( \bar{w}_2 \) is independent of \( \sigma_N \). We then examine the case in which some \( \bar{w}_2 \leq w_1 \) makes the supplier indifferent between the push regime and the pull regime. That is, at this \( \bar{w}_2 \), the difference between the supplier’s profits in the two regimes is

\[
(w_1 - \bar{w}_2) \mu_N + \sigma_N [(w_1 - c) z_r + \bar{w}_2 L_N(z_s) + cz_s] = 0.
\]

Note \((w_1 - c) z_r + \bar{w}_2 L_N(z_s) + cz_s \leq 0\) due to \( \bar{w}_2 \leq w_1 \). Applying the Implicit Function Theorem yields 
\[
\frac{\partial \bar{w}_2}{\partial \sigma_N} = \frac{(w_1 - c) z_r + \bar{w}_2 L_N(z_s) + cz_s}{S(K_s)} \leq 0.
\]

Therefore, \( \bar{w}_2 \) is nonincreasing in \( \sigma_N \).

References


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