Heterogeneous Beliefs, Trading Risk, and the Equity Premium *

Alexander David
John M. Olin School of Business
Washington University in St. Louis
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ABSTRACT

Portfolios of agents who have heterogeneous beliefs about fundamental growth face an exposure to ‘trading risk’, the risk of incurring losses due to equilibrium prices responding to beliefs of other agents. Agents with a coefficient of relative risk aversion of less than one have a ‘speculative’ demand for risky assets and a desire to save more in periods of high dispersion and low disagreement between their models, times of high trading risk. Calibrated to fundamentals and dispersion of expected growth rates from surveys, the model generates time-varying per capita consumption volatility and cross-sectional consumption dispersion with averages close to empirical estimates. The unconditional equity premium generated from conditional Euler equations of our model attains about half its historical value, while the average riskless rate is lower than a benchmark homogeneous beliefs model by about one percent.

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Introduction

The representative agent paradigm with identical agents fails to take into account trading risk among different agents in the economy. In reality, individuals and institutions process new information and decide on their trades with competing models of economic fundamentals, and face the risk that market prices move more in line with the trading models of other agents rather than their own. We call the risk of such losses to other agents in the economy ‘trading risk.’ This exposure to trading risk cancels out across agents, and is thus not captured in the identical agent framework. As such, it underestimates the amount of risk faced by agents. One manifestation of this problem is that when calibrated to aggregate data on fundamentals and stock prices, the framework typically requires individuals to be implausibly highly risk averse to match the observed risk premium on stocks. Moreover, it implies that agents assign very low values to riskless assets, that is, in the model agents require a high riskless rate to be induced to clear bond markets.

At an institutional level, there is substantial evidence that firms build proprietary models that are used for trading different securities, which exposes them to trading risk.

It is often reported in the media that we have increased our trading risk in recent years to offset the decline in investment banking activity levels. In fact, the same shocks and trends in the economy that have led to sharp declines in investment banking have also created significant trading opportunities for our clients and for Goldman Sachs. We believe our willingness to take significant trading risk for appropriate reward is one of the distinguishing features of our firm and gives us a competitive advantage (Goldman Sachs Annual Report 2003 – Letter to Shareholders, Page 2).

As we will see in our analysis, our calibrated model implies that speculative trading opportunities tend to be inversely related to the growth of fundamentals, that is they are counter-cyclical.

We construct a general equilibrium exchange economy in which two types of agents have heterogeneous beliefs. These agents agree to disagree on the parameters of fundamental processes in the economy and update their beliefs about the state of the economy differently even though they observe the same data. While we exogenously specify the models of different agents, their trading profits and survival is endogenous: agents trading with models that do not fit the data will make losses to agents trading with better models, and their share of the market will decline over time. To simplify the terminology, we will say that the two types of agents have different models of the fundamental process, even though in our analysis they simply disagree on its parameter values. In equilibrium, they agree on the prices of assets but disagree on the decomposition of asset returns into expected return and shock components. Due to this disagreement, agents speculate with each
other and as a result, as in models of incomplete markets, individuals’ consumptions are not perfectly correlated.¹ At times of larger dispersion in beliefs or low disagreement between the two types of agents’ models, the premium agents demand to hold stocks is larger.

As suggested, the exposure to trading risk at any given time depends on two key state variables in our model: the dispersion among agents’ beliefs about future states of growth and the disagreement among them as to which model is more likely to generate the current data on fundamentals. While the first is intuitive, the meaning and impact of disagreement merits further clarification. The ‘disagreement value’, as we will refer to it, in equilibrium is the ratio of the two agents’ state price densities. Since agents agree on all prices, it is a ratio of the relative likelihoods that the current data on fundamentals was generated by the model of type 2 agents (for brevity we will simply refer to these agents collectively as agent 2) rather than agent 1. The level of this variable is shown to depend on the past performance of the two agents’ models. After a period in which the models perform comparably, the ratio is near its mean value (we say there is low model disagreement), agents consumptions become about equal, and each agent has a larger risk exposure to price movements due to beliefs of the other type. Conversely, after a period of dominance by say agent 2, the ratio becomes large, agent 2 consumes a larger share of output, and the price variability due to agent 1’s beliefs declines. Therefore, in periods of high model disagreement, there is a low exposure to trading risk.

In their original article, Mehra and Prescott (1985) suggested the lack of perfect insurability due to incomplete markets as a promising direction for the resolution of the puzzle. This direction was followed by Mankiw (1986). However, subsequent articles suggested that if agents are allowed to trade in stocks and bonds, they are able to diversify most of their idiosyncratic risk, since asset prices in equilibrium do respond to changes in the aggregate income distribution (Telmer 1993, Lucas 1994, Heaton and Lucas 1996). Constantinides and Duffie (1996) were able to resolve most aspects of the puzzle with permanent idiosyncratic shocks in an economy where individuals were content not to trade in equilibrium, and are thus unable to hedge the idiosyncratic shocks. Our model has three significant differences: First, individuals do not receive idiosyncratic shocks, but instead, there is idiosyncratic variation in their beliefs. As in the case of incomplete markets, their marginal rates of substitution in equilibrium are not equated because of their inability or unwillingness to insure each other from belief movements. Second, individuals trade in equilibrium. Stock and bond prices do depend on the distribution of beliefs. However, despite the consumption smoothing attained by trading in financial assets, the difference in opinions remains, and agents continue to face an exposure to the trading risk. Therefore, despite trading, the model equity premium does

¹In this sense, even though the number of traded long-lived securities equals the number of shocks in the economy, so that markets are complete, they are effectively incomplete.
not shrink. Finally, trading losses that lead to endogenous shocks to consumption in our model are not permanent, and hence do not cause a trend increase in the cross-sectional standard deviation of consumption growth across agents as in Constantinides and Duffie (1996).

Models of heterogeneous beliefs have become increasingly popular in recent years. As noted by early writers in this field (seminal papers in the field are Lintner 1969, Williams 1979, Varian 1989, Harris and Raviv 1993, Kandel and Pearson 1995), these models are able to generate patterns in trading volume because agents have differing opinions and agree to disagree after observing the same observed information. This is their main advantage over models based on asymmetric information, in which beliefs converge upon observing trades. Among more recent work, Shefrin (2001) and Anderson, Ghysels, and Jurgens (2005) study asset pricing implications of heterogeneous beliefs models. None of these papers explicitly study the equity premium and the volatility of consumption moments of agents, which is the subject of this paper.

More directly, our analysis extends the analysis of continuous time models of heterogeneous beliefs (Detemple and Murthy 1994, Zapatero 1998, Basak 2000, Basak and Croitoru 2000, Buraschi and Jiltsov 2002) to the case of recurrent jumps in the underlying drift of diffusion process. Crucially, the dispersion process in these models monotonically declines over time and asymptotes to zero. Therefore, the dispersion of beliefs across agents will have a temporary effect on the conditional risk premium, but will be unable to match the large risk premium in long samples of data. In contrast, in our analysis, agents have different underlying models of the data generating process as opposed to differing initial priors in the above papers and the dispersion process recurrently fluctuates and leads to a large equity premium over long horizons.

We calibrate our model to fundamental data (aggregate earnings and consumption) and a series of dispersion of earnings forecasts obtained from the Survey of Professional Forecasters. We extend existing methodologies of the maximum likelihood procedure for regime-switching models (Hamilton 1989, Hamilton 1994) to a Generalized Method of Moments (GMM) method that can estimate heterogeneous parameters for two groups of agents. As part of the procedure, we estimate a set of belief series that agents of each type hold of fundamentals being in a strong growth phase. The procedure puts weight on the likelihood of each type of agent observing the fundamental data, as well as the dispersion in beliefs among the two groups.

We find that the conditional expectations of the two types of agents are highly correlated, yet they display some important differences. Unconditionally, agent 1 underestimates earnings growth, while agent 2 is almost unbiased. We will see that in equilibrium, these parameter values imply that agent 2 estimates an unconditional equity premium close to its objective value, while agent 1
underestimates it unconditionally. Agent 1 is not only more pessimistic on average, but his beliefs are also more volatile than agent 2. As we will see, the beliefs of agent 1 overshoot the beliefs of agent 2 in each direction, as strong and weak fundamentals data appear. In addition, the dispersion in agents’ expectations is highly countercyclical. This feature stems from a large disagreement in the two agents’ estimates of the transition probability from weak to strong earnings growth, and significantly lower disagreement on the reverse transition probability. This turns out to be a key factor in causing a larger equity premium during periods of weak growth. The natural question arises: does one agent’s model dominate the other agent’s model? In fitting the process of earnings growth, the two models are close with $R^2$ s of 65 and 67 percent. In addition, the consumptions of agents whose trading strategies are determined by these models grow at about the same rate over the 30-year sample.

We will see that in our calibrated model the Sharpe ratio attains a value in the range of about 9 – 15 percent, a little less than half the value for the aggregate US stock market, stock volatility is generated at plausible levels of about 18 percent, and the equity premium is about 2.5-3 percent when agents have a coefficient of relative risk aversion (CRRA) in the 0.4 – 0.7 range. Moreover, the average riskless return is lower by almost one percent compared to a benchmark model with homogeneous beliefs and has a very low volatility as in the data. The CRRA being less than one is crucial to our analysis, because it implies that both agents view times of increased trading risk as also a time of increased speculative opportunities. The increase in the size of their speculative positions leads to increased stock and consumption volatilities at such times, causing a larger risk premium, and their desire to save more lowers the riskless rate. Most importantly, the risk premium in our model is somewhat paradoxically decreasing in agents’ CRRA. This effect obtains in our model because the exposure to trading risk is endogenous and increases as less risk-averse agents undertake more aggressive strategies and increase both the risk (volatility) of stocks, and the price of risk (the product of their CRRA and consumption volatilities).

Since agents in our model have time-separable preferences and face no trading frictions, standard conditional Euler equations relating conditional risk premia with consumption moments apply, and we study the implications of our equilibrium for moments of individual consumption. We hold the volatility of aggregate consumption growth at about one percent a quarter, as is standard in the literature, and the main binding constraint in the equity premium literature. However, time-variation in trading risk leads to high and variable individual consumption volatilities. We use this degree of freedom relative to the papers using representative agent models. Indeed recent evidence (Attansio, Banks, and Tanner 2002, Brav, Constantinides, and Geczy 2002) suggests that the per

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2Interestingly, in a recent survey of financial economists, Welch (2000) found a wide dispersion in participants’ expected equity premia, including negative premia for some respondents.
capita volatility of quarterly consumption growth ranges between six and twelve percent. As we will see, our model is capable of generating such levels of heterogeneity in consumption growth, and explicitly calibrate to the mid-point of this interval. However, we show that the higher unconditional per capita consumption volatility applied to the unconditional consumption Euler equation justifies only a small fraction of the observed equity premium, and the properties of conditional moments are key to understanding why the model’s equity premium is sizable.

To study the conditional moments of our model, we partition our data into two sub-groups based on the relative optimism of the two agents. As discussed earlier, agent 1 (2) is relatively more optimistic in periods of strong (weak) earnings growth. The groups hold 43 and 57 percent of the quarterly observations in our sample. Our model implies that the distributions of all moments in the two groups are dramatically different, and so are the signs of the equity premia of the two agents. In periods of strong growth, the consumption of agent 1 (2) is positively correlated to stock returns and he is willing to bear a positive (negative) premium to hold long positions in stocks. The reverse holds in periods of weak growth. In line with our earlier comments on the asymmetric dispersion in weak and strong growth periods, consumptions and stocks are more volatile in periods of weak growth. The key features of individuals’ consumptions that help in generating a high unconditional equity premium are (i) close to unit (in absolute value) conditional correlations between individual consumption growth and stock returns, which switch sign over time depending on the relative optimism of the two agents, (ii) large covariances between individuals’ consumption and stock volatilities, which imply higher market prices of risk in periods of high risk, and generate high conditional equity premia, and (iii) high cross-sectional variation in individuals’ consumption growths in periods of weak fundamental growth, which generates a countercyclical equity premium. We note that the average model-generated cross-sectional standard deviation is close to its empirical estimates in Jacobs and Wang (2004) and Storesletten, Telmer, and Yaron (2004), who also report that its variation is consistent with (iii).

We compare our results with those of other authors using models in which agents have time-separable preferences and unrestricted access to capital markets. Reitz (1988) is able to generate a large equity premium with low risk aversion if agents price “peso problem” like events, such as a 70 percent drop in consumption. Longstaff and Piazzesi (2003) get some improvements — half the premium with a CRRA of five — with smaller jumps. In a paper related to ours, Brennan and Xia (2001) model the learning process of homogeneous agents about the dividend process and find that with a CRRA of 10, equity volatility assumes empirically plausible levels and they are able to generate about half the equity premium. Ait-Sahalia, Parker, and Yogo (2003) use a value of 7 to generate the entire risk premium in a model with multiple goods. With the exception of Reitz...
(1988), none of the other papers can lower the riskless rate. In comparison, we generate half the observed equity premium with a CRRA of less than one.

The plan for the remainder of this paper is as follows: In Section I we present the basic structure of the model, and in Section II we characterize the equilibrium and find approximate solutions for asset prices and portfolio choices. A calibration of the model is provided in Section III, and the performance of the model in addressing the equity premium puzzle is discussed in Section IV. We conclude in Section V. Three technical appendixes cover some essential proofs, a detailed description of the projection method used to solve the PDE for asset prices, and the calibration methodology respectively.
I Structure of the Model

In this section, we introduce the assumptions of our economic setting.

**Assumption 1**: Dividends, $q_t$, evolve according to the log-normal process

$$\frac{dq_t}{q_t} = \theta_t \, dt + \sigma_q \, d\tilde{W}_t, \quad (1)$$

where $W_t = (W_{1t}, W_{2t})^\top$ is a two-dimensional vector of independent Weiner processes; the $1 \times 2$ constant vector $\sigma_q$ is assumed known by all investors and is constant over time. The process for $\theta_t$ is described below.

**Assumption 2**: Total output in the economy, $x_t$, evolves according to the log-normal process

$$\frac{dx_t}{x_t} = \kappa_t \, dt + \sigma_x \, d\tilde{W}_t, \quad (2)$$

where the process followed by $\kappa_t$ is described below and $\sigma_x$ is a $1 \times 2$ constant vector known by investors.

It is convenient to stack together the “observation” processes (2), and (1): Let $y = (q, x)^\top$, so that

$$\frac{dy_t}{y_t} = \nu_t \, dt + \Sigma \, d\tilde{W}_t,$$

where $\frac{dy_t}{y_t}$ is to be interpreted as “element-by-element” division, $\nu_t = (\theta_t, \kappa_t)^\top$, and $\Sigma = (\sigma_q^\top, \sigma_x^\top)^\top$. We assume that $\Sigma$ is invertible.

**Assumption 3**: $\nu_t$ follows an $N$-state, continuous-time finite state Markov chain with generator matrix $\Lambda$, that is, over the infinitesimal time interval of length $dt$

$$\lambda_{ij} dt = \text{prob}(\nu_{t+dt} = \nu_j | \nu_t = \nu_i), \quad \text{for} \ i \neq j,$$

$$\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}.$$ 

See, for example, Karlin and Taylor (1982) for a precise definition of the generator matrix. The transition matrix over any finite interval of time, $s$, is obtained from the generator matrix simply as $\exp(\Lambda s)$, where $\exp(A) = \sum_{j=1}^\infty A^j / j!$.

Assumptions 1-3 imply that real dividends and output follow a joint log-normal model with drifts that follow a regime switching model. Following Cecchetti, Lam, and Mark (1990) and Brennan and Xia (2001), dividends are modeled as a part of the entire output of the economy.
Therefore, the claim to dividends, the stock, will be in zero net supply. Aggregate (across agents) consumption in the economy will equal aggregate output. Agents can take ‘bets’ on the stock price as a vehicle for risk sharing. Assumption 3 ensures that agents can use information in dividends to make inferences about the future level of output, and vice versa.

**Assumption 4**: All agents have time separable utility functions over infinitely dated stochastic consumption streams:

\[ U(c) = E^{(m)} \left[ \int_0^\infty \exp(-\rho s) \cdot u(c_s) ds \right], \]

with time discount factor \( \rho \), and felicity \( u(c) = c^\gamma / \gamma \).

The felicity function \( u(c) \) displays constant coefficient of relative risk aversion \( 1 - \gamma \). For all the results in the paper we will use \( 0 < \gamma < 1 \), for which cases the Inada conditions \( \lim_{c \to 0} u'(c) = \infty \), and \( \lim_{c \to \infty} u'(c) = 0 \) hold.

**Assumption 5**: Agents of type \( m \) are collectively endowed with a constant fraction \( e^{(m)} \) of output \( x_t \) in the economy at period \( t \). The endowment at time \( t \) is \( \epsilon_t^{(m)} = e^{(m)} x_t \), where \( x_t \) follows the process in Assumption 2.

**Assumption 6**: Individuals can trade in a short-term (instantaneous) riskless security, and two long term securities, stocks and consol bonds. Stocks pay a continuous dividend described by the process in Assumption 1. Bonds pay a continuous coupon of \( c \) per instant. We denote the prices of stocks and bonds at time \( t \) with the letters \( P_t \) and \( B_t \) respectively. Investors’ portfolio choices in these assets are \( w_{P_t} \) and \( w_{B_t} \) respectively and we impose constraints on admissible strategies that prohibit negative wealth at any future date (see Dybvig and Huang 1989). Both stocks and bonds are in zero net supply. We note that the number of long term assets equals the number of stochastic shocks driving the economy, so markets are dynamically complete.

The last assumption makes it possible to obtain fluctuating aggregate uncertainty and dispersion, which are the objects of the investigation of this paper.

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3 This results because we have a pure exchange economy with no explicit production process or fruit-bearing tree as in Lucas (1978). The Euler equation for asset prices is the same as that in an economy where the stock is in positive net supply, although the market prices of risk will in general differ from the latter case. We note that in Mehra and Prescott (1985) the ratio of aggregate dividends to consumption is identically one in each period, while in our model it is identically zero. As evident from the analysis in Cecchetti, Lam, and Mark (1990) and Brennan and Xia (2001), the equity premium puzzle is robust to the assumption of stocks being in zero net supply. We will see later that retaining all our assumptions but instead assuming that agents have homogeneous beliefs, our model equity premium is very small as well. Empirically, stocks in the U.S. are in ‘small’ positive net supply: Cecchetti, Lam, and Mark (1990) note that the dividend-consumption ratio is of the order of two to five percent. We will also show by generalizing our model that the market prices of risk and riskless rates with such small supply of stock will be very similar to the case modeled here.
Assumption 7: There are $M = 2$ classes of investors. Investors do not observe the realizations of the drifts, $\nu_t$. An investor of class $m$ estimates that the vector of drift parameters, $\nu$, equals $\nu^{(m)}$, and the generator matrix, $\Lambda$, equals $\Lambda^{(m)}$. Investors of each type know the parameter values of all other types.

The assumption captures the notion that investors agree to disagree about the evolution of states if they have different underlying “models” of the state processes. Their perceptions are captured in different filtrations, $\mathcal{F}_t^{(m)}$, and probability measures, $\mathcal{P}_t^{(m)}$, of the states of fundamentals. We note that in the standard heterogeneous beliefs framework (for example Basak 2000) agents disagree on the prior distribution of relevant state variables. Our assumption is similar to that in Harris and Raviv (1993), in which investors have different likelihood functions of the relationship between observed signals and returns on assets, and each investor is absolutely convinced that her model is correct. It is also a form of the over-confidence model of Daniel, Hirshleifer, and Subrahmanyam (2001) in which each investor places an excessively large weight on his personal model.

Since investors do not observe $\nu_t$ (i.e. the drift rates of the process), they need to infer them from the observations of past earnings and output. This will generate a distribution on the set of possible states assessed by investor $m$ to be $\{\nu^{(m)}_1, \ldots, \nu^{(m)}_N\}$ that in turn generate changes in “uncertainty” as they learn about the current state. Given the observation of $y$, investors form the posterior probability

$$
\pi^{(m)}_{it} = \text{prob}(\nu_t = \nu^{(m)}_i|\mathcal{F}_t^{(m)}).
$$

Let $\pi^{(m)}_t = (\pi^{(m)}_{1t}, \ldots, \pi^{(m)}_{Nt})$ be the vector of beliefs.

Lemma 1 Given an initial condition $\pi^{(m)}_0 = \hat{\pi}^{(m)}_0$, with $\sum_{i=1}^N \hat{\pi}^{(m)}_i = 1$ and $0 \leq \hat{\pi}^{(m)}_i \leq 1$ for all $i$, the probabilities $\pi^{(m)}_{it}$ satisfy the N-dimensional system of stochastic differential equations:

$$
d\pi^{(m)}_{it} = \mu^{(m)}_{it} dt + \sigma^{(m)}_{it} d\tilde{W}^{(m)}_t,
$$

where

$$
\mu^{(m)}_{it} = [\hat{\pi}^{(m)}_t \Lambda^{(m)}]_i,
$$

$$
\sigma^{(m)}_{it} = \pi^{(m)}_{it} (\nu^{(m)}_i - \overline{\nu}(\pi^{(m)}_t)) \top (\Sigma^{-1})^{-1},
$$

$$
\overline{\nu}^{(m)}_t = \sum_{i=1}^N \pi^{(m)}_{it} \nu_i = E^{(m)}_t \left[ \frac{dy_t}{y_t} \right],
$$

and

$$
d\tilde{W}^{(m)}_t = \Sigma^{-1} \left( \frac{dy_t}{y_t} - E^{(m)}_t \frac{dy_t}{y_t} \right) = \Sigma^{-1} (\nu_t - \overline{\nu}^{(m)}) dt + d\tilde{W}_t.
$$

Moreover, for every $t > 0$, $\sum_{i=1}^N \pi^{(m)}_{it} = 1$. 

10
Proof. See Wonham (1964), Liptser and Shiryayev (1977), or David (1993).

The filtering theorem for jumps in the underlying drift was first derived (to the best of our knowledge) in Wonham (1964). The first application of this theorem in financial economics as well as several properties of the filtering process are derived in David (1997).

When \( N = 2 \), a case that we analyze in detail in this paper, then \( \pi_{1t}^{(m)} \), a scalar, is sufficient to characterize the distribution of agents of type \( m \). In this case,

\[
\mu_{1t}^{(m)} = (\lambda_{12}^{(m)} + \lambda_{21}^{(m)})[\pi_{1}^{(m)} - \pi_{1t}^{(m)}], \quad \text{and}
\]

\[
\sigma_{1t}^{(m)} = \pi_{1t}^{(m)}(1 - \pi_{1t}^{(m)})(\theta_1^{(m)} - \theta_2^{(m)}, \kappa_1^{(m)} - \kappa_2^{(m)}) \cdot (\Sigma^\top)^{-1},
\]

where \( \pi_1^{(m)} = \lambda_{21}^{(m)}/(\lambda_{12}^{(m)} + \lambda_{21}^{(m)}) \) is the unconditional mean of \( \pi_{1t}^{(m)} \). That is, \( \pi_{1t}^{(m)} \) mean reverts to its unconditional mean with a speed proportional to \( (\lambda_{12}^{(m)} + \lambda_{21}^{(m)}) \), and the volatility of an agent of type \( m \)'s updating process is the product of his uncertainty, \( \pi_{1t}^{(m)}(1 - \pi_{1t}^{(m)}) \) and the signal-to-noise ratio, \( (\theta_1^{(m)} - \theta_2^{(m)}, \kappa_1^{(m)} - \kappa_2^{(m)}) \cdot (\Sigma^\top)^{-1} \).

For later reference, we rewrite the fundamental process \( dy_t/y_t \) in terms of the new innovation process \( d\tilde{W}_t^{(m)} \). The fundamental process \( dy_t \) follows the process

\[
\frac{dy_t}{y_t} = \bar{\nu}_t^{(m)} dt + \Sigma d\tilde{W}_t^{(m)}.
\]

Hence, for example, the dividends process with respect to the filtration of investors of type \( m \) is

\[
\frac{dq_t}{q_t} = \bar{\theta}_t^{(m)} dt + \sigma q d\tilde{W}_t^{(m)}.
\]

Since \( d\tilde{W}_t^{(m)} \) is an “innovations” process under the investors’ filtration, under the separation principle it can be used for dynamic optimization. See David (1997) for a discussion. The difference between the two innovations processes is given by

\[
d\tilde{W}_t^{(2)} = d\tilde{W}_t^{(1)} + \Sigma^{-1}(\bar{\nu}_t^{(1)} - \bar{\nu}_t^{(2)}) dt.
\]

As can be seen, agents’ estimated switching probabilities between drift states, \( \bar{\lambda}_{ij}^{(m)} \), can be substantially different and yet \( d\tilde{W}_t^{(2)} - d\tilde{W}_t^{(1)} \) is a term of the order \( O(dt) \). For reasons to be evident in the next section, we will write

\[
\sigma_{\eta t} = \Sigma^{-1}(\bar{\nu}_t^{(2)} - \bar{\nu}_t^{(1)}),
\]
which is of dimension $2 \times 1$. We will find it useful to write the belief of type 2 agents as a function of type 1’s innovation:

$$d\pi_t^{(2)} = [\mu_t^{(2)} - \sigma_t^{(2)} \eta_t] \, dt + \sigma_t^{(2)} \, d\tilde{W}_t^{(1)}. \quad (7)$$

Let $\nu^*(m) = \pi^*(m)\nu_1^{(m)} + (1 - \pi^*(m))\nu_2^{(m)}$ be the unconditional mean of fundamental growth assessed by investor $m$. We do not restrict $\nu^*(1) = \nu^*(2)$, that is the unconditional mean estimates of the two agents may differ. Since the parameter differences affect only drift rates, we show in the following corollary that the probability measures of the two agents are equivalent over any finite interval.

**Corollary 1** The restriction of agents’ probability measures $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ to the filtration at time $t$, $\mathcal{F}_t$, $\mathcal{P}^{(1)}_t$ and $\mathcal{P}^{(2)}_t$, are equivalent for all $t \in [0, \infty)$. The Radon-Nikodym derivative of $\mathcal{P}^{(2)}_t$ with respect to $\mathcal{P}^{(1)}_t$ is given by $\varrho_t = \varrho_0 \exp \left( -\frac{1}{2} \int_0^t \sigma_{\eta s}^T \sigma_{\eta s} \, ds + \int_0^t \sigma_{\eta s} \, d\tilde{W}_s^{(1)} \right)$ which is a martingale with respect to $\mathcal{P}^{(1)}_t$ on the time interval $[0, t]$ for all $t$.

Corollary 1 implies that the two probability measures are equivalent on the filtrations of agents at any finite time $t$. The measures may be mutually singular over the the infinite horizon when the long-term mean drifts of the two agents are not equal. We will see in the next section that the mutual singularity will not preclude the agreement on security values by the two agents, nor does it imply the existence of arbitrage opportunities.

**II Market Equilibrium**

A rational expectations equilibrium is a set of utility-maximizing consumption choices for each agent and conjectured prices for all securities in each date and state for each agent so that total consumption equals total output in the economy, markets clear and agents agree on prices in all dates and states. Due to the existence of two long-lived securities, markets are dynamically complete and ensure the existence of unique Arrow-Debreu (A-D) security prices for each agent under his own filtration. In equilibrium, agents will agree on these A-D prices as well.

We first examine individuals’ consumption choice problems. Each agent maximizes the utility function in Assumption 4 subject to the budget constraint

$$E^{(m)} \left[ \int_0^\infty c_s^{(m)} \xi_s^{(m)} \, ds \right] \leq E^{(m)} \left[ \int_0^\infty \xi_s^{(m)} \xi_s^{(m)} \, ds \right] \equiv X^{(m)}_0, \quad (8)$$

\[4\]Lemma 1 in Huang and Pagès (1992) shows that the measures are mutually singular if $\int_0^\infty \sigma_{\eta s} = \infty$, almost surely as is true if $\hat{\nu}^{(1)} \neq \hat{\nu}^{(2)}$. As a simple comparison, the objective and risk-neutral measures in the infinite horizon problems in Samuelson (1965) and Merton (1980) are mutually singular.
where \( \xi_t^{(m)} \) is his state-price density (SPD) function for consumption at \( t \), and is determined endogenously in equilibrium, and \( X_0^{(m)} \) is the value of his endowment as specified in Assumption 5 at period 0. The necessary conditions for optimality (see Karatzas, Lehoczky, and Shreve 1987, Cox and Huang 1989) are: 

\[
u'(c_t^{(m)}) = y_m \xi_t^{(m)} \text{ for each } m,
\]

which can also be written as 

\[
e^{(m)} = I_m(y_m \xi_t^{(m)}),
\]

where \( I_m(z) \) is the inverse of \( u_m^{(c)} \), and \( y_m \) is the Lagrange multiplier with respect to the budget constraint. By Assumption 4, the marginal felicity function is monotonically declining and satisfies the Inada conditions implying a unique solution for \( I_m(\cdot) \).

Using the SPD we can write the pricing kernel for an investor of type \( m \) as

\[
d\xi_t^{(m)} / \xi_t^{(m)} = -r_t dt - \phi_t^{(m)} d\tilde{W}_t^{(m)},
\]

in which the real rate of interest, \( r_t \), and market prices of risk, \( \phi_t^{(m)} \), will be determined endogenously. \( \phi_t^{(m)} \equiv (\phi_q^{(m)}, \phi_x^{(m)}) \) is the vector of market of risk at time \( t \) of investors of type \( m \) with respect to the two shocks.

We now define the disagreement value process: 

\[
\eta_t = \xi_t^{(1)} / \xi_t^{(2)},
\]

which is the ratio of the SPDs of the two agents. We first characterize the process followed by \( \eta \) written under the filtration of agent 1. The entire analysis can equivalently be performed under the filtration of agent 2.

**Lemma 2** Given the individual state price densities in (9), \( \eta \) follows the process

\[
\frac{d\eta_t}{\eta_t} = (\phi_t^{(2)} - \phi_t^{(1)}) d\tilde{W}_t^{(1)}.
\]

The proof is in Appendix 1.

Given the pricing kernel in eq. (9), the equilibrium price\(^6\) of a traded security \( i \) with a non-negative payout flow \( \{\delta_{it}\} \) is determined by individuals of type \( m \) as

\[
\xi_t^{(m)} P_{it} = E_t^{(m)} \left[ \int_t^\infty \xi_s^{(m)} \delta_{is} ds | \mathcal{F}_t^{(m)} \right].
\]

For equilibrium to exist, agents must agree on the level of prices at each date and state. This requirement puts restrictions on the risk premium on securities under the measures of the different agents and the objective measure, which we provide below. For security \( i \), we can write the

---

\(^{5}\)The existence of an optimal solution for the complete variational problem including consumption and portfolio choices can be established by verifying a boundedness condition on the felicity function, and Lipschitz and growth conditions on the SPD functions with respect to each state variable (for the finite and infinite horizon cases respectively, see Cox and Huang 1991, Huang and Pagès 1992). For \( 0 < \gamma < 1 \) it is straightforward to verify these conditions from the explicit functional forms for \( r \) and \( \phi^{(m)} \) in Proposition 2 below (see Footnote 9).

\(^{6}\)Throughout this paper we assume that the transversality condition, \( \lim_{t \to \infty} E^{(m)}[\exp(-\int_0^t r_s ds) P_{it}] = 0 \), holds for each agent so that only fundamental valuations are compatible with equilibrium.
dynamics of the price process under the objective measure as
\[
\frac{dP_{it}}{P_{it}} = \mu_{it}dt + \sigma_{it}d\tilde{W}_t,
\]  
(11)
or in terms of the information filtration of agent \(m\),
\[
\frac{dP_{it}}{P_{it}} = \bar{\mu}_{it}(m)dt + \sigma_{it}d\tilde{W}_t(m).
\]  
(12)

Using the definition of \(\tilde{W}_t(m)\) in (4), agreement by all agents on the level of prices at each date implies that
\[
\mu_{it} - \bar{\mu}_{it}(m) = \sigma_{it}\Sigma^{-1}(\nu_t - \bar{\nu}_t(m))
\]  
(13)

for each \(m\), a relationship that we will closely examine in Section IV. In addition, the expected returns of the two different agents are related by
\[
\bar{\mu}_{it}^{(1)} - \bar{\mu}_{it}^{(2)} = \sigma_{it}\Sigma^{-1}(\bar{\nu}_t^{(1)} - \bar{\nu}_t^{(2)}).
\]  
(14)

**Proposition 1** Agents agree on the level of prices at all dates and states if and only if
\[
(\phi_t^{(1)} - \phi_t^{(2)})^\top = \Sigma^{-1}(\bar{\nu}_t^{(1)} - \bar{\nu}_t^{(2)}).
\]  
(15)

The proof in Appendix 1.

It is relevant at this stage to point out why agents agree on prices despite having different probability measures over states of fundamental growth. Essentially, investors take bets on states of fundamental growth ‘trading away’ consumption from states which they think are less likely. Agents’ first order condition for optimization, \(u'(c_t^{(m)}) = \xi_t^{(m)}\), implies that dividends in states are priced so the large marginal utility compensates for the small probability of the state. Differences in unconditional drift rates are compatible with these valuations: at long horizons, each agent consumes nearly the whole endowment in states where the realized mean is close to the agent’s believed long-term mean but far from what other agents believe, since for agents with constant relative risk aversion, no agent can have negative consumption.

Proposition 1 establishes that in equilibrium the ratio of agent’s SPDs, \(\eta_t\), equals \(\varrho_t\), the Radon-Nikodym derivative of agent 2’s probability measure with respect to agent 1’s measure. Since the SPDs are the state prices per unit probabilities assessed by the two types of agents, and agents agree on all prices (including Arrow-Debreu state prices), \(\eta_t\) is the ratio of the likelihoods of observing the fundamentals at date \(t\) as realizations of the models of agent 2 to agent 1. Therefore, \(\eta_t\) will increase when the observed fundamental data at that date are more likely to arise from the model of
agent 2 rather than that of agent 1. Using Lemma 2 and (15) we can write

\[
\frac{d\eta_t}{\eta_t} = (\nu_t^{(2)} - \nu_t^{(1)})^T \Sigma^{-1} d\tilde{W}_t^{(1)} = \sigma_{\eta} d\tilde{W}_t^{(1)}. \tag{16}
\]

This formulation enables us to study the evolution of the disagreement process given the history of beliefs of each agent.

As can be seen, \(\eta_t\) increases under two conditions: (i) when agent 1 has a positive surprise \((d\tilde{W}_t^{(1)} > 0)\) and agent 2 is more optimistic than agent 1, or (ii) when agent 1 has a negative surprise and agent 2 is more pessimistic than agent 1. For example, in the calibrated equilibrium discussed in the next section, we will find that (i) will hold in 57 percent of the sample. In these cases, positive surprises to fundamentals lend more support to the model of agent 2. In the remaining 43 percent of the sample, positive surprises to fundamentals lead to decreases in \(\eta_t\), and lend more support to the model of agent 1. The intuition of the sign of these effects is quite straightforward: for example, if agent 1 receives a positive surprise, and agent 2 is more pessimistic than agent 1, then he would receive an even larger positive surprise, and the likelihood that the data were a realization of agent 1’s model would increase (\(\eta_t\) would decrease). The characterization explains why the correlation between agents’ consumption growths and stock returns is time-varying, and in fact switches sign over time.

As described in the introduction, \(\eta_t\) is a key state variable in our model determining the intertemporal variation in trading risk. In periods when \(\eta_t\) is close to its mean value, we will say there is low disagreement since the fundamentals at that date were as likely to have been a realization of either agent’s model. In such periods, there will be significant trading risk as each group of agents will have a large enough share of the market to move prices. Conversely, \(\eta_t\) gets far from its mean value when fundamental news supports one agent type’s model over the other. In the extreme cases, \(\eta_t \to 0(\infty)\), when news completely supports the model of type 1(2). In these limits, one agent group will dominate, and there will be no trading risk.

To facilitate the analysis of equilibrium, we follow the approach of Cuoco and Hè (1994) to solve for the equilibrium in the effectively incomplete markets model by formulating stochastic weights for the representative agent. The solution method was extended to models with heterogeneous beliefs by Basak (2000).\(^7\)

For given weights \(\lambda_{1t}\) and \(\lambda_{2t}\) for the two agents, the representative agent’s utility function solves:

\[
U(c_t; \lambda_{1t}, \lambda_{2t}) = e^{-\rho t} \max_{c_{1t} + c_{2t} = c_t} \lambda_{1t} \frac{c_{1t}^\gamma}{\gamma} + \lambda_{2t} \frac{c_{2t}^\gamma}{\gamma}.
\]

\(^7\)Some other important applications of this stochastic weights methodology are in Basak and Cuoco (1998) and Detemple and Serrat (1998).
Solving this problem straightforwardly gives the equivalent form:

\[ U(c_t; \lambda_{1t}, \lambda_{2t}) = e^{-\rho t} \frac{c_t^\gamma}{\gamma} \lambda_{1t} \left( 1 + \left( \frac{\lambda_{2t}}{\lambda_{1t}} \right)^{\frac{1}{1-\gamma}} \right)^{1-\gamma}. \] (17)

Following the analysis in Basak (2000), we formulate the equilibrium with the weights \( \lambda_{1t} = 1/y_1 \) and \( \lambda_{2t} = \eta_t/y_2 \), where \( y_1 \) and \( y_2 \) are the Lagrange multipliers associated with the budget constraints of the two agents at time 0 in eq. (8). It is evident that with these weights, consumption allocations coincide with those of competitive equilibrium: that is they satisfy \( u'(c^{(1)}_t)/u'(c^{(2)}_t) = (y_1 \xi_t^{(1)})/(y_2 \xi_t^{(2)}) \), the ratio of individuals’ optimality conditions, and by construction, the goods market clears. The following proposition characterizes equilibrium consumptions, the riskless rate, and the market prices of risk of the two agents. As a prelude, we will require the following lemma, which characterizes the consumption processes of the two agents.

**Lemma 3** To be consistent with utility maximization, the consumption process of an individual of type \( m \) follows the diffusion process \( dc_t^{(m)} = \mu_{ct}^{(m)} dt + \sigma_{ct}^{(m)} d\tilde{W}_t^{(m)}, \) with volatility and drift coefficients

\[
\sigma_{ct}^{(m)} = \frac{1}{a_t^{(m)}} \phi_t^{(m)} \quad \text{and} \quad \mu_{ct}^{(m)} = \frac{1}{a_t^{(m)}} r_t + \frac{1}{2} \frac{b_t^{(m)}}{a_t^{(m)^2}} \phi_t^{(m)} \phi_t^{(m)^\top},
\] (18)

where, \( a_t^{(m)} = -u''_m(c_t^{(m)}/u'_m(c_t^{(m)}) = (1-\gamma)/c_t^{(m)}, \) and \( b_t^{(m)} = -u''_m(c_t^{(m)})/u'''_m(c_t^{(m)}) = (2-\gamma)/c_t^{(m)}. \)

**Proof:** In Appendix 1.

The lemma shows in particular that the volatilities of individuals’ consumption growths are time-varying and equal the product of the inverse of the CRRA and the market prices of risk. Since the norm of the market prices of risk of agent \( m \) at any given time \( t \) will equal the conditional maximal Sharpe ratio (as perceived by agent \( m \)) of all assets in the economy, the volatilities of individuals’ consumption growths will summarize the information about conditional Sharpe ratios attainable by our model. As we will see though, neither the volatility of aggregate consumption growth nor the per capita consumption volatility will be sufficient statistics for conditional Sharpe ratios. We look at \( \phi_t^{(m)} \) carefully below.

**Proposition 2** In equilibrium,
(i) The individual consumption flow rates are

\[
c^1_t = \frac{x_t}{1 + k \eta_t^{1-\gamma}} \quad (20)
\]

\[
c^2_t = \frac{k \eta_t^{1-\gamma} x_t}{1 + k \eta_t^{1-\gamma}} \quad (21)
\]

where \( k = (y_1/y_2)^{\frac{1}{1-\gamma}} \).

(ii) The riskless rate in the economy is given by

\[
r_t = \rho - \frac{1}{2} (2 - \gamma) \left( 1 - \gamma \right) \sigma_x \sigma_x^{\top} + \frac{1 - \gamma}{1 + k \eta_t^{1-\gamma}} \left( \bar{\kappa}^1_t + k \eta_t^{1-\gamma} \bar{\kappa}^2_t \right) \]

\[
- \frac{\gamma k \eta_t^{1-\gamma} \left( (\bar{\theta}^1_t - \bar{\theta}^2_t) \sigma_x - (\bar{\kappa}^1_t - \bar{\kappa}^2_t) \sigma_q \right)^2}{2 (1 - \gamma) \left( 1 + k \eta_t^{1-\gamma} \right)^2 |\Sigma|^2}.
\]

(iii) Finally, the market prices of risk of the two types of agents are

\[
\phi^1_q = (1 - \gamma) \sigma_{x,1} + \frac{k \eta_t^{1-\gamma} \left( (\bar{\theta}^1_t - \bar{\theta}^2_t) \sigma_{x,2} + (\bar{\kappa}^2_t - \bar{\kappa}^1_t) \sigma_{q,2} \right)}{1 + k \eta_t^{1-\gamma} |\Sigma|}, \quad (23)
\]

\[
\phi^2_q = (1 - \gamma) \sigma_{x,1} + \frac{1}{1 + k \eta_t^{1-\gamma} |\Sigma|} \left( (\bar{\theta}^2_t - \bar{\theta}^1_t) \sigma_{x,2} + (\bar{\kappa}^1_t - \bar{\kappa}^2_t) \sigma_{q,2} \right), \quad (24)
\]

\[
\phi^1_x = (1 - \gamma) \sigma_{x,2} + \frac{k \eta_t^{1-\gamma} \left( (\bar{\theta}^2_t - \bar{\theta}^1_t) \sigma_{x,1} + (\bar{\kappa}^1_t - \bar{\kappa}^2_t) \sigma_{q,1} \right)}{1 + k \eta_t^{1-\gamma} |\Sigma|}, \quad (25)
\]

\[
\phi^2_x = (1 - \gamma) \sigma_{x,2} + \frac{1}{1 + k \eta_t^{1-\gamma} |\Sigma|} \left( (\bar{\theta}^1_t - \bar{\theta}^2_t) \sigma_{x,1} + (\bar{\kappa}^2_t - \bar{\kappa}^1_t) \sigma_{q,1} \right). \quad (26)
\]

Proof. In Appendix 1.

It is important to note that the riskless rate and the market prices of risk characterized above depend on the constant \( k = (y_1/y_2)^{\frac{1}{1-\gamma}} \). Therefore, the SPD functions of the agents (eq. 9) are also dependent on \( k \). We will determine \( k \) by ensuring that the budget constraints of the agents (eq. 8) are satisfied with equality with the assumed SPDs. Since this step will involve the solution of a PDE, we defer its discussion to section B. We provide a discussion of the qualitative features of the equilibrium below.

We first make some comments on the riskless rate in eq. (22). The first two terms in this riskless rate are standard, reflecting the time preference and the precautionary demand arising from the noise in the consumption process — with higher consumption volatility, agents’ demand for riskless
assets increases as they desire safer portfolios to offset risk, lowering the equilibrium real rate. The precautionary demand increases in the prudence of agents, captured by the term \((1 - \gamma) \cdot (2 - \gamma)\). The third term is the usual wealth effect on consumption: when the expected growth rate of consumption increases, agents are less willing to save for the future, leading to a higher equilibrium real rate. In our case, the expected growth rates of the two agents are weighted by their respective shares of total consumption.

The last term in the interest rate expression represents the ‘hedging’ demand term that arises due to the dispersions of agents’ expectations of growth rates. We explicitly discuss its two important parts, \(\left(\frac{k \eta_t^{1-\gamma}}{1 + k \eta_t^{1-\gamma}}\right)^2\), and a term that is proportional to \((\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})^2\). When either of these parts is large, agents will have a larger exposure to trading risk, but will also find better speculative opportunities. Consider the first part: It is a concave function of \(\eta\) with a maximum at \(\bar{\eta} = 1/k(1-\gamma) = y_2/y_1\). From (i), it is the product of the shares of consumption of the two types of agents. In periods of low disagreement \((\eta \simeq \bar{\eta})\), the share of each type of agent is near a half, and each group of agents can potentially impact market prices. Therefore, agents face the risk that prices move with the beliefs of each type. Since there is also a larger mass of agents to bet against at such times, agents also perceive higher speculative opportunities. Conversely, in periods of high disagreement \(|\eta - \bar{\eta}|\) large), there are fewer trading risks and opportunities. Similarly, the second term represents speculative opportunities that arise from the dispersion in agents’ beliefs — in periods of higher dispersion, the difference in expected growth rates increases (the square of the term implies that only the size of the difference matters and not its sign). In either case, low disagreement or high dispersion, agents’ savings response depends on their CRRA. For investors with CRRA larger than one, an improvement in opportunities makes them want to consume more currently due to a dominating wealth effect, causing a higher market clearing interest rate. Investors with CRRA less than one want to save and invest more currently due to a dominating substitution effect, leading to a lowering of the riskless rate.\(^8\) To resolve the risk-free rate puzzle we will assume that investors are of the latter type, that is with CRRA less than one.\(^9\) Notably in this case, the impact of trading risk is to drop the riskless rate below that of a benchmark economy in which agents have homogeneous beliefs.

We provide sample plots of the riskless rate in Figure 4 for the set of calibrated parameters (discussed in the next section). The left panel displays the effects of changing the disagreement

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\(^8\)This effect has been well known since the papers of Hakansson (1971) and Merton (1973) for the case of general state variables where investors have homogeneous beliefs. Our incremental contribution is to study the effect of changes in the opportunity set simultaneously for two types of agents, creating buying opportunities for one type of agent, and selling opportunities for the other type.

\(^9\) In addition, the derivative of the function \(\left(\frac{k \eta_t^{1-\gamma}}{1 + k \eta_t^{1-\gamma}}\right)^p\), for \(p = 1, 2\) with respect to \(\eta\) is bounded on \((0, \infty)\) for \(0 < \gamma < 1\), and hence the riskless rate and market prices of risk satisfy the sufficient Lipschitz conditions for the existence of investors’ variational problem in footnote 5.
value $\eta$, under the assumption that the two agents assess the same probability of currently being in an expansion state. The right panel holds the disagreement value constant, and displays the impact of changes in the dispersion of beliefs. In the left panel, given common expansion probabilities, the riskless rate is convex in $\eta$, with a minimum at $\eta = \bar{\eta}$, in line with our comments in the previous paragraph. If agents’ estimates of the drifts of the dividend process differ, then the term $(\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})^2$ can take on various shapes as a function of agents’ common probability of being in an expansion. For the calibrated case, we find that the term is at its lowest when this common belief is near 0.5, the point of maximum uncertainty, and it widens out as the common belief moves in either direction, to zero or to one. Therefore, the riskless rate is a concave function of the common expansion probability. Finally, the most interesting aspect of the riskless rate is shown in the right panel, in which we hold $\eta$ constant and consider the effect of dispersion of beliefs. As agents’ beliefs of the expansion states diverge, the dispersion in expected growth rates increases so that the riskless rate falls as we depart farther from the diagonal line. In our model calibrations, we will find that the two agents beliefs are highly correlated, but in periods of increased dispersion of beliefs, the rate drops significantly, and will help achieve an overall low average riskless rate.

Also as can be seen from eq. (22), the short rate in an economy with heterogeneous beliefs depends not only on current beliefs of agents through the terms $\bar{\kappa}_t^{(m)}$ and $\bar{\theta}_t^{(m)}$, but also on lagged beliefs through the disagreement value $\eta$. From (16), it is evident that $\eta_t$ can be written as

$$\eta_t = \exp \left[ \int_0^t -\frac{1}{2} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)})^\top (\Sigma \Sigma^\top)^{-1} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)}) ds + \int_0^t \Sigma^{-1} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)}) dW_s^{(1)} \right] \right), \quad (27)$$

that is, $\eta_t$ is an integral of weighted averages of past disagreements in drifts of earnings and consumption. In other words, the short rate process displays path dependence with respect to past differences in opinions of the different agents. In the special case that all agents believe that the set of structural parameters in the economy are the same, $\nu^{(1)} = \nu^{(2)}$, and $\Lambda^{(1)} = \Lambda^{(2)}$, since agents observe the same information, and have the same priors, $\kappa_t^{(1)} = \kappa_t^{(2)}$ and $\theta_t^{(1)} = \theta_t^{(2)}$ and $\eta_t = 1$ at each time, and the riskless rate in (22) collapses to

$$r_t = \rho + (1 - \gamma) \bar{\kappa}_t^{(1)} - \frac{1}{2} (2 - \gamma) (1 - \gamma) \sigma_x \sigma_x^\top \sigma^{(1)} \sigma^{(2)} \sigma^{(2)}.$$

Therefore, the dispersion of opinion drives the path dependence of the short rate. The path dependence of the short rate will be useful for obtaining a low volatility of interest rates in our model, a feature of the data.

Turning to the market prices of risk in (iii), we note these are comprised of two terms, the risk in aggregate fundamentals, $(1 - \gamma)\sigma_x$, and the term related to the speculative or trading risk. We focus our attention on the market price of risk with respect to the dividend shock. The reasoning
for the price with respect to the output shock is similar. For a given agent, the second term is the product of two components: the share of the other agent’s consumption of total output and a speculative component, which is proportional to the amount that an agent’s expected growth of earnings exceeds the other agent’s \((\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})\) for agent 1. Both terms have intuitive meanings: the first component implies that when an agent consumes a smaller share of output, he faces greater exposure to prices moving in the direction of the other agent’s beliefs. The second component implies that when an agent is more optimistic relative to the other, he faces potentially a larger correction of prices moving in line with the other agent’s beliefs, and his price of risk increases.\(^{10}\) Indeed the second term will be negative, when an agent is more pessimistic than the other agent. Finally, we note that as \(\gamma\) increases (agents are less risk averse) their consumption shares become more volatile, thereby giving larger weights to the dispersion terms in the trading risk component.

In our empirical section, we will find that for a region of low risk aversion, this effect can dominate the first effect (risk in aggregate fundamentals), causing Sharpe ratios to increase for lower CRRA.

Sample plots of the market price of agent 2 for the set of calibrated parameters are shown in the bottom panel of Figure 4 and display several of the important features of the trading risk component. The left panel shows the effect of the disagreement value on the price of risk when agents hold common beliefs. As we will see, when the two types of agents have common probabilities, agent 2 has a higher expected growth rate of dividends when a recession is more likely. For this case, when \(\eta\) increases, the weight of agent 1 decreases, lowering agent 2’s trading risk, and lowers his market price of risk. In expansionary times, agent 2 has smaller expected growth rate than agent 1, and has a negative price of risk. Once again, the effect of increasing \(\eta\) is to lower his trading risk and make his price of risk less negative. The right panel displays the above noted speculative component as well: in particular, the price of risk of agent 2 reaches a maximum when he is 100 percent sure that the dividend growth is in an expansionary phase, and agent 2 is 100 percent sure it is in a contractionary phase. The prices of risk of agent 1 (not shown) are near mirror opposites of those of agent 1.

We find it useful to consider the sub-case of our model when agents have homogeneous beliefs and the drift of aggregate consumption is non-stochastic. This case was analyzed by David and Veronesi (1999). As was shown by these authors, under these conditions, the equity premium takes the simple form \((1 - \gamma)\sigma_x \sigma_q\). That is, even though stock volatility will fluctuate with the single agent’s beliefs over the dividends drift, the component of volatility arising from belief variation does not command a market risk premium. Because the consumption drift is non-stochastic, the beliefs and consumption are uncorrelated, and only the volatility arising from noise in the dividends

\[^{10}\]While agents trade on their own models as in models of investor over-confidence (for example Daniel, Hirshleifer, and Subrahmanyam 2001), their market price of dividend risk increases when their model predictions are more optimistic relative to traders of the other type.
process is priced in equilibrium. We will use this as a useful benchmark for addressing issues related to the equity premium puzzle since we find in our calibration exercise the drift rates of consumption growth to be non-stochastic.

It is interesting to note that agents’ market prices of dividend risk are non-zero despite stocks (a claim to future dividends) being in zero net supply. The intuition behind this result is that fluctuations in dividend growth lead to divergence in opinions about future growth rates and trading possibilities, which in turn lead to fluctuations in individual consumptions. Therefore, each agent demands a risk premium to bear these shocks. Once again, with zero dispersion in beliefs, this channel disappears.

A natural question that arises is the relationship the risk premium under the objective measure in the model and agents’ beliefs. We summarize this relationship in the following corollary.

**Corollary 2**

\[
\mu_i - r = (1 - \gamma) \sigma_x \sigma_i^T + \left[ \sum_{m=1}^{2} \frac{c_t^{(m)}}{x_t} \Sigma^{-1} (\nu_t - \nu_t^{(m)}) \right] \sigma_i^T. \tag{29}
\]

The premium under the objective measure is thus the sum of the premium in the benchmark economy with homogeneous beliefs and the consumption share weighted estimation errors of the two agents. It follows at once that the conditional premium is \((1 - \gamma) \sigma_x \sigma_i^T\) if both agents are conditionally unbiased. We will see the following case will be particularly relevant in analyzing the premium in our calibrated economy in the next section: \(\Sigma^{-1}(\nu_t - \nu_t^{(1)}) = (\delta, 0)\) and \(\Sigma^{-1}(\nu_t - \nu_t^{(2)}) = (0, 0)\). In this case, agent 1 is pessimistic about earnings growth and agent 2 is unbiased, in which case the premium is \((1 - \gamma)\sigma_x \sigma_i^T + \frac{c_t^{(1)}}{x_t} \delta \sigma_i,1\). The average premium under the objective measure in a long sample will remain large only if the consumption share of agent 1 remains large despite trading using a biased estimate.

Lastly, we comment on the form of the representative agent’s utility function. Using the equilibrium stochastic weights, \(\lambda_{1t} = 1/y_1\), and \(\lambda_{2t} = \eta_t/y_2\), we can write

\[
U_c = e^{-\rho t} x_t^{\gamma-1} \frac{1}{y_1} \left[ 1 + \left( \frac{y_1 \eta_t}{y_2} \right)^{1/(1-\gamma)} \right]^{1-\gamma},
\]

where \(\eta_t\) follows the path-dependent process in (27). Therefore, the marginal utility for consumption of the representative agent at time \(t\) is affected by the entire history of past beliefs in a non-Markovian manner, making this agent similar (albeit with a different functional form) to models of external habit formation such as Campbell and Cochrane (1999). Since \(\eta\) determines the equilibrium consumption shares of the two agents, using (i) of Proposition 2 we can also write

\[
U_c = e^{-\rho t} x_t^{\gamma-1} \frac{1}{y_1} \frac{1}{S_t^{1-\gamma}},
\]
where \( S_{1t} \) is equilibrium consumption share of agent 1, and it follows a path-dependent process. This formulation of the marginal utility reveals why an econometrician who looks only at aggregate consumption data and constructs discount rates as \( e^{-\rho t} x_t^{\gamma-1} \) will be unable to explain asset price movements, which will instead also respond to the distribution of consumption shares in the economy. The equilibrium in such models will have high time-varying risk premia and smooth relatively low real rates. In fact, the equilibrium in a model with complete information and exogenously given state variables \( \{\pi_t^{(m)}\} \), for \( m = 1, 2 \) and \( \{\eta_t\} \), will have the same equilibrium rates and market prices or risk as our incomplete information model. The contribution of our analysis is to show that such an equilibrium arises from heterogeneous learning of agents with standard preferences, and that the endogenously generated dispersion is significantly related to dispersion of forecasts in survey data. Moreover, the \( \eta_t \) process arises due to the incomplete risk sharing among individuals with heterogeneous beliefs.

### A Asset Prices

Since there are two shock processes in the economy, agents require at least two multi-period securities in addition to instantaneous bonds to complete the market (in the traditional sense of market completeness). We implement the equilibrium with stocks paying the dividend process in Assumption 1, and consol (perpetuity) bonds paying a constant coupon flow \( c \). We briefly describe the pricing of these securities below.

The prices are functions of beliefs of each type of agent, and the disagreement value process. We use standard no-arbitrage analysis (see, for example Cochrane 2001) to value these securities. Since agents agree on prices of all assets, we formulate the PDE under the filtration of agents of type 1. The stock price \( P(\pi^{(1)}, \pi^{(2)}, \eta, q) \) is obtained by solving

\[
E^{(m)} \left[ \frac{dP}{P} \right] + \left( \frac{q}{P} - r(\pi^{(1)}, \pi^{(2)}, \eta) \right) dt = -E^{(m)} \left[ \frac{dP}{P} d\xi^{(m)} \right].
\]

We show in Appendix 2 that \( P(\pi^{(1)}, \pi^{(2)}, \eta, q) = p(\pi^{(1)}, \pi^{(2)}, \eta) \cdot q \), where the price-dividend ratio \( p(\cdot) \) satisfies the PDE:

\[
0 = p_{\pi^{(1)}} \left( \mu^{(1)} + \sigma^{(1)} (\sigma_q - \phi^{(1)} \sigma^{(1)}) \right) + p_{\pi^{(2)}} \left( \mu^{(2)} + \sigma^{(2)} (\sigma_q - \phi^{(1)} \sigma^{(2)}) - \sigma_\eta \right) + p_\eta \eta \left( (\sigma_q - \phi^{(1)} \sigma^{(2)}) \sigma_\eta \right) \\
+ p \left( \bar{\theta}^{(1)} - \phi^{(1)} \sigma^{(1)} \right) + 1 - r(\pi^{(1)}, \pi^{(2)}, \eta) p + \frac{1}{2} p_{\pi^{(1)}} \sigma^{(1)} \sigma^{(1)} + p_{\pi^{(2)}} \sigma^{(1)} \sigma^{(2)} + \frac{1}{2} p_\eta \eta \sigma^{(2)} \sigma^{(2)} + p_{\pi^{(1)}} \eta \sigma^{(2)} \sigma^{(1)} + p_{\pi^{(2)}} \eta \sigma^{(2)} \sigma^{(2)} + \frac{1}{2} p_\eta \eta \sigma^{(2)} \sigma^{(2)} \eta \eta,
\]

\[ (31) \]
where the partial derivatives with respect to beliefs of type $m$, are written in vector form, for example, $p_{π^m} = \left( p_{π_1^m}, p_{π_2^m}, \ldots, p_{π_N^m} \right)$. In this paper we solve only for prices in the two-state case, and hence consider only the partial derivatives with respect to $π_1^m$. The bond price-coupon ratio satisfies a similar PDE (eq. (57)). Boundary conditions for these PDEs are discussed next.

When the disagreement value $η → 0(∞)$, equilibrium asset prices converge to the prices in the single agent benchmark economies inhabited by agents solely of type 1 (2). These pricing values are available in closed-form and enable us to pin down the value of the PDEs at the two boundaries of $η$. We provide these boundary conditions in the PDE below.

**Proposition 3** The following pricing multiples obtain in economies inhabited solely by agents of type $m$:

(i) The equity price-dividend ratio:

$$p^{(m)}(π_t^{(m)}) = \sum_{i=1}^{N} C_i^{(m)} \cdot π_{it}^{(m)},$$

in which $C_i^{(m)}$ are positive constants satisfying $\hat{θ}_i^{(m)} \cdot C_i^{(m)} = 1 + \sum_{j=1}^{N} λ_{ij}^{(m)} \cdot C_j^{(m)}$, and $\hat{θ}_i^{(m)} = ρ + (1 - γ)κ_i^{(m)} - \frac{1}{2}(1 - γ)(2 - γ)σ_xσ_x^T$.

(ii) The bond price-coupon ratio:

$$b^{(m)}(π_t^{(m)}) = \sum_{i=1}^{N} D_i^{(m)} \cdot π_{it}^{(m)},$$

in which $D_i^{(m)}$ are positive constants satisfying $\hat{r}_i^{(m)} \cdot D_i^{(m)} = 1 + \sum_{j=1}^{N} λ_{ij}^{(m)} \cdot D_j^{(m)}$, and $\hat{r}_i^{(m)} = ρ + (1 - γ)κ_i^{(m)} - \frac{1}{2}(1 - γ)(2 - γ)σ_xσ_x^T$.

**Proof.** Follows from straightforward extensions of Proposition 1 in David and Veronesi (1999). See also Veronesi (2000).

In the homogeneous agent benchmark economies, pricing multiples are the conditionally expected values of these same multiples in each of the N possible states. For example,

$$C_i^{(m)} = \frac{1}{u_c^{(m)}(c_t^{(m)})q_t} E_t^{(m)} \left[ \int_t^∞ u_c^{(m)}(c_s^{(m)})q_sds \mid ν_t = ν_t^{(m)} \right],$$

is type $m$ agents’ expectation of future dividends conditional on the state at $t$ being $ν_t^{(m)}$, discounted by the marginal utility of consumption. We note that in the homogeneous agent economies, $u_c^{(m)}(c_t^{(m)}) = ξ_t^{(m)}$, where $ξ_t^{(m)}$ is still specified as in (9), but where the interest rate simplifies as in (28), and the market prices of risk collapse to $φ^{(m)} = (1 - γ)σ_x$. 

23
We now specify the conditions at the boundaries of the disagreement value and the belief process for the PDE (31). \( p(\pi^{(1)}, \pi^{(2)}, 0) = p^{(1)}(\pi^{(1)}) \), and \( \lim_{\eta \to \infty} p(\pi^{(1)}, \pi^{(2)}, \eta) = p^{(2)}(\pi^{(2)}) \). The latter condition is imposed at an arbitrarily chosen large value for \( \eta \). The results in the paper use an upper bound for \( \eta \) of 25, while empirically, we estimate the \( \{\eta\} \) process to have been in the interval \((0, 2.5)\) in our sample from 1971-2001. David (1997) shows that 0 and 1 are entrance boundaries for each of the belief processes \( \pi^{(m)} \). That is 0 and 1 cannot be attained from the interior of the belief state space. For such state variables, boundary conditions at 0 and 1 are endogenously generated by substituting \( \pi^{(m)} = 0(1) \), into (31).

Closed form solutions for the PDEs in this section, as well as the one for individuals’ wealth processes (in the next subsection) are not available. However, we are able to provide polynomial approximations to these PDEs using projection methods described for example in Judd (1992) and Judd (1999). In short, each PDE is elliptic and driven by three state variables, each agent’s beliefs, and the disagreement value process. The orthonormal basis is made by compositions of Chebyshev polynomials, of length 15 in each dimension. Overall, the basis for the three-dimensional space is of length 816. The coefficients of the basis elements are obtained by solving the system of equations that arises from Chebyshev collocation. A crucial part of approximating PDE solutions is the verification process, that is checking whether the candidate solution indeed solves the desired functional equation. We find that using the length of polynomials specified, we obtain a mean proportional absolute residual of less than 0.001.  

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Details of the procedure are given in Appendix 2.

Since these solutions are functions of smooth polynomials, we can also use their derivatives to provide approximated solutions for volatilities, risk premia, and, portfolio choices. The volatilities of stock and bond returns with respect to the two shocks are

\[
\sigma_p = \frac{p \pi^{(1)} \sigma^{(1)} + p \pi^{(2)} \sigma^{(2)} + p \eta \eta \sigma_\eta}{p} + \sigma_{\eta}, \quad \text{(32)}
\]

\[
\sigma_B = \frac{b \pi^{(1)} \sigma^{(1)} + b \pi^{(2)} \sigma^{(2)} + b \eta \eta \sigma_\eta}{b}. \quad \text{(33)}
\]

It is useful to note that only the last term of the stock volatility expression arises from the volatility of dividends, the fundamental volatility. The first two terms arise from the volatility of the belief processes of the two types of agents, while the third term is the volatility from the disagreement value process. Since coupons are fixed, the bond price has similar terms but no volatility from fundamentals.

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11 The residual is the value of the right hand side of (31) evaluated for the candidate solution. Upon request, we will supply to the reader the candidate solutions and code to verify the size of the residuals.
B Portfolio Choices and Market Risk Premia

We use the method of Cox and Huang (1989) to solve for portfolio choices of the two types of agents. As pointed out by these authors, using their method, the portfolio choices with \( K \) multi-period assets can be obtained by solving \( K + 1 \) linear PDEs rather than a non-linear PDE that arises in solving directly the representative agent’s lifetime problem.

Let \( X_t^{(1)} \) be the wealth of agents of type 1 when they have made optimal portfolio and consumption choices. First, given their consumption choices in (20), \( X_t^{(1)} \) must satisfy

\[
X_t^{(1)} = \frac{1}{\xi_t^{(1)}} \mathbb{E}^{(1)} \left[ \int_t^\infty c_s^{(1)} \xi_s^{(1)} ds \right].
\] (34)

Due to the linearity of optimal consumption in output, \( x \), we can write

\[
X_t^{(1)}(\pi^{(1)}, \pi^{(2)}, \eta, x) = f^{(1)}(\pi^{(1)}, \pi^{(2)}, \eta) x.
\]

Using the dynamics of output, beliefs and disagreement value, the PDE that must be followed by \( f(\cdot) \) is:

\[
0 = \frac{1}{1 + k \eta \gamma} + f^{(1)}(\bar{\kappa}^{(1)} + f^{(1)}(\mu^{(1)} + \sigma^{(1)} \sigma_x^T) + f^{(1)}(\mu^{(2)} + \sigma^{(2)}(\sigma_x^T - \sigma_\eta)) + f^{(1)}(\eta \sigma_x \sigma_\eta
\]

\[+ \frac{1}{2} f^{(1)}(\pi^{(1)}) \sigma^{(1)} \sigma^{(1)\top} + f^{(1)}(\pi^{(2)}) \sigma^{(2)} \sigma^{(2)\top} + f^{(1)}(\pi^{(1)}) \eta \sigma^{(1)} \sigma_\eta
\]

\[+ \frac{1}{2} f^{(1)}(\pi^{(2)}) \sigma^{(2)} \sigma^{(2)\top} + f^{(1)}(\pi^{(2)}) \eta \sigma^{(2)} \sigma_\eta + \frac{1}{2} f^{(1)}(\eta \sigma_\eta \sigma_\eta^\top \sigma_\eta).
\] (35)

Boundary conditions for the PDE are once again specified using the valuations of homogeneous agents. As in Proposition 3, it is easily shown that the ratio of wealth of agents of type \( m \) to aggregate consumption is given by

\[f^{(m)}(\pi^{(m)}) = \sum_{i=1}^N E_i^{(m)} \cdot \pi_i^{(m)},
\]

in which \( E_i^{(m)} \) are positive constants satisfying \( k_i^{(m)} \cdot E_i^{(m)} = 1 + \sum_{j=1}^N \lambda_{ij}^{(m)} \cdot E_j^{(m)} \), and \( \kappa_i^{(m)} = \rho - \gamma \kappa_i^{(m)} + \frac{1}{2} \gamma (1 - \gamma) \sigma_x \sigma_x^\top. \)

Intuitively, wealth of agent of type \( m \) is formulated as the value of a security paying the entire output flow rate of the economy as a dividend. When solving the PDE for the wealth of agent 1, we use \( f^{(1)}(\pi^{(1)}, \pi^{(2)}, 0) = f^{(1)}(\pi^{(1)}) \), and \( \lim_{\eta \to \infty} f^{(1)}(\pi^{(1)}, \pi^{(2)}, \eta) = 0. \)

The latter condition results because agents 1’s consumption flow tends to zero as \( \eta \) becomes large. Once again it is implemented by imposing the value zero at an arbitrarily chosen large upper bound for \( \eta \).

The volatilities of type 1’s wealth with respect to the two shocks can be written as

\[
\sigma_x^{(1)} = \frac{f^{(1)}(\sigma^{(1)} + f^{(1)}(\sigma^{(2)} + f^{(1)}(\eta \sigma_\eta \sigma_\eta^\top + \sigma_x). \] (36)
Let the dollar positions of investor 1 in stocks and bonds be given by $w^{(1)}_t \equiv (w^{(1)}_{B_t}, w^{(1)}_{P_t})$. With these portfolio choices, the Ito representation of $X^{(1)}_t$ is

$$dX^{(1)}_t = X^{(1)}_t r_t dt + w^{(1)}_t (\mu_{B_t} - r_t, \mu_{P_t} - r_t)^\top dt + w^{(1)}_t (\sigma_{B_t}, \sigma_{P_t})^\top d\tilde{W}^{(1)}_t - c^{(1)}_t dt, \quad (37)$$

where the volatilities $\sigma_{B_t}$ and $\sigma_{P_t}$ are in (33) and (32) respectively. Since the volatilities of wealth in (36) and (37) must be the same, the portfolio choices must satisfy

$$w^{(1)}_t = \sigma^{(1)}_t X_t \cdot (\sigma_{B_t}^\top, \sigma_{P_t}^\top)^{-1} X^{(1)}_t, \quad (38)$$

where $B(\cdot)$ and $p(\cdot)$ solve the PDEs in eqs. (57) and (31) for the inverse of the consol yield and price-dividend ratio respectively, and $f^{(1)}$ solves the PDE (35) for the wealth-consumption ratio. Since output equals consumption in the model, both bonds and stocks are in zero net supply. Therefore, the portfolio choices of agents of type 2 are simply given as $w^{(2)}_t = -w^{(1)}_t$.

We end the characterization of the equilibrium by determining the constant $k = (y_1/y_2)^{1/\gamma}$, which is used in all the pricing formulas above. At the equilibrium level of $k$, the budget constraint of each individual in (8) at time 0 is satisfied with equality. We assume the initial beliefs of each agent to be at their unconditional values $\pi^{(m)}$ given below Lemma 1, and $\eta_0 = 1$. The value of agent 1’s endowment, the right hand side of (8), can be formulated as the solution of the PDE (35) with the flow rate of consumption $1/(1 + k \eta_0^{1-\gamma}) x$ replaced by the endowment flow $e^{(1)}_t x$ given in Assumption 5. Let us call this value $V^{(1)}(\pi^{(1)}_1, \pi^{(2)}_1, \eta_0)$. Then $k$ is the implicit solution to the equation

$$X^{(1)}(\pi^{(1)}_1, \pi^{(2)}_1, \eta_0; k) = V^{(1)}(\pi^{(1)}_1, \pi^{(2)}_1, \eta_0). \quad (39)$$

Notice that the left-hand side value $X^{(1)}(\pi^{(1)}_1, \pi^{(2)}_1, 1; k)$ is decreasing in $k$, while the right-hand side, $V^{(1)}(\pi^{(1)}_1, \pi^{(2)}_1, \eta_0)$ does not depend on $k$. We solve (39) by solving (35) for a set of different values of $k$ and iteratively searching over a grid of values of the left-hand side, $\{X^{(1)}(\pi^{(1)}_1, \pi^{(2)}_1, \eta_0; k_j)\}$ for $j = 1, \cdots, J$.

### III Calibration

In this section, we describe our methodology for estimating parameters for the fundamental processes used by the two sets of agents. The method finds the parameters which jointly maximize the likelihood of each set of agents observing the fundamental processes, as well as matching the dispersion of these agents’ forecasts to those available from surveys.
A Data Series

The fundamental series that we use are the real earnings of S&P 500 companies obtained from Standard and Poor’s, and the real consumption for non-durables and services obtained from the Federal Reserve Board. The regime-switching model with heterogeneous beliefs is fitted to these fundamentals a time series of dispersion in forecasts of earnings.

We construct a measure of dispersion using data from the Survey of Professional Forecasters, available at the Federal Reserve of Philadelphia. Reliable data are available from around 1970. Forecast are for horizons of $\tau = 0, 1, \ldots, 4$ quarters ahead, where $\tau = 0$ indicates a forecast for the current quarter, which typically ends 1.5 months after the deadline to submit the questionnaire back. We will focus our attention on the one quarter ahead forecast. The number of forecasters per quarter varies between 75 and 9 with an average of 34.

We use the cross-sectional dispersion of the earnings growth forecasts. Specifically, at each time $t$, let $F_{D_i}(t, \tau)$ be the forecast of nominal profit growth $\tau$ quarters ahead by forecaster $i$, and $F_{I_i}(t, \tau)$ be the forecast of the price level $\tau$ quarters ahead. We then define $F_{RD_i}(t, \tau) = F_{D_i}(t, \tau)/F_{I_i}(t, \tau)$, as a measure of forecasted real earnings growth. If $n_t$ is the number of individuals at time $t$, we then define the time $\tau$-quarters ahead dispersion of real earnings growth at time $t$ as

$$
    d(t, \tau) = \sqrt{\frac{1}{n_t - 1} \sum_{i=1}^{n_t} \left( \frac{F_{RD_i}(t, \tau)}{F_{RD}(t)} \right)^2}.
$$

(40)

To safeguard us against typos and mistakes, we deleted observations that were four standard deviations away from the mean forecast. The time series of dispersion so obtained is in the top panel of Figure 2.

B Calibration Methodology

As a first step, we test for the number of drift states for the two fundamental series in a standard regime-switching framework. To do this, we use likelihood ratio tests that adjust for the presence of nuisance parameters unidentified under the null hypothesis (for example, under the 1-state null hypothesis, the transition probabilities of a 2-state model are not identified). Exact critical values for the alternative of two states over the null of a single state are available in Table 1A of Garcia (1998). For earnings growth, the log likelihood ratio (LLR) attains a value of 14.1622, which has

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12 Following Longstaff and Piazzesi (2003) we assume that ‘true’ dividends are one half of earnings, an assumption that matches the average historical dividend yield, but has twice its volatility. We might add that the average historical equity premium calculated for our sample with this approximation is very close to the average premium calculated with historical dividends. This assumption on the payout series enables us to match the historical volatility of stock returns, while the pure dividend payout assumption does not.
a P-value of less than one percent. The LLR for a 3-state over a 2-state specification is very small, of the order of $10^{-3}$, suggesting insignificant gains in modeling a third state for the earnings drift. For aggregate consumption growth, the LLR for a 2-state over a 1-state model attains a value of 5.9212, with a P-value that lies between 50 and 70 percent. This is in line with the findings of other authors that aggregate consumption growth is nearly unpredictable (see Hall 1978). Therefore, we fit a model with two drift states of earnings and one for aggregate consumption growth.

The calibration methodology is described in detail in Appendix 3. In brief, we fit a set of drift and generator matrix parameters for each type of agent, and a common set of volatility parameters using the Generalized Method of Moments (GMM). Our method is quite similar to the maximum likelihood approach for regime-switching models in Hamilton (1989) and Hamilton (1994). As part of the procedure, a set or filtered probabilities (beliefs) for the agent over the underlying drift space agent are generated using Bayes’ law. We extend the method to find the best fitting parameters for two sets of agents who each observe the same fundamentals data, and the econometrician in addition observes a set of dispersion in their beliefs. The moments that we fit jointly in our GMM procedure are the scores of the likelihood function for each agent (similar to maximum likelihood), and the dispersion process. We then find the two best sets of parameters that jointly maximize the likelihood of each type of agent observing the fundamental processes, as well as the dispersion among them. The model based dispersion is generated by formulating beliefs for each agent for each quarter, once again by Bayes’ law, and taking the standard deviation cross-sectionally, as is done for the data series in eq. (40). We also use a discretized version of (27) to find the calibrated disagreement value process. We note, that because three volatility parameters are common to each agent, the number of moments exceeds the number of fitted parameters by four, implying that the GMM objective has a chi-squared distribution with four degrees of freedom. The optimized value serves as a specification test of the model.

C Calibrated Parameter Values and Implied Beliefs

The two sets of parameters are shown in Table I. The parameter values are quite compelling. The earnings drift estimates of agent 1 sandwich those of agent 2, suggesting that the former will have more volatile expectations. Agent 1 assesses that earnings growth in its contractionary phase is -26.9 percent, however, the chance of switching out of that state in the following quarter is 11 percent. He assesses growth in the expansion state (8.3 percent growth) to be more stable, and unconditionally, he expects that earnings is in its contraction state for about 40 percent of a long sample. Agent 2 estimates states of -25.6 and 6.8 percent growth, but estimates that the transition probability out of the contractionary state is 27 percent, much higher than agent 1. Due to the low persistence of this state, he expects its unconditional occurrence at 22 percent in long samples. In
summary, agent 2 estimates that the economy shifts out of recessions more rapidly than agent 1, however, he estimates that earnings grows less rapidly in expansions. The unconditional expected real growth rates of earnings of the two agents are about negative two and one percent respectively at an annual rate. Therefore, agent 1 has an unconditional bias of negative three percent, about 0.375 of a standard deviation from the standard deviation of annualized earnings growth. Agent 2 has an almost unbiased unconditional expectation. These biases have implications for the ex-ante expected equity premia of the two agents relative to the equity premium under the objective measure, which we will discuss in the following sub-section.

As we see in the the top panel of Figure 1, agents’ beliefs are highly correlated, although in line with our comments above, agent 1’s beliefs overshoot agent 2’s in each direction. For this reason we will often refer to agent 1 as the more volatile investor, and agent 2 as the more stable investor. The bottom panel shows that this is also true of the agents’ expected growth rate of earnings. A closer look at this panel also reveals that during downturns, agent 2 has higher expected growth rates, that is in these states the stable agent is more optimistic. Similarly, in upturns, the volatile agent is relatively more optimistic. In addition, the absolute difference in the expected growth rates is much higher in downturns. This will be an important observation when discussing the conditional equity premia of our model in the next section. The reason this happens is that the two agents disagree most strongly on the transition probability out of a recession state. Their estimates of transition probabilities out of the expansion states are more similar. Therefore, beliefs of each type are more volatile during recessions, and tend to be more dissimilar, causing higher dispersion at such times. Overall, the dispersion in the two agents’ expected earnings growth is asymmetric and counter-cyclical (higher in downturns).

As mentioned in the introduction, the two models have very similar fits for historical earnings growth. We report the regressions of actual growth on expected growth for each of the two types of agents in Table II, and find the $R^2$'s of the two agents’ models to be 64.7 and 67.3 percent respectively. Both intercepts are insignificantly different from zero, and their beta coefficients are highly statistically significant and very close to one. While agent 2’s model is slightly better in fitting current growth, in forecasting, agent 1’s model is more accurate at horizons of 1 and 2 quarter ahead (results not shown in table).

We create a model-based series of dispersions in forecasts by taking the standard deviation of forecasted growth of the two agents for any given horizon. The construction is analogous to the one from survey data in (40). The actual and model-based dispersions are shown in the top panel of Figure 2. As seen, both data and model-based series tend to increase in and around recessions of the U.S. economy. The two series are strongly positively correlated. A regression of the historical dispersion measure on its model-based counterpart yields an $R^2$ of nearly 20 percent (Table II). The
The beta coefficient of the regression is about 0.71, and is not significantly different from 1, however the model-based dispersion is on average lower, leading to a positive intercept term. Nonetheless, the overall chi-squared statistic for model, which penalizes fitting errors of both agents and the dispersion error, is low with a p-value of more than 0.1.

The calibrated disagreement value process is shown in the bottom panel of Figure 2. As a reminder to the reader, the disagreement value is the ratio of the SPD functions of investors types 1 and 2. Some comments on the dynamics of this process were made following Lemma 2. As discussed, when agent 2 is more optimistic than agent 1, then $\eta$ rises following positive shocks to earnings. In our sample, this happens in 57 percent of the quarters. As pointed out in the discussion below eq. (27), the disagreement value process, $\eta$ is path dependent in beliefs, and is hence slow moving. This should make the disagreement value smoother than expected growth rates which is evident by comparing it with the series in the bottom panel of Figure 2. As seen, it has fewer local peaks and troughs than agents’ beliefs, and plays a role in our model in generating smoother consumption series. Since this process is martingale with a starting value of one, in a long sample, it should exceed one for half the sample periods. In our sample, the process exceeds one in about 63 percent of the sample. Another relevant point is that even though dispersion is strongly countercyclical, the disagreement value process, being slow moving, is not. In fact, the levels of disagreement value during past recessions have differed quite significantly.

Equilibrium consumption levels of the two types of agents are completely determined by the disagreement value process and realized output in the economy as seen in eqs. (20) and (21). We examine the calibrated consumption levels of the two types of agents in the top panel of Figure 3. We show the levels rather than the growth rates, which are more relevant for the equity premium, simply for clearer exposition. Agent 2’s consumption increases in $\eta$, therefore their consumption growth rate is strongly positively correlated to positive earnings shocks (and as we will see stock returns) in periods when agent 2 is relatively more optimistic. Conversely, in periods when agent 1 is more optimistic, the consumption growth of agent 1 (2) will have a positive (negative) correlation with earnings shocks. The figures show that the consumption of agent 1 increases faster in and around most NBER-dated recessions, and other periods of low growth in earnings. Agent 2’s consumptions follow the opposite pattern. Both series have positive trends, and the two are negatively correlated. It is also worth noticing that starting from the same level, the calibrated consumption paths crossed about twelve times in the thirty-year sample, although by the end of the sample, the consumption of agent 1 outpaced that of agent 2. In addition, we notice fairly long swings of the domination of a type 1, followed by rapid declines, that bring their average growth rates about level. As we will see in the next section, the volatilities of the two agents’ consumption growths vary significantly over time, and on average the volatility of agents of type 1 is higher. However, since their mean
growth rates are also higher, we find that the two agents’ welfare is almost identical, and the constant 
k = (y_1/y_2)^{(1/(1-\gamma))}\) determined in eq. (39) is slightly below the one in our calibrated model. We
will shed further light on consumption patterns when discussing portfolio choices and risk premia
of the two types of agents in the next section.

IV Equity Premium and Riskless Rate Results

We use the parameters of fundamental processes calibrated in Section III to study the equity pre-
mium and related statistics implied by the model. In addition, we use different parameters for the
preference of agents, each set of which will have different implications for risk premiums. We use
preference parameters in a fairly narrow range, of \(0.3 \leq \gamma \leq 0.6\), and a time discount, \(\rho\), of two
and three percent. From our discussion in Section II, a CRRA less than one is required to lower
the riskless rate in our model. As we will see, in the specified range of risk aversion, per capita
consumption volatility will lie in a range found in recent papers. In brief, our model can generate
unconditional averages of the real riskless return that are between 17 and 165 basis points above
the time rate of discount, and a risk premium as high as 2.9 percent for these set of parameters. The
conditional moments of consumption growth and asset returns undergo substantial time variation,
which we will describe in detail. We split our analysis into four subsections discussing the pricing
and volatility functions, the properties of unconditional and conditional moments of variables rele-
vant to the puzzles, respectively, and the effects of relaxing our assumption on stocks being in zero
net supply.

A Asset Price, Volatility, and Risk Premium Functions On the State
Space

We start by examining the pricing functions obtained by solving the PDEs in Section A. We look at
a series of plots in Figures 4 through 6 using \(\gamma = 0.3\), which implies a CRRA of 0.7, and \(\rho = 0.02\).
We again provide two plots for each variable with the goal of examining the effects of a changing
disagreement value process with common beliefs for the two agents, and a disagreement value of
one with heterogeneous beliefs.

The price-dividend ratio obtained from the solution of the PDE in (31) is shown in the top
panels of Figure 5. As seen, for a given level of common beliefs the P-D ratio is increasing in
the disagreement value, \(\eta\). As mentioned earlier, when \(\eta\) increases, the relative weight of agent 2
in the economy increases, and the calibrated parameters imply that stocks are valued higher in the
homogeneous economy with only agent 2 (the more stable investor). When beliefs of the two agents
are different, the P-D ratio is increasing in the probability that each investor places of being in an expansion state. The bond price-to-coupon ratio (B-C) ratio is the solution to the PDE (57) and has slightly different shapes, reflecting more closely the movements of the short rate. It is concave in $\eta$ reaching a maximum near $\eta = 1$, the point of lowest disagreement. When disagreement is low, speculative opportunities and trading risk are at their highest levels leading to a low riskless rate and causing an increase in the B-C ratio. When beliefs are allowed to vary heterogeneously, the bond B-C ratio falls in the amount of dispersion (right panel). The intuition is that in periods of higher dispersion, investors perceive higher speculative opportunities in stocks rather than long-term bonds, thus lowering demands and valuations for the latter.

The stock return volatility obtained from eq. (32) is shown in the top panels of Figure 6. The striking feature of the left panel is that volatility is nearly insensitive to the disagreement value $\eta$, but is highly responsive to the probability of an expansion. When the belief of an expansion is around 0.5, the point of maximum uncertainty, stocks are the most volatile. The result stems from the volatility of the belief process in (4). When there is maximum uncertainty, agents have the least confidence in their current estimates and change beliefs rapidly with news. As shown in David (1997), such a volatility process satisfies many of the stylized facts in the GARCH literature. In the right panel we plot volatility as a function of beliefs of each type of agent and as anticipated, find volatility increases in the uncertainty of each type.

Finally, the risk premium on equities demanded by agents of type 2 (stable investors) is obtained from eq. (45) and is shown in the bottom panels of Figure 6. Plots of the risk premium of agent 1 are similar, but with opposite sign and are hence not shown. The premium for each agent is an inner product of the market prices of risk of that agent in Proposition 2 and equity volatility in (32). As such, the risk-premium plots can be visually constructed from the plots of these two variables. The risk premium inherits many of the features of the price of risk discussed below Proposition 2, and we will not repeat these discussions. Notably different though, in periods of high certainty of either state, the risk premiums decline, due to lower volatility in these periods. Looking at the plots, it is evident that in periods when agent 2 is more optimistic of an expansion state, his risk premia are positive. In such periods, the premium of agent 1 is negative. As we will see, agent 2 takes long positions in stocks in most such periods, while agent 1 takes positions of the opposite sign.

We note that the average equity premium under the empirical (objective) measure can be quite different from the ex-ante expected equity premia of each group of agents. The exact relationship is provided in (13). In periods when an agent has higher expected growth of earnings than the drift of the data generating process, he will over estimate the equity premium, and vice versa. It might appear that both agents are ‘biased’, since their equity premia are different, and are in general different from the average premium in the calibration. The difference in expected returns is not
a bias, but instead arises because of different information filtrations of the two agents. Under the different information filtrations, agents disagree on the expected return and shock components of equity returns. If the parameters of the underlying data generating process are known, then the expected return in equilibrium can be calculated from eq. (13), a luxury we do not have for the calibration exercise. Therefore, we took the route of solving the PDE for the price function, finding a calibrated price series, and computed the average of realized returns. As we saw in Table II, agent 1 has a negative unconditional bias of earnings growth, while agent 2 is essentially unconditionally unbiased. Based on this observation, and the relationship between the true and estimated drift rates of agents in eq. (13), we are likely to find the unconditional premium of agent 1 to be below the objective risk premium, and that of agent 2 to be very close to the historical average.

B Unconditional Moments

We use the pricing functions to generate price series and returns from the calibration exercise. For the calibrated results, we use the calibrated belief and disagreement value processes shown in Figures 1 and 2. For example, to obtain a real stock price series, we use the product of the realized proxy for dividends (50% of earnings), and the P-D ratio of \( p(\pi^{(1)}_{tt}, \pi^{(2)}_{tt}, \eta_t) \), whose value is obtained from the solution of eq. (31). Similarly, other variables are generated. Calibrated portfolio weights held in stocks by agent 1 are shown in the bottom panel of Figure 3. The weight varies between -0.2 and 0.2, as the relative optimism of agent 1 changes. As seen in the figure, in most periods, agent 1 holds short positions, consistent with his relative pessimism seen from the bottom panel of Figure 1. The weights are generally small since consumption volatility is lower than stock price volatility.

The results from the calibration exercise are shown in Table III. Column (4) of the table provides the average equity volatility generated by the model. In each case average annualized stock return volatility is between 18 and 20 percent, and increases in \( \gamma \) (we discuss this effect below). This exceeds the volatility of the assumed dividend process in the model of around 7 percent, due to the volatility caused by the beliefs and the disagreement value. Comparing lines 3 and 6, we see that the effect of increasing the time discount from two to three percent is to further increase stock volatility. The intuition is that with higher time discount, greater weight is given to current news on fundamentals.

The average model implied riskless rates is reported in col. (5). As mentioned in the discussion of Proposition 2(ii), the rate declines in periods when dispersion increases, enabling us to match a low average rate. For the sample period, the average riskless rate is 2.75 percent and 2.17 percent for \( \gamma \) of 0.5 and 0.6 respectively. We note that while the model’s average rate is above the one percent average real riskless return, it is 75-90 basis points below the benchmark economy in which agents have homogeneous beliefs. Moreover, the standard deviation of this real rate is extremely low.
between 1 and 2 percent at an annual rate. In the calibration, we have used a time-discount factor, \( \rho \) of two percent. Some authors have argued that a higher time discount is more reasonable. Using instead \( \rho = 0.03 \) raises the average rate by one percent and does not change its standard deviation (lines 3 and 6). With a higher \( \gamma \), the average riskless rate drops, as agents’ savings demand increases in periods of low disagreement and high dispersion. However, with a higher \( \gamma \) the volatility of the short rate as well as individual’s consumption volatilities increase beyond observed levels.

The equity premium under the objective measure calculated as the average annualized excess returns from the model generated price series is in col. (7). For the case when \( \gamma = 0.5 \), the equity premium in col. (7) is about 2.4 percent, compared to a historical equity premium of 6.1 percent in this particular sample. Somewhat remarkably, when we increase \( \gamma \) to 0.6, the equity premium rises to 2.95 percent. The intuition for this result was provided in the discussion of Proposition 2(iii). Essentially, the trading risk component of the market prices of risk of both agent types increases in absolute value as \( \gamma \) increases. Less risk-averse agents take more speculative positions and their relative shares of the economy fluctuate more. The trading risk component in our calibration far outweighs the component of the equity premium due to the aggregate risk in fundamentals, which declines with lower risk aversion. The cost of obtaining a higher risk premium is higher individual consumption volatilities which we will discuss below. For \( \gamma = 0.5 \), the Sharpe ratios generated (column 8) average is about 13 percent, less than half its historical values. Nonetheless, at the low level of risk aversion assumed, this is a substantial improvement on current models.

The main constraint in the equity premium literature is the volatility of aggregate consumption growth, which is around two percent at an annual rate. We have held this constant for each of the calibrated results, but generate higher individual consumption volatilities with heterogeneous beliefs of agents. Recent research with individual consumption data suggests that individual volatilities are far greater. Data on consumption growth at the per capita level are noisy. Moreover, it is not clear which households should be included when studying the equity premium — current shareholders in any period, investors with any assets, households who may in their history have held stocks at some time, etc. Mankiw and Zeldes (1991) found that stockholders’ consumption is almost twice as volatile as the consumption of non-stockholders. More recent evidence (Attansio, Banks, and Tanner 2002, Brav, Constantinides, and Geczy 2002) suggests that if additional filtering criteria are used, one finds the per capita volatility of quarterly consumption growth ranges between six and twelve percent. These estimates are made with generous cutoffs for outliers, and may well be an underestimate of the true level.

In our model the individual consumption functions are in Proposition 2(i). With the calibrated disagreement value process we generate processes for individual consumption and calculate their volatilities. Of the two investors in our model, agent 1 has a higher volatility of consumption, in
line with the larger volatility of his belief process. As we increase $\gamma$, the volatilities of both agents’ consumption growths increase. For the case $\gamma = 0.5$, we find a per-capita volatility of nearly 10 percent, within the range estimated by researchers. The higher per-capita volatilities are essential for generating higher Sharpe ratios in our model relative to the homogeneous agent benchmark. However, as we will see in the next section, this level of volatility is not in itself sufficient in generating our results.

We can also compute the average ex-ante equity premium expected by agents in our model. These are shown in cols. (12) and (13) for the two types of agents. The expected equity premium for each agent is calculated using the market prices of risk in Proposition 2 and the volatilities of asset prices at each date, using the calibrated belief and disagreement processes. As anticipated, we see that the equity premium of agent 1 underestimates the long term historical equity premium in the model by five to six percent, while that of agent 2 is very close to its historical average. In the next subsection when examining conditional equity premia, we will find that the ‘bias’ of agent 1 is consistent with eq. (13). We do note, however, that in periods when agent 1 is more pessimistic he is on average short in equities, and so his strategy earns positive expected returns.

One perhaps surprising aspect of our results is that agent 1, who in the sample underestimates earnings growth and trades based on his model, survives, and in fact has a slightly higher average growth rate of consumption than agent 2. This result may seem at odds with recent results by Blume and Easley (2004) and Yan (2005), who show that ‘irrational’ agents will not survive in the long term. In their analysis, the rational agent exactly knows the data generating process (DGP), while the irrational agent has a mis-specified model, such as a biased estimate of the drift rate. The latter paper reports that when the bias of a type is substantially larger than that of agent 1 in this paper, it may take hundreds of years for his wealth share of the economy to decline noticeably. Nevertheless, we note that there are key differences in our analysis in that both agents face estimation risk and neither agent knows the exact DGP. Agent 2 is ‘more rational’ only in the sense that the econometrician’s estimate of his unconditional mean equals the sample mean. The unconditional mean of the data generating process may be different from its sample mean. Moreover, both agents’ models may be slightly mis-specified relative to the DGP, as every model is. While we have specified both agents’ models without specific constraints on their rationality, we find that each model explains a similar fraction of variation of the realized earnings growth, and when trading with their models, both agents survive.

The last two columns give the equity premiums and riskless rate for the assumed levels of risk aversion and time discount when beliefs are homogeneous. As pointed out earlier, in this case we get back the result in David and Veronesi (1999) that the equity premium does not depend on beliefs and is small, equaling $(1 - \gamma) \cdot \sigma_q \sigma_x^T$. These columns highlight that the equity premium without the
trading risk modeled here is of the order of 0.0005 percent, while the riskless rate is between three and four percent (one to two percentage points higher than the time-discount factor). Therefore, the dispersion of beliefs in our model is the driving force, and none of the other features of the calibration exercise can account for our results.

B.1 Back-of-the-Envelope Approximation of Sample Moments

While calculation of the unconditional moments of asset returns and consumption as described above is cumbersome, we are able to provide some approximate calculations that are easily replicated and show that the premium in our model is close to half its empirically observed level, which is more than 15 times higher than in a benchmark economy with the same level of risk aversion and homogeneous beliefs. The exercise does miss some crucial properties of conditional moments of our model which we will discuss in Section C.

We make use of the following sample moments: First, as pointed out above, agent 1 underestimates earnings growth by 3 percent and agent 2 has an essentially unbiased estimate. Second, the average sample moment of the consumption share of agent 1 is \(0.55\). Third, average stock volatility in the sample is 19 percent.

Using these sample moments we approximate the moments of the equity premium puzzle. First, Corollary 2 implies that the equity premium with a CRRA of 0.5 would average \(0.5 \cdot 0.02 \cdot 0.19 + 0.55 \cdot \frac{0.03}{0.083} \cdot 0.1857 = 0.039\), which is one percent higher than the premium reported above. Second using (22), the approximate sample average of riskless rate is

\[
\bar{\rho} \simeq 0.02 - \frac{1}{2} (2 - \gamma) (1 - \gamma) 0.004 + (1 - \gamma) 0.03 - \frac{\gamma}{2(1 - \gamma)} \cdot 0.55 \cdot 0.45 \cdot \frac{0.03^2}{0.083^2},
\]

where \(\rho = 0.02\), \(\kappa^{(m)} = 0.3\), \(\sigma_{x,1} = 0\), \(\sigma_x \sigma_x^\top = 0.004\), \(\sigma_{q,1} = 0.083\), and \(k \simeq 1\). For \(\gamma = 0.5\), we obtain an \(\bar{\rho} = 0.017\), which is one percent below that from the exact calculation. Similarly from (23) and (24),

\[
\tilde{\phi}_q^{(1)} = 0.5 \cdot 0.02 + 0.45 \cdot \frac{0.03}{0.083} = 0.172, \quad \text{and} \quad \tilde{\phi}_q^{(2)} = 0.5 \cdot 0.02 + 0.55 \cdot \frac{\bar{b}}{0.083} = 0.208.
\]

By Lemma 3, for \(\gamma = 0.5\), the quarterly consumption volatilities of the two agents equal these market prices of risk. We note that these volatilities are about twice as high as those reported from the exact calculation in Table III. In fact, we show in the following subsection, all the moments of our model vary significantly over time and the lower unconditional consumption volatility is consistent with the reported moments of asset prices.
C Conditional Moments

Proposition 2 derived the consumption functions of agents as well as their market prices of risk, conditional on their beliefs and disagreement in a given period. Since agents in the model do not face any trading frictions, as we saw in Lemma 3, agents’ market prices of risk and their conditional consumption volatilities are closely related. Therefore, for a given $\gamma$, statistics of individual consumption growth rates should summarize the information in agents’ prices of risk. In this subsection, we take a closer look at the moments of individual consumptions that are generated by the state variables in our model, and reconcile the equity premia of agents reported in the previous subsection with conventional Euler equations.

We saw in eq. (45) that the conditional risk premium for agent $m$ for any given asset is the conditional covariance between the agent’s market prices of risk and the asset volatility. Eq. (18) of Lemma 3 further showed that the market prices of risk of agent $m$ equals the CRRA times the volatility of the agent’s conditional consumption growth. Combining the two equations we get

$$\mu_{p}^{(m)} + \delta_t - r_t = \rho_t^{(m)} \cdot (1 - \gamma) \cdot ||\sigma_{ct}^{(m)}|| \cdot ||\sigma_{Pt}||,$$

(41)

where $\rho_t^{(m)}$ is the conditional correlation between the consumption growth of an agent of type $m$ and stock returns. If the quantities on the right hand side are time-varying (as is implied by Lemma 3) then care has to be taken to extend the implications of this pricing equation to its unconditional form (see Hansen and Richard 1987, Campbell and Cochrane 2000).

Several authors use the unconditional form of (41) to place bounds on Sharpe ratios and the equity premium. If consumption and stock price volatilities are assumed constant at their expected values, then indeed the unconditional equity premium of investor $m$ must be bounded by $(1 - \gamma) \cdot E[||\sigma_{ct}^{(m)}||] \cdot E[||\sigma_{Pt}||]$. For example, looking at line 3 of Table III, the case where $\rho = 0.02$ and $\gamma = 0.5$, the equity premium of agent 2 would be bounded by $0.5 \cdot (2 \cdot 0.0971) \cdot 0.1857 = 0.018$, clearly smaller than the ex-ante expected premium computed for agent 2 in col. (13) as 2.4 percent (note that the consumption volatility in the table is stated in quarterly units and must be doubled for its implications for the annualized premium). The bound is large because it assumes a correlation between the consumption growth of agent 2 and equity returns of one. If we were to further assume that the correlation was constant at its unconditional value reported in col. (15) of 0.1445, then the premium implied by the unconditional moments would be much smaller at 0.26 percent. This is of course just the correlation puzzle as stated in Cochrane (2001), that the correlation between aggregate consumption and stock returns is no more than twenty percent, thus further deepening the equity premium puzzle. In the remainder of this section, we will point out key features of the time
variation of the moments of individual consumption growths and stock returns that will justify the premiums reported in Table III.

As we will see in the remainder of this section, the conditional distribution of most calibrated variables will be markedly different depending on the relative optimism of the two agents. Therefore, we partition all our series into two parts, the former group having dates in which $\bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}$. Table IV shows the conditional means of various variables in our calibration exercise, and it is useful to also look at the bottom panel of Figure 2, which plots time series of the conditional expectations of dividend drift of the two types of agents. About 43 percent of the observations in the sample fall in the first group. The reason we partition our data into this form is that the sign of the conditional risk premium of agent 1 (2) will be positive (negative) for observations in the first group, and the signs are reversed for observations in the alternative group. The observation can be verified in two ways. First, from Lemma 2 we notice that in periods when the type 2 agents are more optimistic, positive shocks to fundamentals lead to increases in the disagreement value $\eta$. From the agents’ consumption functions in Proposition 2, this increases the consumption share of agent 2, and from Figure 5 leads to higher stock valuations, overall generating a positive correlation between the consumption growth of agent 2 and stock returns. It can also be verified precisely by computing ex-ante correlations as $\rho_t^{(m)} = (\phi_t^{(m)} \cdot \sigma_P^T) / (||\phi_t^{(m)}|| \cdot ||\sigma_P||)$, and using the fact that the market price of risk with respect to the dividend shock of agent 1 (2) is positive (negative) in the first group of observations (see eqs. (23) and (24)). Since in our calibration we find that the volatility of aggregate consumption growth is small, we actually obtain a stronger relationship, that the correlation between the consumption growth of agent 1 (2) and stock returns is close to plus (minus) one in the former group of observations.

Using the calibrated beliefs and disagreement value process, we generate the ex-ante conditional correlations and generate their averages in cols. (1) and (2). As seen, the correlations of the simulated consumption growth of agent 1 and 2 are 0.94 and -0.91 in the first group, and -0.96 and 0.95 in the second group respectively. The correlations differ from one (in absolute value) due to the presence of the small exogenous volatility of aggregate output terms in the market prices of risk. Since the correlations switch sign in the two sub-samples, the unconditional correlations of the two agents are close to -0.14 and 0.14 respectively, similar to estimates in other papers. The high conditional correlations are critical in yielding conditional risk premia close to the product of individual consumption volatility and stock volatility for the two types of agents.

As seen in columns (3) and (4), the average growth rate of consumption of agent 1 (2) is positive (negative) in the first group of observations, that is consumption of each group of agents grows faster in periods when they are relatively more optimistic. Columns (5) and (6) show that the volatilities of consumption growth of both groups of agents are much lower in the top group of observations:

38
the average volatilities are about two percent in the first group and 13 percent in the second group. Intuition from this observation can be obtained from Figure 1, where we notice in the bottom panel that the difference in expected growth rates of the two agents is small when $\bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}$. As noted earlier, agents of type 1 are more optimistic during periods of strong earnings growth, and more pessimistic in periods of weak growth. However, since agents disagree more on the transition probability from weak to strong earnings growth states rather than the reverse transition (see Table I), their beliefs are relatively more volatile during periods of weaker earnings, leading to more volatile disagreement and consumptions. Col. (7) shows the volatility of stock returns in the two states, and once again, the volatility is much higher in the second group. Overall, for both types of agents, market prices of risk (consumption volatilities) are higher in absolute value in the second group of observations, in periods when the volatility of stock returns is also higher. Due to this comovement, risk premiums in the second group of observations rise (in absolute value) more than proportionally than the increase in consumption volatilities.

Within each group, although correlations between agents’ consumption growths and stock returns are stable, there is significant covariation in the norms of the market prices of risk and stock volatilities. This results from the effect of dispersion is agents’ beliefs. In periods of high dispersion, as evident from Figure 4 (bottom-right panel), and Figure 6 (top-right panel), both the market prices of risk and stock volatilities increase rapidly. To separate the effects of changes in uncertainty (common beliefs) that places observations in the first rather than the second group, and changes in dispersion that lead to comovement in the sizes of the market prices of risk and stock volatilities within a group, we decompose the conditional risk premia further. Taking expectations of eq. (45) conditional on being in either of the two groups of observations, and using the fact that correlations between the market prices of risk and stock returns are nearly constant within each sub-group of observations, we can write

$$E \left[ \mu_{Pt}^{(m)} + \delta_t - r_t | \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right]$$

$$\simeq E[\rho^{(m)} | \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}] \cdot E \left[ ||\phi_t^{(m)}|| \cdot ||\sigma_{Pt}|| | \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right],$$

$$\simeq E[\rho^{(m)} | \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}] \cdot \left\{ E \left[ ||\phi_t^{(m)}|| | \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right] \cdot E \left[ ||\sigma_{Pt}|| \cdot \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right] \right\} + \text{Cov} \left[ ||\phi_t^{(m)}||, ||\sigma_{Pt}|| | \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right]$$

(42)

The first component provides the risk premium due to higher average volatilities of consumption growth and stock prices for a given group. The second component measures the risk premium due to changes in dispersion of beliefs within the group of observations, which causes comovement of the two volatilities.
The two components of the ex-ante risk premiums are given in cols. (8) and (10) for agent 1 and (9) and (11) for agent 2 respectively. We note that by Lemma 3, the norm of the market prices of risk must equal the product of the agents’ CRRA and the norm of their consumption volatilities. Indeed, by comparing the norms of the consumption volatilities in (5) and (6), we find that the first component of each agent (cols. (8) and (9)) is very close to that implied by Lemma 3. Due to low volatilities in the first group of observations, these components are very small, on the order of 0.3 percent. In the second group of observations this component is much larger at -3.4 and 2.5 percent for the two agents. The second components, shown in cols. (10) and (11), capture the portion of the risk premiums of the two agents arising from the covariance of the market prices of risk and stock volatility within each of the subgroups. These components are also much larger in absolute value in the second group of observations, and are nearly as large as the first components. Overall, the conditioning shows that about half the generated premium arises from higher average consumption volatilities in periods of higher average stock market volatility. The other half is generated from the covariance between these quantities within each subgroup. The sign of the correlations of the two types of agents discussed earlier, the premia for agents of type 1 are positive in the first group of observations, and negative in the second group, and, hence are negative overall. The reverse holds for agents of type 2. The sum of the components, as shown in cols. (12) and (13)) are again small in the first group of observations, and equal -5.3 percent and 5.2 percent for the second group. The unconditional expectations of these premia equal -2.8 percent and 2.59 percent as in Table III. We come back to this issue after discussing the conditional dispersions of agents’ expected growth rates.

Using the calibrated price process, we next report the ex-post realized excess returns of stocks in col. (14). Somewhat surprisingly, the excess returns are nearly four percent in the first group of observations and much smaller at about one percent in the second group. The unconditional average of the excess returns is about 2.4 percent, as reported in Table III. As we have noted earlier, given the small positive bias of agent 2 in estimating earnings growth, his ex-ante risk premium should be fairly close to the risk premium under the objective measure. For the full sample as reported in the previous sub-section, we indeed found these two premia to be quite similar. Therefore, we can infer that the equity premium under the objective measure is close to -0.8 and 5.2 percent in the two groups of data as reported in col. (13). The relative size of these premia varies inversely with realized excess returns on stocks reported in col. (14), which means that stock prices fall in periods of higher ex-ante forward-looking risk premiums under the objective measure. Indeed, the excess returns over the following one to four quarters are large and positive in the second group, and are negative in the first group of observations (results not shown in table).
We next discuss the impact of dispersion in beliefs on the riskless rate and the cross-sectional dispersion in consumption growth in the two groups of data. As seen in col. (15), the riskless rate is quite high at nearly 3.56 percent in the first group, and 2.13 percent in the second. We noted earlier that since the two types of agents disagree more on the transition probability from the weak to strong dividend state (Table I), their beliefs are more volatile in the second set of observations, which we noted earlier were periods of weaker earning growth. Indeed, as seen in col. (16), the difference in expected growth rates is much larger in the second group of observations, in periods when agents of type 2 are relatively more optimistic. The described equilibrium has this property because we have calibrated parameters of agents estimates to also match the series of forecasters’ dispersions. As seen in cols. (17) and (18) both the historical dispersion and the model-fitted dispersion are about 25 percent higher in the second group relative to the first group of observations. As seen in the right panel of Figure 4, the riskless rate in the economy falls in periods of higher dispersion due to the higher speculative opportunities in such periods (see the discussion following Proposition 2). Therefore, periods of weak growth in fundamentals are also accompanied by lower (real) rates.

We now address the issue as to whether the difference in ex-ante risk premia of the two types of agents is consistent with their expected growth rates of fundamentals. The relationship between these variables can be approximated from eq. (14). Focusing on the second group of observations, the average difference in expected dividend growth rates is about -5 percent, and the volatility of stock returns is 23.7 percent, implying a difference in equity premia of around -14 percent (roughly $-1/0.083 \cdot 0.237 \cdot 0.05$), larger in absolute value than the -10.5 percent difference in the ex-ante premia reported in cols. (12) and (13) from the exact calculation. In the first group of observations, the difference in equity premia is of positive sign and smaller in magnitude, consistent with the smaller difference in estimated fundamentals growth rates.

The higher volatilities of consumption and dispersion in beliefs also lead to more dispersion in growth rates of consumption across agents. As seen in col. (19), the unconditional mean of the cross-sectional standard deviation (cs-sd) is 0.077 at a quarterly rate and it is about twice as high in the second group of observations, which we have seen are periods of low realized returns and high volatility. The negative relationship between the cs-sd in consumption growth and stock returns is consistent with the analysis in Constantinides and Duffie (1996), and in addition is endogenously generated in equilibrium. However, unlike their model, the endogenous shocks to consumption in our model are not permanent, and hence do not cause a trend increase in the cs-sd with the age of agents. Estimates of the cs-sd depend critically on the filtering method used in empirical studies. Jacobs and Wang (2004) provide an estimate of 0.1 based on age and education based cohorts in the Consumer Expenditure Survey (CEX) data, while Storesletten, Telmer, and Yaron (2004) report that it varies between 0.06 and 0.105 (both estimates at a quarterly rate) among cohorts of individuals.
in the Panel Study of Income Dynamics data. This latter study finds that the cs-sd on average increases by 75% as the macroeconomy moves from peak to trough. Our model based average cs-sd lies within the range of these estimates, and even though our conditioning criterion is somewhat different, displays a countercyclical variation as in Storesletten, Telmer, and Yaron (2004).

We note that conditioning our sample on dispersion is just one possible way of parsing our results into periods of high and low trading risk. As suggested earlier, another way to group the data is to separate out periods in which there is low and high disagreement. We created two sub-groups, the first containing periods in which $|\eta_t - \bar{\eta}| < 0.1$. The disagreement value series can be viewed in the bottom panel of Figure 2, and we find that 22 percent of the observations fall into the first group. Our model implies that trading risk is higher within this sub-group of observations. For brevity, we do not present these results in detail, but importantly find that risk (stock volatilities), the price of risk (consumption volatilities), and absolute values of the ex-ante equity premia of the agents are higher in this sub-group. Indeed, this grouping exhibits the positive covariation among these variables as well and leads to the same unconditional moments in Table III.

In summary, by partitioning our data into two sub-groups based on the relative optimism of the two agents we are able to uncover the following three essential conditional properties of agents’ consumption that arise in equilibrium and permit the matching of asset returns: (i) high conditional correlations (in absolute value) between stock returns and consumption growth of each agent, which switch signs depending on the relative optimism of agents; (ii) the positive comovement of the market prices of risk of agents (their CRRA times consumption volatilities) and the volatility of stock returns, and (iii) the negative comovement between the cross-section dispersion in consumption growths and fundamental growth. Property (i) helps address the consumption correlation puzzle by generating high conditional correlations, but unconditional correlations near empirical estimates. Property (ii) implies that higher risk (stock volatility) is also priced higher, boosting conditional equity premia during periods of weak fundamental growth. Property (iii) obtains due to higher dispersion of beliefs in periods of weak growth that provide greater speculative opportunities for investors, lower investors’ demand for safe assets and riskless rates, and at the same time cause them to take greater speculative positions that lead to the higher dispersion in consumption growths. The average per-capita volatilities of consumption growth and the cross-sectional dispersions in consumption growth rates generated by the model are close to their empirical estimates. The interesting aspect of our analysis is that we are able to generate these patterns from the heterogeneous learning of agents about the state of fundamental growth and the resultant disagreement value process.
D Stocks in Positive Net Supply

In the model discussed so far, stocks have been in zero net supply. As claimed in footnote 3, with the alternative assumption of stocks being in ‘small’ positive supply the equilibrium premium and riskless rate are very similar. Now total consumption in any period equals \( x_t + q_t \), the sum of dividends and output in the economy produced from resources not financed by public equity. The additional complication is the introduction of another state variable, \( q/(x_t + q_t) \), the share of dividends of total output. We are once again able to solve for the riskless rate and market prices of risk in this alternative economy. Similar to (22), we find the riskless rate to be

\[
rt = \rho - \frac{1}{2} (2 - \gamma) (1 - \gamma) \left( \frac{x_t}{x_t + q_t} \sigma_x + \frac{q_t}{x_t + q_t} \sigma_q \right)^2 \\
+ \frac{q_t}{x_t + q_t} \frac{1 - \gamma}{1 + k \eta^{-\gamma}} \left( \theta_{t(1)} + k \eta^{-\gamma} \theta_{t(2)} \right) + \frac{x_t}{x_t + q_t} \frac{1 - \gamma}{1 + k \eta^{-\gamma}} \left( \bar{k}_t^{(1)} + k \eta^{-\gamma} \bar{k}_t^{(2)} \right) \\
- \frac{\gamma k \eta^{-\gamma}}{2} \left[ (\theta_{t(1)} - \theta_{t(2)}) \sigma_x - (\bar{k}_t^{(1)} - \bar{k}_t^{(2)}) \sigma_q \right]^2 \\
2 (1 - \gamma) \left( 1 + k \eta^{-\gamma} \right)^2 |\Sigma|^2 
\]

As before, the terms have similar interpretation, where the first term, for precautionary savings, must now incorporate the weighted average of volatility of the two processes, and the wealth effect term similarly incorporates the expected growth of dividend growth as well. As \( q/(x_t + q_t) \to 0 \), we get back (22). Notably, the trading risk component (last term), is identical in the two expressions.

Similarly solving for the market prices of risk (we only provide the equation for \( \phi_{q}^{(1)} \))

\[
\phi_{q}^{(1)} = (1 - \gamma) \left( \frac{q_t}{q_t + x_t} \sigma_{q,1} + \frac{x_t}{q_t + x_t} \sigma_{x,1} \right) + \frac{k \eta^{-\gamma}}{1 + k \eta^{-\gamma}} \left( \theta_{t(1)} - \theta_{t(2)} \right) \sigma_{x,2} + (\bar{k}_t^{(2)} - \bar{k}_t^{(1)}) \sigma_{q,2} \\
|\Sigma|
\]

which is identical to (23) with the sole exception that the market price of aggregate risk contains the weighted average of dividend and consumption volatilities. Once again, the price of the trading risk component is the same.

Solving for asset prices would be similar, but we would have to incorporate the additional state variable, which would be tedious but is still possible using projection methods. Nonetheless, since the equity premium puzzle can be restated as the difficulty of attaining a low riskless return and high Sharpe ratio, it is straightforward to assess if the change in assumption will significantly affect our results. Calibrating our economy to aggregate dividends and output we find that the ratio \( q/(q_t + x_t) \) is on average about 2.6 percent, and for the case \( \gamma = 0.5 \), leads to an average riskless rate which is higher only by four basis points. The values of the four market prices of risk also differ by similar
amounts relative to the zero net supply case. Therefore, the alternative assumption do not affect Sharpe ratios under the measures of the different agents by large amounts.

V Conclusions

We show that a model in which agents have heterogeneous beliefs about the state of fundamental growth has many of the features of a model with incomplete markets. In particular, agents are unable to perfectly share risks, causing individual consumptions to be far from perfectly correlated. In addition, the model has the property that trading in itself is risky for agents, as they face the risks of asset prices being buffeted by the beliefs of other agents (counterparties). The beliefs of both types of agents as well as the disagreement between their models are correlated with individuals’ consumptions and are risk factors that get priced in equilibrium. Trading gains and losses cancel out across agents, and are not detected in the properties of the aggregate consumption process.

We calibrate and examine the asset prices of such a model, and find that we are able to match half the realized equity premium in the data, and lower the riskless rate by one percent relative to a model with homogeneous beliefs, with a coefficient of relative risk aversion of between 0.4 and 0.7. The limited empirical and experimental research done in estimating the risk aversion parameter suggests a value of around one as reasonable. Therefore this paper improves on the results of several recent papers that have attempted to solve the puzzle, and documents improvements by using a CRRA of between 5 and 10. A key feature of our model is that the risk premium increases with lower risk aversion (as long the CRRA is less than one) because the exposure to trading risk is endogenous, and increases as less risk averse agents undertake more aggressive trading strategies. This we believe is a key reason we are able to obtain a high premium with the low level of risk aversion. Further, in our model the average riskless rate is about a percent lower than a benchmark model in which agents have homogeneous beliefs. However, we are unable to completely resolve the low riskless rate puzzle.

The model implies that the equity premium is higher, the riskless rate is lower, and consumption volatilities are higher and more disperse in periods when the dispersion among agents’ expectations of future growth is high and when the disagreement between their models is low. In such periods, agents’ exposure to trading risk is high. The above stylized facts have been noted by several other authors in the literature.

The degree of freedom that we take in our calibration exercise is the higher level of individual consumption volatilities that have been reported in recent papers. In our model, these higher volatilities and heterogeneity in growth rates across agents are endogenously generated in equilibrium due to agents’ speculative motives, while researchers in the past have been unable to do so once trading
in a limited set of securities is permissible. The higher unconditional average of per-capita consumption volatility itself does not justify the observed premium, but its positive covariance with stock volatility and high conditional correlations between individual consumption growth and stock returns together do justify the unconditional moments of risk premia and riskless returns.

While exact properties of individual consumption processes are hard to measure and verify, our results support a growing literature that finds useful information in statistics of cross-sectional consumption growth. The above noted paper by Brav, Constantinides, and Geczy (2002) finds the skewness of the distribution changes over time, and helps explain the equity and value premiums, while Jacobs and Wang (2004) find that the cross-sectional standard deviation is useful in explaining cross-sectional variation in asset returns. While these papers do not explicitly derive the consumption processes as arising endogenously from an equilibrium, we view our results as complimentary to this line of research, providing further empirical predictions to be tested at the individual level. In line with the predictions of our model, Storesletten, Telmer, and Yaron (2004) find that the standard deviation of cross-sectional consumption growth increases substantially during periods of weak fundamental growth.

In the current paper, we have assumed that agents do not update their model parameters over the sample period analyzed, and in this sense agents have dogmatic beliefs about their models. Neither model is “irrational” enough that by trading based on its prediction, an agent loses market share to agents using the other model. The assumption has been made primarily for tractability. An extension to the case where agents update both their models as well as conditional means of fundamentals each period is a problem for future research. From the results of Merton (1980), we know that it takes several years for agents to get good estimates of drifts, even if they are constant. In the current setting, drifts shift at the business cycle frequency, and we would conjecture that it would take even longer to get good estimates of the drift parameters. The trading risk modeled here would be valid as long as significant differences remained between the models of different agents.

The paper has many useful and interesting extensions. It basically introduces a structure where risk-sharing possibilities are endogenous and time varying. Several interesting questions, such as the optimal design of securities and the formation of market arrangements to better hedge the trading risk introduced here can be addressed in such a framework. We will pursue such questions in future work.
Appendix 1

**Proof of Corollary 1.** To establish that \( q_t \) (a zero drift process) is a martingale on \([0, t]\) under \( \mathcal{P}^{(1)} \), it is sufficient to show that the Novikov condition holds, that is, \( E^{(1)} \exp\left[1/2 \int_0^t (\nu_s^{(2)} - \bar{\nu}_s^{(1)})^T (\Sigma \Sigma^T)^{-1} (\nu_s^{(2)} - \bar{\nu}_s^{(1)}) ds\right] < \infty \) (see e.g., Proposition 2.24 in Nielsen 1999). But,

\[
E^{(1)} \exp \left[ 1/2 \int_0^t (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)})^T (\Sigma \Sigma^T)^{-1} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)}) ds \right] < \exp \left[ 1/2 (\bar{\nu} - \bar{\nu})^T (\Sigma \Sigma^T)^{-1} (\bar{\nu} - \bar{\nu}) \cdot t \right] < \infty,
\]

where \( \bar{\nu} = \max_i \nu_i^T (\Sigma \Sigma^T)^{-1} \nu_i \) and, \( \bar{\nu} = \min_i \nu_i^T (\Sigma \Sigma^T)^{-1} \nu_i \). Since \( q_t \) is positive and finite for all \( t \), the measure \( \mathcal{P}_t^{(2)}(A) = E^{(1)}[1_A \cdot q_t] \) is equivalent to \( \mathcal{P}_t^{(1)} \). An application of Girsanov’s Theorem to the relation between the two innovation processes in (5) then implies that the Radon-Nikodym derivative of \( \mathcal{P}_t^{(2)} \) with respect to \( \mathcal{P}_t^{(1)} \) is \( q_t \).

**Proof of Lemma 2.** Let \( \eta_t = g(\xi_t^{(1)}, \xi_t^{(2)}) \). Then, its partial derivatives are \( g\xi^{(1)} = 1/\xi^{(1)}; g\xi^{(2)} = -\xi^{(1)}/\xi^{(2)}; g\xi^{(1)}\xi^{(2)} = 0; g\xi^{(2)}\xi^{(2)} = -1/(\xi^{(2)})^2; g\xi^{(2)}\xi^{(2)} = (\xi^{(2)})^3 \). Using the dynamics of the real kernels in (9), an application of Ito’s lemma implies that

\[
d\eta_t = \frac{1}{\xi^{(2)}} \left( -r_t \xi^{(1)} dt - \phi_t^{(1)} \xi^{(1)} dt \right) + \frac{1}{\xi^{(2)}} \left( -r_t \xi^{(2)} dt - \phi_t^{(2)} \xi^{(2)} dt \right) + \frac{1}{\xi^{(2)}} \left( -r_t \xi^{(2)} dt - \phi_t^{(2)} \xi^{(2)} dt \right) dt.
\]

Therefore,

\[
\frac{d\eta_t}{\eta_t} = (-r_t dt - \phi_t^{(1)} dt) - (-r_t dt - \phi_t^{(2)} dt) + \left( \phi_t^{(2)} \phi_t^{(2)} - \phi_t^{(1)} \phi_t^{(2)} \right) dt.
\]

Now using (5) and (15) and collecting terms completes the proof.

**Proof of Proposition 1.** Suppose agents agree on prices at all dates. By the definition of the market prices of risk, for an asset with current payout flow rate of \( \delta_t \) and volatility \( \sigma_t \), the instantaneous risk premium for agent \( m \) is

\[
\bar{\mu}_t^{(m)} + \delta_t = \sigma_t \phi_t^{(m)^T},
\]

which implies that

\[
\bar{\mu}_t^{(1)} - \bar{\mu}_t^{(2)} = \sigma_t (\phi_t^{(1)} - \phi_t^{(2)})^T.
\]

Since (14) and (46) hold for every asset \( i \) at each time \( t \), (15) must hold.

Now suppose that (15) holds. By Lemma 2, \( \frac{d\eta_t}{\eta_t} = \sigma_t = \frac{d\eta_t}{\bar{\eta}_t} \). By Corollary 1, \( \eta_t \) is the Radon-Nikodym derivative of \( \mathcal{P}_t^{(2)} \) with respect to \( \mathcal{P}_t^{(1)} \). Fix an arbitrary time horizon \( T \). Then,

\[
E_t^{(2)} \left[ \int_t^T \xi_s^{(2)} \eta_t \delta_s ds \right] = E_t^{(1)} \left[ \int_t^T \xi_s^{(2)} \eta_t \delta_s ds \right] = E_t^{(1)} \left[ \int_t^T \xi_s^{(1)} \delta_s ds \right],
\]

where the first equality follows from the definition of a Radon-Nikodym derivative and the second from the definition of \( \eta_t \). Therefore agents agree on the level of the expected discounted value of
fundamentals up to a fixed horizon \( T \). Now, since \( \delta \) and \( \varepsilon_t^{(m)} \) are positive, both discounted values are positive and increase in \( T \). Letting \( T \to \infty \), by the monotone convergence theorem then implies that the agents agree on the discounted value of fundamentals. Under their respective transversality conditions, they agree on the level on prices, as claimed. ■

Proof of Lemma 3. By individual \( m \)'s first order condition for optimal consumption, \( \xi_t^{(m)} = I_m(y_m \xi_t^{(m)}) \), where \( y_m \) is the Lagrange multiplier for agent \( m \). By Itō’s lemma, we can write \( dc_m = \partial I_m(y_m \xi_t^{(m)})/\partial \xi_t^{(m)} \, d\xi_t^{(m)} + 1/2 \partial^2 I_m(y_m \xi_t^{(m)})/\partial \xi_t^{(m)}^2 \, (d\xi_t^{(m)})^2 \). Since the optimum condition can also be written as \( u'_m(c_t^{(m)}) = y_m \xi_t^{(m)} \), we can also write \( \xi_t^{(m)} = I_m(u'_m(c_t^{(m)})) \). Differentiating both sides of the equality implies that \( 1 = I_m(y_m \xi_t^{(m)})u''(c_t^{(m)}) \). By the chain rule, \( \partial I_m(y_m \xi_t^{(m)})/\partial \xi_t^{(m)} = I_m(y_m \xi_t^{(m)})y_m = y_m/u'_m(c_t^{(m)}) \). From individual \( m \)'s state price density in (9), we obtain \( \sigma_t^{(m)} = -y_m \xi_t^{(m)}/u''(c_t^{(m)})\phi^{(m)} = -u''(c_t^{(m)})/u''(c_t^{(m)})\phi^{(m)} \), which equals the statement of the volatility. Similarly, differentiating consumption twice, we obtain \( 0 = I''_m(u'_m(c_t^{(m)})u''(c_t^{(m)})^2 + I'_m(u'_m(c_t^{(m)})u''(c_t^{(m)}), hence I''_m(y_m \xi_t^{(m)}) = -u''(c_t^{(m)})/u''(c_t^{(m)}) \), and \( \partial^2 I_m(y_m \xi_t^{(m)})/\partial \xi_t^{(m)} = -u''(c_t^{(m)})/u''(c_t^{(m)}) \) \( \sigma_t^{(m)} \). Therefore, \( \mu_t^{(m)} = y_m \xi_t^{(m)}/u''(c_t^{(m)}) \), and \( \sigma_t^{(m)} \). Proof of Proposition 2. We define the inverse function of the representative agent’s marginal utility \( U_c(c_t; \lambda_{1t}, \lambda_{2t})^{-1} = I(z_t; \lambda_{1t}, \lambda_{2t}) \equiv I_1(1/z_t) + I_2(1/z_t) \). Now, by the special choice of the weights, \( \lambda_{1t} = 1/y_1 \) and \( \lambda_{2t} = \eta_t/y_2 \), where \( \eta_t = \varepsilon_t^{(1)}/\varepsilon_t^{(2)} \), \( I(z_t; y_1, y_2, \eta_t) = I_1(y_1 z_t) + I_2(1/y_2 \eta_t) \). Therefore, \( I(1/z_t; y_1, y_2, \eta_t) = I_1(1/y_1) + I_2(1/y_2 \eta_t) = c_{1t} + c_{2t} = c_t \). Furthermore, since \( U_c(z_t)^{-1} = I(\cdot), I(\varepsilon_t^{(1)}) = x_t \), or \( U_c(x_t) = \varepsilon_t^{(1)} \). In addition, \( U_c(z_t)/\eta_t = \varepsilon_t^{(2)} \). Now using the form of the representative agent’s utility function in (17) gives the individual consumption processes in (i).

Using the characterization of agent 1’s state price density, \( U_c(x_t) = \xi_t^{(1)} \), for the weights \( \lambda_{1t} = 1/y_1 \) and \( \lambda_{2t} = \eta_t/y_2 \), we obtain

\[
\xi_t^{(1)} = e^{-\rho t} x_t^{-1} \frac{1}{y_1} \left[ 1 + \left( \frac{y_1 \eta_t}{y_2} \right)^{1/(1-\gamma)} \right]^{1-\gamma}.
\] (47)

Since the drift of \( d\xi_t^{(1)}/\xi_t^{(1)} \) equals minus the short rate, an application of Itō’s lemma along with the equations for the processes \( x_t \) and \( \eta_t \) in (2) and (16) implies the riskless rate in (ii).

Market clearing for the consumption good implies that

\[
\sigma_{c,qt}^{(1)} + \sigma_{c,qt}^{(2)} = \sigma_{x,1} x_t, \quad \text{and}
\]

\[
\sigma_{c,qt}^{(1)} + \sigma_{c,qt}^{(2)} = \sigma_{x,2} x_t.
\] (48) (49)

Using the volatilities of consumption from Lemma 3 implies that the equilibrium conditions are:

\[
\frac{1}{a_t^{(1)}} \phi_{qt}^{(1)} + \frac{1}{a_t^{(2)}} \phi_{qt}^{(2)} = \sigma_{x,1} x_t \quad \text{and}
\]

\[
\frac{1}{a_t^{(1)}} \phi_{qt}^{(1)} + \frac{1}{a_t^{(2)}} \phi_{qt}^{(2)} = \sigma_{x,2} x_t.
\] (50) (51)

Now equations (15), (50), and (51) contain four equations in the four market prices of risk, that lead to the unique solution
\[
\begin{align*}
\phi_{qt}^{(1)} &= \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \frac{\sigma_{x,2}(\bar{q}_t^{(1)} - \bar{q}_t^{(2)}) + \sigma_{q,2}(\bar{q}_t^{(2)} - \bar{q}_t^{(1)})}{|\Sigma|} + \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \sigma_{x,1} x, \quad (52) \\
\phi_{qt}^{(2)} &= \frac{a_t^{(2)}}{a_t^{(1)} + a_t^{(2)}} \frac{\sigma_{x,2}(\bar{q}_t^{(2)} - \bar{q}_t^{(1)}) + \sigma_{q,2}(\bar{q}_t^{(1)} - \bar{q}_t^{(2)})}{|\Sigma|} + \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \sigma_{x,1} x, \quad (53) \\
\phi_{xt}^{(1)} &= \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \frac{\sigma_{x,1}(\bar{q}_t^{(2)} - \bar{q}_t^{(1)}) + \sigma_{q,1}(\bar{q}_t^{(1)} - \bar{q}_t^{(2)})}{|\Sigma|} + \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \sigma_{x,2} x, \quad (54) \\
\phi_{xt}^{(2)} &= \frac{a_t^{(2)}}{a_t^{(1)} + a_t^{(2)}} \frac{\sigma_{x,1}(\bar{q}_t^{(1)} - \bar{q}_t^{(2)}) + \sigma_{q,1}(\bar{q}_t^{(2)} - \bar{q}_t^{(1)})}{|\Sigma|} + \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \sigma_{x,2} x, \quad (55)
\end{align*}
\]

Notice that the market prices of risk depend on beliefs of investors of each type through the conditional means of each of the state variables, as well as their risk-aversions and consumption levels, through the coefficients \(a_m\). For the case of CRRA preferences, \(a_m^{(m)} = -u''[e^{(m)}]/u'[e^{(m)}] = (1 - \gamma)/c_t^{(m)}\). Substituting these into eqs. (52) – (55) implies (iii). \(\blacksquare\)

**Proof of Corollary 2.** Equations (20) – (21) and (23) – (26), imply that \(\phi_{t}^{(1)} \frac{\hat{q}(1)}{x_t} + \phi_{t}^{(2)} \frac{\hat{q}(2)}{x_t} = (1 - \gamma)\sigma_x\). Multiplying (14) by \(\frac{\hat{q}(m)}{x}\) for each \(m\), adding across agents, and using the above equality completes the proof. \(\blacksquare\)

**Appendix 2: Solving for Asset Prices Using Projection Methods**

We formulate the PDEs for the N-state case, even though the empirical section of this paper calibrates to a 2-state model. Substituting the dynamics of investors’ beliefs with respect to type 1’s innovation, eqs. (3) and (7), and the definition of the pricing kernel in (9), into (30), and using Ito’s Lemma, leads to the PDE

\[
0 = P_{\pi(1)} (\mu^{(1)} - \sigma^{(1)} \phi^{(1)\top}) + P_{\pi(2)} (\mu^{(2)} - \sigma^{(2)} \phi^{\top}) - P_{\eta} a \phi^{(1)\top} \sigma_{\eta} q + q \frac{\phi^{(1)\top} \sigma_{\eta}}{2} + \frac{1}{2} P_{\pi(1)} \sigma^{(1)\top} + P_{\pi(2)} \sigma^{(2)\top} + P_{\eta} \sigma_{\eta} \sigma_{\eta} \phi^{(1)\top}\phi^{(1)} - \frac{1}{2} \sigma_{\eta}^{2} \sigma_{\eta} \sigma^{(1)\top} (\sigma_{\eta} - \sigma^{(1)} \phi^{(1)\top}) + q P_{\pi(1),q} \sigma^{(1)} \sigma_{\eta}^{\top} + \frac{1}{2} P_{\pi(2),q} \sigma^{(2)} \sigma_{\eta}^{\top} + P_{\pi(2),\eta} \sigma_{\eta} \sigma_{\eta} + q P_{\pi(2),\eta} \sigma_{\eta} \sigma_{\eta}^{\top} + q P_{\eta} \sigma_{\eta} \sigma_{\eta}^{\top} + \frac{1}{2} q^{2} P_{\eta} \sigma_{\eta}^{2} \sigma^{(2)\top} - \frac{1}{2} q^{2} P_{\eta} \sigma_{\eta}^{2} \sigma^{(2)\top}.
\]

The time derivative is omitted since the stock price is an asset with unbounded maturity. Now guessing that the stock price is homogeneous of degree one in dividends, we write \(P(q, \pi^{(1)}, \pi^{(2)}, \eta) = p(\pi^{(1)}, \pi^{(2)}, \eta)q\). Substituting into (56) we find that all terms involving \(q\) cancel, and the price-dividend ratio, \(p(\cdot, \cdot)\), must satisfy (31).

Following similar steps we can show that consol price can be written as \(B(\pi^{(1)}, \pi^{(2)}, \eta) = b(\pi^{(1)}, \pi^{(2)}, \eta) \cdot c\), and \(b(\cdot, \cdot)\) satisfies the PDE

\[
0 = b_{\pi(1)} (\mu^{(1)} - \sigma^{(1)} \phi^{(1)\top}) + b_{\pi(2)} (\mu^{(2)} - \sigma^{(2)} \phi^{(2)\top}) - b_{\eta} \eta \sigma_{\eta} \phi^{(1)\top}\phi^{(1)} - \frac{1}{2} \sigma_{\eta}^{2} \sigma_{\eta} \sigma^{(1)\top} (\sigma_{\eta} - \sigma^{(1)} \phi^{(1)\top}) + q b_{\pi(1),q} \sigma^{(1)} \sigma_{\eta}^{\top} + \frac{1}{2} b_{\pi(2),q} \sigma^{(2)} \sigma_{\eta}^{\top} + b_{\pi(2),\eta} \sigma_{\eta} \sigma_{\eta} + q b_{\pi(2),\eta} \sigma_{\eta} \sigma_{\eta}^{\top} + b_{\eta} \eta \sigma_{\eta}^{2} \sigma^{(2)\top} - \frac{1}{2} b_{\eta} \eta \sigma_{\eta}^{2} \sigma^{(2)\top}.
\]
We use projection methods described in Judd (1992) and Judd (1999) to solve the partial differential equations (PDEs) for the consol bond to coupon ratio, the price-dividend ratio, and the ratio of wealth to aggregate output in eqs. (57), (31), and (35) respectively. We focus our discussion on (57) since each is a 3-dimensional parabolic PDE with the same state variables $\pi^{(1)}$, $\pi^{(2)}$, and $\eta_t$.

We proceed by formulating an ‘approximate’ solution to (57) using projection methods (Judd 1999, Chapter 11).

**STEP 1.** Choice of individual basis functions. We choose the Chebyshev polynomials in each of the 3 dimensions: The Chebysev polynomials on $[-1, 1]$ for the basis for each dimension are given by

$$q_m(x) = \cos (m \cos^{-1} x),$$

for $m = 1, 2, \cdots$, which satisfy the recursive scheme

$$q_{m+1}(x) = 2 x q_m(x) - q_{m-1}(x).$$

These polynomials are restricted for the interval $[a, b]$ using the transformation

$$p_m, (x) = \frac{q_m(\frac{2x-a-b}{b-a})}{||q_m(\frac{2x-a-b}{b-a})||}.$$  

For the belief variables, $a = 0$ and $b = 1$. For $\eta$, we use the interval $[0, 25]$. The family $\{p_n(x)\}_{m=1,2,\cdots}$ are orthonormal polynomials over the chosen intervals.

**STEP 2.** Choose a basis of ‘complete’ polynomials over the space $[0,1]^3 \times [0, 25]$. The basis of degree $M$ over the 3 dimensions is given by

$$P_M = \{p_{i_1,i_2}(\pi_1^{(1)}): p_{2,i_2}(\pi_1^{(2)}): p_{3,i_2}(\eta)\sum_{n=1}^{3} i_n \leq M, 0 \leq i_1, \cdots, i_3\}$$

We will write the generic element of $P_M$ as $\phi_m(\pi_1^{(1)}, \pi_1^{(2)}, \eta)$, $m = 1, 2, \cdots M^c$, where $M^c$ is the length of the complete polynomial basis. The set of complete polynomials for an $N$ dimensional problem grows polynomially in $N$, as opposed to the tensor product basis which would use every possible product of the degree-M individual basis functions, and hence would grow at the rate of $M^N$ (see, e.g., pp. 239 in Judd 1999). The complete polynomials asymptotically, as $M$ becomes large, provide as good an approximation as the tensor product, but with far fewer elements. For example, we solve each PDE using $M = 15$. Using the tensor product basis, we would have a total of 3375 elements, but using the complete basis, we have far fewer, 816 elements. Extending the $L^2$ norm over the 3-dimensional space as the 3-fold integral, it can be verified that the basis of complete polynomials is orthonormal on $[0, 1]^3 \times [0, 25]$.

**STEP 3** Let $\mathcal{D}(y)$ be the differential operator associated with the PDE (57), i.e.

$$\mathcal{D}(y) = y^{(1)}(\mu^{(1)} - \sigma^{(1)}\phi^{(1)}^T) + y^{(2)}(\mu^{(2)} - \sigma^{(2)}(\sigma_{n} + \phi^{(1)}^T)) - \eta \eta \sigma \sigma^T - 1 - r + \frac{1}{2}y^{(1)}\sigma^{(1)}\sigma^{(1)}^T + y^{(2)}\sigma^{(2)}\sigma^{(2)}^T + \frac{1}{2}y^{(2)}\sigma^{(2)}\sigma^{(2)}^T + y^{(1)}\eta \sigma^{(1)}\sigma^{(1)} + y^{(2)}\eta \sigma^{(2)}\sigma^{(2)} + \frac{1}{2}y^{(2)}\eta \sigma^{(2)}\sigma^{(2)}^T \sigma^{(2)}.$$
Write the candidate solution as $\hat{y}(\pi_1^{(1)}, \pi_1^{(2)}, \eta) = \sum_{m=1}^{M^c} a_m \psi_m(\pi_1^{(1)}, \pi_1^{(2)}, \eta)$. Then any solution to the PDE, $\hat{y}$, will be written as $\mathcal{D}(\hat{y}) = 0$.

**Step 4** We appeal to the Chebyshev Interpolation Theorem (see, e.g. Judd 1992, Haan 1997) to find an approximate solution to the PDE. The approximation is made by evaluating the operator $\mathcal{D}(\hat{y})$ at a chosen set of points, and setting it equal to zero at each of these points. Each interpolation point therefore provides us a linear equation in the coefficients $(a_m)_{m=1}^{M^c}$. The chosen points for the 3-dimensional space are the Cartesian product of the zeros of the Chebysev polynomial of the highest degree chosen in Step 2 in each dimension. In general the $m+1$ zeros of the $m$th polynomial are given by

$$x_k = (-\cos \frac{2k-1}{2m} \pi) + 1) \frac{b-a}{2} + a, \quad \text{for } k = 0, 1, \ldots m.$$

For example by choosing polynomials of order 15 in each dimension, we obtain 3375 interpolation points. With $M^T$ interpolation points, we have an overidentified system of equations in $M^T$ unknown coefficients. We note that, since each of the PDEs has a forcing term (for example, the forcing term in (57) is $1 - r(\pi_1^{(1)}, \pi_1^{(2)}, \eta)$), the system of equations is non-homogeneous. Denote the $M^T \times 1$ vector of constants from each equation as $c$, and the $M^T \times M^c$ coefficient matrix as $A$. Analogous to regression coefficients, the best-fitting set of coefficients satisfies:

$$\hat{a} = (A^T A)^{-1} A^T c.$$

### Appendix 3: GMM Estimation of the Regime Switching Model

In this Appendix, we provide the details of the GMM estimation that uses information in both fundamentals and a series of forecasters’ dispersion about future earnings from surveys to estimate the parameter values and time-series of investors beliefs about the hidden states.

First, let $\Psi^{(m)}$ be the set of parameters characterizing the fundamental processes (1) and (2) for each type of investor. The discretized versions of these processes are

$$q_{t+1} = q_t \cdot e^{(\theta^{(m)}_t - \frac{1}{2} \sigma q^{(m)}_t) \Delta t + \sigma q \varepsilon_{t+1}}; \quad x_{t+1} = x_t \cdot e^{(\kappa^{(m)}_t - \frac{1}{2} \sigma x \varepsilon) \Delta t + \sigma x \varepsilon_{t+1}}.$$

Let $G^{(m)}(T)$ be the filtration on the set of unobserved drift states generated by the time series of realized fundamentals for investors of type $m$, $m = 1, 2$. We note that the filtrations on the underlying state spaces will differ even though each agent observes the same set of fundamentals, due to the differences in their parameter estimates $\psi^{(m)}$ and $\Lambda^{(m)}$. We assume that the volatility parameters in the fundamental processes, $\Sigma$, are common to the two agents.

For the estimation procedure we discretized the updating process as follows: Let $\pi^{(m)}(t|t) = \left(\pi_1^{(m)}(t|t), \ldots, \pi_N^{(m)}(t|t)\right)$ be the row vector of probabilities at time $t$ for agents of type $m$, after observing fundamentals at $t$. Let $P^{(m)}(\Delta t) = \exp(\Lambda \cdot \Delta t)$ be the transition matrix over a non-infinitesimal interval between observations, $\Delta t$. In our estimation technique, we estimate $P^{(m)}(0.25)$, the quarterly transition matrix. The implied generator is $\Lambda^{(m)} = \sum_{i=1}^{\infty} (-1)^{i+1} \cdot \left(\left(P^{(m)}(0.25)\right)^i - I_2\right) / i$ (see Israel, Rosenthal, and Wei 2001), whose value we estimate using a series approximation of length 10. A straightforward application of Bayes law implies that the updating rule for the posterior distribution on the state space $\mu^{(m)} = \left(\nu_1^{(m)}, \ldots, \nu_N^{(m)}\right)$ when the time
between observations is $\Delta t$ (see Hamilton 1989, Hamilton 1994):

$$
\pi^{(m)}_i(t | t) = \frac{e^{-\frac{1}{2} \left( \Delta \log(y(t)) - \nu^{(m)}_i \Delta t \right)^\top (\Sigma \Sigma^\top)^{-1} \left( \Delta \log(y(t)) - \nu^{(m)}_i \Delta t \right) \pi^{(m)}_i (t | t - \Delta t) P^{(m)}(\Delta t) \right]_i}{\sum_{j=1}^N e^{-\frac{1}{2} \left( \Delta \log(y(t)) - \nu^{(m)}_j \Delta t \right)^\top (\Sigma \Sigma^\top)^{-1} \left( \Delta \log(y(t)) - \nu^{(m)}_j \Delta t \right) \pi^{(m)}_j (t | t - \Delta t) P^{(m)}(\Delta t) \right]_j},
$$

(59)

where $y(t) = (q(t), x(t))$, $\Sigma = (\sigma_q^\top, \sigma_x^\top)$, and, $\nu^{(m)}_i = \nu^{(m)}_i - \frac{1}{2} (\sigma_x \sigma_x^\top, \sigma_q \sigma_q^\top)$. The beliefs over the next interval with no new information are expected to be:

$$
\pi^{(m)}_i (t + \Delta | t) = \pi^{(m)}_i (t | t) \cdot P^{(m)}(\Delta t).
$$

(60)

David (1993) shows that as the length of time between observations, $\Delta t$, goes to zero, the discrete-time process $\{\pi^{(m)}_i (t | t)\}$ converges almost surely to the diffusion process in (3).

The likelihood function of agent $m$ over fundamentals is then given by:

$$
\mathcal{L}^{(m)} \left( \Psi^{(m)} | \mathcal{G}^{(m)} (T) \right) = \sum_{t=2}^T \log f^{(m)} \left( \Delta \log(y) (t) | \mathcal{G}^{(m)} (t) ; \Psi^{(m)} \right),
$$

(61)

where, $f^{(m)} \left( \Delta \log(y) (t) | \mathcal{G}^{(m)} (t) ; \Psi^{(m)} \right) =

$$
\sum_{i=1}^N \left[ \pi^{(m)}_i (t | t - \Delta t) P^{(m)}(\Delta t) \right]_i \times e^{-\frac{1}{2} \left( \Delta \log(y(t)) - \nu^{(m)}_i \Delta t \right)^\top (\Sigma \Sigma^\top)^{-1} \left( \Delta \log(y(t)) - \nu^{(m)}_i \Delta t \right)}.
$$

The priors $\pi^{(m)} (0 | 0)$ are taken to be the unconditional means of the states implicit in the matrix $\Lambda^{(m)}$.

To obtain heterogeneous parameters of the fundamental processes, we obtain information from the time series of dispersion of earnings growth. Given the beliefs of agents of each type at time $t$, $\pi^{(m)} (t | t)$, we write the beliefs over earnings growth $\tau$ periods ahead (each period of length $\Delta t$) as:

$$
\pi^{(m)} (t, \tau) = \pi^{(m)} (t | t) \cdot (P^{(m)}(\Delta t))^\tau.
$$

(62)

Then we form the expected growth rate of earnings $\tau$ periods ahead for each agent, and the standard deviation of the cross-sectional earnings expectations to obtain a series of model-generated dispersion. Call $d(t, \tau)$, the $\tau$ quarters ahead dispersion in the data from eq. (40), and $d(\pi^{(m)} (t, \tau), \pi^{(m)} (t, \tau))$ the model generated dispersion. We define the dispersion error as:

$$
e(t, \tau) = [d(t, \tau) - d(\pi^{(m)} (t, \tau), \pi^{(m)} (t, \tau))].
$$

We can then estimate $\Psi^{(m)}$ from a GMM procedure of the dispersion variable errors. Let $\epsilon(t)^\top = \{e(t, \tau), \frac{\partial g(t)^\top}{\Psi^{(1)}}(t), \frac{\partial g(t)^\top}{\Psi^{(2)}}(t)\}$ where the second and third terms are the scores of the likelihood function of fundamentals with respect to $\Psi^{(1)}$ and $\Psi^{(2)}$ respectively. In the estimation procedure, we use $\Delta t = 1/4$ and $\tau = 4$. Similar results were obtained using $\tau = 1, \cdots , 4$. We now form the GMM objective:

$$
C = \left( \frac{1}{T} \sum_{t=1}^T \epsilon_t \right)^\top \cdot \Omega^{-1} \cdot \left( \frac{1}{T} \sum_{t=1}^T \epsilon_t \right).
$$

Since we find the processes of dispersion errors, $\{\epsilon_t\}$, to be serially correlated, while the scores are not, we diagonally partition the matrix $\Omega$ into two parts: $\Omega_q$, and $\Omega_d$. $\Omega_d$ is estimated using the
Newey-West correction (see Hamilton 1994, Eq 14.1.19):

\[
\hat{\Omega}_{d,T} = \hat{\Gamma}_{0,T} + \sum_{j=1}^{J} \left[ 1 - j/(J+1) \right] \cdot (\hat{\Gamma}_{j,T} + \hat{\Gamma}_{j,T}^\prime),
\]

where

\[
\hat{\Gamma}_{j,T} = \frac{1}{T-j} \cdot \sum_{t=j+1}^{T} e_t \cdot e_t^\prime.
\]

\(\Omega_\Psi\) is estimated by 

\[
\frac{1}{T} \sum_{t=1}^{T} \left[ \frac{\partial L}{\partial \Psi(1)}(t) \right] \left[ \frac{\partial L}{\partial \Psi(2)}(t) \right]^\top,
\]

where \(\frac{\partial L}{\partial \Psi(1)}(t) = \left( \frac{\partial \phi^{(1)}(t)}{\partial \Psi(1)}(t), \frac{\partial \phi^{(2)}(t)}{\partial \Psi(2)}(t) \right)^\top\).

We will then look for parameters \(\Psi^{(m)}\) that jointly minimize the dispersion error and the scores of the likelihood function from fundamentals. Since each of the likelihood functions depends on distinct drifts and generator elements for each agent, but three common volatility parameters, the overall GMM objective is overidentified and has a \(\chi^2(4)\) distribution. The value of the objective function then serves as an omnibus specification test statistic for the model.

References


Blume, Lawrence, and David Easley, 2004, If you’re so smart, why aren’t you rich? belief selection in complete and incomplete markets, Discussion paper Department of Economics, Cornell University.


Cuoco, Domenico, and Hua He, 1994, Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets, Mimeo The Wharton School, University of Pennsylvania.


Detemple, Jerome B., and Angel Serrat, 1998, An equilibrium analysis of liquidity constraints, University of Chicago, Chicago, IL.


54
Shefrin, Hersh, 2001, On kernels and sentiment, mimeo Santa Clara University.


Table I: 2-State Heterogeneity Model Calibration

| Series Used: Real Earnings, Real Consumption, and Dispersion of Earnings Growth Forecasts |
| Time Span (Quarterly): 1971-2001 |

<table>
<thead>
<tr>
<th>Agent 1:</th>
<th>Drifts:</th>
<th>$\theta_1^{(1)}$</th>
<th>$\theta_2^{(1)}$</th>
<th>$\kappa_1^{(1)}$</th>
<th>$\kappa_2^{(2)}$</th>
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<td>0.0280</td>
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<td>(0.0289)</td>
<td>(0.0103)</td>
<td>(0.0103)</td>
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<td>$\lambda_{21}^{(1)}$</td>
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<td>$P_{21}^{(1)}$</td>
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<td></td>
<td></td>
<td>0.5061</td>
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<td>0.1611</td>
<td>0.0772</td>
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<td></td>
<td></td>
<td>(0.0612)</td>
<td>(0.0444)</td>
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<table>
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<th>Agent 2:</th>
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<td>-0.2305</td>
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<td>(0.0192)</td>
<td>(0.0258)</td>
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<td>(0.0103)</td>
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<tr>
<td>Generator Elements:</td>
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<td>$\lambda_{21}^{(2)}$</td>
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<td></td>
<td></td>
<td>1.3194</td>
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<td>0.2749</td>
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<td></td>
<td></td>
<td>(0.0656)</td>
<td>(0.0462)</td>
<td></td>
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</tr>
</tbody>
</table>

| Volatilities: | $\sigma_{q,1}$ | $\sigma_{q,2}$ | $\sigma_{x,2}$ |
|               | 0.0833         | 0.0169          | 0.0200          |
|               | (0.0003)       | (0.0001)        | (0.0001)        |

GMM estimates of the following (discretized) model for real consumption, $x_t$, and real earnings, $q_t$:

\[ x_{t+1} = x_t e^{(\kappa_t^{(m)} - \frac{1}{2} \sigma_x \sigma_t') \Delta t + \sigma_x \varepsilon_{t+1}} \cdot q_{t+1} = q_t e^{(\theta_t^{(m)} - \frac{1}{2} \sigma_q \sigma_t') \Delta t + \sigma_q \varepsilon_{t+1}}. \]

where $\sigma_q = (\sigma_{q1}, \sigma_{q2})$, $\sigma_x = (0, \sigma_{x,2})$ and where $\left(\theta_t^{(m)}, \kappa_t^{(m)}\right)$ jointly follows a two-state regime switching model. We note that the variance-covariance matrix between the two fundamental shocks is identified by setting $\sigma_{x,1} = 0$. We estimate the quarterly transition probability matrix whose estimates and standard errors are shown. The implied generator is $\Lambda^{(m)} = \sum_{i=1}^{\infty} (-1)^{i+1} \cdot \left(\left(\frac{P^{(m)}(0.25)}{4} \cdot I\right)^i / i\right)$, whose value we estimate using a series approximation of length 10. The GMM errors include the scores of the likelihood function of each type of agent and the difference in model-implied and historical dispersion in forecasts of Professional Forecasters as described in Appendix 3. The $\chi^2(4)$ statistic for the specification test of the model is 7.6341, which has a p-value of 0.1059. Standard errors of parameter estimates are in parentheses.
Table II: Model’s Fit for Earnings Growth and Dispersion in Earnings Growth

| Agent 1:                | $\Delta \log(q) (t) = \alpha + \beta \cdot (\theta_1^{(1)} \pi_1^{(1)} (t|t) + \theta_2^{(1)} \pi_1^{(1)} (t|t)) + \epsilon(t)$ |
|------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| $\hat{\alpha}$        | 0.0932                                                              | $\hat{\beta}$ | 1.4885 | $R^2$ | 0.6476 |
| (0.2224)               | (8.8857)                                                            |                |        |        |        |

| Agent 2:                | $\Delta \log(q) (t) = \alpha + \beta \cdot (\theta_1^{(2)} \pi_1^{(2)} (t|t) + \theta_2^{(2)} \pi_1^{(2)} (t|t)) + \epsilon(t)$ |
|------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| $\hat{\alpha}$        | -0.2116                                                             | $\hat{\beta}$ | 1.8240 | $R^2$ | 0.6737 |
| (-0.5248)              | (10.0718)                                                          |                |        |        |        |

<table>
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<tr>
<th>Dispersion:</th>
<th>$d(t, 4) = \alpha + \beta \cdot d(\pi^{(1)} (t, 4), \pi^{(2)} (t, 4)) + \epsilon(t)$</th>
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<td>$\hat{\alpha}$</td>
<td>4.1301</td>
</tr>
<tr>
<td>(12.1873)</td>
<td>(4.0921)</td>
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</table>

Units of measurement are quarterly and in percentage points. T-statistics are in parentheses. All t-statistics are adjusted for heteroskedasticity and autocorrelation using the methodology of Newey and West (1987). Parameter estimates for the two agents are given in Table I. Beliefs for each agent are updated using the discrete approximation of the belief process in eqs. (59) and (60). Figure 1 (top panel) shows the belief processes of the two agents. The bottom panel shows the 1-quarter ahead expected growth of each agent, and the actual earnings growth process, the variables used for the first two regressions. The model generated dispersion is found by formulating the 4-quarter ahead beliefs of each agent from eq. (62), and formulating the standard deviation of 4-quarter ahead expected growth rates across agents. The top panel of Figure 2 shows the actual and model implied 4-quarter ahead dispersions of earnings growth, which are in the 3rd regression.
Table III: Calibrated Equity Premium and Related Statistics (1971 to 2001)

<table>
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<td>μₚ</td>
<td></td>
<td></td>
<td>σₚ</td>
<td></td>
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<td>μₘ</td>
<td>σₘ</td>
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<td>σ₁</td>
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<td>1</td>
<td>0.02</td>
<td>0.30</td>
<td>0.0537</td>
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<td>2</td>
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<td>0.1829</td>
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<td>4.9428×10⁻⁴</td>
<td>0.0366</td>
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<tr>
<td>3</td>
<td>0.02</td>
<td>0.50</td>
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<td>0.1857</td>
<td>0.0275</td>
<td>0.0112</td>
<td>0.0241</td>
<td>0.1298</td>
<td>0.0971</td>
<td>0.0998</td>
<td>-0.0284</td>
<td>0.0259</td>
<td>-0.1398</td>
<td>0.1445</td>
<td>4.6133×10⁻⁴</td>
<td>0.0338</td>
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<tr>
<td>4</td>
<td>0.02</td>
<td>0.60</td>
<td>0.0524</td>
<td>0.1924</td>
<td>0.0217</td>
<td>0.0176</td>
<td>0.0295</td>
<td>0.1533</td>
<td>0.1381</td>
<td>0.1305</td>
<td>-0.0354</td>
<td>0.0321</td>
<td>-0.1401</td>
<td>0.1447</td>
<td>4.2838×10⁻⁴</td>
<td>0.0310</td>
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<tr>
<td>5</td>
<td>0.03</td>
<td>0.50</td>
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<td>0.1445</td>
<td>4.6133×10⁻⁴</td>
<td>0.0438</td>
<td></td>
</tr>
</tbody>
</table>

The unconditional means of the following variables (annualized units, unless stated) are generated from the model: Col. (3), μₚ, equity cum-dividend return, col. (4), ||σₚ||, volatility of equity returns, col. (5), μₘ, riskless rate, col. (6) σₘ, volatility of riskless rate, col. (7), μₚ−μₘ, equity premium (objective measure), col. (8), (μₚ−μₘ)/||σₚ||, Sharpe ratio, col. (9) and col. (10), σ₁, j = 1, 2, volatilities (quarterly rate) of individual consumption growth, col. (11), σ₂, per capita volatility (quarterly rate) of consumption growth, cols. (12) and (13), μₑ−μₑ, j = 1, 2 equity premium under measure of two different agents, cols. (14) and (15), ρ(1), correlations between consumption growth and stock returns for the two agent types, and cols. (16) and (17) ρ(2), μₑ−μₑ, and ρ(3), equity premium and riskless return under homogeneous beliefs. Equity Premiums and all other statistics are calculated using the solutions to the PDEs in Section II A, using the parameter values in Table I. The belief process, {π(t)}, is generated using the discretized version of the SDE in Lemma 1 as shown in (59) and (59). The disagreement value process is generated using the discretized version of (27). Calibrated belief and disagreement value processes are shown in Figures 1 and 2. The price-dividend ratio is obtained from the solution of PDE (31) as shown in Figure 5 (top panels), and at period t is given by p(π₁(t), π₂(t), η₁). The calibrated price at time t is given by p(π₁(t), π₂(t), η₁)·g(t)/2, where g(t) are S&P 500 earnings per share. Other variables are similarly calculated. The riskless rate is in (22) is shown in Figure 4. Agents’ consumptions are in (20) and (21) and are shown in the top panel of Figure 3. The ex-ante expected premium of each agent is obtained from eq. (45). The risk premium for the homogeneous investor case is (1 − γ)σₓσₓ, and the riskless rate is in (28).
Table IV: Conditional Moments of Model-Generated Variables

<table>
<thead>
<tr>
<th>Cases</th>
<th>Prob.</th>
<th>(1) $\mu_c^{(1)}$</th>
<th>(2) $\mu_c^{(2)}$</th>
<th>(3) $\mu_c^{(1)}$</th>
<th>(4) $\mu_c^{(2)}$</th>
<th>(5) $\sigma_c^{(1)}$</th>
<th>(6) $\sigma_c^{(2)}$</th>
<th>(7) $\sigma_P$</th>
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<tbody>
<tr>
<td>$\tilde{\theta}_t^{(1)} &gt; \tilde{\theta}_t^{(2)}$</td>
<td>0.4308</td>
<td>0.9431</td>
<td>-0.9127</td>
<td>0.0255</td>
<td>-0.0098</td>
<td>0.0416</td>
<td>0.0769</td>
<td>0.1184</td>
</tr>
<tr>
<td>$\tilde{\theta}_t^{(1)} &lt; \tilde{\theta}_t^{(2)}$</td>
<td>0.5691</td>
<td>-0.9596</td>
<td>0.9448</td>
<td>-0.0038</td>
<td>0.0152</td>
<td>0.1482</td>
<td>0.1124</td>
<td>0.2366</td>
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<tr>
<td>Averages</td>
<td>-0.1398</td>
<td>0.1445</td>
<td>0.0088</td>
<td>0.0044</td>
<td>0.1023</td>
<td>0.0971</td>
<td>0.1857</td>
<td></td>
</tr>
</tbody>
</table>

The following (annualized, unless stated) conditional moments are generated from the model for the case when $\rho = 0.02$ and $\gamma = 0.5$ corresponding to line 3 of Table III. Cols. (1) and (2), $\rho^{(m)}$, m=1,2, are the conditional correlations of ex-ante consumption growth with stock returns of the two types of agents respectively, which at time $t$ is calculated as $(\hat{\phi}_t^{(m)} \cdot \sigma_P) / (||\hat{\phi}_t^{(m)}|| \cdot ||\sigma_P||)$, where $\hat{\phi}_t^{(m)}$ are the market prices of risk in Proposition 2 (iii). Cols. (3) and (4), $\mu_t^{(m)}$, are individual ex-post mean consumption growth rates (quarterly), and cols. (5) and (6), $||\sigma_c^{(m)}||$, are individual ex-post consumption growth rate volatilities (quarterly). These are calculated using the agents’ consumption functions in (20) and (21), and the consumption paths are shown in the top panel of Figure 3. The calibrated disagreement value process is in Figure 2. Col. (7) is the volatility of equity returns obtained from the solutions to the PDEs in Section II A as shown in eq. (32), using the parameter values in Table I. For its generation, in addition to the disagreement value process, we use the calibrated beliefs of each agent shown in Figure 1. Cols. (8) and (9), $\rho^{(m)}E[||\phi_t^{(m)}|| \cdot ||\sigma_P||]$, are the products of the conditional correlations in cols. (1) and (2) and expectations of the market prices of risk and stock volatility of the two types of agents respectively. Cols. (10) and (11), $\rho^{(m)}E[\phi_t^{(m)} \cdot ||\sigma_P||]$, are the products of the conditional correlations in cols. (1) and (2) and covariances of the norms of the market prices of risk and stock volatility of the two types of agents respectively, and (12) and (13), $\mu_E^{(m)} - \mu_R$, are the ex-ante expected equity premia of the two agents respectively, calculated as in eq. (45). Col. (14), $\mu_P - \mu_R$, is the ex-post realized cum-dividend realized excess return on equities calculated from the calibrated price and dividend process described in the footnote to Table III. Col. (15), $\mu_R$, is the riskless return calculated as in eq. (22). Col. (16), $E[\delta_t^{(1)} - \delta_t^{(2)}]$, is the difference in expectations of earnings growth of the two types of agents, for the two types of agents, col. (17), $d(t, 4)$, is the four-quarter ahead dispersion of Professional Forecasters, and col. (18), $\delta(t, 4)$, is its model fitted (scaled) counterpart (the two series are shown in Figure 2). Col. (19), $\sigma_{cs}$, is the cross-sectional standard deviation of ex-post consumption growth rates (quarterly) across agents.
The top panel has the time series of filtered beliefs about real earnings growth of the two types of agents. Filtered beliefs of the two agents are obtained from the discretized version of the belief processes in Lemma 1 as shown in eqs. (59) and (60), using the calibrated parameters for each type of agent shown in Table I. The bottom panel has the actual and expected earnings growth of the two types of agents using these filtered beliefs.
The top panel has the model implied dispersion and the dispersion of 1-quarter ahead earnings growth from the Survey of Professional Forecasters. The data series is cross-sectional standard deviation of the panel of forecasts calculated as shown in (40). Model series are analogously calculated as the standard deviation of the forecasts of the two types of agents. The bottom panel shows the disagreement value process, \( \{ \eta \} \), implied by the belief series shown in Figure 1 and eq. (16).
The top panel shows the consumption levels of each agent as calculated using (20) and (21) in Proposition 2. The disagreement value process, \( \{ \eta_t \} \), is obtained from the belief processes of the two agents shown in Figure 1 using eq. (16) and is in the bottom panel of Figure 2. The bottom panel shows the proportion of wealth in the stock held by investors of type 1 (‘volatile’ investors) calculated as \( u_t^{(1)} = \sigma_{X_t}^{(1)} \cdot (\sigma_{B_t}^{T} \sigma_{P_t}^{T})^{-1} \). The volatilities of stocks, bonds, and wealth are calculated as described in Section II A and at time \( t \) are calculated conditional on agents’ beliefs (Figure 1) and disagreement (Figure 2).
The riskless rate is obtained from eq. (22). The effect of variation in these variables by changing the disagreement value is shown in the left panels, when the two agents have the same beliefs. In the right panels, the disagreement value is set to one, and agents’ beliefs are allowed to differ. The parameters of the fundamental processes are shown in Table I. In addition, we use $\rho = 0.02$ and $\gamma = 0.5$. 
The price-dividend ratio on equities (top panels) and the price-coupon ratio (bottom panels) on consol bonds are obtained from the solution of the PDEs in (31) and (57). The effect of variation in these multiples by changing the disagreement value is shown in the left panels, when the two agents have the same beliefs. In the right panels, the disagreement value is set to one, and agents’ beliefs are allowed to differ. The parameters of the fundamental processes are shown in Table I. In addition, we use $\rho = 0.02$ and $\gamma = 0.5$. 
Figure 6: Stock Return Volatility and Agent 2’s Equity Premium

Stock return volatility is obtained from eq. (32). The effect of variation in these variables by changing the disagreement value is shown in the left panels, when the two agents have the same beliefs. In the right panels, the disagreement value is set to one, and agents’ beliefs are allowed to differ. The parameters of the fundamental processes are shown in Table I. In addition, we use $\rho = 0.02$ and $\gamma = 0.3$. 