

# The Term Structure with Semi-credible Targeting<sup>1</sup>

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## **Abstract**

The actions of the monetary authority are seldom taken into account in continuous-time arbitrage pricing models of the term structure of interest rates. This is likely to be problematic in many economies. In the U.S., for example, the Federal Reserve sets targets for interest rates which it enforces through direct market intervention. In this paper we develop a model in which the short rate is subject to a control which keeps it close to a target which changes from time to time. The probability of target changes is not constant in the model but changes as a function of observables. The model performs well at explaining the shifts in the yield curve that accompany target changes.

# 1 Introduction

In U.S. financial markets the Federal Reserve Bank (the Fed) is a unique player. Usually we assume that market participants are price takers in financial markets. However there is evidence that this is not true of the Fed. In fact the Fed sets targets for short-term rates which it attempts to maintain by market intervention, injecting or withdrawing funds from the market to keep the rates near their targets. Targets are changed from time to time and market participants know when such a target change occurs. They may, in fact, have some limited ability to forecast such target changes.

Several recent papers have provided evidence that models which allow the short rate to revert towards a mean which is itself a random process perform better than models in which the mean is constant. Jegadeesh and Pennacchi (1996) suggest the relationship between such a model and the interest rate targeting of the Fed. However their model assumes a continuously varying mean that is unobservable rather than the Fed targets which are observable and change infrequently. Balduzzi, Das, and Foresi (1998) also assume such a mean process but point out that using two bonds one can construct a proxy for the unobserved mean process. By using this proxy they find evidence that the mean is not constant but do not explicitly relate it to observed Fed targets.

Past research on the effect of Fed policy on the term structure focused on the relation between money growth and short term interest rates. By and large the evidence of these studies was that the Fed had little control over interest rates (see Reichenstein (1987) for a survey). More recent papers have focused on the effect of federal funds rate targets on market interest rates. Cook and Hahn (1989) was the first such study. They find that during the 1970's target changes caused significant movements in yields. They also document that market participants are aware of target changes by comparing reports in the *Wall Street Journal* to a record of actual target changes later published by the Fed.

As a result of this evidence several term structure models have been advanced which take account of these targets<sup>1</sup>. These models all rely on the expectations hypothesis to relate longer term yields to short term rates. While expectations of future changes in the target play a major role in each

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<sup>1</sup>see Balduzzi, Bertola, and Foresi (1997), Rudebusch (1995), and Roley and Sellon (1997)

of these models, it is unclear why targets are changed. Balduzzi, Bertola, Foresi, and Klapper (1998) suggest a similar model with the extension that targets are somewhat predictable in the sense that changes follow an AR(1) process. This is consistent with the observation that the direction of target changes is highly persistent.

In addition to these papers there have been several recent papers which examine the effect of central bank rate setting policies on bond prices in countries besides the U.S. Babbs and Webber (1994) examine a model of interest rates in the UK in which the short rate is taken to be equal to the “Band 1 rate” (the rate analogous to the fed funds target rate). The short rate modeled as a pure jump process with time varying jump intensities. Bonds and other interest rate contingent claims can be priced using this model. The obvious shortcoming is that the true short rate does not always equal the Band 1 rate set by the Bank of England. Babbs and Webber (1997) present several models for various countries which focus on the rates at which banks can borrow from the central bank and which serve as a bound on the short rate<sup>2</sup>. They point out that in the U.S. borrowing from the Fed is discouraged and that some change to their model would be necessary to capture the effect of the fed funds target rate. Piazzesi (2000), a recent working paper, improves upon these models by modelling the probability of target changes as varying with several macroeconomic factors which determine the “desired rate” of the Fed<sup>3</sup>.

The purpose of this paper is to cast doubt on models which assume that the short rate mean reverts towards the target. We document that such models perform poorly empirically in explaining the nature of yield curve shifts associated with target changes. In addition mean reversion lacks any theoretical justification as a model of how the short rate would behave in response to enforcement actions by a monetary authority. We propose instead a model in which short rates are subject to a controller such as the Fed who wants to keep the rate near a target. We show that this model does a much better job of explaining yield curves shifts around target changes. This model also has a mathematical feature which is unique in the literature. In the simplest version of the model where there are three factors (the short rate,

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<sup>2</sup>Honoré (1998) has developed econometric tools for dealing with such models and analyzes the case of the German terms structure

<sup>3</sup>One peculiar feature of this model is that when the Fed changes a target it does not change it to the desired rate. So it is unclear what is really meant by the desired rate in this model or what keeps the Fed from achieving its desire.

upward target changes, and downward target changes) claims can be hedged using only two bonds. This is due to a new type of target process which we introduce and which we designate a “semi-credible” target.

## 2 Data and Motivating Evidence

Before turning to the issue of modelling we shall review the history of fed funds targeting and the nature of the data.

During the 1970’s the Fed followed a policy of targeting the overnight fed funds rate. The practice was discontinued after September 1979. There are two datasets of target changes for this early period. The most complete is the one used in Rudebusch (1995) which begins with the target change of 13 September, 1974 and ends with the target change of 19 September, 1979. During this period there were 99 target changes of which 56 were upward and 43 were downward. Rudebusch obtained this data series from notes of the FOMC. The other data set was collected by Cook and Hahn (1989) based on Wall Street Journal reports of what traders thought the Fed was doing. This second data set has 79 target changes of which 50 were upward and 29 were downward. The reason that the two data series are not the same is that during this period the Fed did not announce target changes at the time they occurred<sup>4</sup>. We use the Cook and Hahn data for this period because we are interested in market reactions to target changes and the Cook and Hahn data contains only those changes that market participants knew about<sup>5</sup>. During this period target changes were generally 12.5 basis points in size although some other sizes did occur (12.5 basis points was the median target change during this period).

In order to target interest rates the central bank must inject or drain reserves at various times to offset market movements. The more intervention is done the more reserve levels (and money growth) will fluctuate. In fact it is not possible to simultaneously control interest rates and reserves (at least not perfectly). In October 1979 the Fed, under new chairman Paul Volcker, changed to a policy of targeting non-borrowed reserves<sup>6</sup> in an effort to control money growth. Not surprisingly the volatility of interest rates increased greatly during this time. At its October 1982 meeting the Federal

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<sup>4</sup>See Thornton (1998) for a detailed discussion of the differences between these series.

<sup>5</sup>The results we present here are actually not much affected by which data set we use.

<sup>6</sup>These historical details are found in Meulendyke (1998) Chapter 2

Open Market Committee (FOMC) decided to abandon these targets.

After this period the Fed changed it's official policy to one of targeting borrowed reserves. However there is disagreement as to whether this was really the policy the Fed was following. Transcripts of meetings make it clear that the Fed was setting fed funds targets as early as 1983<sup>7</sup>. At the very least there was (as Meulendyke puts it) an "informal move away from borrowed reserve targets" during the decade of the 1980's and the Fed became more and more explicit about their fed funds targets. Meulendyke (1998) describes this process as being speeded by the stock market crash of 1987. Since December 1984 we have data on 71 target changes. Following Rudebusch (1995) we have elected to use only those following the target change of 4 November 1987. This leaves 52 target changes of which 20 were upward and 32 were downward. During this later time period target changes were usually 25 basis points in size with a few exceptions (25 basis points was the median target change).

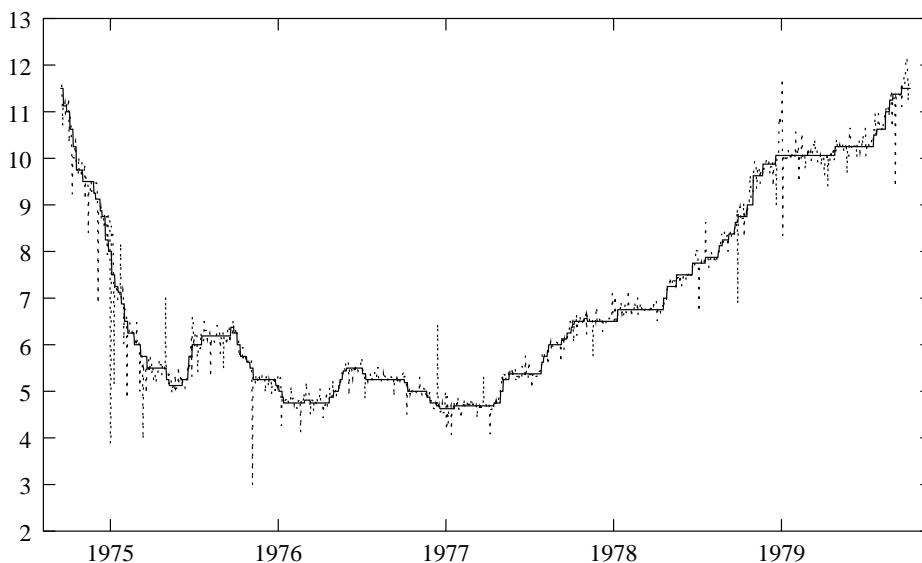


Figure 1: Funds Rate and Targets in the 70's

The fed funds data used in this paper is the rate on overnight fed funds. All data on fed funds rate and yields were obtained from the web site of the

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<sup>7</sup>See Thornton (1998).

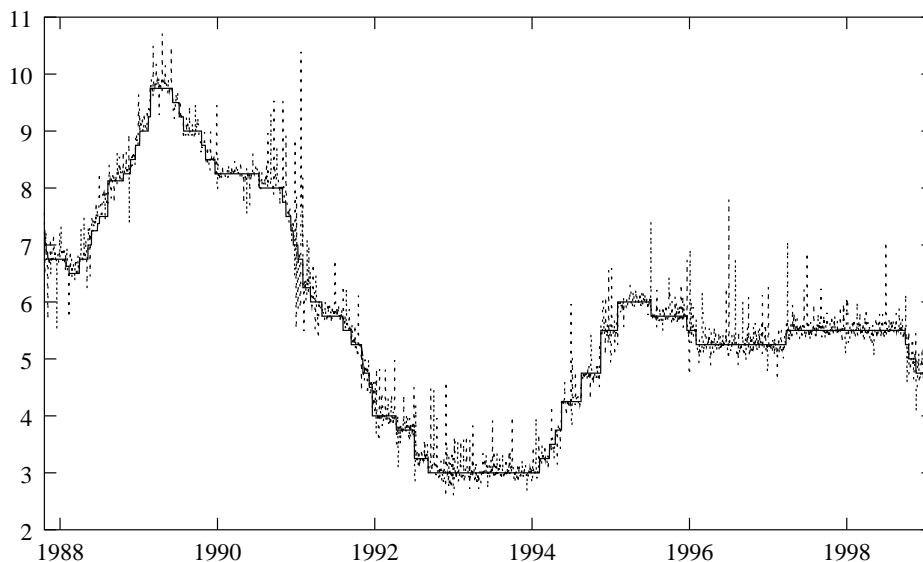


Figure 2: Funds Rate and Targets since 1987

Federal Reserve Bank of St. Louis. The fed funds rate is collected at the end of each day and is the average from trades made during the day through five brokers who report to the Fed. The fed funds rate and targets are plotted for each of these periods in figures 1 and 2. There are a few striking details apparent in these figures. The first is that the fed funds rate stays very close to the target during both periods. The second is that there are some significant deviations from the target but these are remarkably short lived (they show up as one day “spikes” in the data). Most of these correspond to calendar events such as the end of the reserve maintenance period or the end of the calendar year. These spikes are more pronounced in the later period because of the move from lagged to contemporaneous reserve accounting.

Before moving on to modeling it is important to recognize the limitations inherent in using fed funds targets as the basis for a term structure model. In term structure modeling one typically begins with a model for the short rate. As mentioned previously the fed funds rate contains high frequency noise which makes it unsuitable as a short rate proxy. Even if this noise could be filtered out we would not have an adequate short rate proxy because the fed funds rate is not a true riskless interest rate. The parties who trade in the fed funds market are banks and so some default risk exists. The fed

funds rate is higher than the true riskfree rate would be for this reason. In fact general collateral overnight repo rates (which are quoted from the fed funds rate) are typically less than the fed funds rate<sup>8</sup>. The target must therefore reflect this risk premium as well. Our solution to this issue will be to focus on the changes in targets and yields rather than levels. We assume that corresponding to the unobservable short rate there is a target which is related to the observable target in that jumps happen at the same time and are of the same magnitude.

Maturity	1974 – 1979		post 1987	
	Intercept	Target Change	Intercept	Target Change
fed funds	-0.0075 (-0.3630)	0.3555 (4.0877)	-0.0286 (-0.6591)	0.0860 (0.6716)
3-month	0.0162 (1.1274)	0.5421 (8.0174)	-0.0217 (-2.0834)	0.2532 (6.7548)
6-month	0.0138 (1.2657)	0.5315 (10.0678)	-0.0232 (-2.0869)	0.2291 (5.9478)
1-year	0.0174 (1.6084)	0.4745 (9.6382)	-0.0245 (-2.0494)	0.1947 (4.4737)
2-year			-0.0247 (-2.1625)	0.1626 (3.9115)
3-year	0.0126 (1.3611)	0.2860 (6.2343)	-0.0251 (-2.1508)	0.1249 (3.1251)
5-year	0.0053 (0.5951)	0.1938 (4.4196)	-0.0207 (-1.7484)	0.1015 (2.3893)
7-year	0.0030 (0.3916)	0.1747 (4.4941)	-0.0228 (-1.9697)	0.0723 (1.6860)
10-year	0.0059 (0.9306)	0.1198 (3.7117)	-0.0182 (-1.6818)	0.0559 (1.4104)

Table 1: Estimates from a regression of daily yield changes on daily target changes. T-statistics are in parentheses. T-statistics are calculated using White standard errors. Data for the two year bond are not available for the earlier period.

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<sup>8</sup>The natural question this raises is why not use the repo rate rather than the fed funds rate. One reason is that the fed funds rate data is much more readily available. Also the repo rate exhibits the same high frequency noise that the fed funds rate does although to a somewhat lesser extent.

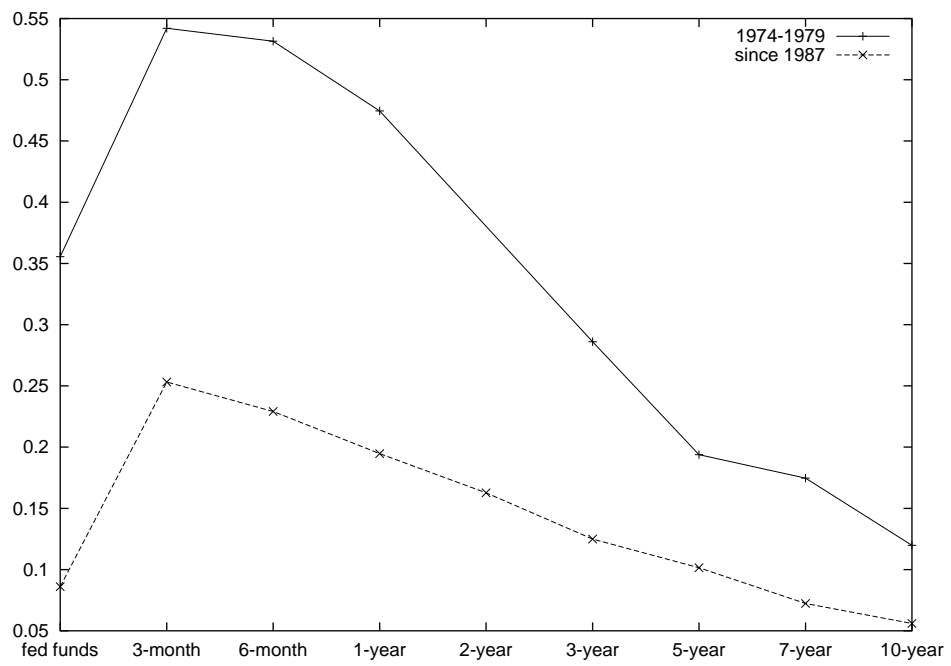


Figure 3: Estimated slopes from Table 1

The first study of the effect of target changes on the yield curve was Cook and Hahn (1989). Using data from the 1974 to 1979 period they regressed daily changes in yields of bonds of various maturities on changes in the target. They found that the response of the yield curve was significant at all maturities but much less than unity. In table 1 we have performed similar regressions for both periods. In doing so we have dropped two observations from each period because they correspond with some of the large spikes in the data. Our criterion for identifying outliers is that a one day change in the fed funds rate of more than 100 basis points will be called an outlier. Accordingly we dropped the target changes of 2 Jan 1975, 28 Sept 1978, 1 Feb 1991, and 6 Jul 1995. The smallest one day move for the fed funds rate was 135 basis points on the last of the four dates. The largest move was on the first date and the fed funds rate moved by 468 basis points that day. Since target changes are separated by several weeks to several months we are not surprised to find that Durbin-Watson tests show no serial correlation in these regressions.

There are several things to notice about Table 1. First note that the slopes are all positive and significantly different from zero except the very long maturities in the later period. The second thing to notice is that all the slopes are much less than unity. This means that while the yields on bonds of all maturities respond to a target change the magnitude of the change is less than the amount of the target change. Finally there is a distinct pattern to the magnitude of these responses. To illustrate this more clearly the slopes from Table 1 are plotted in Figure 3. Notice that the response to a target change is greatest for maturities of about 3 months and much less for the very short and very long maturities. Also from figure 3 note that the responses are much less for all maturities in the later time period, when the size of target changes was greater. These observations will guide us in developing a term structure model which incorporates targets.

### 3 Exponential Affine Models and Targets

One of the first attempts at a term structure model which incorporates targets is Balduzzi, Bertola, and Foresi (1997). This model is perhaps the simplest imaginable. Because the fed funds rate stays close to its target the spread (short rate minus target) is modeled as an AR(1) with a long-run mean of zero while the target is a pure jump process with fixed jump sizes.

In that paper the modeling is done in discrete time. The continuous-time version of this model is

$$\begin{aligned} s_t &= r_t - J_t \\ ds_t &= -\kappa s_t dt + \sigma dW_t \\ dJ_t &= b(dN_t^u - dN_t^d) \end{aligned} \tag{1}$$

where  $r_t$  is the short rate and  $J_t$  is the target. The process  $s_t$  is the spread between short rate and the target. The target process is driven by two jump processes,  $N^u$  and  $N^d$  which represent upward and downward jumps respectively. These are assumed to be independent Poisson processes and  $b$  is the jump size.

An appealing thing about this model is that it falls into the exponential-affine class of Duffie and Kan (1996). Most well-known continuous-time term structure models are in this class. The appeal of models in this class lies in the fact that they all possess zero coupon bond pricing solutions of the form

$$P(x, y, \tau) = \exp(A(\tau) + B_1(\tau)x + B_2(\tau)y)$$

(for a two-factor version) where  $\tau$  is the time to maturity of the bond and  $x$  and  $y$  are state variables. The functions  $A$ ,  $B_1$ , and  $B_2$  are functions of  $\tau$  only and satisfy a particular ODE.

If we identify  $y$  with the target  $J$  then the response to a change in target as is shown in figure 3 is  $-B_2(\tau)/\tau$ . For the model above it turns out that  $B_2(\tau) = -\tau$  so the response to a target change is identically unity for all maturities<sup>9</sup>. It seems quite clear from Table 1 and Figure 3 that this model is not consistent with the data.

Below we introduce and compare three other models which fall into the exponential affine class.

**Model 1** Rather than assuming that the spread is a continuous process which mean reverts toward zero consider the following specification for the short rate

$$dr_t = \kappa(J_{t-} - r_t)dt + \sigma dW_t \tag{2}$$

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<sup>9</sup>This might have been easy to guess since the structure of the problem forces the response at the extreme short end to be unity. This is because the spread does not jump at target changes so the short rate does.

with  $J_t$  as above. In this model the short rate reverts toward the target as in the previous model but the short rate does not jump in response to target changes so the response at the short end is zero. In this model we have that

$$-B_2(\tau)/\tau = 1 + \frac{\exp(-\kappa\tau) - 1}{\kappa\tau} \quad (3)$$

which is monotone and increases from zero at  $\tau = 0$  to unity as  $\tau$  gets large. This does not seem very promising because the response function in Figure 3 does not look monotone. However a statistical test is needed to determine if such a model can explain the shifts in the yield curve at target changes.

The estimation method we employ is related to the regression approach of Cook and Hahn but constrains the responses to agree with the model predicted response. Define  $\eta_n \equiv \Delta y_n + B_2(\tau_n)/\tau \Delta J$  where  $\Delta y_n$  is the daily change on the bond which matures in  $\tau_n$  periods. We use a GMM system with the following moments.

$$\begin{aligned} E(\eta_n) &= 0 \\ E(\eta_n \Delta J) &= 0 \end{aligned} \quad (4)$$

The lack of serial correlation allows us to use Hansen's weighting matrix in the GMM estimation. The estimated parameter  $\kappa$  is extremely small in the first time period at 0.0821. This seems much too small to keep the rate as close to the target as what we observe. However remember that  $\kappa$  is the speed of mean reversion only under the risk-adjusted measure. Risk premia may be included in this estimate. In the later time period the estimate is even smaller at 0.0410. The test for over-identifying restrictions soundly rejects this model in both periods with a  $J$ -statistic of 39.557 in the early period using 16 moments ( $p = 0.0005$ ) and 29.987 in the later period using 18 moments ( $p = 0.0264$ ).

**Model 2** The failure of model 1 is likely due to the fact that the response coefficients rise to unity as time to maturity gets large. This is because of the non-stationary nature of this model. The usual way of inducing stationarity in a short rate model is to have some sort of long-run mean which the short rate reverts to. In this case that would mean changing equation 2 to

$$dr_t = (\kappa_\theta(\theta - r_t) + \kappa_J(J_{t-} - r_t)) dt + \sigma dW_t.$$

Now the drift of the short rate has two components: one draws it toward its target and the other tries to draw it back to the long-run mean of the

process. Since this drift is still linear in the state variables the term structure model is still of exponential affine form. In this case we have

$$-B_2(\tau)/\tau = \kappa_J \left( \frac{1}{\gamma} + \frac{\exp(-\gamma\tau) - 1}{\tau\gamma^2} \right) \quad (5)$$

where  $\gamma = \kappa_J + \kappa_\theta$ . This response function is also monotone and zero at  $\tau = 0$  but increases to  $\kappa_J/\gamma < 1$  at the long end.

Unfortunately this model doesn't fair much better. Using the same type of GMM setup as in equation 4 we estimate in the 1974–1979 period  $\kappa_J = 307.83$  and  $\kappa_\theta = 534.24$ . These estimates look huge and they are. An examination of figure 3 and equation 5 shows why we get these estimates. The response at high maturities implied by these parameter estimates is 0.366 which is close to the average response estimated by OLS across all maturities. This means that  $\kappa_\theta$  has to be a little more than twice what  $\kappa_J$  is. Equation 5 shows that the response function is monotone increasing. But figure 3 shows that the strongest response is for the 3-month bill yield. The larger is the sum  $\kappa_J + \kappa_\theta$  the faster the response function rises before leveling off. Hence the huge estimates. In the post 1987 period we have a similar story except that  $\kappa_\theta$  has to be an even larger multiple of  $\kappa_J$  since the responses are lower than they are in the early period. In this period we estimate  $\kappa_J = 103.94$  and  $\kappa_\theta = 417.47$ . The model is rejected for the early period with a  $J$ -statistic of 35.659 with 16 moment conditions ( $p = 0.001$ ) and in the later period with a  $J$ -statistic of 31.175 with 18 moment conditions ( $p = 0.013$ ).

**Model 3** There is another way to avoid the non-stationarity problem besides adding terms to the drift function. Rather than trying to keep the short rate near a long run mean we can alter the intensities of the target process to keep the target in a certain range. For instance investors probably don't believe that the Fed will ever set the target rate to a negative number. Hence the probability of this event should be zero. Similarly investors may believe that there is an upper limit above which the Fed will not raise rates. The highest the target has ever been is 11.5% so we could model the upward jumps such that the probability of going higher than this level is zero.

This can be accomplished by letting the intensities of  $N^u$  and  $N^d$  depend on the current level of the target. For instance if we let the intensity of downward jumps be  $\lambda^d J_{t-}$  (where  $\lambda^d$  is some constant) then downward jumps get less and less likely the lower the target gets. At a target level of zero further downward jumps are impossible. Similarly the intensity of upward

jumps can be  $\lambda^u(.12 - J_{t-})$ . Fortunately this structure keeps us within the exponential affine class. However in this case we cannot derive the response function,  $-B_2(\tau)/\tau$  in closed form. However we do know that  $B_2(\tau)$  solves the following ODE

$$B_2'(\tau) = \kappa B_1(\tau) - \lambda^u \exp(B_2(\tau)b) + \lambda^d \exp(-B_2(\tau)b) \quad (6)$$

where  $b$  is the size of the target change and  $B_1(\tau)$  is the function

$$B_1(\tau) = \frac{-1}{\kappa} + \frac{\exp(-\kappa\tau)}{\kappa}.$$

The initial condition for this ODE is  $B_2(0) = 0$ . We can solve this numerically for each guess of the parameters and proceed as before. For simplicity we assume that  $\lambda^u = \lambda^d = \lambda$ . The response function for this model is not monotone. Instead the response function rises with maturity to a single peak and then declines for larger maturities. The larger is the value of  $\kappa$  the faster the function rises to this peak. The larger is the value of  $\lambda$  the steeper is the decline for larger maturities.

For the 1974 – 1979 period we estimate  $\kappa = 11.233$  and  $\lambda = 460.47$ . Unfortunately this model is rejected during this period with a  $J$ -statistic of 44.467 ( $p = 0.0025$ ). In the post 1987 period the model fares a bit better. We estimate  $\kappa = 2.020$  and  $\lambda = 500.82$ . The model is not rejected at the 5% level in this period with a  $J$ -statistic of 24.368 ( $p = 0.0818$ ).

The response functions of these three models are plotted in figure 4 for the 1974 – 1979 period and in figure 5 for the post 1987 period. Comparing these figures to figure 3 we can see the strengths and weaknesses of each model. It seems fairly obvious that model 3 is the best model although even that model is rejected by the data in the early period. To get better results than this model we need to look more closely at the data to see what we might be missing<sup>10</sup>.

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<sup>10</sup>We should mention at this point that we have not exhausted the set of two factor affine models. In particular we could have let the short rate have a square root volatility like the CIR model. This change does not change the qualitative performance of the models and hinders exposition because closed form solutions to the response function are not available to our knowledge.

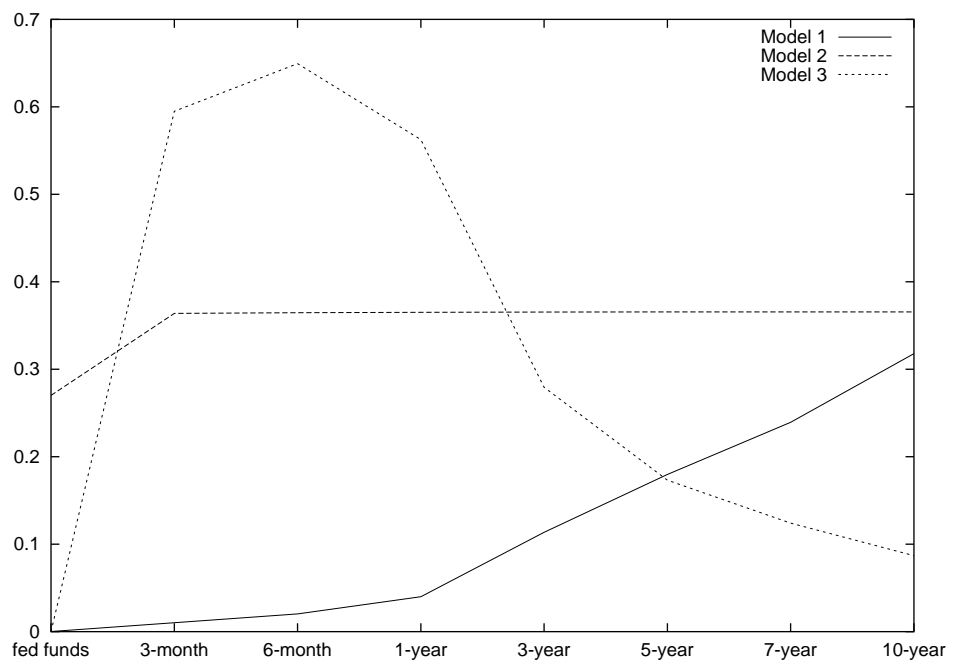


Figure 4: Response functions of Models 1, 2, and 3 evaluated at estimates from the 1974 – 1979 data.

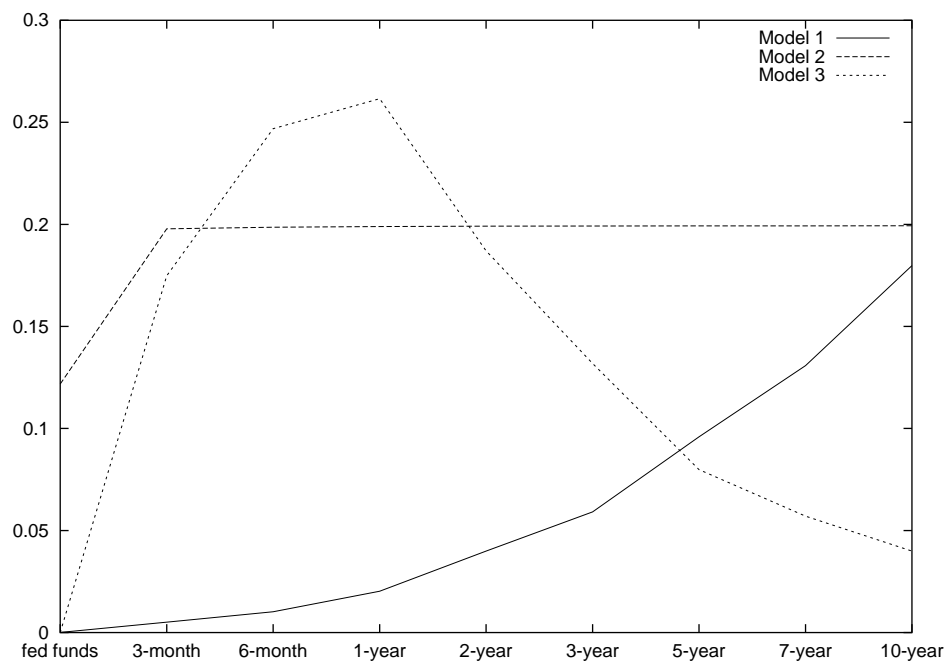


Figure 5: Response functions of Models 1, 2 and 3 evaluated at estimates from the post 1987 data.

### 3.1 Forecasting target changes

One issue we have not addressed is the predictability of target changes. Even a casual observer in financial markets knows that target changes do not come as a complete surprise to investors. In models 1 and 2 target changes are completely unpredictable. In model 3 the probability of a target change in the next instant depends only on the current level of the target and so is constant for weeks or months at a time. Comparing figures 4 and 3 it appears that the failure of model 3 is related to the fact that that model implies too large a response for short maturity assets such as the 3 and 6 month bills. A large response indicates that the market was very surprised by the change. If the change were completely anticipated there should be no response.

In predicting Fed actions a number of economic factors could be considered such as inflation reports, employment numbers, and consumer confidence. It may make sense to include these variables explicitly in a model<sup>11</sup>. We are reluctant to pursue this direction for several reasons. The first is parsimony. We want to try to explain the data with as simple a model as possible. The other reason is that explicitly adding economic variables does not really solve our problem. We would need to ask ourselves how much of the inflation announcement (for instance) was already anticipated by the market. Lastly it may not be necessary to explicitly include these factors since market expectations of them are already present in the data we have. Instead we shall ask how we can use the interest rate data we have to predict target changes.

One interesting question would be whether the funds rate is above or below its target before a target change. This would tell us if investors can use the position of the fed funds rate relative to its target to forecast target changes. To investigate this we performed the the following analysis. We regressed the sign of target changes on two dummy variables: HI and LO. If the fed funds rate was above target on the day before the target change then HI=1 and LO = 0 and the reverse if the funds rate was below target. The regression equation is

$$sgn(\Delta TG_i) = \beta_1 HI_i + \beta_2 LO_i + e_i \quad (7)$$

where  $\Delta TG_i$  is the  $i$ th target change in the sample. The results of this regression are presented in table 2. Notice that both coefficients are highly

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<sup>11</sup>Some progress has been made in this direction in Piazzesi (2000).

significant. Since target changes only occur in a few sizes the standard errors are somewhat suspect. To examine their validity we calculated bootstrap standard errors and found that they were remarkably close to the reported OLS standard errors. Notice that the signs of the coefficients indicate that high rates (relative to target) precede upward realignments and low rates precede downward realignments. Table 2 uses data from both the 1974 – 1979

Variable	Coefficient Estimate	Std Error	P value
HI	0.31373	0.09200	0.0008
LO	-0.46479	0.11027	<0.0001

Table 2: Effect of rate relative to target on target change

period as well as the post 1987 period. Examining these periods separately gives essentially the same result so we do not present those numbers.

It would be tempting to interpret this result as showing that the Fed reacts to market pressure, i.e. the Fed will move the target in the direction the fed funds rate wants to go. There are some who believe that the Fed is more of a market follower than a market leader. Unfortunately we cannot make any such conclusion on the basis of this evidence because other interpretations are possible. What does seem clear is that the position of the fed funds rate relative to its target captures some of the predictability of target changes. This might suggest that in a term structure model we would want upward target changes to follow times when the short rate is high relative to the target and the reverse for downward target changes.

Unfortunately incorporating this into a term structure model takes us outside the exponential affine class. In order to remain in this class the intensities must be affine functions of the state variables. We would want the intensities to depend on the spread between the short rate and the target which cannot be accommodated in an affine function. The reason is that the spread can be both positive and negative and intensities must be positive to be well defined.

## 4 A Random Threshold Model

Stepping outside the exponential affine class does two things. First it allows us to incorporate relationships which are economically or empirically

motivated but not affine. Secondly it increases the computational burden considerably since there are not closed form solutions available.

If we accept the proposition that the Fed may have some control over market rates, then the question which naturally arises is what is the nature of the control policy and can it be inferred from data. If we can answer this question then the next question to address is how this control policy will affect the yield curve. The first thing that we note is that the targeted interest rate is not always equal to the target although it stays close to it as shown in Figures 1 and 2. This would indicate that the costs of keeping the rate exactly equal to the target are prohibitive and that the Fed instead allows the rate to vary but not drift too far from the target. In this regard the findings of Cook and Hahn (1989) are interesting.

Cook and Hahn used reports from the *Wall Street Journal* to identify target changes. While these changes were not announced, they were quickly inferred by market participants from the Fed's actions as shown in the following section from an article quoted in Cook and Hahn (1989).

Friday's maneuver, dealers said, indicated the Fed may have lowered its target range on federal funds to the 11% to 11½ % vicinity . . . Over the past three weeks or so the Fed has used a rate of about 12% as a *trigger* to inject reserves and about an 11½ % rate as a trigger to absorb funds.

In fact most of the data Cook and Hahn were able to collect was about "ranges" of the fed funds rather than actual targets. This may be because during the time period of their study targets were not announced and market participants could only infer them imperfectly. A more recent study was done by Rudebusch (1995) who collected target data up through 1992. The primary source for target data was the weekly "Report of Open Market Operations and Money Market Conditions" from the Trading Desk of the Federal Reserve Bank of New York. In collecting this data he finds that

In addition, a target *range* of about a quarter of a percentage point in size was sometimes specified rather than a precise level. . .

indicating that the Fed itself may not care so much about a target level as about the neighborhood in which the fed funds rate will fluctuate<sup>12</sup>. This

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<sup>12</sup>Subsequent researchers have adopted the convention of using the midpoint of this range as a target.

highlights an aspect to targeting not taken into account in existing models. Namely that there is a range within which the targeted rate is allowed to fluctuate without triggering any response by the Fed. However there are trigger levels which, when reached, evoke a reaction which will push the rate back into the target band.

Admittedly the above evidence is only suggestive. However there are good theoretical reasons to believe that the control policy of the Fed would take the form of a target band. Stochastic control policies which involve intervention only when the object to be controlled has reached the edge of a band have been studied by Harrison and Taksar (1983) and others. The control problem is one of a controller who monitors the level of some variable  $Z$  which in the absence of any control would evolve as the solution to  $dX = \mu(X)dt + \sigma(X)dW$ . The controller may affect the level of the process by *pushing* up or down. Define a control policy as a pair of nonnegative processes  $L_t^u$  and  $L_t^d$  which we interpret as the cumulative amounts of pushing in each direction by the controller. Then we have

$$Z_t = X_t + L_t^d - L_t^u$$

and we say that such a process is subject to a two sided regulator. If the cost function is convex around the target level and the costs of intervention are proportional to the amount of “pushing” then the optimal policy for the controller is to push when the process reaches the edge of some (endogenously determined) band around the target level. The proper amount of control is just enough to keep the process inside this band. The controlled process will exhibit reflection at some points  $B^\ell$  and  $B^u$  which are the lower and upper edges of the band respectively. The cumulative “pushes”,  $L_t^u$  and  $L_t^d$  are the *local times* of the process at  $B^u$  and  $B^\ell$  <sup>13</sup>.

In the current context we imagine that the controller is the “desk”, the trading arm of the Fed’s agent bank, the Federal Reserve Bank of New York. The desk does not change the target but is mandated with enforcing it. The objective thus seems like that described above. There is some question as to what the costs of intervention might be. Normal transaction costs are of course a part of these. In addition the desk may want to avoid destabilizing the money supply by whipsawing the market. So it intervenes only as much and as often as is necessary.

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<sup>13</sup>For more information on reflected processes and local time we refer the reader to Bass (1997).

Such target bands have been studied in the foreign exchange target zone literature beginning with the seminal work of Krugman (1991). In that paper there were no target changes. Other work which built on the Krugman paper incorporated target changes (called *realignments* in the exchange rate literature) of various kinds. In the model of Svensson (1991) realignments may come at any time and are unrelated to the position of the rate within the target band. In contrast the model of Bertola and Caballero (1992) allows realignments to occur only when the rate is at the edge of the band. In this case the rate process would not jump (unless the size of the realignment were greater than the width of the band). To illustrate let us construct a “typical” path for the fed funds rate relative to the target. Figure 6 plots the median of federal funds rate minus target for a window around upward target revisions. Note that at the beginning of the window the rate tends to be near the target but moves further above the target until the realignment occurs. Since the target jumps up (over the current rate) we see that the realignment is followed by a period in which the rate is below the target. Figure 7 shows the same for downward realignments.

Given the evidence of table 2 we would prefer the Bertola and Caballero type realignment. However there are technical problems with the Bertola and Caballero model which make it unsuitable. In that model the rate was a diffusion process which reflected inside a band. Each time the rate hit the edge of the band a decision is made by the bank to either defend or realign. A realignment happens with probability  $p$ . Subsequent decisions are independent of past decisions. However in such a continuous time model if the rate hits once it hits infinitely many times in any  $\Delta t$  increment of time. Hence the bank will never defend the target. Below we derive a model with the behavior that Bertola and Caballero intended but without this technical problem.

We wish the target to move upward (downward) when the short rate is at the top (bottom) of the band. We also know that it is precisely at these times that the controller is intervening. Therefore a natural way to accomplish this is to say that the Fed will change the target when the cumulative intervention by the controller has reached a certain threshold. Even if market participants know that this is the behavior to expect they may not be able to observe the threshold. From the perspective of market participants the threshold is a random variable. They have probabilistic beliefs about what the threshold may be.

We write  $(B_0^\ell, B_0^u)$  for the initial  $(B^\ell, B^u)$  and let  $T_1$  be the first time

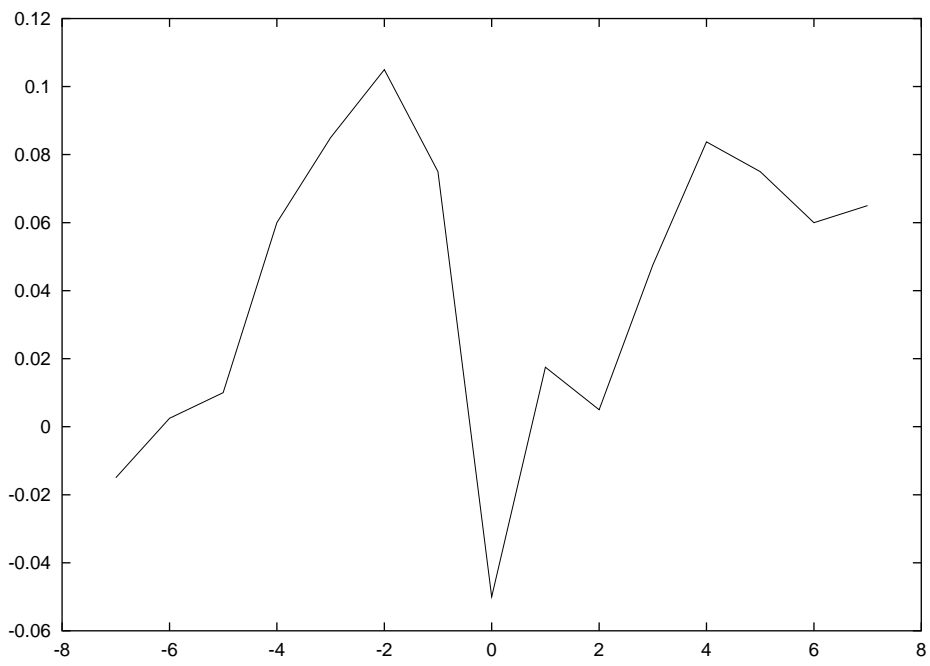


Figure 6: Fed funds rate minus target around an upward realignment

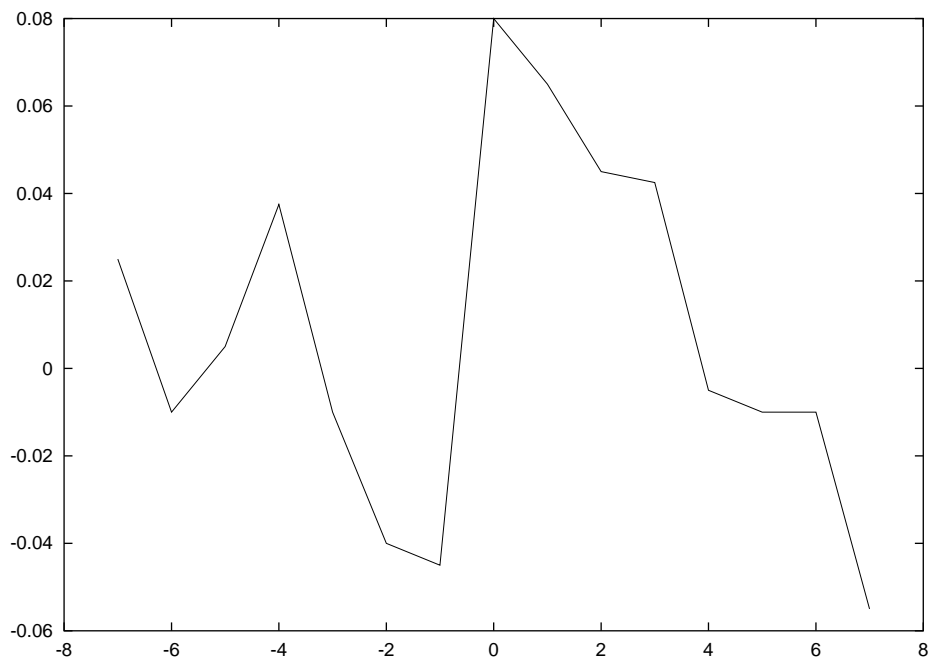


Figure 7: Fed funds rate minus target around a downward realignment

the monetary authority decides to change the target. At time  $T_1$  we shift the band  $(B_0^\ell, B_0^u)$  upwards or downwards an amount  $b > 0$  to  $(B_1^\ell, B_1^u)$ , depending whether  $r_{T_1} = B_0^\ell$  or  $B_0^u$ . The next realignment will result in setting the band to  $(B_2^\ell, B_2^u)$  and so forth.

One way of modeling the change in the bands is to write  $X_t = r_t - B_i^\ell$ ,  $T_i \leq t < T_{i+1}$ , and

$$dX_t = \mu(r_t) dt + \sigma(r_t) dW_t + dL_t^\ell - dL_t^u + dJ_t^u - dJ_t^\ell$$

where now  $J_t^u$  and  $J_t^\ell$  are both pure jump processes that both have jumps of size  $+b$ .  $J_t^u$  jumps only when  $r_t$  is at  $B^u$ ,  $J_t^\ell$  jumps only when  $r_t$  is at  $B^\ell$ . By restricting the band to have width greater than or equal to  $b$  we guarantee that the process  $r_t$  so defined is continuous. If we assume that the random threshold is distributed exponentially with parameter  $\lambda^u$  then  $J_t^u$  is a Poisson process when time is measured by  $L_t^u$ , and similarly for  $J_t^\ell$ . To be more precise,  $J_t^u$  is a pure jump process with jumps of size  $b$ , and the compensator of  $J_t^u$  is  $b\lambda^u L_t^u$ , i.e.,  $M_t^u = J_t^u - b\lambda^u L_t^u$  is a martingale, where  $\lambda^u$  is some fixed parameter. Similarly,  $M_t^\ell = J_t^\ell + b\lambda^\ell L_t^\ell$  is a martingale. Both  $\lambda^u$  and  $\lambda^\ell$  can be functions of the current band location.

Note that although the rate is a continuous process, it is not a diffusion since it is not Markovian. Instead if we let  $J_t = J_t^u - J_t^\ell$  the vector process  $(r_t, J_t)$  is Markovian as is  $(X_t, J_t)$ . When a realignment happens there is no jump in the rate  $r$  (although  $X$  jumps of course) but the stochastic behavior of the rate changes because the set of possible future paths of  $r$  changes.

The fact that the joint process  $(X_t, J_t)$  is Markovian may at first be surprising. One might expect that a knowledge of the past history of interventions (i.e.  $L_t^u$  and  $L_t^\ell$ ) might be informative as to the probability that the central bank will realign in the future. We have eliminated this by assuming that the realignment threshold is exponentially distributed, which implies a lack of memory. So when the rate is on the boundary and the bank is intervening there is a constant probability of a realignment in the next instant. This is another way to characterize the fact that the realignment process is Poisson when indexed by local time. In some ways this may be seen as a disadvantage of the model. It depends on what we believe market participants know about the workings of the central bank and how much the bank is feeling the pressure from the market. The assumption made here seems reasonable and makes the model much more tractable as shall be shown below.

This provides us with a model for the stochastic behavior of the state variable. In the appendix we establish that the market is in fact complete under such a process. Since the market is complete, the absence of arbitrage implies that there is an (unique) equivalent martingale measure under which the deflated claim price must be a martingale. The drift of the process may be different under this equivalent martingale measure. However now there is another difference. The jump intensities will also be different under the risk neutral measure although the jump sizes will of course be the same.

For the remainder of the section we take the process parameters to be those which correspond to the risk neutral measure. We shall derive the price of a claim  $C$  whose payoff depends on the future level of the short rate. If we set  $J_0 = B_0^\ell$  (the initial lower edge of the band) then we have  $r_t = X_t + J_t$ . Under the risk neutral measure we must have

$$C_t = \mathbb{E}[Y_T C_T \mid \mathcal{F}_t]$$

where  $\mathcal{F}_t$  is the information set generated by the path of the vector process  $(X_s, J_s)$  up to time  $t$  and  $Y_t = \exp(-\int_0^t r_s ds)$  is the deflator. By the Markov property we can write  $C_t = C(X_t, J_t, t, L_t^\ell, L_t^u)$ . Because of the stationary independent increments property of a Poisson process, we can in fact write  $C(X_t, J_t, t)$ . In the appendix we show that we must have the following holding in order for  $C$  to be a martingale:

$$\begin{aligned} \mathcal{A}C + \frac{\partial C}{\partial t} - r_t C &= 0 \\ \frac{\partial}{\partial x} C(0, j, t) + \lambda^\ell [C(b, j - b, t) - C(0, j, t)] &= 0 \\ -\frac{\partial}{\partial x} C(D, j, t) + \lambda^u [C(D - b, j + b, t) - C(D, j, t)] &= 0 \end{aligned}$$

where  $D = B^u - B^\ell$ , the width of the band and  $\mathcal{A}$  is the operator which, when applied to a twice differentiable function gives  $\mu(x)f'(x) + 1/2\sigma(x)^2 f''(x)$ . Now adding the terminal value of the claim we have a PDE which the price of any interest rate contingent claim must satisfy.

Note that if  $\lambda^u = \lambda^\ell = 0$  then the bands never move and the derivative of the claim price with respect to  $x$  must vanish at either edge of the band. This is related to Krugman's discussion of "smooth pasting" in his paper when targets are perfectly credible. Since targets do move in our model we refer to them as "semi-credible".

The above model is interesting for purely theoretical reasons as well. In Jones (1984) the pricing of options is examined when there are jumps. It was shown in that paper that it would take three assets to form a riskless hedge if there were jumps of only one size (only two assets are required for models with one state variable which has continuous paths). Here we still only need two assets to form a riskless hedge. A close examination reveals the reason for this. Although asset prices do have jumps we know that they can only occur when the rate is at the boundaries. Hence within the target zone we can hedge as though the process were continuous. At the edges the above boundary conditions guarantee that the same hedge ratios will also hedge against jumps. Here we have a model with three sources of risk that can be hedged with only two assets. As far as we know this is unique in the literature on arbitrage pricing.

## 5 Estimation

Because of the computational difficulty of this model we have elected to estimate only one parameter and fix the others at values that are somewhat arbitrarily chosen. Besides simplifying the estimation this also demonstrates how easily this model fits the data, even without optimizing over all parameters.

The version we shall estimate has the drift of the short rate to be  $\kappa(\theta - r_t)$  and the diffusion function to be  $\sigma$ , a constant. The intensities of the jump processes are taken to be the same as in model 3,  $\lambda^u(.12 - J_{t-})$  and  $\lambda^d J_{t-}$  where in this case  $J_{t-}$  is the lower edge of the band<sup>14</sup>. To further simplify matters we assume that  $\lambda^u = \lambda^d = \lambda$ . We set  $\kappa = 2$ ,  $\theta = .05$ ,  $\sigma = .02$  and assume that the initial band is  $[.0475, .0525]$ . To estimate the response function we calculate the yield curve with the rate at the top of the band and then calculate it again with the rate in the same place but the band shifted up by  $b$ . In the 1974 – 1979 period  $b$  is .00125 and in the post 1987 period it is .0025 (this was done when estimating model 3 as well in which  $b$  enters the response function). The response function is the difference between these yield curves divided by  $b$ .

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<sup>14</sup>An interesting thing about this formulation is that it precludes negative interest rates. The reason is that when the lower edge of the band is at zero it will never go down further. Since the rate is always constrained to be within the band it can never go below zero but exhibits reflection at zero with probability 1.

The estimation proceeded along the same lines as the early estimations. In the early period we estimate  $\lambda = 8377.8$ . This may at first seem high but remember that the Poisson process of interest is indexed by local time which moves much more slowly than calendar time so a higher intensity is required to produce any jumps in a reasonable amount of calendar time. The  $J$ -statistic for this estimation is 14.185 ( $p = .5116$ ). In the later period we estimate  $\lambda = 15333.3$  and a  $J$ -statistic of 18.336 ( $p = .3679$ ). So the model cannot be rejected in either sub-period. The estimated response functions are plotted in figure 8. Notice the striking similarity with figure 3.

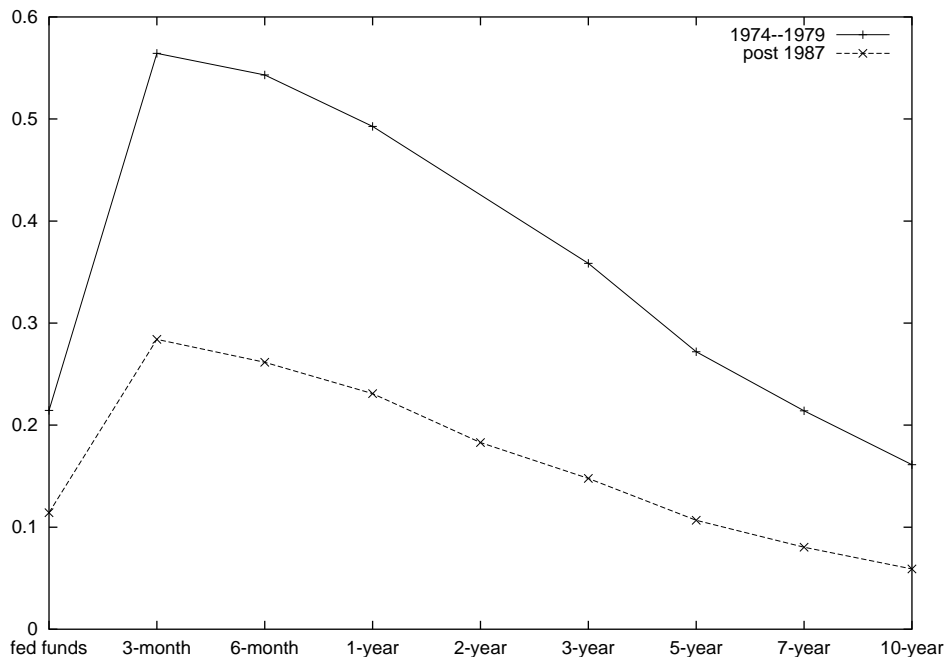


Figure 8: Estimated response functions from the random threshold model.

We cannot solve this model in closed form but we can examine its behavior graphically. The slope of the term structure behaves much the way it does in a one-factor model like Vasicek (1977) or Cox, Ingersoll, and Ross (1985). That is the term structure is upward sloping if the current rate is below its long run mean and the converse if it is above its long run mean. The difference is that here the long run mean is not determined by the parameter  $\theta$  but rather by the intensities of upward and downward jumps. Regardless

of what  $\theta$  is the short rate is confined to stay within its current band until the band moves. Even if  $\theta$  is very high the rate may not obtain that level if the probability of the target moving to that level is very low. In this case note that  $\theta$  was set to 5% but that the long run mean is really 6%. This is because we defined the intensities such that a target of 6% has the least probability of moving away. Lower targets have a higher probability of moving up and higher targets have a higher probability of moving down. Figure 9 shows the term structure using these parameter values and with the initial band set at [4.75%, 5.25%]. Notice how the term structure rises to 6% even though  $\theta$  is 5%. Notice also from figure 9 how the term structure changes as

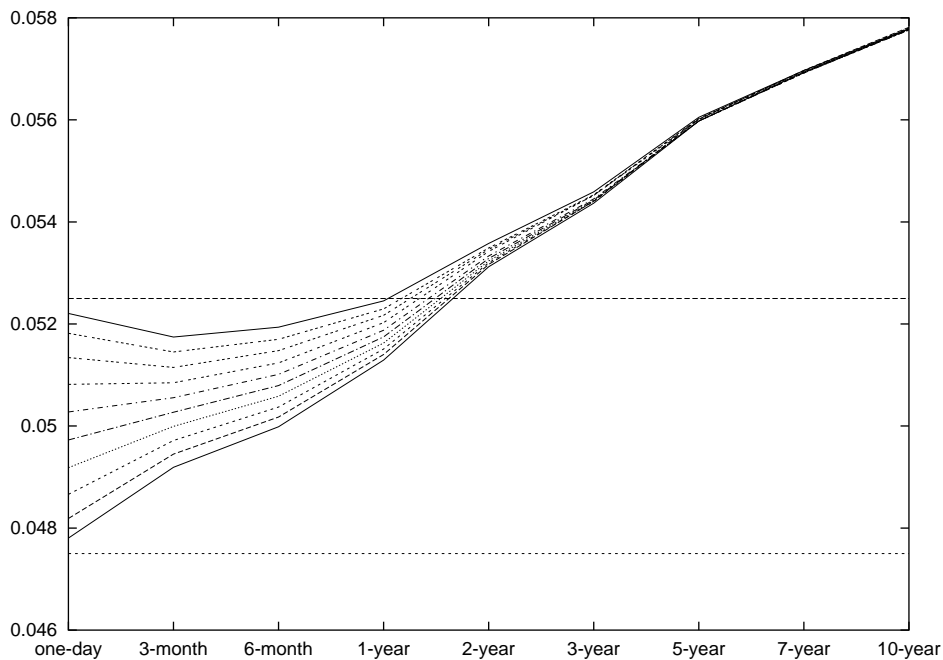


Figure 9: The term structure evaluated at  $\lambda = 12000$  and with short rates set at evenly spaced positions within the band. The band is also shown for reference.

a function of the position within the band. For a given target a change in the short rate causes a rotation of the term structure rather than a parallel shift.

It is tempting to label the target as a “curvature factor” since the response

function is a curve rather than a straight line. This is misleading. The reason is that two affects are combined to make this response function. Recall that the target only moves upward when the short rate is on the upper edge of the band and that the short rate does not move at target changes. So an upward target change has a double affect, the bands move up and the short rate changes from being on the upper edge of the band to being somewhere inside the new band. To isolate the effect of the target we solve the model for several different target levels where in each case the current rate is taken to be at the target (the midrange of the band). The results are plotted in figure 10. Notice the slight downward “bow” in all the yield curve at maturities of one or two years. The is the effect of  $\theta$  begin below 6%, the long run mean determined by the intensities of the jump processes.

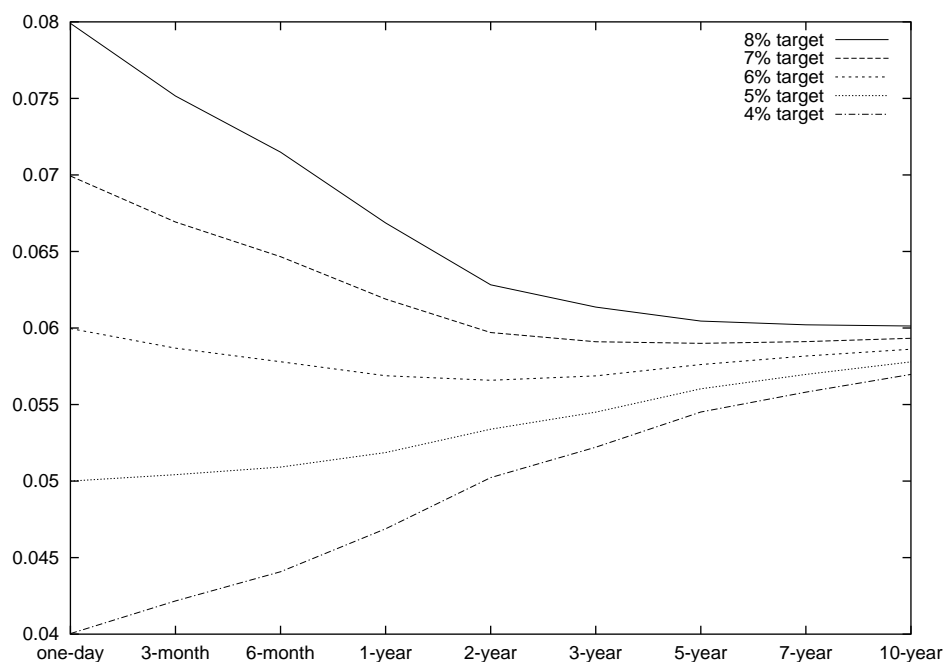


Figure 10: Yield curves for various target levels with the short rate at the target (the midpoint of the band). Parameter values are the same as in the previous figure.

## 6 Conclusion

In this paper we have presented a model which is stationary and allows target changes to depend on the level of the short rate relative to the target. We have shown how this model improves on both stationary and non-stationary exponential-affine models. In so doing we have developed new tools for modeling in the form of semi-credible bands.

One issue which we have not addressed is how to accommodate target changes that can only happen at FOMC meeting dates. In the past target changes could occur at any time but since 1994 the Fed has tried to limit target changes to FOMC meeting dates. The Fed has not always kept to this commitment and recent experience casts doubt on whether the Fed is indeed committed to this policy. For a model which assumes fixed dates for target changes see Piazzesi (2000). Whether our target band approach can be modified to accommodate this is an open question which we leave to future research.

## A

### Proof of Completeness

*Completeness* Let  $r_t$  denote the interest rate process let  $B_t^u$  and  $B_t^\ell$  denote upper and lower band edges processes. Recall that upward jumps in  $B_t^u$  and  $B_t^\ell$  are driven by a jump process  $J_t^u$  which is a Poisson process when indexed by local time on the upper band edge with jump size  $b > 0$ . More formally  $J_t^u$  is a jump process with compensator  $b\lambda^u L_t^u$ , i.e.,  $M_t^u = J_t^u - b\lambda^u L_t^u$  is a martingale, where  $\lambda^u$  is some fixed parameter. Downward jumps are driven by  $J_t^\ell$  which is defined similarly so that  $M_t^\ell = J_t^\ell + b\lambda^\ell L_t^\ell$  is a martingale. There are two steps in showing completeness. The first is to show that every martingale adapted to the filtration generated by  $\mathcal{F}_t = \sigma(r_s, J_s^u, J_s^\ell; s \leq t)$  can be written in terms of stochastic integrals with respect to  $W_t, M_t^u$ , and  $M_t^\ell$ . See Meyer (1976) for information on stochastic integrals and stochastic calculus for not necessarily continuous processes.

**Theorem** If  $Y \in L^2$  is  $\mathcal{F}_t$  measurable, then there exist  $I_s^\ell, I_s^u$ , and  $I_s^W$  predictable such that

$$Y = EY + \int_0^T I_s^\ell dM_s^\ell + \int_0^T I_s^u dM_s^u + \int_0^T I_s^W dW_s.$$

**Proof** It is well-known that a Poisson process has the martingale representation property, that is, every  $L^2$  random variable adapted to the filtration generated by a Poisson process can be represented by a stochastic integral with respect to a certain martingale. A simple time change shows that every  $L^2$  random variable  $Y$  adapted to  $\sigma(J_s^u; s \leq T)$  can be written as  $Y = EY + \int_0^T H_s dM_s^u$  for some predictable integrand  $H_s$ . In particular, this holds for  $Y^u = \exp(i \sum_{j=1}^m v_j J_{s_j}^u)$  if  $0 \leq s_1 \leq \dots \leq s_m \leq T$ . So  $Y^u = EY^u + \int_0^T H_s^u dM_s^u$ . Similarly, we have  $Y^\ell = EY^\ell + \int_0^T H_s^\ell dM_s^\ell$ , and  $Y^W = EY^W + \int_0^T H_s^W dW_s$ , where  $Y^\ell = \exp\left(i \sum_{j=1}^m w_j J_{s_j}^\ell\right)$ ,  $Y^W = \exp\left(i \sum_{j=1}^m x_j W_{s_j}\right)$ . The representation for  $Y^W$  follows because  $W$  has the martingale representation property.

The martingales  $M^u$  and  $M^\ell$  have no continuous parts and no jumps in common. So  $M^u, M^\ell$ , and  $W$  are mutually orthogonal martingales, which means  $[M^\ell, M^u]_t = 0$ ,  $[M^\ell, W]_t = 0$ , and  $[M^u, W]_t = 0$  for all  $t$ . By the product formula,

$$\begin{aligned} Y^u Y^\ell &= (EY^u)(EY^\ell) + (EY^\ell)Y^u + (EY^u)Y^\ell \\ &\quad + \int_0^T \left( \int_0^{s^-} H_s^u dM_s^u \right) H_s^\ell dM_s^\ell \\ &\quad + \int_0^T \left( \int_0^{s^-} H_s^\ell dM_s^\ell \right) H_s^u dM_s^u \\ &= E(Y^u Y^\ell) + \int_0^T K_s^\ell dM_s^\ell + \int_0^T K_s^u dM_s^u \end{aligned}$$

for some predictable  $K_s^\ell, K_s^u$ . By a similar argument,  $Y^u Y^\ell Y^W$  has the form (8). So (8) holds when  $Y = \exp(i \sum_{j=1}^m (v_j J_{s_j}^u + w_j J_{s_j}^\ell + x_j W_{s_j}))$ . Linearity shows that (8) holds when  $Y$  is a linear combination of such random variables. Since such linear combinations are dense in  $L^2$ , a limit argument establishes the theorem.

Since  $M_t^u$  increases only when  $r_t$  is at  $B^u$  and similarly for  $M_t^\ell$ , we can let  $N_t = M_t^u + M_t^\ell + W_t$  and write

$$Y = EY + \int_0^T \left[ I_s^u 1_{(r_s = B_{s^-}^u)} + I_s^\ell 1_{(r_s = B_{s^-}^\ell)} + I_s^W 1_{(r_s \in (B_{s^-}^\ell, B_{s^-}^u))} \right] dN_s. \quad (8)$$

We now show there exists a claim  $C$  such that every other claim can be written in terms of a self-financing strategy with respect to  $C$ . Let  $\{Y_n\}_{n=1}^\infty$

be a subset of  $L^\infty(\mathbb{P})$  such that every random variable in  $L^\infty(\mathbb{P})$  is the almost sure limit of a uniformly bounded subsequence of  $\{Y_n\}$ . Let  $Y_n(t)$  be the price of  $Y_n$  at time  $t$ . There exists an equivalent martingale measure  $\mathbb{Q}$  such that under  $\mathbb{Q}$  each  $Y_n(t)$  is a martingale.

If  $Y \in L^\infty(\mathbb{Q})$ , let  $Y_t = E_{\mathbb{Q}}[Y \mid \mathcal{F}_t]$ . Take a subsequence  $\{Y_{n_j}\}$  of  $\{Y_n\}$  that converges boundedly and almost surely to  $Y$ . Then by the dominated convergence theorem  $Y_{n_j}(t) \rightarrow Y_t$  almost surely. Since  $Y_{n_j}(t)$  is the price of  $Y_{n_j}$  at time  $t$ , it is not hard to see that  $Y_t$  must be the price of  $Y$  at time  $t$ , or else an arbitrage opportunity exists.

By (8) we know every random variable in  $L^2(\mathbb{P})$  can be expressed as  $Z = E_{\mathbb{P}}Z + \int_0^T H_s dN_s$  for some predictable process  $H_s$ . Since  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$ , it is known that every  $Z$  in  $L^2(\mathbb{Q})$  can be expressed as

$$Z = E_{\mathbb{Q}}Z + \int_0^T K_s dM_s \quad (9)$$

for some martingale  $M_t$  and some predictable process  $K_t$ . Choose  $\tilde{M}_t$  to be a martingale that is uniformly bounded and such that if  $\tilde{M}_t = \int_0^t I_s dM_s$ , then  $I_t$  is never 0. This can be accomplished as follows. Let  $R > b$ ,  $S_0 = 0$ , and  $S_{i+1} = \inf\{t > S_i : |M_t - M_{S_i}| \geq R\}$ . Since  $M_t$  has left limits and is right continuous,  $S_i \rightarrow \infty$ . If we let  $I_s = 2^{-i}$  on  $[S_i, S_{i+1})$ , then  $|\tilde{M}_t| \leq 4R$  for all  $t$ . Using (9) we can write any  $Z$  as

$$Z = E_{\mathbb{Q}}Z + \int_0^T \left( \frac{H_s}{I_s} \right) d\tilde{M}_s \quad (10)$$

Define the claim  $C$  by  $C = \tilde{M}_t$ . Since  $C$  is bounded, by the above the price of  $C$  at time  $t$  is  $E_{\mathbb{Q}}[C \mid \mathcal{F}_t] = \tilde{M}_t$ . The equation (10) then asserts that any claim  $Z$  can be attained by a self-financing strategy.

## B

### Derivation of the Pricing Equation

By the product formula and Ito's lemma

$$Y_T C(X_T, J_T, T) = C(X_t, J_t, t) + \int_t^T C_{s-} dY_s + \int_t^T Y_{s-} dC_s$$

$$\begin{aligned}
&= C(X_t, J_t, t) - \int_t^T Y_{s-} C_{s-} r_s ds + \int_t^T Y_{s-} (\mathcal{A}C_{s-} + \frac{\partial C_{s-}}{\partial s}) ds + MG \\
&\quad + \int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial x} dL_s^\ell - \int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial x} dL_s^u + \int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial x} dJ_s + \int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial J} dJ_s \\
&\quad + \sum_{s \leq T} Y_s (C(X_s, J_s, s) - C(X_{s-}, J_{s-}, s-) - \frac{\partial C_{s-}}{\partial x} \Delta X_s - \frac{\partial C_{s-}}{\partial J} \Delta \tilde{J}_s),
\end{aligned}$$

where  $\Delta X_s = X_s - X_{s-}$ . Now recall that all of the jumps in  $X$  come from  $J$  and so we have that

$$\int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial x} dJ_s = \sum_{s \leq T} Y_{s-} \frac{\partial C_{s-}}{\partial x} \Delta X_s$$

and likewise

$$\int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial J} dJ_s = \sum_{s \leq T} Y_{s-} \frac{\partial C_{s-}}{\partial J} \Delta J_s.$$

So these terms cancel and we obtain

$$\begin{aligned}
Y_T C(X_T, J_T, T) &= C(X_t, J_t, t) + \int_t^T C_{s-} dY_s + \int_t^T Y_{s-} dC_s \\
&= C(X_t, J_t, t) - \int_t^T Y_s C r_s ds + \int_t^T Y_s (\mathcal{A}C) ds + MG \\
&\quad + \int_t^T Y_s \frac{\partial C}{\partial x} dL_s^\ell - \int_t^T Y_s \frac{\partial C}{\partial x} dL_s^u \\
&\quad + \sum_{s \leq T} Y_{s-} (C(X_s, J_s, s) - C(X_{s-}, J_{s-}, s-))
\end{aligned}$$

where we have dropped the subscript  $s-$  from the integrals because  $dL_s$  and  $ds$  do not charge points. Now define  $\Delta C_s = C(X_s, J_s, s) - C(X_{s-}, J_{s-}, s-)$  and note that

$$\sum_{s \leq T} \Delta C_s = \sum_{s \leq T} \Delta C_s 1_{\{\Delta J_s^\ell \neq 0\}} + \sum_{s \leq T} \Delta C_s 1_{\{\Delta J_s^u \neq 0\}}$$

If  $\Delta J_s^\ell \neq 0$ , then  $X_{s-} = 0$  (since  $r_{s-} = B^\ell$  and  $X = r - B^\ell$ ), and so  $C(X_s, J_s, s) - C(X_{s-}, J_{s-}, s-) = C(b, J_{s-} - b, s) - C(0, J_{s-}, s)$ .

$$\sum_{s \leq T} \Delta C 1_{\{\Delta J_s^\ell \neq 0\}} = \frac{1}{b} \int_0^T [C(b, J_{s-} - b, s) - C(0, J_{s-}, s)] dJ_s^\ell$$

For jumps at  $B^u$  we have a similar expression.

In order for  $C$  to be a martingale we must have

$$\begin{aligned} \mathcal{A}C + \frac{\partial C}{\partial t} - r_t C &= 0 \\ \frac{\partial}{\partial x} C(0, j, t) + \lambda^\ell [C(b, j - b, t) - C(0, j, t)] &= 0 \\ -\frac{\partial}{\partial x} C(D, j, t) + \lambda^u [C(D - b, j + b, t) - C(D, j, t)] &= 0 \end{aligned}$$

where  $D = B^u - B^\ell$ , the width of the band. Now adding the terminal value of the claim we have a PDE which the price of any interest rate contingent claim must satisfy.

## C

### Monte Carlo Method for Pricing Bonds

The price of the a bond which pays \$1 at time  $T$  and makes no other payments is given by

$$P_t = \mathbb{E} \left[ \exp \left( - \int_t^T r_s ds \right) \middle| r_t, J_t \right]$$

where the expectation is with respect to the risk neutral measure. Consider  $N$  dates equally spaced between  $t$  and  $T$ . Let  $\delta$  be the time between dates. For each date we simulate an interest rate realization and approximate the integral inside the expectation as

$$\int_t^T r_s ds \approx \sum_{i=1}^N r_{t_i} \delta.$$

Clearly as  $\delta \rightarrow 0$  this approximation becomes better and better. We simulate  $M$  such paths and the monte carlo estimate for the bond price is the mean of the exponential of minus this summation.

To simulate the interest rate process begin with an initial value for  $r_t$  and initial band edges  $B_t^u$  and  $B_t^\ell$ . Initialize  $L_t^u$  and  $L_t^\ell$  to zero. Generate two independent exponential random variables,  $e^u$  and  $e^\ell$  with parameters  $\lambda^u$  and  $\lambda^\ell$  respectively. Then use the following scheme to simulate an interest rate path. Begin with  $i = 1$

1. Generate  $r_{t_i}$  from  $r_{t_{i-1}}$  using an appropriate time discrete approximation.
2. Set  $B_{t_i}^u, B_{t_i}^\ell, L_{t_i}^u$  and  $L_{t_i}^\ell$  be equal to their values at the last iteration.
3. If  $r_{t_i} > B_{t_i}^u - L_{t_i}^\ell$  and  $L_{t_i}^\ell - B_{t_i}^u + r_{t_i} > L_{t_i}^u$  then
  - (a) set  $L_{t_i}^u = r_{t_i} + L_{t_i}^\ell - B_{t_i}^u$
  - (b) If  $L_{t_i}^u > e^u$  then
    - i.  $B_{t_i}^u = B_{t_i}^u + b$
    - ii.  $B_{t_i}^u = B_{t_i}^u + b$
    - iii.  $e^u = e^u +$  another exponential with parameter  $\lambda^u$ .
4. If  $r_{t_i} > B_{t_i}^\ell - L_{t_i}^u$  and  $L_{t_i}^u + B_{t_i}^\ell - r_{t_i} > L_{t_i}^\ell$  then
  - (a) set  $L_{t_i}^\ell = r_{t_i} + L_{t_i}^u - B_{t_i}^\ell$
  - (b) If  $L_{t_i}^\ell > e^\ell$  then
    - i.  $B_{t_i}^\ell = B_{t_i}^\ell - b$
    - ii.  $B_{t_i}^\ell = B_{t_i}^\ell - b$
    - iii.  $e^\ell = e^\ell +$  another exponential with parameter  $\lambda^\ell$ .
5. Let  $i = i + 1$  and go back to 1 unless  $i = N$  in which case exit.

This generates an uncontrolled process  $r$  and the controls  $L^u$  and  $L^\ell$ . To get the controlled process define  $r^c = r + L^\ell - L^u$ . To improve efficiency the method of antithetic variables is used which means that if 1000 paths are to be simulated we first generate a matrix  $Z$  of standard normal deviates which has 500 rows and a matrix  $U$  of uniform deviates. The first 500 paths are based on  $Z$  and  $U$ . The next 500 are generated using  $-Z$  and  $1 - U$ .

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