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A New Variance Bound on the Stochastic Discount Factor*

In this paper, we construct a new variance bound on any stochastic discount factor (SDF) of the form $m = m(x)$, with x being a vector of state variables, which tightens the well-known Hansen-Jagannathan bound by a ratio of one over the multiple correlation coefficient between x and the standard minimum variance SDF, m_0. In many applications, the correlation is small, and hence the bound is much improved. For example, when x is the growth rate of consumption, the new variance bound can be 25 times greater than the Hansen-Jagannathan bound, making it much more difficult to explain the equity-premium puzzle.

I. Introduction

Hansen and Jagannathan (1991) provide a lower bound on the variance of a stochastic discount factor (SDF). As many asset pricing models can be represented by using an SDF (see, e.g., Cochrane [2001] and references therein), this bound became instantly known as the Hansen-Jagannathan bound and has been applied widely in a variety of finance problems. On developing related bounds, Snow (1991) derives a bound in terms of higher moments, Stutzer (1995) obtains a bound using Bayesian information criterion, Bansal and Lehmann (1997) investigate a growth form of the bound, Balduzzi and Kallal (1997) relate the bound to risk premia, and Chrétien (2003) derives a bound on the autocorrelation of SDFs. Moreover, Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000) derive similar bounds in incomplete

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markets. The role of conditional information was first explored by Hansen and Richard (1987) and further investigated by Gallant, Hansen, and Tauchen (1990). Recently, Ferson and Siegel (2003) and Bekaert and Liu (2004) have shown how conditional information might be used to optimally tighten the original Hansen-Jagannathan bound. Rosenberg and Engle (2002) and references therein provide empirical estimates for the related SDF. However, none of these studies have analyzed the role of state variables in the determination of the bound, although most SDFs are functions of some observable state variables.

This paper studies the role of state variables in the determination of the Hansen-Jagannathan bound. We show that the Hansen-Jagannathan bound can be improved by a factor of $1/\rho^2_{x,m_0}$, where ρ_{x,m_0} is the multiple correlation coefficient between the state variables and the standard minimum variance SDF m_0. In many applications, the correlations between the state variables and the returns are small, and hence our bound is substantially tighter than Hansen-Jagannathan's. For example, when x is the gross growth rate of consumption, the correlation is usually less than 30% and our bound is more than 10 times larger. Notice that our bound, like the original Hansen-Jagannathan one, is still an unconditional bound and hence is easily estimated in practice. In contrast, estimation of the conditional bounds of Ferson and Siegel (2003) and Bekaert and Liu (2004) is more difficult, and these bounds often offer very small improvements over the original Hansen-Jagannathan one.

We also apply the new bound to examine consumption-based asset pricing models. In general, it offers a much sharper bound on the variance of the marginal rate of substitution. As a result, it makes the *equity premium* and *correlation* puzzles more difficult to explain.

The rest of the paper is organized as follows. The bound is presented in Section II, applications of the bound to consumption-based asset pricing models are provided in Section III, and Section IV presents conclusions.

II. An Improved Bound on the Stochastic Discount Factor

Under the law of one price, it is well known (see, e.g., Cochrane 2001) that there exists a random variable m_{t+1}, called the stochastic discount factor, the state price density, or the pricing kernel, such that

$$E[R_{t+1}m_{t+1}|I_t] = 1_N, \tag{1}$$

where 1_N is an N-vector of ones, R_{t+1} is the gross returns on N assets at time $t+1$, and I_t is the information available at time t.

As conditional moments are very difficult to estimate in practice, one is often interested in the unconditional form of (1). If the the time subscript is suppressed, the unconditional pricing equation is given by

$$E[Rm] = 1_N. \tag{2}$$

While (2) is the restriction on the SDF of an asset pricing model, it is well known that the return on a particular portfolio can also serve as an SDF:

$$m_0 = \mu_m + (1_N - \mu_m \mu)' \Sigma^{-1} (R - \mu), \tag{3}$$

where $\mu_m = E[m]$ is the mean of m that can be set as an arbitrary value, and μ and Σ are the mean and the covariance matrix of the asset returns. We assume that μ is not proportional to 1_N in order to avoid the trivial case. The N assets are risky and assumed to be nonredundant here so that Σ is nonsingular. For easier reference, we call m_0 the default SDF since it always prices the N assets correctly (satisfying eq. [1] regardless of the validity of any asset pricing model). If there is a risk-free asset with constant gross return R_f, equation (2) implies that $\mu_m = 1/R_f$. This puts a restriction on the mean of all SDFs. However, in the presence of a risk-free asset, it is easy to see that the default SDF is still defined in the same way as above in terms of the risky assets, except for requiring further $\mu_m = 1/R_f$.

Besides m_0, there is a countless number of SDFs that satisfy (2). The celebrated Hansen-Jagannathan bound places a lower bound on the variance of *all* such SDFs, with mean $E[m] = \mu_m$:

$$\mathrm{Var}[m] \geq \mathrm{Var}[m_0] = (1_N - \mu_m \mu)' \Sigma^{-1} (1_N - \mu_m \mu), \tag{4}$$

where m_0 is as defined in (3). As m_0 is an SDF and it attains the minimum, the Hansen-Jagannathan bound is optimal in the sense that one cannot find a better lower bound for all the SDFs.

How can one improve on the Hansen-Jagannathan bound? The idea is to put a certain structure on the SDFs. A good structure will restrict the class of SDFs and yet remain general enough to include many interesting SDFs. The structure we impose is

$$m = m(x), \tag{5}$$

where $x = (x_1, \ldots, x_K)'$ is a vector of K state variables. The SDFs of many well-known theoretical asset pricing models have such a form. For example, factor models, such as the capital asset pricing model (CAPM) and Fama and French's (1993) three-factor model, all specify m as a linear function of factors. In nonlinear models, Bansal and Viswanathan (1993) specify m as a nonlinear function of the equity market return, the Treasury bill yield, and the term spread (the x here), and Dittmar (2002) specifies $m(x)$ as a cubic function of aggregate wealth. If one takes a stand that the state variable x is unobservable or unknown, a projection of the pricing kernel on known variables may be done to yield a new kernel in terms of observables. For instance, Aït-Sahalia and Lo (2000) project the pricing kernel onto equity returns, avoiding the use of aggregate consumption data, and Rosenberg and Engle (2002) expand further on both the projection and the associated estimation methodology.

The question we ask is whether there exists such a constant $c = c(x,$

m_0), which depends only on x and m_0 (and hence is estimable in empirical studies), but is *independent of* the particular functional form of m and satisfies

$$\mathrm{Var}[m(x)] \geq c(x, m_0) \times \mathrm{Var}[m_0], \tag{6}$$

where $c = c(x, m_0) \geq 1$. If so, this clearly offers an improvement over the Hansen-Jagannathan bound.

As it turns out, we can find such a constant $c = c(x, m_0) \geq 1$ as follows. Consider the linear regression of m_0 on x:

$$m_0 = \alpha + \beta' x + \epsilon_0. \tag{7}$$

It is well known by construction that $E[\epsilon_0] = 0$ and $\mathrm{Cov}[\epsilon_0, x] = 0$. To obtain the new bound, we impose a slightly stronger assumption of $E[\epsilon_0|x] = 0$. Under this regression condition, we present the key result of this paper in our first proposition.

Proposition 1. Suppose that a stochastic discount factor $m = m(x)$ is a function of K state variables x and we have $E[\epsilon_0|x] = 0$ in the regression of $m_0 = \alpha + \beta' x + \epsilon_0$, where $m_0 = \mu_m + (1_N - \mu_m \mu)' \Sigma^{-1}(R - \mu)$ is a linear combination of asset returns. Then, for all $m(x)$ with $E[m(x)] = \mu_m$, we have

$$\mathrm{Var}[m(x)] \geq \frac{1}{\rho^2_{x,m_0}} \mathrm{Var}[m_0], \tag{8}$$

where ρ_{x,m_0} is the multiple correlation coefficient between x and m_0, and the equality holds if and only if $m(x) = \alpha + \beta' x$.

Proof. First, it is important to note that the SDF places a strong restriction on the covariance between m and m_0 so that

$$\mathrm{Cov}[m, m_0] = \mathrm{Var}[m_0]. \tag{9}$$

This follows (see, e.g., Ferson and Siegel 2003) through simple algebra:

$$\begin{aligned} E[mm_0] &= \mu_m E[m] + (1_N - \mu_m \mu)' \Sigma^{-1} E[m(R - \mu)] \\ &= \mu_m^2 + (1_N - \mu_m \mu)' \Sigma^{-1} (1_N - \mu_m \mu) \\ &= E[m_0^2] \end{aligned} \tag{10}$$

and the fact that both m and m_0 have the same mean μ_m. Under the assumption that $E[\epsilon_0|x] = 0$, we have

$$\mathrm{Cov}[\epsilon_0, m(x)] = E[\epsilon_0 m(x)] = E[E[\epsilon_0|x] m(x)] = 0 \tag{11}$$

and hence

$$\begin{aligned} \mathrm{Var}[m_0] &= \mathrm{Cov}[m_0, m(x)] = \mathrm{Cov}[\beta' x, m(x)] = \beta' \Sigma_{xm} \\ &= \beta' \Sigma_{xx}^{1/2} \Sigma_{xx}^{-1/2} \Sigma_{xm}, \end{aligned} \tag{12}$$

where $\Sigma_{xm} = \mathrm{Cov}[x, m(x)]$ and $\Sigma_{xx} = \mathrm{Var}[x]$. Applying the Cauchy-Schwarz inequality to the vectors $\Sigma_{xx}^{1/2}\beta$ and $\Sigma_{xx}^{-1/2}\Sigma_{xm}$, we have

$$\mathrm{Var}[m_0]^2 = (\beta'\Sigma_{xx}^{1/2}\Sigma_{xx}^{-1/2}\Sigma_{xm})^2 \leq (\beta'\Sigma_{xx}\beta)(\Sigma_{xm}'\Sigma_{xx}^{-1}\Sigma_{xm}). \tag{13}$$

Now, from the regression of $m(x)$ on x, we have

$$\mathrm{Var}[m(x)] \geq \Sigma_{xm}'\Sigma_{xx}^{-1}\Sigma_{xm}. \tag{14}$$

A combination of (13) and (14) and using the expression

$$\rho_{x,m_0}^2 = \frac{\beta'\Sigma_{xx}\beta}{\mathrm{Var}[m_0]} \tag{15}$$

yields the desired inequality on $\mathrm{Var}[m(x)]$. For (8) to be an equality, we need both (13) and (14) to be equalities. Expression (13) is an equality if and only if $\Sigma_{xx}^{-1}\Sigma_{xm}$ is proportional to β. Expression (14) is an equality if and only if $m(x)$ is a linear function of x. Together with the fact that m and m_0 have the same mean, these two conditions are satisfied if and only if $m(x) = \alpha + \beta'x$. QED

Before we analyze the implications of proposition 1, it is useful to discuss its assumptions. First, in the spirit of the original Hansen-Jagannathan bound, m here is an *arbitrary* function of state variables. Similarly to the popular Hansen-Jagannathan bound of (4), which is derived under the law of one price, we do not restrict m to be strictly positive, although our bound also works for positive m. It should be noted that Hansen and Jagannathan (1991) also provide a tighter bound on $\mathrm{Var}[m]$ by imposing an additional assumption of no arbitrage that $m > 0$. Although the tighter bound is not analytically available, Hansen and Jagannathan (1991) find that it is close to the standard Hansen-Jagannathan bound in their applications. Therefore, to the extent that our new bound can substantially improve on the standard Hansen-Jagannathan bound, it will also be tighter than the Hansen-Jagannathan no-arbitrage bound.

Second, in comparison with the assumptions underlying the Hansen-Jagannathan bound, the only additional one that we impose is the regression assumption that $E[\epsilon_0|x] = 0$. A sufficient condition for $E[\epsilon_0|x] = 0$ to hold is when the returns and the state variables are jointly elliptically distributed (see, e.g., Muirhead 1982, 36). So, we have the following corollary.

Corollary 1. Suppose that a stochastic discount factor $m = m(x)$ is a function of K state variables x, and x and the asset returns are jointly elliptically distributed. Then, for all $m(x)$ with $E[m(x)] = \mu_m$, we have

$$\mathrm{Var}[m(x)] \geq \frac{1}{\rho_{x,m_0}^2}\mathrm{Var}[m_0], \tag{16}$$

where ρ_{x,m_0} is the multiple correlation coefficient between x and m_0.

The usual multivariate normality assumption is a special case of the elliptical assumption. The normality assumption is common in both theory and empirical studies. For example, many asset pricing tests assume that stock returns and factors are jointly normal. Theoretically, diffusion models imply locally lognormal distributions that are well approximated by normal ones. Hence, the corollary covers many cases of practical relevance. However, the elliptical assumption is far more general than the normality assumption. It contains multivariate t, Kotz, mixture normal, and many other useful distributions that may provide for a better description of the return data. When one is interested in the consumption CAPM or in SDFs that are based on the Fama and French (1993) factors, the multivariate elliptical distribution seems to be a good first-order approximation of the data. For example, Zhou (1993) shows that the multivariate t-distribution is a good model for the size and industry portfolios, and Kan and Zhou (2003b) and Tu and Zhou (2004) demonstrate that it also models the Fama and French portfolios and factors well. It should be emphasized that even though corollary 1 makes the multivariate elliptical distribution assumption on x, it does not imply that m has an elliptical distribution. In fact, m can be an arbitrary function of x, and there is no distributional assumption imposed on m. In particular, m can be strictly positive for all values of x.

Finally, if the state variables x are not elliptically distributed, but if a suitable transformation of $y = g(x)$ and the asset returns are jointly elliptically distributed, proposition 1 still applies to $m(y) = m(g(x))$ to yield an improved bound by replacing the earlier multiple correlation of x with m_0 with the multiple correlation of y with m_0. A related point is that the projection of m_0 on x is not necessarily linear as long as the residual has expectation zero conditional on x.[1] Theoretically, the condition $E[\epsilon_0|x] = 0$ might not be satisfied in the linear regression of m_0 on x, $m_0 = \alpha + \beta' x + \epsilon_0$, but might be so in a certain nonlinear regression $m_0 = f(x) + \epsilon_0$. Then, following the proof of proposition 1, we have the following corollary.

Corollary 2. Suppose that a stochastic discount factor $m = m(x)$ is a function of K state variables x and we have $E[\epsilon_0|x] = 0$ in the nonlinear regression of $m_0 = f(x) + \epsilon_0$. Then, for all $m(x)$ with $E[m(x)] = \mu_m$, we have

$$\mathrm{Var}\,[m(x)] \geq \frac{1}{\rho^2_{f(x),m_0}} \mathrm{Var}\,[m_0], \tag{17}$$

where $\rho_{f(x),m_0}$ is the multiple correlation coefficient between $f(x)$ and m_0.

Proposition 1 looks amazingly simple. Like the Hansen-Jagannathan bound, it places an (often much stricter) restriction on the variance of the SDF with the minimum knowledge of the functional form of the SDF. Because the bound is formed with moments of only observables, it has the same appealing features of the Hansen-Jagannathan bound. In particular, it can often shed light on why a particular class of asset pricing models fails to explain asset returns and indicate what steps may be taken to improve them. As $\rho^2_{x,m_0} \leq 1$, the bound must be no worse than the Hansen-Jagannathan bound. In fact,

1. We thank an anonymous referee for this interesting point.

ρ^2_{x,m_0} is often small in practice, so the bound can be much sharper than the Hansen-Jagannathan bound.[2] However, it is important to note that our improved bound comes at a cost. Unlike the Hansen-Jagannathan bound, which works for all SDFs, our bound is not universal and works only for a class of asset pricing model that is in the form of $m = m(x)$. Therefore, for a different choice of state variables, we need a different bound. Nevertheless, the fact that our bound is specialized to a given class of asset pricing models does not prevent us from using it as a tool for model diagnostic.

In almost every application in the literature in which one uses the Hansen-Jagannathan bound, one needs to specify x and check whether an SDF $m(x)$ violates the Hansen-Jagannathan bound. Our point is that if one is willing to specify x to check the Hansen-Jagannathan bound, one can be better off by comparing the variance of $m(x)$ with our tighter new bound instead of the Hansen-Jagannathan bound. Although the use of our new bound requires additional computational cost since we cannot use the same bound on all SDFs, the advantage is that we are able to detect some invalid SDFs that pass the test of the Hansen-Jagannathan bound.

When a proposed m fails our new bound, the interpretation is the same as when it fails the Hansen-Jagannathan bound. We can conclude that either the choice of the set of state variables or the functional form is wrong. Our bound, however, allows us to focus on the question of what functional form is needed to make the SDF feasible given a choice of the state variables. For example, if one believes that the SDF is a polynomial of the market return, one can use the new bound to find out what order of the polynomial is necessary for the SDF to be acceptable. One may suggest that given the choice of x, it may be possible to use a nonparametric technique to come up with an estimate of the functional form $m(x)$ and directly test the moment condition of $E[m(x)R] = 1_N$ instead of using our bound. The problem is that it is unclear how a nonparametric method can be used to estimate the functional form $m(x)$. Furthermore, even if a nonparametric estimate of the SDF is available, it is a difficult task to establish the distribution theory for the specification test. As a result, in the spirit of the original Hansen-Jagannathan bound, the use of our new bound provides a simple and fast specification test for detecting invalid SDFs.

By specifying a parametric functional form and the state variables for an SDF, the traditional specification test allows us to directly test the validity of the SDF using return data. This approach imposes stringent limits on the class of asset pricing models, but it can result in a very sharp prediction on the validity of the SDF when we have sufficient data. On another extreme, the Hansen-Jagannathan bound imposes almost no structure on the SDF other than the law of one price. The result is that it can deliver a variance bound

2. Shanken (1987) derives a bound similar to the Hansen-Jagannathan bound with the use of a multiple correlation coefficient. However, in that case, the correlation coefficient is between m and a proxy. In contrast, our correlation coefficient here is between x and m_0. Hence, our bound differs from Shanken's bound.

that is applicable for all SDFs. The price to pay for this generality is that the bound may not be very tight and informative. Our approach stands between these two extremes. We limit the class of SDF to a function of a set of state variables x, but yet we do not need to specify its parametric functional form. The result is that we can deliver a tighter bound than the Hansen-Jagannathan bound. There is always a trade-off between the broadness of the class of asset pricing models and the tightness of the bound. We consider all three approaches to have their respective merits, and one is definitely not superior to the other. Which approach is more appropriate depends on the context of the problem. For example, if a proposed m fails the Hansen-Jagannathan bound, there is no need to use our new bound. However, if the proposed m passes the Hansen-Jagannathan bound, one may like to compare the variance of the proposed m with our new bound to gather more information about its validity.

In comparison with the Hansen-Jagannathan bound, our proposed new bound has an additional advantage of being robust to measurement errors in the state variables. Intuitively, if the true state variables are measured with errors, this will increase the variance of the SDFs that are based on the noisy proxy of the state variables. In fact, the larger the measurement errors, the larger the variance of the proposed SDF and, hence, the easier for the proposed SDF to pass the Hansen-Jagannathan bound that is completely independent of the state variables and their measurement errors. This observation suggests that, with other things held constant, a wrong SDF that is based on a noisy state variable stands a better chance of satisfying the Hansen-Jagannathan bound. In contrast, our new bound does not reward noisy state variables because if a state variable x is measured with a lot of noise, the resulting ρ^2_{x,m_0} is small and our new bound for such an SDF will be tighter. As a result, it is not any easier for a wrong SDF to pass our new bound by simply introducing a noisy state variable.

Finally, it is important to note that while the Hansen-Jagannathan bound is a quadratic function of μ_m, this is not the case for our new bound. This is so because m_0 is a function of μ_m, so ρ^2_{x,m_0} is also a function of μ_m. In the following corollary, we give an explicit expression of our new bound as a function of μ_m.

Corollary 3. For a stochastic discount factor of the form $m = m(x)$ with mean μ_m, we have

$$\mathrm{Var}\,[m(x)] \geq \frac{(a\mu_m^2 - 2b\mu_m + c)^2}{a_1\mu_m^2 - 2b_1\mu_m + c_1}, \tag{18}$$

where

$$a = \mu'\Sigma^{-1}\mu, \tag{19}$$

$$b = \mu'\Sigma^{-1}1_N, \tag{20}$$

$$c = 1_N' \Sigma^{-1} 1_N, \tag{21}$$

$$a_1 = \mu' \Sigma^{-1} \Sigma_{xR}' \Sigma_{xx}^{-1} \Sigma_{xR} \Sigma^{-1} \mu, \tag{22}$$

$$b_1 = \mu' \Sigma^{-1} \Sigma_{xR}' \Sigma_{xx}^{-1} \Sigma_{xR} \Sigma^{-1} 1_N, \tag{23}$$

$$c_1 = 1_N' \Sigma^{-1} \Sigma_{xR}' \Sigma_{xx}^{-1} \Sigma_{xR} \Sigma^{-1} 1_N, \tag{24}$$

and $\Sigma_{xR} = \mathrm{Cov}[x, R']$.

Proof. From (3), we have

$$\mathrm{Var}[m_0] = (1_N - \mu_m \mu)' \Sigma^{-1} (1_N - \mu_m \mu) = a\mu_m^2 - 2b\mu_m + c \tag{25}$$

and

$$\mathrm{Cov}[x, m_0] = \mathrm{Cov}[x, (R - \mu)'] \Sigma^{-1} (1_N - \mu_m \mu) = \Sigma_{xR} \Sigma^{-1} (1_N - \mu_m \mu). \tag{26}$$

Then using (15), we have

$$\begin{aligned} \rho_{x,m_0}^2 &= \frac{\beta' \Sigma_{xx} \beta}{\mathrm{Var}[m_0]} \\ &= \frac{\mathrm{Cov}[x,\ m_0]' \Sigma_{xx}^{-1} \mathrm{Cov}[x,\ m_0]}{a\mu_m^2 - 2b\mu_m + c} \\ &= \frac{(1_N - \mu_m \mu)' \Sigma^{-1} \Sigma_{xR}' \Sigma_{xx}^{-1} \Sigma_{xR} \Sigma^{-1} (1_N - \mu_m \mu)}{a\mu_m^2 - 2b\mu_m + c} \\ &= \frac{a_1 \mu_m^2 - 2b_1 \mu_m + c_1}{a\mu_m^2 - 2b\mu_m + c}. \end{aligned} \tag{27}$$

Dividing (25) by (27), we prove the corollary. QED

The corollary shows that the lower bound of the variance of an SDF with the form of $m(x)$ is actually a fourth-order polynomial of μ_m over a second-order polynomial of μ_m. There are two cases in which we can rule out the validity of $m(x)$ as an SDF. In the first case, Σ_{xR} is a zero matrix. In this case, we have $\rho_{x,m_0}^2 = 0$ for any value of μ_m and the lower bound on $\mathrm{Var}[m(x)]$ is infinity, so there is no feasible SDF of the form $m(x)$ that can price all the N assets correctly. This suggests that for x to be valid state variables in an SDF, it cannot be uncorrelated with returns on all the assets.

In the second case, $K = 1$ and $\mu_m = b_1/a_1$. When $K = 1$, we have $a_1 c_1 = b_1^2$, and as a result $\rho_{x,m_0}^2 = 0$ for $\mu_m = b_1/a_1$, which implies that the lower bound on $\mathrm{Var}[m(x)]$ is infinity. Therefore, there is no $m(x)$ with mean b_1/a_1 that can price all the N assets correctly. Note that the minimum-variance

SDF m_0 is a function of μ_m. When μ is not proportional to 1_N, m_0 for different values of μ_m are not perfectly correlated, so there is no single state variable that can be perfectly correlated with m_0 for every choice of μ_m. Therefore, for a given state variable x, there will always be one choice of μ_m such that m_0 is uncorrelated with x. This suggests that when the SDF is a function of only one state variable, there will always be a value of μ_m such that $m(x)$ is an infeasible SDF, regardless of the functional form of $m(x)$.[3]

The $K = 1$ case of proposition 1 is of particular interest. In this case, ρ_{x,m_0} is the simple correlation coefficient between two univariate random variables x and m_0. If $\rho_{x,m_0} = \pm 1$, the above bound reduces to the Hansen-Jagannathan bound. Moreover, if $x = m = m_0$, both our new bound and the Hansen-Jagannathan one are identical. However, our new bound can in general be much tighter than the Hansen-Jagannathan bound. Consider two examples. The first is the extreme case in which x is uncorrelated with m_0. Our new bound says that it is impossible to find such an SDF or its variance must be infinity if found. The Hansen-Jagannathan bound, however, still states that $\mathrm{Var}[m_0]$ is the lower bound with no use of the zero correlation information. Therefore, one may not be able to detect that $m(x)$ is in fact an invalid SDF using the Hansen-Jagannathan bound alone.

In the second example, $m = m(x)$, where x is the growth rate of consumption. If x has a correlation of 30% with m_0, then the new bound is more than 10 times higher than the Hansen-Jagannathan bound! The 30% correlation is in fact an optimistic assumption. Ferson and Harvey (1992) report sample correlations of various consumption growth measures and the stock returns, and they find that none of them exceeds 30%. Further applications of proposition 1 to consumption-based asset pricing models are detailed in Section III.

Some numerical illustrations may be illuminating. Consider the well-known 25 size and book-to-market sorted portfolios used by Fama and French (1993).[4] In figure 1, we plot the standard Hansen-Jagannathan bound for any m that prices the 25 assets correctly using a solid line. The bound is estimated as

$$\hat{\sigma}_0^2(\mu_m) = \left(1 - \frac{N+2}{T}\right)(1_N - \mu_m\hat{\mu})'\hat{\Sigma}^{-1}(1_N - \mu_m\hat{\mu}) - \frac{N}{T}\mu_m^2, \qquad (28)$$

where $\hat{\mu}$ and $\hat{\Sigma}$ are the sample mean and variance of the returns on the 25 portfolios, estimated using monthly data over the period 1952/1–2002/12. Under normality assumption, Ferson and Siegel (2003, proposition 4) show that $\hat{\sigma}_0^2(\mu_m)$ is an unbiased estimator of $\mathrm{Var}[m_0]$, and it is superior to the unadjusted bound, especially when N is large relative to T.

Now, suppose that we propose a class of asset pricing models in which m is a (possibly nonlinear) function of the excess return on the market portfolio,

3. When $K > 1$ and μ is not proportional to 1_N, we have $a_1c_1 > b_1^2$ in general, so the denominator of (18) will not be equal to zero for any choice of μ_m.

4. We are grateful to Ken French for making these data available on his Web site.

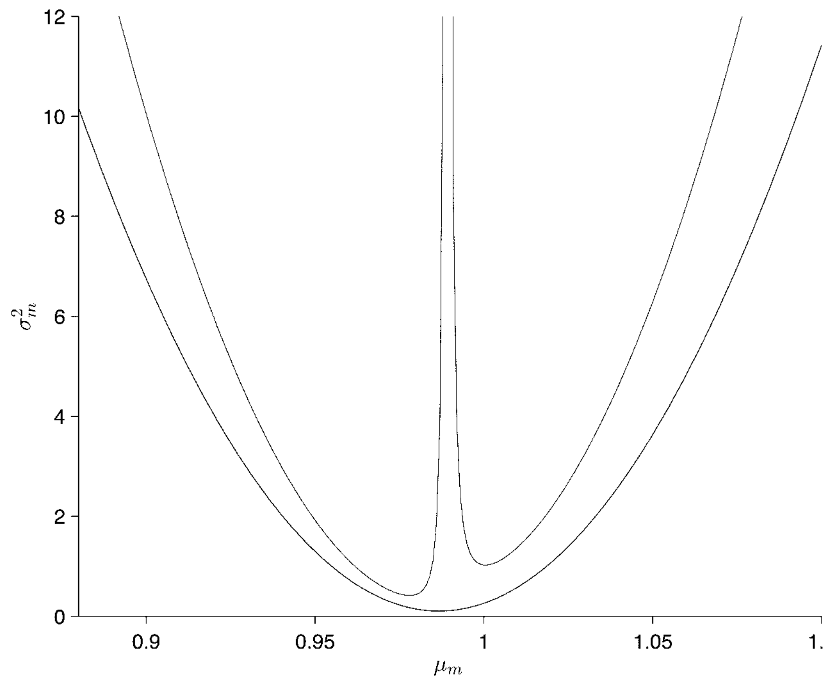

FIG. 1.—Variance bounds of stochastic discount factors. The figure plots three variance bounds on the stochastic discount factors when the test assets are 25 size and book-to-market ranked portfolios. The solid line is the Hansen-Jagannathan bound for all stochastic discount factors. The dashed line is the variance bound for $m(x)$, where x is the excess return on the value-weighted market portfolio. The dotted line is the variance bound for $m(x)$, where x is the three Fama-French factors (excess return on the value-weighted market portfolio, return difference between large and small size portfolios, and return difference between high and low book-to-market portfolios). The three variance bounds are estimated using monthly data over the period 1952/1–2002/12.

R_M, a particular case of which is the well-known CAPM. For this choice of state variable R_M, we plot the lower bound of $\mathrm{Var}[m(R_M)]$ using a dashed line in figure 1. This variance bound is estimated on the basis of

$$\hat{\sigma}^2_{m(x)}(\mu_m) = \frac{\hat{\sigma}^2_0(\mu_m)}{\hat{\rho}^2_{x,m_0}}, \tag{29}$$

where $\hat{\rho}_{x,m_0}$ is the sample multiple correlation coefficient between x and m_0. Note that since we have only one state variable, for some choice of μ_m (0.9901, corresponding to a monthly interest rate of 1%) we have the lower bound of $\mathrm{Var}[m(R_M)]$ equal to infinity. With this value of μ_m, we can see that our new bound provides a substantial improvement over the standard Hansen-Jagannathan bound. Outside this value, R_M is in general fairly highly correlated with m_0, so our new bound provides less an improvement, though still substantial.

Increasing the number of state variables will in general reduce the variance bound because ρ^2_{x,m_0} can only increase with a larger set of state variables. We illustrate this by expanding the set of state variables to the Fama-French three factors $x = (R_M, R_{\text{SMB}}, R_{\text{HML}})'$, where R_{SMB} is the return difference between small and large size portfolios, and R_{HML} is the return difference between high and low book-to-market portfolios. In figure 1, we plot our new bound on $\text{Var}\,[m(R_M, R_{\text{SMB}}, R_{\text{HML}})]$ using a dotted line. Comparing this bound with the dashed one for the case of $x = R_M$, we can see that with more state variables included in the SDF, the new bound is closer to the Hansen-Jagannathan one. Nevertheless, the new bound can still be substantially higher than the Hansen-Jagannathan bound. Over the range of values of μ_m that we plot in figure 1, our new bound offers at least a 34% increase over the Hansen-Jagannathan bound and as much as a 646% increase for some values of μ_m.

III. Impact on Consumption-Based Models

Cochrane (2001) provides an excellent survey of the standard consumption-based asset pricing models originated by Breeden (1979). The well-known first-order condition (Euler equation) for an investor's expected utility maximization problem is

$$u'(C_t) = E_t[\delta u'(C_{t+1})R_{t+1}], \tag{30}$$

where u is the utility function, δ is the subjective time discount factor of the investor, C_t is the consumption at time t, and R_{t+1} is the gross return of an asset at time $t + 1$. So the basic asset pricing equation is

$$1 = E_t[mR_{t+1}], \qquad m = \delta \frac{u'(C_{t+1})}{u'(C_t)}, \tag{31}$$

where m is the well-known SDF or the intertemporal marginal rate of substitution.

Applying proposition 1 to some well-known utility functions is straightforward. For example, consider the power utility

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \qquad m(x) = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} = \delta e^{-\gamma x}, \tag{32}$$

where $x = \ln (C_{t+1}/C_t)$ is consumption growth. If we are sure that the utility function is indeed a power utility function and the true values of δ and γ are known, we can directly test (31). However, researchers are often not equipped with the knowledge of the exact functional form of the utility function. In that case, if we are willing to assume joint elliptical distribution of x and

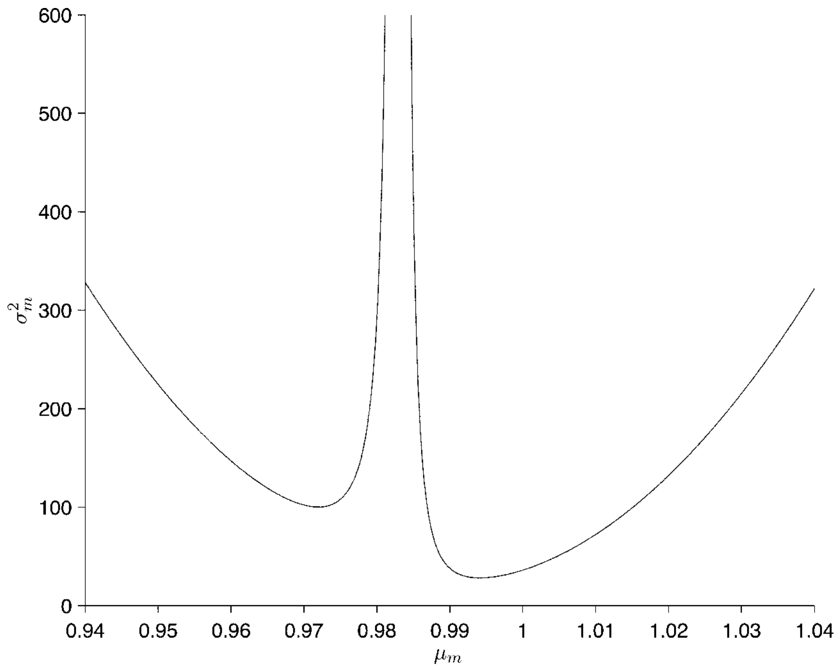

FIG. 2.—Variance bound of stochastic discount factors that are functions of growth rate of consumption. The figure plots the variance bound on the stochastic discount factors when the test assets are 25 size and book-to-market ranked portfolios and the stochastic discount factor is a function of the continuously compounded growth rate of durable consumption. The variance bound is estimated using monthly data over the period 1959/2–2002/12.

returns and assume that the intertemporal marginal rate of substitution can be written as a function of x, then proposition 1 says

$$\mathrm{Var}\,[m(x)] \geq \frac{\mathrm{Var}\,[m_0]}{\rho^2_{x,m_0}}. \tag{33}$$

In figure 2, we plot this lower bound for $\mathrm{Var}\,[m(x)]$ using the same 25 portfolio returns as before, where the consumption growth per capita is measured using nondurable consumption data from the Citibase available from February 1952 to December 2002. Comparing this with the Hansen-Jagannathan bound in figure 1, we find that the bound for $\mathrm{Var}\,[m(x)]$ in figure 2 is much higher, in fact, at least 128 times higher than the Hansen-Jagannathan bound. The reason is that the highest ρ^2_{x,m_0} that we can find within the range of μ_m that we plot is 0.0078. Therefore, in order for $m(x)$ to price the 25 size and book-to-market ranked portfolios correctly, it has to be extremely volatile.

This high required volatility on $m(x)$ here also has important implications on the parameters of the utility function. For example, substituting (31) into (32), we observe that for a fixed value of γ, the investor's subjective time discount factor, δ, must satisfy

$$\delta \geq \frac{1}{|\rho_{x,m_0}|}\sqrt{\frac{\mathrm{Var}[m_0]}{\mathrm{Var}[e^{-\gamma x}]}}. \tag{34}$$

As δ discounts the future utility to present, it measures the investor's impatience. The smaller the δ is, the more impatient the investor. Using the Hansen-Jagannathan bound, one considers a value of $\delta = \sqrt{\mathrm{Var}[m_0]/\mathrm{Var}[e^{-\gamma x}]}$ to be acceptable. However, even at a very high correlation level of the consumption growth with the asset returns that results in $|\rho_{x,m_0}| = 0.3$, equation (34) suggests that the investor has to be at least 3.33 times more patient in order for $m(x)$ to be a valid SDF.

Applying proposition 1 is straightforward if the marginal rate of substitution can be written as a function of the ratio of consumption C_{t+1}/C_t or the first difference of consumption $C_{t+1} - C_t$, since these two terms can be reasonably assumed to have an elliptical distribution. However, not every utility function has such a simple representation. Nevertheless, as long as we are willing to make an elliptical distribution on C_{t+1}/C_t, which can be justified theoretically under constant absolute risk aversion (CARA) utility,[5] and its conditional mean and variance are constant over time, the following proposition shows that the bound in proposition 1 continues to hold.

Proposition 2. Suppose that a stochastic discount factor $m = m(C_t, C_{t+1}) = \delta u'(C_{t+1})/u'(C_t)$. Let $x = \ln(C_{t+1}/C_t)$. Suppose, conditional on C_t, that x and m_0 are multivariate elliptically distributed with constant mean and variance. Then

$$\mathrm{Var}[m(x)] \geq \frac{\mathrm{Var}[m_0]}{\rho^2_{x,m_0}}, \tag{35}$$

where ρ_{x,m_0} is the correlation between x and m_0.

Proof. Write $m = m(C_t, C_t e^x)$. Conditional on C_t, proposition 1 can be applied to yield equation (35), except the terms on both sides are conditional on C_t:

$$\mathrm{Var}[m(x)|C_t] \geq \frac{\mathrm{Var}[m_0|C_t]}{\rho^2_{x,m_0|C_t}}. \tag{36}$$

However, under the assumption that the conditional mean and variance of x and m_0 are constant, the conditional moments on the right-hand side are

5. For example, Merton (1973) and Cochrane (2001) show that the optimal consumption growth for a CARA investor is lognormal in the standard diffusion setup for the asset returns.

the same as the unconditional moments. As for the left-hand side, using the iterated law of expectations, we have

$$\begin{aligned}\mathrm{Var}\,[m(x)] &= E[\mathrm{Var}\,[m(x)|C_t]] + \mathrm{Var}\,[E[m(x)|C_t]] \\ &\geq E[[\mathrm{Var}\,[m(x)|C_t]] \\ &\geq E\left[\frac{\mathrm{Var}\,[m_0]}{\rho^2_{x,m_0}}\right] \\ &= \frac{\mathrm{Var}\,[m_0]}{\rho^2_{x,m_0}}. \end{aligned} \tag{37}$$

This completes the proof. QED

Now let us examine the implications of proposition 1 on the *equity premium* and *correlation* puzzles. Since Mehra and Prescott (1985), the equity premium puzzle became well known: the consumption-based SDF is not volatile enough to explain the risk premium of equity. As put by Cochrane (2001, 456), it follows from the definition of an SDF that

$$\frac{\sigma(m)}{E[m]} \geq \frac{|E[R^e]|}{\sigma(R^e)}, \tag{38}$$

where σ is the standard deviation operator and R^e is the excess return on the market index. Alternatively, (38) is the result of applying the Hansen-Jagannathan bound to the two-asset case: the risk-free asset and the market index. Using data that the postwar excess return on the New York Stock Exchange (NYSE) value-weighted index is, on average, approximately 8% per year and the standard deviation is approximately 16% per year, and assuming $E[m] = 1/R_f = 0.99$, Cochrane shows that $\sigma(m) > 0.50$. To justify this, a very large risk aversion parameter is required. Under either power utility or exponential utility, the state variable of the SDF can be taken as either the consumption growth or the change of consumption, and it is reasonable, to at least a first-order approximation, to assume that x and R_t^e have a multivariate elliptical distribution. Then, on the basis of Cochrane's (2001, 457) estimate of a value of $\rho_{x,R^e} = 0.2$, proposition 1, together with the fact that m_0 is a linear function of R^e, implies that

$$\begin{aligned}\sigma(m) &\geq \frac{1}{|\rho_{x,m_0}|}E[m]\frac{|E[R^e]|}{\sigma(R^e)} = \frac{1}{|\rho_{x,R^e}|}E[m]\frac{|E[R^e]|}{\sigma(R^e)} \\ &= 5 \times 0.5 = 2.5, \end{aligned} \tag{39}$$

which demands an even greater risk aversion parameter (in terms of variance, this bound is 25 times greater than the Hansen-Jagannathan bound). Further empirical study of this and related models based on recent data is provided later in this section.

After manipulation of $E(mR^e) = 0$, it is simple to show that

$$\sigma(m) = \frac{1}{|\rho_{m,R^e}|} E[m] \frac{|E[R^e]|}{\sigma(R^e)}. \tag{40}$$

The key difference between this bound and (39) is that ρ_{m,R^e} in general depends on the choice of a utility function, but ρ_{x,R^e} of (39) is known or is easily estimated from the data (independent of the special functional form of m). In the special case in which $m(x)$ is a linear function of x, (39) and (40) are the same. The much stricter bound (40) is termed as the correlation puzzle by Cochrane (2001, 457). The proof there applies only to the case in which $m(x)$ is a linear function of x. In contrast, $m(x)$ here can be an arbitrary nonlinear function of the state variable. Therefore, it generalizes the correlation puzzle to potentially many utility functions. In a setting with multiple assets, Cochrane and Hansen (1992) show that

$$\mathrm{Var}[m] = \frac{\mathrm{Var}[m_0]}{\rho^2_{m,m_0}}, \tag{41}$$

which follows directly from (9). However, this bound (which is actually an identity) can be calculated only if we know ρ^2_{m,m_0}. This in turn requires us to specify m explicitly, which often is difficult because we may have doubts on its functional form. In addition, the resulting variance bound is applicable only to that particular choice of m. In contrast, our variance bound makes use of the multiple correlation coefficient of the default SDF with the state variables and studies its impact on the variance bound of an arbitrary $m(x)$.

Finally, let us examine applications of proposition 1 to some recent asset pricing models. Owing to the failure to explain the equity premium puzzle, models of SDFs with multiple state variables have been developed. The addition of more variables in general should increase the multiple correlation between x and m_0, making the new bound closer to the Hansen-Jagannathan one. Abel (1990), for example, provides a model in which the investor's power utility depends not only on the consumption but also on a time-varying benchmark. Under some simplifying assumptions (see, e.g., Kirby 1998), this results in an SDF of

$$m = \frac{\delta(C_{t+1}/C_t)^{\gamma}}{(C_{t-1}/C_{t-2})^{1-\gamma}}. \tag{42}$$

In this case, we can take $x_1 = \ln(C_{t+1}/C_t)$ and $x_2 = \ln(C_{t-1}/C_{t-2})$. The innovations of consumption growth can be assumed to be multivariate elliptically distributed, and then proposition 1 easily applies to yield a bound on $\sigma(m)$.

Out of the models with multiple state variables, the Campbell and Cochrane

(1999, 2000) model seems getting the most attention. They propose a model with an SDF

$$m_{CC} = \delta\left(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma}, \tag{43}$$

where S_t is the surplus consumption ratio. In their model, they assume that the two ratios in m are conditionally lognormal, and we can take $x = (\ln(C_{t+1}/C_t),\ \ln(S_{t+1}/S_t))$ as the state variables in the model. Therefore, we can apply proposition 1 to yield a bound on $\sigma(m_{CC})$. In what follows, we will focus our empirical study on this model.

At the outset, it should be noted that $S_t = (C_t - X_t)/C_t$ is unobservable since the level of habit X_t is latent. Following Li (2001) as well as Liu (2003), we extract S_t from a model and then compute the moments and bounds on the basis of the extracted series. The underlying data-generating process for S_t is the nonlinear square root model of Campbell and Cochrane (1999, 2000). They assume that the log surplus consumption ratio evolves according to

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \lambda(s_t)(c_{t+1} - c_t - g), \tag{44}$$

where $s_t = \log(S_t)$, $c_t = \log(C_t)$, and ϕ, g, and $\bar{s}$ are parameters. The sensitivity function $\lambda(s_t)$ is given by

$$\lambda(s_t) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1-2(s_t-\bar{s})} - 1 & \text{if } s_t < \bar{s} + \frac{1}{2}(1-\bar{S}^2) \\ 0 & \text{if } s_t \geq \bar{s} + \frac{1}{2}(1-\bar{S}^2), \end{cases} \tag{45}$$

where $\bar{S} \equiv \sigma_c\sqrt{\gamma/(1-\phi)}$ is the steady-state surplus consumption ratio and $\bar{s} = \log(\bar{S})$. Notice that g and σ_c are the mean and standard deviation of the log consumption growth and hence can be easily estimated from the data (as the sample mean and standard deviation following Campbell and Cochrane's independently and identically distributed assumption on the log consumption). However, other parameters (ϕ, γ, and δ) have to be specified exogenously. In our applications, we choose parameters following the same approach as in Campbell and Cochrane (1999, 2000). The value of ϕ is chosen to be 0.989 (0.87 annualized). For γ, we choose over a range of two to 20. Finally, for each value of γ, δ is chosen such that the log risk-free rate is 0.0783% (0.94% annualized), where the log risk-free rate is given by

$$r_f = -\ln(\delta) + \gamma g - \frac{\gamma}{2}(1-\phi). \tag{46}$$

Campbell and Cochrane show that their model with these choices of parameters, even for γ as low as two, matches a wide variety of phenomena including the predictability of stock returns from price-dividend ratios and the leverage effect by which low prices imply more volatile returns. However, they did not carry out a diagnostic test using the Hansen-Jagannathan bound, nor has

TABLE 1 **Variance Bound Test of the Campbell and Cochrane Habit Model**

			Market Portfolio			Fama-French Portfolios		
γ	$\sigma(m_C)$	$\sigma(m_{CC})$	$\hat{\sigma}_0$	$\hat{\rho}_{x,m_0}$	$\hat{\sigma}_{m(x)}$	$\hat{\sigma}_0$	$\hat{\rho}_{x,m_0}$	$\hat{\sigma}_{m(x)}$
2	.0148	.1358	.1886	.146	1.2931	.4603	.095	4.8279
4	.0293	.2072	.1886	.143	1.3182	.4603	.097	4.7664
6	.0435	.2627	.1886	.142	1.3313	.4603	.097	4.7599
8	.0575	.3113	.1886	.141	1.3397	.4603	.097	4.7707
10	.0713	.3563	.1886	.140	1.3456	.4603	.096	4.7880
12	.0849	.3991	.1886	.140	1.3499	.4603	.097	4.8081
14	.0984	.4408	.1886	.139	1.3532	.4603	.095	4.8294
16	.1118	.4819	.1886	.139	1.3557	.4603	.095	4.8510
18	.1251	.5228	.1886	.139	1.3576	.4603	.095	4.8727
20	.1383	.5638	.1886	.139	1.3590	.4603	.094	4.8941

NOTE.—The models are the standard consumption CAPM and Campbell and Cochrane (1999, 2000). The variance bound test is based on monthly data over the period 1959/2–2002/12. The γ column is the curvature parameter of the utility function; $\sigma(m_C)$ is the standard deviation of the stochastic discount factor of the consumption CAPM, $\sigma(m_{CC})$ is that of the Campbell and Cochrane model, $\hat{\sigma}_0$ is the Hansen-Jagannathan bound, ρ_{x,m_0} is the multiple correlation coefficient between the state variables and the default stochastic discount factor m_0, and $\hat{\sigma}_{m(x)}$ is the new bound, when the value-weighted market portfolio of the NYSE is used as the test asset. The last three columns of the table reports the bounds and correlation when the Fama-French 25 size and book-to-market ranked portfolios are used as the test assets.

this been carried out by others, especially when multiple portfolios are used as test assets.

As a result, it is of interest here to see how their model performs in terms of the Hansen-Jagannathan and our new bounds on the basis of the market portfolio and the Fama-French portfolios used earlier. Table 1 provides the results using monthly data over the period 1959/2–2002/12. The bounds are computed using two different sets of test assets. The first set is a single-asset case, the value-weighted market index of the NYSE. The second set is the Fama-French 25 size and book-to-market ranked portfolios.

We discuss the results using the market portfolio first. When the utility curvature parameter γ is equal to two, the standard deviation of the SDF of the consumption CAPM, $\sigma(m_C)$, is only 0.0148. As a result, the traditional consumption CAPM has a hard time satisfying the Hansen-Jagannathan bound. Even if we raise γ to as high as 20, $\sigma(m_C)$ still cannot satisfy the Hansen-Jagannathan bound. In contrast to the consumption CAPM, the SDF of the Campbell and Cochrane model is much more volatile. For example, when $\gamma = 2$, we find that the standard deviation of the SDF in the Campbell and Cochrane model has an impressive standard deviation of $\sigma(m_{CC}) = 0.1358$, about 9.2 times larger than that of the consumption CAPM. Although $\sigma(m_{CC})$ still cannot satisfy the Hansen-Jagannthan bound, which is estimated to be $\hat{\sigma}_0 = 0.1886$, the values are fairly close to each other. So, it is not surprising that if we allow γ to increase to four or above, $\sigma(m_{CC})$ easily satisfies the Hansen-Jagannathan bound. Therefore, on the basis of the Hansen-Jagannthan bound alone, one might conclude that the Campbell and Cochrane model is a superb model, even with a relatively small value of γ. However, comparison of $\sigma(m_{CC})$ with our new bound raises new issues on this model. Because the state variables (growths of consumption and surplus consumption ratio) have a fairly low multiple correlation coefficient with the market return (it ranges from 0.139 to 0.146, depending on the values of γ), our new bound is, on average, about six to seven times larger than the Hansen-Jagannathan bound. Therefore, even if we increase γ to 20, the Campbell and Cochrane model still cannot pass our new bound.

We now turn our attention to the results using the Fama-French 25 size and book-to-market sorted portfolios as the test assets. It is apparent from the table that all the bounds now have greater values than before. This is intuitive. When more test assets are used, it becomes more difficult for the model to explain the asset prices. Indeed, when $\gamma = 2$, the models fail more significantly than before in passing the bounds. Interestingly, despite using more assets, the Campbell and Cochrane model can still pass the Hansen-Jagannathan bound at the high end of γ ($\gamma \geq 16$). Nevertheless, our new bound is in the range of 4.8–4.9, making it almost impossible for the consumption and Campbell and Cochrane models to satisfy when $\gamma \leq 20$.

Because the kernel variance is an increasing function of γ, the Campbell and Cochrane model can eventually satisfy our new bound if γ is large enough. The question is how large it must be. It can be verified that, in order for the Campbell and Cochrane model to pass our new bound, we will need γ to be 129 or above, which is quite an unreasonably high value.

In summary, while the Campbell and Cochrane (1999, 2000) model has remarkable power in explaining the asset prices and can pass the Hansen-Jagannathan bound with a reasonably high risk version parameter, it still fails to pass the proposed new bound of this paper. With data from 1959/2–2002/12, the new bound is at least more than six times higher than the Hansen-Jagannathan bound. The reason for such a higher bound is that the state variables have low correlations with the asset returns. This seems to suggest that future asset pricing models should focus on state variables that are highly correlated with the returns on the assets. An increase of the volatility of the pricing kernel alone may not be sufficient to explain the expected returns of the assets if the state variables have low correlations with the returns of the assets.

IV. Conclusions

In this paper, we derive a new variance bound on any stochastic discount factor of the form $m = m(x)$, where x is a set of state variables. In contrast to the well-known Hansen-Jagannathan bound, our bound tightens it by a ratio of $1/\rho^2_{x,m_0}$, where ρ_{x,m_0} is the multiple correlation between x and the standard minimum variance SDF, m_0. In many applications, the correlation is small, and hence our bound is substantially tighter than Hansen and Jagannathan's. We show that if x is the gross growth rate of consumption and if we use Cochrane's (2001) estimates of market volatility and ρ_{x,m_0}, the new bound is 25 times greater, making it much more difficult to explain the equity premium puzzle on the basis of existing asset pricing models. Moreover, applying the new bound, with the growth rate of consumption as a state variable, to the 25 size and book-to-market sorted portfolios used by Fama and French (1993) can even yield a variance bound that is more than 100 times greater than the Hansen-Jagannathan one. As the Hansen-Jagannathan bound poses significant challenges for existing asset models to meet, our new sharply improved bound seems to raise this challenge onto a new plateau. In particular, we show that the recent model of Campbell and Cochrane (1999, 2000) can pass the Hansen-Jagannathan bound easily when the market is the only test asset and can also pass the bound for a relatively high value of the risk aversion parameter when the Fama-French 25 portfolios are used as the test assets, but fails to do so for our new bound with any reasonable risk aversion parameter.

The key insight of this paper is that in order for us to successfully explain asset prices using a theoretical pricing kernel, the state variables must have high correlations with the asset returns. This suggests that a potential direction for improving models, such as that of Campbell and Cochrane (1999, 2000), is to identify state variables that are highly correlated with the stock market. In addition, motivated by Ferson and Siegel (2003), Bekaert and Liu (2004), and others, it is of interest to examine how conditional information might be used to tighten the bound even further. Another important issue, inspired by Hansen and Jagannathan (1997), is to develop SDF-based distance measures for competing misspecified asset pricing models. Hodrick and Zhang (2001), Dittmar (2002), and Kan and Zhou (2003a), among others, show the wide usefulness of the Hansen-Jagannathan distance. In contrast to these applications, the distance measure can be refined to be dependent on state variables. This would likely shed new insights on the roles played by the state variables in an asset pricing model, a topic of interest for future research.

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