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Asset-pricing Tests under Alternative Distributions

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ABSTRACT

Given the normality assumption, we reject the mean-variance efficiency of the Center for Research in Security Prices value-weighted stock index for three of the six consecutive ten-year subperiods from 1926 to 1986. However, the normality assumption is strongly rejected by the data. Under plausible alternative distributional assumptions of the elliptical class, the efficiency can no longer be rejected. When the normality assumption is violated but the ellipticity assumption is maintained, many tests tend to be biased toward overrejection and both the accuracy of estimated beta and R^2 are usually overstated.

MANY ASSET-PRICING MODELS predict a linear relationship between the expected return on an asset and the covariance between the asset's return and one or more factors. It is this mean-variance framework that plays a central role in modern theories of asset pricing. However, Chamberlain (1983) showed that the mean-variance analysis is consistent with investor's portfolio decision making if and only if the returns are elliptically distributed. Moreover, in the case of elliptical returns, the capital asset-pricing model (CAPM) of Sharpe (1964) and Lintner (1965) and multibeta models will remain valid theoretically.[1] Therefore, it is important to test asset-pricing models for the case where the returns are elliptically distributed. And yet, Gibbons, Ross, and Shanken (1989), among others, provide tests that are valid only under the normality assumption, a special case of the elliptical distributions. Affleck-Graves and McDonald (1989) and MacKinlay and Richardson (1991) examine tests without the normality assumption, but their approaches are difficult to apply to obtain exact tests in the elliptical case.

Complementing the existing studies, we propose exact tests for both the case where the returns are elliptically distributed and the case where the residuals are elliptically distributed. When the normality assumption is violated but the ellipticity assumption is maintained, our results show that the usual tests can be biased and the widely used beta and R^2 (estimated

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[1] See Owen and Rabinovitch (1983) and Ingersoll (1987, chapter 4) for a discussion of portfolio choice and multibeta pricing models in the case where the returns are elliptically distributed.

from the market model) are not as accurate as commonly believed. To assess whether the returns are elliptically distributed, we provide exact tests based on measures of multivariate skewness and kurtosis proposed by Mardia (1970), complementing studies on the distributional properties of stock returns, of which Fama (1965) and Affleck-Graves and McDonald (1989) summarize most of the univariate approaches.[2]

The paper is organized as follows. In Section I, we derive the exact tests for both the case of elliptical residuals and the case of elliptical returns. We analyze also how the accuracy of the estimated beta and R^2 may be affected when residuals or returns are elliptically distributed. In Section II, we introduce measures of multivariate skewness and kurtosis and show how they can be used to test for ellipticity. Then, by using monthly data for every consecutive ten-year period from 1926 to 1986, we apply the tests to study the multivariate normality of the market model residuals and that of the excess returns. In Section III, we test the efficiency of the Center for Research in Security Prices (CRSP) value-weighted index under plausible alternative distributional assumptions on both the residuals and the returns. Section IV concludes the paper.

I. Exact Asset-pricing Tests under Elliptical Distributions

In this section, we focus our analysis on testing the mean-variance efficiency of a given portfolio. We consider first the normality case by presenting the standard multivariate framework of Gibbons, Ross, and Shanken (1989). Then, we test the mean-variance efficiency in the case where the model residuals are elliptically distributed and the case where the returns are elliptically distributed. Finally, we analyze how the accuracy of estimated beta and R^2 may be affected when residuals or returns are elliptically distributed. Because there are no analytical solutions for both of the elliptical cases, a simple numerical approach based on Monte Carlo integration is proposed to obtain the exact p-values. Despite the generality of our approach, we will consider only the market model in what follows. This is because it is an important model and it is the simplest case of the multivariate regression. The simplicity of the model allows us to better illuminate the central ideas and the econometric theory being employed. Once the simple case is understood, the results for the general case are straightforward and thus only a few remarks are provided for the generalizations.

A. Tests under Normality

Assume that there is a riskless rate of interest, r_{ft}, for each time period. Consider the returns on N assets in excess of the riskless rate. As in many

[2] Richardson and Smith (1991) provide multivariate normality tests based on the generalized method of moments approach.

studies, we assume the market model regression for the excess returns:

$$r_{it} = \alpha_i + \beta_i r_{pt} + \varepsilon_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T, \tag{1}$$

where r_{it} is the excess return on asset i in period t, r_{pt} the excess return on the given portfolio, ε_{it} the disturbance or random error, N the number of assets, and T the number of periods or sample size. Model (1) is a special case of the multivariate regression model. Throughout this subsection and the next, we assume that the model residuals, $\mathbf{E}_t = (e_{1t}, \dots, \varepsilon_{Nt})'$, are independent and identically distributed (i.i.d.) over time with zero mean and nonsingular covariance matrix $\mathbf{\Sigma}$.

Given the portfolio p, the most widely asked question is whether this portfolio is mean-variance efficient. It is well known that efficiency implies the following restrictions on the parameters:

$$H_0\colon \alpha_i = 0, \quad i = 1, \dots, N. \tag{2}$$

If the model residuals follow a multivariate normal distribution, Gibbons, Ross, and Shanken (1989) provide an exact test (the GRS test) for the efficiency hypothesis H_0:

$$W_T \equiv \left[(T - N - 1)/\left(N\hat{\phi}_p^2\right)\right]\hat{\boldsymbol{\alpha}}'\hat{\mathbf{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}} \tag{3}$$

where $\hat{\phi}_p^2 = 1 + \bar{r}_p^2/s_p^2$, $\bar{r}_p$ is the sample mean of r_{pt}, s_p^2 the sample variance of r_{pt} without adjusting for degrees of freedom, and $\hat{\boldsymbol{\alpha}}$ and $\hat{\mathbf{\Sigma}}$ are the maximum likelihood estimators of the corresponding parameters in (1). The GRS test has rich economic interpretations and attractive statistical properties. Under the null hypothesis that the given portfolio is mean-variance efficient, W_T follows an F distribution with degrees of freedom N and $T - N - 1$. The efficiency hypothesis is rejected for large values of W_T. The GRS test is fundamental for testing efficiency under normality.

B. Tests under Elliptical Residuals

A random vector $\mathbf{X}$ is said to have an *elliptical distribution* with parameters $\boldsymbol{\theta}$ $(N \times 1)$ and $\mathbf{\Sigma}(\mathrm{N} \times N)$ if its density function is of the form

$$f(\mathbf{X}) = C_N|\mathbf{\Sigma}|^{-1/2}g[(\mathbf{X} - \boldsymbol{\theta})'\mathbf{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\theta})], \tag{4}$$

where C_N is a constant and $g(\cdot)$ some function. If $\mathbf{X}$ is elliptical, it can be shown that the mean and the covariance matrix are linked to the parameters by

$$E(\mathbf{X}) = \boldsymbol{\theta} \quad \text{and} \quad \operatorname{cov}(\mathbf{X}) = c^2\mathbf{\Sigma}, \tag{5}$$

where c^2 is some constant that depends only on the specific functional form of $g(\cdot)$. The class of elliptical distributions is large, containing as special cases the multivariate normal, mixture normal, multivariate t, multivariate stable, Kotz and Pearson II distributions, and is the largest class of distributions that possess linear conditional expectations (Kelker (1970)).

The GRS test will not in general have an exact F distribution when the model residuals follow an elliptical distribution other than the normal. Although analytically intractable, the exact distribution can be computed numerically by using an important property of the W_T statistic that it is invariant to any nonsingular linear transformation of the residuals (see Appendix A for a proof). In other words, if every $\mathbf{E}_t$, an $N \times 1$ vector of the residuals at t, is replaced by $\mathbf{CE}_t$ where $\mathbf{C}$ is any $N \times N$ nonsingular matrix, the value of the test statistic will remain the same. In particular, we can multiply the residuals by $\Sigma^{-1/2}$ to get new residuals that follow an elliptical distribution with Σ being the identity matrix. Therefore, as far as the distribution of W_T is concerned, a three-step approach may be used to compute the p-value $\text{Prob}(W_T > x)$. First, the new residuals are drawn from the elliptical distribution which is straightforward because no unknown parameters of the market model are involved. Second, at each of the draws, the statistic W_T is computed and compared against the observed value x. Third, we repeat this process say 10,000 times, the percentage for which W_T is greater then x is readily computed. This is the numerical approximation to the exact p-value $\text{Prob}(W_T > x)$. This method is in fact a Monte Carlo integration approach applied to compute the integral $\text{Prob}(W_T > x)$. The numerical error is independent of both the sample size T and the number of assets N. The accuracy improves as the number of draws increases. Throughout the paper, we use 10,000 draws. Then the approach often generates values that are accurate to 2 or 3 decimal points. For our inference purposes, this level of accuracy seems to be very satisfactory.

Table I illustrates how the p-values of the GRS test will change if the model residuals follow a multivariate t, a mixture-normal, and a Kotz

Table I

The GRS Test under Alternative Residual Distribution

Table I provides a comparison of the p-values of the Gibbons, Ross, and Shanken (1989) test (GRS test) under alternative distributional assumptions on the model residuals. The first column is the number of assets, N, and the second is the p-values of the GRS test under the multivariate normality assumption. The rest of the columns show how this p-value will change if the residuals follow a multivariate t, a mixture multivariate normal, and a Kotz distribution, respectively. For each of the three alternative distributions, the p-values are with three choices of the degrees of freedom (γ is fixed at a value of 10 for the mixture multivariate normal distribution).

		Multivariate t			Mixture Normal			Multivariate Kotz		
N	Normal	$\nu = 5$	$\nu = 8$	$\nu = 24$	$\epsilon = 5\%$	$\varepsilon = 25\%$	$\varepsilon = 50\%$	$\nu = 5$	$\nu = 10$	$\nu = 20$
10	0.050	0.065	0.054	0.052	0.063	0.084	0.072	0.050	0.044	0.042
20	0.050	0.068	0.064	0.055	0.061	0.090	0.082	0.047	0.050	0.049
40	0.050	0.065	0.059	0.052	0.054	0.069	0.086	0.046	0.050	0.053
58	0.050	0.051	0.052	0.051	0.050	0.052	0.051	0.050	0.050	0.048
10	0.100	0.113	0.104	0.103	0.108	0.132	0.125	0.101	0.096	0.094
20	0.100	0.122	0.117	0.104	0.115	0.142	0.144	0.097	0.101	0.095
40	0.100	0.123	0.113	0.101	0.105	0.123	0.150	0.099	0.099	0.099
58	0.100	0.100	0.104	0.101	0.101	0.108	0.099	0.100	0.098	0.099

distribution, respectively. The p-values are computed for three choices of the degrees of freedom, different numbers of assets (N varies from 10 to 58), and two significance levels (5 and 10 percent). However, for the ease of computation, the sample size is fixed at $T = 60$. It is interesting that the p-values are not much different from one another. As the degree of freedom increases from 5 to 24, the t distribution becomes closer to the normal distribution, and so the p-values are closer to either 5 or 10 percent. In the Kotz case, the larger the degree of freedom, the more the distribution differs from the normal. Nevertheless, as ν varies from 5 to 20, the p-values do not change substantially. Of all the p-values, the notable differences occur in the mixture-normal case. At the 5 percent level, the largest difference is 4 percent when $\varepsilon = 25$ percent and $N = 20$. At the 10 percent level, the largest difference is 5 percent. Notice that there are always differences between the p-values obtained under normality and those obtained under other elliptical distributions. The differences may or may not be economically important in a particular application, but our approach helps to assess it.

C. Tests under Elliptical Returns

Our analysis of the GRS test has so far been focused on the model residuals. We now change our focus to the returns. Let $\mathbf{X} = (\mathbf{r}', r_p)'$ be an $N + 1$ vector of the individual excess returns and the excess return of the given benchmark portfolio. Assume that r_p contains some assets which are not in $\mathbf{r}$ so that the covariance matrix of $\mathbf{X}$ is nonsingular. Partition the mean and the covariance matrix with respect to $\mathbf{r}$ and r_p:

$$\mu = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix}. \tag{6}$$

Under the usual normality assumption, the distribution of $\mathbf{r}$ conditional on r_p must also be normal. The mean is a linear function of r_p, but the covariance matrix does not depend on r_p:

$$E(\mathbf{r} \mid r_p) = \mu_1 + \mathbf{V}_{12}\mathbf{V}_{22}^{-1}(r_p - \boldsymbol{\mu}_2), \tag{7}$$

$$\mathrm{Var}(\mathbf{r} \mid r_p) = \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21}. \tag{8}$$

It is clear that (7) and (8) imply the market model regression where $\mathbf{V}_{12}\mathbf{V}_{22}^{-1}$ plays the role of the betas and $\mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21}$ plays the role of the $\boldsymbol{\Sigma}$ matrix. It is thus true that testing efficiency under the multivariate normality assumption on the excess returns can be regarded as a special case of the market model parameterization. The latter may allow r_p to be fixed or to follow a distribution other than the normal.

However, the market model parameterization does not include the case where the excess returns are elliptically distributed but nonnormal. Let $\boldsymbol{\mu}$ and $\mathbf{V}$ be the parameters of the elliptical distribution. The expectation of $\mathbf{r}$ conditional on r_p will be exactly the same as (7), but the conditional covariance matrix is no longer independent of r_p:[3]

$$\operatorname{Var}(\mathbf{r} \mid r_p) = k(r_p)(\mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21}), \tag{9}$$

where $k(\cdot)$ is some function of r_p. It can also be shown that if the conditional covariance does not depend on r_p, $\mathbf{X} = (\mathbf{r}', r_p)'$ must be normal. Hence, the model residuals of (1) must depend on r_p for elliptical returns other than the normal. This states that the residual covariance matrix is time varying. In contrast, the residual covariance matrix $\mathbf{\Sigma}$ of the market model is not time varying, always being a function of the parameters alone. Hence, asset-pricing tests in the case of elliptical returns do not fall into the previous case of elliptical residuals.

Nevertheless, we can adapt our previous approach to obtain the exact p-value. It can be shown (Fang and Zhang, 1990, pp. 67–70) that $k(r_p)$ is a function of the quadratic form $(r_p - \boldsymbol{\mu}_2)'\mathbf{V}_{22}^{-1}(r_p - \boldsymbol{\mu}_2)$ alone. Other than this, it depends neither on any components of $\mathbf{X} = (\mathbf{r}', r_p)'$ nor $\mathbf{V}$ and $\boldsymbol{\mu}$. Therefore, because of the invariance property of the GRS test, we can assume $\mathbf{\Sigma}_t = k(r_{pt})\mathbf{I}$ for the computation of the p-value. Conditional on r_{pt}, μ_2, and $\mathbf{V}_{22}$, both $k(r_{pt})$ and $\mathbf{\Sigma}_t$ are completely determined. As a result, samples from the residual distributions are easily generated. Despite the time-varying nature of the residuals, each draw for the entire period, i.e., a draw of all the residuals, is still i.i.d. Therefore, the Monte Carlo integration approach remains a valid method for obtaining the p-value. However, this is the p-value conditional on the realizations of r_{pt}. Because r_{pt} is treated here as a random variable rather than a fixed constant, we are interested in the p-value unconditional on the realizations of r_{pt}, implying that r_{pt} should be integrated out from the GRS statistic along with the residuals. To do so, we need only to generate r_{pt} from its marginal distribution in each of the sample draws. An alternative method is to draw r_{pt} and the residuals together from their joint elliptical distribution. Because of the invariance property, we can, for purpose of computing the p-value, assume that the joint elliptical distribution has parameters $\boldsymbol{\mu}_1 = \mathbf{0}$, $\mathbf{V}_{12} = \mathbf{0}$, and $\mathbf{V}_{11} = \mathbf{I}$ with the additional parameters $\boldsymbol{\mu}_2$ and $\mathbf{V}_{22}$. This latter approach is preferred to the first one because neither marginal nor conditional distributions are required, which may be difficult to obtain.

Notice that the true values of the parameters μ_2 and $\mathbf{V}_{22}$ are unknown and have to be estimated from data. This is the limitation of the above procedure. Nevertheless, our experiments show that small perturbations of the estimates have only negligible effects. As its extension, we notice that the procedure is applicable not only to the GRS test, but also to four other widely used tests; Wilks' Λ test, Hotelling-Lawley's T_0^2 test, the Pillai trace test and Roy's largest root test, of which the GRS test is a special case.[4] These tests

[3] I am especially grateful to one of the referees for pointing this out in addition to many other detailed suggestions that substantially improved the paper.

[4] See Muirhead (1982) for a discussion of these four tests. An additional multivariate test is given in Butler and Frost (1992).

are useful for testing general linear hypotheses on the regression coefficient matrix. Since many asset-pricing tests are about the regression coefficient matrix and are equivalent to one of the four tests, the suggested approach should have wide applications. For example, any of the four exact small sample tests can be used to examine the mean-variance spanning hypothesis of Huberman and Kandel (1987). Our approach may also be applied to time series models where a vector autoregression model conditional on initial observations is a multivariate regression.

Finally, consider parameter estimates for elliptical residuals and elliptical returns. Notice first that the usual estimates are consistent. The accuracy as measured by the variances is asymptotically the same for elliptical residuals (or i.i.d. residuals with finite fourth moments), but this is not true for elliptical returns. For example, the variances of estimates for β and R^2 obtained under the normality assumption should be adjusted to $\operatorname{var}(\hat{\beta})(1+\kappa)$ and $\operatorname{var}(\hat{R}^2)(1+\kappa)$ where $\kappa = \Delta_2/N(N+2) - 1$ and Δ_2 is the kurtosis (Section II). The adjustments can be justified asymptotically. Table II provides simulation results in small samples. The first panel reports ratios of the variances for normal and t residuals (the degree of freedom is 6). The ratios are close to one when the sample size is as small as 20, confirming the asymptotic results. The second panel reports the same ratios for normal and t returns. The asymptotic results imply that the ratios should be close to $(1+\kappa) = 2$. This is supported by the simulations. Thus, the adjustments seem to be adequate even for small samples, suggesting that the accuracy of widely used estimates for such parameters as β and R^2 may be overstated for i.i.d. elliptical returns other than the normal.

II. Multivariate Skewness, Kurtosis, and Ellipticity Tests

In this section, we introduce first measures of multivariate skewness and kurtosis. Then we show how they can be used to obtain exact tests for ellipticity, and finally, we apply the measures to test whether the market model residuals and stock returns follow a multivariate normal distribution.

A. Definitions and Tests

Most asset-pricing tests assume normality for either the residuals or excess returns. We have examined in the first section how to obtain exact tests when the normality assumption is removed but the ellipticity assumption is maintained. The question remains whether or not the residuals or the excess returns indeed follow a multivariate normal distribution or a distribution of the elliptical class. To answer this question, we need tests for ellipticity.

Let $\mathbf{X}_1, \ldots, \mathbf{X}_T$ be the observations on an $N \times 1$ random vector $\mathbf{X}$ over T periods. Following Mardia (1970), the *multivariate skewness* and *kurtosis* can be defined as

$$D_1 = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} r_{ts}^3 \quad \text{and} \quad D_2 = \frac{1}{T} \sum_{t=1}^{T} r_{tt}^2, \tag{10}$$

Table II

Estimation under Alternative Distributions

We examine the accuracy of estimation for beta and R^2 in the market model:

$$r_t = \alpha + \beta r_{pt} + \varepsilon_t, \quad t = 1, \ldots, T; \quad \varepsilon_t \text{ is i.i.d. over time;}$$

where r_t and r_{pt} are returns (or excess returns) on an asset and a benchmark portfolio. The accuracy is measured by the variance and the ratio of the variance obtained under one distributional assumption to that obtained under another distributional assumption indicates the sensitivity of the accuracy of the alternative distributional assumptions. Panel A reports the variance ratios in the case where r_{pt}'s are treated as fixed constants and the residuals follow the normal and t distributions respectively. Panel B reports the variance ratios in the case where r_{pt}'s are treated as random variable and r_t and r_{pt} are jointly normal and t distributed respectively. The computations are based upon 10,000 simulated data sets and the degree of freedom of the t distribution is fixed at six.

Sample Size	$\text{var}(\hat{\beta})_t/\text{var}(\hat{\beta})_n$	$\text{var}(\hat{R}^2)_t/\text{var}(\hat{R}^2)_n$
	Panel A. Alternative Residual Distributions	
$T = 20$	0.991	1.455
$T = 40$	0.995	1.371
$T = 60$	1.010	1.441
$T = 80$	1.040	1.409
$T = 120$	0.987	1.293
$T = 600$	1.001	1.099
	Panal B. Alternative Return Distributions	
$T = 20$	1.523	1.368
$T = 40$	1.590	1.546
$T = 60$	1.699	1.619
$T = 80$	1.764	1.729
$T = 120$	1.847	1.824
$T = 600$	1.961	1.931

where $r_{ts} \equiv (\mathbf{X}_t - \overline{\mathbf{X}})'\mathbf{S}^{-1}(\mathbf{X}_s - \overline{\mathbf{X}})$, $\overline{\mathbf{X}}$ and $\mathbf{S}$ are the sample mean and sample covariance matrix respectively. As sample size increases, the central limit theorem implies that D_1 and D_2 converge to their population counterparts

$$\Delta_1 \equiv E\Big([(\mathbf{X} - \boldsymbol{\theta})'\Sigma^{-1}(\mathbf{Y} - \boldsymbol{\theta})]^3\Big), \tag{11}$$

and

$$\Delta_2 \equiv E\Big([(\mathbf{X} - \boldsymbol{\theta})'\Sigma^{-1}(\mathbf{X} - \boldsymbol{\theta})]^2\Big), \tag{12}$$

where $\mathbf{Y}$ is independent of $\mathbf{X}$ but has the same distribution. When $\mathbf{X}$ follows a multivariate normal distribution, it can be shown that: $TD_1/6 \sim \chi_f^2$, where $f = N(N+1)(N+2)/6$, and $D_2 \sim N(N+2)/[8N(N+2)/T]^{1/2} \sim N(0,1)$. Therefore both D_1 and D_2 can be used to test multivariate normality. If $N = 1$, D_1 and D_2 are tests for univariate normality. The size of the tests can be determined from the asymptotic chi-squared and normal distributions.

Notice that both D_1 and D_2 are invariant to shifts and nonsingular linear transformations of the data. With our analysis in Section I, it is clear that a

Monte Carlo integration approach can be used to compute the exact p-values where the samples can easily be drawn from the standard normal distribution. Because no specifics of the normal distribution other than the availability of samples from it are required, the procedure also applies to test the hypothesis that the data come from a given member of the elliptical class. In this case, the samples are drawn from the corresponding distribution of the elliptical class with parameters $\boldsymbol{\theta} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}_N$.

B. Residual and Return Normality

Consider the returns on twelve industry portfolios formed by following the groupings procedure of Sharpe (1982) and others. The industry groups are: petroleum, finance/real estate, consumer durables, basic industries, food/tobacco, construction, capital goods, transportation, utilities, textiles/trade, services, and leisure. The portfolio returns are value weighted. The benchmark portfolio return is the value-weighted New York Stock Exchange return available from CRSP at the University of Chicago. All returns are in excess of the 30-day Treasury bill rate available from Ibbotson Associates. The monthly data span February 1926 to January 1986.

The first panel of Table III reports the results by using the skewness and kurtosis tests to test univariate normality of each of the residuals and multivariate normality of all the residuals together, where the residuals are obtained by running ordinary least square regressions of the industry excess returns on the index excess returns.[5] The number of rejections of univariate normality at the 5 percent significance level is summarized in the second and fourth columns. We cannot reject the univariate normality assumption at the 5 percent level for most industries in most periods. However, there is strong evidence against multivariate normality and we reject it for all of the twelve periods at the 5 percent significance level (or a much lower level). On testing normality of the returns, similar results are provided in the second panel of Table III. There is little evidence against univariate normality, but normality is strongly rejected from a multivariate point of view.

III. Efficiency Tests

Given that we have rejected multivariate normality of both the residuals and the excess returns, it is of interest to determine which distributions of the elliptical class, if any, are reasonable alternatives. Furthermore, under reasonable alternative distributional assumptions on either the residuals or the returns, we want to examine how the conclusions of the mean-variance efficiency test may change.

Fang, Kotz, and Ng (1990) provide a number of examples of elliptical distributions. However, for ease of implementation, we consider only the

[5] The p-values are computed by using the exact approach as discussed in the previous subsection. However, strictly speaking, the p-values are not exact in the residual case because the true disturbance terms are unobservable and it is their estimates, the ordinary least squares residuals, that are used in the computation of the p-values.

Table III

Residual and Return Normality

The residuals are from the market model

$$r_{it} = \alpha_i + \beta_i r_{pt} + \varepsilon_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T; \quad \varepsilon_{it} \text{ is i.i.d. over time;}$$

where r_{it} and r_{pt} are the excess returns on the ith industry portfolio and on the CRSP value-weighted index, $N = 12$ the number of industries and $T = 120$ the sample size in each ten-year period. The residual normality is tested by using the skewness and kurtosis tests and the results are provided in the first panel of the table. The second column reports the range of observed univariate skewness values for all the residuals. The third column summarizes the number of rejections to univariate normality of each of the 13 residuals at the 5 percent level. The fourth column provides the multivariate skewness statistic and the associated p-value (in parentheses) of the test for multivariate normality. The last three columns report similar results by using the kurtosis tests. In the second panel of the table, we report the results of testing the normality of the 13 excess returns. (the 12 industry excess returns plus the index excess return) by using the same procedure.

Period	Range	Rejections	Multivariate	Range	Rejections	Multivariate
Panel A. Residual Normality						
1926/2–1936/1	[0.00, 1.32]	6	53.24 (0.000)	[2.98, 6.74]	8	228.72 (0.000)
1936/2–1946/1	[0.01, 2.52]	7	53.50 (0.000)	[2.48, 10.93]	8	229.01 (0.000)
1946/2–1956/1	[0.00, 1.78]	4	35.47 (0.000)	[2.77, 7.90]	5	201.88 (0.000)
1956/2–1966/1	[0.00, 0.42]	2	22.69 (0.001)	[2.62, 3.97]	1	178.94 (0.000)
1966/2–1976/1	[0.00, 1.11]	2	33.25 (0.000)	[2.62, 9.45]	6	199.14 (0.000)
1976/2–1986/1	[0.00, 0.19]	1	24.59 (0.000)	[2.41, 4.27]	1	178.14 (0.001)
Panal B. Return Normality						
1926/2–1936/1	[0.02, 2.97]	9	82.18 (0.000)	[5.14, 11.34]	13	286.31 (0.000)
1936/2–1946/1	[0.02, 1.12]	6	71.91 (0.000)	[5.26, 8.57]	13	272.16 (0.000)
1946/2–1956/1	[0.00, 0.64]	2	42.60 (0.000)	[2.55, 4.34]	2	229.99 (0.000)
1956/2–1966/1	[0.00, 0.33]	4	30.10 (0.000)	[2.83, 4.99]	1	207.58 (0.000)
1966/2–1976/1	[0.00, 0.43]	2	48.49 (0.000)	[2.78, 5.30]	6	239.53 (0.000)
1976/2–1986/1	[0.00, 0.17]	0	30.99 (0.000)	[3.07, 3.95]	0	207.60 (0.000)

multivariate t and the multivariate mixture-normal distributions as the alternative distributions to the multivariate normal. We want to find the appropriate degrees of freedom that allow the two alternative distributions to best fit the residuals or the returns. Consider first the residuals. By trial and error, we obtain $\nu = 8$ for the multivariate t, and $\gamma = 2$, 3, and 5 for the mixture-normal distribution (ε is fixed at 50 percent). The first panel of Table IV provides the results testing the hypothesis that the residuals follow a given distribution of the alternatives. Because elliptical distributions have zero population skewness, one would expect that the chosen alternatives are perhaps plausible judged by the kurtosis test and not so by the skewness test. However, it is striking that the variations of the sample skewness are large and match those from the data, perhaps due to the difficulty of measuring the third moments. Notice that, to fit the data, we have made three choices of the gamma parameter of the mixture-normal distribution and only one choice of

Table IV

Alternative Residual and Return Distributions

The residuals are from the market model

$$r_{it} = \alpha_i + \beta_i r_{pt} + \varepsilon_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T; \quad \varepsilon_{it} \text{ is i.i.d. over time;}$$

where r_{it} and r_{pt} are the excess returns on the ith industry portfolio and the CRSP value-weighted index, $N = 12$ the total number of industries and $T = 120$ the sample size for each of the ten-year periods. In the first panel of the table, we test whether the residuals follow a multivariate t distribution or a multivariate mixture-normal distribution with the degrees of freedom ν and γ as specified in the second and fifth columns. The third and fourth columns report the p-values obtained by using the multivariate skewness and kurtosis tests for the hypothesis that the residuals are multivariate t. The last two columns report the same p-values but for the hypothesis that the residuals are multivariate mixture normal. In the second panel of the table, we report the results of testing whether the returns follow a multivariate t distribution or a multivariate mixture normal distribution by using the same procedure.

Period	ν	S-test	K-test	γ	S-test	K-test
Panel A. Residual Distributions						
1926/2–1936/1	8	0.003	0.001	5	0.030	0.620
1936/2–1946/1	8	0.409	0.824	5	0.020	0.679
1946/2–1956/1	8	0.008	0.001	3	0.020	0.980
1956/2–1966/1	8	0.773	0.879	2	0.440	0.860
1966/2–1976/1	8	0.097	0.054	3	0.090	0.411
1976/2–1986/1	8	0.722	0.877	2	0.160	0.980
Panel B. Return Distributions						
1926/2–1936/1	5	0.254	0.747	8	0.014	0.822
1936/2–1946/1	5	0.409	0.824	8	0.054	0.184
1946/2–1956/1	7	0.703	0.243	4	0.434	0.029
1956/2–1966/1	7	0.997	0.001	3	0.302	0.113
1966/2–1976/1	7	0.472	0.655	4	0.085	0.409
1976/2–1986/1	7	0.995	0.001	3	0.210	0.112

the degree of freedom of the multivariate t. This is due to the fact that, with a given gamma, the variations of the sample skewness and kurtosis are relatively small compared to those of the multivariate t distribution. Some of the p-values in 1926/2–1936/1 and 1946/2–1956/1 periods are small, but almost all others are greater than 5 percent and are substantially large in most cases. Hence, the two alternative distributions with the chosen degrees of freedom are very good candidates for the distribution of the residuals. In contrast, all the p-values computed by imposing the normality assumption are virtually zero. Consider now the returns. Similar to the residual case, we find the alternative distributions are good candidates for the distribution of the (excess) returns if $\nu = 5$ and $\nu = 7$, and $\gamma = 3$, 4, and 8. The second panel of Table IV provides the results supporting the choices of the degrees of freedom.

With the chosen alternative distributions of the residuals we examine now the mean-variance efficiency of the CRSP value-weighted index. Under the normality assumptions, Gibbons, Ross, and Shanken's (1989) exact test is reported in Table V as GRS which has an F distribution with degrees of freedom 12 and 107. The p-value is reported as P_0^*, computed analytically from the F distribution. Efficiency is rejected in three of the six subperiods at the 5 percent level. To assess the accuracy of our proposed numerical approach, the same p-value is also computed numerically and reported as P_0 in the third column of the table. A comparison of P_0^* and P_0 indicates the anticipated accuracy. The numerical errors are all below 1 percent and in some cases the results agree with one another up to 3 decimal points. Under the alternative distributions, the p-values are reported as P_1 and P_2 com-

Table V

Test of Efficiency of the CRSP Value-weighted Index under Alternative Distributions

The efficiency is examined by using the market model

$$r_{it} = \alpha_i + \beta_i r_{pt} + \varepsilon_{it}, \quad i = 1, \ldots, N, \quad \varepsilon_{it} \text{ is i.i.d over time;}$$

where r_{it} is the excess return on the ith industry portfolio, r_{pt} the excess return on the CRSP value-weighted index and $N = 12$ is the number of industries. The data are monthly returns and the sample size it $T = 120$ for each of the ten-year periods. The efficiency implies the following hypothesis to be tested:

$$H_0: \quad \alpha_i = 0, \quad i = 1, \ldots, N.$$

We test H_0 under the usual normality assumption and two alternative distributional assumptions, multivariate t, and multivariate mixture normal, for both the residuals and reutrns. In the table, GRS is the Gibbons, Ross, and Shanken (1989) test statistic. P_0^* is the p-value under normality assumption which is computed analytically by using an F distribution with degrees of freedom 12 and 107. P_0 is the same p-value but computed by using our proposed numerical approach. P_1 and P_2 are the p-values when the residuals or returns follow the alternative distributions.

Period	GRS	P_0^*	P_0	P_1	P_2
Panel A. Alternative Residual Distributions					
1926/2–1936/1	1.574	0.110	0.110	0.107	0.104
1936/2–1946/1	1.195	0.296	0.295	0.298	0.296
1946/2–1956/1	2.370	0.010	0.010	0.009	0.010
1956/2–1966/1	2.790	0.002	0.003	0.002	0.003
1966/2–1976/1	3.378	0.000	0.000	0.000	0.000
1976/2–1986/1	1.544	0.120	0.121	0.118	0.111
Panel B. Alternative Return Distributions					
1926/2–1936/1	1.574	0.110	0.110	0.210	0.177
1936/2–1946/1	1.195	0.296	0.295	0.315	0.399
1946/2–1956/1	2.370	0.010	0.010	0.103	0.241
1956/2–1966/1	2.790	0.002	0.003	0.068	0.110
1966/2–1976/1	3.378	0.000	0.000	0.036	0.062
1976/2–1986/1	1.544	0.120	0.121	0.216	0.191

puted by the proposed approach. They are strikingly close to those p-values obtained under the normality assumption. In almost all cases, the differences are no more than 1 percent, suggesting that alternative distributional assumptions on the model residuals matter little.[6] This is also what one may expect from Table I. Therefore, with the given data, conclusions reached under the multivariate normality assumption remain valid under the alternative distributions. However, as demonstrated below, the test will no longer be robust when the index returns are treated as random.

Under the chosen alternative distributions of the excess returns, the p-values are reported in the second panel of Table V. They are larger than those obtained under the normality assumption. Furthermore, except for a value of 3.5 percent, all the p-values are greater 5 percent. Therefore, the normality assumption tends to results in lower p-values. Under the normality assumption, we reject efficiency in three of the six periods. However, under the alternative distributional assumptions, we can no longer reject efficiency in all of the six periods.

To obtain some intuition on why the p-values are higher under the alternative distributions, consider the geometric interpretation of the GRS test. Gibbons, Ross, and Shanken (1989) show that:

$$\hat{\theta}^{*2} = \hat{\theta}_p^2 + \hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}, \tag{13}$$

where $\hat{\theta}^*$ is the Sharpe measure of the ex post efficient portfolio (ratio of the expected excess return to the standard deviation of the excess return), and $\hat{\theta}_p$ the Sharpe measure of the given portfolio. It follows that the expectation of $\hat{\theta}^{*2}$ is asymptotically determined by the righthand side with $\hat{\boldsymbol{\Sigma}}^{-1}$ replaced by its limit. In the case where the excess returns have greater kurtosis than the normal, i.e., $\Delta_2 > N(N + 2)$, we have $\kappa > 0$. Similar to the beta estimate, there will be more variations of the alpha estimate, yielding greater variance for $\hat{\theta}^*$. Intuitively, this says that we are more uncertain about the ex post efficient portfolio being ex ante efficient. Now, by Gibbons, Ross, and Shanken (1989),

$$W_T = c\left[(1 + \hat{\theta}^{*2})/\left(1 + \hat{\theta}_p^2\right) - 1\right], \tag{14}$$

where $c = (T - N - 1)/N$. As it is the ex ante efficiency of p that is being tested, we ideally want to compare p with the ex ante efficient portfolio. Because the ex ante efficient portfolio is unobservable, (14) states that we compare the sample Sharpe measure of p with that of the ex post efficient portfolio. If the difference is large (W_T or the GRS statistic is large), we reject efficiency. However, there will always be sample variations between the Sharpe measures even if p is ante efficient. The p-value of the test reflects the probability of observing a given level of the variations. With normality assumption, the p-value is computed from the F distribution. With multivari-

[6] A related robust result (see Fang and Zhang (1990)) is that, if the residuals are not i.i.d., but jointly (across assets and time) elliptical, the p-values will be *exactly* equal to those obtained under the usual normality assumption.

ate t assumption, we expect more variations of the ex post efficient portfolio and so there is greater probability of observing a given level of the variations. This says that, if the excess returns are i.i.d. elliptical and have greater kurtosis than the normal, the p-value of the GRS test tends to be greater than that obtained under the normality assumption. As it is usually the case that stock returns have higher kurtosis than those from the normal distribution, empirical studies which ignore the nonnormality of the returns are likely to overreject the theory being tested (assuming the returns are elliptically distributed).[7]

In summary, testing the mean-variance efficiency of the CRSP value-weighted index yields very similar results under plausible alternative distributional assumptions on the model residuals (treating the index returns as constants), but not so on the returns (treating the index returns as random variables). Under normality assumption, the efficiency of the index is rejected in half of the periods at the 5 percent level. However, under alternative assumptions on the returns, the efficiency can no longer be rejected.

IV. Conclusions

In this paper, we propose exact tests for many asset-pricing models under the theoretically consistent assumption that the returns are independent, identical, and elliptically distributed. Since almost all empirical work in finance is conducted under the multivariate normality assumption, our results tell exactly how the usual conclusions may be affected by a violation of the normality assumption. To examine further the robustness of many commonly used tests, we also provide exact inference for alternative distributional assumptions on the model residuals. In addition, we provide measures of multivariate skewness and kurtosis and apply them to examine the ellipticity of stock returns and the market model residuals. Using monthly industry returns for every consecutive ten-year period from 1926 to 1986, we find strong evidence against the usual multivariate normality assumption. We also test the mean-variance efficiency of the CRSP value-weighted index. Under the usual normality assumption, we reject the efficiency for half of the periods, but the efficiency can no longer be rejected under plausible alternative assumptions on the stock returns. Our results suggest that, if the returns are elliptically distributed, empirical studies that ignore the nonnormality are likely to overreject the theory being tested, but the proposed approach can be used to detect the magnitude of the overrejection.

Appendix A

In this appendix, we prove that under the null hypothesis the GRS statistic is invariant to nonsingular linear transformation of the model residuals and

[7] If the returns are not elliptically distributed, we will be unable to tell whether there will be overrejection or underrejection of the null hypothesis because the true p-value is unknown. What remains true is that tests based on the normality assumption tend to overreject the null as compared with tests based on the ellipticity assumption.

hence it can be computed once samples from the residuals are drawn. To simplify the presentation, we rewrite the market model in matrix form:

$$\mathbf{R} = \mathbf{R}_p\mathbf{B} + \mathbf{E}, \tag{A1}$$

where $\mathbf{R}$ is a T (observations) by N (assets) matrix of returns, $\mathbf{R}_p$ is a T by 2 matrix with the first column a vector of ones and the second the portfolio returns, $\mathbf{B}$ is a 2 by N coefficient matrix with the N alphas, $\boldsymbol{\alpha}$, in the first row and the N betas, $\boldsymbol{\beta}$, in the second and $\mathbf{E}$ is the T by N model residual matrix.

We want to prove that W_T is invariant if $\mathbf{E}$ is replaced by $\mathbf{EC}$, where $\mathbf{C}$ is any $N \times N$ nonsingular matrix. To do so, let $\boldsymbol{\Pi}$ be the following 2×2 matrix:

$$\boldsymbol{\Pi} \equiv (\hat{\mathbf{B}} - \mathbf{B})\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\mathbf{B}} - \mathbf{B})', \tag{A2}$$

where $\hat{\mathbf{B}}$ and $\hat{\boldsymbol{\Sigma}}$ are the ML estimators of $\mathbf{B}$ and $\boldsymbol{\Sigma}$:

$$\hat{\mathbf{B}} = (\mathbf{R}'_p\mathbf{R}_p)^{-1}\mathbf{R}'_p\mathbf{R} = \mathbf{B} + (\mathbf{R}'_p\mathbf{R}_p)^{-1}\mathbf{R}'_p\mathbf{E}, \tag{A3}$$

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\left(\mathbf{R} - \mathbf{R}_p\hat{\mathbf{B}}\right)'\left(\mathbf{R} - \mathbf{R}_p\hat{\mathbf{B}}\right) = \mathbf{E}'\mathbf{ME} \tag{A4}$$

with $\mathbf{M} \equiv \mathbf{I}_T - \mathbf{R}_p(\mathbf{R}'_p\mathbf{R}_p)^{-1}\mathbf{R}'_p$. Since $\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}$ is the (1, 1) element of $\boldsymbol{\Pi}$ under the null, it is sufficient to show that $\boldsymbol{\Pi}$ is invariant. Indeed, if $\mathbf{E}$ is replaced by $\mathbf{EC}$, the inverse of the covariance estimator is

$$\hat{\boldsymbol{\Sigma}}^{-1} = (\mathbf{C}'\mathbf{E}'\mathbf{MEC})^{-1} = \mathbf{C}^{-1}(\mathbf{E}'\mathbf{ME})^{-1}\mathbf{C}^{-1\prime}$$

and the coefficients estimator minus the true parameter is

$$\hat{\mathbf{B}} - \mathbf{B} = (\mathbf{R}'_p\mathbf{R}_p)^{-1}\mathbf{R}'_p\mathbf{EC}.$$

It follows then that $\boldsymbol{\Pi}$ remains unchanged after the transformation. Finally, by (A3) and (A4), we know that, under the null, the GRS statistic is a function of the residuals alone. If samples from the residuals are drawn, the GRS statistic can be computed. Q.E.D.

Appendix B

In this appendix, we provide the methods for drawing samples from multivariate t, mixture normal, and Kotz distributions. Because $\mathbf{Y} = \boldsymbol{\theta} + \mathbf{LX}$ will have arbitrary parameters $\boldsymbol{\theta}$ and $\Sigma = \mathbf{LL}'$, it is enough to show only how to generate $\mathbf{X}$ where the parameters $\boldsymbol{\theta} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}_N$.

A. Drawing Samples from a Multivariate t Distribution

We draw $Z_i \sim N(0, 1)$ for $i = 1, \ldots, N$, and $c \sim X_\nu^2$, where c is independent of $\mathbf{Z}$, then, by Fang, Kotz, and Ng (1990, p. 85), $\mathbf{X} = \mathbf{Z}/(c/\nu)^{1/2}$ follows a multivariate t distribution with degree of freedom ν.

B. Drawing Samples from a Mixture-Normal Distribution

We draw U from a uniform distribution over [0, 1], and $\mathbf{X}$ from $N(\mathbf{0}, \mathbf{I}_N)$. If $U < (1 - \varepsilon)$, done. Otherwise, let $\mathbf{X} = \sqrt{\gamma}\mathbf{X}$. Then, from Johnson (1987, p. 56), the resulting $\mathbf{X}$ has a mixture-normal distribution.

C. Drawing Samples from a Kotz Distribution

We draw $\mathbf{U}$ from an N-uniform distribution, $U_i = Z_i/(Z_1^2 + \cdots + Z_N^2)^{1/2}$, where $Z_i \sim N(0,1), i = N$, and g^2 from a gamma distribution $\Gamma(\nu + N/2 - 1)$, then, by Fang, Kotz, and Ng (1990, pp. 76, 77) and Johnson (1987, p. 127), $\mathbf{X} = r^{-1/2}g\mathbf{U}$, $r = 1/2$, follows a Kotz distribution with degree of freedom ν.

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