A More Detailed and Complete Appendix for
“Macroeconomic Volatilities and Long-run Risks of Asset Prices”

This is an on-line appendix with more details and analysis for the readers of the paper.

B.1 Derivation for the $A_i$’s, risk-free rate and market price of risk

First, we re-write the normalized aggregator $f$ defined in Equation (5) as

$$f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}} (1 - \gamma) J [G - 1],$$

where

$$G \equiv \left(\frac{C}{(1 - \gamma) J}\right)^{1 - \frac{1}{\psi}}. \quad (B1)$$

Then, taking partial derivatives of $f(C, J)$ with respect to $J$ and $C$, we have

$$f_J = (\theta - 1) \beta G - \beta \theta \quad (B2)$$

and

$$f_C = \beta G (1 - \gamma) J. \quad (B3)$$

where we use the notation $\theta = \frac{1 - \gamma}{1 - \frac{1}{\psi}}$. Theoretically, the aggregator $f(C, J)$ should be an increasing function of the value function $J$ (see, e.g., Skiadas, 2009, Chapter 6.3). Otherwise, the monotonicity axiom of preferences will be violated. This places joint restrictions on $\gamma$ and $\psi$ such that $\theta \geq 1$ or $\theta < 0$. This is because $f_J > 0$ implies that

If $\theta > 1$: $G > \frac{\theta}{\theta - 1}$

If $\theta < 1$: $G < \frac{\theta}{\theta - 1}$.

If $\theta > 1$, the first inequality is possible to have solutions. However, if $0 < \theta < 1$, the second inequality is impossible as $G > 0$ always. Hence, the necessary restriction on $\gamma$ and $\psi$ is either $\theta > 1$ or $\theta < 0$. If $\theta = 1$, as shown by Duffie and Epstein (1992), we obtain the standard additive expected utility of constant relative risk aversion (CRRA). So $\theta > 1$ can be extended to $\theta \geq 1$. 

1
Conjecturing a solution for $J$ of the following form,

$$
J(W_t, X_t, V_{1t}, V_{2t}) = \exp(A_0 + A_1X_t + A_2V_{1t} + A_3V_{2t}) \frac{W_t^{1-\gamma}}{1-\gamma},
$$  \tag{B4}

and using the standard envelope condition $f_C = J_W$, we have

$$
C = J_W^{-\psi}(1-\gamma)J^{\frac{1-\gamma}{1-\psi}}. \tag{B5}
$$

Substituting (B3) and (B4) into (B5), we obtain

$$
\frac{C}{W} = \beta^\psi \exp\left[(A_0 + A_1X_t + A_2V_{1t} + A_3V_{2t}) \frac{1-\psi}{1-\gamma}\right]. \tag{B6}
$$

and hence $J$ can be re-written as

$$
J(C_t, X_t, V_{1t}, V_{2t}) = \beta^{-\psi(1-\gamma)} \exp[\psi(A_0 + A_1X_t + A_2V_{1t} + A_3V_{2t})] C_t^{1-\gamma} \frac{1}{1-\gamma}. \tag{B7}
$$

Further substituting (B6) and (B4) into (B1), we get

$$
\beta G = \frac{C_t}{W_t}.
$$

Applying the log-linear approximation, we obtain

$$
\beta G = \frac{C_t}{W_t} \approx g_1 - g_1 \log g_1 + g_1 \log(\beta G). \tag{B8}
$$

This implies that

$$
f = \theta J(\beta G - \beta) \approx \theta J \left[ g_1 \frac{1-\psi}{1-\gamma} (A_0 + A_1X_t + A_2V_{1t} + A_3V_{2t}) + \xi \right], \tag{B9}
$$

where $\theta = \frac{1-\gamma}{1-\psi}$ and

$$
\xi = g_1 - g_1 \log g_1 + g_1 \psi \log \beta - \beta. \tag{B10}
$$

Substituting (B9) into the HJB Equation (6),

$$
f(C, J) + C \cdot (\mu + X) J_C + \frac{1}{2} [\delta_c V_1 + (1 - \delta_c) V_2] C^2 J_{CC} + J_X \cdot (-\alpha X) + \frac{1}{2} \varphi_x^2 [\delta_x V_1 + (1 - \delta_x) V_2] J_{XX} + J_{V_1} \cdot \kappa_1 (\tilde{V}_1 - V_1) + \frac{1}{2} \sigma_v^2 J_{V_1V_1} + J_{V_2} \cdot \kappa_2 (\tilde{V}_2 - V_2) + \frac{1}{2} \sigma_v^2 J_{V_2V_2} = 0, \tag{B11}
$$

where $\{C_t\}$ is the optimal consumption process, and we have used the definition of

$$
\mathcal{A}^c J = \sum_i b(z) \frac{\partial J(z)}{\partial z} + \sum_{i,j} (\sigma \sigma^T)_{i,j}(z) \frac{\partial^2 J}{\partial z_i \partial z_j}.
$$
with \( z = (C, X, V_1, V_2) \) and \( b(z) \) and \( \sigma(z) \) the drift and diffusive terms for \( z \) defined in Equation (2). Collecting the terms containing constant, \( X_t, V_{1t} \) and \( V_{2t}, \) resp, we have

\[
\theta g_1 \frac{1 - \psi}{1 - \gamma} A_0 + \theta \xi + (1 - \gamma) \mu + \kappa_1 V_1 \psi A_2 + \kappa_2 V_2 \psi A_3 = 0
\]
\[
X : \quad \theta g_1 \frac{1 - \psi}{1 - \gamma} A_1 + (1 - \gamma) - \alpha \psi A_1 = 0
\]
\[
V_1 : \quad \theta g_1 \frac{1 - \psi}{1 - \gamma} A_2 - \frac{1}{2} \gamma (1 - \gamma) \delta c + \frac{1}{2} \phi_x^2 \delta_x \psi^2 A_1^2 - \kappa_1 \psi A_2 + \frac{1}{2} \sigma_1^2 \psi^2 A_2^2 = 0
\]
\[
V_2 : \quad \theta g_1 \frac{1 - \psi}{1 - \gamma} A_3 - \frac{1}{2} \gamma (1 - \gamma) (1 - \delta c) + \frac{1}{2} \phi_x^2 (1 - \delta_x) \psi^2 A_1^2 - \kappa_2 \psi A_3 + \frac{1}{2} \sigma_2^2 \psi^2 A_3^2 = 0.
\]

Solving the above algebraic equations, we obtain

\[
A_0 = \frac{1}{g_1 \psi} [\theta \xi + (1 - \gamma) \mu + \kappa_1 V_1 \psi A_2 + \kappa_2 V_2 \psi A_3],
\]
\[
A_1 = \frac{1 - \gamma}{(g_1 + \alpha) \psi},
\]
\[
A_2 = \frac{-b_1 - \sqrt{b_1^2 - 4a_1 c_1}}{2a_1},
\]
\[
A_3 = \frac{-b_2 - \sqrt{b_2^2 - 4a_2 c_2}}{2a_2},
\]

with

\[
a_1 = \frac{1}{2} \sigma_1^2 \psi^2, \quad b_1 = -(g_1 + \kappa_1) \psi, \quad c_1 = -\frac{1}{2} \gamma (1 - \gamma) \delta c + \frac{1}{2} \phi_x^2 \delta_x \frac{(1 - \gamma)^2}{(g_1 + \alpha)^2},
\]
\[
a_2 = \frac{1}{2} \sigma_2^2 \psi^2, \quad b_2 = -(g_1 + \kappa_2) \psi, \quad c_2 = -\frac{1}{2} \gamma (1 - \gamma) \delta c (1 - \delta c) + \frac{1}{2} \phi_x^2 (1 - \delta_x) \frac{(1 - \gamma)^2}{(g_1 + \alpha)^2}.
\]

We then derive the risk-free rate and market prices of risks. Recall that the pricing kernel is given by Equation (A6). Based on the definition for \( f, \) we have

\[
f_J = \xi_1 - g_1 (A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{1 - \gamma \psi}{1 - \gamma},
\]
\[
f_C = \beta^{\psi \gamma} \exp \left[ (B + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{1 - \gamma \psi}{1 - \gamma} \right] C_t^{-\frac{\gamma}{1 - \gamma}},
\]

where

\[
\xi_1 = (\theta - 1) \xi - \beta - g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_0.
\]

Applying Ito’s Lemma to \( \pi_t \) in Equation (A6), we have

\[
\frac{d \pi_t}{\pi_t} = -(r_f dt + \lambda_1 dZ_{1t} + \lambda_2 dZ_{2t} + \lambda_3 dw_{1t} + \lambda_4 dw_{2t}),
\]

where the risk-free rate \( r_f \) and the market prices of risks, \( \lambda_i, i = 1, 2, 3, 4, \) are given below.
First, the risk-free rate is

\[ r_f = r_0 + r_1 X_t + r_2 V_{1t} + r_3 V_{2t}, \quad (B15) \]

where

\[ r_0 = -(\xi_1 + (\kappa_1 A_2 V_1 + \kappa_2 A_3 V_2) \frac{1 - \gamma \psi}{1 - \gamma} - \gamma \mu), \]

\[ r_1 = \frac{1}{\psi}, \]

\[ r_2 = (g_1 + \kappa_1) A_2 \frac{1 - \gamma \psi}{1 - \gamma} - \frac{1}{2} \left( \frac{1 - \gamma \psi}{1 - \gamma} \right)^2 (A_1^2 \varphi_x^2 \delta_x + A_2^2 \sigma_1^2) - \frac{1}{2} \gamma (\gamma + 1) \delta_c, \quad (B16) \]

\[ r_3 = (g_1 + \kappa_2) A_3 \frac{1 - \gamma \psi}{1 - \gamma} - \frac{1}{2} \left( \frac{1 - \gamma \psi}{1 - \gamma} \right)^2 [A_1^2 \varphi_x^2 (1 - \delta_x) + A_3^2 \sigma_2^2] - \frac{1}{2} \gamma (\gamma + 1) (1 - \delta_c). \]

Second, the market prices of risks are

\[ \lambda_1 = \gamma \sqrt{V_{1t} \delta_c + V_{2t}(1 - \delta_c)}, \]

\[ \lambda_2 = -\frac{1 - \gamma \psi}{1 - \gamma} A_1 \varphi_x \sqrt{V_{1t} \delta_x + V_{2t}(1 - \delta_x)}, \]

\[ \lambda_3 = -\frac{1 - \gamma \psi}{1 - \gamma} A_2 \sigma_1 \sqrt{V_{1t}}, \]

\[ \lambda_4 = -\frac{1 - \gamma \psi}{1 - \gamma} A_3 \sigma_2 \sqrt{V_{2t}}. \quad (B17) \]

Q.E.D.

**B.2 Derivation for the** $A_{im}$’s

Let

\[ \frac{D_t}{P_t} = \exp \{ (A_{0m} + A_{1m} X_t + A_{2m} V_{1t} + A_{3m} V_{2t}) \}. \quad (B18) \]

A key step in the derivation is to use the following pricing relation given in

\[ E_t \left( \frac{dP_t}{P_t} \right) + \frac{D_t}{P_t} dt = r_f dt - E_t \left[ \frac{d\pi_t}{\pi_t} \frac{dP_t}{P_t} \right]. \quad (B19) \]

With similar loglinear approximation as Equation (B8), we can approximate the ratio as

\[ \frac{D_t}{P_t} \approx g_{0m} + g_{1m} \log \frac{D_t}{P_t} = g_{0m} + g_{1m} ((A_{0m} + A_{1m} X_t + A_{2m} V_{1t} + A_{3m} V_{2t})), \quad (B20) \]

where

\[ g_{0m} = g_{1m} - g_{1m} \log g_{1m}. \]
Applying Ito’s lemma to (B18), we have
\[
\frac{dP_t}{P_t} = \frac{dD_t}{D_t} - (A_{1m} dX_t + A_{2m} dV_{1t} + A_{3m} dV_{2t}) + \frac{1}{2} A_{1m}^2 (dX_t)^2 + \frac{1}{2} A_{2m}^2 (dV_{1t})^2 + \frac{1}{2} A_{3m}^2 (dV_{2t})^2.
\]

Hence,
\[
E_t(\frac{dP_t}{P_t}) = dt = \mu_d + \phi X_t + \alpha A_{1m} X_t - \kappa_1 A_{2m} (\bar{V}_1 - V_{1t}) - \kappa_2 A_{3m} (\bar{V}_2 - V_{2t})
\]
\[+ \frac{1}{2} A_{1m}^2 \varphi_x^2 [V_{1t} \delta_x + V_{2t} (1 - \delta_x)] + \frac{1}{2} A_{2m}^2 \sigma_1^2 V_{1t} + \frac{1}{2} A_{3m}^2 \sigma_2^2 V_{2t}. \tag{B21}
\]

The risk premium term in Equation (B19) can thus be written as
\[
-E_t \left[ \frac{d\pi_t}{\pi_t} \frac{dP_t}{P_t} \right] / dt = \sigma_{dc} \lambda_1 \sqrt{V_{1t} \delta_c} + V_{2t} (1 - \delta_c) - (A_{1m} \varphi_x - \sigma_{dx}) \lambda_2 \sqrt{V_{1t} \delta_x + V_{2t} (1 - \delta_x)}
\]
\[- (A_{2m} \sigma_1 - \sigma_{dc}) \lambda_3 \sqrt{V_{1t}} - (A_{3m} \sigma_2 - \sigma_{dc}) \lambda_4 \sqrt{V_{2t}}, \tag{B22}
\]

where \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are market prices of risks defined in Equation (B17).

Now, substituting (B20), (B21), (B22), and risk-free rate (B15) into Equation (B19), and collecting terms containing \( X_t \), we obtain
\[
A_{1m} = - \frac{\phi - \frac{1}{\varphi}}{g_{1m} + \alpha}. \tag{B23}
\]

Collecting terms containing \( V_{1t} \) and \( V_{2t} \) resp, we obtain an equation for \( A_{2m} \),
\[
a_{2m} A_{2m}^2 + b_{2m} A_{2m} + c_{2m} = 0
\]
with
\[
a_{2m} = \frac{1}{2} \sigma_1^2, \quad b_{2m} = g_{1m} + \kappa_1 - \frac{1 - \gamma \psi}{1 - \gamma} A_2 \sigma_1^2, \quad c_{2m} = (\frac{1}{2} A_{1m}^2 - \frac{1 - \gamma \psi}{1 - \gamma} A_1 A_{1m}) \varphi_x^2 \delta_x + r_2.
\]

Solving it, we have
\[
A_{2m} = -\frac{b_{2m} \pm \sqrt{b_{2m}^2 - 4a_{2m}c_{2m}}}{2a_{2m}}. \tag{B24}
\]

We choose the root that goes to zero when \( \sigma_1 \) goes to zero. This is because when \( \sigma_1 \), or \( a_{2m} \) goes to zero, the price sensitivity to \( V_1 \) should be zero.

Similarly, we obtain an equation for \( A_{3m} \),
\[
a_{3m} A_{3m}^2 + b_{3m} A_{3m} + c_{3m} = 0
\]
with
\[ a_{3m} = \frac{1}{2} \sigma^2, \quad b_{2m} = g_{1m} + \kappa_2 - \frac{1 - \gamma \psi}{1 - \gamma} A_3 \sigma^2, \quad c_{3m} = \left( \frac{1}{2} A_{1m}^2 - \frac{1 - \gamma \psi}{1 - \gamma} A_1 A_{1m} \right) \varphi_x^2 (1 - \delta_x) + r_3. \]

The solution is
\[ A_{3m} = \frac{-b_{3m} \pm \sqrt{b_{3m}^2 - 4a_{3m}c_{3m}}}{2a_{3m}}, \tag{B25} \]
where we choose the root in a similar fashion as for \( A_{2m} \) above.

Finally, collecting the constant terms in Equation (B19), we obtain
\[ \mu_d - \kappa_1 A_{2m} \bar{V}_1 - \kappa_2 A_{3m} \bar{V}_2 + g_{0m} + g_{1m} A_{0m} + r_0 = 0, \]
and re-arrange terms to get
\[ A_{0m} = -\frac{1}{g_{1m}} \left[ \mu_d - \kappa_1 A_{2m} \bar{V}_1 - \kappa_2 A_{3m} \bar{V}_2 + g_{1m} - g_{1m} \log g_{1m} + r_0 \right]. \]
So far, we obtain all the \( A_{im} \) coefficients.

To obtain the market return volatility, we apply Ito’s Lemma to Equation (B18) and obtain
\[
\frac{dP_t}{P_t} = \left[ \mu_d - (A_{2m} \kappa_1 \bar{V}_1 + A_{3m} \kappa_2 \bar{V}_2) + (\phi + \alpha A_{1m}) X_t \right. \\
\left. + \frac{1}{2} A_{1m}^2 \varphi_x^2 \delta_x + \frac{1}{2} A_{2m} \sigma_1^2 + A_{2m} \kappa_1 - A_{1m} \sigma_{dx} \varphi_x \delta_x - A_{2m} \sigma_1 \sigma_{dv} \right] \bar{V}_t \\
+ \frac{1}{2} A_{1m}^2 \varphi_x^2 (1 - \delta_x) + \frac{1}{2} A_{3m} \sigma_2^2 + A_{3m} \kappa_2 - A_{1m} \sigma_{dx} \varphi_x (1 - \delta_x) - A_{3m} \sigma_2 \sigma_{dv} \bar{V}_t \right] dt \\
+ \varphi_d \sqrt{V_1} \delta_d + V_2 (1 - \delta_d) dB_t + \sigma_{dc} \sqrt{V_1} \delta_c + V_2 (1 - \delta_c) dZ_{1t} \\
+ \sigma_{dx} - A_{1m} \varphi_x \sqrt{V_1} \delta_x + V_2 (1 - \delta_x) dZ_{2t} \\
+ \sigma_{dv} - A_{2m} \sigma_1 \sqrt{V_1} \delta_1 + (\sigma_{dv} - A_{2m} \sigma_2) \sqrt{V_2} dW_{2t} \\
= \left[ c_3 + c_4 X_t + c_5 V_1 + c_6 V_2 \right] dt + \sqrt{c_1 V_1 + c_2 V_2} dZ_t,
\]
where \( c_i \) (\( i = 1 \) to 6) are constants, \( dZ_t \) is a new Brownian motion defined accordingly, and hence the variance of the price process is
\[ V_t = c_1 V_1 + c_2 V_2, \]
with
\[
c_1 = \varphi_d^2 \delta_d + \sigma_{dc}^2 \delta_c + (\sigma_{dx} - A_{1m} \varphi_x)^2 \delta_x + (\sigma_{dv} - A_{2m} \sigma_1)^2, \\
c_2 = \varphi_d^2 (1 - \delta_d) + \sigma_{dc}^2 (1 - \delta_c) + (\sigma_{dx} - A_{1m} \varphi_x)^2 (1 - \delta_x) + (\sigma_{dv} - A_{2m} \sigma_2)^2. \tag{B26}
\]
and the parameters for the drift term are

\[ c_4 = \phi + \alpha A_{1m}, \]

\[ c_5 = \left( \frac{1}{2} A^2_{1m} \varphi^2 \delta_x + \frac{1}{2} A_{2m} \sigma_1^2 + A_{2m} \kappa_1 - A_{1m} \sigma_{dx} \varphi_x \delta_x - A_{2m} \sigma_1 \sigma_{dx} \right), \quad (B27) \]

\[ c_6 = \left( \frac{1}{2} A^2_{1m} \varphi^2 (1 - \delta_x) + \frac{1}{2} A_{3m} \sigma_2^2 + A_{3m} \kappa_2 - A_{1m} \sigma_{dx} \varphi_x (1 - \delta_x) - A_{3m} \sigma_2 \sigma_{dx} \right). \]

Q.E.D.

**B.3 Solutions to \( g_1 \) and \( g_{1m} \)**

Note that the derived solutions depend on the approximation constant \( g_1 \), which can be solved endogenously. Given the model parameters, we can compute the unconditional mean of consumption-wealth ratio as a function of the parameters,

\[ g_1 = E \left( \frac{C}{W} \right) = \beta^\psi \exp \left\{ A_{0a} \exp \left\{ \frac{1}{4} A^2_{1a} \varphi_x^2 \left( \frac{V_1 \delta_x + V_2 (1 - \delta_x)}{\alpha} \right) \right\} \right. \]

\[ \cdot \exp \left\{ - \frac{2 \kappa_1 V_1}{\sigma_1^2} \log \left( 1 - \frac{A_{2a}}{2 \kappa_1 / \sigma_1^2} \right) \right\} \cdot \exp \left\{ - \frac{2 \kappa_2 V_2}{\sigma_2^2} \log \left( 1 - \frac{A_{3a}}{2 \kappa_2 / \sigma_2^2} \right) \right\}. \quad (B28) \]

Note that the \( A_{ia} \)'s on the right hand side are also functions of \( g_1 \). Substituting \( A_{ia} \) as function of \( g_1 \) into Equation (B28), we obtain a nonlinear function in terms of \( g_1 \) only, and hence \( g_1 \) can be solved in terms of the fundamental parameters of the model, and can be computed numerically with many available algorithms.

Similarly, we can solve \( g_{1m} \) endogenously based on dividend-price ratio given as

\[ g_{1m} = E \left( \frac{D}{P} \right) = \exp \left\{ A_{0m} \exp \left\{ \frac{1}{4} A^2_{1m} \varphi_x^2 \left( \frac{V_1 \delta_x + V_2 (1 - \delta_x)}{\alpha} \right) \right\} \right. \]

\[ \cdot \exp \left\{ - \frac{2 \kappa_1 V_1}{\sigma_1^2} \log \left( 1 - \frac{A_{2m}}{2 \kappa_1 / \sigma_1^2} \right) \right\} \cdot \exp \left\{ - \frac{2 \kappa_2 V_2}{\sigma_2^2} \log \left( 1 - \frac{A_{3m}}{2 \kappa_2 / \sigma_2^2} \right) \right\}. \quad (B29) \]

This can be solved numerically as above. Q.E.D.

**B.4 Predictability of variables**

The regressors of the three regressions given in Equation (14)-(16) all have the generic functional form of

\[ dY_t = [a_0 + a_1 X_t + a_2 V_{1t} + a_3 V_{2t}] dt + \sqrt{b_1 V_{1t} + b_2 V_{2t}} dZ_t, \]
given in Equation (17) where \( dY_t \) corresponds to excess return \( d\ln P_t + \frac{D_t}{P_t} - r_f dt \), consumption growth \( d\ln C_t \) and dividend growth \( d\ln D_t \), respectively. For stock market excess return, we have

\[
\begin{align*}
a_1 &= c_4 + r_1 + g_{1m}A_{1m}, \\
a_2 &= c_5 - \frac{c_1}{2} + r_2 + g_{2m}A_{2m}, \\
a_3 &= c_6 - \frac{c_2}{2} + r_3 + g_{3m}A_{3m},
\end{align*}
\]  
(B30)

where \( c_1, c_2, c_4, c_5 \) and \( c_6 \) are defined in Equations (B26) and (B27).

For consumption growth, we have

\[
\begin{align*}
a_1 &= 1, \\
a_2 &= -\frac{\delta_c}{2}, \\
a_3 &= -\frac{1 - \delta_c}{2}.
\end{align*}
\]  
(B31)

For dividend growth, we have

\[
\begin{align*}
a_1 &= \varphi, \\
a_2 &= -\frac{\varphi^2 \sigma_d^2 \delta_d + \sigma_d^2 \delta_c + \sigma_c^2 \delta_c}{2}, \\
a_3 &= -\frac{\varphi^2 (1 - \delta_d) + \sigma_d^2 (1 - \delta_c) + \sigma_c^2 + \sigma_c^2 (1 - \delta_c)}{2}.
\end{align*}
\]  
(B32)

We want to show Equations (A13). Given Equations (B30) and (B18), and denoting \( \text{Cov}(x,y) \equiv <x,y> \), and \( pd_t \equiv p_t - d_t \), we have

\[
\begin{align*}
\left< \int_t^{t+\tau} dy_s, p_t - d_t \right> &= \int_t^{t+\tau} ds a_0 + a_1 X_s + a_2 V_{1s} + a_3 V_{2s}, pd_t \\
&= \int_t^{t+\tau} ds [a_1 <x_s, pd_t> + a_2 < V_{1s}, pd_t> + a_3 < V_{2s}, pd_t>] \\
&= -\int_0^\tau ds \left[ a_1 A_{1m} \frac{\sigma_x^2}{2\alpha} e^{-\alpha s} + a_2 A_{2m} \frac{\sigma_1^2 V_1}{2\kappa_1} e^{-\kappa_1 s} + a_3 A_{3m} \frac{\sigma_2^2 V_2}{2\kappa_2} e^{-\kappa_2 s} \right]
\end{align*}
\]

where

\[
\sigma_x^2 = \varphi_x^2 [V_1 \delta_x + V_2 (1 - \delta_x)].
\]

Integrating the above equation, we obtain Equation (A13), where

\[
\begin{align*}
\text{Cov}(\Delta r, p-d) &= -\left[ a_1 A_{1m} \frac{\sigma_x^2}{2\alpha} (1 - e^{-\alpha \tau}) + a_2 A_{2m} \frac{\sigma_1^2 V_1}{2\kappa_1} (1 - e^{-\kappa_1 \tau}), \\
&\quad \quad \quad + a_3 A_{3m} \frac{\sigma_2^2 V_2}{2\kappa_2} (1 - e^{-\kappa_2 \tau}) \right], \\
\text{Var}(p-d) &= A_{1m}^2 \frac{\sigma_x^2}{2\alpha} + A_{2m}^2 \frac{\sigma_1^2 V_1}{2\kappa_1} + A_{3m}^2 \frac{\sigma_2^2 V_2}{2\kappa_2},
\end{align*}
\]  
(B33)  
(B34)
with
\[ \sigma^2 = \varphi^2 \left[ V_1 \delta_x + V_2 (1 - \delta_x) \right] \]
which are the same given in the text. We have used the unconditional covariance:
\[ <X_t, X_x> = \frac{\sigma^2}{2\alpha} e^{-\alpha|t-s|} \]
\[ <V_{it}, V_{is}> = \frac{\sigma^2 V_i}{2\kappa_i} e^{-\kappa_i|t-s|} \]
for \( i = 1, 2 \).

Similar computation applies to obtain Equation (A17). Q.E.D.

### B.5 Predictability of volatilities

First, we prove Equation (A16). To do so, we apply the following approximation:
\[ \frac{1}{\tau} \int_0^\tau \exp(x_s)ds \approx \exp\left(\frac{1}{\tau} \int_0^\tau x_s ds\right) \tag{B35} \]
for any process \( x_s \). This is equivalent to approximating the arithmetic mean by the geometric mean. The approximation is good when the variation of \( x_t \) is small, which is true for our variance processes because the magnitude is generally in the order of \( 10^{-3} \sim 10^{-4} \), and the variation of \( \log V_t \) is within 1. Applying the approximation to \( \log V_t \), we have
\[ \frac{1}{\tau} \int_0^\tau \sqrt{V_t} dt = \frac{1}{\tau} \int_0^\tau \exp\left(\frac{1}{2} \ln V_t \right) dt \approx \exp\left(\frac{1}{2\tau} \int_0^\tau \ln V_t dt \right). \tag{B36} \]
Hence,
\[ \ln \frac{1}{\tau} \int_0^\tau \sqrt{V_t} dt \approx \frac{1}{2\tau} \int_0^\tau \ln V_t dt \]
\[ = \frac{1}{2\tau} \left[ \int_0^\tau \ln \bar{V} + \int_0^\tau \ln (1 + \frac{V_t - \bar{V}}{V}) dt \right] \]
\[ \approx \frac{1}{2} \ln \bar{V} + \frac{1}{2\tau} \int_0^\tau \frac{V_t - \bar{V}}{V} dt \]
\[ = Const + \frac{1}{2\tau V} \int_0^\tau V_t dt, \tag{B37} \]
which is Equation (A16).

Because of the approximation above, we can express the volatilities as an integral of \( b_1 V_{1s} + b_2 V_{2s} \) over \((t, t + \tau)\). Plugging these terms into the definition of the covariance, we then obtain Equation (A17).
Then we provide the derivation of the AR(1) coefficient. Consider a stochastic process of the form
\[ b_1 V_{1t} + b_2 V_{2t}, \]
where \( b_1 \) and \( b_2 \) are constants. Due to independence between \( V_1 \) and \( V_2 \), the unconditional auto-covariance can be evaluated as
\[ b_1^2 \sigma_1^2 V_1 \exp(-\kappa_1 \tau) + b_2^2 \sigma_2^2 V_2 \exp(-\kappa_2 \tau) \]
and the unconditional variance can be evaluated as
\[ b_1^2 \sigma_1^2 V_1^2 + b_2^2 \sigma_2^2 V_2^2. \]
Hence, the AR(1) coefficient can be computed easily based on above. Q.E.D.

\section*{B.6 Derivation of VRP}

We derive the time \( t \) expected future realized variance over time period \( \tau \) under the risk-neutral probability. The market prices of risk for \( V_{1t} \) and \( V_{2t} \) are \( \lambda_3 \) and \( \lambda_4 \) of Equation (B17), hence the risk premia associated with \( V_{1t} \) and \( V_{2t} \) are
\[ \lambda_3 \sigma_1 \sqrt{V_{1t}} = -\nu_1 V_{1t}, \quad \text{and} \quad \lambda_4 \sigma_2 \sqrt{V_{2t}} = -\nu_2 V_{2t}, \quad \text{(B38)} \]
where
\[ \nu_1 = \frac{1 - \gamma \psi}{1 - \gamma} A_2 \sigma_1^2, \quad \text{and} \quad \nu_2 = \frac{1 - \gamma \psi}{1 - \gamma} A_3 \sigma_2^2. \quad \text{(B39)} \]
Hence, the risk-neutral processes for \( V_{1t} \) and \( V_{2t} \) are
\[ dV_{1t} = \kappa_1^Q \left( \frac{\kappa_1^Q}{\kappa_1} V_1 - V_{1t} \right) dt + \sigma_1 \sqrt{V_{1t}} d\omega_{1t}^Q, \]
\[ dV_{2t} = \kappa_2^Q \left( \frac{\kappa_2^Q}{\kappa_2} V_2 - V_{2t} \right) dt + \sigma_2 \sqrt{V_{2t}} d\omega_{2t}^Q, \quad \text{(B40)} \]
where the risk-neutral mean-reversion coefficients for \( V_{it} \) are defined as
\[ \kappa_i^Q = \kappa_i - \nu_i \quad \text{(B41)} \]
for \( i = 1, 2 \). In order for well-defined risk-neutral processes in Equation (B40), we need to have \( \kappa_i^Q \)'s to be positive such that
\[ \nu_1 < \kappa_1 \quad \text{and} \quad \nu_2 < \kappa_2, \quad \text{(B42)} \]
Now we compute the squared VIX, or more generally, variance swap rate $VS_t$ with maturity $\tau_0$, defined as the risk neutral expectation of the variance. Because the risk-neutral process and the physical process of Equation are both Heston (1993) processes, we obtain Equation

$$VS_t = \sum_{i=1}^{2} c_i (A_i^Q + B_i^Q V_{it}),$$

where the constants $A_i^Q$ and $B_i^Q$ $(i = 1, 2)$ are given by

$$A_i^Q = \frac{\kappa_i \bar{V}_i}{\kappa_i^Q} \left[ 1 - \frac{1 - e^{-\kappa_i^Q \tau_0}}{\kappa_i^Q \tau_0} \right], \quad B_i^Q = \frac{1 - e^{-\kappa_i^Q \tau_0}}{\kappa_i^Q \tau_0}. \quad (B43)$$

Q.E.D.

B.7 The GMM test

First, it will be useful to see why we can assume $\bar{V}_1 = \bar{V}_2$. In our model, the combination, $V_1 \delta_c + V_2 (1 - \delta_c)$, is the variance of the consumption growth. The relative importance of the two volatility factors in driving consumption variance is characterized by $\delta_c$, hence without loss of generality, we can assume $\bar{V}_1 = \bar{V}_2$. This is because, if we have $\bar{V}_1 \neq \bar{V}_2$, we can redefine another latent variable $V_{2t}' \equiv b V_{2t}$, where $b = \frac{\bar{V}_1}{\bar{V}_2}$, such that

$$dV_{2t}' = \kappa_2 (\bar{V}_1 - V_{2t}') dt + \sigma_2' \sqrt{V_{2t}'} dw_{2t} \quad (B44)$$

with $\sigma_2' = \sqrt{b} \sigma_2$. By adjusting $\delta_c$ accordingly, the new process match exactly the same variance of the consumption growth.

Denote $h(\theta)$ as the vector of target moments implied by the model given parameter set $\theta$. We choose 23 target moments as described in the text. Let $h_T$ be sample vector from data with size $T$ corresponding to the target moments, and expressed as

$$h_T = \phi(g_T) \quad (B45)$$

with

$$g_T \equiv \frac{1}{T} \sum_{t=1}^{T} x_t \quad (B46)$$

where $x_t$ is a vector representing market data, the details are given below. The GMM estimator $\{\theta_T : T \geq 1\}$ is defined as

$$\min_{\theta_T} [h(\theta_T) - h_T]' W [h(\theta_T) - h_T] \quad (B47)$$
for some positive definite weighting matrix $W$. If the model is true and data is stationary, then the GMM estimator must be consistent (Hansen 1982).

By optimizing the quadratic form of Equation (B47), and substituting Equation (B45) into the first order condition, we obtain

$$A_T[h(\theta_T) - h_T] = A_T[\phi(g(\theta_T)) - \phi(g_T)] = 0,$$

(B48)

with

$$A_T = \frac{\partial h'(\theta_T)}{\partial \theta_T} W;$$

(B49)

and

$$D_T = \frac{\partial h(\theta_T)}{\partial \theta_T}.$$

(B50)

For a consistent estimator $\theta_T$, asymptotically we have Taylor expansion

$$\text{plim}[\phi(g(\theta_T)) - \phi(g_T)] = \frac{d\phi(\theta_0)}{d\theta} \times \text{plim}[g(\theta_T) - g_T]$$

(B51)

Let $A \equiv \text{plim}A_T$ and $D \equiv \text{plim}D_T$, following Zhou (1994), the covariance matrix for the target moments is

$$\Lambda_T = \frac{1}{T}(I - D(AD)^{-1}A) \begin{bmatrix} \frac{d\phi}{d\theta} \end{bmatrix} S \begin{bmatrix} \frac{d\phi}{d\theta} \end{bmatrix}' (I - D(AD)^{-1}A)'$$

where $S$ is the spectral matrix defined as

$$S \equiv \sum_{j=-\infty}^{\infty} E x_t x_{t-j}.$$

Denote

$$J = (h(\theta_T) - h_T)\Lambda_T(h(\theta_T) - h_T)'.$$

which measures the sum of squared errors of target moments,

$$J \sim \chi^2(\# \text{ of moments} - \# \text{ of parameters}),$$

In addition, if $J_r$ is J-statistics with the same covariance matrix for a restricted version of the model, then

$$J_r - J \sim \chi^2(\# \text{ of restrictions})$$

where the number of restrictions is the number of parameters that is restricted in one-factor model. Q.E.D.
B.8 Moment conditions for GMM test

In this section, we present the moment conditions for GMM estimation. The 23-dimensional vector $h_T(\theta)$ in the quadratic form $h'_T(\theta)W_T(\theta)h_T(\theta)$ that we choose to minimize are the differences between the model functions and their sample values. The first 15 moment functions are

\[
\begin{array}{llll}
E(\Delta c) & \sigma(\Delta c) & AC1(\Delta c) \\
E(\Delta d) & \sigma(\Delta d) & AC1(\Delta d) \\
E(r_e) & \sigma(r_e) & AC1(r_e) \\
E(r_f) & \sigma(r_f) & AC1(r_f) \\
E(p - d) & \sigma(p - d) & AC1(p - d)
\end{array}
\]

Denote $r_x$ as consumption growth $\Delta c$, dividend growth $\Delta d$, excess return $r_e$, risk free rate $r_f$, and price-dividend ratio $p - d$, the above moments are given as

\[
\sigma(r_x) = \sqrt{E[\{r_x\}^2] - E(r_x)^2}
\]

\[
AC1(r_x) = \frac{E[r_{x,t+1} r_x; t] - E[r_x]^2}{E[r_x^2] - E[r_x]^2}
\]

where $E(r_x)$ are easy to compute analytically given the processes in the paper.

The 16th and 17th moments are $E[\text{VRP}]$, $\sigma(\text{VRP})$, the expectation and standard deviation of variance risk premium (VRP) defined in the text.

The 18th to 20th moments are the regression coefficients $\beta$’s. All the 3 regression coefficients are of the form of

\[
\beta = \frac{\text{Cov}(r_{x,t+1}, (p_t - d_t))}{\text{Var}(p_t - d_t)}
\]

\[
= \frac{E[r_{x,t+1}, (p_t - d_t)] - E[r_{x,t+1}] \cdot E[p_t - d_t]}{E[(p_t - d_t)^2] - E[p_t - d_t]^2}
\]

(B52)

where $r_x$ are consumption growth, dividend growth, excess return, resp.

The 20th to 23rd moments are the three regression $\beta$’s of volatility regressions for $\Delta t = 1$ year. Specifically, they are

\[
\beta_{vol} = \frac{\text{Cov}(\ln \text{Vol}_{t+\tau}, (p_t - d_t))}{\text{Var}(p_t - d_t)}
\]

\[
= \frac{E[\ln \text{Vol}_{t+\tau}, (p_t - d_t)] - E[\ln \text{Vol}_{t+\tau}] \cdot E[p_t - d_t]}{E[(p_t - d_t)^2] - E[p_t - d_t]^2}
\]

where Vol$_{t+\tau}$ is given in Equation (A15) and stands for volatility of consumption, dividend, and excess return, resp. With specification of all the moment conditions, and the analyt-
ical formula of the moments implied by the model that is solved in the paper, the GMM estimation and tests can be carried out as usual (see, e.g., Singleton, 2006).

We show the 26 elements of the moments in $g_{T}$ as follows. The first 15 moments are:

$$E[r_{x}], E[r_{x}^{2}], E[r_{x,t+1}, r_{xt}]$$

where $r_{x}$ stands for consumption growth, dividend growth, excess return, risk free rate, and price-dividend ratio.

Moments 16 and 17 are:

$$E[VRP_{t}], E[VRP^{2}_{t}]$$

where VRP is the variance risk premium.

Moments 18 to 20 are:

$$E[r_{x,t+1}, (p_{t} - d_{t})]$$

where $r_{x}$ stands for consumption growth, dividend growth, and excess return.

Moments 21 to 26 are:

$$E[\log \text{Vol}_{xt}], E[(\log \text{Vol}_{xt}), (p_{t} - d_{t})]$$

where $x$ stands for consumption, dividend, and excess return. The data can be obtained through quarterly data regression

$$r_{x,t+1} = \alpha + \beta r_{x,t} + \epsilon_{x,t}$$

and annual expected volatility $\text{Vol}_{xt}$ are obtained from

$$\text{Vol}_{xt} = \sum_{k=1}^{4} |\epsilon_{t+k}|$$

Finally, the form of function $h_{T} = \phi(g_{T})$ that links the target functions $h_{T}$ and the moments $g_{T}$, as well as its first-order derivative matrix are elementary, and can be obtained from authors upon request. Q.E.D.

**B.9 Accuracy of the Log-linear Approximation**

To show that the log-linear approximation (which is accurate when $\psi = 1$) is accurate enough for the parameter values of interest, we take a three step approach. First, we show
the standard deviation of the log consumption-wealth ratio is small. Second, we show that
the second factor of the two-factor volatility model contributes less than 2% to it. Third,
we show that the exact solution of a one-factor with the same magnitude of the standard
development is very close to the log-linear approximation.

First, extending the approximation in discrete-time models by many, Chacko and Viceira
(2005) show that the approximation works for continuous models too as long as the standard
development of the log consumption-wealth ratio does not vary too much around its uncondi-
tional mean. Specifically, the approximation is a Taylor expansion of the consumption-wealth
ratio around its unconditional mean level, denoted as $g_1$,\
\[
\frac{C_t}{W_t} = e^{c_t - \omega_t} \approx e^{E[c_t - \omega_t]} + e^{E[c_t - \omega_t]} \cdot \left[ (c_t - \omega_t) - E(c_t - \omega_t) \right],
\]
\[
\approx g_1 - g_1 \log g_1 + g_1 \log \left( \frac{C_t}{W_t} \right). \tag{B54}
\]
This implies that a small enough standard deviation of $\log(C_t/W_t)$ yields a good approxi-
mation. In Panel A of Table 1, we, like Chacko and Viceira (2005), show that the standard
development of this ratio is indeed small at less than 1.8% for a range of preference parameters,
and is much smaller for $\psi$ closer to 1 (this is not surprising as the approximation is accurate
when $\psi = 1$). In particular, for our model parameterization, the standard deviation is 1.6%.

Second, the relative contribution of the second volatility factor to the total standard
development of the log consumption wealth ratio is small, as shown by the results in Panel B
of Table 1. This means that the approximation error for our two factor model is almost the
same as a one-factor model.

Finally, we have to show that the approximation error of a one-factor model is indeed
small for the parameter values of interest. To do so, we design a one-factor version of our
model with the one factor calibrated to the first volatility in our model and provide the
exact solution. (Ideally, we want to compare the exact solution of our two-factor model to
the linear approximation. But that is too complex to solve.)

The one-factor model is a non-trivial version of the two-factor one,
\[
\frac{dC_t}{C_t} = \mu dt + \sqrt{V_t} dZ_t, \quad dV_t = \kappa (\bar{V} - V_t) + \sigma \sqrt{V_t} dw_t.
\]
We calibrate the parameters to match the first two moments of consumption growth. The
value function in steady state can be written as

\[ J(V_t, C_t) = e^{G(V_t)} \frac{C_t^{1-\gamma}}{1-\gamma}. \]  

(B55)

It can be shown that the solution for \( G(V) \) follows an ODE as

\[
\left[ \frac{1}{\epsilon} (\beta - \beta e^{G(V)}) + (1 - \gamma)(\mu - \frac{\gamma}{2} V) \right] + \kappa(V-V) \frac{dG(V)}{dV} + \frac{1}{2} \sigma^2 V \left( \frac{dG(V)}{dV} \right)^2 + \frac{1}{2} \sigma^2 V \frac{d^2 G(V)}{dV^2} = 0,
\]

(B56)

where \( \epsilon = \frac{1}{1-\gamma} \). We solve this ODE numerically and report the results in Figure 1 with both the function values and their differences (errors). It is seen that the solution is almost the same as the linear approximation for the parameter range we consider.

B.10 Monotonicity of the Aggregator

Theoretically, it is very important to note that the log-linear approximation of the aggregator \( f \) should be an increasing function of \( J \). Otherwise, it will be in violation of the monotonicity axiom of preferences (see, e.g., Skiadas, 2009, Chapter 6.3). But this is not always the case for all possible parameter values, which is a drawback of certain approximations. However, it should and must be so in the domain of interest of the state variables. Indeed, based on the partial derivative \( f_J \) of (B2),

\[ f_J = (\theta - 1)\beta G - \beta \theta, \]

we know that the variation of \( f_J \) is driven only by \( \beta G \), which is the consumption wealth ratio based on (B8). In order to check whether \( f_J > 0 \) for the relevant state variables, we need to verify whether \( f_J \) is positive for the reasonable range of consumption wealth ratio, \( \beta G \), which has mean value equal to \( g_1 \). Based on (B8), the standard deviation of \( \beta G \) can be computed as

\[
\sigma_{cw} = \sqrt{A_1^2 \sigma_x^2 + A_2^2 \sigma_v^2 + A_3^2 \sigma_{v1}^2 + A_4^2 \sigma_{v2}^2} \cdot g_1 \frac{|1-\psi|}{1-\gamma}
\]

where \( \sigma_x, \sigma_{v1}, \sigma_{v2} \) are unconditional standard deviations of state variables \( X, V_1, V_2 \). As a result, we only need to check the positivity of \( f_J \) in the range of \( \beta G \in (g_1 - 2\sigma_{cw}, g_1 + 2\sigma_{cw}) \) of interest. Figure 1 shows the numerical values \( f_J \) in terms of the number of standard deviation from the mean of \( \beta G \). Within the range of our interest, \( f_J \) is indeed positive as it should.
Table 1: Unconditional Standard Deviation of the Log Consumption-Wealth Ratio

This table shows the consumption-wealth ratio variability around its long term mean level as well as the percentage contribution of the components to the total variation. It shows that the standard deviation of log consumption wealth ratio is less than 2%, hence the approximation of log-linearization is a good one. In addition, the new factor contribution to this variability is small, with less than 1%, due to its short-run nature.

<table>
<thead>
<tr>
<th>EIS $\psi$</th>
<th>1.5</th>
<th>1.45</th>
<th>1.4</th>
<th>1.35</th>
<th>1.3</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard Deviation of log C-W Ratio (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.610 1.504 1.378 1.237 1.103 0.777</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.684 1.572 1.443 1.304 1.152 0.807</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.761 1.639 1.514 1.355 1.209 0.848</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.650 1.544 1.423 1.355 1.148 0.833</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Contribution by the New Factor (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.839 0.883 0.757 0.832 0.848 0.908</td>
</tr>
<tr>
<td>9</td>
<td>0.841 0.890 0.763 0.837 0.932 0.937</td>
</tr>
<tr>
<td>8</td>
<td>0.847 0.906 0.970 0.867 0.905 0.976</td>
</tr>
<tr>
<td>5</td>
<td>1.298 1.301 1.346 1.158 1.453 1.480</td>
</tr>
</tbody>
</table>
Figure 1: Log-linear Approximation vs Numerical Solution

The figure plots the log-linear approximated vs. exact numerical solution for $G(V)$ of Equation (B56). The parameters are:

$$\psi = 1.5, \gamma = 10, \beta = 0.01, \mu = 0.02, \kappa = 0.035, \bar{V} = 0.0004, \sigma = 0.0026,$$

which are designed to match the unconditional moments of consumption growth and its volatility.
Figure 2: Monotonicity of the Aggregator

The figure plots the partial derivative of aggregator $f$ with respect to $J$, $f_J$, vs. the number of standard deviation from the mean value of $f_J$. The parameters are taken from Table 3 of the paper.