A More Detailed and Complete Appendix for "Macroeconomic Volatilities and Long-run Risks of Asset Prices"

This is an on-line appendix with more details and analysis for the readers of the paper.

B.1 Derivation for the A_i's, risk-free rate and market price of risk

First, we re-write the normalized aggregator f defined in Equation (5) as

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J[G-1],$$

where

$$G \equiv \left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}}. \tag{B1}$$

Then, taking partial derivatives of $f(C,J)$ with respect to J and C , we have

$$f_J = (\theta-1)\beta G - \beta\theta \tag{B2}$$

and

$$f_C = \beta\frac{G}{C}(1-\gamma)J. \tag{B3}$$

where we use the notation $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. Theoretically, the aggregator $f(C,J)$ should be an increasing function of the value function J (see, e.g., Skiadas, 2009, Chapter 6.3). Otherwise, the monotonicity axiom of preferences will be violated. This places joint restrictions on γ and ψ such that $\theta \geq 1$ or $\theta < 0$. This is because $f_J > 0$ implies that

$$\text{If } \theta > 1: \quad G > \frac{\theta}{\theta-1}$$
$$\text{If } \theta < 1: \quad G < \frac{\theta}{\theta-1}.$$

If $\theta > 1$, the first inequality is possible to have solutions. However, if $0 < \theta < 1$, the second inequality is impossible as $G > 0$ always. Hence, the necessary restriction on γ and ψ is either $\theta > 1$ or $\theta < 0$. If $\theta = 1$, as shown by Duffie and Epstein (1992), we obtain the standard additive expected utility of constant relative risk aversion (CRRA). So $\theta > 1$ can be extended to $\theta \geq 1$.

Conjecturing a solution for J of the following form,

$$J(W_t, X_t, V_{1t}, V_{2t}) = \exp(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{W_t^{1-\gamma}}{1-\gamma}, \tag{B4}$$

and using the standard envelope condition $f_C = J_W$, we have

$$C = J_W^{-\psi} [(1-\gamma)J]^{\frac{1-\gamma\psi}{1-\gamma}} \beta^{\psi}. \tag{B5}$$

Substituting (B3) and (B4) into (B5), we obtain

$$\frac{C}{W} = \beta^{\psi} \exp\left[(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{1-\psi}{1-\gamma} \right]. \tag{B6}$$

and hence J can be re-written as

$$J(C_t, X_t, V_{1t}, V_{2t}) = \beta^{-\psi(1-\gamma)} \exp[\psi(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t})] \frac{C_t^{1-\gamma}}{1-\gamma}. \tag{B7}$$

Further substituting (B6) and (B4) into (B1), we get

$$\beta G = \frac{C_t}{W_t}.$$

Applying the log-linear approximation, we obtain

$$\beta G = \frac{C_t}{W_t} \approx g_1 - g_1 \log g_1 + g_1 \log(\beta G). \tag{B8}$$

This implies that

$$f = \theta J(\beta G - \beta) \approx \theta J \left[g_1 \frac{1-\psi}{1-\gamma}(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) + \xi \right], \tag{B9}$$

where $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$ and

$$\xi = g_1 - g_1 \log g_1 + g_1 \psi \log \beta - \beta. \tag{B10}$$

Substituting (B9) into the HJB Equation (6),

$$\begin{aligned} & f(C,J) + C \cdot (\mu + X) J_C + \frac{1}{2}[\delta_c V_1 + (1-\delta_c)V_2] C^2 J_{CC} + J_X \cdot (-\alpha X) + \frac{1}{2}\varphi_x^2 [\delta_x V_1 + (1-\delta_x)V_2] J_{XX} \\ & + J_{V_1} \cdot \kappa_1(\bar{V}_1 - V_1) + \frac{1}{2}\sigma_1^2 v_1 J_{V_1 V_1} + J_{V_2} \cdot \kappa_2(\bar{V}_2 - V_2) + \frac{1}{2}\sigma_2^2 V_2 J_{V_2 V_2} = 0, \end{aligned} \tag{B11}$$

where $\{C_t\}$ is the optimal consumption process, and we have used the definition of

$$\mathcal{A}^c J \equiv \sum_i b(z) \frac{\partial J(z)}{\partial z} + \sum_{i,j} (\sigma\sigma^T)_{i,j}(z) \frac{\partial^2 J}{\partial z_i z_j},$$

with $z = (C, X, V_1, V_2)$ and $b(z)$ and $\sigma(z)$ the drift and diffusive terms for z defined in Equation (2). Collecting the terms containing constant, X_t, V_{1t} and V_{2t}, *resp*, we have

$$
\begin{aligned}
& \theta g_1 \frac{1-\psi}{1-\gamma} A_0 + \theta\xi + (1-\gamma)\mu + \kappa_1 \bar{V}_1 \psi A_2 + \kappa_2 \bar{V}_2 \psi A_3 = 0 \\
X: \quad & \theta g_1 \frac{1-\psi}{1-\gamma} A_1 + (1-\gamma) - \alpha\psi A_1 = 0 \\
V_1: \quad & \theta g_1 \frac{1-\psi}{1-\gamma} A_2 - \frac{1}{2}\gamma(1-\gamma)\delta_c + \frac{1}{2}\varphi_x^2 \delta_x \psi^2 A_1^2 - \kappa_1 \psi A_2 + \frac{1}{2}\sigma_1^2 \psi^2 A_2^2 = 0 \\
V_2: \quad & \theta g_1 \frac{1-\psi}{1-\gamma} A_3 - \frac{1}{2}\gamma(1-\gamma)(1-\delta_c) + \frac{1}{2}\varphi_x^2 (1-\delta_x) \psi^2 A_1^2 - \kappa_2 \psi A_3 + \frac{1}{2}\sigma_2^2 \psi^2 A_3^2 = 0.
\end{aligned}
$$

Solving the above algebraic equations, we obtain

$$
\begin{aligned}
A_0 &= \frac{1}{g_1 \psi}[\theta\xi + (1-\gamma)\mu + \kappa_1 \bar{V}_1 \psi A_2 + \kappa_2 \bar{V}_2 \psi A_3], \\
A_1 &= \frac{1-\gamma}{(g_1+\alpha)\psi}, \\
A_2 &= \frac{-b_1 - \sqrt{b_1^2 - 4a_1 c_1}}{2a_1}, \\
A_3 &= \frac{-b_2 - \sqrt{b_2^2 - 4a_2 c_2}}{2a_2},
\end{aligned} \tag{B12}
$$

with

$$
\begin{aligned}
a_1 = \tfrac{1}{2}\sigma_1^2 \psi^2, \quad b_1 = -(g_1+\kappa_1)\psi, \quad c_1 = -\tfrac{1}{2}\gamma(1-\gamma)\delta_c + \tfrac{1}{2}\varphi_x^2 \delta_x \tfrac{(1-\gamma)^2}{(g_1+\alpha)^2}, \\
a_2 = \tfrac{1}{2}\sigma_2^2 \psi^2, \quad b_2 = -(g_1+\kappa_2)\psi, \quad c_2 = -\tfrac{1}{2}\gamma(1-\gamma)\delta_c(1-\delta_c) + \tfrac{1}{2}\varphi_x^2 (1-\delta_x) \tfrac{(1-\gamma)^2}{(g_1+\alpha)^2}.
\end{aligned}
$$

We then derive the risk-free rate and market prices of risks. Recall that the pricing kernel is given by Equation (A6). Based on the definition for f, we have

$$
\begin{aligned}
f_J &= \xi_1 - g_1(A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{1-\gamma\psi}{1-\gamma}, \\
f_C &= \beta^{\psi\gamma} \exp\left[(B + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{1-\gamma\psi}{1-\gamma}\right] C_t^{-\gamma},
\end{aligned}
$$

where

$$
\xi_1 = (\theta - 1)\xi - \beta - g_1 \frac{1-\gamma\psi}{1-\gamma} A_0. \tag{B13}
$$

Applying Ito's Lemma to π_t in Equation (A6), we have

$$
\frac{d\pi_t}{\pi_t} = -(r_f dt + \lambda_1 dZ_{1t} + \lambda_2 dZ_{2t} + \lambda_3 dw_{1t} + \lambda_4 dw_{2t}), \tag{B14}
$$

where the risk-free rate r_f and the market prices of risks, $\lambda_i, i = 1, 2, 3, 4$, are given below.

First, the risk-free rate is

$$r_f = r_0 + r_1 X_t + r_2 V_{1t} + r_3 V_{2t}, \tag{B15}$$

where

$$\begin{aligned}
r_0 &= -(\xi_1 + (\kappa_1 A_2 \bar{V}_1 + \kappa_2 A_3 \bar{V}_2)\frac{1-\gamma\psi}{1-\gamma} - \gamma\mu), \\
r_1 &= \frac{1}{\psi}, \\
r_2 &= (g_1 + \kappa_1) A_2 \frac{1-\gamma\psi}{1-\gamma} - \frac{1}{2}\left(\frac{1-\gamma\psi}{1-\gamma}\right)^2 (A_1^2 \varphi_x^2 \delta_x + A_2^2 \sigma_1^2) - \frac{1}{2}\gamma(\gamma+1)\delta_c, \\
r_3 &= (g_1 + \kappa_2) A_3 \frac{1-\gamma\psi}{1-\gamma} - \frac{1}{2}\left(\frac{1-\gamma\psi}{1-\gamma}\right)^2 [A_1^2 \varphi_x^2 (1-\delta_x) + A_3^2 \sigma_2^2] - \frac{1}{2}\gamma(\gamma+1)(1-\delta_c).
\end{aligned} \tag{B16}$$

Second, the market prices of risks are

$$\begin{aligned}
\lambda_1 &= \gamma\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}, \\
\lambda_2 &= -\frac{1-\gamma\psi}{1-\gamma} A_1 \varphi_x \sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}, \\
\lambda_3 &= -\frac{1-\gamma\psi}{1-\gamma} A_2 \sigma_1 \sqrt{V_{1t}}, \\
\lambda_4 &= -\frac{1-\gamma\psi}{1-\gamma} A_3 \sigma_2 \sqrt{V_{2t}}.
\end{aligned} \tag{B17}$$

Q.E.D.

B.2 Derivation for the A_{im}'s

Let

$$\frac{D_t}{P_t} = \exp\{(A_{0m} + A_{1m} X_t + A_{2m} V_{1t} + A_{3m} V_{2t})\}. \tag{B18}$$

A key step in the derivation is to use the following pricing relation given in

$$E_t\left(\frac{dP_t}{P_t}\right) + \frac{D_t}{P_t} dt = r_f dt - E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}\right]. \tag{B19}$$

With similar loglinear approximation as Equation (B8), we can approximate the ratio as

$$\frac{D_t}{P_t} \approx g_{0m} + g_{1m} \log\frac{D_t}{P_t} = g_{0m} + g_{1m}((A_{0m} + A_{1m} X_t + A_{2m} V_{1t} + A_{3m} V_{2t})), \tag{B20}$$

where

$$g_{0m} = g_{1m} - g_{1m} \log g_{1m}.$$

Applying Ito's lemma to (B18), we have

$$\frac{dP_t}{P_t} = \frac{dD_t}{D_t} - (A_{1m}dX_t + A_{2m}dV_{1t} + A_{3m}dV_{2t}) + \frac{1}{2}A_{1m}^2(dX_t)^2 + \frac{1}{2}A_{2m}^2(dV_{1t})^2 + \frac{1}{2}A_{3m}^2(dV_{2t})^2.$$

Hence,

$$\begin{aligned} E_t(\frac{dP_t}{P_t})/dt &= \mu_d + \phi X_t + \alpha A_{1m}X_t - \kappa_1 A_{2m}(\bar{V}_1 - V_{1t}) - \kappa_2 A_{3m}(\bar{V}_2 - V_{2t}) \\ &\quad + \frac{1}{2}A_{1m}^2\varphi_x^2[V_{1t}\delta_x + V_{2t}(1-\delta_x)] + \frac{1}{2}A_{2m}^2\sigma_1^2 V_{1t} + \frac{1}{2}A_{3m}^2\sigma_2^2 V_{2t}. \end{aligned} \quad \text{(B21)}$$

The risk premium term in Equation (B19) can thus be written as

$$\begin{aligned} -E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}\right]/dt &= \sigma_{dc}\lambda_1\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)} - (A_{1m}\varphi_x - \sigma_{dx})\lambda_2\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)} \\ &\quad -(A_{2m}\sigma_1 - \sigma_{dv})\lambda_3\sqrt{V_{1t}} - (A_{3m}\sigma_2 - \sigma_{dv2})\lambda_4\sqrt{V_{2t}}, \end{aligned} \quad \text{(B22)}$$

where λ_1, λ_2, λ_3 and λ_4 are market prices of risks defined in Equation (B17).

Now, substituting (B20), (B21), (B22), and risk-free rate (B15) into Equation (B19), and collecting terms containing X_t, we obtain

$$A_{1m} = -\frac{\phi - \frac{1}{\psi}}{g_{1m} + \alpha}. \quad \text{(B23)}$$

Collecting terms containing V_{1t} and V_{2t}, *resp*, we obtain an equation for A_{2m},

$$a_{2m}A_{2m}^2 + b_{2m}A_{2m} + c_{2m} = 0$$

with

$$a_{2m} = \frac{1}{2}\sigma_1^2, \quad b_{2m} = g_{1m} + \kappa_1 - \frac{1-\gamma\psi}{1-\gamma}A_2\sigma_1^2, \quad c_{2m} = (\frac{1}{2}A_{1m}^2 - \frac{1-\gamma\psi}{1-\gamma}A_1A_{1m})\varphi_x^2\delta_x + r_2.$$

Solving it, we have

$$A_{2m} = \frac{-b_{2m} \pm \sqrt{b_{2m}^2 - 4a_{2m}c_{2m}}}{2a_{2m}}. \quad \text{(B24)}$$

We choose the root that goes to zero when σ_1 goes to zero. This is because when σ_1, or a_{2m} goes to zero, the price sensitivity to V_1 should be zero.

Similarly, we obtain an equation for A_{3m},

$$a_{3m}A_{3m}^2 + b_{3m}A_{3m} + c_{3m} = 0$$

with

$$a_{3m} = \frac{1}{2}\sigma_2^2, \ b_{2m} = g_{1m} + \kappa_2 - \frac{1-\gamma\psi}{1-\gamma}A_3\sigma_2^2, \ c_{3m} = (\frac{1}{2}A_{1m}^2 - \frac{1-\gamma\psi}{1-\gamma}A_1A_{1m})\varphi_x^2(1-\delta_x) + r_3.$$

The solution is

$$A_{3m} = \frac{-b_{3m} \pm \sqrt{b_{3m}^2 - 4a_{3m}c_{3m}}}{2a_{3m}}, \tag{B25}$$

where we choose the root in a similar fashion as for A_{2m} above.

Finally, collecting the constant terms in Equation (B19), we obtain

$$\mu_d - \kappa_1 A_{2m}\bar{V}_1 - \kappa_2 A_{3m}\bar{V}_2 + g_{0m} + g_{1m}A_{0m} + r_0 = 0,$$

and re-arrange terms to get

$$A_{0m} = -\frac{1}{g_{1m}}\left[\mu_d - \kappa_1 A_{2m}\bar{V}_1 - \kappa_2 A_{3m}\bar{V}_2 + g_{1m} - g_{1m}\log g_{1m} + r_0\right].$$

So far, we obtain all the A_{im} coefficients.

To obtain the market return volatility, we apply Ito's Lemma to Equation (B18) and obtain

$$\begin{aligned}
\frac{dP_t}{P_t} &= [\mu_d - (A_{2m}\kappa_1\bar{V}_1 + A_{3m}\kappa_2\bar{V}_2) + (\phi + \alpha A_{1m})X_t \\
&\quad +(\frac{1}{2}A_{1m}^2\varphi_x^2\delta_x + \frac{1}{2}A_{2m}\sigma_1^2 + A_{2m}\kappa_1 - A_{1m}\sigma_{dx}\varphi_x\delta_x - A_{2m}\sigma_1\sigma_{dv})V_{1t} \\
&\quad +(\frac{1}{2}A_{1m}^2\varphi_x^2(1-\delta_x) + \frac{1}{2}A_{3m}\sigma_2^2 + A_{3m}\kappa_2 - A_{1m}\sigma_{dx}\varphi_x(1-\delta_x) - A_{3m}\sigma_2\sigma_{dv2})V_{2t}]dt \\
&\quad +\varphi_d\sqrt{V_{1t}\delta_d + V_{2t}(1-\delta_d)}dB_t + \sigma_{dc}\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}dZ_{1t} \\
&\quad +(\sigma_{dx} - A_{1m}\varphi_x)\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}dZ_{2t} \\
&\quad +(\sigma_{dv} - A_{2m}\sigma_1)\sqrt{V_{1t}}dw_{1t} + (\sigma_{dv2} - A_{3m}\sigma_2)\sqrt{V_{2t}}dw_{2t} \\
&= [c_3 + c_4X_t + c_5V_{1t} + c_6V_{2t}]dt + \sqrt{c_1V_{1t} + c_2V_{2t}}dZ_t,
\end{aligned}$$

where c_i ($i = 1$ to 6) are constants, dZ_t is a new Brownian motion defined accordingly, and hence the variance of the price process is

$$V_t = c_1V_{1t} + c_2V_{2t},$$

with

$$\begin{aligned}
c_1 &= \varphi_d^2\delta_d + \sigma_{dc}^2\delta_c + (\sigma_{dx} - A_{1m}\varphi_x)^2\delta_x + (\sigma_{dv} - A_{2m}\sigma_1)^2, \\
c_2 &= \varphi_d^2(1-\delta_d) + \sigma_{dc}^2(1-\delta_c) + (\sigma_{dx} - A_{1m}\varphi_x)^2(1-\delta_x) + (\sigma_{dv2} - A_{3m}\sigma_2)^2,
\end{aligned} \tag{B26}$$

and the parameters for the drift term are

$$\begin{aligned} c_4 &= \phi + \alpha A_{1m}, \\ c_5 &= (\frac{1}{2}A_{1m}^2\varphi_x^2\delta_x + \frac{1}{2}A_{2m}\sigma_1^2 + A_{2m}\kappa_1 - A_{1m}\sigma_{dx}\varphi_x\delta_x - A_{2m}\sigma_1\sigma_{dv}), \\ c_6 &= (\frac{1}{2}A_{1m}^2\varphi_x^2(1-\delta_x) + \frac{1}{2}A_{3m}\sigma_2^2 + A_{3m}\kappa_2 - A_{1m}\sigma_{dx}\varphi_x(1-\delta_x) - A_{3m}\sigma_2\sigma_{dv2}). \end{aligned} \tag{B27}$$

Q.E.D.

B.3 Solutions to g_1 and g_{1m}

Note that the derived solutions depend on the approximation constant g_1, which can be solved endogenously. Given the model parameters, we can compute the unconditional mean of consumption-wealth ratio as a function of the parameters,

$$\begin{aligned} g_1 &= E\left(\frac{C}{W}\right) = \beta^{\psi}\exp\{A_{0a}\}\exp\left\{\frac{1}{4}A_{1a}^2\varphi_x^2\frac{(\bar{V}_1\delta_x + \bar{V}_2(1-\delta_x))}{\alpha}\right\} \\ &\cdot\exp\left\{-\frac{2\kappa_1\bar{V}_1}{\sigma_1^2}\log(1-\frac{A_{2a}}{2\kappa_1/\sigma_1^2})\right\}\cdot\exp\left\{-\frac{2\kappa_2\bar{V}_2}{\sigma_2^2}\log(1-\frac{A_{3a}}{2\kappa_2/\sigma_2^2})\right\}. \end{aligned} \tag{B28}$$

Note that the A_{ia}'s on the right hand side are also functions of g_1. Substituting A_{ia} as function of g_1 into Equation (B28), we obtain a nonlinear function in terms of g_1 only, and hence g_1 can be solved in terms of the fundamental parameters of the model, and can be computed numerically with many available algorithms.

Similarly, we can solve g_{1m} endogenously based on dividend-price ratio given as

$$\begin{aligned} g_{1m} &= E\left(\frac{D}{P}\right) = \exp\{A_{0m}\}\exp\left\{\frac{1}{4}A_{1m}^2\varphi_x^2\frac{(\bar{V}_1\delta_x + \bar{V}_2(1-\delta_x))}{\alpha}\right\} \\ &\cdot\exp\left\{-\frac{2\kappa_1\bar{V}_1}{\sigma_1^2}\log(1-\frac{A_{2m}}{2\kappa_1/\sigma_1^2})\right\}\cdot\exp\left\{-\frac{2\kappa_2\bar{V}_2}{\sigma_2^2}\log(1-\frac{A_{3m}}{2\kappa_2/\sigma_2^2})\right\}. \end{aligned} \tag{B29}$$

This can be solved numerically as above. Q.E.D.

B.4 Predictability of variables

The regressors of the three regressions given in Equation (14)-(16) all have the generic functional form of

$$dY_t = [a_0 + a_1X_t + a_2V_{1t} + a_3V_{2t}]dt + \sqrt{b_1V_{1t} + b_2V_{2t}}dZ_t,$$

given in Equation (17) where dY_t corresponds to excess return $d\ln P_t+\frac{D_t}{P_t}-r_f dt$, consumption growth $d\ln C_t$ and dividend growth $d\ln D_t$, respectively. For stock market excess return, we have

$$\begin{aligned} a_1 &= c_4 + r_1 + g_{1m}A_{1m}, \\ a_2 &= c_5 - \frac{c_1}{2} + r_2 + g_{2m}A_{2m}, \\ a_3 &= c_6 - \frac{c_2}{2} + r_3 + g_{3m}A_{3m}, \end{aligned} \tag{B30}$$

where c_1, c_2, c_4, c_5 and c_6 are defined in Equations (B26) and (B27).

For consumption growth, we have

$$a_1 = 1,\ a_2 = -\frac{\delta_c}{2},\ a_3 = -\frac{1-\delta_c}{2}. \tag{B31}$$

For dividend growth, we have

$$a_1 = \varphi,\ a_2 = -\frac{\varphi_d^2\delta_d + \sigma_{dc}^2\delta_c + \sigma_{dv}^2 + \sigma_{dx}^2\delta_x}{2}, a_3 = -\frac{\varphi_d^2(1-\delta_d) + \sigma_{dc}^2(1-\delta_c) + \sigma_{dv2}^2 + \sigma_{dx}^2(1-\delta_x)}{2}. \tag{B32}$$

We want to show Equations (A13). Given Equations (B30) and (B18), and denoting $\mathrm{Cov}(x,y) \equiv < x, y >$, and $pd_t \equiv p_t - d_t$, we have

$$\begin{aligned} & < \int_t^{t+\tau} dy_s, p_t - d_t > \\ = & \int_t^{t+\tau} ds < a_0 + a_1X_s + a_2V_{1s} + a_3V_{2s}, pd_t > \\ = & \int_t^{t+\tau} ds[a_1 < x_s, pd_t > + a_2 < V_{1s}, pd_t > + a_3 < V_{2s}, pd_t >] \\ = & -\int_0^{\tau} ds\big[a_1A_{1m}\frac{\sigma_x^2}{2\alpha}e^{-\alpha s} + a_2A_{2m}\frac{\sigma_1^2\bar{V}_1}{2\kappa_1}e^{-\kappa_1 s} + a_3A_{3m}\frac{\sigma_2^2\bar{V}_2}{2\kappa_2}e^{-\kappa_2 s}\big] \end{aligned}$$

where

$$\sigma_x^2 = \varphi_x^2[\bar{V}_1\delta_x + \bar{V}_2(1-\delta_x)]$$

Integrating the above equation, we obtain Equation (A13), where

$$\begin{aligned} \mathrm{Cov}(\Delta_\tau y, p-d) &= -\Big[a_1A_{1m}\frac{\sigma_x^2}{2\alpha^2}(1-e^{-\alpha\tau}) + a_2A_{2m}\frac{\sigma_1^2\bar{V}_1}{2\kappa_1^2}(1-e^{-\kappa_1\tau}), \\ & \qquad + a_3A_{3m}\frac{\sigma_2^2\bar{V}_2}{2\kappa_2^2}(1-e^{-\kappa_2\tau})\Big] \end{aligned} \tag{B33}$$

$$\mathrm{Var}(p-d) = A_{1m}^2\frac{\sigma_x^2}{2\alpha} + A_{2m}^2\frac{\sigma_1^2\bar{V}_1}{2\kappa_1} + A_{3m}^2\frac{\sigma_2^2\bar{V}_2}{2\kappa_2}, \tag{B34}$$

with

$$\sigma_x^2 = \varphi_x^2[\bar{V}_1\delta_x + \bar{V}_2(1-\delta_x)]$$

which are the same given in the text. We have used the unconditional covariance:

$$\begin{aligned} < X_t, X_s > &= \frac{\sigma_x^2}{2\alpha} e^{-\alpha|t-s|} \\ < V_{it}, V_{is} > &= \frac{\sigma_i^2 \bar{V}_i}{2\kappa_i} e^{-\kappa_i|t-s|} \end{aligned}$$

for $i = 1, 2$.

Similar computation applies to obtain Equation (A17). Q.E.D.

B.5 Predictability of volatilities

First, we prove Equation (A16). To do so, we apply the following approximation:

$$\frac{1}{\tau}\int_0^\tau \exp(x_s)ds \approx \exp(\frac{1}{\tau}\int_0^\tau x_s ds) \tag{B35}$$

for any process x_s. This is equivalent to approximating the arithmetic mean by the geometric mean. The approximation is good when the variation of x_t is small, which is true for our variance processes because the magnitude is generally in the order of $10^{-3} \sim 10^{-4}$, and the variation of $\log V_t$ is within 1. Applying the approximation to $\log V_t$, we have

$$\frac{1}{\tau}\int_0^\tau \sqrt{V_t}dt = \frac{1}{\tau}\int_0^\tau \exp(\frac{1}{2}\ln V_t)dt \approx \exp(\frac{1}{2\tau}\int_0^\tau \ln V_t dt). \tag{B36}$$

Hence,

$$\begin{aligned} \ln \frac{1}{\tau}\int_0^\tau \sqrt{V_t}dt &\approx \frac{1}{2\tau}\int_0^\tau \ln V_t dt \\ &= \frac{1}{2\tau}\left[\int_0^\tau \ln \bar{V} + \int_0^\tau \ln(1 + \frac{V_t - \bar{V}}{\bar{V}})dt\right] \\ &\approx \frac{1}{2}\ln \bar{V} + \frac{1}{2\tau}\int_0^\tau \frac{V_t - \bar{V}}{\bar{V}}dt \\ &= Const + \frac{1}{2\tau\bar{V}}\int_0^\tau V_t dt, \end{aligned} \tag{B37}$$

which is Equation (A16).

Because of the approximation above, we can express the volatilities as an integral of $b_1V_{1s} + b_2V_{2s}$ over $(t, t+\tau)$. Plugging these terms into the definition of the covariance, we then obtain Equation (A17).

Then we provide the derivation of the AR(1) coefficient. Consider a stochastic process of the form

$$b_1 V_{1t} + b_2 V_{2t},$$

where b_1 and b_2 are constants. Due to independence between V_1 and V_2, the unconditional auto-covariance can be evaluated as

$$b_1^2 \frac{\sigma_1^2 \bar{V}_1}{2\kappa_1} \exp(-\kappa_1 \tau) + b_2^2 \frac{\sigma_2^2 \bar{V}_2}{2\kappa_2} \exp(-\kappa_2 \tau)$$

and the unconditional variance can be evaluated as

$$b_1^2 \frac{\sigma_1^2 \bar{V}_1}{2\kappa_1} + b_2^2 \frac{\sigma_2^2 \bar{V}_2}{2\kappa_2}.$$

Hence, the AR(1) coefficient can be computed easily based on above. Q.E.D.

B.6 Derivation of VRP

We derive the time t expected future realized variance over time period τ_0 under the risk-neutral probability. The market prices of risk for V_{1t} and V_{2t} are λ_3 and λ_4 of Equation (B17), hence the risk premia associated with V_{1t} and V_{2t} are

$$\lambda_3 \sigma_1 \sqrt{V_{1t}} = -\nu_1 V_{1t}, \quad \text{and} \quad \lambda_4 \sigma_2 \sqrt{V_{2t}} = -\nu_2 V_{2t}, \tag{B38}$$

where

$$\nu_1 = \frac{1-\gamma\psi}{1-\gamma} A_2 \sigma_1^2, \quad \text{and} \quad \nu_2 = \frac{1-\gamma\psi}{1-\gamma} A_3 \sigma_2^2. \tag{B39}$$

Hence, the risk-neutral processes for V_{1t} and V_{2t} are

$$\begin{aligned} dV_{1t} &= \kappa_1^Q \left(\frac{\kappa_1}{\kappa_1^Q} \bar{V}_1 - V_{1t}\right) dt + \sigma_1 \sqrt{V_{1t}} dw_{1t}^Q, \\ dV_{2t} &= \kappa_2^Q \left(\frac{\kappa_2}{\kappa_2^Q} \bar{V}_2 - V_{2t}\right) dt + \sigma_2 \sqrt{V_{2t}} dw_{2t}^Q, \end{aligned} \tag{B40}$$

where the risk-neutral mean-reversion coefficients for V_{it} are defined as

$$\kappa_i^Q = \kappa_i - \nu_i \tag{B41}$$

for $i = 1, 2$. In order for well-defined risk-neutral processes in Equation (B40), we need to have κ_i^Q's to be positive such that

$$\nu_1 < \kappa_1 \quad \text{and} \quad \nu_2 < \kappa_2. \tag{B42}$$

Now we compute the squared VIX, or more generally, variance swap rate VS_t with maturity τ_0, defined as the risk neutral expectation of the variance. Because the risk-neutral process and the physical process of Equation are both Heston (1993) processes, we obtain Equation

$$\mathrm{VS}_t = \sum_{i=1}^{2} c_i (A_i^Q + B_i^Q V_{it}),$$

where the constants A_i^Q and B_i^Q $(i = 1, 2)$ are given by

$$A_i^Q = \frac{\kappa_i \bar{V}_i}{\kappa_i^Q}\left[1 - \frac{1 - e^{-\kappa_i^Q \tau_0}}{\kappa_i^Q \tau_0}\right], \qquad B_i^Q = \frac{1 - e^{-\kappa_i^Q \tau_0}}{\kappa_i^Q \tau_0}. \tag{B43}$$

Q.E.D.

B.7 The GMM test

First, it will be useful to see why we can assume $\bar{V}_1 = \bar{V}_2$. In our model, the combination, $V_{1t}\delta_c + V_{2t}(1-\delta_c)$, is the variance of the consumption growth. The relative importance of the two volatility factors in driving consumption variance is characterized by δ_c, hence without loss of generality, we can assume $\bar{V}_1 = \bar{V}_2$. This is because, if we have $\bar{V}_1 \neq \bar{V}_2$, we can redefine another latent variable $V'_{2t} \equiv bV_{2t}$, where $b = \frac{\bar{V}_1}{\bar{V}_2}$, such that

$$dV'_{2t} = \kappa_2(\bar{V}_1 - V'_{2t})dt + \sigma'_2\sqrt{V'_{2t}}dw_{2t} \tag{B44}$$

with $\sigma'_2 = \sqrt{b}\sigma_2$. By adjusting δ_c accordingly, the new process match exactly the same variance of the consumption growth.

Denote $h(\theta)$ as the vector of target moments implied by the model given parameter set θ. We choose 23 target moments as described in the text. Let h_T be sample vector from data with size T corresponding to the target moments, and expressed as

$$h_T = \phi(g_T) \tag{B45}$$

with

$$g_T \equiv \frac{1}{T}\sum_{t=1}^{T} x_t \tag{B46}$$

where x_t is a vector representing market data, the details are given below. The GMM estimator $\{\theta_T : T \geq 1\}$ is defined as

$$\min_{\theta_T} [h(\theta_T) - h_T]' W [h(\theta_T) - h_T] \tag{B47}$$

for some positive definite weighting matrix W. If the model is true and data is stationary, then the GMM estimator must be consistent (Hansen 1982).

By optimizing the quadratic form of Equation (B47), and substituting Equation (B45) into the first order condition, we obtain

$$A_T[h(\theta_T) - h_T] = A_T[\phi(g(\theta_T)) - \phi(g_T)] = 0, \tag{B48}$$

with

$$A_T = \frac{\partial h'(\theta_T)}{\partial \theta_T} W, \tag{B49}$$

and

$$D_T = \frac{\partial h(\theta_T)}{\partial \theta'_T} \tag{B50}$$

For a consistent estimator θ_T, asymptotically we have Taylor expansion

$$\text{plim}[\phi(g(\theta_T)) - \phi(g_T)] = \frac{d\phi(\theta_0)}{d\mu} \times \text{plim}[g(\theta_T) - g_T] \tag{B51}$$

Let $A \equiv \text{plim}A_T$ and $D \equiv \text{plim}D_T$, following Zhou (1994), the covariance matrix for the target moments is

$$\Lambda_T = \frac{1}{T}(I - D(AD)^{-1}A)\left[\frac{d\phi}{d\mu}\right] S \left[\frac{d\phi}{d\mu}\right]' (I - D(AD)^{-1}A)'$$

where S is the spectral matrix defined as

$$S \equiv \sum_{j=-\infty}^{\infty} Ex_t x_{t-j}.$$

Denote

$$J = (h(\theta_T) - h_T)\Lambda_T(h(\theta_T - h_T))'$$

which measures the sum of squared errors of target moments,

$$J \sim \chi^2(\# \text{ ofmoments} - \# \text{ of parameters}),$$

In addition, if J_r is J-statistics with the same covariance matrix for a restricted version of the model, then

$$J_r - J \sim \chi^2(\# \text{ of restrictions})$$

where the number of restrictions is the number of parameters that is restricted in one-factor model. Q.E.D.

B.8 Moment conditions for GMM test

In this section, we present the moment conditions for GMM estimation. The 23-dimensional vector $h_T(\theta)$ in the quadratic form $h'_T(\theta)W_T(\theta)h_T(\theta)$ that we choose to minimize are the differences between the model functions and their sample values. The first 15 moment functions are

$$\begin{array}{ccc} E(\Delta c) & \sigma(\Delta c) & \mathrm{AC1}(\Delta c) \\ E(\Delta d) & \sigma(\Delta d) & \mathrm{AC1}(\Delta d) \\ E(r_e) & \sigma(r_e) & \mathrm{AC1}(r_e) \\ E(r_f) & \sigma(r_f) & \mathrm{AC1}(r_f) \\ E(p-d) & \sigma(p-d) & \mathrm{AC1}(p-d) \end{array}$$

Denote r_x as consumption growth Δc, dividend growth Δd, excess return r_e, risk free rate r_f, and price-dividend ratio $p-d$, the above moments are given as

$$\begin{aligned} \sigma(r_x) &= \sqrt{E[(r_x)^2] - E(r_x)^2} \\ \mathrm{AC1}(r_x) &= \frac{E[r_{x,t+1}r_{x,t}] - E[r_x]^2}{E[r_x^2] - E[r_x]^2} \end{aligned}$$

where $E(r_x)$ are easy to compute analytically given the processes in the paper.

The 16th and 17th moments are $E[\mathrm{VRP}]$, $\sigma(\mathrm{VRP})$, the expectation and standard deviation of variance risk premium (VRP) defined in the text.

The 18th to 20th moments are the regression coefficients β's. All the 3 regression coefficients are of the form of

$$\begin{aligned} \beta &= \frac{\mathrm{Cov}(r_{x,t+1}, (p_t - d_t))}{\mathrm{Var}(p_t - d_t)} \\ &= \frac{E[r_{x,t+1}, (p_t - d_t)] - E[r_{x,t+1}] \cdot E[p_t - d_t]}{E[(p_t - d_t)^2] - E[p_t - d_t]^2} \end{aligned} \tag{B52}$$

where r_x are consumption growth, dividend growth, excess return, *resp.*

The 20th to 23rd moments are the three regression β's of volatility regressions for $\Delta t = 1$ year. Specifically, they are

$$\begin{aligned} \beta_{vol} &= \frac{\mathrm{Cov}(\ln \mathrm{Vol}_{t,t+\tau}, (\mathrm{p_t} - \mathrm{d_t}))}{\mathrm{Var}(p_t - d_t)} \\ &= \frac{E[\ln \mathrm{Vol}_{t,t+\tau} \cdot (\mathrm{p_t} - \mathrm{d_t})] - \mathrm{E}[\ln \mathrm{Vol}_{t,t+\tau}] \cdot \mathrm{E}[\mathrm{p_t} - \mathrm{d_t}]}{E[(p_t - d_t)^2] - E[p_t - d_t]^2} \end{aligned}$$

where $\mathrm{Vol}_{t,t+\tau}$ is given in Equation (A15) and stands for volatility of consumption, dividend, and excess return, *resp.* With specification of all the moment conditions, and the analytical formula of the moments implied by the model that is solved in the paper, the GMM estimation and tests can be carried out as usual (see, e.g., Singleton, 2006).

We show the 26 elements of the moments in g_T as follows. The first 15 moments are:

$$E[r_x],\ E[r_x^2],\ E[r_{x,t+1}, r_{xt}]$$

where r_x stands for consumption growth, dividend growth, excess return, risk free rate, and price-dividend ratio.

Moments 16 and 17 are:

$$E[\mathrm{VRP_t}],\ \mathrm{E}[\mathrm{VRP_t^2}]$$

where VRP is the variance risk premium.

Moments 18 to 20 are:

$$E[r_{x,t+1}, (p_t - d_t)]$$

where r_x stands for consumption growth, dividend growth, and excess return.

Moments 21 to 26 are:

$$E[\log \mathrm{Vol_{xt}}],\ \mathrm{E}[(\log \mathrm{Vol_{xt}}), (\mathrm{p_t} - \mathrm{d_t})]$$

where x stands for consumption, dividend, and excess return. The data can be obtained through quarterly data regression

$$r_{x,t+1} = \alpha + \beta r_{x,t} + \epsilon_{x,t}$$

and annual expected volatility $\mathrm{Vol_{xt}}$ are obtained from

$$\mathrm{Vol_{xt}} = \sum_{\mathrm{k=1}}^{4} |\epsilon_{\mathrm{t+k}}| \tag{B53}$$

Finally, the form of function $h_T = \phi(g_T)$ that links the target functions h_T and the moments g_T, as well as its first-order derivative matrix are elementary, and can be obtained from authors upon request. Q.E.D.

B.9 Accuracy of the Log-linear Approximation

To show that the log-linear approximation (which is accurate when $\psi = 1$) is accurate enough for the parameter values of interest, we take a three step approach. First, we show the standard deviation of the log consumption-wealth ratio is small. Second, we show that the second factor of the two-factor volatility model contributes less than 2% to it. Third, we show that the exact solution of a one-factor with the same magnitude of the standard deviation is very close to the log-linear approximation.

First, extending the approximation in discrete-time models by many, Chacko and Viceira (2005) show that the approximation works for continuous models too as long as the standard deviation of the log consumption-wealth ratio does not vary too much around its unconditional mean. Specifically, the approximation is a Taylor expansion of the consumption-wealth ratio around its unconditional mean level, denoted as g_1,

$$\begin{aligned} \frac{C_t}{W_t} &= e^{c_t-\omega_t} \approx e^{E[c_t-\omega_t]} + e^{E[c_t-\omega_t]} \cdot [(c_t-\omega_t) - E(c_t-\omega_t)], \\ &\approx g_1 - g_1 \log g_1 + g_1 \log(C_t/W_t). \end{aligned} \tag{B54}$$

This implies that a small enough standard deviation of $\log(C_t/W_t)$ yields a good approximation. In Panel A of Table 1, we, like Chacko and Viceira (2005), show that the standard deviation of this ratio is indeed small at less than 1.8% for a range of preference parameters, and is much smaller for ψ closer to 1 (this is not surprising as the approximation is accurate when $\psi = 1$). In particular, for our model parameterization, the standard deviation is 1.6%.

Second, the relative contribution of the second volatility factor to the total standard deviation of the log consumption wealth ratio is small, as shown by the results in Panel B of Table 1. This means that the approximation error for our two factor model is almost the same as a one-factor model.

Finally, we have to show that the approximation error of a one-factor model is indeed small for the parameter values of interest. To do so, we design a one-factor version of our model with the one factor calibrated to the first volatility in our model and provide the exact solution. (Ideally, we want to compare the exact solution of our two-factor model to the linear approximation. But that is too complex to solve.)

The one-factor model is a non-trivial version of the two-factor one,

$$\begin{aligned} \frac{dC_t}{C_t} &= \mu dt + \sqrt{V_t} dZ_t, \\ dV_t &= \kappa(\bar{V} - V_t) + \sigma \sqrt{V_t} dw_t. \end{aligned}$$

We calibrate the parameters to match the first two moments of consumption growth. The value function in steady state can be written as

$$J(V_t, C_t) = e^{\mathcal{G}(V_t)} \frac{C_t^{1-\gamma}}{1-\gamma}. \tag{B55}$$

It can be shown that the solution for $\mathcal{G}(V)$ follows an ODE as

$$\left[\frac{1}{\epsilon}(\beta - \beta e^{\epsilon \mathcal{G}(V)}) + (1-\gamma)(\mu - \frac{\gamma}{2}V)\right] + \kappa(\bar{V} - V)\frac{d\mathcal{G}(V)}{dV} + \frac{1}{2}\sigma^2 V \left(\frac{d\mathcal{G}(V)}{dV}\right)^2 + \frac{1}{2}\sigma^2 V \frac{d^2\mathcal{G}(V)}{dV^2} = 0, \tag{B56}$$

where $\epsilon = \frac{1/\psi - 1}{1-\gamma}$. We solve this ODE numerically and report the results in Figure 1 with both the function values and their differences (errors). It is seen that the solution is almost the same as the linear approximation for the parameter range we consider.

B.10 Monotonicity of the Aggregator

Theoretically, it is very important to note that the log-linear approximation of the aggregator f should be an increasing function of J. Otherwise, it will be in violation of the monotonicity axiom of preferences (see, e.g., Skiadas, 2009, Chapter 6.3). But this is not always the case for all possible parameter values, which is a drawback of certain approximations. However, it should and must be so in the domain of interest of the state variables. Indeed, based on the partial derivative f_J of (B2),

$$f_J = (\theta - 1)\beta G - \beta\theta,$$

we know that the variation of f_J is driven only by βG, which is the consumption wealth ratio based on (B8). In order to check whether $f_J > 0$ for the relevant state variables, we need to verify whether f_J is positive for the reasonable range of consumption wealth ratio, βG, which has mean value equal to g_1. Based on (B8), the standard deviation of βG can be computed as

$$\sigma_{cw} = \sqrt{A_1^2\sigma_x^2 + A_2^2\sigma_{v1}^2 + A_3^2\sigma_{v2}^2} \cdot g_1 \frac{|1-\psi|}{|1-\gamma|}$$

where $\sigma_x, \sigma_{v1}, \sigma_{v2}$ are unconditional standard deviations of state variables X, V_1, V_2. As a result, we only need to check the positivity of f_J in the range of $\beta G \in (g_1 - 2\sigma_{cw}, g_1 + 2\sigma_{cw})$ of interest. Figure 1 shows the numerical values f_J in terms of the number of standard deviation from the mean of βG. Within the range of our interest, f_J is indeed positive as it should.

Table 1: Unconditional Standard Deviation of the Log Consumption-Wealth Ratio

This table shows the consumption-wealth ratio variability around its long term mean level as well as the percentage contribution of the components to the total variation. It shows that the standard deviation of log consumption wealth ratio is less than 2%, hence the approximation of log-linearization is a good one. In addition, the new factor contribution to this variability is small, with less than 1%, due to its short-run nature.

	EIS ψ					
	1.5	1.45	1.4	1.35	1.3	1.2
γ	Standard Deviation of log C-W Ratio (%)					
10	1.610	1.504	1.378	1.237	1.103	0.777
9	1.684	1.572	1.443	1.304	1.152	0.807
8	1.761	1.639	1.514	1.355	1.209	0.848
5	1.650	1.544	1.423	1.355	1.148	0.833
γ	Contribution by the New Factor (%)					
10	0.839	0.883	0.757	0.832	0.848	0.908
9	0.841	0.890	0.763	0.837	0.932	0.937
8	0.847	0.906	0.970	0.867	0.905	0.976
5	1.298	1.301	1.346	1.158	1.453	1.480

Figure 1: Log-linear Approximation vs Numerical Solution

The figure plots the log-linear approximated vs. exact numerical solution for $\mathcal{G}(V)$ of Equation (B56). The parameters are:

$$\psi = 1.5, \gamma = 10, \beta = 0.01, \mu = 0.02, \kappa = 0.035, \bar{V} = 0.0004, \sigma = 0.0026,$$

which are designed to match the unconditional moments of consumption growth and its volatility.

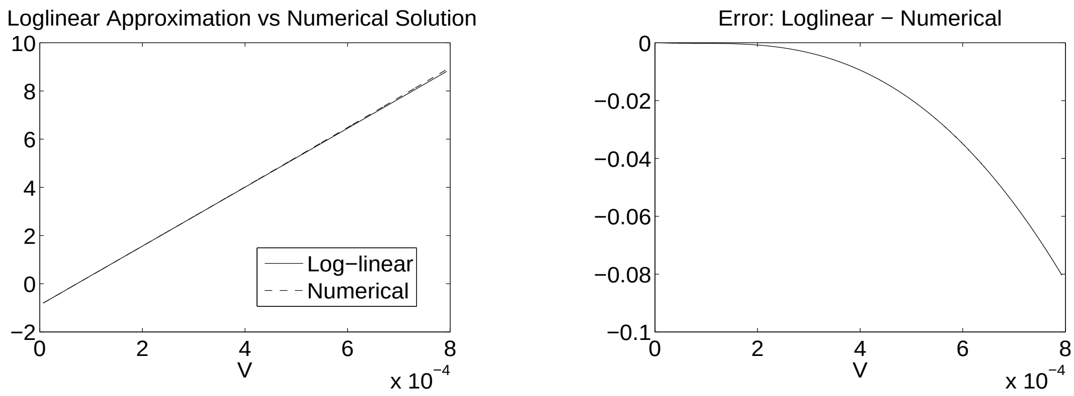

Figure 2: Monotonicity of the Aggregator

The figure plots the partial derivative of aggregator f with respect to J, f_J, vs. the number of standard deviation from the mean value of f_J. The parameters are taken from Table 3 of the paper.