Going to Extremes: Correcting Simulation Bias in Exotic Option Valuation

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Monte Carlo simulation is widely used in practice to value exotic options for which analytical formulas are not available. When valuing those options that depend on extreme values of the underlying asset, convergence of the standard simulation is slow as the time grid is refined, and even a daily simulation interval produces unacceptable errors. This article suggests approximating the extreme value on a subinterval by a random draw from the known theoretical distribution for an extreme of a Brownian bridge on the same interval. This approach provides reliable option values and retains the flexibility of simulations, in that it allows great freedom in choosing a price process for the underlying asset or a joint process for the asset price, its volatility, and other asset prices.

Options on the extreme values of an asset price or index over a specified time interval are widely traded in over-the-counter markets. One approach to valuing this type of option is to sample the asset price path at discrete time intervals and then value the option as an option on the maximum of the discretely sampled points. For any nonzero time interval, this method induces a bias because many time points at which the maximum could occur are ignored. Narrowing the time interval between observations of the price path shrinks the bias toward zero, but a very small time interval may be needed to produce accurate answers.[1] We propose a more efficient simulation algorithm that draws on the theory of the Brownian bridge.

To understand the practical significance of the bias, consider the case of a max call option, which is an option on the maximum price of an asset over the life of the option. For a representative example with a daily simulation of an option with three months to maturity, the option is truly worth $4.57 and the simulation bias is about 42 cents. Because the bias is on the order of the square root of the time subdivision, further subdividing the time interval improves matters very slowly. By comparison, a daily simulation with our adjustment has a bias of only about 2.5 cents. Although a daily simulation using our approach requires more computation than a daily simulation using the traditional approach (and may take as long as a traditional simulation using half-days), the vast increase in precision more than compensates for the extra effort.

The basic idea of our approach is to approximate the price process over an interval by a Brownian bridge on the interval. First, simulate the asset price process at a grid of points, as in the traditional simulation procedure. Then, draw the maximum (or, as relevant, the minimum) of the stock price process on the interval using the known theoretical distribution of a Brownian bridge on an interval. The remaining bias is the result of the cumulative effect of the discretization and of the counterfactual assumption of a constant variance on the interval. This approach retains the flexibility of simulations, in that great freedom exists in choosing the joint process for the asset price and its variance.[2]

BROWNIAN BRIDGE SIMULATION

To illustrate the general methodology, consider the valuation of a max call (or lookback call) option on a stock. The max option is path dependent because it provides its holder with settlement proceeds equal to the difference between the highest stock price over the life of the option and the strike price set at the beginning.

Although most of what we have to say remains valid more generally, assume for concreteness a constant riskless rate, r, and the following diffusion process for the risk-neutralized stock price:

$$dS(t) = rS(t)dt + \sigma[t, S(t)]dW(t), \qquad (1)$$

where $S(0) = S$ is today's stock price and $W(t)$ is the standard Wiener process. There are many variations on the standard simulation approach. Typically, the

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continuous interval is replaced by a finite set of times, and the distribution of changes on each finite interval approximates what would happen in the continuous model.

Let T be the time to maturity. Divide $(0,T)$ into n small subintervals,

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = T,$$

with length $\Delta t \equiv t_{i+1} - t_i = T/n$ for $i = 1,\ldots, n$. The discretized model then is

$$S(t_{i+1}) - S(t_i) = rS(t_i)\Delta t + \sigma[t_i, S(t_i)] [W(t_{i+1}) - W(t_i)]. \quad (2)$$

The standard simulation approach (or more precisely, the variant we studied) is to simulate $[W(t_{i+1}) - W(t_i)]$ from a normal distribution with a mean of zero and variance Δt and then use Equation 2 to obtain a sample path of the stock price. At each such path, the option payoff is evaluated as

$$X = \max [0, \max_{0 \le i \le n} S(t_i) - K], \quad (3)$$

where K is the strike price.

Let X_m be the option payoff given the mth simulated path, and let M be the number of simulated paths. Then,

$$\hat{C} = e^{-rT} \frac{1}{M} \sum_{m=1}^{M} X_m \quad (4)$$

is the approximate call price, where r is the risk-free rate.

The maximum of the price process on a discrete set of times is always lower than its maximum for all times, suggesting that the above procedure tends to underestimate the true option price. Notice that over each time interval (t_i, t_{i+1}), both the drift and diffusion coefficient are approximately constant. Then, a straightforward application of Girsanov's theorem (see, e.g., Karatzas and Shreve 1988, p. 302) implies that over the time interval (t_i, t_{i+1}), the approximate continuous-time stock price process can be written as

$$S(t) = S(t_i) + \sigma[t_i, S(t_i)]B_t, \text{ with } t_i \le t \le t_{i+1}, \quad (5)$$

where B_t is the standard Brownian bridge connecting zero and $b = [S(t_{i+1}) - S(t_i)]/\sigma[t_i, S(t_i)]$ over the time interval (t_i, t_{i+1}).

If we draw the maximum of the Brownian bridge over (t_i, t_{i+1}), $i = 1,\ldots, n$ and compute the global maximum over i, the resulting maximum is clearly larger than the maximum of the simulated prices (see Equation 3). This result suggests that the option price based on the Brownian bridge maximums should overcome the underestimation bias of the standard simulation method. We will see that, in practice, the Brownian bridge approach is indeed much more accurate than the standard simulation method.

The Brownian bridge approach is straightforward to implement if a draw of the Brownian bridge maximum is easy to accomplish. In fact, the distribution of the maximum is well known:

$$P(\max_{t_i \le t \le t_{i+1}} B_t \le x | B_{t_{i+1}} = b) = 1 - e^{-2x(x-b)/h}, \quad (6)$$

where $h = t_{i+1} - t_i$. (See, e.g., Karatzas and Shreve 1988, p. 265.) Hence, a draw of the maximum is easy. Indeed, if u is taken from the standard uniform distribution, then by Equation 6, $\left[b + \sqrt{b^2 - 2h \log(1-u)}\right]/2$ will have the same distribution as the Brownian bridge maximum.

The above procedure simulates the prices directly from the price process. An alternative is to perform the simulation in a transformation of the state variable (and perhaps a transformation of the time variable). A good choice of transform makes the model converge more quickly and prevents the discrete simulation from moving into parts of the state space that are inaccessible in the continuous limit. In our later examples, the volatility is assumed to be proportional to the stock price, that is, $\sigma[t, S(t)] = \sigma S(t)$, where σ is a constant. In this case, a log transform of the state variable would be natural because it makes the stock price a homoscedastic random walk with constant drift and prevents the stock price from going negative. The discretized model can be written as

$$\log S(t_{i+1}) - \log S(t_i) = r\Delta t + \sigma[W(t_{i+1}) - W(t_i)], \quad (7)$$

and hence a draw of the maximum price over the interval (t_i, t_{i+1}) is given by

$$S_i^* = S(t_i)\exp(\sigma B_{\max}), \quad (8)$$

where $B_{\max}$ is a draw from the distribution of the Brownian bridge maximum.

Most of our simulations assume a lognormal price process to allow comparison with known analytical values. For these examples, the log transform would reduce the bias to zero, even with only one time subinterval, because the resulting maximum price would be a draw from its true distribution. Hence, we use no transformations in our simulations in order to avoid painting an unreasonably favorable picture of the success of our procedure. In general, given lognormal prices with constant volatility and a terminal option payoff depending only on the maximum, simulation is not needed at all because the density of the maximum is known and the max option price can be computed either directly or by evaluating a onedimensional integral.

APPLICATIONS AND NUMERICAL COMPARISONS

We applied the Brownian bridge approach to four examples. Although the methodology is equally applicable to a general diffusion process, we assumed a lognormal stock price for the first three examples, allowing comparison of the numerical values with the analytical ones for the first two examples. The third is an example for which there are no known analytical formulas. In the fourth example, we assumed the volatility is a mean-reverting stochastic process instead of a constant, showing that the Brownian bridge approach is easily generalized to a multivariate framework of stochastic processes.

The Max Option

Assume the underlying stock price is log-normal and

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad (9)$$

where both $\mu = r$ and σ are constants. Assume further for simplicity that the strike price is the stock price today. Then, the analytical formula for the max option price is

$$C_{\max} = S(0)e^{-rT} N(d_1)(1 - \sigma^2/2r) - S(0) + S(0)[1 - N(d_2)](1 + \sigma^2/2r), \quad (10)$$

where

$d_1 = -(r - \sigma^2/2)T/\sqrt{\sigma^2 T}$, $d_2 = -(r + \sigma^2/2)T/\sqrt{\sigma^2 T}$, and $N(\cdot)$ is the normal distribution function (Goldman, Sosin, and Shepp 1979).

Let $S(0)$ equal 50, μ equal 15 percent, σ equal 25 percent, and r equal 10 percent. Table 1 reports the analytical prices and the numerical values for the times to maturity, T, of 0.25 (three months) and 1.00 (one year). For the three-month option, the analytical price is 4.567. With 10,000 draws, the simulated prices from the conventional approach are 3.748, 4.008, and 4.126 for three-, two-, and one-day discretizing intervals, respectively. These prices are clearly far away from 4.567, a result that is not attributable to the number of draws, because one million draws yield similar values. In contrast, at the three-, two- and one-day discretizing intervals, the Brownian bridge approach yields 4.405, 4.486, and 4.523, which are much closer than the conventionally determined price to the analytical

Table 1. Max Call Option
(standard errors in parentheses)

		Number of Draws: Simulation					
Time to Maturity	Number of Periods	1,000	5,000	10,000	100,000	1,000,000	Analytical Values
Standard simulation							
0.25	30	3.717	3.735	3.748	3.817	3.823	4.567
		(0.145)	(0.063)	(0.045)	(0.014)	(0.004)	
0.25	60	3.972	4.025	4.008	4.030	4.036	4.567
		(0.134)	(0.062)	(0.044)	(0.014)	(0.004)	
0.25	90	4.245	4.188	4.126	4.131	4.144	4.567
		(0.142)	(0.063)	(0.044)	(0.014)	(0.004)	
1.00	54	6.956	6.814	6.991	7.081	7.072	8.240
		(0.329)	(0.140)	(0.102)	(0.032)	(0.010)	
1.00	250	7.657	7.624	7.758	7.691	7.718	8.240
		(0.322)	(0.145)	(0.103)	(0.032)	(0.010)	
1.00	360	7.582	7.901	7.839	7.840	7.793	8.240
		(0.326)	(0.148)	(0.104)	(0.033)	(0.010)	
Brownian bridge simulation							
0.25	30	4.368	4.391	4.405	4.477	4.484	4.567
		(0.067)	(0.030)	(0.021)	(0.007)	(0.002)	
0.25	60	4.451	4.506	4.486	4.510	4.518	4.567
		(0.065)	(0.029)	(0.021)	(0.007)	(0.002)	
0.25	90	4.646	4.584	4.523	4.528	4.542	4.567
		(0.066)	(0.030)	(0.021)	(0.007)	(0.002)	
1.00	54	8.020	7.863	8.041	8.137	8.127	8.240
		(0.098)	(0.043)	(0.031)	(0.010)	(0.003)	
1.00	250	8.183	8.145	8.278	8.207	8.234	8.240
		(0.097)	(0.043)	(0.031)	(0.010)	(0.003)	
1.00	360	8.022	8.335	8.272	8.273	8.225	8.240
		(0.097)	(0.044)	(0.031)	(0.010)	(0.003)	

Notes: The max or lookback call option pays its holder with settlement proceeds equal to the difference between the highest stock price over the life of the option and the strike price set at the beginning. Assume the stock price is lognormal with $S(0) = 50$, $\mu = 15$ percent, $\sigma = 25$ percent, $r = 10$ percent, and $X = 50$.

price of 4.567; in fact, the daily discretizing interval produces prices within pennies of the analytical one.

As the option maturity lengthens, the conventional simulation approach worsens. For example, at the daily discretizing interval and with 10,000 draws, it produces a value of 7.839, about 40 cents away from the analytical price 8.240. In contrast, the Brownian bridge approach produces a reliable numerical value of 8.272.

The Knock-Out Option

The knock-out option represents a wide range of barrier options. Consider a down-and-out call option. Technical traders who believe the market will go down once a certain support level is penetrated are particularly interested in this type of option. In addition, traders who have a bullish short-term and long-term outlook on the stock and would like a cheap call would be interested in a down-and-out call. The payoff of a simple down-and-out call is the payoff of an otherwise identical standard European call option if the price does not touch a predetermined level, H. If that level is touched, the option is worthless at expiration. The analytical value, C_{do}, is

$$C_{do} = B[S(0), X] - \left[\frac{S0}{H}\right]^{-\gamma} B\left[\frac{H^2}{S(0)}, X\right], \qquad (11)$$

where $\gamma = 2r/\sigma^2$ and $B(S,X)$ is the standard Black–Scholes formula for a call with stock price S and strike X (see, e.g., Cox and Rubinstein 1985, p. 410).

The implementation of both the standard simulation approach and the Brownian bridge approach for the down-and-out call option is similar to the max option. For the standard simulation, the only change is that the payoff, previously Equation 3, is replaced by

$$X = S_T - K, \text{ if } S_T > K \text{ and } \min_{0 \le i \le n} S(t_i) > H; \ 0, \text{ otherwise.} \qquad (12)$$

For the Brownian Bridge simulation, instead of the maximum, it is the minimum over each of the discretizing intervals that needs to be drawn and used to evaluate whether the option has gone down and out. To draw the minimum of the Brownian bridge B_t connecting zero and $b = [S(t_{i+1}) - S(t_i)]/\sigma[t_i, S(t_i)]$ over time $(0,h)$, we need only to draw the maximum of the Brownian bridge B_t connecting zero and $-b$ as before. Then, the reflection principle suggests that the minus of the maximum yields the desired minimum.

Assume the same option parameter values as in the max option case. In addition, let the predetermined level H equal 45. Table 2 provides the results that show gains in accuracy similar to the case of the max option. For example, at the daily discretizing interval and with 10,000 draws, the conventional simulation approach produces a value of 4.344, about 30 cents away from the analytical price of 4.032. In contrast, the Brownian bridge approach produces a reliable numerical value of 4.037.

The Swing Option

The swing option pays the difference between the maximum and minimum stock prices minus a fixed strike. This option is useful for investors who expect the stock to have a large trading range. Although the swing option has no known analytical solutions, the Brownian bridge approach is straightforward to apply to the payoff:[3]

$$X = \max[0, \max_{0 \le i \le n} S(t_i) - \min_{0 \le i \le n} S(t_i) - K]. \qquad (13)$$

Assume the same option parameter values as in the max option case. Table 3 provides the results. We do not have analytical prices because of the lack of analytical formulas for the swing option. Nevertheless, one can verify that the Brownian bridge approach produces reliable numerical values because the values converge and change little by further decreasing the discretizing interval. These values, however, differ greatly from those computed by using the conventional simulation approach. For example, at the daily discretizing interval and with 10,000 draws, the difference is about 30 cents for the three-month option, suggesting again the inadequacy of the conventional simulation approach.

Lookback Call

In this example, which is a lookback call as in the first example, the volatility is no longer a constant and follows a mean-reverting stochastic process. The risk-neutralized joint process of the stock price and the volatility can be written as

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dW_1(t) \qquad (14)$$

and

$$dv(t) = \kappa[\bar{v} - v(t)]dt + \sigma_v v(t) dW_2(t), \qquad (15)$$

where κ, $\bar{v}$, and σ_v are constants. Assume for simplicity that $dW_1(t)$ and $dW_2(t)$ are independent.

In this case, the standard options have no explicit solutions; hence, solutions to the max option are even more intractable. The simulation procedure, however, is straightforward to apply.[4] The only change will be one additional draw of the volatility at each of the discretized intervals (t_i, t_{i+1}).

Assume the same option parameter values as before. In addition, let the volatility parameters have the following values: v_0 = 25 percent, $\kappa = 2$, $\bar{v}$ = 25 percent, and σ_v = 10 percent. Table 4 provides the results. As in the previous cases, the Brownian bridge procedure greatly reduces the bias.

CONCLUSIONS

We have proposed and demonstrated a "Brownian bridge" adjustment to simulation procedures for computing prices of options whose payoffs depend on extreme values. The approach significantly reduces the bias in pricing lookback and barrier options.

Other practical issues should be addressed in the implementation of these results. How do we include the bid–ask spread? More generally, how do we formulate and estimate the volatility process? The best answers to these questions vary across markets with different institutional structures and trading patterns and also depend on the details of the contract under consideration.

Table 2. Knock-Out Option
(standard errors in parentheses)

		Number of Draws: Simulation					
Time to Maturity	Number of Periods	1,000	5,000	10,000	100,000	1,000,000	Analytical Values
Standard simulation							
0.25	30	4.503	4.407	4.448	4.570	4.550	4.032
		(0.093)	(0.041)	(0.029)	(0.009)	(0.003)	
0.25	60	4.208	4.347	4.289	4.375	4.392	4.032
		(0.088)	(0.041)	(0.029)	(0.009)	(0.003)	
0.25	90	4.438	4.382	4.344	4.312	4.348	4.032
		(0.093)	(0.041)	(0.029)	(0.009)	(0.003)	
1.00	54	7.382	6.498	6.947	7.032	7.005	5.477
		(0.134)	(0.057)	(0.041)	(0.013)	(0.004)	
1.00	250	5.920	6.159	6.296	6.251	6.233	5.477
		(0.123)	(0.057)	(0.041)	(0.013)	(0.004)	
1.00	360	6.120	5.929	6.005	6.127	6.098	5.477
		(0.130)	(0.056)	(0.040)	(0.013)	(0.004)	
Brownian bridge simulation							
0.25	30	3.771	3.817	3.887	4.062	4.044	4.032
		(0.091)	(0.040)	(0.028)	(0.009)	(0.003)	
0.25	60	3.735	3.920	3.892	4.009	4.018	4.032
		(0.086)	(0.040)	(0.028)	(0.009)	(0.003)	
0.25	90	4.215	4.048	4.037	4.012	4.037	4.032
		(0.092)	(0.041)	(0.029)	(0.009)	(0.003)	
1.00	54	6.494	5.219	5.480	5.599	5.576	5.477
		(0.132)	(0.055)	(0.039)	(0.013)	(0.004)	
1.00	250	5.061	5.416	5.439	5.536	5.491	5.477
		(0.119)	(0.056)	(0.040)	(0.013)	(0.004)	
1.00	360	5.574	5.353	5.379	5.520	5.490	5.477
		(0.128)	(0.055)	(0.040)	(0.013)	(0.004)	

Notes: The knock-out option pays nothing when a predetermined level H is touched by the stock price and offers the payoff of a standard call option otherwise. Assume the stock price is lognormal with $S(0) = 50$, $\mu = 15$ percent, $\sigma = 50$ percent, $r = 10$ percent, $X = 50$, and $H = 45$.

Table 3. Swing Option
(standard errors in parentheses)

Time to Maturity	Number of Periods	Number of Draws: Simulation 1,000	5,000	10,000	100,000	1,000,000
Standard simulation						
0.25	30	0.814	0.755	0.783	0.791	0.785
		(0.063)	(0.027)	(0.019)	(0.006)	(0.002)
0.25	60	0.801	0.912	0.911	0.913	0.919
		(0.057)	(0.029)	(0.021)	(0.007)	(0.002)
0.25	90	1.043	0.987	0.967	0.971	0.983
		(0.069)	(0.030)	(0.021)	(0.007)	(0.002)
1.00	54	8.312	8.284	8.409	8.496	8.479
		(0.245)	(0.104)	(0.076)	(0.024)	(0.008)
1.00	250	9.518	9.557	9.619	9.537	9.552
		(0.244)	(0.111)	(0.079)	(0.025)	(0.008)
1.00	360	9.546	9.858	9.777	9.750	9.705
		(0.250)	(0.114)	(0.080)	(0.025)	(0.008)
Brownian bridge simulation						
0.25	30	1.216	1.164	1.203	1.215	1.209
		(0.048)	(0.021)	(0.015)	(0.005)	(0.001)
0.25	60	1.124	1.242	1.241	1.246	1.252
		(0.045)	(0.021)	(0.015)	(0.005)	(0.002)
0.25	90	1.320	1.269	1.251	1.255	1.268
		(0.049)	(0.022)	(0.015)	(0.005)	(0.002)
1.00	54	10.087	10.074	10.194	10.271	10.251
		(0.085)	(0.037)	(0.027)	(0.008)	(0.003)
1.00	250	10.393	10.417	10.481	10.397	10.412
		(0.084)	(0.038)	(0.027)	(0.008)	(0.003)
1.00	360	10.254	10.577	10.496	10.471	10.425
		(0.085)	(0.038)	(0.027)	(0.009)	(0.003)

Notes: The swing option pays the difference between the maximum and minimum stock price minus a fixed strike. Assume the stock price is lognormal with $S(0) = 50$, $\mu = 15$ percent, $\sigma = 25$ percent, $r = 10$ percent, and $X = 50$.

Table 4. Lookback Option with Stochastic Volatility
(standard errors in parentheses)

Time to Maturity	Number of Periods	Number of Draws: Simulation 1,000	5,000	10,000	100,000	1,000,000
Standard simulation						
0.25	30	3.616	3.738	3.784	3.819	3.819
		(0.140)	(0.062)	(0.044)	(0.014)	(0.004)
0.25	60	4.314	4.114	4.048	4.055	4.048
		(0.151)	(0.063)	(0.044)	(0.014)	(0.004)
0.25	90	4.344	4.147	4.134	4.132	4.146
		(0.149)	(0.063)	(0.045)	(0.014)	(0.004)
1.00	54	6.716	6.996	7.013	7.099	7.084
		(0.328)	(0.144)	(0.102)	(0.032)	(0.010)
1.00	250	8.229	7.725	7.761	7.688	7.713
		(0.325)	(0.145)	(0.102)	(0.032)	(0.010)
1.00	360	7.943	7.697	7.654	7.798	7.799
		(0.313)	(0.143)	(0.102)	(0.032)	(0.010)
Brownian bridge simulation						
0.25	30	4.268	4.399	4.443	4.481	4.480
		(0.142)	(0.063)	(0.045)	(0.014)	(0.005)
0.25	60	4.807	4.598	4.528	4.536	4.529
		(0.152)	(0.064)	(0.045)	(0.014)	(0.005)
0.25	90	4.752	4.549	4.535	4.531	4.544
		(0.150)	(0.064)	(0.045)	(0.014)	(0.004)
1.00	54	7.768	8.049	8.070	8.157	8.139
		(0.333)	(0.147)	(0.104)	(0.033)	(0.010)
1.00	250	8.763	8.241	8.278	8.202	8.228
		(0.328)	(0.147)	(0.103)	(0.033)	(0.010)
1.00	360	8.375	8.134	8.087	8.231	8.231
		(0.316)	(0.144)	(0.103)	(0.033)	(0.010)

Notes: The max or lookback call option pays its holder with settlement proceeds equal to the difference between the highest stock price over the life of the option and the strike price set at the beginning. Assume the stock price is lognormal $S(0) = 50$, $\mu = 15$ percent, $\sigma = 25$ percent, $r = 10$ percent, and $X = 50$ but the volatility follows the standard mean-reverting process with $v_0 = 25$ percent, $\kappa = 2$, $\bar{v} = 25$ percent, and $\sigma_v = 10$ percent.

NOTES

1. For a general analysis of simulation methods in option pricing, see, for example, Campbell, Lo, and MacKinlay (1997, Chapter 9).
2. Mathematically, our analysis is similar to that of Rogers and Satchell (1991), who looked at a different but related problem of estimating variance from high, low, and close data when the high and low are based on discrete observations rather than a continuous trading record. Hobson (1995) and Carr, Ellis, and Gupta (forthcoming) have a completely different approach, which bounds lookback prices, given call prices. Broadie, Glasserman, and Kou (1996) suggest bias correction terms for a discrete model without simulation.
3. A bias is introduced by drawing the maximum and the minimum independently on each subinterval. The bias tends to zero, however, as all the subintervals shrink, because it becomes rarer for the global maximum and minimum to lie in the same subinterval.
4. Formally, this is a case for which the diffusion assumption and notation we used earlier for convenience do not apply. Application of the procedure is obvious, however, because of the state dependency of the variance. Given that variance is a smooth function of time, using the variance at the start of the time interval is a sensible procedure.

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