Technical Appendix
for
Bayesian Inference in Asset Pricing Tests

Inference on $\boldsymbol{\alpha}$

1. Diffuse Prior:

$$(pdf) \qquad P(\boldsymbol{\alpha}) \propto [v + (\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})'\mathbf{H}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})]^{-\frac{v+N}{2}}, \tag{8}$$

where $v = T - 1 - N$, $\mathbf{H} = v\mathbf{S}^{-1}/a$. Note also that the exponent $-(v+N)/2 = -(T-1)/2$.

$$E[\boldsymbol{\alpha}] = \hat{\boldsymbol{\alpha}}$$

$$\mathrm{Var}[\boldsymbol{\alpha}] = \frac{v}{v-2}\mathbf{H}^{-1} = \frac{a}{v-2}\mathbf{S} = \frac{a}{T-3-N}\mathbf{S}$$

Random samples of $\boldsymbol{\alpha} \sim$ can be generated by drawing:

$$(i) \quad \mathbf{X} \sim N(0, \boldsymbol{\Omega}). \quad \boldsymbol{\Omega} \equiv a\mathbf{S}$$

$$(ii) \quad y \sim \chi^2(v) \quad \text{or} \quad y = 2y^*, \; y^* \sim \text{gamma}(\frac{v}{2})$$

$$\Rightarrow \boldsymbol{\alpha} \equiv \frac{\mathbf{X}}{\sqrt{y}} + \hat{\boldsymbol{\alpha}}$$

The proof that this is distributed multivariate Student t (MVT) given by (8) follows from:

Proposition:

With $\mathbf{X}\sim N(0, \boldsymbol{\Omega})$, $y \sim \chi^2(n)$ then

$$\mathbf{Z} \equiv \frac{\mathbf{X}}{\sqrt{y}} \sim MVT \begin{cases} \mathbf{H} = n\boldsymbol{\Omega}^{-1} \\ \hat{\boldsymbol{\alpha}} = 0 \end{cases}$$

Proof:

$$y \text{ has density} \qquad P(y) = \frac{y^{\frac{n}{2}-1}e^{-\frac{y}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \qquad \text{and} \qquad \mathbf{Z}|y \sim N(0, \frac{\boldsymbol{\Omega}}{y})$$

thus the joint density of y and $\mathbf{Z}$ is:

$$P(\mathbf{Z}, y) = P(\mathbf{Z}|y)P(y) \propto$$

$$\left|\frac{\boldsymbol{\Omega}}{y}\right|^{-\frac{1}{2}} exp[-\frac{1}{2}\mathbf{Z}'\left(\frac{\boldsymbol{\Omega}}{y}\right)^{-1}\mathbf{Z}]y^{\frac{n}{2}-1}exp(-\frac{y}{2}) \propto$$

$$y^{\frac{N}{2}} exp[-\frac{1}{2}\mathbf{Z}'\boldsymbol{\Omega}^{-1}\mathbf{Z}y]y^{\frac{n}{2}-1}exp[-\frac{1}{2}y]$$

which can be written:

$$y^{\frac{n+N}{2}-1}exp[-\frac{1}{2}qy], \quad q \equiv 1 + \mathbf{Z}'\boldsymbol{\Omega}^{-1}\mathbf{Z}$$

or

$$q^{-\frac{n+N}{2}+1}\underbrace{(qy)^{\frac{n+N}{2}-1}exp[-\frac{1}{2}qy]}_{\text{(a } \chi^2 \text{ density)}}$$

Integrating y out, we get the density of $\mathbf{Z}$

$$P(\mathbf{Z}) \propto q^{-\frac{n+N}{2}}$$

Notice $[\int P(\mathbf{Z}, y)dy = q^{-1}\int P(\mathbf{Z}, y)d(qy)]$. ▮

2. With the investigator's prior: $P_0(\boldsymbol{\alpha})$

$$(pdf) \qquad P(\boldsymbol{\alpha}) \propto P_0(\boldsymbol{\alpha})[v + (\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})'\mathbf{H}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})]^{-\frac{v+N}{2}}$$
$$= \text{constant} \times P_0(\boldsymbol{\alpha}) \times \text{standardized MVT}$$

where

$$\text{constant} = \left[\int P_0(\boldsymbol{\alpha}) \times \text{standardized MVT} d\boldsymbol{\alpha}\right]^{-1}$$

so that

$$\int P(\boldsymbol{\alpha}) d(\boldsymbol{\alpha}) = 1$$

The mean or function of interest is $g(\boldsymbol{\alpha})$ is

$$\int g(\boldsymbol{\alpha})\text{standardized}\, P(\boldsymbol{\alpha}) d\boldsymbol{\alpha}$$
$$= \text{constant} \int g(\boldsymbol{\alpha}) P_0(\boldsymbol{\alpha}) \times \text{standardized}\, MVT\, d\boldsymbol{\alpha}$$
$$= \left[\int g(\boldsymbol{\alpha}) P_0(\boldsymbol{\alpha}) \times \text{standardized}\, MVT\, d\boldsymbol{\alpha}\right] / \text{contant}_1$$

where

$$\text{constant}_1 \equiv \int P_0(\boldsymbol{\alpha}) \times \text{standardized}\, MVT\, d\boldsymbol{\alpha}$$
$$= 1/\text{constant}$$

Inference on λ

$$\lambda \equiv \boldsymbol{\alpha}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\alpha}$$
$$E[\lambda] = \int \lambda(\text{standardized } P(\boldsymbol{\alpha}, \boldsymbol{\Sigma})) d\boldsymbol{\alpha} d\boldsymbol{\Sigma}$$
$$P(\boldsymbol{\alpha}, \boldsymbol{\Sigma}) = P(\boldsymbol{\alpha}|\boldsymbol{\Sigma}) P(\boldsymbol{\Sigma})$$
$$P(\boldsymbol{\Sigma}) \propto |\boldsymbol{\Sigma}|^{-\frac{T-1+N}{2}} exp[-\frac{1}{2} tr \boldsymbol{\Sigma}^{-1} \mathbf{S}]$$

which is inverted Wishart, with deg.$=T-2$.

$$P(\boldsymbol{\alpha}|\boldsymbol{\Sigma}) \propto |\boldsymbol{\Sigma}|^{-\frac{1}{2}} exp[-\frac{1}{2}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})'(a\boldsymbol{\Sigma})^{-1}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})]$$

which is normal with mean $\hat{\boldsymbol{\alpha}}$ and variance $a\boldsymbol{\Sigma}$.

Draw samples from IW$(T-2, N)$:

(i) Decompose (once) $\mathbf{S}^{-1} = \mathbf{L}\mathbf{L}'$, where $\mathbf{L}$ is lower triangular.

(ii) Get $U_{ij} \sim N(0,1)$, $i > j$ and $U_{ii} \sim \sqrt{\chi^2(T-1-i)}$, $i = 1, \ldots, N$

(iii) Form the lower triangular $\mathbf{U}$ matrix so [Geweke (1988)]

$$\boldsymbol{\Sigma} = \mathbf{R}'\mathbf{R} \sim IW(\mathbf{S}, T-2)$$

with

$$\mathbf{R} = (\mathbf{L}\mathbf{U})^{-1}$$

and

$$\boldsymbol{\Sigma}^{-1} = (\mathbf{L}\mathbf{U})(\mathbf{L}\mathbf{U})' \sim W(\mathbf{S}^{-1}, T-2)$$

Odds Ratio under Cauchy Prior

Prior:

$$H_0: \quad P(\cdot|H_0) \propto |\boldsymbol{\Sigma}|^{-\frac{N+1}{2}}$$
$$H_A: \quad P(\cdot|H_A) \propto P(\boldsymbol{\alpha}|\boldsymbol{\Sigma})|\boldsymbol{\Sigma}|^{-\frac{N+1}{2}}$$

with $P(\boldsymbol{\alpha}|\boldsymbol{\Sigma})$ being Cauchy:

$$P(\boldsymbol{\alpha}|\boldsymbol{\Sigma}) = \frac{c|k\boldsymbol{\Sigma}|^{-\frac{1}{2}}}{(1+\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}/k)^{\frac{N+1}{2}}}$$

Odds Ratio:

$$K_c = \frac{\int\int L(\boldsymbol{\beta},\boldsymbol{\Sigma}|H_0)P(\boldsymbol{\beta},\boldsymbol{\Sigma}|H_0)d\boldsymbol{\beta}d\boldsymbol{\Sigma}}{\int\int\int L(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Sigma}|H_A)P(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\Sigma}|H_A)d\boldsymbol{\alpha}d\boldsymbol{\beta}d\boldsymbol{\Sigma}} \equiv \frac{I_1}{I_2}, \tag{19}$$

(i) The Numerator

Since $\boldsymbol{\alpha} = 0$, we can write:

$$\mathbf{Y} - \mathbf{X}\mathbf{B} = \begin{pmatrix} y_{11} & \dots & Y_{N1} \\ \vdots & \ddots & \vdots \\ y_{1T} & \dots & y_{NT} \end{pmatrix} - \begin{pmatrix} X_1 \\ \vdots \\ X_T \end{pmatrix} \begin{pmatrix} \beta_1 & \dots & \beta_N \end{pmatrix} \equiv \mathbf{Y} - \mathbf{X}_0\boldsymbol{\beta}'.$$

Let

$$\hat{\boldsymbol{\beta}}_0' = [(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{X}_0'\mathbf{Y}]'.$$
$$\mathbf{S}_0 \equiv (\mathbf{Y} - \mathbf{X}_0\hat{\boldsymbol{\beta}}_0')'(\mathbf{Y} - \mathbf{X}_0\hat{\boldsymbol{\beta}}_0'),$$

and

$$b \equiv \left(\sum_{i=1}^{T} X_i^2\right)^{-1}$$

then

$$I_1 = \int\int (2\pi)^{-\frac{TN}{2}}|\boldsymbol{\Sigma}|^{-\frac{T}{2}} exp[-\frac{1}{2}tr(\mathbf{Y} - \mathbf{X}_0\boldsymbol{\beta}')'(\mathbf{Y} - \mathbf{X}_0\boldsymbol{\beta}')\boldsymbol{\Sigma}^{-1}]|\boldsymbol{\Sigma}|^{-\frac{N+1}{2}}d\boldsymbol{\beta}d\boldsymbol{\Sigma}$$

Notice that

$$(\mathbf{Y} - \mathbf{X}_0\boldsymbol{\beta}')'(\mathbf{Y} - \mathbf{X}_0\boldsymbol{\beta}') = \mathbf{S}_0 + (\boldsymbol{\beta}' - \hat{\boldsymbol{\beta}}_0')'\mathbf{X}_0'\mathbf{X}_0(\boldsymbol{\beta}' - \hat{\boldsymbol{\beta}}_0')$$

and

$$tr(\boldsymbol{\beta}' - \hat{\boldsymbol{\beta}}_0')'\mathbf{X}_0'\mathbf{X}_0(\boldsymbol{\beta}' - \hat{\boldsymbol{\beta}}_0')\boldsymbol{\Sigma}^{-1} = \frac{1}{b}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0).$$

$$\begin{aligned}
\Rightarrow \quad I_1 &= (2\pi)^{-\frac{TN}{2}} \int \left\{ \int |\boldsymbol{\Sigma}|^{-\frac{1}{2}} exp[-\frac{1}{2b}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)]d\boldsymbol{\beta} \right\} |\boldsymbol{\Sigma}|^{-\frac{T+N}{2}} exp[-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}_0]d\boldsymbol{\Sigma} \\
&= (2\pi)^{-\frac{TN}{2}} \int (2\pi)^{\frac{N}{2}} b^{\frac{N}{2}} |\boldsymbol{\Sigma}|^{-\frac{T+N}{2}} exp[-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}_0]d\boldsymbol{\Sigma} \\
&= (2\pi)^{-\frac{TN}{2}} (2\pi)^{\frac{N}{2}} b^{\frac{N}{2}} |\mathbf{S}_0|^{-\frac{v}{2}} \int \frac{|\mathbf{S}_0|^{-\frac{v}{2}}}{|\boldsymbol{\Sigma}|^{\frac{v+N+1}{2}}} exp[-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}_0]d\boldsymbol{\Sigma} \\
&= (2\pi)^{-\frac{TN}{2}} (2\pi)^{\frac{N}{2}} b^{\frac{N}{2}} |\mathbf{S}_0|^{-\frac{v}{2}} C(v)
\end{aligned}$$

where $v = T - 1$ and

$$C(v) \equiv 2^{\frac{vN}{2}} \pi^{\frac{N(N+1)}{4}} \prod_{i=1}^{N} T[(v+1-i)/2]$$

The denominator

$$I_2 = \int\int\int (2\pi)^{-\frac{TN}{2}} |\boldsymbol{\Sigma}|^{-\frac{T}{2}} exp[-\frac{1}{2} tr(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB})] |\boldsymbol{\Sigma}|^{-\frac{N+1}{2}} P(\boldsymbol{\alpha}|\boldsymbol{\Sigma}) d\boldsymbol{\alpha} d\boldsymbol{\beta} d\boldsymbol{\Sigma}$$

Notice that

$$(\mathbf{B} - \hat{\mathbf{B}})'\mathbf{X}'\mathbf{X}(\mathbf{B} - \hat{\mathbf{B}}) = \frac{1}{a}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})' + \frac{1}{b}(\boldsymbol{\beta} - \overline{\boldsymbol{\beta}})(\boldsymbol{\beta} - \overline{\boldsymbol{\beta}})'$$

so, conditional on $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}$, $\boldsymbol{\beta}$ has mean $\overline{\boldsymbol{\beta}}$ and covariance $b\boldsymbol{\Sigma}$. Integrating $\boldsymbol{\beta}$ out we get:

$$I_2 = \int\int (2\pi)^{-\frac{TN}{2}} |\boldsymbol{\Sigma}|^{-\frac{T}{2}} (2\pi)^{\frac{N}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}} b^{\frac{N}{2}} exp[-\frac{1}{2a}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})] |\boldsymbol{\Sigma}|^{-\frac{N+1}{2}} P(\boldsymbol{\alpha}|\boldsymbol{\Sigma}) exp[-\frac{1}{2} tr\boldsymbol{\Sigma}^{-1}\mathbf{S}] d\boldsymbol{\alpha} d\boldsymbol{\Sigma}$$

$$= (2\pi)^{-\frac{TN}{2}} (2\pi)^{\frac{N}{2}} b^{\frac{N}{2}} \int\int exp[-\frac{1}{2a}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})] |\boldsymbol{\Sigma}|^{-\frac{T+N}{2}} exp[-\frac{1}{2} tr\boldsymbol{\Sigma}^{-1}\mathbf{S}] d\boldsymbol{\alpha} d\boldsymbol{\Sigma}$$

$$= (2\pi)^{-\frac{TN}{2}} (2\pi)^{\frac{N}{2}} b^{\frac{N}{2}} \mathbf{S}^{-\frac{v}{2}} C(v) \int\int exp[-\frac{1}{2a}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})] P(\boldsymbol{\alpha}|\boldsymbol{\Sigma}) P(\boldsymbol{\Sigma}) d\boldsymbol{\alpha} d\boldsymbol{\Sigma}$$

with

$$P(\boldsymbol{\Sigma}) = \frac{1}{C(v)} \frac{|\mathbf{S}|^{\frac{v}{2}}}{|\boldsymbol{\Sigma}|^{\frac{v+N+1}{2}}} exp[-\frac{1}{2} tr\boldsymbol{\Sigma}^{-1}\mathbf{S}]$$

which is the standard inverted Wishart (IW) density. Therefore,

$$K_c = \left(\frac{|\mathbf{S}|}{|\mathbf{S}_0|}\right)^{\frac{v}{2}} / Q, \qquad v = T - 1 \tag{20}$$

and

$$Q = \int\int exp[-\frac{1}{2a}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})] \underbrace{P(\boldsymbol{\alpha}|\boldsymbol{\Sigma})}_{\text{Cauchy}} \underbrace{P(\boldsymbol{\Sigma})}_{\text{IW}} d\boldsymbol{\alpha} d\boldsymbol{\Sigma}$$

Odds Ratio under Normal Prior

Prior:

$$H_0: \quad P(\cdot|H_0) \propto |\boldsymbol{\Sigma}|^{-\frac{N+1}{2}}$$

$$H_A: \quad P(\cdot|H_A) \propto P(\boldsymbol{\alpha}|\boldsymbol{\Sigma})|\boldsymbol{\Sigma}|^{-\frac{N+1}{2}}$$

with $P(\boldsymbol{\alpha}|\boldsymbol{\Sigma})$ being normal:

$$P(\boldsymbol{\alpha}|\boldsymbol{\Sigma}) = (2\pi)^{-\frac{N}{2}}|k\boldsymbol{\Sigma}|^{-\frac{1}{2}}exp[-\frac{1}{2}\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}/k]$$

The Numerator

Same as the Cauchy case.

The Denominator

$$I_2 = \iiint (2\pi)^{-\frac{TN}{2}}|\boldsymbol{\Sigma}|^{-\frac{T}{2}}exp[-\frac{1}{2}tr(\mathbf{Y}-\mathbf{XB})'(\mathbf{Y}-\mathbf{XB})]|\boldsymbol{\Sigma}|^{-\frac{N+1}{2}}P(\boldsymbol{\alpha}|\boldsymbol{\Sigma})d\boldsymbol{\alpha}d\boldsymbol{\beta}d\boldsymbol{\Sigma}$$

$$= (2\pi)^{-\frac{TN}{2}}(2\pi)^{\frac{N}{2}}b^{\frac{N}{2}}\iint P(\boldsymbol{\alpha}|\boldsymbol{\Sigma})exp[-\frac{1}{2a}(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}})]|\boldsymbol{\Sigma}|^{-\frac{T+N}{2}}exp[-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}]d\boldsymbol{\alpha}d\boldsymbol{\Sigma}$$

$$= (2\pi)^{-\frac{TN}{2}}(2\pi)^{\frac{N}{2}}b^{\frac{N}{2}}\iint (\frac{1}{k})^{\frac{N}{2}}(2\pi)^{-\frac{N}{2}}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}exp[-\frac{1}{2}\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}/k]exp[-\frac{1}{2a}(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}})]|\boldsymbol{\Sigma}|^{-\frac{T+N}{2}}exp[-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}]d\boldsymbol{\alpha}d\boldsymbol{\Sigma}$$

$$= (2\pi)^{-\frac{TN}{2}}(2\pi)^{\frac{N}{2}}b^{\frac{N}{2}}(\frac{a}{k})^{\frac{N}{2}}\iint exp[-\frac{1}{2}\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}/k](2\pi)^{-\frac{N}{2}}|a\boldsymbol{\Sigma}|^{-\frac{1}{2}}exp[-\frac{1}{2a}(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}})]|\boldsymbol{\Sigma}|^{-\frac{T+N}{2}}exp[-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}]d\boldsymbol{\alpha}d\boldsymbol{\Sigma}$$

$$= (2\pi)^{-\frac{TN}{2}}(2\pi)^{\frac{N}{2}}b^{\frac{N}{2}}(\frac{a}{k})^{\frac{N}{2}}|\mathbf{S}|^{-\frac{v}{2}}C(v)\iint exp[-\frac{1}{2}\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}/k]\underbrace{f(\boldsymbol{\alpha}|\boldsymbol{\Sigma})}_{\text{normal}}\underbrace{P(\boldsymbol{\Sigma})}_{\text{std. IW}}d\boldsymbol{\alpha}d\boldsymbol{\Sigma}$$

Therefore:

$$K_n = \left(\frac{|\mathbf{S}|}{|\mathbf{S}_R|}\right)^{\frac{T-1}{2}}\left(\frac{k}{a}\right)^{\frac{N}{2}}/Q$$

where

$$Q = \iint exp[-\frac{1}{2}\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}/k]f(\boldsymbol{\alpha}|\boldsymbol{\Sigma})P(\boldsymbol{\Sigma})d\boldsymbol{\alpha}d\boldsymbol{\Sigma}$$

Odds Ratio using the Savage Density

1. Prior under the alternative

$$P_A = P(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}|H_A) \propto \underbrace{P(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{\Sigma})}_{\text{normal}} \underbrace{P(\boldsymbol{\Sigma})}_{\text{IW}}$$

2. Prior under the null hypothesis

$$P(\boldsymbol{\beta}, \boldsymbol{\Sigma}|H_0) = P_A|_{\alpha=0} \propto P(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{\Sigma})P(\boldsymbol{\Sigma})|_{\alpha=0}$$

3. Odds ratio

$$K_s = \frac{\text{marginal posterior density of } \boldsymbol{\alpha} \text{ at } \boldsymbol{\alpha} = 0}{\text{marginal prior density of } \boldsymbol{\alpha} \text{ at } \boldsymbol{\alpha} = 0}$$

Proof:

$$K_s = \frac{\int\int L(\boldsymbol{\beta}, \boldsymbol{\Sigma}|H_0)P(\boldsymbol{\beta}, \boldsymbol{\Sigma}|H_0)d\boldsymbol{\beta}d\boldsymbol{\Sigma}}{\int\int\int L(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}|H_A)P(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}|H_A)d\boldsymbol{\alpha}d\boldsymbol{\beta}d\boldsymbol{\Sigma}}$$

$$= \frac{\int\int L(\boldsymbol{\beta}, \boldsymbol{\Sigma}|H_0)\left[\frac{P(\alpha,\beta|\Sigma)P(\Sigma)|_{\alpha=0}}{\int\int P(\alpha,\beta|\Sigma)P(\Sigma)|_{\alpha=0}\, d\beta d\Sigma}\right]d\boldsymbol{\beta}d\boldsymbol{\Sigma}}{\int\int\int L(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}|H_A)\left[\frac{P(\alpha,\beta|\Sigma)P(\Sigma)}{\int\int\int P(\alpha,\beta|\Sigma)P(\Sigma)d\alpha d\beta d\Sigma}\right]d\boldsymbol{\alpha}d\boldsymbol{\beta}d\boldsymbol{\Sigma}}$$

Note that:

$$\int\int\int P(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{\Sigma})P(\boldsymbol{\Sigma})d\boldsymbol{\alpha}d\boldsymbol{\beta}d\boldsymbol{\Sigma} \equiv 1$$

So we can express:

$$= \frac{1}{\int\int P(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{\Sigma})P(\boldsymbol{\Sigma})|_{\alpha=0}\, d\boldsymbol{\beta}d\boldsymbol{\Sigma}} \times \frac{\int\int L(\boldsymbol{\beta}, \boldsymbol{\Sigma}|H_0)P(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{\Sigma})P(\boldsymbol{\Sigma})|_{\alpha=0}\, d\boldsymbol{\beta}d\boldsymbol{\Sigma}}{\int\int\int L(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}|H_A)P(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{\Sigma})P(\boldsymbol{\Sigma})d\boldsymbol{\alpha}d\boldsymbol{\beta}d\boldsymbol{\Sigma}}$$

where the first term is the marginal prior density of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha} = 0$ and the second term is the marginal posterior density of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha} = 0$.

4. If we choose prior as:

$$P(\mathbf{B}|\boldsymbol{\Sigma}) \propto |\boldsymbol{\Sigma}|^{-1} exp[-\frac{1}{2}tr(\mathbf{B} - \hat{\mathbf{B}}_0)'\mathbf{X}_0'\mathbf{X}_0(\mathbf{B} - \hat{\mathbf{B}}_0)\boldsymbol{\Sigma}^{-1}]$$

$$P(\boldsymbol{\Sigma}) \propto |\boldsymbol{\Sigma}|^{-\frac{\mu_0}{2}} exp[-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}_0]$$

where $\mu_0 = T_0 - 1 + N$.

(i) The marginal prior density of $\boldsymbol{\alpha}$:

$$P_0(\boldsymbol{\alpha}) = C_0(v_0\pi)^{-\frac{N}{2}}|\mathbf{H}_0|^{\frac{1}{2}}[1 + (\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}_0)'\mathbf{H}_0(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}_0)/v_0]^{-\frac{v_0+N}{2}}$$

where

$$C_0 = \frac{T((v_0 + N)/2)}{T(v_0/2)}$$

$$v_0 = T_0 - 1 - N, \quad T_0 \text{ is the number of periods}$$

$$\mathbf{H}_0 = v_0 \mathbf{S}_0^{-1}/a_0$$

and a_0 is the (1,1) element of $(\mathbf{X}_0'\mathbf{X}_0)^{-1}$,

$$|\mathbf{H}_0| = v_0^N a_0^{-N}/|\mathbf{S}_0|$$

(ii) The marginal posterior of $\boldsymbol{\alpha}$:

$$P_1(\boldsymbol{\alpha}) = C_1(v_1\pi)^{-\frac{N}{2}}|\mathbf{H}_1|^{\frac{1}{2}}[1+(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}_1)'\mathbf{H}_0(\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}_1)/v_1]^{-\frac{v_1+N}{2}}$$

$$C_1 = \frac{T((v_1+N)/2)}{T(v_1/2)}$$

$$v_1 = T-1-N$$

$$\mathbf{H}_1 = v_1\mathbf{S}_1^{-1}/a_1$$

and a_1 is the (1,1) element of $\mathbf{A}^{-1}$ (which is defined below),

$$\mathbf{S}_1 \equiv \mathbf{S}_0 + \mathbf{S}_{11} + \mathbf{S}_{12}$$

Proof:

Consider the likelihood:

$$L \propto |\boldsymbol{\Sigma}|^{-\frac{T-T_0}{2}} exp\{-\frac{1}{2}tr[(\mathbf{B}-\hat{\mathbf{B}}_1)'\mathbf{X}_1'\mathbf{X}_1(\mathbf{B}-\hat{\mathbf{B}}_1)]\boldsymbol{\Sigma}^{-1}\}exp\{-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}_{11}\}$$

where OLS estimator $\hat{\mathbf{B}}_1$ and $\mathbf{S}_{11}$ are obtained by using data from T_0+1 to T, i.e. $T-T_0$ periods. Then

$$\text{Posterior} \propto |\boldsymbol{\Sigma}|^{-1}exp\{-\frac{1}{2}tr[(\mathbf{B}-\hat{\mathbf{B}}_0)'\mathbf{X}_0'\mathbf{X}_0(\mathbf{B}-\hat{\mathbf{B}}_0)+(\mathbf{B}-\hat{\mathbf{B}}_1)'\mathbf{X}_1'\mathbf{X}_1(\mathbf{B}-\hat{\mathbf{B}}_1)]\boldsymbol{\Sigma}^{-1}\}|\boldsymbol{\Sigma}|^{-\frac{\mu_1}{2}}exp\{-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}(\mathbf{S}_0+\mathbf{S}_{11})\}$$

where

$$\mu_1 = (T-T_0)+\mu_0 = T-1+N$$

As

$$(\mathbf{B}-\hat{\mathbf{B}}_0)'\mathbf{X}_0'\mathbf{X}_0(\mathbf{B}-\hat{\mathbf{B}}_0)+(\mathbf{B}-\hat{\mathbf{B}}_1)'\mathbf{X}_1'\mathbf{X}_1(\mathbf{B}-\hat{\mathbf{B}}_1)$$
$$= \mathbf{B}'(\mathbf{X}_0'\mathbf{X}_0+\mathbf{X}_1'\mathbf{X}_1)\mathbf{B}-\mathbf{B}'\mathbf{X}_0'\mathbf{X}_0\hat{\mathbf{B}}_0-\hat{\mathbf{B}}_0'\mathbf{X}_0'\mathbf{X}_0\mathbf{B}+\hat{\mathbf{B}}_0'\mathbf{X}_0'\mathbf{X}_0\hat{\mathbf{B}}_0-\mathbf{B}'\mathbf{X}_1'\mathbf{X}_1\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_1'\mathbf{X}_1'\mathbf{X}_1\mathbf{B}+\hat{\mathbf{B}}_1'\mathbf{X}_1'\mathbf{X}_1\hat{\mathbf{B}}_1$$
$$= (\mathbf{B}-\tilde{\mathbf{B}})'\mathbf{A}(\mathbf{B}-\tilde{\mathbf{B}})+\mathbf{S}_{12}$$

where

$$\mathbf{A} \equiv \mathbf{X}_0'\mathbf{X}_0+\mathbf{X}_1'\mathbf{X}_1$$
$$\tilde{\mathbf{B}} \equiv \mathbf{A}^{-1}(\mathbf{X}_0'\mathbf{X}_0\hat{\mathbf{B}}_0+\mathbf{X}_1'\mathbf{X}_1\hat{\mathbf{B}}_1)$$
$$\mathbf{S}_{12} \equiv \hat{\mathbf{B}}_0'\mathbf{X}_0'\mathbf{X}_0\hat{\mathbf{B}}_0+\hat{\mathbf{B}}_1'\mathbf{X}_1'\mathbf{X}_1\hat{\mathbf{B}}_1-\tilde{\mathbf{B}}'\mathbf{A}\tilde{\mathbf{B}}$$
$$= \hat{\mathbf{B}}_0'\mathbf{X}_0'\mathbf{X}_0\hat{\mathbf{B}}_0+\hat{\mathbf{B}}_1'\mathbf{X}_1'\mathbf{X}_1\hat{\mathbf{B}}_1-(\mathbf{X}_0'\mathbf{X}_0\hat{\mathbf{B}}_0+\mathbf{X}_1'\mathbf{X}_1\hat{\mathbf{B}}_1)'\mathbf{A}^{-1}(\mathbf{X}_0'\mathbf{X}_0\hat{\mathbf{B}}_0+\mathbf{X}_1'\mathbf{X}_1\hat{\mathbf{B}}_1)$$

So the posterior:

$$\text{Posterior } \propto |\boldsymbol{\Sigma}|^{-1}exp\{-\frac{1}{2}tr[(\mathbf{B}-\tilde{\mathbf{B}})'\mathbf{A}(\mathbf{B}-\tilde{\mathbf{B}})]\boldsymbol{\Sigma}^{-1}\}|\boldsymbol{\Sigma}|^{-\frac{\mu_1}{2}}exp\{-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\mathbf{S}_1\}$$

with

$$\mathbf{S}_1 = \mathbf{S}_0+\mathbf{S}_{11}+\mathbf{S}_{12} \qquad \blacksquare$$