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Two-Sex Demographic Models

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Classical stable population theory, the standard model of population age structure and growth, is ill suited to addressing many issues that concern economists and demographers because it is a “one-sex” theory. This paper investigates the existence, uniqueness, and dynamic stability of equilibrium in the birth matrix–mating rule (BMMR) model, a new model of age structure and growth for two-sex, monogamously mating, populations. The paper shows, by means of examples, that the BMMR model can have multiple nontrivial equilibria and establishes sufficient conditions for uniqueness. It generalizes a theorem of W. Brian Arthur to nonlinear systems and uses it to establish sufficient conditions for local dynamic stability.

The relevance of the economics of the family to mainstream economic concerns is now well established. Examples are numerous. In addition to old favorites such as labor force participation, investment in human capital, and the intergenerational transmission of wealth, they include discussions of saving behavior (see Kotlikoff 1988; Modigliani 1988) and the burgeoning literature on “Ricardian equivalence” (Barro 1974; Bernheim 1987; Feldstein 1988). Gary Becker (1988), in his 1987 presidential address to the American Economic Association, argues the importance of family economics for understanding such

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macro issues as cyclical fluctuations, economic development, and economic growth.

As the family moves toward the center of the economic stage, demography cannot be far behind. Yet "classical stable population theory," the standard demographic model of population age structure and growth, is ill suited to addressing these and many other issues that concern demographers and economists.

The fundamental problem is that classical stable population theory is a "one-sex" theory: only females matter. Its two building blocks are an age-specific fertility schedule and an age-specific mortality schedule for the female population. Calculating these schedules requires very little data: it suffices to observe for a single period the ages at which females give birth and the ages at which females die. Classical stable population theory imbues these observed, age-specific vital rates with significance by assuming that they remain constant over time. This assumption allows classical stable population theorists to calculate a population's equilibrium age structure and growth rate and to predict its evolution over time. Demography's two-sex problem is to generalize classical stable population theory to monogamously mating, age-structured populations.¹

In the terminology of Thomas Kuhn (1970), demographers have generally regarded the two-sex problem as a *puzzle* rather than as a fundamental *anomaly* whose resolution might require recasting the *paradigm*. Even viewed as a puzzle, demography's two-sex problem cannot be solved by introducing constant, age-specific fertility and mortality schedules for males as well as for females. The male fertility and mortality schedules play no role in classical stable population theory, and introducing them in this way yields two incompatible one-sex models: the "female dominance" model based on the female mortality and maternity schedules and the "male dominance" model based on the male mortality and paternity schedules. The incompati-

¹ The two-sex problem has a long history in demography. Alfred Lotka, the founder of classical stable population theory, discussed it in 1922. Keyfitz (1968), Coale (1972), Pollard (1973), and Charlesworth (1980) provide surveys of the classical theory. Goodman (1953), Fredrickson (1971), Keyfitz (1971), Yellin and Samuelson (1977), Das Gupta (1978), Schoen (1981), Caswell and Weeks (1986), and Pollak (1986, 1987b, 1990) discuss the two-sex problem; Das Gupta, Schoen, and Caswell and Weeks provide further reference to the literature. In an unpublished doctoral dissertation, Feeney (1972) presents a two-sex model with the same basic structure as mine, but without a satisfactory proof of the existence of equilibrium. Biologists recognize two other two-sex problems. The most fundamental one is why some species, including our own, reproduce sexually (see Maynard Smith 1978; Bernstein et al. 1985). The second takes sexual reproduction as given and seeks to explain why the sex ratio for a species or a population assumes a particular value; R. A. Fisher developed the classical theory of sex ratio determination; for modern views see Maynard Smith (1980), Charnov (1982), and Samuelson (1985). Demographers take both sexual reproduction and the sex ratio of newborns as given.

bility of the two mirror-image models becomes evident when the intrinsic or equilibrium growth rates implied by the female dominance and the male dominance models differ. Unless these two implied growth rates happen to coincide, the implied sex ratio of the population approaches zero or infinity.

According to Coale (1972, p. 56), "The greatest imbalance of the sexes found in a search of recent Demographic Year Books of the U.N. was in West Berlin in 1950, where $r_m = -0.0001$, and $r_f = -.0115$." Pollard (1973) begins his chapter on the two-sex problem by citing Kuczynski's calculation of male and female net reproduction rates for France in the years immediately following World War I. Using the female model, Kuczynski found that the average number of daughters that would be born to a female then aged 0 was 0.977; using the male model, he found that the average number of sons that would be born to a male then aged 0 was 1.194. Thus the "use of a one-sex model with the female component of the population would predict a continually decreasing population for France whilst the same model applied to the male component would predict a continually increasing population" (Pollard 1973, p. 82).

Demography's two-sex problem is a fundamental anomaly that can be resolved only by replacing classical stable population theory with a model that recognizes that the observed rates for both females and males are in disequilibrium. In Pollak (1986) I propose a model of monogamously mating, age-structured population, the birth matrix-mating rule (BMMR) model. In the BMMR model the fertility of a representative female of a particular age is not a constant but a function whose value depends on the population's age-sex composition. There are two reasons for this dependence. First, the probability that a female of age i will find a mate depends on the number of females in each age category and the number of males in each age category. Second, the number of offspring produced by a mated female may depend not only on her age but also on the age of her mate (see Goldman and Montgomery, in press). Thus the BMMR model avoids the contradictions of classical stable population theory by allowing fertility rates to adjust to the population's age-sex structure. The BMMR model, unlike classical stable population theory, provides a theoretical framework capable of analyzing the effects on marriage patterns of a "marriage squeeze."² A marriage squeeze can arise when a population initially in equilibrium—that is, a population maintaining an unchanging age structure and growing at a constant rate—is disturbed by a sudden change in the birth rate. For example, suppose

² Schoen (1983) and Goldman, Westoff, and Hammerslough (1984) provide discussions of the marriage squeeze and references to the literature.

that females generally marry older males and that an equilibrium is disturbed by a baby boom. Consider what happens when young females from the leading edge of the baby boom cohort enter the marriage market. These females find that, compared with the situation faced by their older sisters when they entered the marriage market, there is a surplus of young females relative to appropriately older males. A mating rule or marriage function—a function mapping the female and male populations by age into unions identified by the ages of both partners—is the appropriate construct for analyzing the adjustments in marriage patterns induced by a marriage squeeze.

This paper begins by briefly describing classical stable population theory. Section II describes the BMMR model and sketches a proof of the existence of a nontrivial equilibrium. Section III establishes a sufficient condition for local dynamic stability and shows that this condition is satisfied in four demographically interesting cases. It then establishes a sufficient condition for uniqueness of equilibrium, shows that it is satisfied in demographically interesting cases, and demonstrates by example that the BMMR model can have multiple nontrivial equilibria. Section IV is a brief conclusion.

I. Classical Stable Population Theory

The mathematics of classical stable population theory is straightforward. Let \mathbf{F} denote the female population vector by age, $\mathbf{F} = (F_1, \dots, F_n)$, where n is the greatest age that any individual can attain. Let \mathbf{d} denote the female mortality schedule, $\mathbf{d} = (d_1, \dots, d_n)$, where $d_n = 1$, and \mathbf{b} the female fertility (maternity) schedule, $\mathbf{b} = (b_1, \dots, b_n)$, where b_i is the number of female offspring born to a female of age i .

These two schedules define a mapping or projection of the female population in period t into the female population in period $t + 1$: the age-specific fertility rates determine the number of newborns in period $t + 1$, and the mortality schedule determines the number in each of the other age categories. Applying the age-specific fertility schedule to the number of females of each age in period t determines the number of newborns in period $t + 1$:

$$F_1^{t+1} = \sum_{i=1}^n b_i F_i^t. \quad (1)$$

Applying the age-specific mortality schedule to the number of females of each age in period t determines the number in the successor category in period $t + 1$:

$$F_i^{t+1} = (1 - d_{i-1})F_{i-1}^t. \quad (2)$$

Matrix notation allows us to express this compactly as

$$\mathbf{F}^{t+1} = \mathbf{L}\mathbf{F}^t, \tag{3}$$

where the $n \times n$ projection matrix \mathbf{L} ,

$$\mathbf{L} = \begin{bmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \\ (1 - d_1) & 0 & \dots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & (1 - d_{n-1}) & 0 \end{bmatrix}, \tag{4}$$

is called the “Leslie” matrix.

Linear algebra provides powerful tools for investigating the existence of equilibrium and dynamic stability in the classical model. An equilibrium is defined as an age distribution, $\hat{\mathbf{F}}$, and an equilibrium or intrinsic growth rate, \hat{r} , which satisfy the matrix equation

$$(1 + \hat{r})\hat{\mathbf{F}} = \mathbf{L}\hat{\mathbf{F}}. \tag{5}$$

Thus the equilibrium age distribution is an eigenvector of the Leslie matrix and $1 + \hat{r}$ is the corresponding eigenvalue. Standard demographic terminology calls an age structure that reproduces itself up to a scale factor a “stable age distribution” rather than an “equilibrium age distribution.” I have departed from the standard terminology because it blurs the distinction between the existence problem and the dynamic stability problem.³

A mathematical problem that arises in the one-sex model when the vector $\mathbf{F} = 0$ requires special attention because it foreshadows more serious problems in the two-sex model. In mathematical usage, it is conventional to say that an equilibrium is a nonzero vector $\hat{\mathbf{F}}$ that satisfies (5). From a demographic standpoint it is preferable to call $\mathbf{F} = 0$, as well as any other vector \mathbf{F} that maps into zero, a “trivial” equilibrium. In the one-sex model, trivial equilibria are uninteresting both demographically and mathematically. In the two-sex model, trivial equilibria are the major obstacles blocking the “natural” fixed-point proof of the existence of a nontrivial equilibrium. In the one-sex model, if females beyond a certain age do not reproduce, then population vectors consisting entirely of such females will, after a finite number of periods, map into the zero vector. In the one-sex model, it is often convenient to drop such nonreproductive females

³ Ecologists may find my terminology misleading because the model contains no concept of an equilibrium population size.

from the model, analyze the reduced model, and reinterpret the results in terms of the original model. In the two-sex model we cannot drop nonreproductive individuals because they may mate with potentially reproductive individuals.

II. The Birth Matrix–Mating Rule (BMMR) Model

The BMMR model has three essential building blocks: a birth matrix, a mating rule, and the female and male mortality schedules. Thus the fertility of a female of age i is not specified directly but is derived from the underlying birth matrix and the mating rule and depends on the population's age-sex composition. An element of the birth matrix such as b_{ij} represents the expected number of female offspring born in a period to an “(i, j) union,” that is, the union of a female of age i with a male of age j . More precisely, each (i, j) union formed in period t produces b_{ij} female offspring and σb_{ij} male offspring who appear as newborns in period $t + 1$. The parameter σ denotes the secondary sex ratio—the ratio of male to female newborns—and I assume that it is a constant, independent of the population's age-sex structure and independent of the ages of the parents. I denote the birth matrix by $\mathbf{B} = \{b_{ij}\}$, the number of (i, j) unions by u_{ij} , and the corresponding unions matrix by \mathbf{U} .

The mating rule shows the number of unions of each type—that is, identified by age of female and age of male—as a function of the number of individuals in each age-sex category. A mating rule is thus a mapping, $\mu(\mathbf{F}, \mathbf{M})$, of the population vector (\mathbf{F}, \mathbf{M}) into the matrix of unions: $\mathbf{U} = \mu(\mathbf{F}, \mathbf{M})$; it is often convenient to write $u_{ij} = \mu^{ij}(\mathbf{F}, \mathbf{M})$, where $\mu^{ij}(\mathbf{F}, \mathbf{M})$ denotes the function mapping (\mathbf{F}, \mathbf{M}) into the variable u_{ij} . Any model of a monogamously mating, age-structured population requires an assumption about the durability of unions. In the BMMR model I assume that unions last for a single period. The advantage of this “southern California” assumption of serial monogamy is that it exposes the model's logical structure, simplifying substantially both the notation and the analysis. The assumption that matings persist for one period means that the length of the time period plays a double role in the model, as Parlett (1972) points out. Pollak (1987*b*) analyzes the BMMR model with “persistent unions.”

The dynamics of the BMMR model are straightforward to describe.

a) The initial population vector, together with the mating rule, determines the number of unions of each type; the number of unions of each type, together with the birth matrix, determines the number of newborns in the next period. The number of newborn females is

given by the “newborns” function, $\phi^1(\mathbf{F}^t, \mathbf{M}^t)$, and newborn males by $\sigma\phi^1(\mathbf{F}^t, \mathbf{M}^t)$:

$$F_1^{t+1} = \phi^1(\mathbf{F}^t, \mathbf{M}^t) = \sum_i \sum_j b_{ij}\mu^{ij}(\mathbf{F}^t, \mathbf{M}^t), \tag{6a}$$

$$M_1^{t+1} = \sigma\phi^1(\mathbf{F}^t, \mathbf{M}^t).$$

Thus the number of female offspring born to an average female of age i , $B^i(\mathbf{F}, \mathbf{M})$ —the analogue of the parameter b_i of classical stable population theory—is a function of the population vector (\mathbf{F}, \mathbf{M}) and is equal to a weighted sum (across male age classes) of the birth matrix parameters $\{b_{i1}, \dots, b_{in}\}$:

$$B^i(\mathbf{F}, \mathbf{M}) = \frac{\sum_j b_{ij}\mu^{ij}(\mathbf{F}, \mathbf{M})}{F_i}. \tag{7}$$

b) The initial population vector, together with the mortality schedules, determines the number of individuals in each of the other age-sex categories, just as in classical stable population theory:

$$F_i^{t+1} = \phi^{iF}(\mathbf{F}^t, \mathbf{M}^t) = (1 - d_{i-1}^F)F_{i-1}^t, \tag{6b}$$

$$M_j^{t+1} = \phi^{jM}(\mathbf{F}^t, \mathbf{M}^t) = (1 - d_{j-1}^M)M_{j-1}^t.$$

Thus the BMMR model defines a transformation that maps the population in period t into the population in period $t + 1$:

$$(\mathbf{F}^{t+1}, \mathbf{M}^{t+1}) = \phi(\mathbf{F}^t, \mathbf{M}^t). \tag{8}$$

Formally, an equilibrium is defined as an age distribution and a growth rate, $(\hat{\mathbf{F}}, \hat{\mathbf{M}}, \hat{r})$, that satisfy the equation

$$[(1 + \hat{r})\hat{\mathbf{F}}, (1 + \hat{r})\hat{\mathbf{M}}] = \phi(\hat{\mathbf{F}}, \hat{\mathbf{M}}). \tag{9}$$

When the *equilibrium* female and male fertility rates of the BMMR model are used to construct the female dominance model and the mirror-image male dominance model, the two one-sex models are consistent with each other in the sense that they imply identical growth rates for the female and male populations. Furthermore, the equilibrium age structure and growth rates corresponding to these two one-sex models are identical to the equilibrium age structure and growth rates of the BMMR model. Away from equilibrium, however, these two one-sex models and the BMMR model generate different predictions.

Establishing the existence of equilibrium and analyzing dynamic behavior in the BMMR model is more complex than in classical stable population theory because the properties of the mating rule make the BMMR model inherently nonlinear. Any mating rule for monogamous unions must satisfy two accounting requirements: a *nonnegativity* condition ensuring that the number of unions of each type is positive or zero and an *adding-up* condition ensuring that the number of mated females of each age does not exceed the total number of females of that age, with a similar requirement holding for males. This adding-up requirement is the fundamental source of nonlinearity in the BMMR model.

Despite its nonlinearity, we can establish the existence of equilibrium in the BMMR model under suitable assumptions. In addition to the two accounting axioms, I impose three substantive axioms on the mating rule. (a) Universal scope: The mating rule must be defined for all nonnegative population vectors. (b) Continuity: The function μ must be continuous in (\mathbf{F}, \mathbf{M}) . (c) Homogeneity: The function μ must be homogeneous of degree one in (\mathbf{F}, \mathbf{M}) : that is, $\mu(\lambda\mathbf{F}, \lambda\mathbf{M}) = \lambda\mu(\mathbf{F}, \mathbf{M})$ for all $\lambda > 0$. The homogeneity axiom implies that a 1 percent increase in the number of individuals in every age-sex category results in a 1 percent increase in the number of unions of every type. Individuals' searching for mates in a restricted and increasingly crowded region suggests a "density dependent" mating rule, but the homogeneity axiom precludes dependence of mating on population density. Because the elements of the birth matrix and the mortality schedules are constants, homogeneity of the mating rule guarantees the homogeneity of the mapping $\phi(\mathbf{F}, \mathbf{M})$. In classical stable population theory, the corresponding mapping is linear as well as homogeneous. Because we are trying to construct a two-sex model capable of maintaining an unchanging age structure while growing at a constant rate, homogeneity is an attractive assumption. These five axioms on the mating rule, together with a condition that I call "*r*-productivity," are sufficient to ensure the existence of equilibrium in the BMMR model.

An equilibrium of the BMMR model is a fixed point of the mapping $\phi(\mathbf{F}, \mathbf{M})$ or, more precisely, a fixed point of a related mapping in which the population vector is suitably normalized. A "natural" proof strategy would attempt to apply a fixed-point theorem to this mapping. The difficulty with this strategy is that the mapping carries some points in the domain into $(0, 0)$. Although $(0, 0)$ and any initial population vectors $(\mathbf{F}^0, \mathbf{M}^0)$ that map into $(0, 0)$ satisfy equation (9) and thus are equilibria, this strategy fails to establish the existence of a nontrivial equilibrium. An alternative proof strategy avoids this difficulty by drastically limiting the domain of the mapping, reducing the problem

to one dimension. A threshold observation is that the only nonlinear component of the BMMR model is the mating rule. Hence, the only nonlinear component of the mapping $\phi(\mathbf{F}, \mathbf{M})$ is the newborns function $\phi^1(\mathbf{F}, \mathbf{M})$. To exploit this fact the existence proof decomposes the argument into two parts: the determination of an equilibrium age structure and the determination of an equilibrium growth rate. We begin by disconnecting the birth matrix and the mating rule from the mortality schedules and imagining that the number of newborns grows at a constant rate r . (For definiteness, suppose that in each period storks take away all newborns and bring replacements: in the first period they bring N newborn females and σN males; in the second, $[1 + r]N$ females and $[1 + r]\sigma N$ males; in the third, $[1 + r]^2 N$ females and $[1 + r]^2 \sigma N$ males, and so on.) After n periods the age structure of the population is uniquely determined by the mortality schedules, the secondary sex ratio (σ), the growth rate of newborns (r), and the initial number of newborn females (N). For example, the age structure of the female population at time t is given by

$$F_\tau^t = F_1^1 (1 + r)^{t-\tau} \prod_{k=1}^{\tau-1} (1 - d_k^F), \tag{10}$$

where $\tau < n$ and $F_1^1 = N$. For $n = 3$ and no early mortality, this implies $(F_1, F_2, F_3) = [(1 + r)^2 N, (1 + r)N, N]$. In Pollak (1986) I call a population with this structure an r -equilibrium. If the model has a nontrivial equilibrium $(\hat{\mathbf{F}}, \hat{\mathbf{M}}, \hat{r})$, then $\hat{\mathbf{F}}$ and $\hat{\mathbf{M}}$ are r -equilibrium populations corresponding to \hat{r} . A *normalized* r -equilibrium female population, $\mathbf{F}(r)$, is an r -equilibrium population in which the number of newborn females is one; the normalized r -equilibrium male population, $\mathbf{M}(r)$, is one in which the number of newborn males is σ . Since the newborn females are a cohort of size one, the 1-year-olds are the survivors of a cohort of $1/(1 + r)$, the 2-year-olds the survivors of a cohort of $1/(1 + r)^2$, and so on:

$$F_\tau^* = \frac{\prod_{k=1}^{\tau-1} (1 - d_k^F)}{(1 + r)^{\tau-1}}. \tag{11}$$

For $n = 3$ and no early mortality, this implies $\mathbf{F}(r) = [1, 1/(1 + r), 1/(1 + r)^2]$.

If the BMMR model has a nontrivial equilibrium $(\hat{\mathbf{F}}, \hat{\mathbf{M}}, \hat{r})$, then the equilibrium age structure $(\hat{\mathbf{F}}, \hat{\mathbf{M}})$ is an r -equilibrium population for $r = \hat{r}$. Hence, when we search for a nontrivial equilibrium, it suffices to restrict our attention to r -equilibrium populations, thus reducing the problem to a single dimension: an equilibrium of the BMMR

model corresponds to a value of r for which the r -equilibrium population satisfies equation (9):

$$[(1 + r)\mathbf{F}(r), (1 + r)\mathbf{M}(r)] = \phi[\mathbf{F}(r), \mathbf{M}(r)]. \quad (12)$$

By the definition of an r -equilibrium population, the number of individuals in every age-sex category except newborns grows at the rate r . Hence, we need only find a value of r for which the birth matrix and the mating rule imply that the number of newborn females corresponding to a normalized r -equilibrium population is $1 + r$. In the example with no early mortality, this implies that the new female population vector is given by $[(1 + r), 1, 1/(1 + r)]$. Defining the function $\psi(r)$ by

$$\psi(r) = \phi^1[\mathbf{F}(r), \mathbf{M}(r)] - 1, \quad (13)$$

we can express this equilibrium condition as

$$\psi(r) = r. \quad (14)$$

In terms of figure 1, establishing the existence of a nontrivial equilibrium requires showing that the function $\psi(r)$ crosses the 45° line. Pollak (1986) proves that the BMMR model has a nontrivial equilibrium provided that the model is r -productive. The r -productivity condition requires that there exist a value of r for which a normalized r -equilibrium population produces at least $1 + r$ newborn females. The purpose of the r -productivity condition is to ensure that there is some value of r for which the function $\psi(r)$ lies above the 45° line. The fixed birth matrix implies that there is some value of r for which the function ψ lies below the 45° line. For any normalized r -equilibrium population, a larger value of r implies a smaller number of individuals in each age group except newborns; furthermore, as r approaches infinity, the normalized r -equilibrium female population vector approaches $(1, 0, 0, \dots, 0)$ and the corresponding male vector $(\sigma, 0, 0, \dots, 0)$. For sufficiently large r , only $(1, 1)$ unions can form, and the adding-up condition implies that the number of such unions cannot exceed $\min\{1, \sigma\}$. Thus for sufficiently large r , the number of newborn females must approach or be less than $b_{11} \min\{1, \sigma\}$. For sufficiently large r , this upper bound on the number of newborn females must be less than $1 + r$. Hence, for sufficiently large r , the function $\psi(r)$ lies below the 45° line. (In Pollak [1986] I assume that newborns do not enter unions, so that in the next period the number of newborns approaches zero and $\psi(r)$ approaches negative one as r approaches infinity.)

Since the function $\psi(r)$ lies above the 45° line for some values of r and below it for others, continuity implies that it crosses the 45° line at least once. Hence, the BMMR model has a nontrivial equilibrium.

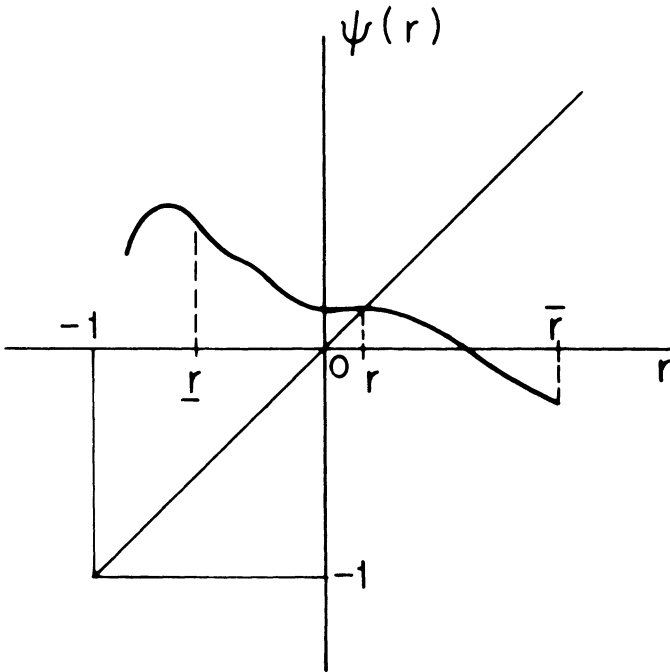


FIG. 1.—Existence of equilibrium in the BMMR model

III. Dynamic Stability and Uniqueness

Dynamic stability, unlike existence, cannot be reduced to one dimension. In the Appendix I extend the elegant argument of Arthur (1981, 1982) to show that an equilibrium of the BMMR model is locally stable if, in a neighborhood of that equilibrium, the number of newborns is a nondecreasing function of the number of individuals in every age category and a strictly increasing function of the number of individuals in at least two adjacent categories.⁴ I show by example that, unless additional axioms are imposed, the BMMR model need not satisfy this condition. Four sets of demographically meaningful conditions imply local stability.

a) Collective maximizing mating: Suppose that the mating rule is such that the configuration of unions formed are those that maximize the total number of newborns. With collective maximizing mating, an increase in the number of individuals in any age-sex category cannot decrease the number of newborns. Thus maximizing mating implies

⁴ Caswell and Weeks (1986) use both analytic and simulation techniques to investigate dynamic stability in a related two-sex model.

that the newborns function is nondecreasing in all its arguments and strictly increasing in some. To ensure stability in this case (and in the three remaining cases), we must also assume that the newborns function is strictly increasing for individuals of two adjacent ages. It might be thought that evolution would favor collective maximizing mating, but this is incorrect. Although collective maximizing mating favors the interest of the group, evolutionary arguments run not in terms of groups but in terms of individuals or genes (see Dawkins 1976). In the vocabulary of economics, natural selection provides no mechanism for internalizing externalities.

b) Individual maximizing mating: Suppose that each individual seeks to maximize the number of his or her offspring subject to the constraint that all matings must be voluntary and monogamous. Suppose that the birth matrix contains no "ties," except perhaps for unions that produce zero offspring. Then the equilibrium mating pattern can be found using a straightforward algorithm. First select the largest element in the birth matrix and form the maximal number of unions of that type (the maximal number of (i, j) unions is the minimum of the number of females of age i and the number of males of age j in the population). Now select the largest remaining element in the birth matrix and form the maximal number of unions of that type from the unmated population (i.e., the population remaining after eliminating those individuals mated at previous stages). Proceed in this way through the birth matrix or until the only remaining elements are zero. Individual maximizing mating need not correspond to collective maximizing mating, but local dynamic stability is ensured because an increase in the number of individuals in any age category cannot decrease the number of newborns.

Proof. Suppose, for definiteness, that a female of age i_0 is added to the population. In the algorithm for individual maximizing mating, suppose, without loss of generality, that the new female is the last female of age i_0 to be mated. Adding a female of age i_0 to the population has no effect on unions with higher fertility than the one at which such females become the binding constraint in the algorithm. If the new female mates with a previously unmated male, then the number of newborns increases at this stage and no lower-order union is displaced. If she mates with a previously mated male, then (1) the number of offspring increases at this stage of the algorithm because the new union produces more offspring than the union it displaces and (2) a female of age i_1 is displaced. Thus if we introduce a female of age i_0 and withdraw a female of age i_1 , the net effect is an increase in the number of newborns. But now we can proceed sequentially, reintroducing the female of age i_1 and withdrawing a female of age i_2 , and

so on, at each step increasing (or not decreasing) the number of newborns.

The following example illustrates the difference between collective and individual maximizing mating. Suppose that there are two age groups and $b_{22} = 4, b_{12} = b_{21} = 3,$ and $b_{11} = 0.$ With equal numbers in all age-sex categories, collective maximizing mating calls for mixed unions, while with individual maximizing mating those in the older age category would mate with each other.

If there are ties between nonzero elements in the birth matrix, individual maximizing mating may fail to yield a unique set of equilibrium mating patterns or a unique number of newborns. For example, suppose that there are two age groups and $b_{22} = b_{21} = 4, b_{12} = 3,$ and $b_{11} = 0.$ Suppose that $F_2 < M_1 + M_2.$ To determine uniquely the number of newborns (a prerequisite to investigating whether newborns are a nondecreasing function of the number of individuals in every age category), we need a tie-breaking rule to determine which males mate with females of age 2 and which are left. Among the tie-breaking rules guaranteeing that the newborns function is nondecreasing are random selection and priority by age (e.g., oldest first, youngest first).

c) Zero spillover mating: Suppose that the mating rule is such that the number of unions involving females of age i and males of age j depends only on F_i and M_j and is independent of the number of individuals in the other age-sex categories:

$$u_{ij} = \mu^{ij}(\mathbf{F}, \mathbf{M}) = \mu^{ij}(F_i, M_j), \quad \text{for all } i, j. \tag{15}$$

Provided that these functions are nondecreasing in F_i and $M_j,$ an increase in the number of individuals in any age-sex category cannot decrease the number of newborns.

Schoen’s “harmonic mean” mating rule (Schoen 1981),

$$\mu^{ij}(\mathbf{F}, \mathbf{M}) = \frac{\alpha_{ij} F_i M_j}{F_i + M_j}, \quad \alpha_{ij} > 0, \sum_j \alpha_{ij} \leq 1 \quad \forall i \text{ and } \sum_i \alpha_{ij} \leq 1 \quad \forall j, \tag{16}$$

is an example of a zero spillover mating rule involving only a single parameter for each type of union.

The constant elasticity of substitution (CES) mating rule,

$$\mu^{ij}(\mathbf{F}, \mathbf{M}) = [(\alpha_{ijf})^{-\rho_y} F_i^{-\rho_y} + (\alpha_{ijm})^{-\rho_y} M_j^{-\rho_y}]^{-1/\rho_y}, \tag{17}$$

where $\alpha_{ijf} > 0, \alpha_{ijm} > 0,$ and $\rho_y > 0$ for all $ij, \sum_j \alpha_{ijf} \leq 1$ for all $i,$ and $\sum_i \alpha_{ijm} \leq 1$ for all $j,$ is a more general zero spillover rule. The summation conditions and the requirement that $\rho_y > 0$ for all ij ensure that the

adding-up axiom is satisfied. When $\rho_{ij} = 1$ and $\alpha_{ijf} = \alpha_{ijm} = \alpha_{ij}$, for all i, j , the CES mating rule reduces to

$$\mu^{ij}(\mathbf{F}, \mathbf{M}) = (\alpha_{ij}^{-1} F_i^{-1} + \alpha_{ij}^{-1} M_j^{-1})^{-1}, \quad (18)$$

which, after some manipulation, reduces to Schoen's harmonic mean mating rule.⁵

d) The IMEX model with males in surplus: Pollak (1986) defines the "identical males-exhaustive mating" (IMEX) model as a special case of the BMMR model. We begin by distinguishing between *eligible* and *ineligible* individuals and defining an *exhaustive mating rule*. Eligible individuals are eligible to mate; ineligible individuals are noncandidates. Those in the ineligible population are not paired with mates even when there are surplus members of the opposite sex and all eligible members of their own sex have mates. Thus the eligible population may exclude the sick, the very young, and the very old. It may also, however, exclude various fractions of the individuals in each age-sex category. A mating rule is said to be exhaustive if it never leaves both unmated females and unmated males in the eligible population. Unless the number of eligible females happens to equal the number of eligible males, an exhaustive mating rule does leave unmated either some eligible females or some eligible males. An exhaustive mating rule, however, guarantees the formation of the maximum number of unions involving members of the eligible population.

In the IMEX model all males in the eligible population are identical in the sense that the fertility of an (i, j) union is independent of the age of the male. Provided that males are in surplus in equilibrium, the IMEX model is essentially equivalent to the classical model, and in the neighborhood of such an equilibrium, an increase in the number of individuals in any age-sex category cannot decrease the number of newborns. The IMEX model with females in surplus, on the other hand, provides an example of a specification in which additional individuals in some age-sex categories can reduce the number of newborns. More specifically, additional low-fertility females can reduce average fertility per union without increasing the number of unions, thus reducing the number of newborns.

My discussion of dynamics has focused on local rather than global stability for three reasons. First, some initial population vectors must converge to the trivial equilibrium: consider, for example, an initial population vector with no females. Second, an example due to Brian

⁵ Caswell and Weeks (1986) use the CES mating rule; Jere Behrman, Samuel Preston, and I are now estimating the CES and other "marriage functions" using U.S. and Japanese data.

Arthur cited in Pollak (1986) shows that, instead of converging to an equilibrium, the BMMR model can oscillate; the example violates the condition that the number of newborns be strictly increasing for individuals of at least two adjacent ages. Third, a model with multiple equilibria cannot be globally stable, and as the following example shows, the BMMR model can have multiple nontrivial equilibria.

Specifying the example requires mortality schedules, a birth matrix, and a mating rule. (a) Mortality schedules: All individuals live two periods. (b) Birth matrix: Only unions in which both the female and the male are newborns are fertile, so b_{11} is the only nonzero element of the birth matrix; let $b_{11} = b$. (c) Mating rule: All individuals are in the eligible population. Females of age 2 have first priority in mating and prefer males of age 1. Females of age 1 have second priority in mating and also prefer males of age 1.

Applying the mating rule to r -equilibrium populations, $\{1, 1/(1+r)\}$, we find that for $r \leq 0$, $F_2 \geq M_1$; hence, for $r \leq 0$, all potentially reproductive males are mated with females of age 2 and no newborns are produced; for $r \leq 0$, $\psi(r) = -1$. For $r > 0$ we have $u_{11} = 1 - [1/(1+r)]$, and hence $\psi(r) = b - 1 - [b/(1+r)]$. It is easily verified that this function increases monotonically and is asymptotic to $b - 1$. For $b < 4$, the function $\psi(r)$ does not cross the 45° line (see fig. 2), and the model has no nontrivial equilibrium. For $b = 4$, the function $\psi(r)$ is tangent to the 45° line at $r = 1$ and the model has a single nontrivial equilibrium. For $b > 4$ the function $\psi(r)$ intersects the 45° line twice and the model has two nontrivial equilibria. For example, for $b = 9/2$ the equilibria are $\hat{r} = 1/2$ and $\hat{r} = 2$. This example not only shows that multiple equilibria are possible in the BMMR model but also illustrates the crucial role of the r -productivity condition in ensuring the existence of a nontrivial equilibrium.

A sufficient condition for the BMMR model to have a unique nontrivial equilibrium is easy to obtain: it is clear from the geometry of figure 1 that if the function $\psi(r)$ decreases monotonically, then it can cross the 45° line only once, and hence the BMMR model can have only one nontrivial equilibrium. It is plausible that the function $\psi(r)$ could be downward sloping: the larger the value of r used to calculate the r -equilibrium, the smaller the number of individuals in each age-sex category except newborns; with fewer individuals in each age-sex category, one might expect fewer newborns in the next period. Nevertheless, the five axioms imposed on the mating rule and the r -productivity condition do not imply that the function $\psi(r)$ is downward sloping, and the example just presented shows that the nontrivial equilibrium need not be unique. An overly strong sufficient condition for uniqueness is that at every population vector an increase

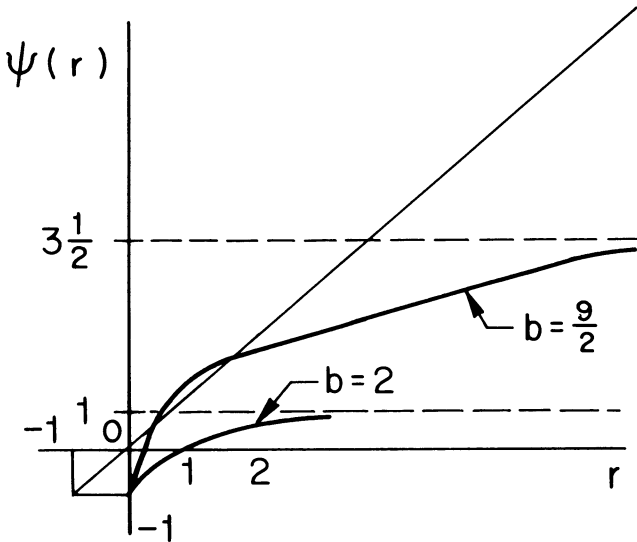


FIG. 2.—Cases of no equilibrium and multiple equilibria

in the number of individuals in each age-sex category not decrease the number of newborns in the next period.⁶ Thus the first three of the four demographically meaningful conditions for local dynamic stability—the two types of maximizing mating and zero spillover mating—provide conditions for uniqueness; if any one of these local stability conditions holds globally, then the nontrivial equilibrium is unique.⁷

IV. Conclusion

This paper has described the BMMR model, sketched a proof of the existence of equilibrium, established sufficient conditions for uniqueness and local dynamic stability, and shown by example that the model can have multiple nontrivial equilibria. In a model with multiple equilibria, initial conditions determine which, if any, of several equilibria will be realized, and small differences in initial conditions can cause large differences in long-run behavior. Thus uniqueness and dynamic stability are intimately related. Further work—both theoretical and empirical—is required to determine whether any of the sufficient conditions for uniqueness and dynamic

⁶ A weaker but still overly strong sufficient condition is that at every r -equilibrium population vector, an increase in the number of individuals in each age-sex category not decrease the number of newborns in the next period.

⁷ The IMEX model with males in surplus cannot hold globally.

stability is satisfied and whether multiple equilibria or dynamic instability is a realistic possibility in the BMMR model.

Simulation is a possible approach to investigating these issues, but the BMMR model illustrates the difficulty of using simulation techniques when a model lacks a parametric specification. Simulating the BMMR model requires us first to specify a functional form for the mating rule and then to specify the appropriate parameter values. An analogy with economics is useful. Classical stable population theory is like the input-output production model: both are so highly parameterized that data from a single period are enough to identify all the model's structural parameters and to allow us to predict its evolution. The BMMR model is like the neoclassical production model: both involve functions that are not specified parametrically, and the issue of specifying "plausible" or "realistic" parameter values for the model does not arise until functional forms are specified. The power of the BMMR model, like that of the neoclassical production model, arises from its generality: in both cases, the model's failure to specify a parametric functional form is not a weakness but a strength.⁸

Further work is required to transform the BMMR model from a merely formal into a substantive model of population age structure and growth. The transformation requires importing behavioral theories from social science into the BMMR model to explain its three primitives: the birth matrix, the mating rule, and the mortality schedules.⁹ From the standpoint of social science, however, these three primitives are different kinds of analytical constructs. The mortality schedule is often regarded as a biological datum, although there is ample precedent (from Malthus to recent concern about excess female infant mortality rates in India) for regarding mortality as endogenous. The elements of the birth matrix reflect the decisions of indi-

⁸ Both classical stable population theory and the input-output production model are linear, but the more significant similarity is that both have simple parametric specifications. Classical stable population theory is not the only demographic model that allows us to calculate structural parameters and predict the evolution of a population from a single period's data, just as the input-output model is not the only production model whose entire structure is revealed by a snapshot. If the good fairy who helps demographers revealed that each mating rule belonged to a particular one-parameter family (e.g., the harmonic mean or some other suitably restricted subset of the CES class), that revelation would enable us to calculate the parameters from one period's data. Similarly, if the good fairy who helps econometricians revealed that the underlying technology belonged to a particular one-parameter family (e.g., Cobb-Douglas or some other CES with a known elasticity of substitution), we could calculate the parameters from a snapshot. In each case, however, different revelations used to analyze the same data yield different predictions.

⁹ In the generalized version of the model in which unions can persist for more than one period (Pollak 1987*b*), there is a fourth primitive requiring a behavioral explanation: the schedule specifying the probabilities that unions of each type will end in desertion or divorce.

viduals or families facing economic and biological constraints; the analysis of these decisions is the subject matter of the economic theory of fertility. The mating rule is more complex analytically than the mortality schedules or the birth matrix because it reflects not only individual behavior but also the interactions of individuals in the marriage market. Because the mating rule is a “reduced form” representing the equilibrium that corresponds to some unspecified set of “structural” equations, treating it as a primitive in the BMMR model is especially problematic.

One might think that treating the mating rule as a primitive in the BMMR model is like treating aggregate excess demand functions as primitives in general equilibrium analysis, but the analogy is misleading. Economics possesses a highly developed theory relating reduced-form excess demand functions to a structural model in which maximizing economic agents make choices subject to appropriate constraints. Demography, on the other hand, lacks highly developed behavioral theories relating reduced-form mating rules to a structural model of marriage market equilibrium.¹⁰

To summarize: Classical stable population theory is parsimonious both because it allows us to use well-known, powerful mathematical techniques to investigate existence and dynamic stability and because it allows us to infer a population’s equilibrium age structure and its dynamic behavior from very little data. The BMMR model, which allows fertility rates to depend on the population’s age-sex structure, is more complex analytically and more demanding in its data requirements. In return for these extravagances, the BMMR model solves demography’s two-sex problem and provides a framework for addressing the marriage squeeze and other important issues in demography and economics that require a two-sex model.

Appendix

Local Stability

An equilibrium of the BMMR model is locally stable if, in a neighborhood of that equilibrium, the mapping $\phi(\mathbf{F}, \mathbf{M})$ is nondecreasing in all its arguments

¹⁰ The foundation for a structural theory of the marriage market has only recently been laid. A substantial literature now exists on matching models, following the line of analysis begun by Gale and Shapley (1962) in their celebrated paper and Becker (1973, 1981). Mortensen (1988) provides an accessible recent survey and references to the matching literature. Lam (1988) and Stapleton (1988) analyze marriage in models with household public goods. Assuming “transferable utility,” Lam examines the differing effects of gains from marriage attributable to specialization (in household production or the market) and to joint consumption (of household public goods). Using a “hedonic price” approach, Stapleton analyzes marriage market equilibrium under the assumption that individual characteristics vary continuously, in contrast to matching models, which assume a discrete distribution of individual characteristics.

and strictly increasing for individuals of at least two adjacent ages. This proposition is a corollary to a global stability theorem that Arthur (1981) establishes for a one-sex, linear demographic model. The stability argument sketched here is not self-contained, but only indicates the necessary connections with Arthur's. I focus on newborn females; a parallel argument holds for males.

Continuity implies that, if the initial population vector is sufficiently close to an equilibrium, then the population vector n periods later will also be close to that equilibrium. Thus it suffices to make the stability argument in terms of the population vector $(\mathbf{F}^n, \mathbf{M}^n)$ instead of $(\mathbf{F}^1, \mathbf{M}^1)$. This is advantageous because, regardless of how unbalanced the sexes were in the initial population vector, after n periods the sex ratio within each cohort must be "balanced," that is, consistent with the mortality schedules and the secondary sex ratio. Hence, the female population vector is a sufficient statistic for the entire population vector, and we can reduce the domain of the newborns function from $2n$ to n dimensions:

$$F_1^t = \Lambda[F_1^{t-1}, F_2^{t-1}, \dots, F_n^{t-1}].$$

Using the mortality schedules, we can express female births as a function not of the current female population vector but of newborn females in the previous n periods:

$$F_1^t = \Lambda^*[F_1^{t-1}, F_1^{t-2}, \dots, F_1^{t-n}].$$

The function Λ^* is homogeneous of degree one in its arguments. Hence, we may adopt Arthur's device of dividing both sides of the function by $(1 + r)^t$:

$$\frac{F_1^t}{(1 + r)^t} = \Lambda^*\left[\frac{F_1^{t-1}}{(1 + r)^t} \frac{1 + r}{1 + r}, \frac{F_1^{t-2}}{(1 + r)^t} \frac{(1 + r)^2}{(1 + r)^2}, \dots, \frac{F_1^{t-n}}{(1 + r)^t} \frac{(1 + r)^n}{(1 + r)^n}\right].$$

Replacing $F_1^t/(1 + r)^t$ by the new variables \tilde{F}^t , we can write

$$\tilde{F}^t = \Lambda^{**}[\tilde{F}^{t-1}, \dots, \tilde{F}^{t-n}, r].$$

Arthur calls $\{\tilde{F}^t, \tilde{F}^{t-1}, \dots\}$ a "growth-corrected" birth sequence. The function Λ^* and, hence, the function Λ^{**} are homogeneous of degree one in $(\tilde{F}^{t-1}, \dots, \tilde{F}^{t-n})$, so Euler's theorem implies

$$\tilde{F}^t = \sum_{k=1}^n \frac{\partial \Lambda^{**}}{\partial \tilde{F}^{t-k}} \tilde{F}^{t-k}.$$

Treating the coefficients in the Euler's theorem expression as constants, we have an expression analogous to Arthur's linear expression (8), which we can use to investigate local stability.

To establish stability, it suffices to show that, for some value of r , the growth-corrected birth sequence becomes constant over time. The essence of Arthur's proof is a demonstration that the dynamic process can be viewed as one of averaging growth-corrected birth sequences and that averaging causes a contraction of the extreme values in past cohorts. Following Arthur, we pick r to be a nontrivial equilibrium, \hat{r} , which, in the BMMR model, need not be unique. For an equilibrium r , a constant growth-corrected birth sequence will reproduce itself; letting $\tilde{F}^{t-1} = \tilde{F}^{t-2} = \dots = \tilde{F}^{t-n} = z$, we have

$$z = \sum_{k=1}^n \frac{\partial \phi^{**}}{\partial \tilde{F}^{t-k}} z,$$

so the coefficients in the Euler's theorem expression sum to one. Applied to this averaging process, Arthur's argument implies that an equilibrium of the BMMR model is *locally* stable provided that, in a neighborhood of the equilibrium, these coefficients are all nonnegative and that they are strictly positive for females of at least two adjacent ages.¹¹ This will be the case if, in a neighborhood of equilibrium, the function $\phi^1(\mathbf{F}^1, \mathbf{M}^1)$ is nondecreasing in all its arguments and strictly increasing for individuals of at least two adjacent ages.¹²

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¹¹ Because we have used the female population as a sufficient statistic for the entire population, this condition is satisfied if an increase in the number of individuals in two adjacent cohorts causes an increase in the number of newborns; positive coefficients for one sex can outweigh negative coefficients for the other.

¹² Arthur also establishes weaker conditions for stability, but those presented here are the ones of greatest demographic interest.

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