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Habit Formation and Dynamic Demand Functions

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Most economists would agree that past consumption patterns are an important determinant of present consumption patterns, and that one ought to distinguish between long-run and short-run demand functions. But although the distinction between long-run and short-run behavior is traditional in the theory of the firm, it is seldom made in the theory of consumer behavior.

If we regard demand theory as a theory of how a given amount of money (expenditure, called income) is allocated among goods, then—in a world without consumer durables—there are three reasons why long- and short-run demand functions might differ. (i) The consumer may have contractually fixed commitments which prevent him from adjusting some portion of his consumption (for example, housing) in response to changes in prices or income. When these fixed commitments lapse, he is able to adjust to his long-run equilibrium. (ii) The consumer may be ignorant of consumption possibilities or of his own tastes outside the range of his past consumption experience. In this case his adjustment to a new price-income situation will involve a time-consuming learning process. (iii) Finally, goods may be "habit forming" so that an individual's current preferences depend on his past consumption patterns. In this case a change in prices or income will cause a change in consumption which will induce a change in tastes, which will cause a further change in consumption.

In this paper I formulate a model of consumer behavior based on habit formation, beginning with a specific class of demand functions derived from the "modified Bergson family" of utility functions. The properties of these utility functions and the corresponding demand functions are briefly summarized in Section 1. I then postulate that the parameters

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of these functions (and hence the corresponding demand functions) depend in a specific way on past consumption; I introduce a specific habit-formation hypothesis and consider the short-run properties of the implied system of dynamic demand functions in Section 2. In the following section the long-run properties of the dynamic system are examined, and in Section 4 it is shown that the dynamic system is locally stable. In the concluding section I survey the literature on dynamic demand functions and consider several possible directions in which the model can be generalized.

1. A Family of Static Utility Functions

In an earlier paper (Pollak 1967) I considered the family of utility functions defined by

$$U(X) = \sum_{k=1}^{n} a_k \log (x_k - b_k) \qquad a_i > 0, (x_i - b_i) > 0, \sum a_k = 1. \quad (1.1)$$

$$U(X) = -\sum_{k=1}^{n} a_k(x_k - b_k)^c \qquad c < 0, a_i > 0, (x_i - b_i) > 0, \quad (1.2)$$

$$U(X) = \sum_{k=1}^{n} a_k(x_k - b_k)^c \qquad 0 < c < 1, a_i > 0, (x_i - b_i) > 0, \quad (1.3)$$

$$U(X) = -\sum_{k=1}^{n} a_k(b_k - x_k)^c \qquad c > 1, a_i > 0, (b_i - x_i) > 0, \quad (1.4)$$

$$U(X) = -\sum_{k=1}^{n} a_k e^{(b_k - x_k)/a_k} \qquad a_i > 0, \quad (1.5)$$

where x_i denotes the level of consumption of the ith good. The demand functions corresponding to (1.1) are of the form

$$h^i(P, \mu) = b_i - \frac{a_i}{p_i} \sum_k b_k p_k + \frac{a_i}{p_i} \mu, \quad (1.6)$$

where p_i denotes the price of the ith good and μ denotes total expenditure, henceforth referred to as "income." It is convenient to write (1.6) as

$$h^i(P, \mu) = b_i - \gamma_i(P) \sum_{k=1}^{n} b_k p_k + \gamma_i(P)\mu, \quad (1.7)$$

where

$$\gamma_i(P) = \frac{a_i}{p_i}. \quad (1.8)$$

The demand functions corresponding to (1.2), (1.3), and (1.4) are of the form (1.7), where

$$\gamma_i(P) = \left[\left(\frac{p_i}{a_i}\right)^{1/(c-1)}\right] \Big/ \left[\sum_{k=1}^{n} p_k \left(\frac{p_k}{a_k}\right)^{1/(c-1)}\right]. \quad (1.9)$$

Those corresponding to (1.5) are of the form

$$h^i(P, \mu) = b_i - \gamma_i(P) \sum_{k=1}^{n} b_k p_k + \gamma_i(P)\mu + \omega_i(P), \tag{1.10}$$

where

$$\gamma_i(P) = a_i \Big/ \Big(\sum_k a_k p_k\Big) \tag{1.11a}$$

and

$$\omega_i(P) = -a_i \log p_i + \gamma_i(P) \sum_k a_k p_k \log p_k. \tag{1.11b}$$

The utility functions (1.1)–(1.5) are additive; that is,

$$F[U(X)] = \sum_{k=1}^{n} u^k(x_k),$$

where $F' > 0$. The corresponding demand functions, (1.7) and (1.10), are locally linear in income; that is, they are of the form $h^i(P, \mu) = \chi_i(P) + \gamma_i(P)\mu$.[1]

The same paper showed that, under strong regularity conditions, the utility functions (1.1)–(1.5) are the only additive direct utility functions which yield demand functions locally linear in income.[2]

If the b's are all equal to 0, (1.1), (1.2), and (1.3) become

$$U(X) = \sum_{k=1}^{n} a_k \log x_k \qquad a_i > 0 \qquad \sum a_k = 1, \tag{1.12}$$

$$U(X) = \sum_{k=1}^{n} a_k x_k^c \qquad a_i > 0 \qquad c < 0, \tag{1.13}$$

$$U(X) = -\sum_{k=1}^{n} a_k x_k^c \qquad a_i > 0 \qquad 0 < c < 1. \tag{1.14}$$

[1] The qualification "locally" is necessary because (i) in some cases the utility function is not defined over the entire commodity space and hence the demand functions are not defined over the entire price-income space, and (ii) nonnegativity constraints on the x's were ignored in deriving the demand functions. Therefore, the demand functions (1.7) and (1.10) are appropriate only in price-income situations in which they imply that optimal consumption of each good is nonnegative, and in which the appropriate regularity conditions are satisfied at the implied optimum. For example, in the case of (1.1), (1.2), and (1.3), the regularity condition $x_i - b_i > 0$ is satisfied if and only if $\gamma_i(P)(\mu - \sum b_k p_k) > 0$. Since $\gamma_i(P) > 0$, this holds if and only if $\mu - \sum b_k p_k > 0$. If $b_i > 0$, then the regularity condition $x_i - b_i > 0$ implies the nonnegativity condition, $x_i > 0$. If $b_i < 0$, then the nonnegativity conditions imply an additional constraint on admissible price-income situations: $b_i - \gamma_i(P) \sum b_k p_k + \gamma_i(P)\mu > 0$. The conditions corresponding to (1.4) and (1.5) can be derived in a similar manner. We consider only regions of the price-income space in which these restrictions hold, so the demand functions (1.7) and (1.10) are appropriate.

[2] By insisting that the utility function be differentiable I ruled out the "fixed coefficient" utility function

$$U(X) = \min_k \frac{(x_k - b_k)}{a_k}, \qquad a_i > 0.$$

That is, they reduce to the "Bergson Family" of utility functions whose indifference maps correspond to the isoquant maps of the CES class of production functions (Bergson 1936; Samuelson 1965, p. 787–88). The demand functions corresponding to (1.12)–(1.14) exhibit "expenditure proportionality": $h^i(P, \mu) = \gamma_i(P)\mu$. The utility function (1.4) is inadmissible when all of the b's are 0.

The demand functions (1.6) correspond to the well-known Klein-Rubin linear expenditure system which has been estimated by Stone (1954), Powell (1966), Yoshihara (1969), and others. The indifference map of the corresponding utility function (1.1) is homothetic to the point $(b_1, \ldots, b_n)$. If the b's are all positive and income is greater than $\sum_k b_k p_k$, it is legitimate to describe the individual as purchasing necessary quantities of the goods, $(b_1, \ldots, b_n)$, and then dividing his remaining or "supernumerary" income, $\mu - \sum b_k p_k$, among the goods in fixed proportions, $(a_1, \ldots, a_n)$.[3] The income-consumption curves are straight lines radiating upward from the point $(b_1, \ldots, b_n)$. If the b's are all negative it makes no sense to describe the individual as purchasing a necessary (negative) collection of goods and dividing his supernumerary income (which is greater than his actual income) in constant proportions $(a_1, \ldots, a_n)$ among the goods. When the b's are all negative, the utility function (1.1) is defined over the entire commodity space. But the demand functions (1.6) were derived without regard to nonnegativity constraints on consumption and therefore coincide with the true demand functions only when they imply nonnegative consumption of all goods.

When all goods are consumed in positive quantities, the income-consumption curves are linear, and the linear extensions of these income-consumption curves pass through the point $(b_1, \ldots, b_n)$. Thus, the income-consumption curves can be described as radiating upward from the point $(b_1, \ldots, b_n)$, regardless of the signs of the b's.[4]

The indifference maps corresponding to (1.2) and (1.3) are homothetic to the point $(b_1, \ldots, b_n)$ and the income-consumption curves (or their linear extensions) radiate upward from this point. The indifference map corresponding to (1.4) is homothetic to $(b_1, \ldots, b_n)$, but this point must lie in the first quadrant to satisfy the regularity condition $(b_i - x_i) > 0$. The income-consumption curves may be described as converging to this point, which may be interpreted as a "bliss point."

The indifference map of (1.5) may be thought of as homothetic to the point $(-\infty, \ldots, -\infty)$; the income-consumption curves are parallel straight lines.

[3] The utility function (1.1) is not defined for commodity bundles in which any x_i is less than the corresponding b_i, so the demand functions (1.6) are not appropriate when income is less than $\sum b_k p_k$; in this case, we can say nothing about behavior.

[4] The b's need not all be of the same sign. In (1.6), if b_i is positive (negative) the demand for the ith good is inelastic (elastic), so if the x's are taken to be broad commodity groups, one would expect positive b's.

2. Habit Formation: Short-Run Utility Functions and Short-Run Demand Functions

In this Section I modify the static utility functions (1.1)–(1.5) to allow past consumption to influence current tastes.

It is best to begin with a simple example based on the utility function (1.1) with positive b's:

$$U(X) = \sum_{k=1}^{n} a_k \log(x_k - b_k), \qquad a_i > 0, b_i > 0, (x_i - b_i) > 0, \sum_k a_k = 1.$$

Although the b's can be interpreted as a "necessary" collection of goods, there is no presumption that they are physiologically rather than psychologically necessary. Indeed, it seems plausible that the "necessary" quantity of a good should depend—at least in part—on past consumption of that good. The simplest assumption is that the necessary quantity of each good is proportional to consumption of that good in the previous period: that is

$$b_{it} = \beta_i x_{it-1}, \qquad 0 \leqq \beta_i < 1, \tag{2.1}$$

where b_{it} is the value of b_i in period t, x_{it} the value of x_i in period t, and β_i a "habit formation coefficient."[5] A more general assumption is that the necessary quantity of each good is a linear function of consumption of that good in the previous period. That is

$$b_{it} = b_i^* + \beta_i x_{it-1}, \qquad 0 \leqq \beta_i < 1. \tag{2.2}$$

Here b_i^* can be interpreted as a "physiologically necessary" component of b_{it} and $\beta_i x_{it-1}$ as the "psychologically necessary" component.

If all goods are subject to habit formation of the type described by (2.2), the utility function (1.1) becomes

$$U^t(X_t) = \sum_{k=1}^{n} a_k \log(x_{kt} - b_{kt}) \qquad a_i > 0, (x_{it} - b_{it}) > 0, \sum_k a_k = 1, \tag{2.3}$$

where b_{it} is defined by (2.2).

In period t the individual is supposed to choose $x_{1t}, \ldots, x_{nt}$ which maximize (2.3) subject to the budget constraint

$$\sum_{k=1}^{n} p_{kt} x_{kt} = \mu_t.$$

[5] The requirement $\beta < 1$ is a stability condition.

The resulting demand functions (suppressing the time subscripts of the p's and μ) are of the form

$$h^{it}(P, \mu, X_{t-1}) = b_i^* - (a_i/p_i) \sum_k p_k b_k^* + (a_i/p_i)\mu + \beta_i x_{it-1} - (a_i/p_i) \sum_k p_k \beta_k x_{kt-1}.^6 \quad (2.4)$$

These short-run demand functions, like their static counterparts (1.6), are locally linear in income. Since the b's are linear in past consumption and since current consumption depends linearly on the b's, present consumption of each good is a linear function of past consumption of all goods. Since the β's are positive, there is a positive relation between past and current consumption of each good, $[\partial h^{it}(P, \mu, X_{t-1})]/(\partial x_{it-1}) = \beta_i - a_i\beta_i > 0$.[7]

The "habit formation" assumptions of (2.1) and (2.2) imply that consumption in the previous period influences current preference and demand, but that consumption in the more distant past does not. This assumption may be generalized by allowing the necessary quantity of each good to depend on a geometrically weighted average of all past consumption of that good. The analogues of (2.1) and (2.2) are

$$b_{it} = \beta_i y_{it-1} \quad (2.5)$$

and

$$b_{it} = b_i^* + \beta_i y_{it-1}, \quad (2.6)$$

where

$$y_{it-1} = (1 - \delta) \sum_{j=0}^{\infty} \delta^j x_{it-1-j}, \qquad 0 \leqq \delta < 1. \quad (2.7)$$

I assume that the "memory" coefficient, δ, is the same for all goods.[8] If $\delta = 0$, (2.5) and (2.6) reduce to (2.1) and (2.2), respectively.

If (2.6) is substituted into (2.3) we obtain a dynamic utility function which depends on all past levels of consumption, not just on consumption in the previous periods. The corresponding demand functions are of the same form as (2.4), except that x_{it-1} is replaced by y_{it-1}. Since y_{it-1} depends linearly on past consumption of the ith good, it is easy to show that, *ceteris paribus*, a higher level of past consumption of a good implies a higher level of present consumption of that good.

[6] The regularity conditions for (2.3) and (1.1) are identical, and as in the case of (1.1), they imply $\mu_t - \sum_k b_{kt}p_{kt} > 0$. Hence, a decline in income (price constant) may result in an inadmissible price-income situation (for example, one in which the demand functions are undefined). Furthermore, with habit formation, an increase in income from an initially admissible price-income situation may change the b's in such a way that the initial price-income situation becomes inadmissible.

[7] And a negative relation between past consumption of a good and current consumption of every other good: $[\partial h^{kt}(P, \mu, X_{t-1})]/\partial x_{it-1} = (-a_k p_i \beta_i)/p_k < 0, i \neq k$.

[8] I have not yet been able to establish stability when different goods have different δ's.

The requirement that the b's are positive permits us to interpret them as a necessary collection of goods, but the utility function (2.3) and the demand functions (2.4) are defined for negative as well as positive b's. Even if b_i^* and b_{it} are negative, the short-run demand functions retain all of the general properties described above. The only casualty is the interpretation of b_i^* and b_{it} as components of a necessary basket. But the habit hypothesis does not require this interpretation. The essence of the habit hypothesis is (i) that past consumption influences current preferences and hence, current demand and (ii) that a higher level of past consumption of a good implies, *ceteris paribus*, a higher level of present consumption of that good. It is easily verified that this is true for (2.4) regardless of the signs of b_{it} and b_i^*.

The four habit-formation hypotheses can be applied to the utility functions (1.2)–(1.5) with little further difficulty. The demand functions (1.7) and (1.3) can be written as

$$h^{it}(P, \mu) = b_{it} - \gamma_i(P) \sum b_{kt}p_k + \gamma_i(P)\mu, \tag{2.8}$$

$$h^{it}(P, \mu) = b_{it} - \gamma_i(P) \sum b_{kt}p_k + \gamma_i(P)\mu + \omega_i(P), \tag{2.9}$$

where b_{it} is given in terms of past consumption by the appropriate habit-formation hypothesis and time subscripts on prices and income have been suppressed. However, regularity conditions make certain habit-formation hypotheses incompatible with certain utility functions; the regularity conditions specified for (1.1)–(1.5) in terms of the b_is must be satisfied by the b_{it}s implied by past consumption.

These short run-demand functions are, of course, similar to their static counterparts: consumption of each good is an increasing linear function of income and its own past consumption, and a decreasing linear function of the past consumption of all other goods.

3. Long-Run Demand Functions and Utility Functions

This section considers the existence and characteristics of the long-run equilibria associated with the habit-formation model of Section 2. The dynamics of the model (that is, the stability of equilibrium) are considered in Section 4.

Again I begin with an example based on the utility function (1.1) and the habit formation postulate (2.2). The implied short-run demand functions (2.4) have already been described in detail.

Given the consumption vector of period 0, and given prices and income of period 1, the short-run demand functions yield a consumption vector for period 1. In a "steady state" or "long-run equilibrium," the optimal consumption vector for period 1 will be identical with the consumption vector of period 0. And, if prices and income remain constant over time, the optimal consumption vector in every subsequent period will also be equal to the consumption vector of period 0.

The long-run equilibrium consumption vector could be found by solving the short-run demand functions (2.4) under the assumption that $x_{it} = x_{it-1} = x_i$ for all i. But an alternative route is somewhat simpler. The first-order maximization conditions corresponding to (2.3) are

$$a_i/(x_{it} - b_{it}) = (-\lambda)p_i \qquad i = 1, \ldots, n, \tag{3.1}$$
$$\sum_k p_k x_{kt} = \mu.$$

In the short run, the utility maximizing x's must satisfy (3.1), where the b's are determined by past consumption. But in the long-run equilibrium, the b's are given by $b_i = b_i^* + \beta_i x_i$, where x_i is the long-run equilibrium value of x_{it}. Thus, in the long-run equilibrium the x's must satisfy

$$a_i/(x_i - b_i^* - \beta_i x_i) = (-\nu)p_i \qquad i = 1, \ldots, n, \tag{3.2}$$
$$\sum_k p_k x_k = \mu.^9$$

Solving (3.2) for x_i yields

$$x_i = \left(\frac{b_i^*}{1 - \beta_i}\right) + \left(\frac{a_i}{1 - \beta_i}\right)\left(\frac{1}{-\nu}\right)\left(\frac{1}{p_i}\right). \tag{3.3}$$

Multiplying (3.3) by p_i, summing over all goods, solving for $(1/-\nu)$, and substituting into (3.3), we obtain the "long-run" or "equilibrium" demand functions:

$$h^i(P, \mu) = B_i - \frac{A_i}{p_i}\sum_k p_k B_k + \frac{A_i}{p_i}\mu, \tag{3.4}$$

where

$$A_i = \left[\frac{a_i}{(1 - \beta_i)}\right] \Big/ \left[\sum_k \frac{a_k}{(1 - \beta_k)}\right], \qquad B_i = \frac{b_i^*}{1 - \beta_i}. \tag{3.5}$$

Equation (3.4) can be written as

$$h^i(P, \mu) = B_i - \Gamma_i(P)\sum_k p_k B_k + \Gamma_i(P)\mu, \tag{3.6}$$

where

$$\Gamma_i(P) = \frac{A_i}{p_i}. \tag{3.7}$$

[9] In the short run, the value of the Lagrangian multiplier, λ, depends on the values of the b's and hence on past consumption. Thus, it is necessary to introduce a new symbol, ν, for the "long-run" value of the Lagrangian multiplier.

$\Gamma_i(P)$ is the partial derivative of the long-run demand function with respect to income, and can be written as

$$\Gamma_i(P) = \left[\frac{\gamma_i(P)}{(1-\beta_i)}\right] \Big/ \left[\sum_k \frac{p_k \gamma_k(P)}{1-\beta_k}\right], \tag{3.7a}$$

where $\gamma_i(P)$ is the partial derivative of the short-run demand function with respect to income (1.8): $\gamma_i(P) = a_i/p_i$.

The "long-run demand functions" (3.6) show the steady-state consumption patterns consistent with the short-run demand functions (2.4). These long-run demand functions were not derived by maximizing a long-run utility function, and they are *not* the demand functions implied by the utility function $\sum_k a_k \log [x_k - (b_k^* + \beta_k x_k)]$.

Demand functions derived from utility functions satisfy the Slutsky symmetry conditions. But the long-run demand functions (3.4) were not derived from a utility function; they were defined as steady-state or equilibrium values corresponding to the short-run demand functions (2.4). In general, there is no guarantee that long-run demand functions defined in this way will satisfy the Slutsky symmetry conditions, or that they can be "rationalized" by a "long-run utility function."[10] But in this case it is obvious that the long-run demand functions (3.4) can be rationalized by the long-run utility function

$$U(X) = \sum_k A_k \log (x_k - B_k),\ A_i > 0,\ (x_i - B_i) > 0,\ \sum_k A_k = 1, \tag{3.8}$$

where A_i and B_i are defined by (3.5).[11] And since these long-run demand functions (3.5) can be derived from a utility function, they must satisfy the Slutsky symmetry conditions.

The procedure used to find the long-run demand functions corresponding to (2.4) can be used in the other cases as well. The long-run demand functions corresponding to (2.8) are of the form (3.6) where

$$\Gamma_i(P) = \left[\left(\frac{p_i}{A_i}\right)^{1/(c-1)}\right] \Big/ \left[\sum_k p_k \left(\frac{p_k}{A_k}\right)^{1/(c-1)}\right] \tag{3.9a}$$

and

$$A_i = \frac{a_i}{(1-\beta_i)^{1-c}}, \qquad B_i = \frac{b_i^*}{1-\beta_i}. \tag{3.10}$$

These long-run demand functions can be rationalized by the long-run utility functions corresponding to (1.2), (1.3), and (1.4) where the a's and b's are replaced by the corresponding A's and B's. Equation (3.9a) can be written as

$$\Gamma_i(P) = \left[\frac{\gamma_i(P)}{1-\beta_i}\right] \Big/ \left[\sum_k \frac{p_k \gamma_k(P)}{1-\beta_k}\right], \tag{3.9b}$$

[10] This is not quite true (see Gorman 1967).

[11] Clearly $A_i > 0$ and $\sum A_k = 1$. If $\bar{x}_i$ is an admissible long-run equilibrium, then $[\bar{x}_i - (b_i^* + \beta_i \bar{x}_i)] > 0$, so $(\bar{x}_i - B_i) > 0$.

where $\gamma_i(P)$ is defined by (1.8). The long-run demand functions corresponding to (2.9) are of the form

$$h^i(P, \mu) = B_i - \Gamma_i(P) \sum_k p_k B_k + \Gamma_i(P)\mu + \Omega_i(P), \tag{3.11}$$

where

$$\Gamma_i(P) = A_i \Big/ \Big(\sum_k p_k A_k\Big), \tag{3.12a}$$

$$\Omega_i(P) = -A_i \log p_i + \Gamma_i(P) \sum_k p_k A_k \log p_k, \tag{3.13}$$

and

$$A_i = \frac{a_i}{1 - \beta_i}, \qquad B_i = \frac{b_i^*}{1 - \beta_i}. \tag{3.14}$$

These long-run demand functions can be rationalized by a long-run utility function obtained from (1.5) by replacing the a's and b's by the corresponding A's and B's. Equation (3.12a) can be written in the form (3.9b) where $\gamma_i(P)$ is defined by (1.10).[12]

These results can immediately be extended to the habit hypothesis (2.6). In this case a long-run equilibrium or steady-state consumption vector is one which, if it prevailed in period 0 and in every previous period, would be optimal in period 1 (and, hence, in every future period). But if consumption of each good has been constant since time out of mind, y_{it-1} is equal to x_{it-1}. Therefore, the long-run equilibrium determined for the habit hypothesis (2.4) applies to (2.6) as well.

Thus, the long-run demand functions and utility functions corresponding to the short-run utility functions and demand functions of Section 2 are of the same general form as their short-run counterparts. The value of the parameter c is the same in both the long run and the short run, but the other parameters differ, and the habit-formation coefficients (the β's) enter into these parameters. The long-run equilibrium does not depend on the value of the "memory" coefficient, δ.

Not every habit-formation assumption is compatible with every utility function; some combinations necessarily violate the regularity conditions and hence are inadmissible. For example, if the utility function is (1.4), and habit formation is defined by (2.1), the regularity condition

[12] In the fixed coefficient case described in footnote 2, the long-run demand functions are of the form

$$h^i(P, \mu) = B_i - \Big[A_i \Big/ \Big(\sum_k p_k A_k\Big)\Big] \sum_k P_k B_k + \Big[A_i \Big/ \Big(\sum_k P_k A_k\Big)\Big] \mu,$$

where A_i and B_i are defined by (3.14). These demand functions can be rationalized by the long-run utility function

$$U(X) = \min_k \frac{(x_k - B_k)}{A_k}.$$

$(b_{it} - x_{it}) > 0$ is necessarily violated in the long-run equilibrium, since $\beta_i x_i - x_i = x_i(\beta_i - 1) < 0$ for all $x_i > 0$, and $0 < \beta_i < 1$. In this case the habit-formation assumption (2.1), with $\beta_i < 1$, implies that the bliss point is less than past consumption, so the system is incompatible with constant prices and income.

4. Dynamics

This section shows that the system of dynamic demand functions introduced in Section 2 are locally stable. To prove this, it is convenient to write the demand functions in matrix form. Let b_t, b^* and X_t denote the column vectors whose elements are b_{it}, b_i^*, and x_{it}, respectively, and let $\hat{\beta}$ denote the diagonal matrix, diag $(\beta_i, \ldots, \beta_n)$. Then the habit-formation hypothesis (2.4) can be written as

$$b_t = b^* + \hat{\beta} X_{t-1}, \tag{4.1}$$

and the habit-formation hypothesis (2.6) as

$$b_t = b^* + \hat{\beta} Y_{t-1}, \tag{4.2}$$

where Y_{t-1} denotes the column vector whose elements are y_{it-1}. The vector Y_{t-1} is defined by the matrix analogue of (2.7):

$$Y_{t-1} = (1 - \delta) \sum_{j=0}^{\infty} \delta^j X_{t-1-j}. \tag{4.3}$$

Let P_t and γ_t denote the column vectors whose elements are p_{it} and $\gamma_{it}(P_t)$, respectively, and let μ_t denote the level of income in period t. Using this notation, the system of demand functions (2.8) becomes

$$X_t = b_t - \gamma_t P_t' b_t + \gamma_t \mu_t, \tag{4.4}$$

where P_t' denotes the transpose of P_t, and (2.9) becomes

$$X_t = b_t - \gamma_t P_t' b_t + \gamma_t \mu_t + \omega_t, \tag{4.5}$$

where ω_t is the column vector whose elements are $\omega_i(P_t)$.

I first show that the system of dynamic demand function (4.4) and (4.5) are locally stable under the habit-formation hypothesis (4.1) and then that they are stable under (4.2).

If X_0 (that is, the consumption vector of period 0) is given, then (4.4) determines X_1 as a function of P_1, μ_1, and X_0. In the same way, X_2 is determined by (4.4) as a function of P_2, μ_2, and X_1, or, more conveniently, as a function of X_0, P_1, μ_1, P_2, μ_2. Thus, for any initial consumption vector X_0 and any price-income sequence $\{(P_1, \mu_1), (P_2, \mu_2), (P_3, \mu_3), \ldots\}$, (4.4) determines the corresponding consumption sequence $\{X_1, X_2, \ldots\}$.

Section 3 identified the equilibrium consumption vector X^* corresponding to the price-income situation (P^*, μ^*). Clearly, if $X_0 = X^*$ and

$\{P_1, \mu_1), (P_2, \mu_2), \ldots\} = \{(P^*, \mu^*), (P^*, \mu^*), \ldots\}$, then $\{X_1, X_2, \ldots\} = \{X_1^*, X_2^*, \ldots\}$. The present Section shows that if X_0 is sufficiently close of X^*, then the consumption sequence corresponding to $\{(P^*, \mu^*), (P^*, \mu^*), \ldots\}$ will converge to X^*. The discussion of dynamic stability is inevitably complicated by regularity conditions, for example $(x_{it} - b_{it}) > 0$, and nonnegativity conditions which must be satisfied in every time period. We begin by ignoring these conditions.

Since prices and income are assumed constant over time, we drop the time subscription on P_t, μ_t, and γ_t. Substituting (4.1) into (4.4) yields

$$X_t = MX_{t-1} + d, \tag{4.6}$$

where

$$M = (I - \gamma P')\hat{\beta} \tag{4.7}$$

and

$$d = (I - \gamma P')b^* + \gamma\mu. \tag{4.8}$$

It is easily verified that X_t is given by

$$X_t = M^t X_0 + \left[\sum_{j=0}^{t-1} M^j\right] d. \tag{4.9}$$

Thus the stability of the system of difference equations (4.6) rests on the following theorem.

Theorem: Let M be the matrix defined by (4.7) where γ and P are $n \times 1$ vectors with positive elements such that $P'\gamma = 1$, and $\hat{\beta}$ is the diagonal matrix diag $(\beta_1, \ldots, \beta_n)$ where $0 \leqq \beta_i < 1$; then the characteristic roots of M are all less than 1 in modulus. (The proof of this theorem is in the Appendix.)

If the characteristic roots of M are less than 1 in modulus, it is well known that

$$\lim_{t \to \infty} M^t = 0$$

and

$$\lim_{t \to \infty} \sum_{j=0}^{t-1} M^j = (I - M)^{-1},$$

so the system of difference equations (4.6) converge to

$$X = (I - M)^{-1} d. \tag{4.10}$$

But the stability of the system of dynamic demand functions requires more than the stability of the difference equation system (4.6). It is also necessary to show that the nonnegativity and regularity conditions inherent in the consumption problem are satisfied in every time period. For example, in the case of the dynamic version of (1.1), (1.2), and (1.3) we must have $(x_{it} - b_{it}) > 0$ and $x_{it} \geqq 0$ for all i and t. I shall show that corresponding to every admissible price-income situation there exists a neighborhood of the long-run equilibrium such that, for initial values of X in this neighborhood (i) the system (4.4) converges to the long-run equilibrium and (ii) the sequence of consumption vectors $\{X_1, X_2, \ldots\}$ satisfies the nonnegativity and regularity conditions in each time period. This is shown only for the regularity conditions relevant to (1.1), (1.2), and (1.3), leaving the other cases to the reader.

With no loss of generality, the initial consumption vector X_0 can be written as

$$X_0 = X^* + \epsilon Z, \tag{4.11a}$$

where X^* is the long-run equilibrium consumption vector, Z is an $n \times 1$ vector satisfying

$$\sum_{j=1}^{n} z_j^2 = 1, \tag{4.11b}$$

and ϵ is a nonnegative scalar. It is assumed that X^* is strictly positive and that is satisfies the regularity conditions $X^* - b^* - \hat{\beta}X^* > 0$. It must be shown that there exists an $\epsilon^* > 0$ such that for all ϵ, $0 \leqq \epsilon < \epsilon^*$, both the nonnegativity conditions and the regularity conditions are satisfied in each time period for all Z satisfying (4.11b).

Substituting (4.11) into (4.6) and making use of the identity $X^* = MX^* + d$, we find

$$X_t = X^* + \epsilon M^t Z. \tag{4.12}$$

Hence,

$$X_t - b_t = X^* - b^* - \hat{\beta}X^* + \epsilon(M^t Z - \hat{\beta}M^{t-1}Z). \tag{4.13}$$

The nonnegativity condition requires (4.12) to be nonnegative for all t, and the regularity conditions require (4.13) to be strictly positive for all t.

Let m_{ij}^t denote the ijth element of M^t. Since

$$\lim_{t\to\infty} M^t = 0,$$

there exists a number m such that $|m_{ij}^t| < m$ for all i, j, and t. Hence, the elements of $M^t Z$ are each less (in absolute value) than mn and the elements of $M^t Z - \hat{\beta}M^{t-1}Z$ are each less than $2mn$. Therefore, for all Z satisfying (4.11b) and all sufficiently small ϵ, (4.12) will be arbitrarily close to X^* (which is strictly positive), and (4.13) will be arbitrarily close to $X^* - b^* - \hat{\beta}X^*$ (which is also strictly positive).

If the demand functions are given by (4.5) rather than (4.6) where M is defined by (4.8) and d by

$$d = (I - \gamma P')b^* + \gamma\mu + \omega, \tag{4.14}$$

the above stability argument requires no modification. Thus, the dynamic demand functions (4.4) and (4.5) are locally stable under the habit-formation hypothesis (4.1).

If habit formation is described by (4.2) rather than (4.1), the initial conditions must specify consumption vectors for period 0 and all previous periods. As before, prices and income are supposed to remain at their period 1 levels for all subsequent periods. The system of dynamic demand functions (4.4) and (4.5) may be written as

$$X_t = MY_{t-1} + d, \qquad t = 1, 2, \ldots, \tag{4.15a}$$

where M is given by (4.7) and d by (4.8) or (4.14). It is convenient to write (4.15a) in the equivalent form,

$$X_{t+1} = MY_t + d, \qquad t = 0, 1, \ldots. \tag{4.15b}$$

From the definition of Y_t, (4.3)

$$Y_t = (1 - \delta)X_t + \delta Y_{t-1}, \qquad t = 1, 2, \ldots. \tag{4.16}$$

Substituting (4.16) into (4.15b), multiplying (4.15a) by δ, and subtracting, we obtain

$$X_{t+1} = NX_t + (1 - \delta)d, \qquad t = 1, 2, \ldots, \tag{4.17}$$

where

$$N = \delta I + (1 - \delta)M. \tag{4.18}$$

Thus, (4.15) determines X_1 as a function of Y_0 and (4.17) determines $X_2, X_3, \ldots$. The difference equation system (4.17) is stable if and only if the characteristic roots of N lie within the unit circle. But the characteristic roots of N are related to characteristic roots of M by

$$\lambda_i(N) = \delta + (1 - \delta)\lambda_i(M),$$

so—neglecting regularity and nonnegativity conditions—the stability of the system of difference equations is guaranteed. The long-run equilibrium implied by (4.17) is given by

$$X = (I - N)^{-1}(1 - \delta)d, \tag{4.19a}$$

and since $I - N = (1 - \delta)(I - M)$, this is equivalent to

$$X = (I - M)^{-1}d. \tag{4.19b}$$

That is, the long-run equilibrium is independent of the value of the memory coefficient δ.

It is now necessary to show that, for Y_0 sufficiently close to the long-run equilibrium, X^*, the regularity and nonnegativity conditions are satisfied in every time period.

Substituting (4.15a) into (4.16) and solving for Y_t,

$$Y_t = NY_{t-1} + (1 - \delta)d, \qquad t = 1, 2, \ldots. \tag{4.20}$$

Without loss of generality, we may write Y_0 as

$$Y_0 = X^* + \epsilon Z, \tag{4.21}$$

where X^* is the long-run equilibrium consumption vector and Z is an arbitrary vector satisfying (4.11b). Since $X^* = NX^* + (1 - \delta)d$, it is easily shown that

$$Y_t = X^* + \epsilon N^t Z, \qquad t = 0, 1, \ldots. \tag{4.22}$$

Substituting (4.22) into (4.15b),

$$X_{t+1} = X^* + \epsilon M N^t Z, \qquad t = 0, 1, \ldots. \tag{4.23}$$

To show that nonnegativity and regularity conditions are satisfied in every time period (provided they are satisfied at the long-run equilibrium), it must be shown that for all ϵ smaller than some ϵ^*, X_{t+1} and $X_{t+1} - b^* - \hat{\beta} Y_t$ are positive for all Z satisfying (4.11b). The fact that X_{t+1} is positive (provided X^* is positive) follows immediately from an argument similar to that used for the simpler habit hypothesis (4.1). To show that the regularity, conditions are satisfied we observe that

$$X_{t+1} - b^* - \hat{\beta} Y_t = X^* - b^* - \hat{\beta} X^* + (MN^t - \hat{\beta} N^t) Z\epsilon. \tag{4.24}$$

Provided the long-run equilibrium satisfies the regularity conditions, $X^* - b^* - \hat{\beta} X^*$ is strictly positive, an argument almost identical with that used for (4.1) establishes the required result.

5. Conclusion

In conclusion we survey the literature on dynamic demand functions and discuss several possible generalizations.

1. Although it is frequently mentioned in passing that long-run and short-run demand functions differ, little has been done to incorporate this fact into the theory of demand. In empirical work, a lagged adjustment hypothesis is often invoked (Nerlove 1958; Houthakker and Taylor 1966, pp. 5–21). But this approach is unsatisfactory from a theoretical viewpoint unless it can be justified in a framework of utility maximization. Richard Stone, who has worked extensively with the linear expenditure system (1.6), has pointed out that it is not reasonable to suppose that the a's and b's remain constant over time. He suggests (Stone 1966, pp. 192–93) two ways of introducing systematic changes in these parameters. The first is the introduction of time trends—either linear or quadratic—for both a's and the b's. The second is to allow the a's and b's to depend on "the past history of the branch of demand to which they relate." Perhaps because his primary interest is empirical rather than theoretical, Stone does not investigate the long-run behavior implied by dynamic demand functions of the form he suggests.

Maurice Peston (1967) considers a "Cobb-Douglas" utility function (1.12) of the form

$$x_{1t}^{\alpha_t} x_{2t}^{1-\alpha_t},$$

where the parameter α_t depends on the ratio x_{2t-1}/x_{1t-1}. Peston discusses the effect of the imposition of a tax on one of the goods and investigates the existence and stability of the long-run equilibrium. W. M. Gorman (1967) assumes that short-run behavior is determined by a short-utility function, $f(x, \alpha)$, where α is a vector whose elements are assumed to be functions of past consumption of all goods: $\alpha_r = a_r(X)$, $r = 1, \ldots, t$. Gorman considers the conditions on the utility function, f, and the habit functions, $\alpha_r(X)$, which imply the existence of a long-run utility function or "choice indicator" from which the long-run demand functions could be derived.[13]

2. Any utility function can be made dynamic by allowing some or all of its parameters to depend on past consumption. Tractable results were obtained from the utility functions (1.1)–(1.5) and the four habit-formation assumptions used in this paper because (i) the demand functions were linear in the b's, and (ii) the b's were linear in past consumption. This resulted in dynamic demand functions which were linear in past consumption. If the a's or c's in (1.1)–(1.5) were allowed to depend on past consumption, the results would be far less tractable.[14]

3. In the theory of demand it is usually assumed that an individual's utility function depends on his own consumption, but not on the consumption of others. By allowing some or all of the parameters of an individual's utility function to depend on the consumption of others, interdependence can be incorporated into the theory of consumer behavior. In the case of the utility functions (1.1)–(1.5), particularly simple results are obtained if the b's are assumed to depend linearly on other people's consumption. But if everyone behaves in this manner, the derivation of market-demand functions is likely to be quite difficult.

In a dynamic model, it is possible to introduce interdependence in a more tractable way by postulating that the parameters of an individual's utility function depend on other people's *past* consumption. Because other people's current consumption does not influence current preferences, there is no difficulty deriving short-run market demand functions. And presumably interdependence could be incorporated into the habit-formation model by assuming that the parameters of an individual's utility function depend on other people's past consumption as well as his own.

4. In empirical work, and in some theoretical problems as well, it is necessary to specify stochastic demand functions. The procedure suggested

[13] As Gorman points out, he does not show that his "choice indicator" is a well-behaved utility function, only that it satisfies the appropriate first-order conditions.

[14] The a's appear to enter the demand functions (1.6) linearly, but in fact they do not; they have been normalized so that they sum to unity, and if this normalization rule is dropped, the a's must be replaced by $a_i/\sum a_k$.

here for making demand functions dynamic can also be used to make them stochastic. In general, a system of demand functions derived from a utility function can be made stochastic by assuming that some of the parameters of the utility function are random variables. This procedure yields particularly simple results when the parameters of the utility function enter the demand functions linearly. Thus, if the b's are assumed to be of the form $b_i = b_i^* + u_i$, where u_i is a random variable with 0 mean, then the demand functions will be stochastic and the disturbance term in each demand equation will be a linear combination of the u's. The system of stochastic demand functions generated in this way will satisfy both the budget constraint and the Slutsky symmetry conditions.[15]

5. At first glance it might seem possible to convert the habit-formation model of Section 2 into a consumer-durable model by allowing the β's to be negative. This procedure has no theoretical standing. A fundamental assumption of the habit-formation model is that the individual does not take account of the effect of his current purchase on his future preferences and future consumption. In the case of habit formation, this assumption is plausible; in the case of consumer durables, it is not. A model of demand for consumer durables must explicitly recognize the intertemporal nature of the problem.

Appendix[16]

0. It is necessary to show that the characteristic roots of the matrix M, (4.7),

$$M = (I - \gamma P')\hat{\beta} \tag{A.1}$$

lie within the unit circle. To show this, we define a matrix S which is similar to M, and show that the characteristic roots of S lie within the unit circle.

Let $\hat{P}$ denote the diagonal matrix diag$(p_1, \ldots, p_n)$. We define the matrix S by

$$S = \hat{P}M\hat{P}^{-1}. \tag{A.2a}$$

Then, $\hat{P}\gamma$ is an $n \times 1$ matrix whose elements sum to unity and $P'\hat{P}^{-1}$ is the $1 \times n$ row vector $(1, 1, \ldots, 1)$, which we denote by e. We may write (A.2a) as

$$S = (I - T)\hat{\beta}, \tag{A.2b}$$

where T is the $n \times n$ matrix $\hat{P}\gamma P'\hat{P}^{-1} = \hat{P}\gamma e$. Hence.

$$T = \begin{bmatrix} t_1 & t_1 \ldots t_1 \\ t_2 & t_2 \quad t_2 \\ \vdots & \\ t_n & t_n \quad t_n \end{bmatrix}$$

[15] Wales and I (1969) have estimated the linear expenditure system using this procedure to specify the error structure. We estimated a number of dynamic versions of the linear expenditure system including proportional habit (2.1) and linear habit formation (2.2).

[16] I am grateful to David B. Wales for providing this proof.

where $t_i = p_i\gamma_i$. We remark that the elements of T are strictly positive and that all column sums are unity.

1. Define the row vectors $r_1, \ldots, r_n$ by

$$\begin{aligned} r_1 &= e \\ r_2 &= (t_2, -t_1, 0, 0, \ldots, 0) \\ r_3 &= (t_3, 0, -t_1, 0\ldots, 0) \\ r_i &= (t_i, 0, \ldots, 0, -t_1, 0, \ldots, 0) \\ r_n &= (t_n, 0, 0, \ldots, 0, -t_1). \end{aligned}$$

Then $r_1T = e$ and $r_iT = 0$, $i \neq 1$. Hence $[r_1, \ldots, r_n]$ is a basis of characteristic vectors for T. Since $r_1(I - T) = 0$ and $r_i(I - T) = r_i$, $i \neq 1$, $[r_1, \ldots, r_n]$ is a basis of characteristic vectors for $I - T$. Furthermore, $r_1(I - T)\hat{\beta} = 0$, so r_1 is a characteristic vector of S with characteristic value 0.

2. Let V be the subspace spanned by $[r_2, \ldots, r_n]$. Then V is a subspace of dimension $n - 1$ and any vector w in R^n can be written uniquely as

$$w = ar_1 + v, \tag{A.3}$$

where a is a scalar and $v \in V$.

3. Let w be a characteristic vector of S with characteristic value λ.

$$wS = \lambda w. \tag{A.4}$$

But

$$wS = (ar_1 + v)S = vS = v(I - T)\hat{\beta} = v\hat{\beta},$$

so

$$v\hat{\beta} = \lambda(ar_1 + v) = \lambda ar_1 + \lambda v. \tag{A.5}$$

Let $v = (\xi_1, \ldots, \xi_n)$. Then $v\hat{\beta} = (\beta_1\xi_1, \beta_2\xi_2, \ldots, \beta_n\xi_n) = \lambda(a, a, \ldots, a) + \lambda(\xi_1, \ldots, \xi_n)$, so

$$\beta_i\xi_i = \lambda a + \lambda\xi_i = \lambda(a + \xi_i), \qquad i = 1, \ldots, n. \tag{A.6}$$

4. If $\xi_i = 0$ for some i, then $0 = \lambda a$. There are two cases: (i) If $\lambda = 0$, $|\lambda| < 1$. (ii) If $a = 0$, from (A.5), $v\hat{\beta} = \lambda v$ so λ is a characteristic value of $\hat{\beta}$. But the characteristic values of $\hat{\beta}$ are $[\beta_1, \ldots, \beta_n]$ and, by hypothesis, $0 \leqq \beta_i < 1$

5. If $\xi_i \neq 0$ for any i, and $a = 0$, then $\beta_i\xi_i = \lambda\xi_i$, so $\lambda = \beta_i$ and $|\lambda| < 1$.

6. If $\xi_i \neq 0$ for any i and $a \neq 0$, we may multiply w by a scalar changing a and v. In particular, we may take $a = 1$ without loss of generality, so $\beta_i\xi_i = \lambda(1 + \xi_i)$.

7. We now show that if (i) $\xi_i \neq 0$ for any i and (ii) $a \neq 0$ and (iii) $|\lambda| > 1$; then the real part of ξ_i, $Re\ \xi_i$, is negative for all i.

If $\xi_i = -1$; then $Re\ \xi_i = -1 < 0$.

If $\xi_i \neq -1$, then $\lambda = \dfrac{\beta_i\xi_i}{1 + \xi_i}$,

so

$$1 < |\lambda| = \frac{\beta_i|\xi_i|}{|1 + \xi_i|}.$$

Hence,

$$|\xi_i| > |1 + \xi_i|$$

and

$$Re\ \xi_i < 0, \qquad i = 1, \ldots, n.$$

8. In terms of the basis $[r_2, \ldots, r_n]$ we may write v as

$$v = \sum_{i=2}^{n} v_i r_i .$$

Then

$$\xi_1 = v_2 t_2 + v_3 t_3 + \ldots v_n t_n = \sum_{i=2}^{n} v_i t_i ,$$

$\xi_i = -v_i t_1$, $i = 2, \ldots, n$. So $Re\ \xi_i = -t_1\ Re\ v_i < 0$, $i = 2, \ldots, n$, and, hence, $Re\ v_i > 0$, $i = 2, \ldots, n$. But

$$Re\ \xi_1 = \sum_{i=2}^{n} t_i\ Re\ v_i > 0,$$

which contradicts the result established in 7.

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