Unspanned Stochastic Volatility, Is It There After All? Evidence from Hedging Interest Rate Caps

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ABSTRACT

There are conflicting views in the literature on whether bonds span interest rate derivatives. While Heidari and Wu (2003) and Collin-Dufresne and Goldstein (2002a) argue that there are unspanned stochastic volatility factors in caps and swaptions markets, Fan, Gupta, and Ritchken (2003) show that the benefit of including unspanned stochastic volatility for hedging swaptions is minor. We study the ability of multifactor quadratic term structure models estimated using bond data in hedging interest rate caps. We find that although these models capture bond yields well, they have serious difficulties in hedging caps, especially cap straddles. Furthermore, straddle hedging errors are highly correlated with at-the-money cap implied volatilities and can explain a large fraction of hedging errors of all caps across moneyness and maturity. Our results suggest that there are indeed systematic unspanned factors related to stochastic volatility in caps market. The different conclusions for caps and swaptions are consistent with the fact that cap implied volatilities are much more volatile than swaption implied volatilities, and caps are much more sensitive to changes in correlations of bond yields than swaptions. Term structure models that explicitly incorporate stochastic volatility and correlation will be important for pricing and hedging caps and in resolving the relative mispricing between caps and swaptions.

Interest rate caps and swaptions are the most widely traded interest rate derivatives in the world. According to the Bank of International Settlement, their combined notional values are more than 10 trillion dollars in recent years. Because of the size of these markets, accurate and efficient pricing and hedging of caps and swaptions have enormous practical importance. Prices of interest rate derivatives also contain information that may not be available in Libor and swap rates for testing existing term structure models. As pointed out by Dai and Singleton (2003) "With the growing availability of time series data on the implied volatilities of fixed-income derivatives, comparisons of DTSM (dynamic term structure model)-implied to market prices of derivatives is increasingly being used in assessing goodness-of-fit."

One key question of the fast growing literature on interest rate derivatives is the so-called unspanned stochastic volatility puzzle.[1] Interest rate caps and swaptions are derivatives written on Libor and swap rates, and their prices should be determined by the same set of factors that determine Libor and swap rates. However, several recent studies have shown that there seem to be risk factors that affect the prices of caps and swaptions but not the underlying Libor and swap rates. In other words, bonds do not seem to span interest rate derivatives. For example, Heidari and Wu (2003) show that while the three common term structure factors (i.e., the level, slope and curvature of the yield curve) can explain 99.5% of the variations of bond yields, they explain less than 60% of swaption implied volatilities. By including three additional volatility factors, the explanatory power is increased to over 97%. Similarly, Collin-Dufresne and Goldstein (2002a) show that there is a very weak correlation between changes in swap rates and returns on at-the-money (ATM) cap straddles: the R^2s of regressions of straddle returns on changes of swap rates are typically less than 20%. They find that one principal component explains 80% of regression residuals of straddles with different maturities. As straddles are approximately delta neutral and mainly exposed to volatility risk, they refer to the factor that drives straddle returns but is not affected by the term structure factors as "unspanned stochastic volatility" (hereafter USV).

The presence of USV has important implications for term structure modeling. It shows that to price and hedge caps and swaptions, models that explicitly incorporate USV are needed. In contrast, some popular term structure models, although have been reasonably successful in fitting yield data, may not be very useful for pricing interest rate derivatives. Subsequent to Heidari and Wu (2003)

[1]Another question is the relative pricing between caps and swaptions. Although both caps and swaptions are derivatives on Libor rates, exsiting models calibrated to one set of prices tend to significantly misprice the other set of prices.

and Collin-Dufresne and Goldstein (2002a), several term structure models that explicitly capture stochastic volatility and correlation have been developed (see e.g., Collin-Dufresne and Goldstein 2002b, and Han 2002). These new term structure models resemble the stochastic volatility models for equity option pricing (see, e.g., Heston 1993, and Hull and White 1987). While pricing and hedging interest rate derivatives become more complicated in these new models, the additional cost is justified if USV is economically significant.

However, the relevance of USV has been challenged in the recent literature by Fan, Gupta and Ritchken (2003) (hereafter FGR). These authors argue that "if USV is important, then it should not be possible to hedge swaptions efficiently using a model based on state variables limited to the set of swap rates. More importantly, such models should certainly not be able to hedge contracts which have extreme sensitivity to volatilities, such as straddles." They show that contrary to this prediction, multifactor models with state variables linked solely to underlying Libor and swap rates, can hedge swaptions and even swaption straddles very well. Therefore, they conclude that "the potential benefits of looking outside the Libor market for factors that might impact swaptions prices without impacting swap rates" are minor. According to FGR (2003), one important reason for their different conclusions is that linear regression used in Heidari and Wu (2003) and Collin-Dufresne and Goldstein (2002a) could give misleading results due to the highly nonlinear dependence of straddle returns on the underlying interest rates.

While FGR (2003) show that the benefit of USV for hedging swaptions is not significant, they acknowledge that "it may be the case that unspanned stochastic volatility is more important in the cap market." Such a result would be consistent with the findings of Collin-Dufresne and Goldstein (2002b) that cap implied volatilities are much more volatile than swaption implied volatilities and caps are much more sensitive to changes in correlations of bond yields than swaptions. Indeed models calibrated to swaption prices but ignore time varying correlations, such as Longstaff, Schwartz, and Santa-Clara (2001) have relatively large pricing errors for caps. Therefore, USV might be more relevant for pricing and hedging caps.

Our paper contributes to the literature by re-examining the issue of USV in interest rate caps market. We adopt the approach of FGR (2003) to study whether multifactor term structure models estimated using bond data alone can successfully hedge caps and cap straddles. There are several innovations that distinguish our paper from FGR (2003).

First, we use an unique dataset from SwapPX on interest rate caps with different strikes and maturities in our analysis. As shown in Jarrow, Li and Zhao (2003), there is a pronounced volatility skew in cap implied volatilities. Therefore our data make it possible to study the cross-sectional performance of different term structure models in hedging caps. Second, the term structure models considered in our analysis are quite different from that of FGR (2003). In the past decade, a huge literature has emerged in finance that studies the performance of various term structure models in explaining both the time-series and cross-sectional properties of bond yields (see Dai and Singleton 2003 for an excellent review of the huge literature). The most well-known models, such as the affine term structure models (ATSMs) of Duffie and Kan (1996) and the quadratic term structure models (QTSMs) of Ahn, Dittmar and Gallant (2002) (hereafter ADG), have been reasonably successful in fitting term structure data. By studying whether these models can successfully price and hedge caps, we relate our analysis of USV more closely to the existing term structure literature. Third, the models considered in our paper are estimated using bond data alone. In contrast, some parameters of FGR's model need to be recalibrated constantly using swaption prices to improve model fit. This practice, however, may implicitly use information from swaption prices in designing hedging strategies.

We choose the QTSMs in our analysis for two reasons. First, as demonstrated by ADG (2002), the QTSMs have obvious advantages over the ATSMs in fitting both the conditional mean and volatility of bond yields, which are important for pricing and hedging interest rate derivatives. Second, there is also some preliminary evidence that the ATSMs may not be able to price interest rate derivatives well (see e.g., Jagannathan, Kaplin and Sun 2001, and Collin-Dufresne and Goldstein 2002b).

We estimate the canonical forms of the three-factor QTSMs using Libor bond yields via extended Kalman filter. We find that consistent with the existing literature, the QTSMs can capture term structure dynamics very well. The maximal flexible model has a great fit of different aspects of bond yields, and model-based hedging in all the QTSMs can reduce more than 95% of the variations of bond yields. On the other hand, the QTSMs have serious difficulties in hedging long-term and out-of-the-money (OTM) caps. Principle component analysis of hedging errors of caps across moneyness and maturity shows that there are additional factors affecting cap prices that are not spanned by the yield factors. To focus on the issue of USV, we find that the QTSMs can explain little variations of at-the-money (ATM) cap straddle returns. The strong correlation between straddle hedging errors and changes in ATM cap implied volatilities indicates that the unspanned factors are mainly related to stochastic volatility. The first few principle components of straddle hedging errors can explain a large percentage of hedging errors of all caps across moneyness and maturity. To the extent that USV can be proxied by straddle hedging errors, our results show that the impacts of USV on cap prices are systematic.

Therefore both linear regression and model-based hedging show that USV plays a much more important role for pricing and hedging caps than swaptions. This is consistent with existing findings that models that do not include USV tend to have much larger pricing errors for caps than swaptions. Term structure models that explicitly incorporate stochastic volatility and correlation, such as Collin-Dufresne and Goldstein (2002b) and Han (2002), will be important for pricing and hedging caps and in resolving the relative mispricing between caps and swaptions.

The rest of this paper is organized as follows. In Section I, we introduce the data and provide some preliminary analysis of USV in interest rate caps using linear regression. Section II introduces the QTSMs and their empirical performance for capturing term structure dynamics. In Section III, we study the pricing and hedging of caps in the QTSMs. Section IV concludes the paper and discusses directions of future research.

I. Unspanned Stochastic Volatility: Some Preliminary Analysis

In this section, we provide some preliminary analysis of USV in caps market using the linear regression approach of Collin-Dufresne and Goldstein (2002a). The data used in our analysis are obtained from SwapPX and contain Libor and swap rates, and prices of interest rate caps with different strikes and maturities.[2] The data cover the period from August 1, 2000 to November 7, 2002. After excluding weekends, holidays and missing data, in total we have 557 trading days in our sample. The data are collected everyday when the market is open between 3:30 and 4:00 pm.

Using daily three-month Libor spot and forward rates at 9 maturities (3 and 6 month, 1, 2, 3, 4, 5, 7, and 10 year), we construct the yield curve of Libor zero-coupon bonds. As shown in Figure 1, the yield curve is relatively flat at the beginning of the sample and declines over time, with the short end declining more than the long end. As a result, the yield curve becomes upward sloping in later part of the sample. Table I reports the summary statistics of the levels and changes of 6 month, 1, 2, 5, 7, and 10 year yields. Consistent with the upward sloping yield curve, long-term bonds tend to have higher yields than short-term bonds. On average, all yields exhibit negative changes, consistent with the declining interest rates during our sample period. Changes of short-term yields

[2]Jointly developed by GovPX and Garban-ICAP, SwapPX is the first widely distributed service delivering 24 hour real-time rates, data and analytics for the world-wide interest rate swaps market. GovPX was established in early 1990s by the major U.S. fixed-income dealers as a response to regulators' demands to increase the transparency of the fixed-income markets. It aggregates quotes from most of the largest fixed-income dealers in the world. Garban-ICAP is the world's leading swap broker specializing in trades between dealers and between dealers and large customers. According to Harris (2003), "Its securities, derivatives, and money brokerage businesses have daily transaction volumes in excess of 200 billion dollars".

have higher standard deviation and kurtosis and are more negatively skewed than changes of long-term yields. Yield levels are highly persistent with first-order autoregressive coefficients being close to one. In contrast, yield changes are much less persistent and the autoregressive coefficients decline with maturity. Principle component analysis shows that consistent with the existing literature, the first three principle components can explain more than 99% of the variations in both the levels and changes of bond yields.

Interest rate caps are portfolios of call options on Libor rates. Specifically a cap gives its holder a series of European call options, called caplets, on Libor forward rates. Each caplet has the same strike price as the others, but with different expiration dates. The caps in our data are written on three-month Libor. Suppose $L(t,T)$ is the 3-month Libor forward rate at $t \leq T$, for the interval from T to $T+\frac{1}{4}$. A caplet for the period $\left[T, T+\frac{1}{4}\right]$ struck at K pays $\frac{1}{4}\left(L(T,T)-K\right)^{+}$ at $T+\frac{1}{4}$. Note that while the cash flow on this caplet is received at time $T+\frac{1}{4}$, the Libor rate is determined at time T. This means that there is no uncertainty about the caplet's cash flow after the Libor rate is set at time T.[3] A cap is just a portfolio of these caplets whose maturities are three months apart. For example, a five-year cap on three-month Libor struck at six percent represents a portfolio of 19 separately exercisable caplets with quarterly maturities ranging from 6 month to 5 years, where each caplet has a strike price of 6%.

Existing literature on interest rate derivatives has mainly focused on ATM contracts. One advantage of our data is that we observe prices of caps over a wide range of strikes and maturities.[4] For example, every day for each maturity, there are ten different strike prices, which are 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 8.0, 9.0, and 10.0 percent between August 1, 2000 and October 17, 2001, and 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, and 7.0 percent between November 2, 2001 and November 1, 2002.[5] Throughout the whole sample, caps have fifteen different maturities, which are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0 years.

[3]Standard industry practice is to use Black's (1976) formula to price the caplet. Brace, Gatarek, and Musiela (1997) and Miltersen, Sandmann, and Sondermann (1997) show that market practice is consistent with arbitrage-free pricing if the LIBOR rates follow a log-normal distribution under the appropriate forward measure.

[4]To our knowledge, the only existing study that considers caps with different strikes is Gupta and Subrahmanyam (2001). Their data, obtained from Tullett and Tokoyo Liberty, covers a shorter time period (March 1 to December 31, 1998), has a narrower spectrum of strikes and maturities (four choices for each), and the maximum maturity is only five years. Their sample also covers the turbulent periods of the Russian financial crisis and the collapse of LTCM in the summer and fall of 1998, which might make their data less reliable.

[5]The strike prices are lowered to 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0 and 5.5 percent between October 18 and November 1, 2001.

As shown in Jarrow, Li, and Zhao (2003), there is a pronounced volatility skew in cap implied volatilities during our sample period. Ideally we would like to use caplet prices to back out the implied volatilities. Unfortunately, we only observe cap prices. To simplify computation, we consider the difference between prices of caps with adjacent maturities. Thus, our analysis deals with only the sum of the few caplets between two neighboring maturities. For example, for the rest of the paper, 1.5 year caps represent the sum of the 1.25 and 1.5 year caplets. Because of changing market conditions, throughout our analysis, we focus on caps with fixed moneyness. Therefore each day, we interpolate cap prices with respect to strike price to obtain prices at fixed moneyness. After eliminating all observations that violate various arbitrage restrictions, we plot the average implied volatilities of caps across moneyness and maturity over the whole sample period in Figure 2. Consistent with the existing literature, the implied volatilities of close-to-the-money caps have a humped shape. However, the implied volatilities of all other caps decline with maturity. There is also a pronounced volatility skew for caps with all maturities, with the skew being stronger for short-term caps. The pattern is similar to that of equity options, i.e., ITM caps have higher implied volatilities than OTM caps. The implied volatilities of the very short-term caps are more like a symmetric smile than a skew. Figure 3 plots the time series of Black implied volatilities for 2 and 8 year caps across moneyness. It is clear that the implied volatilities are much higher in the second half of our sample due to the more uncertain economic environment. As a result of changing interest rates and strike prices, there are more ITM caps in the second half of our sample.

If the caps market is well integrated with the Libor and swap market, then the three common term structure factors that explain more than 99% of bond yields should also explain cap prices well. Low explanatory power would suggest that there could be factors affecting cap prices that are not spanned by bonds. To test this hypothesis, we regress weekly returns of caps with fixed moneyness and maturity on weekly changes of the three yield factors.

As caps are traded over the counter, we only observe their prices with fixed time to maturity, but not fixed maturity dates. To calculate weekly returns at a fixed moneyness, we need the price of a cap one week later that has the same strike price and a maturity that is two years minus one week. Following previous studies, such as FGR (2003) and Collin-Dufresne and Goldstein (2002a), we interpolate with respect to maturity the prices of caps with the same strike price a week later.

Through the above interpolation, we obtain a series of weekly cap returns for each moneyness and maturity. Table II reports the R^2s of regressions of weekly returns of caps on weekly changes of the three yield factors for each moneyness/maturity group. Because of changing interest rates and strike prices, we do not have the same number of observations throughout the whole sample for all moneyness/maturity groups. The bold entries represent observations with less than 10% of missing values and the rest with 10-50% of missing values. In total we have 111 weeks of nonoverlapping observations if there are no missing data.

The R^2s in Table II show that the three yield factors can explain a large percentage of returns of ITM and short-term caps. But the explanatory power is significantly worsened for OTM and long-term caps, suggesting that bonds may not be able to span caps. Collin-Dufresne and Goldstein (2002a) argue that the unspanned factor is mainly related to stochastic volatility, because changes of swap rates can explain little variations of ATM straddle returns which are mostly sensitive to volatility risk. We also regress ATM straddle returns on changes in the three yield factors and obtain very similar results. With a few exceptions, the R^2s of our straddle regressions are typically in single digit. Therefore the results from our linear regression analysis are consistent with that of Collin-Dufresne and Goldstein (2002a). As pointed out by FGR (2003), however, linear regression could be misleading due to the nonlinear dependence of straddle returns on underlying yield factors. To rigorously address this issue, we examine the performance of multifactor term structure models in hedging caps.

II. Quadratic Term Structure Models

A. Quadratic Term Structure Models

FGR (2003) show that linear regression and model-based hedging provide very different answers on USV in swaptions market. To understand the importance of USV for caps, we adopt the approach of FGR (2003) by testing whether multifactor term structure models estimated using bond data can hedge caps well. To relate more closely to the existing literature, in our analysis we consider term structure models that have been widely studied recently for fitting both the time-series and cross-sectional behavior of bond yields. These models have the advantages that most fixed-income securities can be priced in (essentially) closed-form and model parameters can be estimated using bond data alone. On the other hand, in FGR's model, pricing relies on simulation and some parameters need to be re-calibrated constantly using swaption prices to improve model performance.

The ATSMs of Duffie and Kan (1996) and the QTSMs of ADG (2002) are probably the most widely studied models in the existing literature. In our empirical analysis, we choose the QTSMs of ADG (2002), because they have several advantages over the ATSMs. First, since interest rate is a quadratic function of the state variables, it is guaranteed to be positive in the QTSMs. On the other hand, in the ATSMs, the spot rate, an affine function of the state variables, is guaranteed to be positive only when all state variables follow square-root processes. Second, the QTSMs do not have the limitations facing the ATSMs in simultaneously fitting interest rate volatility and correlations among the state variables. That is in the ATSMs, increasing the number of factors that follow square-root processes improves the modeling of volatility clustering in bond yields, but reduces the flexibility in modeling correlations among state variables. Third, the QTSMs have the potential to capture observed nonlinearity in term structure data (see e.g., Ahn and Gao 1999). Indeed, ADG show that the QTSMs have better empirical performance than the ATSMs in capturing both the conditional mean and volatility of bond yields.

For the rest of this section, we briefly introduce the QTSMs, the estimation method, and their performance in capturing term structure dynamics. The economy is represented by the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P\right)$, where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the augmented filtration generated by an N-dimensional standard Brownian motion, W, on this probability space. We assume $\{\mathcal{F}_t\}_{0\leq t\leq T}$ satisfies the usual hypothesis (see Protter 1990). The QTSMs rely on the following assumptions:

- The instantaneous interest rate r_t is a quadratic function of the N-dimensional state variables X_t,

$$r\left(X_t\right) = X_t' \Psi X_t + \beta' X_t + \alpha. \tag{1}$$

- The state variables follow a multivariate Gaussian process,

$$dX_t = \left[\mu + \xi X_t\right] dt + \Sigma dW_t. \tag{2}$$

- The market price of risk is an affine function of the state variables,

$$\zeta(X_t) = \eta_0 + \eta_1 X_t. \tag{3}$$

Note that in the above equations Ψ, ξ, Σ, and η_1 are N-by-N matrices, β, μ and η_0 are vectors of length N and α is a scalar. The quadratic relation between r_t and X_t has the desired property that r_t is guaranteed to be positive if Ψ is positive semidefinite and $\alpha - \frac{1}{4}\beta'\Psi\beta \geq 0$. Although X_t follows a Gaussian process in (2), interest rate r_t exhibits conditional heteroskedasticity because of the quadratic relationship between r_t and X_t. As a result, the QTSMs are more flexible in modeling volatility clustering in bond yields and correlations among the state variables than the ATSMs.

We assume that ξ permits the following eigenvalue decomposition,

$$\xi = U \Lambda U^{-1}$$

where Λ is the diagonal matrix of the eigenvalues, $\Lambda \equiv diag\left[\lambda_i\right]_N$, and U is the matrix of the eigenvectors of ξ, $U \equiv \left[u_1\ u_2\ ...\ u_N\right]$. The eigenvalues are assumed to be negative to ensure the stationarity of the state variables. The conditional distribution of the state variables X_t is multivariate Gaussian with conditional mean

$$E\left[X_{t+\Delta t}|X_t\right] = U\Lambda^{-1}\left[\Phi - I_N\right]U^{-1}\mu + U\Lambda^{-1}\left[\Phi - I_N\right]U^{-1}X_t \tag{4}$$

and conditional variance

$$var\left[X_{t+\Delta t}|X_t\right] = U\Theta U' \tag{5}$$

where Φ is a diagonal matrix with elements $\exp(\lambda_i \Delta t)$ for $i = 1, ..., N$, Θ is a N-by-N matrix with elements $\left[\frac{v_{ij}\left(e^{\Delta t\left(\lambda_i+\lambda_j\right)}-1\right)}{\lambda_i+\lambda_j}\right]$, where $\left[v_{ij}\right]_{N\times N} = U^{-1}\Sigma\Sigma' U'^{-1}$.

With the specification of market price of risk, we can relate the risk-neutral measure Q to the physical one P as follows,

$$E\left[\frac{dQ}{dP}|\mathcal{F}_t\right] = \exp\left[-\int_0^t \zeta(X_s)' dW_s + \frac{1}{2}\int_0^t \zeta(X_s)'\zeta(X_s)ds\right],\ \text{for } t \leq T.$$

Applying Girsanov's theorem, we obtain the risk-neutral dynamics of the state variables

$$dX_t = \left[\delta + \gamma X_t\right]dt + \Sigma dW_t^Q$$

where $\delta = \mu - \Sigma\eta_0, \gamma = \xi - \Sigma\eta_1$, and W_t^Q is an N-dimensional standard Brownian motion under measure Q.

Under the above assumptions, a large class of fixed-income securities can be priced in (essentially) closed-form (see Leippold and Wu 2002). We discuss the pricing of zero-coupon bonds below and the pricing of caps in Appendix A. Let $V(t,\tau)$ be the time-t value of a zero-coupon bond that pays 1 dollar at time $t+\tau$. In the absence of arbitrage, the discounted value process $e^{\left(-\int_0^t r(X_s)ds\right)}V(t,\tau)$ is a $Q-$martingale. Thus the value function must satisfy the fundamental PDE, which requires the bond's instantaneous return equals the risk-free rate,

$$\frac{1}{2}tr\left(\Sigma\Sigma'\frac{\partial^2 V(t,\tau)}{\partial X_t \partial X_t'}\right) + \frac{\partial V(t,\tau)}{\partial X_t'}\left(\delta + \gamma X_t\right) + \frac{\partial V(t,\tau)}{\partial t} = r_t V(t,\tau)$$

with the terminal condition $V(t,0) = 1$. The solution takes the form

$$V(t,\tau) = \exp\left[-X_t' A(\tau) X_t - b(\tau)' X_t - c(\tau)\right],$$

where $A(\tau), b(\tau)$ and $c(\tau)$ satisfy the following system of ordinary differential equations (ODEs),

$$\begin{aligned} \frac{\partial A(\tau)}{\partial \tau} &= \Psi + A(\tau)\gamma + \gamma' A(\tau) - 2A(\tau)\Sigma\Sigma' A(\tau); && (6) \\ \frac{\partial b(\tau)}{\partial \tau} &= \beta + 2A(\tau)\delta + \gamma' b(\tau) - 2A(\tau)\Sigma\Sigma' b(\tau); && (7) \\ \frac{\partial c(\tau)}{\partial \tau} &= \alpha + b(\tau)'\delta - \frac{1}{2} b(\tau)'\Sigma\Sigma' b(\tau) + tr\left[\Sigma\Sigma' A(\tau)\right]; && (8) \\ \text{with } A(0) &= 0_{N\times N}; b(0) = 0_N; c(0) = 0. \end{aligned}$$

Consequently, the yield-to-maturity, $y(t, \tau)$, is a quadratic function of the state variables

$$y(t, \tau) = \frac{1}{\tau}\left[X_t' A(\tau) X_t + b(\tau)' X_t + c(\tau)\right]. \quad (9)$$

In contrast, in the ATSMs the yields are linear in the state variables and therefore the correlations among the yields are solely determined by the correlations of the state variables. Although the state variables in the QTSMs follow multivariate Gaussian process, the quadratic form of the yields helps to model the complicated volatility and correlation structure of bond yields.

B. Estimation Method

To price and hedge caps in the QTSMs, we need to estimate both model parameters and latent state variables. Due to the quadratic relationship between bond yields and the state variables, the state variables are not identified by the observed yields even in the univariate case in the QTSMs. Previous studies, such as ADG (2002) have used the efficient method of moments (EMM) of Gallant and Tauchen (1996) to estimate the QTSMs. However, some recent studies, such as Duffee and Stanton (2001), suggest that EMM may not perform very well in small samples for estimating term structure models. Hence, following Duffee and Stanton's (2001) suggestions, we choose the extended Kalman filter to estimate model parameters and extract latent state variables. Previous studies that have used the extended Kalman filter in estimating the ATSMs include Duan and Simonato (1995), De Jong and Santa-Clara (1999), and Lund (1997), among others.

To implement the extended Kalman filter, we first recast the QTSMs into a state-space representation. Suppose we have a time series of observations of yields of L zero-coupon bonds with maturities $\Gamma = (\tau_1, \tau_2, ..., \tau_L)$. Let $Y_k = f(X_k, \Gamma)$ be the vector of the L observed yields at the discrete time points $k\Delta t$, for $k = 1, 2, ..., K$, where Δt is the sample interval (one day in our case). After the following change of variable,

$$Z_k = U^{-1}(\xi^{-1}\mu + X_k),$$

we have the state equation:

$$Z_k = \Phi Z_{k-1} + w_k, \qquad w_k \sim N(0, \Theta) \quad (10)$$

where Φ and Θ are first introduced in (4) and (5), and measurement equation:

$$Y_k = d_k + H_k Z_k + v_k, \quad v_k \sim N(0, R^v) \tag{11}$$

where the innovations in the state and measurement equation w_k and v_k follow serially independent Gaussian processes and are independent from each other. The time-varying coefficients of the measurement equation d_k and H_k are determined at the ex ante forecast of the state variables,

$$\begin{aligned} H_k &= \frac{\partial f(Uz - \xi^{-1}\mu, \Gamma)}{\partial z} |_{z=Z_{k|k-1}} \\ d_k &= f(UZ_{k|k-1} - \xi^{-1}\mu, \Gamma) - H_k Z_{k|k-1} + B_k, \end{aligned}$$

where $Z_{k|k-1} = \Phi Z_{k-1}$.

In the QTSMs, the transition density of the state variables is multivariate Gaussian under both the physical and risk-neutral measure. Thus the transition equation in the Kalman filter is exact. The only source of approximation error is due to the linearization of the quadratic measurement equation. As our estimation uses daily data, the approximation error, which is proportional to one-day ahead forecast error, is likely to be minor. In Appendix B, we further discuss how to minimize the approximation error by introducing the correction term B_k.[6] The Kalman filter starts with the initial state variable $Z_0 = E(Z_0)$ and variance-covariance matrix P_0^Z,

$$P_0^Z = E\left[(Z_0 - E(Z_0))(Z_0 - E(Z_0))'\right].$$

Given the set of filtering parameters, Ξ, we can write down the log-likelihood of observations based on the Kalman filter

$$\begin{aligned} \log \mathcal{L}(Y; \Xi) &= \sum_{k=1}^{K} \log f(Y_k; \mathcal{Y}_{k-1}, \Xi) \\ &= -\frac{LK}{2}\log(2\pi) - \frac{1}{2}\sum_{k=1}^{K}\log\left|P_{k|k-1}^Y\right| - \frac{1}{2}\sum_{k=1}^{K}\left[\left(Y_k - \hat{Y}_{k|k-1}\right)'\left(P_{k|k-1}^Y\right)^{-1}\left(Y_k - \hat{Y}_{k|k-1}\right)\right] \end{aligned}$$

with $\mathcal{Y}_{k-1}$ is the information set at time $(k-1)\Delta t$, and $P_{k|k-1}^Y$ is the time $(k-1)\Delta t$ conditional variance of Y_k,

$$P_{k|k-1}^Y = H_k' P_{k|k-1}^Z H_k + R^v.$$

Parameters are obtained by maximizing the above likelihood function. To avoid local minimum, in our estimation procedure, we use many different starting values and search for the optimal point

[6] The differences between parameter estimates with and without the correction term are very small and we report those estimates with the correction term B_k.

using Simplex method. Then we use gradient-based method to refine those estimates, until they cannot be further improved. This is the standard technique in the literature (see e.g., Duffee 2002).

C. Parameter Estimates and Model Performance

Following ADG (2002), we consider the canonical forms of the three-factor QTSMs given that they are quite successful in explaining term structure data. In all models, the following restrictions are imposed for identification purpose. We assume that Ψ is a symmetric semi-positive definite matrix with diagonal elements of 1:

$$\Psi = \begin{bmatrix} 1 & \Psi_{12} & \Psi_{13} \\ \Psi_{12} & 1 & \Psi_{23} \\ \Psi_{13} & \Psi_{23} & 1 \end{bmatrix}.$$

We also assume that $\mu \geq 0$, $\alpha > 0$, $\beta = 0_N$, ξ and δ_1 are lower triangular matrices, and Σ is a diagonal matrix. We consider the following three models with a decreasing order of complexity:

- QTSM1. This is the maximal flexible model that has a fully specified covariance matrix of the state variables and allow interactions among the state variables in the determination of r_t. For this model, we need to estimate α, three off-diagonal elements of Ψ, three elements of μ, six elements of ξ, three elements of Σ, three elements of δ_0, and six elements of δ_1. The total number of parameters is 25.

- QTSM2. This model has orthogonal state variables, but allow interactions among the state variables in determining r_t. That is in QTSM2, ξ and δ_1 are diagonal, so the state variables are orthogonal under both P and Q measure. However, Ψ is non-diagonal, allowing interactions in the determination of r_t. The total number of parameters of this model is 19.

- QTSM3. It has orthogonal state variables and no interactions in the determination of r_t. Thus the additional restriction in this model relative to QTSM2 is that Ψ is diagonal. In total we have 16 parameters for QTSM3.

We estimate the above three-factor models using 6 month, 1, 2, 5, 7 and 10 year yields. Over the sample period, we have 557 observations and we drop likelihood of the first one for initializing the Kalman filter and the one after September 11, 2001 as an extreme outlier. In implementing the extended Kalman filter, we assume that all yields are observed with independent measurement errors, which follow normal distribution of zero mean and standard deviation of $\sigma_{1/2}, \sigma_1, \sigma_2, \sigma_5, \sigma_7$, and σ_{10} for each maturity respectively. We thus have 6 additional parameters for each of the three models.

Parameter estimates and log-likelihood values for each model are reported in Table III. The likelihood ratios between different models indicate that correlations among the state variables and their interactions in determining r_t are important for better model performance. We examine the performance of the QTSMs in capturing yield curve dynamics from several different perspectives. Figure 4 plots the time series observations of QTSM1 model-implied state variables and the three yield factors obtained from principle component analysis. There are clear differences between the two sets of variables due to the nonlinear relationship between bond yields and the state variables in the QTSMs. Figure 5 shows that QTSM1 model-implied yields are almost indistinguishable from the corresponding observed yields. Table IV reports the summary statistics of the levels and changes of QTSM1 model-implied yields. A comparison with the summary statistics of the actual yields in Table I shows that QTSM1 can capture the mean, standard deviation, skewness, kurtosis and first order autocorrelations of bond yields very well.

In later sections, we will study whether bonds can span caps by testing whether the QTSMs estimated using bond data alone can hedge caps well. For comparison, we first look at the performance of the QTSMs in hedging zero-coupon bonds. We assume that the filtered state variables are traded and use them as hedging instruments. We conduct delta-neutral hedge for the six zero-coupon bonds on a daily basis. Hedging performance is measured by variance ratio, which is the percentage of the variations of an unhedged position that can be reduced by hedging. The results on the hedging performance in Panel C of Table IV show that in most cases the variance ratios are higher than 95%. This should not be surprising given the good performance of the QTSMs in capturing term structure data.

III. Pricing and Hedging Interest Rate Caps in QTSMs

If the Libor and swap market and the caps market are well integrated, then the estimated three-factor QTSMs should be able to price and hedge caps well. Otherwise, it would be a strong indication that there are risk factors affecting cap prices that are not spanned by bonds. With the estimated parameters and the state variables of the three QTSMs, we re-examine the issue of USV in caps market.

A. Pricing Interest Rate Caps

We first study the performance of the QTSMs in pricing caps. In contrast to the numerous studies that fit the ATSMs and QTSMs to bond yields, there is little work testing their performance for pricing interest rate derivatives. In a recent paper, Jagannathan, Kaplin and Sun (2001) show that a three-factor Cox, Ingersoll and Ross (1985) model has large pricing errors for caps and swaptions. To the best of our knowledge, our paper is probably the first one that empirically studies the performance of the QTSMs in pricing and hedging caps.

Panel A, B, and C of Table V report the RMSE of percentage pricing errors of caps with different moneyness and maturity for QTSM3, QTSM2, and QTSM1 respectively. Percentage pricing error, defined as the difference between market and model price divided by market price, is a better measure of model performance, because caps with different moneyness and maturity can have dramatically different prices. This measure has been used in previous studies, such as Longstaff, Santa-Clara and Schwartz (2001) and FGR (2003). As pointed out before, we interpolate cap prices with respect to strike price to obtain prices at fixed moneyness. Similar to Table II, the bold entries are moneyness/maturity groups that have less than 10% of missing values and the rest have between 10 to 50% of missing values.

All three QTSMs have smaller percentage pricing errors for ITM and long-term caps than OTM and short-term caps. For example, in QTSM1, while the percentage pricing errors for ITM caps are less than 10%, they can be over 40% for short-term OTM caps. QTSM2 and QTSM3 have especially high percentage pricing errors for short-term and OTM caps. In general, QTSM1 has smaller percentage pricing errors across moneyness and maturity than the other two models, except that QTSM2 has slightly lower percentage pricing errors for deep ITM caps. This indicates that the additional flexibility provided by the correlations among the state variables improves the model's pricing performance. QTSM2 has lower pricing errors than QTSM3 for short-term caps, but higher pricing errors for long-term caps.

While the QTSMs have significant pricing errors, the RMSE of percentage pricing error does not tell the direction of mispricing. Panel D of Table V reports the average percentage pricing errors of the best model, QTSM1. It is clear that QTSM1 underprices ITM caps and overprices OTM caps. This is consistent with Jarrow, Li and Zhao (2002) who show that it is difficult to capture the pronounced volatility skew in caps data.

The pricing analysis shows that although the QTSMs can capture the level of bond yields pretty well, they still have systematic biases for pricing caps, especially OTM caps. A deep ITM option behaves almost like the underlying asset, because the probability that the option will be eventually in the money is high. Thus it is not surprising that the QTSMs have relatively better performance in pricing ITM caps. However, OTM caps are much more sensitive to the tail behavior of the distribution of the underlying interest rates. Therefore, to accurately price OTM caps, the QTSMs need to capture not only the level but also the whole distribution, especially the tail distribution, of the bond yields.

B. Hedging Interest Rate Caps

Pricing analysis mainly focuses on whether a model can capture the distribution of underlying asset price on the maturity date of an option. On the other hand, hedging analysis also reveals whether a model can capture the dynamics of the evolution of the underlying price process. In this section, we study the performance of the three-factor QTSMs in hedging caps.

Based on the estimated model parameters, we conduct delta-neutral hedge of weekly changes of cap prices using filtered state variables as hedging instruments. We could also use Libor zero-coupon bonds as hedging instruments by matching the hedge ratios of a cap with that of zero-coupon bonds. However, using deltas of zero-coupon bonds introduces one additional layer of potential model misspecification. To improve hedging performance, we allow daily rebalance, i.e., we adjust the hedge ratios everyday given changes in market conditions. Therefore daily changes of a hedged position is the difference between daily changes of the unhedged position and the hedging portfolio. The latter equals to the sum of the products of a cap's hedge ratios with respect to the state variables and changes in the corresponding state variables. Weekly changes are just the accumulation over daily positions. Over the sample period, there are 111 nonoverlapping hedged and unhedged changes for each moneyness/maturity group if there are no missing data.

Again, we measure hedging effectiveness by variance ratio, the percentage of the variations of an unhedged position that can be reduced by hedging. This measure is similar in spirit to R^2 in linear regression.[7] The variance ratios of the three QTSMs in Table VI show that all models have better hedging performance for ITM, short-term (maturities from 1.5 to 4 years) caps than OTM, medium and long-term caps (maturities longer than 4 years) caps. There is a high percentage of variations in long-term and OTM cap prices that cannot be hedged. The maximal flexible model QTSM1 again has better hedging performance than the other two models.

Interestingly, the variance ratios of model-based hedging in Table VI are very similar to the R^2s of linear regressions in Table II. In fact, linear regression has higher explanatory power than model-based hedging. This is mainly because we run separate regression with different parameters for caps within each moneyness/maturity group. On the other hand, the hedge ratios of all caps in model-based hedging are determined by the same set of parameters estimated using bond data. Thus the number of parameters and degrees of freedom are much larger in regression analysis in Table

[7]FGR (2002) also consider RMSE of hedging errors because their hedging errors have significant biases. Since the hedging bias in our case is very small, we only report the variance ratios.

II. Therefore both regression and model-based hedging suggest that bond market factors cannot satisfactorily hedge interest rate caps, especially OTM and long-term caps.

To control for the fact that the QTSMs maybe misspecified, in Panel D of Table VI, we further regress hedging errors of each moneyness/maturity group on changes of the three yield factors. While the three yield factors can explain some additional hedging errors, their incremental explanatory power is not very significant. Thus even combined with the three yield factors, there is still a large fraction of cap prices that cannot be explained by the QTSMs. We conduct principle component analysis of hedging errors of caps with different moneyness in Table VII, focusing on those moneyness groups for which we have enough observations throughout the whole sample period. We also repeat the analysis by combining these caps together. It is clear that the first principle component explains about 50-60% of the hedging errors of all caps and caps within each moneyness group. Each of the next two components explains about additional 10% of hedging errors. Our analysis of hedging errors suggests that there could be multiple unspanned factors in caps data.

C. Hedging Cap Straddles: Evidence of Unspanned Stochastic Volatility

Hedging analysis based on the QTSMs confirms the findings of Collin-Dufresne and Goldstein (2002a) that there are unspanned factors in caps market. Collin-Dufresne and Goldstein (2002a) show that changes in swap rates in general can explain less than 20% of ATM cap straddle returns which are most sensitive to volatility risk. Therefore, they argue that the unspanned factor is a stochastic volatility factor that significantly affects cap prices but not bond yields. However, as pointed out by FGR (2003), linear regression results could be misleading because straddle returns are highly nonlinear in underlying yield factors. They show that although linear regression can explain little variations in swaption straddle returns, a three-factor Heath, Jarrow and Morton (1992) model can hedge swaption straddles pretty well.

In Table VIII, we re-examine the issue of USV in caps market by testing the performance of the QTSMs in hedging ATM cap straddles. We obtain ATM floor prices from cap prices using the put-call parity and construct weekly straddle returns. As straddles are highly sensitive to volatility risk, we conduct both delta and gamma neutral hedge. The variance ratios of QTSM1 are as low as the R^2s of linear regressions of straddle returns on the yield factors in Table II, suggesting that neither approach can explain much variations of straddle returns. While FGR (2003) show that linear regression and model-based hedging have dramatically different performance for swaption straddles, we find that the difference between the two approaches for cap straddles is very small. Collin-Dufresne and Goldstein (2002a) show that 80% of straddle regression residuals can be explained by one additional factor.

Principle component analysis of straddle hedging errors in Panel B of Table VIII shows that the first factor can explain about 60% of the total variations of hedging errors. The second and third factor each explains about 10% of hedging errors and two additional factors combined can explain about another 10% of hedging errors. The correlation matrix of the hedging errors across maturities in Panel C shows that the hedging errors of short-term (2, 2.5, 3, 3.5, and 4 year), medium-term (4.5 and 5 year) and long-term (8, 9, and 10 year) straddles are highly correlated within each group, again suggesting that there could be multiple unspanned factors.

To further understand whether the unspanned factors are related to stochastic volatility, we study the relationship between ATM cap implied volatilities and straddle hedging errors. Principle component analysis in Panel A of Table IX shows that the first component explains 85% of the variations of cap implied volatilities. In Panel B, we regress straddle hedging errors on changes of the three yield factors and obtain R^2s that are close to zero. However, if we include the first few principle components of cap implied volatilities, the R^2s increase significantly: for some maturities, the R^2s are above 90%. In the extreme case in which we regress straddle hedging errors of each maturity on changes of the yield factors and cap implied volatilities with the same maturity, the R^2s in most cases are above 90%. These results show that straddle returns are mainly affected by volatility risk but not term structure factors.

Thus the poor hedging performance of the QTSMs is mainly due to the USV in caps data. If the USV is indeed systematic, including this factor should significantly improve the hedging performance of all caps. As ATM straddles are mainly exposed to volatility risk, their hedging errors can serve as a proxy of the USV. Table IX reports the R^2s of regressions of hedging errors of caps across moneyness and maturity on changes of the three yield factors and the first five principle components of straddle hedging errors. In contrast to the regressions in Panel D of Table VI, which only include the three yield factors, the additional factors from straddle hedging errors significantly improve the R^2s of the regressions: for most moneyness/maturity groups, the R^2s are above 90%. Interestingly for long-term caps, the R^2s of ATM and OTM caps are actually higher than that of ITM caps. Therefore a combination of the yield factors and the USV factors can explain cap prices across moneyness and maturity very well.

Our analysis reaches quite different conclusions from that of FGR (2003). FGR (2003) show that allowing time varying volatility (i.e., allowing volatility parameters to be recalibrated from swaption prices) increase the variance ratio of swaption straddles by about 10%, which they argue is not significant enough. In contrast the first few principle components of straddle hedging errors can explain a large percentage of hedging errors of all caps. Therefore incorporating USV is much more important for hedging caps than swaptions. This result is consistent with the findings of Collin-Dufresne and Goldstein (2002b) that the implied volatilities of caps are much more volatile than that of swaptions: the variance of cap implied volatilities is about two times as that of swaption implied volatilities. The main reason for such a big difference is that while caps and swaptions are equally sensitive to changes in volatilities, caps are much more sensitive to changes in correlations than swaptions, even though swaptions are options on portfolios of bonds. Collin-Dufresne and Goldstein (2002b) show that the big difference between the implied volatilities of caps and swaptions simply can not be reconciled within term structure models with constant volatility and correlation, instead they develop random field models (see, e.g., Goldstein 2000, Santan-Clara and Sornette 2001) that explicitly consider stochastic and correlation to address this issue. Therefore, it is clear that the USV we document in caps market is due to the combined effects of stochastic volatility and correlation of bond yields. These two factors are important not only for pricing and hedging caps but also for resolving the relative mispricing between caps and swaptions.

IV. Conclusion

In this paper, we re-examine the issue of USV in caps market. While FGR (2003) show that the benefit of including USV for hedging swaptions is minor, we find that USV plays much more important roles for pricing and hedging caps. Our results reconfirm the findings of Collin-Dufresne and Goldstein (2002a) that there are unspanned stochastic volatility factors affecting caps but not Libor and swap rates. The different conclusions for caps and swaptions are consistent with the fact that cap implied volatilities are much more volatile that swaption implied volatilities, and caps are much more sensitive to changes in correlations than swaptions. Therefore, term structure models that explicitly incorporate stochastic volatility and correlation would be important for pricing and hedging caps and have the potential to resolve the relative mispricing between caps and swaptions. The empirical literature on Libor-based interest rate derivatives is far from being fully developed and there are many interesting open questions. One obvious one is to test whether existing models with stochastic volatility and correlation can price caps, especially the pronounced volatility skew in caps data well. Another important issue is to understand the economic foundations of the USV. In this regard, Back's (1993) work on stochastic volatility in equity options market may prove to be useful.

REFERENCES

Ahn, D., and B. Gao, 1999, A Parametric Nonlinear Model of Term Structure Dynamics, *Review of Financial Studies* 12, 721-762.

Ahn, D., R. Dittmar, and R. Gallant, 2002, Quadratic Term Structure Models: Theory and Evidence, *Review of Financial Studies* 15, 243-288.

Back, K., 1993, Asymmetric Information and Options, *Review of Financial Studies* 6, 435-472.

Black, F., 1976, The Pricing of Commodity Contracts, *Journal of Financial Economics* 3, 167-179.

Brace, A., D. Gatarek, and M. Musiela, 1997, The Market Model of Interest Rate Dynamics, *Mathematical Finance* 7, 127-155.

Collin-Dufresne, P. and R.S. Goldstein, 2002a, Do Bonds Span the Fixed Income Markets? Theory and Evidence for Unspanned Stochastic Volatility, *Journal of Finance* 57, 1685-1729.

Collin-Dufresne, P. and R.S. Goldstein, 2002b, Stochastic Correlation and the Relative Pricing of Caps and Swaptions in a Generalized Affine Framework. Working paper, Carnegie Mellon University and Washington University.

Cox, J.C., J.E. Ingersoll and S.A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385-407.

Dai, Q., and K. Singleton, 2003, Term Structure Dynamics in Theory and Reality, *Review of Financial Studies* 16, 631-678.

De Jong, F., P. Santa-Clara, 1999, The Dynamics of the Forward Interest Rate Curve: A Formulation with State Variables, *Journal of Financial and Quantitative Analysis* 34, 131-57.

Duan, J., J. Simonato, 1999, Estimating and Testing Exponential-Affine Term Structure Models by Kalman Filter, *Review of Quantitative Finance and Accounting* 13, 111-35.

Duffee, G., 2002, Term Premia and Interest Rate Forecasts in Affine Models, *Journal of Finance* 57, 405-443.

Duffee, G., and R. Stanton, 2001, Estimations of Dynamic Term Structure Models, Working paper, University of California, Berkeley.

Duffie, D., and R. Kan, 1996, A Yield-Factor Model of Interest Rates, *Mathematical Finance* 6, 379-406.

Duffie, D., J. Pan, and K. Singleton, 2001, Transform Analysis and Asset Pricing for Affine Jump-Diffusions, *Econometrica* 68, 1343-1376.

Fan, R., A. Gupta, and P. Ritchken, 2003, Hedging in the Possible Presence of Unspanned Stochastic Volatility: Evidence from Swaption Markets, *Journal of Finance* 58, 2219-2248.

Grewal, M., A. Andrews, 2001, Kalman filtering : theory and practice using MATLAB, New York, Wiley.

Gupta, A. and M. Subrahmanyam, 2001, An Examination of the Static and Dynamic Performance of Interest Rate Option Pricing Models in the Dollar Cap-Floor Markets, Working paper, Case Western Reserve University.

Goldstein, R.S., 2000, The Term Structure of Interest Rates as a Random Field, *Review of Financial Studies* 13, 365-384.

Harris, L., 2003, *Trading and Exchanges: Market Microstructure for Practitioners*, Oxford University Press.

Heath, D., R. Jarrow, and A. Morton, 1992, Bond Pricing and the Term Structure of Interest Rates: A New Methodology, *Econometrica* 60, 77-105.

Heidari, M. and L. Wu, 2003, Are Interest Rate Derivatives Spanned by the Term Structure of Interest Rates?, *Journal of Fixed Income Market* 13, 75-86.

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327-343.

Hull, J. and A. White, 1987, The Pricing of Options on Assets with Stochastic Volatilities, *Journal of Finance* 42, 281-300.

Jagannathan, R., A. Kaplin, and S. Sun, 2001, An Evaluation of Multi-factor CIR Models Using LIBOR, Swap Rates, and Cap and Swaption Prices, *Journal of Econometrics* forthcoming.

Jarrow, R., H. Li, and F. Zhao, 2003, Interest Rate Caps "Smile" Too! But Can the LIBOR Market Models Capture It?, Working paper, Cornell University.

Leippold, M. and L. Wu, 2002, Asset Pricing Under the Quadratic Class, *Journal of Financial and Quantitative Analysis* 37, 271-295.

Longstaff, F., P. Santa-Clara, and E. Schwartz, 2001, The Relative Valuation of Caps and Swaptions: Theory and Evidence, *Journal of Finance* 56, 2067-2109.

Lund, J., 1997, Non-linear Kalman filtering techniques for term-structure models, Working paper. Aarhus School of Business.

Miltersen, M., K. Sandmann, and D. Sondermann, 1997, Closed-form Solutions for Term Structure Derivatives with Lognormal Interest Rates, *Journal of Finance* 52, 409-430.

Protter, P., 1990, *Stochastic Integration and Differential Equations*, Springer, New York.

Santa-Clara, P. and D. Sornette, 2001, The Dynamics of the Forward Interest Rate Curve with Stochastic String Shocks, *Review of Financial Studies* 14, 2001.

Appendix A. Closed-form Pricing Formula for Interest Rate Caps

Leippold and Wu (2002) show that a large class of fixed-income securities can be priced in closed-form in the QTSMs using the transform analysis of Duffie, Pan, and Singleton (2001). They show that the time-t value of a contract that has an exponential quadratic payoff structure at terminal time T, i.e.

$$\exp\left(-q(X_T)\right) = \exp\left(-X_T'\overline{A}X_T - \overline{b}'X_T - \overline{c}\right)$$

has the following form

$$\begin{aligned}\psi\left(q, X_t, t, T\right) &= E_Q\left(e^{-\int_t^T r(X_s)ds}e^{-q(X_T)}|\mathcal{F}_t\right) \\ &= \exp\left[-X_t A(T-t)X_t - b(T-t)'X_t - c(T-t)\right].\end{aligned}$$

where $A(.)$, $b(.)$ and $c(.)$ satisfy the ODEs (4)-(6) with the initial conditions $A(0) = \overline{A}, b(0) = \overline{b}$ and $c(0) = \overline{c}$.

The time-t price a call option with payoff $\left(e^{-q(X_T)} - y\right)^+$ at $T = t + \tau$ equals

$$\begin{aligned}C\left(q, y, X_t, \tau\right) &= E_Q\left(e^{-\int_t^T r(X_s)ds}\left(e^{-q(X_T)} - y\right)^+|\mathcal{F}_t\right) \\ &= E_Q\left(e^{-\int_t^T r(X_s)ds}\left(e^{-q(X_T)} - y\right)\mathbf{1}_{\{-q(X_T)\geq \ln(y)\}}|\mathcal{F}_t\right) \\ &= G_{q,q}\left(-\ln\left(y\right), X_t, \tau\right) - yG_{0,q}\left(-\ln\left(y\right), X_t, \tau\right),\end{aligned}$$

where $G_{q_1,q_2}\left(y, X_t, \tau\right) = E_Q\left[e^{-\int_t^T r(X_s)ds}e^{-q_1(X_T)}\mathbf{1}_{\{q_2(X_T)\leq y\}}|\mathcal{F}_t\right]$ and can be computed by the inversion formula,

$$G_{q_1,q_2}\left(y, X_t, \tau\right) = \frac{\psi\left(q_1, X_t, t, T\right)}{2} - \frac{1}{\pi}\int_0^\infty \frac{e^{ivy}\psi\left(q_1 + ivq_2\right) - e^{-ivy}\psi\left(q_1 - ivq_2\right)}{iv}dv.$$

Similarly, the price of a put option is

$$P\left(q, y, \tau, X_t\right) = yG_{0,-q}\left(\ln\left(y\right), X_t, \tau\right) - G_{q,-q}\left(\ln\left(y\right), X_t, \tau\right).$$

We are interested in pricing a cap which is portfolio of European call options on future interest rates with a fixed strike price. For simplicity, we assume the face value is 1 and the strike price is $\overline{r}$. At time 0, let $\tau, 2\tau, ..., n\tau$ be the fixed dates for future interest payments. At each fixed date $k\tau$, the $\overline{r}$-capped interest payment is given by $\tau\left(\mathcal{R}\left(\left(k-1\right)\tau, k\tau\right) - \overline{r}\right)^+$, where $\mathcal{R}\left(\left(k-1\right)\tau, k\tau\right)$ is the τ-year floating interest rate at time $\left(k-1\right)\tau$, defined by

$$\begin{aligned}\frac{1}{1+\tau\mathcal{R}\left(\left(k-1\right)\tau, k\tau\right)} &= \varrho\left(\left(k-1\right)\tau, k\tau\right) \\ &= E^Q\left(\exp\left(-\int_{(k-1)\tau}^{k\tau} r\left(X_s\right)ds\right)|\mathcal{F}_{(k-1)\tau}\right).\end{aligned}$$

The market value at time 0 of the caplet paying at date $k\tau$ can be expressed as

$$\begin{aligned} Caplet\,(k) &= E^Q\left[\exp\left(-\int_0^{k\tau} r\,(X_s)\,ds\right)\tau\left(\mathcal{R}\left((k-1)\,\tau, k\tau\right)-\overline{r}\right)^+\right] \\ &= (1+\tau\overline{r})\,E^Q\left[\exp\left(-\int_0^{(k-1)\tau} r\,(X_s)\,ds\right)\left(\frac{1}{(1+\tau\overline{r})}-\varrho\left((k-1)\,\tau, k\tau\right)\right)^+\right]. \end{aligned}$$

Hence, the pricing of the $k-th$ caplet is equivalent to the pricing of an $(k-1)\,\tau$-for-τ put struck at $K=\frac{1}{(1+\tau\overline{r})}$. Therefore,

$$Caplet(k) = G_{0,-q_\tau}\left(\ln K, X_{(k-1)\tau}, (k-1)\,\tau\right) - \frac{1}{K}G_{q_\tau,-q_\tau}\left(\ln K, X_{(k-1)\tau}, (k-1)\,\tau\right).$$

Similarly for the $k-th$ floorlet

$$Floorlet(k) = -G_{0,q_\tau}\left(-\ln K, X_{(k-1)\tau}, (k-1)\,\tau\right) + \frac{1}{K}G_{q_\tau,q_\tau}\left(-\ln K, X_{(k-1)\tau}, (k-1)\,\tau\right).$$

Appendix B. Quadratic Measurement Bias Correction

The linearized measurement equation generally introduces a bias term. For quadratic measurement equation, the bias term could be corrected (see Grewal and Andrews 2001). Specifically, the yield with maturity τ_j, Y_{jk} is a quadratic function of the state variables Z_k in the form

$$Y_{jk} = Z_k' A Z_k + b' Z_k + c \equiv q(Z_k)$$

for some parameters A, b, and c.Using Taylor series expansion at the ex ante forecast of the state variables $Z_{k|k-1,}$

$$\begin{aligned} Y_{jk} &= q(Z_{k|k-1}) + \left[b' + Z_{k|k-1}'\left(A + A'\right)\right]\left(Z_k - Z_{k|k-1}\right) \\ &\quad + \left(Z_k - Z_{k|k-1}\right)' A \left(Z_k - Z_{k|k-1}\right). \end{aligned}$$

The extended Kalman filter omits the quadratic term in the above expression and thus introduces the bias term B_k to the measurement equation, i.e.,

$$\begin{aligned} B_k &= E_{k-1}\left[\left(Z_k - Z_{k|k-1}\right)' A \left(Z_k - Z_{k|k-1}\right)\right] \\ &= E_{k-1}\left[trace\left(\left(Z_k - Z_{k|k-1}\right)' A \left(Z_k - Z_{k|k-1}\right)\right)\right] \\ &= \left[trace\left(A\left(Z_k - Z_{k|k-1}\right)'\left(Z_k - Z_{k|k-1}\right)\right)\right] \\ &= trace\left\{AE_{k-1}\left[\left(Z_k - Z_{k|k-1}\right)'\left(Z_k - Z_{k|k-1}\right)\right]\right\} \\ &= trace\left\{AP_{k|k-1}^Z\right\}. \end{aligned}$$

However, we should note that this does not eliminate the linearization approximation error of the measurement equation since the Kalman gain is still computed with first derivatives of the measurement function.

Table I. Summary Statistics of Libor Zero-Coupon Bond Yields

This table reports the summary statistics and principle component analysis of the levels and changes of yields on Libor zero-coupon bonds, which are constructed using three-month Libor forward rates provided by SwapPX. The sample is from August 1, 2000 to November 7, 2002. Excluding holidays, weekends, and missing data, we have 557 trading days in total.

Panel A: Yield Levels.

	Maturity (yr)					
	0.5	1	2	5	7	10
Mean (%)	3.536	3.691	4.159	5.049	5.361	5.666
Std. Dev. (%)	1.791	1.631	1.359	0.924	0.777	0.638
Skewness	0.613	0.601	0.461	0.082	-0.011	-0.055
Kurtosis	1.876	2.003	2.155	2.518	2.672	2.799
First-order Partial Autocorrelation	0.998	0.998	0.997	0.997	0.996	0.994

Panel B: Yield Changes.

	Maturity (yr)					
	0.5	1	2	5	7	10
Mean (%)	-0.010	-0.010	-0.009	-0.007	-0.006	-0.005
Std. Dev. (%)	0.048	0.061	0.070	0.072	0.069	0.068
Skewness	-8.388	-3.320	-0.981	-0.101	0.184	0.371
Kurtosis	130.51	44.748	14.980	6.491	4.839	3.921
First-order Partial Autocorrelation	0.240	0.141	0.116	0.066	0.085	0.082

Panel C: Percentage of Variance Explained by the Principle Components.

	Principle Component					
	1	2	3	4	5	6
Level	96.83%	3.10%	0.045%	0.019%	0.002%	0.000%
Change	87.72%	9.84%	1.51%	0.74%	0.13%	0.07%

Table II. Regression Analysis of USV in Caps Market

This table reports the R^2s of regressions of weekly returns of caps across moneyness and maturity and at-the-money cap straddles on weekly changes of the three yield factors. Due to changes in interest rates and strike prices, we do not have the same number of observations for each moneyness/maturity group. The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the ones with 10-50% of missing values.

Moneyness (K/F)	Maturity												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	-	0.918	0.896	0.683	-	0.904	0.789	0.784	0.594	0.566
0.65	-	-	0.930	0.926	0.919	0.902	0.686	0.589	0.895	0.742	0.678	0.528	0.452
0.70	-	-	0.926	0.914	0.902	**0.860**	**0.673**	0.497	0.829	0.661	0.681	0.491	0.380
0.75	-	0.926	**0.916**	**0.914**	**0.904**	**0.856**	0.691	0.431	0.809	0.681	0.658	0.466	0.343
0.80	-	0.902	**0.920**	**0.904**	**0.875**	**0.869**	**0.667**	**0.493**	**0.780**	**0.647**	**0.642**	0.441	0.339
0.85	0.964	**0.881**	**0.907**	**0.894**	**0.861**	**0.825**	**0.644**	**0.529**	**0.756**	**0.632**	**0.601**	**0.422**	0.295
0.90	0.954	**0.860**	**0.875**	**0.874**	**0.834**	**0.813**	**0.628**	**0.517**	**0.699**	**0.611**	**0.539**	**0.425**	**0.318**
0.95	**0.919**	**0.801**	**0.851**	**0.841**	**0.811**	**0.793**	**0.592**	**0.481**	**0.623**	**0.602**	**0.504**	**0.403**	**0.314**
1.00	**0.871**	**0.663**	**0.807**	**0.786**	**0.785**	**0.766**	**0.570**	**0.453**	**0.570**	**0.545**	**0.494**	**0.369**	**0.300**
1.05	**0.824**	**0.584**	**0.757**	**0.745**	**0.751**	**0.763**	**0.527**	**0.416**	**0.512**	0.506	**0.491**	0.359	0.261
1.10	**0.761**	**0.542**	**0.714**	**0.729**	**0.714**	**0.723**	**0.507**	0.406	0.446	0.487	0.461	0.309	0.240
1.15	**0.666**	**0.490**	**0.664**	**0.707**	**0.673**	**0.632**	0.483	0.376	0.388	0.420	0.431	-	-
1.20	0.642	**0.430**	**0.599**	**0.624**	**0.592**	0.603	0.457	0.424	0.298	0.367	0.421	-	-
1.25	0.413	**0.356**	**0.512**	**0.540**	0.532	0.551	0.598	0.555	0.213	0.340	-	-	-
1.30	0.802	**0.279**	**0.428**	0.453	0.427	0.487	0.504	0.504	0.126	-	-	-	-
1.35	0.759	**0.149**	0.344	0.372	0.367	0.416	0.438	-	-	-	-	-	-
1.40	0.726	0.086	0.265	0.284	0.298	0.274	-	-	-	-	-	-	-
Straddle	**0.300**	**0.061**	**0.209**	**0.160**	**0.083**	**0.031**	**0.044**	**0.021**	**0.025**	**0.014**	**0.046**	**0.024**	**0.035**

Table III. Parameter Estimates of Three-Factor QTSMs

This table reports parameter estimates and standard errors (in parentheses) of the three canonical three-factor QTSMs using Kalman Filter.

Parameter	QTSM3		QTSM2		QTSM1	
α	0.0001*	(0.0051)	0.2584*	(4.0608)	0.0034*	(0.0563)
Ψ_{12}			-0.9033	(0.0066)	0.8373	(0.0078)
Ψ_{13}			-0.2723	(0.0287)	-0.5525	(0.0024)
Ψ_{23}			0.0745	(0.0177)	-0.7585	(0.0073)
μ_1	0.1120	(0.0184)	0.7359	(0.0822)	0.3566	(0.0017)
μ_2	0.0059	(0.0419)	0.1153	(0.0167)	0.2265	(0.0071)
μ_3	0.1565	(0.0382)	1.5268	(0.0150)	0.6136	(0.0088)
ξ_{11}	-1.2234	(0.0428)	-0.0468	(0.0049)	-0.0144	(0.0004)
ξ_{21}					4.5979	(0.1614)
ξ_{31}					1.0728	(0.1238)
ξ_{22}	-0.6142	(0.1631)	-0.7578	(0.0789)	-3.4952	(0.0978)
ξ_{32}					3.2078	(0.3519)
ξ_{33}	-0.0083	(0.0020)	-0.0002	(0.0000)	-2.2678	(0.1608)
Σ_{11}	0.0479	(0.0008)	0.0519	(0.0025)	0.0222	(0.0003)
Σ_{22}	0.0725	(0.0042)	0.0801	(0.0016)	0.0728	(0.0003)
Σ_{33}	0.0468	(0.0019)	0.0235	(0.0024)	0.0220	(0.0004)
δ_1	0.0094	(0.0007)	0.0359	(0.0002)	0.0104	(0.0001)
δ_2	-0.1903	(0.0019)	0.0190	(0.0041)	-0.0021	(0.0005)
δ_3	-0.0438	(0.0034)	-0.0108	(0.0009)	-0.0378	(0.0003)
γ_{11}	-0.0530	(0.0039)	-0.1295	(0.0003)	-0.0518	(0.0004)
γ_{21}					1.0130	(0.0073)
γ_{31}					0.0276	(0.0012)
γ_{22}	-1.1378	(0.0200)	-1.1219	(0.0068)	-1.1698	(0.0001)
γ_{32}					0.3018	(0.0013)
γ_{33}	-0.5544	(0.0155)	0.0133	(0.0044)	-0.0558	(0.0023)
$\sigma_{1/2}$	4.2749*	(0.2026)	0.0128*	(0.6710)	0.0002*	(0.1130)
σ_1	2.8876*	(0.1358)	2.2165*	(0.1003)	2.0417*	(0.1043)
σ_2	1.5794*	(0.1481)	1.8485*	(0.3384)	1.9916*	(0.0660)
σ_5	1.7779*	(0.0677)	2.1867*	(0.0496)	1.9257*	(0.0744)
σ_7	0.8568*	(0.0861)	0.0049*	(0.2917)	0.0005*	(0.0958)
σ_{10}	2.9730*	(0.1153)	2.7741*	(0.2328)	2.7830*	(0.0998)
Log-Likelihood	21243		22043		22300	

* 1e-4.

Table IV. The Performance of QTSMs in Modeling Bond Yields

This table reports the performance of the three-factor QTSMs in capturing bond yields.

Panel A. Summary statistics of QTSM1 model-predicted levels of bond yields.

	Maturity (yr)					
	0.5	1	2	5	7	10
Mean (%)	3.529	3.683	4.154	5.049	5.358	5.666
Std. Dev. (%)	1.787	1.632	1.350	0.918	0.774	0.622
Skewness	0.616	0.598	0.468	0.093	-0.017	-0.059
Kurtosis	1.882	1.997	2.163	2.534	2.676	2.783
First-order Partial Autocorrelation	0.998	0.998	0.998	0.997	0.996	0.995

Panel B. Summary statistics of QTSM1 model-predicted changes of bond yields.

	Maturity (yr)					
	0.5	1	2	5	7	10
Mean (%)	-0.010	-0.009	-0.009	-0.007	-0.006	-0.005
Std. Dev. (%)	0.047	0.055	0.065	0.072	0.069	0.062
Skewness	-7.463	-3.222	-0.653	0.280	0.343	0.366
Kurtosis	111.082	42.979	12.058	4.714	4.279	4.132
First-order Partial Autocorrelation	0.286	0.262	0.186	0.109	0.094	0.080

Panel C. Variance ratios of model-based hedging of zero-coupon bonds in QTSMs using filtered state variables as hedging instruments. Variance ratio measures the percentage of the variations of an unhedged position that can be reduced through hedging.

	Maturity (yr)					
	0.5	1	2	5	7	10
QTSM3	0.717	0.948	0.982	0.98	0.993	0.93
QTSM2	0.99	0.956	0.963	0.975	0.997	0.934
QTSM1	0.994	0.962	0.969	0.976	0.997	0.932

Table V. The Performance of QTSMs in Pricing Interest Rate Caps

This table reports the performance of the three QTSMs in pricing interest rate caps. Percentage pricing error is measured as the difference between market and model-impled price divided by market price. The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the ones with 10-50% of missing values.

Panel A. RMSE of percentage pricing errors of QTSM3.

Money-ness K/F	**Maturity**												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	11.3	10.5	9.7	9.2	11.0	10.9	10.3	9.4	8.8	9.4
0.65	-	-	13.4	11.8	10.9	10.1	9.6	11.2	10.8	10.5	9.5	9.2	10.2
0.70	-	15.6	13.8	11.4	10.3	**9.5**	**9.5**	11.3	*11.1*	11.3	*10.1*	9.9	10.9
0.75	-	15.9	**13.9**	**11.8**	**11.2**	**10.2**	10.7	12.4	12.2	12.0	11.0	10.8	11.2
0.80	-	16.7	**15.5**	**13.2**	**12.7**	**11.5**	**11.3**	**12.6**	**12.6**	**12.1**	**11.2**	10.6	11.3
0.85	16.4	**20.0**	**18.5**	**15.4**	**15.0**	**13.2**	**12.1**	**13.0**	**13.6**	**12.4**	**12.2**	**11.0**	11.4
0.90	25.3	**27.2**	**22.4**	**17.9**	**17.3**	**15.1**	**13.7**	**13.9**	**14.9**	**12.9**	**13.6**	**11.8**	**12.0**
0.95	**42.6**	**37.7**	**27.7**	**21.6**	**20.0**	**17.3**	**16.0**	**15.5**	**16.4**	**13.7**	**14.8**	**12.9**	**12.7**
1.00	**69.0**	**50.9**	**35.0**	**26.1**	**23.6**	**19.1**	**18.9**	**17.5**	**18.3**	**15.2**	**15.7**	**13.1**	**13.6**
1.05	**105.9**	**66.1**	**43.9**	**31.4**	**28.1**	**21.6**	**21.9**	**19.6**	**20.7**	17.5	**16.7**	16.1	14.8
1.10	**174.0**	**84.6**	**54.6**	**38.0**	**33.5**	**26.8**	**25.1**	22.2	24.5	19.8	20.3	20.4	17.5
1.15	**291.5**	**107.5**	**67.1**	**46.3**	**40.1**	**33.4**	29.4	25.8	30.2	22.9	22.9	23.6	-
1.20	411.1	**135.5**	**81.7**	**55.7**	48.9	40.2	34.8	30.7	36.2	25.7	24.6	25.1	19.4
1.25	253.1	**194.2**	**97.3**	**67.2**	59.2	48.6	38.5	32.5	44.0	29.1	25.2	27.8	21.6
1.30	207.6	**364.1**	**116.7**	80.9	70.5	57.2	42.9	34.5	53.5	32.6	27.3	-	-
1.35	193.2	**587.8**	138.3	98.0	81.7	66.1	45.5	36.2	-	-	30.5	-	-
1.40	200.0	344.5	167.2	116.9	94.5	76.3	50.6	-	-	-	-	-	-

Panel B. RMSE of percentage pricing errors of QTSM2.

Money-ness K/F	**Maturity**												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	5.0	4.0	3.8	4.5	4.9	4.3	4.2	4.4	6.0	7.2
0.65	-	-	6.3	4.7	4.1	4.1	4.9	5.0	4.2	4.6	5.7	6.9	8.1
0.70	-	9.0	6.0	5.0	4.5	**5.0**	**5.8**	5.6	5.1	5.4	7.4	8.7	10.1
0.75	-	8.9	**6.1**	**5.8**	**5.7**	**6.5**	7.2	6.7	6.7	6.9	9.5	11.6	13.9
0.80	-	9.1	**6.8**	**7.5**	**8.0**	**9.5**	**9.9**	**8.9**	**9.5**	**9.8**	**12.6**	14.6	16.5
0.85	8.4	**10.2**	**8.4**	**10.3**	**11.3**	**13.2**	**13.1**	**12.0**	**12.7**	**13.2**	**16.1**	**17.8**	19.7
0.90	11.5	**13.7**	**11.1**	**13.5**	**14.6**	**16.6**	**16.9**	**15.7**	**16.4**	**16.7**	**20.0**	**21.3**	**23.0**
0.95	**18.6**	**18.7**	**14.7**	**17.3**	**18.2**	**20.5**	**21.5**	**19.9**	**20.5**	**20.6**	**23.9**	**25.1**	**26.5**
1.00	**30.9**	**25.4**	**19.5**	**21.6**	**22.5**	**25.0**	**26.1**	**24.3**	**24.6**	**24.6**	**27.3**	**28.6**	**30.2**
1.05	**46.6**	**32.0**	**24.9**	**26.5**	**27.6**	**30.2**	**30.8**	**28.6**	**29.0**	28.1	**30.5**	32.4	32.8
1.10	**75.8**	**39.7**	**31.2**	**32.5**	**33.6**	**36.3**	**35.5**	32.0	32.8	30.8	33.0	34.8	33.3
1.15	123.6	49.0	38.4	39.7	40.3	41.5	37.8	33.6	36.6	33.8	36.8	39.0	-
1.20	169.2	**59.5**	**46.8**	**47.5**	**45.5**	44.7	37.8	32.5	42.0	38.8	40.6	43.6	40.1
1.25	102.1	**79.0**	**56.2**	**53.2**	50.9	48.4	39.4	34.0	50.1	44.2	44.6	46.9	43.4
1.30	86.9	**134.6**	**65.5**	59.6	56.8	53.7	43.0	37.1	59.3	51.3	48.0	-	-
1.35	87.2	**210.2**	76.4	67.3	64.8	61.0	47.8	41.2	-	-	52.0	-	-
1.40	91.2	130.6	91.6	77.6	75.5	70.8	53.3	-	-	-	-	-	-

Panel C. RMSE of percentage pricing errors of QTSM1.

Money-ness K/F	Maturity												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	8.1	6.9	6.1	5.6	7.1	6.5	5.5	4.2	4.0	4.7
0.65	-	-	9.9	7.7	6.6	6.0	5.7	7.1	6.1	5.6	4.5	4.5	5.6
0.70	-	13.0	9.6	7.1	6.2	**5.5**	**5.6**	7.0	5.9	5.9	4.6	4.9	6.0
0.75	-	13.1	**9.3**	**7.1**	**6.2**	**5.4**	6.0	7.3	5.9	5.5	4.8	5.4	6.6
0.80	-	12.7	**9.4**	**6.9**	**6.1**	**5.6**	**6.1**	**7.0**	**5.3**	**5.1**	**4.9**	5.8	7.2
0.85	10.2	**13.0**	**9.3**	**6.7**	**6.4**	**6.2**	**6.6**	**7.2**	**5.5**	**5.4**	**5.9**	**6.7**	8.1
0.90	10.7	**14.2**	**9.4**	**7.1**	**7.0**	**7.2**	**7.7**	**7.9**	**6.2**	**6.1**	**7.5**	**7.9**	**9.2**
0.95	**12.1**	**15.2**	**9.7**	**7.9**	**7.9**	**8.3**	**9.3**	**9.0**	**7.4**	**7.3**	**8.8**	**9.3**	**10.2**
1.00	**16.2**	**17.3**	**10.7**	**9.1**	**9.2**	**10.0**	**11.0**	**10.4**	**8.7**	**8.6**	**9.6**	**10.2**	**11.5**
1.05	**21.7**	**18.8**	**12.0**	**10.7**	**11.1**	**12.3**	**12.8**	**12.0**	**10.2**	10.1	**10.3**	12.2	13.1
1.10	**34.3**	**20.9**	**13.8**	**12.8**	**13.5**	**14.8**	**14.5**	13.2	11.7	11.0	11.4	13.9	14.8
1.15	**55.4**	**23.6**	**16.2**	**15.6**	**16.3**	**16.9**	15.3	13.7	13.5	12.2	13.0	16.2	-
1.20	77.1	**27.4**	**19.7**	**19.0**	**18.6**	17.9	14.4	11.7	15.7	14.1	13.8	-	17.8
1.25	53.8	**33.9**	**24.3**	**21.6**	21.3	19.1	14.5	13.2	19.5	16.1	14.3	18.7	18.9
1.30	45.2	**55.2**	**29.1**	24.7	24.2	21.4	16.2	15.5	24.5	19.1	14.9	-	-
1.35	46.0	**86.5**	35.1	28.7	28.2	24.6	18.5	17.8	-	-	-	-	-
1.40	47.2	57.4	43.2	34.0	33.4	28.8	20.9	-	-	-	-	-	-

Panel D. Average percentage pricing errors of QTSM1.

Money-ness K/F	Maturity												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	6.6	5.5	4.4	3.4	5.1	4.7	4.0	2.6	1.5	1.4
0.65	-	-	7.9	5.7	4.8	4.1	3.5	5.2	4.3	4.1	2.1	1.2	1.3
0.70	-	11.2	7.4	5.1	4.5	**3.7**	**3.1**	4.9	3.8	4.0	1.3	0.4	0.1
0.75	-	11.0	**7.2**	**5.1**	**4.2**	**3.1**	2.9	4.6	3.1	3.1	0.5	-0.6	-1.5
0.80	-	10.1	**7.1**	**4.2**	**3.3**	**1.7**	**1.7**	**3.5**	**1.8**	**1.8**	**-0.9**	-1.9	-2.5
0.85	7.5	**9.5**	**6.4**	**2.9**	**1.9**	**0.1**	**0.3**	**2.1**	**0.4**	**0.4**	**-2.4**	**-3.4**	-3.9
0.90	6.3	**8.3**	**5.3**	**1.8**	**0.6**	**-1.2**	**-1.3**	**0.6**	**-1.1**	**-1.0**	**-4.0**	**-4.6**	**-4.6**
0.95	**3.3**	**6.4**	**4.3**	**0.7**	**-0.4**	**-2.4**	**-3.1**	**-1.1**	**-2.7**	**-2.5**	**-5.3**	**-5.8**	**-5.5**
1.00	**-0.7**	**4.6**	**3.0**	**-0.4**	**-1.6**	**-3.7**	**-4.6**	**-2.6**	**-4.0**	**-3.8**	**-6.3**	**-6.6**	**-6.4**
1.05	**-4.7**	**3.2**	**1.7**	**-1.7**	**-3.0**	**-5.3**	**-6.0**	**-3.7**	**-5.2**	-4.6	**-6.9**	-8.0	-7.3
1.10	**-10.9**	**1.4**	**0.3**	**-3.4**	**-4.6**	**-7.1**	**-6.9**	-3.8	-6.0	-4.9	-7.5	-8.6	-6.5
1.15	**-18.8**	**-0.8**	**-1.1**	**-5.5**	**-6.3**	**-8.1**	-6.0	-2.6	-6.5	-5.0	-8.6	-9.6	-
1.20	-23.0	**-3.1**	**-3.1**	**-7.7**	**-7.1**	-7.9	-3.8	-0.5	-7.5	-5.9	-8.9	-	-6.2
1.25	-14.9	**-7.5**	**-5.1**	**-8.3**	-7.3	-7.3	-2.5	1.6	-9.4	-6.4	-9.1	-9.8	-5.7
1.30	-11.6	**-14.8**	**-6.4**	-8.6	-7.0	-7.0	-1.7	3.0	-10.6	-7.7	-8.6	-	-
1.35	-7.1	**-22.3**	-7.5	-8.8	-7.3	-7.4	-1.4	3.8	-	-	-	-	-
1.40	-2.6	-13.7	-9.6	-9.3	-8.5	-8.9	-0.9	-	-	-	-	-	-

Table VI: The Performance of QTSMs in Hedging Interest Rate Caps

This table reports the performance of the three QTSMs in hedging interest rate caps. Hedging effectiveness is measured by variance ratio, the percentage of the variations of an unhedged position that can be reduced through hedging. The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the ones with 10-50% of missing values.

Panel A. Variance ratios of QTSM3.

Money-ness (K/F)	Maturity											
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	-	0.909	-	-	-	0.857	0.574	0.688	0.485	0.290
0.65	-	-	0.885	0.909	0.912	-	0.650	0.530	0.859	0.650	0.645	0.423	0.292
0.70	-	-	0.884	0.906	0.916	**0.866**	**0.647**	0.455	0.811	0.547	0.592	0.347	0.250
0.75	-	0.849	**0.872**	**0.906**	**0.899**	**0.858**	0.663	0.390	0.805	0.623	0.577	0.344	0.230
0.80	-	0.821	**0.880**	**0.897**	**0.871**	**0.868**	**0.649**	**0.474**	**0.781**	**0.597**	**0.581**	0.331	0.216
0.85	0.872	**0.812**	**0.870**	**0.889**	**0.859**	**0.835**	**0.625**	**0.513**	**0.768**	**0.586**	**0.544**	**0.315**	0.182
0.90	0.878	**0.810**	**0.838**	**0.870**	**0.836**	**0.832**	**0.598**	**0.488**	**0.732**	**0.565**	**0.504**	**0.323**	**0.187**
0.95	**0.874**	**0.779**	**0.810**	**0.847**	**0.830**	**0.830**	**0.548**	**0.446**	**0.685**	**0.558**	**0.476**	**0.293**	**0.181**
1.00	**0.855**	**0.662**	**0.781**	**0.809**	**0.801**	**0.802**	**0.521**	**0.416**	**0.649**	**0.502**	**0.447**	**0.250**	**0.162**
1.05	**0.828**	**0.592**	**0.735**	**0.775**	**0.758**	**0.778**	**0.469**	**0.377**	**0.604**	0.470	**0.437**	0.210	0.131
1.10	**0.805**	**0.538**	**0.694**	**0.741**	**0.713**	**0.733**	**0.440**	0.353	0.539	0.451	0.315	0.189	-
1.15	**0.675**	**0.480**	**0.631**	**0.704**	**0.661**	**0.675**	0.392	0.368	0.500	0.357	-	-	-
1.20	0.630	**0.440**	**0.566**	**0.628**	**0.566**	0.602	0.436	0.412	0.359	0.372	-	-	-
1.25	0.591	**0.377**	**0.508**	**0.557**	0.458	0.532	0.470	0.493	0.387	-	-	-	-
1.30	0.597	**0.328**	**0.424**	0.488	0.277	0.446	0.421	-	0.170	-	-	-	-
1.35	0.555	**0.249**	0.320	0.415	0.185	0.305	0.298	-	-	-	-	-	-
1.40	-	0.180	0.220	0.277	0.049	-	-	-	-	-	-	-	-

Panel B. Variance ratios of QTSM2.

Money-ness (K/F)	Maturity											
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	-	0.916	-	-	-	0.853	0.665	0.668	0.487	0.246
0.65	-	-	0.900	0.914	0.916	-	0.660	0.537	0.852	0.691	0.597	0.400	0.236
0.70	-	-	0.899	0.908	0.914	**0.856**	**0.657**	0.478	0.818	0.612	0.557	0.332	0.207
0.75	-	0.866	**0.883**	**0.907**	**0.901**	**0.843**	0.680	0.437	0.802	0.617	0.541	0.311	0.204
0.80	-	0.834	**0.891**	**0.898**	**0.873**	**0.857**	**0.661**	**0.494**	**0.781**	**0.591**	**0.537**	0.296	0.175
0.85	0.895	**0.823**	**0.881**	**0.885**	**0.866**	**0.827**	**0.641**	**0.521**	**0.770**	**0.586**	**0.514**	**0.287**	0.153
0.90	0.893	**0.815**	**0.852**	**0.868**	**0.849**	**0.827**	**0.624**	**0.505**	**0.745**	**0.571**	**0.491**	**0.304**	**0.158**
0.95	**0.888**	**0.782**	**0.831**	**0.852**	**0.844**	**0.828**	**0.587**	**0.472**	**0.714**	**0.576**	**0.480**	**0.291**	**0.169**
1.00	**0.875**	**0.682**	**0.802**	**0.819**	**0.820**	**0.807**	**0.566**	**0.446**	**0.695**	**0.533**	**0.475**	**0.277**	**0.153**
1.05	**0.855**	**0.621**	**0.761**	**0.791**	**0.784**	**0.793**	**0.523**	**0.413**	**0.672**	0.514	**0.492**	0.239	0.125
1.10	**0.844**	**0.571**	**0.722**	**0.763**	**0.742**	**0.755**	**0.510**	0.402	0.625	0.491	0.416	0.189	-
1.15	**0.784**	**0.512**	**0.662**	**0.726**	**0.687**	**0.698**	0.471	0.424	0.610	0.426	-	-	-
1.20	0.741	**0.446**	**0.593**	**0.645**	**0.604**	0.639	0.519	0.480	0.531	0.421	-	-	-
1.25	0.700	**0.361**	**0.524**	**0.585**	0.512	0.590	0.594	0.571	0.516	-	-	-	-
1.30	0.681	**0.274**	**0.410**	0.516	0.343	0.539	0.548	-	0.338	-	-	-	-
1.35	0.628	**0.160**	0.288	0.426	0.295	0.426	0.441	-	-	-	-	-	-
1.40	-	0.063	0.189	0.330	0.178	-	-	-	-	-	-	-	-

Panel C. Variance ratio of QTSM1.

Money-ness (K/F)	Maturity												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	-	0.917	-	-	-	0.862	0.679	0.665	0.494	0.257
0.65	-	-	0.904	0.919	0.920	-	0.671	0.549	0.861	0.704	0.609	0.431	0.255
0.70	-	-	0.903	0.913	0.916	**0.862**	**0.666**	0.487	0.822	0.619	0.565	0.355	0.218
0.75	-	0.865	**0.884**	**0.911**	**0.902**	**0.852**	0.689	0.447	0.807	0.620	0.544	0.326	0.198
0.80	-	0.831	**0.890**	**0.900**	**0.876**	**0.864**	**0.670**	**0.504**	**0.785**	**0.594**	**0.537**	0.305	0.185
0.85	0.894	**0.818**	**0.880**	**0.893**	**0.869**	**0.833**	**0.649**	**0.531**	**0.773**	**0.590**	**0.516**	**0.296**	0.159
0.90	0.890	**0.810**	**0.853**	**0.872**	**0.851**	**0.832**	**0.631**	**0.514**	**0.748**	**0.577**	**0.491**	**0.314**	**0.171**
0.95	**0.888**	**0.779**	**0.832**	**0.855**	**0.847**	**0.833**	**0.596**	**0.481**	**0.716**	**0.578**	**0.481**	**0.303**	**0.182**
1.00	**0.875**	**0.677**	**0.803**	**0.824**	**0.826**	**0.815**	**0.575**	**0.456**	**0.695**	**0.533**	**0.476**	**0.287**	**0.164**
1.05	**0.856**	**0.619**	**0.767**	**0.799**	**0.797**	**0.805**	**0.536**	**0.424**	**0.671**	0.512	**0.492**	0.245	0.138
1.10	**0.851**	**0.575**	**0.737**	**0.779**	**0.763**	**0.773**	**0.523**	0.411	0.623	0.490	0.415	0.204	-
1.15	**0.789**	**0.529**	**0.692**	**0.755**	**0.724**	**0.722**	0.483	0.422	0.611	0.426	-	-	-
1.20	0.756	**0.489**	**0.645**	**0.692**	**0.654**	0.673	0.521	0.470	0.533	0.415	-	-	-
1.25	0.733	**0.438**	**0.603**	**0.645**	0.575	0.634	0.587	0.551	0.514	-	-	-	-
1.30	0.724	**0.393**	**0.534**	0.591	0.444	0.602	0.540	-	0.334	-	-	-	-
1.35	0.691	**0.324**	0.449	0.539	0.408	0.515	0.436	-	-	-	-	-	-
1.40	-	0.260	0.373	0.464	0.319	-	-	-	-	-	-	-	-

Panel D. R^2s of regressions of QTSM1 hedging errors on changes of the three yield factors.

Money-ness (K/F)	Maturity												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	-	0.921	-	-	-	0.912	0.788	0.815	0.658	0.579
0.65	-	-	0.929	0.928	0.921	-	0.676	0.573	0.904	0.804	0.765	0.611	0.471
0.70	-	-	0.927	0.922	0.919	**0.872**	**0.675**	0.507	0.847	0.679	0.666	0.462	0.353
0.75	-	0.914	**0.908**	**0.922**	**0.903**	**0.861**	0.697	0.450	0.835	0.687	0.646	0.429	0.312
0.80	-	0.886	**0.915**	**0.912**	**0.882**	**0.873**	**0.675**	**0.510**	**0.811**	**0.653**	**0.639**	0.417	0.324
0.85	0.951	**0.870**	**0.905**	**0.899**	**0.872**	**0.839**	**0.654**	**0.543**	**0.802**	**0.647**	**0.610**	**0.397**	0.286
0.90	0.942	**0.853**	**0.876**	**0.882**	**0.855**	**0.837**	**0.633**	**0.524**	**0.776**	**0.632**	**0.573**	**0.412**	**0.310**
0.95	**0.935**	**0.825**	**0.860**	**0.864**	**0.853**	**0.837**	**0.597**	**0.488**	**0.738**	**0.623**	**0.554**	**0.395**	**0.306**
1.00	**0.923**	**0.746**	**0.841**	**0.836**	**0.839**	**0.820**	**0.578**	**0.462**	**0.709**	**0.566**	**0.541**	**0.361**	**0.269**
1.05	**0.906**	**0.694**	**0.816**	**0.816**	**0.819**	**0.814**	**0.539**	**0.428**	**0.679**	0.537	**0.545**	0.338	0.210
1.10	**0.895**	**0.659**	**0.799**	**0.804**	**0.794**	**0.786**	**0.530**	0.421	0.630	0.508	0.480	0.278	-
1.15	**0.857**	**0.624**	**0.770**	**0.789**	**0.773**	**0.742**	0.491	0.439	0.623	0.434	-	-	-
1.20	0.848	**0.611**	**0.741**	**0.746**	**0.729**	0.701	0.530	0.486	0.572	0.427	-	-	-
1.25	0.824	**0.577**	**0.712**	**0.716**	0.673	0.668	0.612	0.598	0.541	-	-	-	-
1.30	0.796	**0.560**	**0.680**	0.687	0.626	0.679	0.559	-	0.378	-	-	-	-
1.35	0.777	**0.511**	0.634	0.662	0.603	0.636	0.464	-	-	-	-	-	-
1.40	-	0.455	0.582	0.638	0.573	-	-	-	-	-	-	-	-

Table VII. Principle Component Analysis of Cap Hedging Errors

This table reports the percentage of variance of cap hedging errors with different moneyness that can be explained by the principle components.

Moneyness (K/F)	Principle Component				
	1	2	3	4	5
0.80	60.6%	11.9%	9.4%	4.6%	4.5%
0.85	58.3%	11.8%	9.6%	6.2%	4.4%
0.90	57.6%	11.9%	10.0%	5.5%	5.2%
0.95	57.5%	10.7%	9.8%	6.4%	4.7%
1.00	56.0%	12.1%	9.8%	7.0%	5.3%
1.05	40.0%	25.0%	21.1%	5.9%	3.8%
1.10	49.0%	31.7%	8.6%	5.2%	3.3%
1.15	67.1%	14.9%	8.5%	6.5%	1.8%
Overall	51.5%	12.1%	9.7%	6.6%	5.6%

Table VIII. Hedging Interest Rate Cap Straddles

Panel A. The performance of QTSM1 in hedging ATM cap straddles measured by variance ratio.

	Maturity												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
QTSM1	0.29	0.03	0.19	0.13	0.14	0.11	0.04	0.02	0.06	0.02	0.04	0.01	0.00

Panel B. Percentage of variance of ATM straddles hedging errors explained by the principle components.

Principle Component					
1	2	3	4	5	6
59.3%	12.4%	9.4%	6.7%	4.0%	2.8%

Panel C. Correlation matrix of straddles hedging errors across maturity.

Maturity	Maturity												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
1.5	1.00	-	-	-	-	-	-	-	-	-	-	-	-
2	0.38	1.00	-	-	-	-	-	-	-	-	-	-	-
2.5	0.28	0.66	1.00	-	-	-	-	-	-	-	-	-	-
3	0.03	0.33	0.73	1.00	-	-	-	-	-	-	-	-	-
3.5	0.27	0.52	0.63	0.59	1.00	-	-	-	-	-	-	-	-
4	0.13	0.44	0.37	0.37	0.77	1.00	-	-	-	-	-	-	-
4.5	0.20	0.21	-0.04	-0.08	-0.05	-0.06	1.00	-	-	-	-	-	-
5	0.10	0.11	-0.12	-0.13	-0.16	-0.15	0.96	1.00	-	-	-	-	-
6	0.21	0.16	0.19	0.13	0.25	0.05	0.27	0.23	1.00	-	-	-	-
7	0.30	0.34	0.33	0.35	0.46	0.38	0.28	0.22	0.08	1.00	-	-	-
8	0.10	0.12	0.30	0.30	0.25	0.11	0.36	0.34	0.29	0.29	1.00	-	-
9	0.14	0.11	0.25	0.29	0.26	0.12	0.39	0.37	0.32	0.38	0.83	1.00	-
10	0.08	-0.01	0.17	0.14	0.12	0.01	0.32	0.35	0.26	0.28	0.77	0.86	1.00

Table IX. Straddle Hedging Errors and Cap Implied Volatilities

This table reports the relation between straddle hedging errors and ATM Cap implied volatilities.
Panel A. Percentage of variance of ATM Cap implied volatilities explained by the principle components.

Principle Component					
1	2	3	4	5	6
85.73%	7.91%	1.85%	1.54%	0.72%	0.67%

Panel B. R^2s of the regressions of ATM straddles hedging errors on changes of the three yield factors (row one); changes of the three yield factors and the first four principle components of the ATM Cap implied volatilities (row two); and changes of the three yield factors and maturity-wise ATM Cap implied volatility (row three).

Maturity												
1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.10	0.06	0.02	0.01	0.01	0.04	0.00	0.00	0.01	0.01	0.00	0.01	0.04
0.29	0.49	0.54	0.43	0.63	0.47	0.95	0.96	0.21	0.70	0.68	0.89	0.96
0.68	0.70	0.81	0.87	0.85	0.90	0.95	0.98	0.95	0.98	0.97	0.98	0.99

Table X. Straddle Hedging Error As a Proxy of Systematic USV

This table reports the contribution of USV proxied by the first few principle components of straddle hedging errors in explaining the hedging errors of caps across moneyness and maturity. It reports the R^2s of regressions of hedging errors of caps across moneyness and maturity on changes of the three yield factors and the first five principle components of straddle hedging errors. .The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the ones with 10-50% of missing values.

Money-ness (K/F)	Maturity												
	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60	-	-	-	-	0.945	-	-	-	0.948	0.880	0.884	0.786	0.880
0.65	-	-	0.938	0.949	0.954	-	0.947	0.952	0.960	0.928	0.871	0.807	0.838
0.70	-	-	0.934	0.944	0.943	**0.911**	**0.934**	0.936	0.940	0.885	0.839	0.791	0.776
0.75	-	0.934	**0.926**	**0.945**	**0.943**	**0.910**	0.936	0.919	0.950	0.899	0.862	0.814	0.791
0.80	-	0.917	**0.934**	**0.938**	**0.935**	**0.909**	**0.950**	**0.946**	**0.951**	**0.898**	**0.862**	0.821	0.840
0.85	0.958	**0.909**	**0.927**	**0.928**	**0.928**	**0.889**	**0.956**	**0.959**	**0.959**	**0.906**	**0.861**	**0.818**	0.843
0.90	0.949	**0.900**	**0.908**	**0.922**	**0.924**	**0.896**	**0.961**	**0.969**	**0.969**	**0.920**	**0.871**	**0.856**	**0.871**
0.95	**0.943**	**0.886**	**0.905**	**0.918**	**0.936**	**0.906**	**0.966**	**0.976**	**0.980**	**0.967**	**0.889**	**0.882**	**0.893**
1.00	**0.932**	**0.859**	**0.905**	**0.909**	**0.939**	**0.902**	**0.988**	**0.989**	**0.984**	**0.973**	**0.910**	**0.894**	**0.907**
1.05	**0.919**	**0.821**	**0.897**	**0.902**	**0.937**	**0.897**	**0.986**	**0.985**	**0.980**	0.969	**0.908**	0.917	0.885
1.10	**0.913**	**0.793**	**0.890**	**0.894**	**0.928**	**0.880**	**0.979**	0.976	0.974	0.967	0.913	0.921	-
1.15	**0.879**	**0.763**	**0.871**	**0.880**	**0.915**	**0.860**	0.970	0.968	0.966	0.963	-	-	-
1.20	0.881	**0.749**	**0.844**	**0.848**	**0.894**	0.846	0.966	0.963	0.954	0.957	-	-	-
1.25	0.870	**0.742**	**0.818**	**0.817**	0.870	0.819	0.945	0.943	0.941	-	-	-	-
1.30	0.861	**0.702**	**0.802**	0.808	0.836	0.802	0.920	-	0.908	-	-	-	-
1.35	0.855	**0.661**	0.764	0.774	0.801	0.758	0.884	-	-	-	-	-	-
1.40	-	0.640	0.725	0.743	0.761	0.536	-	-	-	-	-	-	-

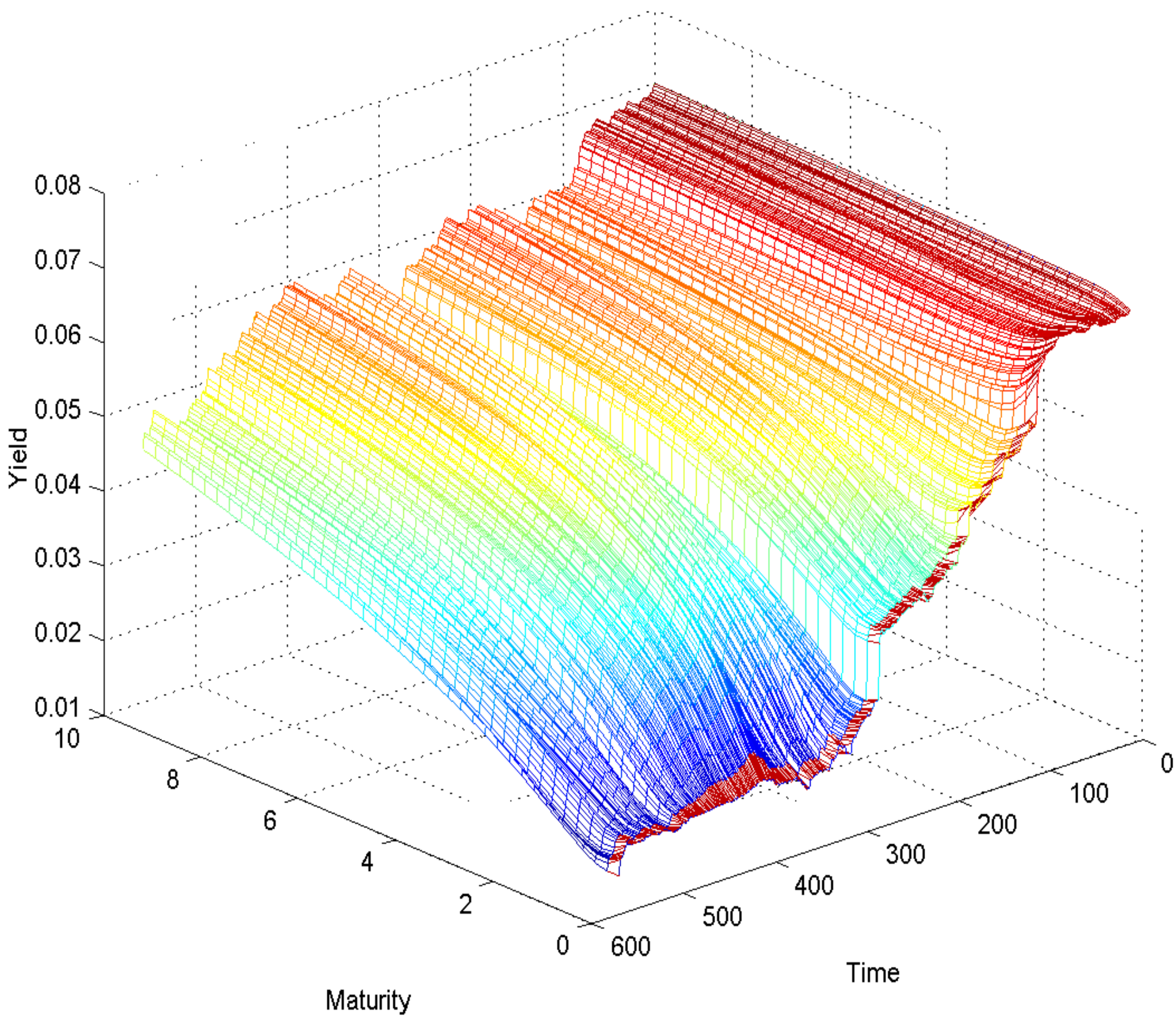

Figure 1: The yield-to-maturity of the LIBOR bonds

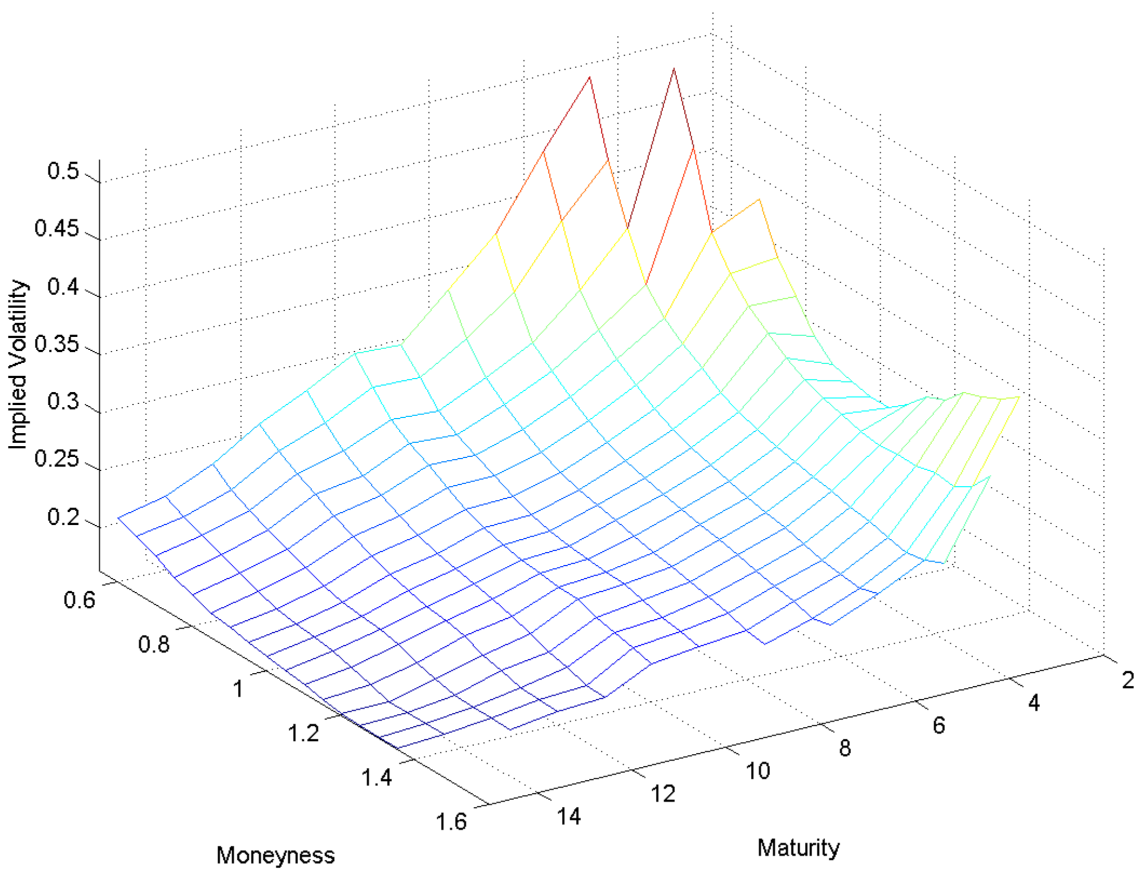

Figure 2: The average Black's implied volatility of the interest caps.

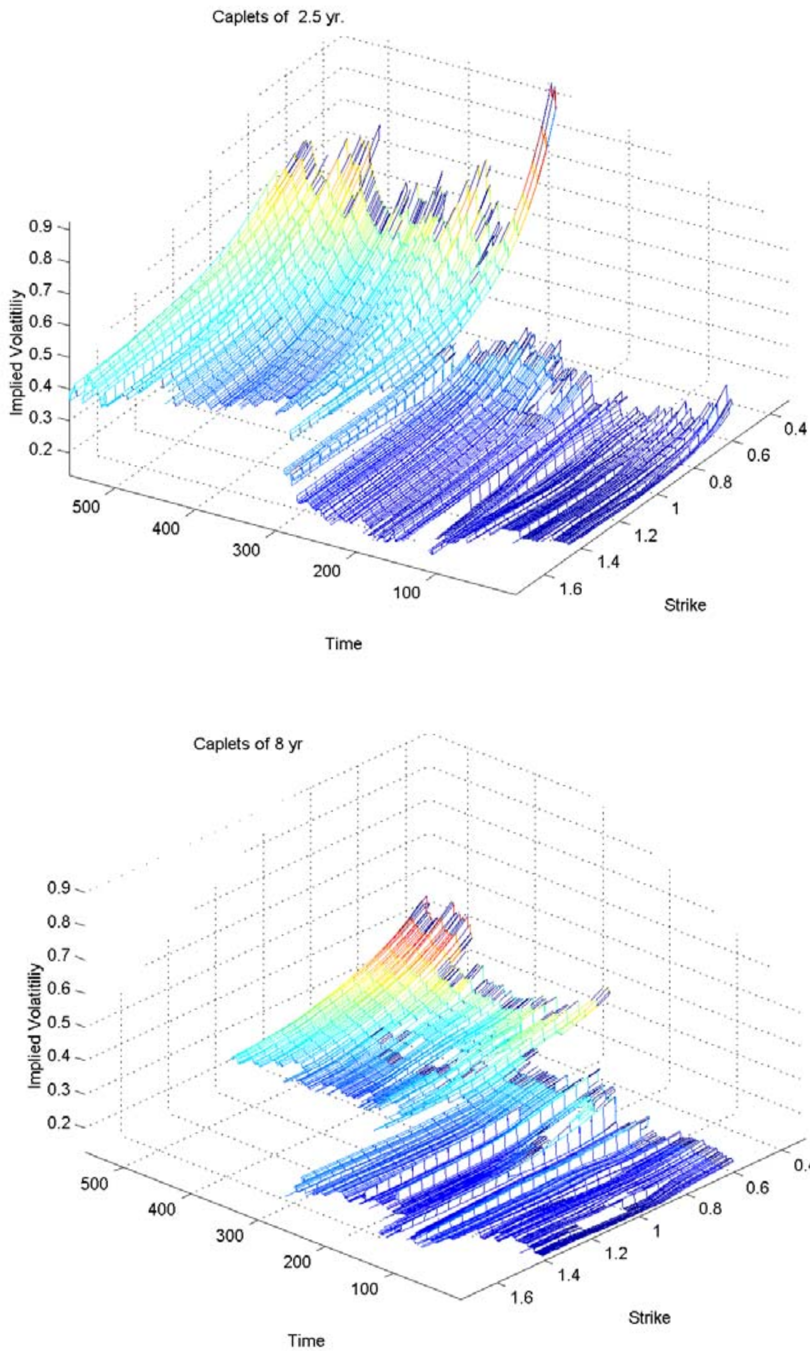

Figure 3: The implied volatility of the interest caps across the sample period.

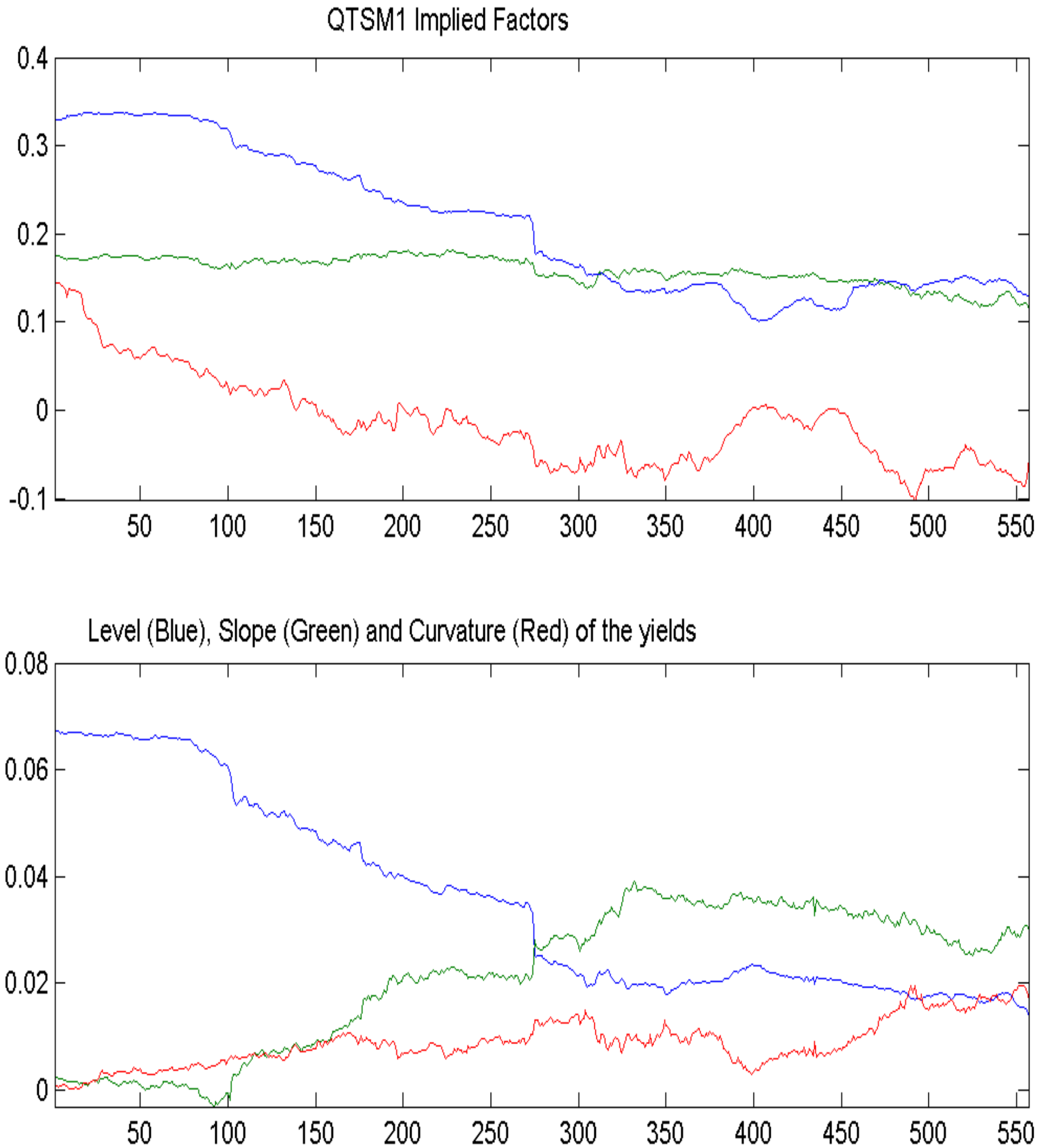

Figure 4: The QTSM1 implied factors and the three yield level factors

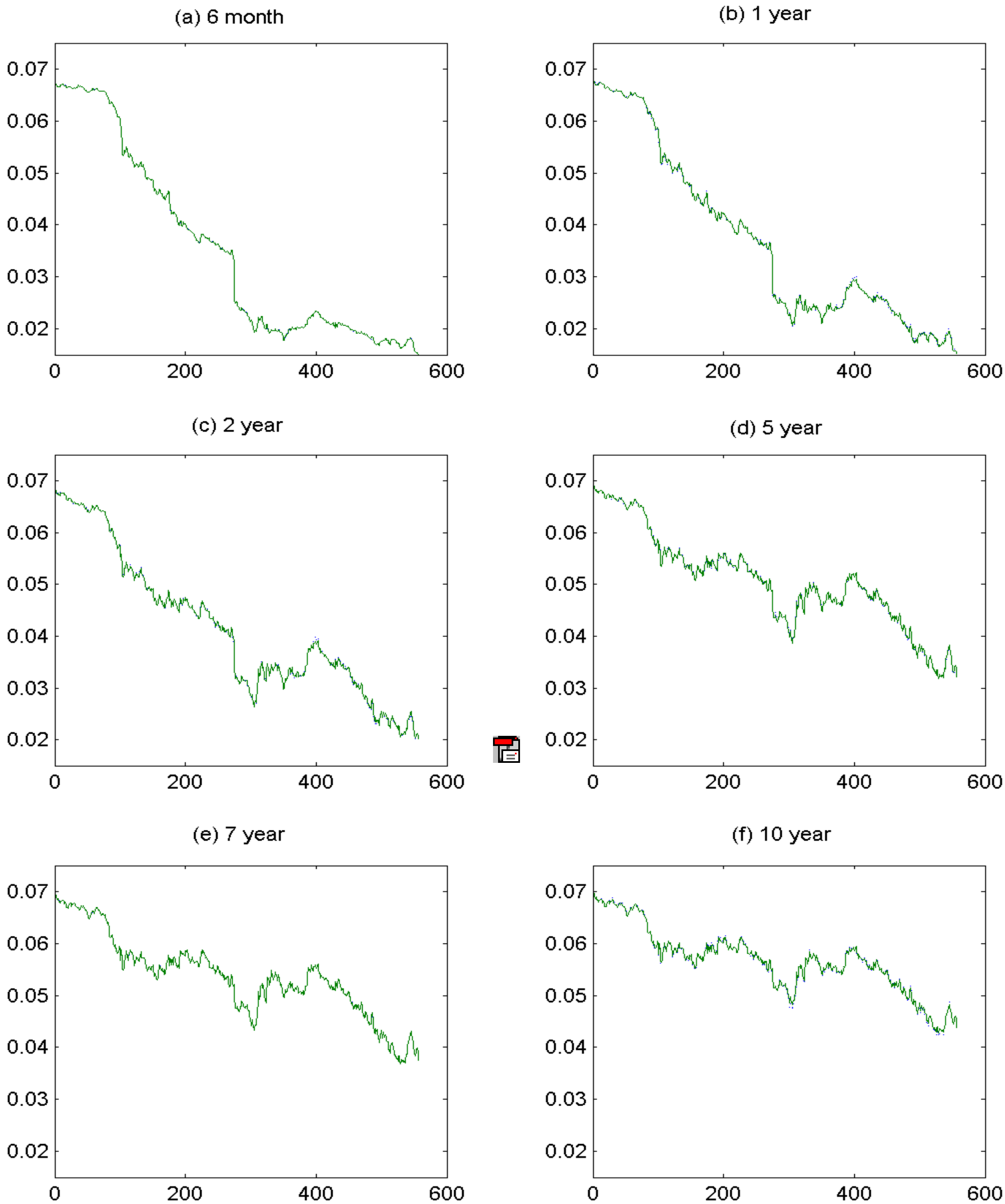

Figure 5: The observed yields (dot) and the QTSM1 projected yields (solid).