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# Modeling the term structure of interest rates: A new approach

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Abstract

The term structure of interest rates is modeled as a random field with conditional volatility. Random field models allow consistency with the current shape of the term structure without the need for recalibration. However, most such models are Gaussian, with no conditional volatility. State-dependent volatility is introduced while a key property of Gaussian random field models is retained. Each forward rate is part of a low-dimensional diffusion process, simplifying estimation and derivatives pricing. The modeling approach also implies that, in general, the set of zero coupon bonds does not complete the market, and term structure derivatives cannot always be priced by arbitrage.

*JEL classification:* G12, G13

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## 1 Introduction

Traditional term structure models (a few examples from a vast literature are Vasicek, 1977; Cox, Ingersoll, and Ross, 1985; Jamshidian, 1989; Constantinides, 1992; Jamshidian, 1995; Chen, 1996; Balduzzi, Das, Foresi, and Sundaram, 1996; and Ahn and Gao, 1999) typically specify the instantaneous interest rate as a function of a small set of state variables, which follow a time-homogeneous Markov process (usually a diffusion). Such models have many desirable analytical properties but are generally not consistent with the observed term structure of bond prices. So-called arbitrage-free models (examples include Ho and Lee, 1986; and Black, Derman, and Toy, 1990) match the observed bond prices by introducing explicitly time-varying parameters. Heath, Jarrow, and Morton (1992) match observed bond prices by having the state variables follow a non-Markovian process. However, such approaches are not consistent with the time series behavior of the term structure, in the sense that no possible series of innovations is consistent with several subsequent observations of the term structure. Consequently, these models require frequent recalibration, in which the allegedly constant parameters of the model are changed to suit current market conditions. The need for recalibration represents a hidden source of risk that the model ignores.

So-called random field or stochastic string models, pioneered by Kennedy (1994), eliminate this problem by letting each bond have some variation that is independent of the variation of any other bond. Because a continuum of bond prices exists (indexed by maturity), such models are driven by infinitely many sources of innovation. In contrast to arbitrage-free models, which have a few state variables and many parameters, random field models have a continuum of state variables (corresponding to either the yields of zero coupon bonds or instantaneous forward rates of all maturities), but typically only a few parameters. The dynamics of the state variables are not specified individually, but instead all at once by smooth functions of maturity. Kennedy (1994) considers one particular model with only three parameters to be estimated. Such models are trivially able to match the observed yield curve and do so without explicitly time-varying parameters and without recalibration. Although the term structure is driven by an infinite set of underlying factors, each forward rate follows a scalar diffusion. Consequently, common tasks such as derivatives pricing are tractable. The infinite-dimensional nature of random field models avoids some unpleasant implications of other models. For example, as Goldstein (2000) points out, three-factor term structure models imply that a 30-year bond can be perfectly hedged with one- two-, and three-month bills. Random field models avoid this problem.

The Kennedy models, however, do not have conditional volatility. The variances of the innovations to forward rates (as well as the correlations between innovations to different forward rates) are constant functions of maturity, whereas much empirical evidence suggests that volatility is state-dependent. Chan, Karolyi, Longstaff, and Sanders (1992), for example, offer evidence on the volatility of the instantaneous interest rate. Goldstein (2000) develops a general class of random field models with conditional volatility and derives restrictions needed to make such models free of arbitrage. However, although such models are generally specified as solutions to a set of stochastic differential equations, proving existence and uniqueness of such a solution is often difficult or impossible. Furthermore, most current random field models with conditional volatility do not have a key property of the Kennedy family of models: Individual forward rates are no longer part of a low-dimensional diffusion process, but instead follow a complex, infinite-dimensional process.

I develop a class of random field models in which the volatility of bond yields and forward rates depends on a set of latent variables, not the level or shape of the yield or forward curve. The latent variables themselves follow a diffusion process. With this approach, each forward rate is part of a low-dimensional diffusion, even if an extremely large or infinite number of factors drive term structure evolution as a whole. This property greatly simplifies tasks such as estimation and derivatives pricing. Furthermore, it is relatively easy to characterize necessary and sufficient criteria for existence and uniqueness of a forward rate process and for absence of arbitrage.

An interesting property of my class of models is that the set of zero coupon bonds does not necessarily complete the market. Consequently, derivative securities may have volatility risk that can be hedged only with other derivative securities. This property, although common among stochastic volatility models for equities or other securities, is rare in term structure models.

The rest of this paper is organized as follows. In Section 2, I develop a new class of random field models, called latent variable (LV) term structure models, which introduce conditional volatility (as well as conditional risk premia) in an analytically tractable way. Section 3 discusses conditions necessary for existence and uniqueness of a forward rate process and for absence of arbitrage. Once a latent variable process is chosen, such conditions are essentially technical regularity criteria. Section 4 specifies a subclass of particularly analytically tractable models, called affine latent variable (ALV) models. In Section 5, I characterize the types of behavior LV and ALV models can exhibit, using several specific models as examples, and discuss the relationship between ALV models and affine yield models. In Section 6, on derivatives pricing, I derive a simple, low-dimensional differential equation satisfied by European-style term structure derivatives, even when the number of factors driving the term structure as a whole is infinite. Section 7 concludes the paper.

## 2 Latent variable term structure models

I now develop a new class of random field models, introducing conditional volatility in a way that retains a key property of the Kennedy (1994) and Kennedy (1997) family of models; i.e., each forward rate is part of a finite-dimensional diffusion process. This property simplifies many term structure applications, such as estimation and derivatives pricing, and makes it relatively straightforward to construct specific versions of the model. This goal is achieved through a latent variable approach as follows.

As with all random field term structure models, my modeling approach uses a continuum of state variables to represent forward rates (or, alternately, zero coupon bond prices or yields) of all maturities and specifies the instantaneous covariance between any pair of forward rates (or yields or bond prices) directly, not deriving them as consequences of interest rate dynamics. A covariance surface is constructed by integrating a function of maturity over a two-parameter Brownian sheet. A Brownian sheet $W_{s,t},\ s,t \geq 0$ is a two-dimensional surface of random variables, continuous in both $s$ and $t$ almost surely, with the properties

$$E\left[W_{s,t}\right] = 0 \text{ and} \tag{1}$$

$$Cov\left[W_{s_1,t_1}, W_{s_2,t_2}\right] = \left(s_1 \wedge s_2\right) \cdot \left(t_1 \wedge t_2\right). \tag{2}$$

The existence of a two-parameter Brownian sheet is shown in Čentsov (1956) and is presented as an exercise in Karatzas and Shreve (1991). Rogers and Williams (1994) also discuss a class of Gaussian processes that includes a two-parameter Brownian sheet, and Nualart (1995) contains a general treatment of random fields. For $s \neq 0$, each $W_{s,t}$ can be viewed as a scaled Brownian motion:

$$W_{s,t} = \sqrt{s} B_{s,t}, \tag{3}$$

where $B_{s,t}$ is a standard Brownian motion for each fixed value of $s$. Furthermore, the processes $W_{s_1,t}$ and $W_{s_2,t}$ have instantaneous covariance of $(s_1 \wedge s_2)$. Stochastic integration may be extended to the two-dimensional case, taking $W_{s,t}$ as an integrator. See, for example, Ivanov and Leonenko (1989). The Brownian sheet provides enough independent sources of innovation to allow construction of term structure models in which each forward rate has its own source of innovation. Instantaneous forward rates are denoted by $f_t(T)$, where $t$ is the current time and $T$ is the time of maturity. I take as state variables forward rates with fixed time of maturity (i.e., fixed $T$) instead of fixed time until maturity (i.e., fixed $T - t$).

Kennedy (1997) discusses a model in which forward rates have the covariance structure

$$Cov\left[df_t\left(T_1\right), df_t\left(T_2\right)\right] = \sigma^2 e^{-\kappa(T_1-t)-\kappa(T_2-t)-\rho|T_1-T_2|} dt. \tag{4}$$

A forward rate process satisfying this condition can be constructed in the state space of a Brownian sheet by integrating over the sheet in the $s$-dimension

$$df_t(T) = \mu(T-t)\,dt + \int_{s=T-t}^{+\infty} \sigma\sqrt{2\rho}e^{-\kappa(T-t)-\rho[s-(T-t)]}dW_{s,t}. \tag{5}$$

(Restrictions on the drift are needed to prevent arbitrage, but are not discussed for purposes of this example.) The resulting surface $f_t(T)$ is continuous in both $t$ and $T$, almost surely, and each forward rate has its own source of innovation; i.e., there exists no nonzero linear combination of forward rates with zero variance. Furthermore, each forward rate follows a scalar diffusion process. The diffusion property of forward rates has considerable advantages. For some model specifications of this type, Kennedy (1994) is able to derive closed-form prices of options on zero coupon bonds.

However, a model such as that specified in Eq. (5) has no conditional volatility, because covariances are deterministic functions of the maturities of the two forward rates. One could extend this approach to incorporate state-dependent volatility by having the covariances be functions of certain benchmark yields, such as the instantaneous interest rate. However, with this approach, the future evolution of each forward rate depends on an entire continuum of state variables, making traditional approaches to solving term structure problems difficult or impossible. Often, it is not clear that a forward rate process with a particular covariance surface even exists. [1] By contrast, I make the covariance surface a function of a finite set of state variables that follow a joint diffusion process. I refer to these variables as latent variables, because, in general, their values cannot be inferred from a cross-section of bond prices or forward rates. I also require that the risk premium of each forward rate depend only on the same set of latent variables. The latent variables therefore play a role analogous to that of stochastic volatility (and stochastic risk premium) in equity models. Each individual forward rate is then part of a finite-dimensional diffusion, even when the specified covariance surface implies that the term structure as a whole is driven by infinitely many factors. Because each state variable of either type (the latent variables or the instantaneous forward rates) is part of a finite-dimensional diffusion, many traditional techniques for estimation, derivatives pricing, etc., can be applied even when the term structure as a whole evolves according to an infinite-dimensional process.

[1] Papers taking this approach generally assume existence and prove absence of arbitrage conditional on this assumption. For example, Goldstein (2000) specifies the forward rate process as a solution to a stochastic differential equation and derives conditions for absence of arbirage. However, restrictions on the class of models needed to ensure existence of a forward rate process are not derived. Longstaff, Santa-Clara, and Schwartz (1999) use a random field model for the term structure of swap rates instead of forward rates and yields.

The two sets of state variables in a latent variable term structure model with $N$ variables (LV or LV-$N$ for short, with $N = 0$ permitted) are thus the set of instantaneous forward rates and the latent variables. An LV-$N$ model is defined as any model with the following properties.

**Definition 1** *A latent variable term structure model is any model in which an $N \times 1$ vector of latent variables $X_t$ and a continuum of forward rates $f_t(T)$, $T \geq t$ satisfy the following conditions:*

*LV-1 The initial term structure of forward rates is integrable for all $T \geq 0$:*

$$y_0(T) = \int_0^T f_0(u)\, du. \tag{6}$$

*LV-2 The latent variables $X_t$ follow a joint diffusion process:*

$$dX_t = \mu_X(X_t)\, dt + \sigma_X(X_t)\, dZ_t, \tag{7}$$

*where $Z_t$ is an $N$-dimensional standard Brownian motion, $\mu_X$ is an $N \times 1$ vector-valued function, and $\sigma_X$ is an $N \times N$ matrix-valued function.*

*LV-3 Each forward rate process solves a stochastic differential equation for all $t \leq T$:*

$$\begin{aligned} df_t(T) = {} & \mu_f(X_t, T-t)\, dt + \sigma_{fZ}^T(X_t, T-t)\, dZ_t \\ & + \textstyle\int_{s=0}^{+\infty} \sigma_{fW}(X_t, T-t, s)\, dW_{s,t}, \end{aligned} \tag{8}$$

*where $\mu_f$ and $\sigma_{fW}$ are scalar-valued functions, $\sigma_{fZ}$ is an $N \times 1$ vector-valued function, and $W_{s,t}$ is a two-parameter Brownian sheet, independent of $Z_t$. The $\sigma_{fZ}$ and $\sigma_{fW}$ functions must be $L^2$-integrable for all values of $t \geq 0$ and $T \geq t$, almost surely:*

$$\int_{\substack{u\in[0,t]\\ v\in[u,T]\\ w\in[u,T]}} \left|\sigma_{fZ}(X_u, v-u)\, \sigma_{fZ}^T(X_u, w-u)\right| d(u \times v \times w) < \infty \text{ and} \tag{9}$$

$$\int_{\substack{s\in[0,+\infty)\\ u\in[0,t]\\ v\in[u,T]\\ w\in[u,T]}} \left| \begin{array}{l} \sigma_{fW}(X_u, v-u, s) \\ \cdot\sigma_{fW}(X_u, w-u, s) \end{array} \right| d(s \times u \times v \times w) < \infty. \tag{10}$$

*LV-4 The $\mu_f$ function has the following form:*

$$\begin{aligned} \mu_f(X_t, T-t) = {} & [\sigma_{fZ}^T(X_t, T-t) \textstyle\int_t^T \sigma_{fZ}(X_t, u-t)\, du \\ & + \textstyle\int_{\substack{s\in[0,+\infty)\\ u\in[t,T]}} [\sigma_{fW}(X_t, T-t, s) \\ & \qquad \cdot\sigma_{fW}(X_t, u-t, s)]\, d(s \times u) \\ & + \sigma_{fZ}^T(X_t, T-t)\, \lambda_Z(X_t) \\ & + \textstyle\int_0^{+\infty} \sigma_{fW}(X_t, T-t, s)\, \lambda_W(X_t, s)\, ds], \end{aligned} \tag{11}$$

*where $\lambda_Z$ is an $N \times 1$ vector-valued function and $\lambda_W$ is a scalar-valued function. These two functions must be $L^2$-integrable for all $t \geq 0$, almost surely:*

$$\int_0^t \left|\lambda_Z\left(X_u\right)\lambda_Z^T\left(X_u\right)\right| du < \infty \text{ and} \tag{12}$$

$$\int_{\substack{s\in[0,+\infty)\\ u\in[0,t]}} \left|\lambda_W\left(X_u, s\right)\right|^2 d\left(s \times u\right) < \infty. \tag{13}$$

*The integrability restrictions on $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$ ensure that the $\mu_f$ function is $L^1$-integrable, almost surely.*

*LV-5 For all $t \geq 0$, the functions $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$ satisfy:*

$$\begin{aligned}
E[\exp\{ & -\tfrac{1}{2}\textstyle\int_0^t \left[\lambda_Z\left(X_v\right) + \int_v^T \sigma_{fZ}\left(X_v, u-v\right) du\right]^T \\
& \qquad \cdot \left[\lambda_Z\left(X_v\right) + \textstyle\int_v^T \sigma_{fZ}\left(X_v, u-v\right) du\right] dv \\
& -\tfrac{1}{2}\textstyle\int_{\substack{s\in[0,+\infty)\\ v\in[0,t]}} [\lambda_W\left(X_v, s\right) \\
& \qquad + \textstyle\int_v^T \sigma_{fW}\left(X_v, u-v, s\right) du]^2 d\left(s \times v\right) \\
& - \textstyle\int_0^t \left[\lambda_Z\left(X_v\right) + \int_v^T \sigma_{fZ}\left(X_v, u-v\right) du\right]^T dZ_v \\
& - \textstyle\int_{\substack{s\in[0,+\infty)\\ v\in[0,t]}} [\lambda_W\left(X_v, s\right) \\
& \qquad + \textstyle\int_v^T \sigma_{fW}\left(X_v, u-v, s\right) du] dW_{s,v}\}] = 1 \text{ and}
\end{aligned} \tag{14}$$

$$\begin{aligned}
E\{\exp[ & -\tfrac{1}{2}\textstyle\int_0^t \lambda_Z^T\left(X_u\right)\lambda_Z\left(X_u\right) du \\
& -\tfrac{1}{2}\textstyle\int_{\substack{s\in[0,\infty)\\ u\in[0,t]}} \left[\lambda_W\left(X_u, s\right)\right]^2 d\left(s \times u\right) \\
& - \textstyle\int_0^t \lambda_Z^T\left(X_u\right) dZ_u - \int_{\substack{s\in[0,\infty)\\ u\in[0,t]}} \lambda_W\left(X_u, s\right) dW_{s,u}]\} = 1.
\end{aligned} \tag{15}$$

LV-1 is essentially a technical requirement that guarantees the existence of yields and zero coupon bond prices at time $t = 0$. Requirement LV-2 defines the latent variable process. The latent variables determine the level of volatility and risk premium associated with each forward rate. LV-3 ensures that forward rates have an important property: Any finite set of forward rates, together with the latent variables, follows a diffusion process, because forward rate dynamics depend only on the latent variables. LV-4 and LV-5 guarantee the absence of arbitrage. As discussed in Section 3, there exists an equivalent martingale measure, under which the drift of each forward rate is given by the first two terms on the right-hand side of Eq. (11). The remaining two terms are the risk premia associated with the sources of innovation $Z_t$ and $W_{s,t}$. Eq. (15) ensures that the equivalent measure is well defined, and Eq. (14) ensures that bond prices are martingales under that measure.

An LV model is therefore characterized entirely by the initial values of the state variables $X_0$ and $f_0\left(T\right)$ and the six functions $\mu_X$, $\sigma_X$, $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$. In Section 3, I show that, once an integrable initial forward curve and a unique latent variable process are specified by appropriate choice of $f_0(T)$, $X_0$, $\mu_X$, and $\sigma_X$, the restrictions in LV-3 through LV-5 guarantee the existence of a unique forward rate process and that this process offers investors no arbitrage opportunities. For now, I take these implications as given and consider some properties of forward rates, yields, and bond prices under an LV model specification.

The instantaneous variances and covariances of all state variables are found from Eqs. (7) and (8). The auxiliary function $c_{WW}$ is defined as

$$\begin{aligned} c_{WW}(X_t, T_1 - t, T_2 - t) = \textstyle\int_0^{+\infty} & [\sigma_{fW}(X_t, T_1 - t, s) \\ & \cdot \sigma_{fW}(X_t, T_2 - t, s)]ds. \end{aligned} \tag{16}$$

The covariances of the state variables are then

$$Var[dX_t] = c_{XX}(X_t)\,dt = \sigma_X(X_t)\,\sigma_X^T(X_t)\,dt, \tag{17}$$

$$\begin{aligned} Cov[dX_t, df_t(T)] &= c_{Xf}(X_t, T - t)\,dt \\ &= \sigma_X(X_t)\,\sigma_{fZ}(X_t, T - t)\,dt, \text{ and} \end{aligned} \tag{18}$$

$$\begin{aligned} Cov[df_t(T_1), df_t(T_2)] &= c_{ff}(X_t, T_1 - t, T_2 - t)\,dt \\ &= [\sigma_{fZ}^T(X_t, T_1 - t)\,\sigma_{fZ}(X_t, T_2 - t) \\ &\quad + c_{WW}(X_t, T_1 - t, T_2 - t)]dt. \end{aligned} \tag{19}$$

As shown, the variance of any state variable or the covariance between any two state variables depends only on time to maturity (for forward rates) and the latent variables $X_t$. Furthermore, as seen from Eqs. (7), (8) and (11), the conditional mean of any state variable depends only on the latent variables as well. The latent variables themselves follow a joint diffusion process. As seen from the specification of the dynamics of the forward rates, each forward rate, or finite set of forward rates, together with the latent variables also follows a joint diffusion process. Section 3 shows how this property holds under an equivalent martingale measure as well.

Any finite set of forward rates therefore follows a process that is driven by finitely many sources of innovation. The term structure of forward rates as a whole may be driven by infinitely many innovations. From Eq. (8), note that the same innovations that drive the latent variables can also affect forward rates directly (i.e., shocks to the volatility of forward rates can be correlated to the shocks to forward rates themselves). However, each forward rate also has its own source of innovation, through the two-parameter Brownian sheet $W_{s,t}$. For some specifications of $\sigma_{fW}$, the set of all forward rates will have finite rank. In this case, there exists an integer $K$ such that, for any $T_1, ..., T_M$ with $M > K$, the instantaneous covariance matrix of $f_t(T_1), ..., f_t(T_M)$ is singular. For a trivial example, take $\sigma_{fW}(X_t, T-t, s) = 0$. For this specification, the surface $f_t(T)$ has rank at most $N$ (because forward rates can also depend on the $N$ sources of innovations to the latent variables). For a nontrivial example, take

$$\sigma_{fW}(X_t, T-t, s) = \sum_{i=1}^{M} g_i(s)\, h_i(X_t, T_i - t) \tag{20}$$

for some functions $h_i(X_t, T-t)$ and $g_i(s)$. In this specification, the set of all forward rates $f_t(T)$ has rank at most $M+N$. However, for some specifications of $\sigma_{fW}$, the instantaneous covariance matrix of any $M$ forward rates is not singular, no matter how large $M$ is. See the example in Eq. (5) from Kennedy (1997).

The diffusion property of instantaneous forward rates does not hold, if, for example, forward rate dynamics are defined instead as

$$df_t(T) = \mu(r_t, T-t)\, dt + \int_{s=0}^{+\infty} \sigma(r_t, T-t, s)\, dW_{s,t}, \tag{21}$$

where the instantaneous interest rate is defined as

$$r_t = f_t(t). \tag{22}$$

Restrictions on the drift in Eq. (21) are needed to ensure absence of arbitrage, but they are not specified for purposes of this example. The instantaneous interest rate takes on the identity of a different state variable (i.e., a different forward rate) at each instant. Consequently, the process followed by the interest rate is complex and potentially infinite-dimensional. Given that the volatility of each forward rate depends on the interest rate, the forward rates also follow a complex and possibly infinite-dimensional process. Apart from complicating derivative pricing (the approach presented in Section 6 cannot be applied to this type of model), it is difficult to determine whether a process obeying Eq. (21) even exists. A model of this type is specified in the derivative pricing section of Goldstein (2000), who derives sufficient restrictions on $\mu$ to guarantee absence of arbitrage. However, it remains to derive the conditions on $\sigma$ needed to guarantee existence of a unique forward rate process.

Under the LV-N model specification, the diffusion property of instantaneous forward rates (together with the latent variables) is shared by noninstantaneous forward rates, defined as

$$F_t(T_1, T_2) = \int_{T_1}^{T_2} f_t(u)\, du. \tag{23}$$

The process followed by such a forward rate can be found by integrating Eq. (8) over maturity. The integrability assumptions on $\lambda_f$, $\lambda_W$, $\sigma_{fZ}$, and $c_{WW}$

allow a change of the order of integration resutling in a stochastic differential equation for forward rates. Some needed auxiliary functions are defined as

$$\sigma_{FZ}\left(X_t, T-t\right) = \int_t^T \sigma_{fZ}\left(X_t, u-t\right) du, \tag{24}$$

$$\sigma_{FW}\left(X_t, T-t, s\right) = \int_t^T \sigma_{fW}\left(X_t, u-t, s\right) du, \text{ and} \tag{25}$$

$$\begin{aligned} c_{VV}\left(X_t, T_1-t, T_2-t\right) &= \textstyle\int_{\substack{u\in[t,T_1]\\ v\in[t,T_2]}} c_{WW}\left(X_t, u-t, v-t\right) d\left(u\times v\right) \\ &= \textstyle\int_0^{+\infty} \sigma_{FW}\left(X_t, T_1-t, s\right) \\ &\quad \cdot \sigma_{FW}\left(X_t, T_2-t, s\right) ds. \end{aligned} \tag{26}$$

The dynamics of forward rates over discrete time intervals can then be written as

$$\begin{aligned} dF_t\left(T_1, T_2\right) = \{&\left[\mu_F\left(X_t, T_2-t\right) - \mu_F\left(X_t, T_1-t\right)\right] dt \\ &+ \left[\sigma_{FZ}\left(X_t, T_2-t\right) - \sigma_{FZ}\left(X_t, T_1-t\right)\right]^T dZ_t \\ &+ \textstyle\int_{s=0}^{+\infty} \left[\sigma_{FW}\left(X_t, T_2-t, s\right) - \sigma_{FW}\left(X_t, T_1-t, s\right)\right] dW_{s,t}\}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \mu_F\left(X_t, T-t\right) &= \textstyle\int_t^T \mu_f\left(X_t, u-t\right) du \\ &= [\tfrac{1}{2}\sigma_{FZ}^T\left(X_t, T-t\right)\sigma_{FZ}\left(X_t, T-t\right) \\ &\quad + \tfrac{1}{2}c_{VV}\left(X_t, T-t, T-t\right) \\ &\quad + \sigma_{FZ}^T\left(X_t, T-t\right)\lambda_Z\left(X_t\right) \\ &\quad + \textstyle\int_0^{+\infty} \sigma_{FW}\left(X_t, T-t, s\right)\lambda_W\left(X_t, s\right) ds]. \end{aligned} \tag{28}$$

The instantaneous covariances of forward rates with each other and with the latent variables can now be found:

$$
\begin{aligned}
Cov\left[dF_t\left(T_1,T_2\right),dF_t\left(T_3,T_4\right)\right] &= c_{FF}\left(X_t,T_1-t,T_2-t,\right.\\
&\qquad \left.T_3-t,T_4-t\right)dt\\
&= [\sigma_{FZ}^T\left(X_t,T_2-t\right)\sigma_{FZ}\left(X_t,T_4-t\right)\\
&\quad - \sigma_{FZ}^T\left(X_t,T_1-t\right)\sigma_{FZ}\left(X_t,T_4-t\right)\\
&\quad - \sigma_{FZ}^T\left(X_t,T_2-t\right)\sigma_{FZ}\left(X_t,T_3-t\right)\\
&\quad + \sigma_{FZ}^T\left(X_t,T_1-t\right)\sigma_{FZ}\left(X_t,T_3-t\right)\\
&\quad + c_{VV}\left(X_t,T_2-t,T_4-t\right)\\
&\quad - c_{VV}\left(X_t,T_1-t,T_4-t\right)\\
&\quad - c_{VV}\left(X_t,T_2-t,T_3-t\right)\\
&\quad + c_{VV}\left(X_t,T_1-t,T_3-t\right)]dt \text{ and}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
Cov\left[dF_t\left(T_1,T_2\right),dX_t\right] &= c_{XF}\left(X_t,T_1-t,T_2-t\right)dt\\
&= \sigma_X\left(X_t\right)\,[\sigma_{FZ}\left(X_t,T_2-t\right)\\
&\qquad -\sigma_{FZ}\left(X_t,T_1-t\right)]dt.
\end{aligned} \tag{30}
$$

Forward rates over a discrete time interval share the diffusion property: The evolution of a forward rate depends only on the latent variables. Zero coupon bond yields, however, do not share this characteristic. Non-annualized yields are defined in the usual way:

$$
y_t\left(T\right) = F_t\left(t,T\right). \tag{31}
$$

Yields then follow the process

$$
\begin{aligned}
dy_t\left(T\right) &= \left[-r_t+\mu_F\left(X_t,T-t\right)\right]dt\\
&\quad +\sigma_{FZ}^T\left(X_t,T-t\right)dZ_t\\
&\quad +\textstyle\int_{s=0}^{+\infty}\sigma_{FW}\left(X_t,T-t,s\right)dW_{s,t}.
\end{aligned} \tag{32}
$$

The covariances of yields, and the covariances between yields and the latent variables, are as follows:

$$
\begin{aligned}
Cov\left[dy_t\left(T_1\right),dy_t\left(T_2\right)\right] &= c_{YY}\left(X_t,T_1-t,T_2-t\right)dt\\
&= [\sigma_{FZ}^T\left(X_t,T_1-t\right)\sigma_{FZ}\left(X_t,T_2-t\right)\\
&\quad + c_{VV}\left(X_t,T_1-t,T_2-t\right)]dt \text{ and}
\end{aligned} \tag{33}
$$

$$Cov\left[dy_t\left(T\right), dX_t\right] = c_{XY}\left(X_t, T-t\right)dt = \sigma_X\left(X_t\right)\sigma_{FZ}\left(X_t, T-t\right)dt. \quad (34)$$

The process followed by yields is potentially complex. Although the covariance structure of yields depends only on time to maturity and the latent variables, from Eq. (32), the drift of each yield depends on the instantaneous interest rate. Because the instantaneous interest rate takes on the identity of a different forward rate each instant, it follows a potentially complex and infinite-dimensional process, and yield dynamics inherit this complexity. However, many problems of estimation and derivatives pricing can be formulated in terms of forward rates instead of yields. Consequently, such problems often have tractable solutions, despite the complexity of the process followed by yields. In Section 6, derivative pricing problems are solved by expressing the derivative payoff in terms of forward rates, not yields. Kimmel (2001) estimates several LV models using a method of moments technique on forward rates.

Zero coupon bond prices are nonlinear functions of nonannualized yields:

$$B_t\left(T\right) = e^{-y_t(T)}. \quad (35)$$

Application of Ito's Lemma allows bond price dynamics to be expressed as follows:

$$\begin{aligned} dB_t\left(T\right) = \{&[r_t - \sigma_{FZ}^T\left(X_t, T-t\right)\lambda_Z\left(X_t\right) \\ &- \textstyle\int_0^{+\infty}\sigma_{FW}\left(X_t, T-t, s\right)\lambda_W\left(X_t, s\right)ds]B_t\left(T\right)dt \\ &- \sigma_{FZ}^T\left(X_t, T-t\right)B_t\left(T\right)dZ_t \\ &- B_t\left(T\right)\textstyle\int_{s=0}^{+\infty}\sigma_{FW}\left(X_t, T-t, s\right)dW_{s,t}\}. \end{aligned} \quad (36)$$

The instantaneous (proportional) drift is equal to the instantaneous interest rate $r_t$, plus risk premia associated with the innovations $Z_t$ and $W_{s,t}$, which is the necessary (but not sufficient) form for a model that is free from arbitrage. In Section 3, I show that the integrability restrictions in the LV model definition are sufficient as well. The drift of an instantaneous forward rate, shown in Eq. (11), has two similar risk premia terms. The other two terms in the forward rate drift are convexity adjustments that arise because the relationship between forward rates and bond prices is nonlinear.

## 3 Existence of the forward rate process and absence of arbitrage

As discussed in Section 2, an LV model is completely defined by its initial state, $X_0$ and $f_0\left(T\right)$, and the six functions $\mu_X$, $\sigma_X$, $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$. I now characterize the restrictions needed on the initial state and the six functions to ensure that unique latent variable and forward rate processes satisfying LV-1 through LV-5 exist and that these processes offers no arbitrage opportunities.

Requirement LV-1 imposes restrictions on the initial term structure of forward rates. The initial forward curve need not be continuous, but it must be integrable. If it is not integrable, zero coupon bond yields and prices are not defined. The future evolution of forward rates and of latent variables does not depend on the initial forward curve, so LV-1 is the only constraint on $f_0(T)$.

The latent variable process defined in Eq. (7) is a generic diffusion, requiring appropriate specification of $X_0$, $\mu_X$, and $\sigma_X$. Existence and uniqueness of multivariate diffusions is a well-studied issue, but a simple characterization of conditions that are both necessary and sufficient remains elusive. Multivariate versions of the growth and Lipschitz conditions (see Stroock and Varadhan, 1979, for a thorough treatment) are sufficient, but not necessary. For example, the square-root process of Feller (1951) fails even to be locally Lipschitz but exists nonetheless. Full treatment of the existence and uniqueness of the latent variable process must therefore take place in the context of a specific subfamily of the general LV model. For example, in Section 4 the requirements for affine diffusions are completely characterized.

Given an initial forward curve $f_0(T)$ that is integrable for all $T \geq 0$ and specifications of $X_0$, $\mu_X$, and $\sigma_X$ such that a unique latent variable process exists, the only restrictions on the functions $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$ needed for the existence of a unique forward rate process are the integrability constraints described in Eqs. (9), (10), (12), and (13). Forward rates solve Eq. (8), with the drift coefficient specified in Eq. (11). The drift and diffusion coefficients depend only on the latent variables. Neither the forward rate itself nor any other function of the level or shape of the term structure appears on the right-hand side of Eq. (8). Consequently, once the latent variable process and the $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$ functions are specified, forward rates can be calculated by evaluating the integrals on the right-hand side. For the diffusion terms, $L^2$-integrability (almost surely) is both necessary and sufficient. See references on stochastic integration with continuous integrators, such as Karatzas and Shreve (1991) or Revuz and Yor (1994). For the drift terms, $L^1$-integrability (almost surely) tautologically suffices.

Existence of a unique forward rate process therefore imposes only integrability restrictions on the functions $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$. Now consider absence of arbitrage, which requires the stronger restrictions of Eqs. (14) and (15). The following process $Y_t$ is a deflator; i.e., a strictly positive process:

$$Y_t = \exp\left[-\int_0^t r_u du\right]. \tag{37}$$

From its exponential form, $Y_t$ is strictly positive, provided the exponent is finite. Rewriting Eq. (8) in stochastic integral form, and setting $T = t$, the instantaneous interest rate can be expressed as

$$\begin{aligned} r_t = [&f_0(t) + \int_{\substack{u\in[0,t] \\ v\in[u,t]}} \sigma_{fZ}^T(X_u, t-u)\,\sigma_{fZ}(X_u, v-u)\,d(u\times v) \\ &+ \int_{\substack{s\in[0,+\infty) \\ u\in[0,t] \\ v\in[u,t]}} \sigma_{fW}(X_u, t-u, s)\,\sigma_{fW}(X_u, v-u, s)\,d(s\times u\times v) \\ &+ \int_0^t \sigma_{fZ}^T(X_u, t-u)\,\lambda_Z(X_u)\,du \\ &+ \int_{\substack{s\in[0,+\infty) \\ u\in[0,t]}} \sigma_{fW}(X_u, t-u, s)\,\lambda_W(X_u, s)\,d(s\times u) \\ &+ \int_0^t \sigma_{fZ}^T(X_u, t-u)\,dZ_u + \int_{\substack{s\in[0,+\infty) \\ u\in[0,t]}} \sigma_{fW}(X_u, t-u, s)\,dW_{s,u}]. \end{aligned} \tag{38}$$

Integrating over time results in:

$$\begin{aligned} \int_0^t r_u du = [&\int_0^t f_0(u)\,du \\ &+ \int_{\substack{u\in[0,w] \\ v\in[u,w] \\ w\in[0,t]}} \sigma_{fZ}^T(X_u, w-u)\,\sigma_{fZ}(X_u, v-u)\,d(u\times v\times w) \\ &+ \int_{\substack{s\in[0,+\infty) \\ u\in[0,w] \\ v\in[u,w] \\ w\in[0,t]}} \sigma_{fW}(X_u, w-u, s) \\ &\qquad \cdot\,\sigma_{fW}(X_u, v-u, s)\,d(s\times u\times v\times w) \\ &+ \int_{\substack{u\in[0,v] \\ v\in[0,t]}} \sigma_{fZ}^T(X_u, v-u)\,\lambda_Z(X_u)\,d(u\times v) \\ &+ \int_{\substack{s\in[0,+\infty) \\ u\in[0,v] \\ v\in[0,t]}} \sigma_{fW}(X_u, v-u, s)\,\lambda_W(X_u, s)\,d(s\times u\times v) \\ &+ \int_{\substack{u\in[0,v] \\ v\in[0,t]}} \sigma_{fZ}^T(X_u, v-u)\,d(Z_u\times v) \\ &+ \int_{\substack{s\in[0,+\infty) \\ u\in[0,v] \\ v\in[0,t]}} \sigma_{fW}(X_u, v-u, s)\,d(W_{s,u}\times v)]. \end{aligned} \tag{39}$$

By the integrability assumptions in Eqs. (9), (10), (12), and (13), every term on the right-hand side of Eq. (39) is finite, almost surely. The process $Y_t$, as defined in Eq. (37), is therefore positive, almost surely, and is consequently a deflator. If a money market asset is available, its deflated price is trivially equal to one. For zero coupon bonds, Eqs. (35) and (32) allow prices to be written as

$$\begin{aligned} B_t(T) = \exp[\, &\int_0^t r_u du - \int_0^t \mu_F(X_u, T-u)\,du \\ &- \int_0^t \sigma_{FZ}^T(X_u, T-u)\,dZ_u \\ &- \int_{\substack{s\in[0,+\infty) \\ u\in[0,t]}} \sigma_{FW}(X_u, T-u, s)\,dW_{s,u}]. \end{aligned} \tag{40}$$

Deflated bond prices, denoted by $B_t^Y(T)$, are then

$$\begin{aligned}
B_t^Y(T) &= B_t(T)\cdot Y_t \\
&= \exp[-\textstyle\int_0^t \mu_F(X_u, T-u)\,du \\
&\qquad - \textstyle\int_0^t \sigma_{FZ}^T(X_u, T-u)\,dZ_u \\
&\qquad - \textstyle\int_{\substack{s\in[0,+\infty)\\ u\in[0,t]}} \sigma_{FW}(X_u, T-u, s)\,dW_{s,u}] \\
&= \exp[-\tfrac{1}{2}\textstyle\int_0^t \sigma_{FZ}^T(X_u, T-u)\,\sigma_{FZ}(X_u, T-u)\,du \\
&\qquad -\tfrac{1}{2}\textstyle\int_0^t c_{VV}(X_t, T-u, T-u)\,du \\
&\qquad - \textstyle\int_0^t \sigma_{FZ}^T(X_u, T-u)\,\lambda_Z(X_u)\,du \\
&\qquad - \textstyle\int_{\substack{s\in[0,+\infty)\\ u\in[0,t]}} \sigma_{FW}(X_u, T-u, s)\,\lambda_W(X_u, s)\,d(s\times u) \\
&\qquad - \textstyle\int_0^t \sigma_{FZ}^T(X_u, T-u)\,dZ_u \\
&\qquad - \textstyle\int_{\substack{s\in[0,+\infty)\\ u\in[0,t]}} \sigma_{FW}(X_u, T-u, s)\,dW_{s,u}].
\end{aligned} \tag{41}$$

There exists an equivalent martingale measure for this deflated bond price process. First, the process $\xi_t$ is defined as

$$\begin{aligned}
\xi_t = \exp[&-\tfrac{1}{2}\textstyle\int_0^t \lambda_Z^T(X_u)\,\lambda_Z(X_u)\,du - \tfrac{1}{2}\int_{\substack{s\in[0,\infty)\\ u\in[0,t]}} [\lambda_W(X_u, s)]^2\,d(s\times u) \\
&- \textstyle\int_0^t \lambda_Z^T(X_u)\,dZ_u - \int_{\substack{s\in[0,\infty)\\ u\in[0,t]}} \lambda_W(X_u, s)\,dW_{s,u}].
\end{aligned} \tag{42}$$

By the integrability assumptions in Eqs. (12) and (13), this process is strictly positive. From Eq. (15), it is a martingale as well. If each bond price is multiplied by the process $\xi_t$, the result is:

$$\begin{aligned}
B_t^Y(T)\,\xi_t = \exp\{&-\tfrac{1}{2}\textstyle\int_0^t [\sigma_{FZ}(X_u, T-u) + \lambda_Z(X_u)]^T \\
&\qquad \cdot[\sigma_{FZ}(X_u, T-u) + \lambda_Z(X_u)]\,du \\
&-\tfrac{1}{2}\textstyle\int_{\substack{s\in[0,+\infty)\\ u\in[0,t]}} [\sigma_{FW}(X_u, T-u, s) + \lambda_W(X_u, s)]^2\,d(s\times u) \\
&- \textstyle\int_0^t [\sigma_{FZ}(X_u, T-u) + \lambda_Z(X_u)]^T\,dZ_u \\
&- \textstyle\int_{\substack{s\in[0,+\infty)\\ u\in[0,t]}} [\sigma_{FW}(X_u, T-u, s) + \lambda_W(X_u, s)]\,dW_{s,u}\}.
\end{aligned} \tag{43}$$

From Eq. (14), the product of each deflated bond price and $\xi_t$ is also an exponential martingale. For some arbitrary finite time horizon $\tau$, an equivalent measure $Q$ is defined by specifying its Radon-Nikodym derivative:

$$\frac{dQ}{dP} = \xi_\tau. \tag{44}$$

Under the measure $Q$, deflated bond prices (prior to time $\tau$) are martingales:

$$\begin{aligned} E_{t_1}^{Q}\left[B_{t_2}^{Y}(T)\right] &= \frac{E_{t_1}^{P}\left[\xi_\tau B_{t_2}^{Y}(T)\right]}{E_{t_1}^{P}[\xi_\tau]} = \frac{E_{t_1}^{P}\left[B_{t_2}^{Y}(T) E_{t_2}^{P}[\xi_\tau]\right]}{\xi_{t_1}} \\ &= \frac{E_{t_1}^{P}\left[B_{t_2}^{Y}(T)\xi_{t_2}\right]}{\xi_{t_1}} = \frac{B_{t_1}^{Y}(T)\xi_{t_1}}{\xi_{t_1}} = B_{t_1}^{Y}(T). \end{aligned} \tag{45}$$

Consequently, the measure $Q$ is an equivalent martingale measure.

The measure $Q$ is not necessarily unique. For example, taking an LV model in which some elements of the $\sigma_{fZ}$ function are uniformly zero, the corresponding elements of the $\sigma_{FZ}$ function are also uniformly zero. The corresponding elements of the $\lambda_Z$ function can be changed arbitrarily, provided the integrability restrictions in Eqs. (12) and (15) are still satisfied, and deflated bond prices are still martingales under this alternate specification of the $Q$ measure. Changing these elements of the $\lambda_Z$ function has no observable effect on forward rate or bond price dynamics in this case. The $\lambda_Z$ function is a market price of risk specification for the innovations that drive the latent variables. If some elements of the $\sigma_{fZ}$ function are uniformly zero, the set of zero coupon bonds does not offer investors any exposure to these sources of risk. Consequently, the market price of risk for these innovations cannot be determined simply by observing bond price dynamics, because bond prices do not span this source of risk. This issue is discussed in greater detail in Section 5.

Given that the latent variables do not, in general, correspond to traded assets, even if the $\sigma_{fZ}$ function is not equal to zero, it is useful to describe their dynamics under the measure $Q$. An application of Girsanov's theorem allows construction of a set of processes that are Brownian motions under the measure $Q$:

$$Z_t^Q = Z_t + \int_0^t \lambda_Z(X_u)\,du. \tag{46}$$

The latent variable process can then be written as

$$dX_t = \mu_X^Q(X_t)\,dt + \sigma_X(X_t)\,dZ_t^Q, \tag{47}$$

where

$$\mu_X^Q(X_t) = \mu_X(X_t) - \sigma_X(X_t)\,\lambda_Z(X_t). \tag{48}$$

Similarly, a Brownian sheet under the measure $Q$ can be defined as:

$$W_{s,t}^Q = W_{s,t} + \int_0^t \lambda_W(X_u, s)\,du. \tag{49}$$

The dynamics of forward rates (both instantaneous and discrete) under the measure $Q$ are then

$$df_t(T) = \mu_f^Q(X_t, T-t)\,dt + \sigma_{fZ}^T(X_t, T-t)\,dZ_t^Q \\ + \int_{s=0}^{+\infty} \sigma_{fW}(X_t, T-t, s)\,dW_{s,t}^Q \text{ and} \tag{50}$$

$$\begin{aligned} dF_t(T_1, T_2) = \{ & \left[\mu_F^Q(X_t, T_2 - t) - \mu_F^Q(X_t, T_1 - t)\right] dt \\ & + \left[\sigma_{FZ}(X_t, T_2 - t) - \sigma_{FZ}(X_t, T_1 - t)\right]^T dZ_t^Q \\ & + \int_{s=0}^{+\infty} \left[\sigma_{FW}(X_t, T_2 - t, s) - \sigma_{FW}(X_t, T_1 - t, s)\right] dW_{s,t}^Q\}, \end{aligned} \tag{51}$$

where the functions $\mu_f^Q$ and $\mu_F^Q$ are defined as

$$\begin{aligned} \mu_f^Q(X_t, T-t) = & \,\sigma_{fZ}^T(X_t, T-t) \int_t^T \sigma_{fZ}(X_t, u-t)\,du \\ & + \int_{\substack{s\in[0,+\infty) \\ u\in[t,T]}} [\sigma_{fW}(X_t, T-t, s) \\ & \quad \cdot \sigma_{fW}(X_t, u-t, s)]\,d(s \times u) \text{ and} \end{aligned} \tag{52}$$

$$\begin{aligned} \mu_F^Q(X_t, T-t) &= \int_t^T \mu_f^Q(X_t, u-t)\,du \\ &= \tfrac{1}{2}\sigma_{FZ}^T(X_t, T-t)\,\sigma_{FZ}(X_t, T-t) \\ & \quad + \tfrac{1}{2} c_{VV}(X_t, T-t, T-t)\,. \end{aligned} \tag{53}$$

Forward rates retain the diffusion property under the measure $Q$; i.e., the dynamics of forward rates depend only on time to maturity and the latent variables $X_t$, so each forward rate, together with the latent variables, follows a diffusion process. As with the $P$ measure, this property is not shared by yields and bond prices:

$$dy_t(T) = \left[-r_t + \mu_F^Q(X_t, T-t)\right] dt + \sigma_{FZ}^T(X_t, T-t)\,dZ_t^Q \\ + \int_{s=0}^{+\infty} \sigma_{FW}(X_t, T-t, s)\,dW_{s,t}^Q \text{ and} \tag{54}$$

$$dB_t(T) = r_t B_t(T)\,dt - \sigma_{FZ}^T(X_t, T-t)\,B_t(T)\,dZ_t^Q \\ - B_t(T) \int_{s=0}^{+\infty} \sigma_{FW}(X_t, T-t, s)\,dW_{s,t}^Q. \tag{55}$$

Bond prices have drift equal to their current prices times the instantaneous interest rate, as one would expect under an equivalent martingale measure. However, because the interest rate follows a complex, and possibly infinite-dimensional process, yields and bond prices will in general not be part of any finite-dimensional diffusion process.

The relation between absence of arbitrage and existence of an equivalent martingale measure, beginning with the pioneering work of Harrison and Kreps

(1979) and Harrison and Pliska (1981), and subsequently studied and expanded by many others, such as Delbaen and Schachermeyer (1994), is well known. However, it is usually stated for economies with finitely many independent assets. Some classical results on absence of arbitrage, market completeness, and existence and uniqueness of an equivalent martingale measure fail to hold in economies with infinitely many assets that have discontinuous sample paths. See, for example, Artzner and Heath (1995) or Battig and Jarrow (1999). However, in the setting of an LV model, the classical results regarding absence of arbitrage continue to apply. For example, the textbook proof that existence of an equivalent martingale measure implies absence of arbitrage presented in Duffie (1996) applies without modification. The key property is that deflated bond prices are continuous martingales under the measure $Q$. The deflated wealth process associated with any measurable, adapted, self-financing, and $L^2$-integrable trading strategy is therefore a local martingale. Consider a trading strategy over the period $[0, \tau]$ that trades in countably many zero coupon bonds, with $\theta_t(i)$ denoting the number of bonds with maturity $T_i$ at time $t$, with $\theta_t(i) = 0$ for $t > T_i$. From the self-financing requirement, the deflated wealth process is

$$\sum_{i=1}^{\infty} \theta_\tau(T_i) B_\tau^Y(T_i) = \sum_{i=1}^{\infty} \theta_0(T_i) B_0^Y(T_i) + \int_0^\tau \sum_{i=1}^{\infty} \theta_t(T_i) dB_t^Y(T_i). \tag{56}$$

The second term on the right is a local martingale. Harrison and Pliska (1981) require the deflated wealth process to be bounded below, and a local martingale that is bounded below is a supermartingale. Consequently, the expectation of the deflated wealth process at time $\tau$ is well defined and finite, and it satisfies

$$E\left[\sum_{i=1}^{\infty} \theta_\tau(T_i) B_\tau^Y(T_i)\right] \le \sum_{i=1}^{\infty} \theta_0(T_i) B_0^Y(T_i). \tag{57}$$

Therefore, for any trading strategy with a wealth process that is bounded below, the expected value of the time $\tau$ payoff cannot be positive if the initial wealth is zero or negative. Thus, no such trading strategy is an arbitrage. The argument presented here applies also to trading strategies involving uncountably many bonds. The summation signs in Eqs. (56) and (57) need only be changed to integrals in this case. The deflated gains process remains a local martingale and, if bounded below, is a supermartingale, so the corresponding trading strategy cannot be an arbitrage.

Construction of an LV model requires selection of the initial state $f_0(T)$ and $X_0$, and the functions $\mu_X$, $\sigma_X$, $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$. To ensure existence of a latent variable and forward rate processes, care must be taken when specifying $X_0$, $\mu_X$, and $\sigma_X$. However, a rich literature exists on multivariate diffusions, providing candidate latent variable processes. Existence considerations impose only technical integrability constraints on $f_0(T)$ and on $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$ and $\lambda_W$. Absence of arbitrage imposes the somewhat stronger constraints in Eqs. (14)

and (15). However, constructing specific LV models is easy. For example, if $f_0(T)$ is chosen to be integrable, any multivariate diffusion is chosen for the latent variable process, and the $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$ functions are bounded for each value of $T$, with the integrals of $\sigma_{fW}$ and $\lambda_W$ with respect to $s$ also bounded, one readily verifies that all constraints in the definition of an LV model are satisfied. In Section 4, I develop a family of models in which the latent variables follow an affine diffusion, but the $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$ functions are not necessarily bounded.

## 4 Affine latent variable term structure models

As discussed in Section 2, LV models have the property that forward rates, together with the latent variables, follow a finite-dimensional diffusion process. Section 3 shows that this property holds under an equivalent martingale measure $Q$ as well. In principle, quantities such as the conditional probability densities or conditional moments of the state variables can be found by solving finite-dimensional partial differential equations. In practice, solving such equations can be computationally intensive. However, if the diffusion is affine, such partial differential equations decompose into systems of ordinary differential equations, which can be solved rapidly on modern computers. The popularity of the affine yield models of Duffie and Kan (1996) is no doubt due in large part to this property. I therefore introduce a subset of the LV family of models, called affine latent variable models (ALV or ALV-N for short), in which each forward rate, together with the latent variables, follows an affine diffusion.

**Definition 2** *An ALV-N model is an LV-N model that satisfies the following conditions.*

*ALV-1 The instantaneous drift of each latent variable is a linear function of $X_t$:*

$$\mu_X(X_t) = \kappa(\theta - X_t), \tag{58}$$

*where $\theta$ is an $Nx1$ vector and $\kappa$ is an $NxN$ matrix.*

*ALV-2 The instantaneous covariance between each pair of latent variables is a linear function of $X_t$:*

$$c_{XX}(X_t) = \sigma_X(X_t)\,\sigma_X^T(X_t) = \sigma_{XX}^2(0) + \sum_{k=1}^{N} \sigma_{XX}^2(k)\,X_t(k), \tag{59}$$

*where $\sigma_{XX}^2(k)$ is an $NxN$ matrix for each $0 \le k \le N$.*

*ALV-3 The instantaneous covariance between any latent variable and any instan-*

*taneous forward rate is a linear function of $X_t$:*

$$\begin{aligned} c_{Xf}\left(X_t, T-t\right) &= \sigma_X\left(X_t\right)\sigma_{fZ}\left(X_t, T-t\right) \\ &= \sigma^2_{Xf}\left(T-t, 0\right) + \textstyle\sum_{k=1}^{N}\sigma^2_{Xf}\left(T-t, k\right)\cdot X_t\left(k\right), \end{aligned} \tag{60}$$

*where $\sigma^2_{Xf}\left(T-t, k\right)$ is an $Nx1$ vector-valued function for each $0 \leq k \leq N$.*

*ALV-4 The instantaneous covariance between any two instantaneous forward rates is a linear function of $X_t$:*

$$\begin{aligned} c_{ff}\left(X_t, T_1-t, T_2-t\right) &= [\sigma^T_{fZ}\left(X_t, T_1-t\right)\sigma_{fZ}\left(X_t, T_2-t\right) \\ &\quad + c_{WW}\left(X_t, T_1-t, T_2-t\right)] \\ &= \sigma^2_{ff}\left(T_1-t, T_2-t, 0\right) \\ &\quad + \textstyle\sum_{k=1}^{N}\sigma^2_{ff}\left(T_1-t, T_2-t, k\right)X_t\left(k\right), \end{aligned} \tag{61}$$

*where $\sigma^2_{ff}\left(T_1-t, T_2-t, k\right)$ is a scalar function for $0 \leq k \leq N$.*

*ALV-5 The instantaneous drift of each latent variable under the equivalent martingale measure $Q$ (defined in Section 3) is a linear function of $X_t$:*

$$\mu^Q_X\left(X_t\right) = \mu_X\left(X_t\right) - \sigma_X\left(X_t\right)\lambda_Z\left(X_t\right) = \widetilde{\kappa}\left(\widetilde{\theta} - X_t\right), \tag{62}$$

*where $\widehat{\theta}$ is an $Nx1$ vector and $\widehat{\kappa}$ is an $NxN$ matrix.*

*ALV-6 The instantaneous drift of each forward rate is a linear function of $X_t$:*

$$\begin{aligned} \mu_f\left(X_t, T-t\right) &= [\sigma^T_{fZ}\left(X_t, T-t\right)\textstyle\int_t^T\sigma_{fZ}\left(X_t, u-t\right)du \\ &\quad + \textstyle\int_{\substack{s\in[0,+\infty)\\ u\in[t,T]}}\sigma_{fW}\left(X_t, T-t, s\right) \\ &\qquad\qquad \sigma_{fW}\left(X_t, u-t, s\right)d\left(s\times u\right) \\ &\quad + \sigma^T_{fZ}\left(X_t, T-t\right)\lambda_Z\left(X_t\right) \\ &\quad + \textstyle\int_0^{+\infty}\sigma_{fW}\left(X_t, T-t, s\right)\lambda_W\left(X_t, s\right)ds] \\ &= \gamma\left(T-t, 0\right) + \textstyle\sum_{k=1}^{N}\gamma\left(T-t, k\right)X_t\left(k\right), \end{aligned} \tag{63}$$

*for some $N+1$ scalar functions $\gamma\left(T-t, k\right)$, $0 \leq k \leq N$.*

Given any ALV model, another fully equivalent ALV model can be constructed by replacing the vector of latent variables with a nonsingular linear combination of itself. If $\Gamma$ is an $N\times 1$ matrix of constants, and $\Sigma$ is a nonsingular $N\times N$ matrix of constants, the latent variables $X_t$ can be replaced by an alternate set $\widehat{X_t}$, and the functions $\mu_X$, $\sigma_X$, $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$ can be replaced by the functions $\widehat{\mu}_X$, $\widehat{\sigma}_X$, $\widehat{\sigma}_{fZ}$, $\widehat{\sigma}_{fW}$, $\widehat{\lambda}_Z$, and $\widehat{\lambda}_W$, defined as

$$\widehat{X_t} = \Gamma + \Sigma\cdot X_t, \tag{64}$$

$$\widehat{\mu}_X\left(\widehat{X}_t\right) = \Sigma \cdot \mu_X\left(\Sigma^{-1}\left(\widehat{X}_t - \Gamma\right)\right), \tag{65}$$

$$\widehat{\sigma}_X\left(\widehat{X}_t\right) = \Sigma \cdot \sigma_X\left(\Sigma^{-1}\left(\widehat{X}_t - \Gamma\right)\right), \tag{66}$$

$$\widehat{\sigma}_{fZ}\left(\widehat{X}_t, T-t\right) = \sigma_{fZ}\left(\Sigma^{-1}\left(\widehat{X}_t - \Gamma\right), T-t\right), \tag{67}$$

$$\widehat{\sigma}_{fW}\left(\widehat{X}_t, T-t, s\right) = \sigma_{fW}\left(\Sigma^{-1}\left(\widehat{X}_t - \Gamma\right), T-t, s\right), \tag{68}$$

$$\widehat{\lambda}_Z\left(\widehat{X}_t\right) = \lambda_Z\left(\Sigma^{-1}\left(\widehat{X}_t - \Gamma\right)\right), \text{ and} \tag{69}$$

$$\widehat{\lambda}_W\left(\widehat{X}_t, s\right) = \lambda_W\left(\Sigma^{-1}\left(\widehat{X}_t - \Gamma\right), s\right). \tag{70}$$

If the original specification satisfies requirements LV-1 through LV-5, the alternate specification does also, and by construction, the alternate specification satisfies ALV-1 through ALV-6. The family of ALV-N models is therefore closed with respect to linear transformations of the latent variables. To take advantage of this fact, I always express ALV-N models in a canonical form.

Simply choosing the drift and covariance of the latent variables to have the linear form specified in Eqs. (58) and (59) does not guarantee that a unique solution to Eq. (7) exists. Specifically, the choice of the $\kappa$, $\theta$, and $\sigma^2_{XX}$ parameters must ensure that the instantaneous covariance matrix $c_{XX}$ is positive definite for all achievable values of the state vector. [2] Generally, this condition involves a trade-off between flexibility in the drift specification and flexibility in the diffusion specification. Duffie and Kan (1996) discuss admissibility conditions for general affine diffusions, and Dai and Singleton (2000) discuss in detail the admissibility conditions for affine diffusions with three state variables. Any affine diffusion has a canonical representation (obtained by applying an appropriate linear transformation to the state vector) in which the $\sigma_X(X_t)$ function is diagonal and in which the unconditional mean of any unbounded state variable is normalized to zero. For a diffusion of the $A_M(N)$ class, the canonical form drift is

$$\mu_X(X_t) = \kappa(\theta - X_t) \tag{71}$$

[2] Positive definiteness is sufficient to guarantee not only existence of a unique latent variable process, but also that those latent variables that are bounded (in one direction) cannot achieve the boundary value. If the covariance matrix can become positive semidefinite, then the boundary can be achieved with positive probability. In this case, a number of commonly used results, such as the Kolmogorov backward equation for conditional moments and the forward or backward equations for conditional densities, do not hold at the boundary.

where

$$\kappa = \begin{bmatrix} \kappa_{(1,1)} & \cdots & \kappa_{(1,M)} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \kappa_{(M,1)} & \cdots & \kappa_{(M,M)} & 0 & \cdots & 0 \\ \kappa_{(M+1,1)} & \cdots & \kappa_{(M+1,M)} & \kappa_{(M+1,M+1)} & \cdots & \kappa_{(M+1,N)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \kappa_{(N,1)} & \cdots & \kappa_{(N,M)} & \kappa_{(N,M+1)} & \cdots & \kappa_{(N,N)} \end{bmatrix} \text{ and} \tag{72}$$

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_M \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{73}$$

The canonical form $\sigma_X$ is a diagonal matrix, with

$$\left[\sigma_X\left(X_t\right)\right]_{ii} = \sqrt{X_t\left(i\right)},\ 1 \leq i \leq M \text{ and} \tag{74}$$

$$\left[\sigma_X\left(X_t\right)\right]_{ii} = \sqrt{\alpha_i + \sum_{j=1}^{M} \beta_{i,j} X_t\left(j\right)},\ M+1 \leq i \leq N, \tag{75}$$

where $\alpha_i \in \{0,1\}$, $M+1 \leq i \leq N$.[3] (The latent variables have been ordered so that the $M$ bounded variables have indices 1 through $M$.) The following restrictions on the parameters and initial state are then required to guarantee existence of a unique latent variable process:

$$X_0\left(i\right) > 0,\ 1 \leq i \leq M, \tag{76}$$

$$\left(\kappa\theta\right)_i \geq \frac{1}{2},\ 1 \leq i \leq M, \tag{77}$$

$$\kappa_{i,j} \leq 0, 1 \leq i, j \leq M, i \neq j, \text{ and} \tag{78}$$

$$\beta_{i,j} \geq 0,\ M+1 \leq i \leq N,\ 1 \leq j \leq M. \tag{79}$$

[3] This specification is slightly more general than in Dai and Singleton (2000), whose canonical specification requires that $\alpha_i = 1$, $M+1 \leq i \leq N$. Some affine diffusions cannot be expressed in canonical form unless the definition of the canonical form is extended to allow $\alpha_i = 0$.

These constraints ensure that a unique latent variable process exists and is linear; i.e., conditions LV-2, ALV-1, and ALV-2 are satisfied. (Some of these constraints are vacuous if $M = 0$ or $M = N$.) Stationarity of this process is not required to prevent arbitrage, but may nonetheless be considered a desirable modeling property. The latent variable process is stationary if the real part of each eigenvalue of the $\kappa$ matrix is positive.

In an affine yield model, the instantaneous interest rate is a linear function of the state variables. The ALV family of models imposes the weaker restriction that the instantaneous variances and covariances of forward rates (and the covariances between forward rates and latent variables) be linear functions of the latent variables themselves. From Eqs. (60) and (61), we can see that the $\sigma_{fZ}$ function must have the following form:

$$\sigma_{fZ}\left(X_t, T-t\right) = \sigma_X\left(X_t\right) \begin{bmatrix} f_1\left(T-t\right) \\ \vdots \\ f_N\left(T-t\right) \end{bmatrix}. \tag{80}$$

The covariance between forward rates and latent variables is then

$$\begin{aligned} c_{Xf}\left(X_t, T-t\right) &= \sigma_X\left(X_t\right)\sigma_{fZ}\left(X_t, T-t\right) \\ &= \sigma^2_{Xf}\left(T-t, 0\right) + \textstyle\sum_{k=1}^{N} \sigma^2_{Xf}\left(T-t, k\right)\cdot X_t\left(k\right) \\ &= \begin{bmatrix} f_1\left(T-t\right)X_t\left(1\right) \\ \vdots \\ f_M\left(T-t\right)X_t\left(M\right) \\ f_{M+1}\left(T-t\right)\left[\alpha_{M+1} + \sum_{i=1}^{M}\beta_{M+1,i}X_t\left(i\right)\right] \\ \vdots \\ f_N\left(T-t\right)\left[\alpha_N + \sum_{i=1}^{M}\beta_{N,i}X_t\left(i\right)\right] \end{bmatrix}. \end{aligned} \tag{81}$$

When an ALV model is expressed in the canonical form, the covariance of any latent variable with any forward rate depends only on the first $M$ latent variables and is proportional to the corresponding diagonal element of $c_{XX}$.

From Eqs. (61) and (81), the $\sigma_{fW}\left(X_u, w-u, s\right)$ function must have the form

$$\sigma_{fW}\left(X_t, T-t, s\right) = h\left(T-t\right)\sqrt{g_0\left(s\right) + \sum_{i=1}^{M} g_i\left(s\right)X_t\left(i\right)}, \tag{82}$$

with $g_0\left(s\right) \geq 0$ for each $0 \leq i \leq M$.

Neither the $\sigma_{fZ}$ function nor the $\sigma_{fW}$ function can depend on the latent variable $X_t(i)$ for any $M+1 \leq i \leq N$. These latent variables are unbounded, whereas the first $M$ latent variables are strictly positive. Any linear function of the latent variables that has a nonzero coefficient on $X_t(i)$ for some $M+1 \leq i \leq N$ is also unbounded. Because variances are bounded below by zero, no forward rate can have a variance that is linearly dependent on an unbounded latent variable. Similarly, covariances between forward rates and covariances between forward rates and latent variables cannot depend on an unbounded latent variable, because the absolute value of a covariance is bounded above by the product of the standard deviations of the two variables. Unbounded latent variables thus do not affect any of the variances or covariances in $c_{XX}$, $c_{Xf}$, or $c_{ff}$.

The integrability restrictions in Eqs. (9) and (10) translate directly into integrability restrictions on the $f$, $g$, and $h$ functions above:

$$\int_{\substack{u\in[0,t]\\ v\in[u,T]\\ w\in[u,T]}} \left|f_i^2(v-u)\right| d(u\times v\times w) < \infty,\ 1\leq i\leq N \text{ and} \tag{83}$$

$$\int_{\substack{s\in[0,+\infty)\\ u\in[0,t]\\ v\in[u,T]\\ w\in[u,T]}} \left|h(v-u)\,h(w-u)\,g_i^2(s)\right| d(s\times u\times v\times w) < \infty,\ 0\leq i\leq M. \tag{84}$$

These conditions ensure that LV-3, ALV-3, and ALV-4 are satisfied.

Turning to the specifications of the market price of risk processes $\lambda_Z$ and $\lambda_W$, the requirements LV-4, ALV-5, and ALV-6 are satisfied if these functions have the form

$$\lambda_Z(X_t) = \sigma_X^{-1}(X_t)\begin{bmatrix} \lambda_{1,0}+\sum_{i=1}^N \lambda_{1,i}X_t(i) \\ \vdots \\ \lambda_{N,0}+\sum_{i=1}^N \lambda_{N,i}X_t(i)\end{bmatrix} \text{ and} \tag{85}$$

$$\lambda_W(X_t,s) = \frac{\Lambda_0(s)+\sum_{i=1}^N \Lambda_i(s)\,X_t(i)}{\sqrt{g_0(s)+\sum_{i=1}^M g_i(s)\,X_t(i)}}, \tag{86}$$

where $\lambda_{i,j}$ is a constant for each $0\leq i\leq N$ and $1\leq j\leq N$ and $\Lambda_i(s)$ is a scalar function for each $0\leq i\leq N$. Some additional restrictions prevent the market price of risk from growing without bound as the latent variables approach their boundaries:

$$\lambda_{i,j}=0,\ 1\leq i\leq M,\ 0\leq j\leq N,\ i\neq j. \tag{87}$$

An analogous restriction for affine yield models is usually imposed: A bounded state variable must have market price of risk that is proportional to its square root. However, an aditional restriction is applied for ALV models. Given that unbounded state variables may have volatility that can approach zero, their market price of risk is similarly restricted. For each unbounded latent variable with $\alpha_i = 0$,

$$\lambda_{i,0} = 0, M+1 \leq i \leq N, \alpha_i = 0, \tag{88}$$

$$\lambda_{i,j} = 0,\ M+1 \leq i \leq N,\ M+1 \leq j \leq N,\ \alpha_i = 0, \text{ and} \tag{89}$$

$$\lambda_{i,j} = \beta_{i,j},\ M+1 \leq i \leq N,\ 1 \leq j \leq M,\ \alpha_i = 0. \tag{90}$$

These restrictions are sufficient to ensure that Eq. (12) is satisfied. The second moment of any variable that is part of an affine diffusion is not only finite, but also known in closed-form. Furthermore, the square of each element of the market price of risk specification in Eq. (85) is a quadratic function of the latent variables, divided by a function that is bounded below by a positive number. By a dominated convergence argument, the integral in Eq. (12) is finite. The corresponding conditions on $\Lambda_i(s)$, $0 \leq s \leq N$ required to satisfy Eq. (13) are more complex and must be verified for each specific model considered. In particular, I cannot use a dominated convergence argument, because $g_0(s)$ is never bounded below by any positive number. If it were, Eq. (84) would be violated.

The conditions of Eqs. (14) and (15) are automatically satisfied in the case of ALV models. The criteria of Novikov and Kazamaki provide sufficient (but not necessary) conditions for satisfaction of both equations. Neither criterion is satisfied globally (for all $t$) for all values of the market price of risk parameters. However, both criteria are satisfied locally (for some positive value of $t$, where $t$ depends on the parameters but not on the latent variables), provided Eqs. (83), (84), and (13) are satisfied. Local satisfaction of these criteria is sufficient, because by the law of iterated expectations (see Karatzas and Shreve, 1991, for an application to verification of the martingale condition for exponential supermartingales), Eqs. (14) and (15) are satisfied for all values of $t$. Eqs. (83), (84), and (13) therefore stand as the only integrability conditions (other than initial forward rate integrability) that must be verified for an ALV model.

For affine yield models, Duffee (2002) defines the terms completely affine and essentially affine. In a completely affine yield model, the market price of risk specification depends only on bounded state variables, but in essentially affine models, the market price of risk can depend on all state variables. The market price of risk for ALV models specified above allows models that are analogous to the essentially affine yield models. Some state variables are bounded away from zero and have essentially arbitrary (linear) risk premia. Others can approach zero and are constrained to have risk premia proportional to their volatility.

Constructing an ALV model therefore requires specification of an initial forward curve, selection of an affine diffusion of the $A_N(N)$ class, and selection of $\sigma_{fZ}$, $\sigma_{fW}$, $\lambda_Z$, and $\lambda_W$ functions of the form given in Eqs. (80), (82), (85), and (86) that satisfy only mild technical regularity conditions. In Section 5, I construct several different ALV models exhibiting a variety of interesting types of behavior.

## 5 Characterizing LV models

LV models can exhibit many different types of behavior. Some are indistinguishable from affine yield models, whereas others exhibit non-Markovian or infinite factor behavior. Furthermore, in some LV models, the set of zero coupon bonds completes the market, allowing hedging of term structure derivatives using only bond portfolios. However, in other models, zero coupon bonds do not complete the market, and hedging of term structure derivatives requires other derivative instruments. I shall now examine how LV models exhibit each of these characteristics.

Every completely affine yield model of the $A_M(N)$ family is an ALV-M model, in which the latent variables follow an affine diffusion of the $A_M(M)$ class. Consider the state variables $Y_t$ of the canonical version of the affine yield model. The notation is similar to that of Dai and Singleton (2000).

$$dY_t = \begin{bmatrix} dY_t^B \\ dY_t^D \end{bmatrix} = \begin{bmatrix} \kappa^{BB} & 0_{M\times(N-M)} \\ \kappa^{DB} & \kappa^{DD} \end{bmatrix} \left( \begin{bmatrix} \theta^B \\ 0_{(N-M)\times 1} \end{bmatrix} - \begin{bmatrix} Y_t^B \\ Y_t^D \end{bmatrix} \right) dt + \sqrt{\begin{bmatrix} S_t^B & 0_{M\times(N-M)} \\ 0_{(N-M)\times M} & S_t^D \end{bmatrix}} \begin{bmatrix} dZ_t^B \\ dZ_t^D \end{bmatrix}, \tag{91}$$

where $\kappa^{BB}$ , $\kappa^{DB}$, and $\kappa^{DD}$ are an $M \times M$, $(N-M)\times M$, and $(N-M)\times(N-M)$ matrices of constants, respectively, and $\theta^B$ is an $M \times 1$ vector of constants. $S_t^B$ and $S_t^B$ are $M \times M$ and $(N-M)\times(N-M)$ diagonal matrices of the form

$$S_t^B = \begin{bmatrix} Y_t^B(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Y_t^B(M) \end{bmatrix} \text{ and} \tag{92}$$

$$S_t^D = \begin{bmatrix} \alpha_{M+1} + \beta_{M+1}^T Y_t^B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_N + \beta_N^T Y_t^B \end{bmatrix}, \tag{93}$$

where each $\alpha_i \in \{0,1\}$, $M+1 \le i \le N$ and each $\beta_i$, $M+1 \le i \le N$ is an $M \times 1$ vector of constants.

In any completely affine yield model, the link between the canonical state variables and the dynamics of bond prices, yields, and forward rates is through the instantaneous interest rate and the market price of risk specification:

$$r_t = d_0 + d_Y^T Y_t = \begin{bmatrix} d_Y^B \\ d_Y^D \end{bmatrix}^T \begin{bmatrix} Y_t^B \\ Y_t^D \end{bmatrix} \text{ and} \tag{94}$$

$$\Lambda_t = \left[ \sqrt{\begin{bmatrix} S_t^B & 0_{M\times(N-M)} \\ 0_{(N-M)\times M} & S_t^D \end{bmatrix}} \right] \begin{bmatrix} \lambda^B \\ \lambda^D \end{bmatrix}, \tag{95}$$

where $d_Y^B$ and $\lambda^B$ are $M \times 1$ vectors of constants, $d_Y^D$ and $\lambda^D$ are $(N-M)\times 1$ vectors of constants, and $d_0$ is a constant. As per Duffie and Kan (1996), bond prices are then exponential-affine in the state vector

$$B_t(T) = e^{-q_0(T-t)-q_Y^T(T-t)Y_t}, \tag{96}$$

where $q_0(T-t)$ is a scalar function and $q_Y(T-t)$ is an $N \times 1$ vector-valued function. These two functions jointly solve the differential equations with boundary conditions

$$\begin{aligned} q_0'(T-t) = d_0 - q_Y^T(T-t) &\begin{bmatrix} \kappa^{BB} \\ \kappa^{DB} \end{bmatrix} \theta^B \\ &- \textstyle\sum_{i=1}^M \alpha_i \left[\frac{q_Y(T-t)}{2} + \lambda\right]_i [q_Y(T-t)]_i, \end{aligned} \tag{97}$$

$$q_Y'(T-t) = d_Y - \begin{bmatrix} \kappa^{BB} & 0_{M\times(N-M)} \\ \kappa^{DB} & \kappa^{DD} \end{bmatrix}^T q_Y(T-t) - q_2(T-t), \tag{98}$$

$$q_0(0) = 0, \text{ and} \tag{99}$$

$$q_Y(0) = 0_{N\times 1}, \tag{100}$$

where the $N \times 1$-valued $q_2$ function is defined as

$$q_2(T-t) = \begin{bmatrix} \{\left[\frac{q_Y(T-t)}{2} + \lambda\right]_1 [q_Y(T-t)]_1 \\ +\sum_{i=M+1}^N \beta_i(1) [q_Y(T-t)]_i \left[\frac{q_Y(T-t)}{2} + \lambda\right]_i\} \\ \vdots \\ \{\left[\frac{q_Y(T-t)}{2} + \lambda\right]_M [q_Y(T-t)]_M \\ +\sum_{i=M+1}^N \beta_i(M) [q_Y(T-t)]_i \left[\frac{q_Y(T-t)}{2} + \lambda\right]_i\} \\ 0_{(N-M)x1} \end{bmatrix}. \tag{101}$$

Instantaneous forward rates can be expressed as

$$f_t(T) = q_0'(T-t) + [q_Y'(T-t)]^T Y_t. \tag{102}$$

To express such a model in ALV-M form, I take the latent variables to be the $Y_t^B$ subset of the affine yield state variables, because these latent variables follow an affine diffusion of the $A_M(M)$ class, and only these state variables enter into the variances, covariances, and risk premia of forward rates. The innovations associated with the latent variables can affect forward rates through the $\sigma_{fZ}$ term in Eq. (8). The innovations associated with the remaining N-M state variables of the affine yield model will come from the Brownian sheet (i.e., the $\sigma_{fW}$ term) in the ALV-$M$ representation. First the auxilary function $S_{ij}(X_t)$ is defined:

$$S_{ij}(X_t) = \sqrt{\alpha_j + \sum_{i=1}^{M} \beta_j(i) X_t(i)}. \tag{103}$$

The ALV-$M$ representation is then

$$X_0 = Y_0^B, \tag{104}$$

$$f_0(T) = q_0'(T) + [q_Y'(T)]^T Y_0, \tag{105}$$

$$\mu_X(X_t) = \kappa^{BB}\left(\theta^B - X_t\right), \tag{106}$$

$$\sigma_X(X_t) = \sqrt{\begin{matrix} X_t(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_t(M) \end{matrix}}, \tag{107}$$

$$\sigma_{fZ}(X_t, T-t) = \sigma_X(X_t)\, q_Y'^B(T-t), \tag{108}$$

$$\sigma_{fW}(X_t, T-t, s) = \sum_{j=M+1}^{N} 1_{s\in[j-1,j)} [q_Y'(T-t)]_j S_{ij}(X_t), \tag{109}$$

$$\lambda_Z(X_t) = \sigma_X(X_t)\lambda^B, \text{ and} \tag{110}$$

$$\lambda_W(X_t, s) = \sum_{j=M+1}^{N} 1_{s\in[j-1,j)}\lambda^D(j) [q_Y'(T-t)]_j S_{ij}(X_t), \tag{111}$$

where $q_Y^B$ denotes the $M \times 1$ vector containing the first M elements of $q_Y$. One verifies immediately that the conditions ALV-1 through ALV-6 are satisfied, because all variances, covariances, and drifts of all state variables (both latent variables and forward rates) are linear in the latent variables under both the $P$ and $Q$ measures. The latent variables have the same initial values as the bounded state variables of the affine yield model and, furthermore, have the same drift and diffusion specification. Consequently, the process followed by the latent variables $X_t$ is equivalent to the process followed by the bounded state variables $Y_t^B$ of the affine yield model. Furthermore, one can verify in a straightforward if somewhat tedious way that the drift and diffusion coefficients of forward rates in the ALV-M representation are equivalent to the corresponding coefficients in the affine yield representation. Because both the initial forward rates and their subsequent dynamics are the same in the two models, the process followed by forward rates under the ALV-M model is equivalent to the process followed in the affine yield model. Verification of Eq. (13) follows immediately, because, over any finite time interval, the squared market price of risk is a linear function of the state variables, and the coefficients of that linear function are bounded (because the $q_Y$ function is differentiable, it must also be continuous and therefore bounded on a compact set). By a similar argument, Eqs. (83) and (84) are also satisfied. Because, as discussed in Section 4, these three conditions are the only integrability restrictions that must be verified for an ALV model, I conclude that any completely affine yield model has an ALV-M representation.

The essentially affine yield models of Duffee (2002) cannot be expressed in ALV-M form, as per the above construction, because the market price of risk may depend on all $N$ state variables, instead of the subset of $M$ taken as the latent variables for completely affine yield models. Using the canonical representation of an affine yield model, but extending the market price of risk as per Duffee (2002), I can replace Eq. (95) with

$$\Lambda_t = \begin{bmatrix} \sqrt{S_t^B}\lambda^B \\ \left[\sqrt{S_t^D}\right]^{-1}\left[\lambda_0^D + \lambda_Y^D Y_t^D\right] \end{bmatrix}, \tag{112}$$

where $\lambda_0^D$ is an $(N-M)\times 1$ vector of constants and $\lambda_Y^D$ is an $(N-M)\times M$ matrix of constants. However, every essentially affine yield model has an ALV-N representation.[4] The construction above can be modified to include all $N$ state variables of the affine yield models as latent variables. The specifications of $\sigma_{fZ}$ and $\lambda_Z$ must be extended to include all $N$ variables, but $\sigma_{fW}$ and $\lambda_W$ can be taken to be uniformly zero:

$$X_0 = Y_0, \tag{113}$$

$$f_0(T) = q_0'(T) + \left[q_Y'(T)\right]^T Y_0, \tag{114}$$

$$\mu_X(X_t) = \begin{bmatrix} \kappa^{BB} & 0_{M\times(N-M)} \\ \kappa^{DB} & \kappa^{DD} \end{bmatrix} \left( \begin{bmatrix} \theta^B \\ 0_{(N-M)\times 1} \end{bmatrix} - \begin{bmatrix} Y_t^B \\ Y_t^D \end{bmatrix} \right), \tag{115}$$

[4] Some essentially affine yield models can be expressed as ALV-K models for some $M < K < N$. This is the case when some but not all of the state variables of the model appear in the market price of risk specification. However, such models still have an ALV-N representation, as described in the text.

$$\sigma_X(X_t) = \sqrt{\begin{bmatrix} S_t^{XB} & 0_{M\times(N-M)} \\ 0_{(N-M)\times M} & S_t^{XD} \end{bmatrix}}, \tag{116}$$

$$\sigma_{fZ}(X_t, T-t) = \sigma_X(X_t)\, q_Y'(T-t), \tag{117}$$

$$\sigma_{fW}(X_t, T-t, s) = 0, \tag{118}$$

$$\lambda_Z(X_t) = \begin{bmatrix} \sqrt{S_t^{XB}}\lambda^B \\ \left[\sqrt{S_t^{XD}}\right]^{-1}\left[\lambda_0^D + \lambda_Y^D Y_t^D\right] \end{bmatrix}, \text{ and} \tag{119}$$

$$\lambda_W(X_t, s) = 0, \tag{120}$$

where the functions $S_t^{XB}$ and $S_t^{XD}$ are defined as

$$S_t^{XB} = \begin{bmatrix} X_t(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_t(M) \end{bmatrix} \text{ and} \tag{121}$$

$$S_t^{XD} = \begin{bmatrix} \alpha_{M+1} + \sum_{i=1}^M \beta_{M+1}(i)\, X_t(i) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_N + \sum_{i=1}^M \beta_N(i)\, X_t(i) \end{bmatrix}. \tag{122}$$

As in the completely affine case, one can verify that the process followed by forward rates is identical under the two representations.

An interesting situation arises by taking the ALV-N (or ALV-M, in the completely affine case) representation of an $A_M(N)$ model, but modifying the initial term structure of forward rates. Referring to the ALV-N initial forward rates as $f_0(T)$ and those of the affine yield model as $f_0^A(T)$, consider whether the following condition holds:

$$\lim_{T\uparrow\infty}\left[f_0(T) - f_0^A(T)\right] = 0. \tag{123}$$

First, if, for some affine yield model, this limit converges to a constant different than zero, another affine yield model can be found for which the condition holds. Changing the $d_0$ parameter of affine yield models generates parallel (and permanent) shifts in the yield and forward curves but has no effect on forward rate drifts or volatilities. From Eqs. (97) and (102), changing $d_0$ has the effect of changing all forward rates by a constant. Therefore, the $d_0$ parameter can be adjusted so that it converges to zero.

Second, because the initial forward rates have no effect on the dynamics of either the latent variables or the forward rates, such a model eventually converges to the corresponding affine yield model. Intuitively, the initial term structure of forward rates of the ALV-N model is equal to those implied by some affine yield model, plus a perturbation that dies out with increasingly large maturity. Because the two models have the same drift and volatility of forward rates, this perturbation will die out with the passage of time. Eventually, the maturity date of the longest forward rates with a perturbation larger than some arbitrarily chosen number will be reached and passed, and all remaining forward rates will then have perturbations smaller than this amount. In fact, such convergence is uniform.

It is relatively straightforward to derive similar results for the LV family of models. For example, consider any term structure model in which a finite set of state variables satisfy a time-homogeneous diffusion process. Provided the instantaneous interest rate and the risk premia of the innovations are functions only of this finite set of state variables, an LV representation of this model will exist.

Affine yield models impose strong restrictions on the family of ALV models. Consider examples of ALV models that do not correspond to any affine yield model, starting with $N = 0$. The only completely affine yield model corresponding to this choice is the model of Vasicek (1977), and the essentially affine extension of this model adds a single parameter. By contrast, a wide variety of ALV-0 models can be constructed, even if the model is single factor. Because there are no latent variables, the functions $\mu_X$, $\sigma_X$, $\sigma_{fZ}$, and $\lambda_Z$ are irrelevant. The only requirements that $\sigma_{fW}$ and $\lambda_W$ must satisfy are the integrability requirements of Eqs. (83), (84), and (13). These requirements are all satisfied if, for example, the $\sigma_{fW}$ and $\lambda_W$ are integrable with respect to $s$ and the integrals are continuous functions of maturity (because continuous functions are bounded on compact sets). Suppose the specification of $\sigma_{fW}$ is such that the model has a finite factor decomposition; i.e., an integer $K$ exists such that the rank of the instantaneous covariance matrix of any set of yields is not greater than $K$:

$$\det \begin{bmatrix} c_{YY}(X_t, T_1 - t, T_1 - t) & \cdots & c_{YY}(X_t, T_1 - t, T_M - t) \\ \vdots & \ddots & \vdots \\ c_{YY}(X_t, T_M - t, T_1 - t) & \cdots & c_{YY}(X_t, T_M - t, T_M - t) \end{bmatrix} = 0 \tag{124}$$

for all $M > K$. An ALV-0 model of this type (every LV-0 model is also an ALV-0 model) can also be expressed as a Heath, Jarrow, and Morton (1992) model. An LV-0 model without a finite factor representation is equivalent to a Kennedy (1994) model (except that Kennedy considers the dynamics of forward rates only under a martingale measure). In this case, an infinite set of zero coupon bonds exists such that no zero variance portfolio can be formed from this set (other than the trivial portfolio with zero weight on all bonds).

As the covariances and risk premia of forward rates in an $ALV-0$ model can be specified by an almost arbitrary choice of $\sigma_{fW}$ and $\lambda_W$, the $ALV-0$ class of models includes many models that do not correspond to any affine yield model. Neither the $LV-N$ nor $ALV-N$ family of models requires that the volatility of very long maturity forward rates approach zero. At first, this may seem to contradict the results of Dybvig, Ingersoll, and Ross (1996), who show that arbitrage considerations prohibit the limiting (with increasing maturity) forward rate and limiting zero coupon bond yield from decreasing with positive probability. However, the following model is a valid LV-0 model, according to the definition

$$\begin{aligned} \sigma_{fW}\left(X_t, T-t, s\right) &= \sigma\sqrt{2}e^{-s} \text{ and} \\ \lambda_W\left(X_t, s\right) &= 0. \end{aligned} \tag{125}$$

Given any integrable initial forward curve, one can easily verify that all the integrability restrictions in the definition of an LV model are satisfied. By the arguments of Section 3, the specified forward rate process exists and offers no arbitrage opportunities. However, the instantaneous variance of each forward rate is $\sigma^2$, and the volatility of (annualized) zero coupon bond yield is also $\sigma^2$, regardless of maturity, which is in seeming contradiction of the Dybvig et al. (1996) results, because, if volatility remains positive, any forward rate or yield of any maturity can decline with positive probability. However, if the drift of a forward rate is examined, then

$$df_t\left(T\right) = \sigma^2\left(T-t\right)dt + \sigma dW_t, \tag{126}$$

$$dy_t\left(T\right) = \left[-r_t + \sigma^2\frac{\left(T-t\right)^2}{2}\right]dt + \sigma\left(T-t\right)dW_t, \text{ and} \tag{127}$$

$$d\widehat{y}_t\left(T\right) = d\left[\frac{y_t\left(T\right)}{T-t}\right] = \left[\frac{y_t\left(T\right)}{\left(T-t\right)^2} - \frac{r_t}{T-t} + \sigma^2\frac{\left(T-t\right)}{2}\right]dt + \sigma dW_t, \tag{128}$$

where $W_t$ is the standard Brownian motion defined by

$$W_t = \sqrt{2}\int_0^{+\infty} e^{-s}dW_{s,t} \tag{129}$$

and $\widehat{y}_t\left(T\right)$ denotes an annualized yield. As shown, the drift of both forward rates and annualized yields increases without bound as maturity increases, whereas their volatilities are constant (not dependent on maturity). The probability of a forward rate or yield falling therefore decreases to zero as maturity increases, not because of vanishing volatility, but because of increasingly large drift. LV-0 models therefore conform to the predictions of Dybvig, Ingersoll, and Ross (1996) without necessarily having volatility of long maturity forward rates approach zero. $LV-N$ models with $N>0$ can exhibit similar behavior. However, depending on the latent variables specification, Eq. (14) may impose restrictions on the volatilities of long maturity forward rates.

$LV - N$ models with $N > 0$ can exhibit behavior similar to $LV - 0$ models. Some are equivalent to affine yield models, others are equivalent to Heath, Jarrow, and Morton (1992) models, and still others cannot be expressed as either. However, with latent variables, there can also be stochastic volatility and stochastic risk premium and the possibility of market incompleteness. For example, consider the LV-1 model

$$\mu_X (X_t) = \kappa (\theta - X_t), \tag{130}$$

$$\sigma_X (X_t) = \sqrt{X_t}, \tag{131}$$

$$\sigma_{fZ} (X_t, T - t) = 0, \tag{132}$$

$$\sigma_{fW} (X_t, T - t, s) = \left\{ \begin{array}{cc} \sigma\sqrt{2\rho} e^{-k(T-t)-\rho[s-(T-t)]}\sqrt{X_t} & s \geq T - t \\ 0 & s < T - t \end{array} \right\}, \tag{133}$$

$$\lambda_Z (X_t) = \lambda_0 \sqrt{X_t}, \text{ and} \tag{134}$$

$$\lambda_W (X_t, s) = \lambda_1 e^{-s} \sqrt{X_t}, \tag{135}$$

with the initial value of the single latent variable $X_0$ chosen to be any arbitrary positive number and the initial forward rates $f_0 (T)$ chosen to be any integrable function of $T$. This model is similar to the Kennedy (1997) model discussed in Section 2, except that the forward rate variances and covariances are proportional to the latent variable, instead of constant. For existence of a stationary latent variable process that does not achieve the boundary value, $\kappa > 0$ and $2 \cdot \kappa \cdot \theta \geq 1$ are sufficient. The latent variable process is a square-root process of Feller (1951) (used in term structure modeling by Cox et al. (1985)), and has variance that is finite and known in closed-form:

$$Var [X_{t+\Delta} \mid X_t] = \frac{\theta}{2\kappa} \left(1 - e^{-\kappa\Delta}\right)^2 + \frac{1}{\kappa} e^{-\kappa\Delta} \left(1 - e^{-\kappa\Delta}\right). \tag{136}$$

Because the coefficients on the square root of the latent variable in $\sigma_{fW}$ and $\lambda_W$ are bounded, and they are clearly $L^2$- integrable in expectation (and therefore almost surely), Eqs. (83), (84), and (13) are all satisfied. This model is therefore well defined and free from arbitrage. Like the model of Kennedy (1997), it is infinite dimensional, but each individual forward rate, together with the latent variable, follows a two-dimensional diffusion process. Unlike the Kennedy model, volatility is state dependent. Furthermore, there is volatility risk that cannot be hedged by any portfolio of zero coupon bonds, because any portfolio of bonds is instantaneously uncorrelated with the latent variable (i.e., $\sigma_{fZ}$ is uniformly zero). Derivative instruments such as options, caps, etc., most likely depend on the value (and dynamics) of the latent variable (see Section 6). In particular, the value of $\lambda_0$ does not affect the process followed by bond prices or forward rates at all. However, derivative prices depend on $\lambda_0$, which is the market price of risk of the latent variable. Derivatives are not necessarily redundant assets with respect to zero coupon bonds in this model.

The incompleteness of the market (when only zero coupon bonds are traded) holds even if the model is changed to allow

$$\sigma_{fZ}(X_t, T-t) = q(T-t)\sqrt{X_t} \tag{137}$$

for some function $q$, which must be chosen so that the integrability restrictions continue to hold. Although portfolios of zero coupon bonds can now have nonzero instantaneous covariances with the latent variables, no bond portfolio (except the trivial portfolio with no bonds at all) has $c_{WW}$ equal to zero. Consequently, a derivative asset with price that depends only on the latent variable could be partially hedged with a portfolio of zero coupon bonds. However, because such a portfolio cannot have perfect instantaneous correlation with the derivative security, any such hedge would be less than perfect.

In an LV model, the volatility and risk premia of forward rates must depend only on the latent variables and not directly on quantities such as the instantaneous interest rate or other characteristics of the shape of the term structure. However, volatility and risk premia can still be highly correlated with the interest rate, yields, etc., because these the latent variables and forward rates can covary. However, forward rate risk premia can also evolve essentially independently of other aspects of the term structure. For example, consider the following model:

$$\mu_X(X_t) = \kappa(\theta - X_t), \tag{138}$$

$$\sigma_X(X_t) = 1, \tag{139}$$

$$\sigma_{fZ}(X_t, T-t) = 0, \tag{140}$$

$$\sigma_{fW}(X_t, T-t, s) = \sigma\sqrt{2\rho}e^{-k(T-t)-\rho[s-(T-t)]}, \tag{141}$$

$$\lambda_Z(X_t) = \lambda_0 + \lambda_1 \cdot X_t, \text{ and} \tag{142}$$

$$\lambda_W(X_t, s) = (\lambda_2 + \lambda_3 \cdot X_t)\, e^{-s}. \tag{143}$$

The latent variable can take any arbitrary initial value, and the initial forward rates need only be integrable. Verification of the other integrability restrictions is straightforward, and the only restrictions on any of the model parameters are $k > 0$, $\rho \geq 0$. The latent variable follows an Ornstein-Uhlenbeck type process, used in term structure modeling by Vasicek (1977). Under the equivalent martingale measure specified in Eq. (44), the dynamics of forward rates do not depend on the latent variable at all. In fact, under this measure, this model is equivalent to the model of Kennedy (1997), and inclusion of the latent variable in the model is entirely superfluous if the risk-neutral dynamics are the only concern. However, under the physical measure, forward rates have risk premia that depend on the latent variable, which is completely independent (under the equivalent martingale measure) of the forward rates themselves.

Many of the features exhibited by the examples considered so far can be combined into a single model. For example, consider the ALV-2 model

$$\mu_X(X_t) = \begin{bmatrix} \kappa_{11} & 0 \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \left( \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix} - \begin{bmatrix} X_t(1) \\ X_t(2) \end{bmatrix} \right), \tag{144}$$

$$\sigma_X(X_t) = \begin{bmatrix} \sqrt{X_t(1)} & 0 \\ 0 & \sqrt{1+\beta_2 X_t(1)} \end{bmatrix}, \tag{145}$$

$$\sigma_{fZ}(X_t, T-t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{146}$$

$$\sigma_{fW}(X_t, T-t, s) = \sigma\sqrt{\alpha_0 + \beta_0 X_t(1)}\sqrt{2\rho}e^{-k(T-t)-\rho[s-(T-t)]}, \tag{147}$$

$$\lambda_Z(X_t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and} \tag{148}$$

$$\lambda_W(X_t, s) = (\lambda_0 + \lambda_1 \cdot X_t(2))\, e^{-s}. \tag{149}$$

The integrability restrictions of Eqs. (83), (84), and (13) are readily verified for any values of $\lambda_0$ and $\lambda_1$, with $2 \cdot \kappa_{11} \cdot \theta_1 \geq 1$ and $\alpha_0$, $\beta_0$, $\beta_2$, $\kappa_{11}$, $k$, $\rho > 0$. In this model, the forward rates are driven by infinitely many independent factors; forward rates do not span the innovations to either of the latent variables; and the risk premia of forward rates depend on the second latent variable, which is independent of forward rate dynamics under the equivalent martingale measure.

Many of the examples shown above have a $\sigma_{fW}$ that is based on the density function for an exponential distribution. I now choose a two-parameter family of distributions, the Gaussian family, and use the density functions of these distributions to generate $\sigma_{fW}$ functions in which the correlations between different forward rates can be interpreted geometrically. Consider the square root of the density function of a Gaussian distribution:

$$K(s, \mu, \sigma) = \frac{e^{-\frac{(s-\mu)^2}{4\sigma^2}}}{\sqrt[4]{2\pi\sigma^2}}. \tag{150}$$

If the product of two such kernels, with different values of $\mu$ and $\sigma$, is integrated and the square root of the negative logarithm of the result is taken, then

$$\sqrt{-Log\int_{-\infty}^{+\infty} K(s, \mu_1, \sigma_1)\, K(s, \mu_2, \sigma_2)\, ds} = \sqrt{\begin{array}{l} \frac{(\mu_1-\mu_2)^2}{4\left(\sigma_1^2+\sigma_2^2\right)} \\ +\frac{1}{2}Log\left[\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1\sigma_2}\right] \end{array}}. \tag{151}$$

The right-hand side of this equation defines a norm on the two-dimensional space occupied by all possible values of $\{u, \sigma\}$, with $\sigma > 0$ (note that it is not the usual Euclidean norm). In an ALV-0 model, covariance surfaces can be generated by choosing

$$\sigma_{fW}\left(X_t, T-t, s\right) = \phi\left(T-t\right) K\left(s, \mu_0\left(T-t\right), \sigma_0\left(T-t\right)\right), \tag{152}$$

where $\phi$, $\mu_0$, and $\sigma_0$ are functions subject only to the technical integrability restrictions in the LV model definition. Provided $\mu_0\left(T-t\right)/\sigma_0\left(T-t\right)$ is suffiently large for all values of $T-t$, because the $K$ function in Eq. (150) is truncated for $s < 0$, the instantaneous variance of any forward rate is then approximately

$$c_{ff}\left(X_t, T-t, T-t\right) \approx \phi^2\left(T-t\right). \tag{153}$$

(The approximate equality can be replaced by exact equality if, for example, the state space is extended to include a two-parameter Brownian sheet $W_{s,t}$ defined for all real values of $s$, with $t \geq 0$.) The instantaneous correlation of two forward rates (assuming neither has zero variance) is then approximately

$$\frac{c_{ff}\left(X_t, T_1-t, T_2-t\right)}{\sqrt{\begin{array}{l} c_{ff}\left(X_t, T_1-t, T_1-t\right) \\ \cdot c_{ff}\left(X_t, T_2-t, T_2-t\right)\end{array}}} \approx \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2}} e^{-\frac{\left(\mu_1-\mu_2\right)^2}{4\left(\sigma_1^2+\sigma_2^2\right)}}. \tag{154}$$

This quantity is a convex transformation of the norm defined above. A correlation structure can therefore be defined by choosing essentially arbitrary functions $\mu_0\left(T-t\right)$ and $\sigma_0\left(T-t\right)$. Two forward rates for which these two functions are close will have a high correlation, but when either or both of the two functions are far apart, the two forward rates will have a low correlation. Correlation structures can therefore be thought of as a curve through a two-dimensional space, providing somewhat more intuition than the formulaic approach of specifying $\sigma_{fW}$ directly. Each forward rate corresponds to a point on the curve through the two-dimensional space, and the correlation between two forward rates is related to their distance.

For ALV-N models, the approach can be extended:

$$\begin{aligned}\sigma_{fW}\left(X_t, T-t, s\right) = [&\phi\left(X_t, T-t\right) \\ &\cdot K\left(s, \mu_X\left(X_t, T-t\right), \sigma_X\left(X_t, T-t\right)\right)].\end{aligned} \tag{155}$$

With this approach, the distance between two forward rates can change as the values of the latent variables change. Using this technique, one can easily construct models in which, for example, one latent variable governs the volatility of the forward curve, whereas a second latent variable governs the correlations between different forward rates.

LV models can therefore exhibit several types of interesting behavior. While encompassing many traditional models as special cases, other LV models are similar to the models of Heath, Jarrow, and Morton (1992). Some LV models exhibit infinite-factor behavior and can be consistent with both the current cross-sectional shape of the term structure and the series of innovations needed to match its behavior over time. Volatility does not have to be a function only of zero coupon bond yields, but can also depend on additional latent variables, so that the set of bonds does not complete the market. Finally, innovations to state-dependent risk premia can be correlated with innovations to forward rates, but they can also be independent of term structure innovations. [5]

## 6 Derivatives pricing

The family of LV models has the unusual property that the prices of term structure derivative instruments cannot always be expressed as functions only of the prices of zero coupon bonds. Pricing and hedging of derivative securities may therefore require as many as $N$ other derivatives, in addition to zero coupon bonds. However, provided the time series of sufficiently many derivative prices is observed to infer the market price of risk and current values of the latent variables, finding prices of other derivatives is relatively straightforward.

If the prices of derivative instruments are not observed, then the dynamics of the latent variables still can be estimated from the time series of zero coupon bond yields. In principle, the current values of the latent variables could also be estimated using filtering techniques, although whether such techniques can be implemented successfully in practice remains an open issue. However, the dynamics estimated from time series behavior are under the physical measure, not the equivalent martingale measure. Some knowledge of investor preferences is therefore required to identify the equivalent martingale measure uniquely and calculate derivative prices. Although this is certainly a reasonable field of enquiry, derivatives pricing by equilibrium or agent preference arguments is left for future study. Throughout the remainder, I assume that time series observations of sufficiently many derivative instruments have been made to identify the equivalent martingale measure uniquely and that these prices have been used to determine the current values of the latent variables.

If term structure derivatives are to be priced in an analytically simple fashion, two key conditions must be satisfied. First, the payoff of the derivative must be first-order homogeneous in some set of bond prices. This requirement is satisfied trivially for derivatives with European-style exercise. Second, the

[5] Collin-Dufresne and Goldstein (2001) show that traditional finite-factor term structure models can sometimes exhibit this type of behavior as well.

specification of volatility must be such that the price of the derivative inherits the homogeneity of its payoff. LV-N models satisfy this condition. However, models in which yield or forward volatilities depend on the level or shape of the term structure generally fail to do so.

In general, the evolution of the term structure under an LV-N model can depend on the entire history of the evolution of the latent variables, or even on infinitely many underlying factors. However, a consequence of first-order homogeneity is that prices of term structure derivatives can be expressed as solutions to finite-dimensional partial differential equations. In particular, European-style derivatives with payoffs that are functions of $K$ zero coupon bond prices satisfy an $(N+K+1)$-dimensional differential equation.

If a derivative with price process $G_t$ has a time $T$ payoff that is a function only of the time $T$ values of cash flows at times $T_1, ..., T_K$, with $T < T_i$, $1 \leq i \leq K$ and the time $T$ values of the latent variables, the payoff of the derivative in terms of forward rates instead of bond prices is

$$\begin{aligned} G_T(X_T, B_T(T), B_T(T_1), \\ \dots, B_T(T_K)) = B_T(T)\, G_T(X_T, e^{-F_T(T,T)}, e^{-F_T(T,T_1)}, \\ \dots, e^{-F_T(T,T_K)}). \end{aligned} \tag{156}$$

The derivative payoff can be rewritten in the form shown on the right-hand side because the price at time $T$ of a zero coupon bond with maturity at time $T$ is known with certainty to be equal to one. European-style derivatives thus trivially have first-order homogeneous payoffs. For any $LV-0$ model, the derivative price prior to time $t$ inherits the homogeneity the final payoff and can be expressed as

$$\begin{aligned} G_t(X_t, B_t(T), B_t(T_1), \\ \dots, B_t(T_K)) = B_t(T)\, h(T-t, X_t, F_t(T,T_1), \\ \dots, F_t(T,T_K)), \end{aligned} \tag{157}$$

with boundary condition

$$\begin{aligned} h(0, X_T, F_T(T,T_1), \\ \dots, F_T(T,T_K)) = G_T(X_T, e^{-F_T(T,T)}, e^{-F_T(T,T_1)}, \\ \dots, e^{-F_T(T,T_K)}). \end{aligned} \tag{158}$$

To see this, the Feynman-Kac partial differential equation is applied to this expression for the option price:

$$
\begin{aligned}
&\{B_t(T)\tfrac{\partial h}{\partial t} + rB_t(T)h + B_t(T)\textstyle\sum_{i=1}^{N}\tfrac{\partial h}{\partial X_t(i)}\left[\mu_X^Q(X_t)\right]_i \\
&+ B_t(T)\textstyle\sum_{i=1}^{K}\tfrac{\partial h}{\partial F_t(T,T_i)}\left[\mu_F^Q(X_t,T_i-t) - \mu_F^Q(X_t,T-t)\right] \\
&+ \tfrac{1}{2}B_t(T)\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N}\tfrac{\partial^2 h}{\partial X_t(i)\partial X_t(j)}\left[c_{XX}(X_t)\right]_{ij} \\
&+ B_t(T)\textstyle\sum_{i=1}^{N}\sum_{j=1}^{K}\tfrac{\partial^2 h}{\partial X_t(i)\partial F_t(T,T_j)}\left[c_{XF}(X_t,T-t,T_j-t)\right]_i \\
&+ \tfrac{1}{2}B_t(T)\textstyle\sum_{i=1}^{K}\sum_{j=1}^{K}\tfrac{\partial^2 h}{\partial F_t(T,T_i)\partial F_t(T,T_j)}c_{FF}(X_t,T-t,T_i-t, \\
&\qquad\qquad\qquad\qquad\qquad\qquad T-t,T_j-t) \\
&- B_t(T)\textstyle\sum_{i=1}^{N}\tfrac{\partial h}{\partial X_t(i)}\left[c_{XY}(X_t,T-t)\right]_i \\
&- B_t(T)\textstyle\sum_{i=1}^{M}\tfrac{\partial h}{\partial F_t(T,T_i)}\left[c_{YY}(X_t,T-t,T_i-t)\right. \\
&\qquad\qquad\left. - c_{YY}(X_t,T-t,T-t)\right] - rB_t(T)h\} = 0.
\end{aligned}
\tag{159}
$$

The terms containing the instantaneous interest rate cancel, and, after dividing both sides by $B_t(T)$, the equation can be written as

$$
\begin{aligned}
&\{\tfrac{\partial h}{\partial t} + \textstyle\sum_{i=1}^{N}\tfrac{\partial h}{\partial X_t(i)}\left[\mu_X^Q(X_t) - c_{XY}(X_t,T-t)\right]_i \\
&+ \textstyle\sum_{i=1}^{K}\tfrac{\partial h}{\partial F_t(T,T_i)}c_{FF}(X_t,T-t,T_i-t) \\
&+ \tfrac{1}{2}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N}\tfrac{\partial^2 h}{\partial X_t(i)\partial X_t(j)}\left[c_{XX}(X_t)\right]_{ij} \\
&+ \textstyle\sum_{i=1}^{N}\sum_{j=1}^{K}\tfrac{\partial^2 h}{\partial X_t(i)\partial F_t(T,T_j)}\left[c_{XF}(X_t,T-t,T_j-t)\right]_i \\
&+ \tfrac{1}{2}\textstyle\sum_{i=1}^{K}\sum_{j=1}^{K}\tfrac{\partial^2 h}{\partial F_t(T,T_i)\partial F_t(T,T_j)}c_{FF}(X_t,T-t,T_i-t, \\
&\qquad\qquad\qquad\qquad\qquad\qquad T-t,T_j-t)\} = 0.
\end{aligned}
\tag{160}
$$

Inspection of this differential equation shows that the coefficients of the partial derivatives depend only on the latent variables, because the instantaneous interest rate does not appear anywhere. The price of a European derivative in an LV model therefore can be expressed without direct reference to the interest rate. For some models, the value of the instantaneous interest rate can be inferred from the value of the latent variables and forward rates. For example, from Section 5, the model of Cox, Ingersoll, and Ross (1985) can be expressed as an ALV-1 model. In this case, the derivative price can be rewritten to include dependency on the interest rate. However, for other models, one cannot necessarily determine the value of the instantaneous interest rate from the latent variables and the forward rates. In this case, one can nonetheless price European derivatives from Eq. (160), which is $(N+K+1)-$dimensional. Consider an LV model in which the instantaneous interest rate follows an infinite-dimensional process (see Section 5 for examples). For an option on a zero coupon bond, $K=1$, so an $(N+2)-$dimensional partial differential equation must be solved to find its price, despite the infinite-dimensional interest rate process. Because affine yield models are special cases of ALV models, the same result applies. In an $A_M(N)$ affine yield model, the price of an option on a zero coupon bond solves an $(M+2)-$dimensional partial differential equation. Completely affine yield $A_M(N)$ models can be expressed in $ALV-M$ form, whereas essentially affine yield $A_M(N)$ models can be expressed only as $ALV-J$ models for some $J>M$. However, the solution to the above differential equation depends only on $M$ of the latent variables. This finding is intuitively appealing, because the only reason for including more than $M$ latent variables is the dependence of risk premia on these variables. Because derivatives pricing takes place under an equivalent martingale measure, these are irrelevant.

For random field models other than $LV-N$ models, it may not be possible to derive a finite-dimensional differential equation. For example, if the variances and covariances of forward rates were to depend on the instantaneous interest rate, the terms containing the interest rate would not cancel in the above differential equation, which would have no solution. In this case, the derivative price fails to inherit the first-order homogeneity of its payoff and depends explicitly on the instantaneous interest rate, which in general follows a complex and infinite-dimensional process. Kennedy (1994) specifies forward rate variances and covariances as deterministic functions of maturity and uses an approach very similar to mine. Goldstein (2000) has forward rate variances and covariances depend on the instantaneous interest rate and must take a completely different derivative pricing approach, using path integrals instead.

A few additional points are worth noting. First, Eq. (160) resembles the Kolmogorov backward equation for conditional expectations, but the coefficients on the first derivatives are not the drifts under either the physical or equivalent martingale measure. These drifts are in fact those obtained under a change of measure in which the zero coupon bond $B_t(T)$ is taken as the numeraire. Second, in the case of ALV models, the partial differential equation is linear in the latent variables (provided the final payoff of the derivative does not depend nonlinearly on the latent variables) and can be decomposed into a system of ordinary differential equations. Such a system of equations can be solved by numeric techniques far more rapidly than a generic nonlinear partial differential equation with the same number of dimensions. Consequently, ALV models share the analytically attractive properties of affine yield models with respect to derivative pricing. The only additional complexity that arises in the ALV case is the explicit time-dependency of the coefficients on the partial derivatives in Eq. (160), and commonly used numeric techniques readily adapt to this case.

Thus, when the dimensionality of the pricing equation (i.e., $N+M+1$) is relatively small, as would be the case, for example, with an LV-1 model when the underlying asset is a zero coupon bond, solutions to the equation can be found quickly with numeric approximation techniques. When the number of variables is large, either because of a large number of latent variables or because the underlying asset has many cash flows (e.g., a coupon bond), Monte Carlo simulation is probably a better technique. Duffie and Kan (1996) consider numeric techniques for valuing derivatives under an affine yield model, which can readily be adapted for ALV-N models. Santa-Clara and Sornette (2001) use Monte Carlo simulation for a model that is equivalent to an ALV-0 model, to value the "cheapest to deliver" option in the Chicago Board of Trade Treasury Bond Futures contract.

Pricing of American exercise options is far more complicated under $LV - N$ models, as it is under most models. In particular, the payoff of such an option is not first-order homogeneous in a finite set of bond prices, because the option may be exercised at any time. Therefore, no particular reason exists to expect this property from the price prior to expiry. In general, pricing American options requires modeling the entire term structure, which may require infinitely many factors. Monte Carlo simulation techniques (with approximate exercise decision rules) can perhaps be adapted to the LV-N case. Such techniques for other models are examined in, for example, Bossaerts (1989), Carr (1998), and Longstaff and Schwartz (2001).

## 7 Conclusion

The original Kennedy (1994) random field model has several advantages, namely the ability to match both the observed cross-section of bond prices and the time series of innovations to those prices, with each forward rate following a simple diffusion process. Most methods for introducing conditional volatility lose this property, requiring forward rates to follow a complex, infinite-dimensional diffusion, so that is it difficult even to verify existence conditions for the proposed models. I therefore propose a new method for introducing conditional volatility, the family of LV models, such that each forward rate remains part of a low-dimensional diffusion. Verifying existence of a unique forward rate process, as well as other tasks, such as derivatives pricing, are thereby vastly simplified. Within the general class of LV models, I specify requirements for an ALV model, which is particularly simple to analyze analytically. I proceed to describe methods for constructing such models and specify a market price of risk process that allows extension of this family of models from a martingale measure to a physical measure. After describing the types of behavior that different versions of the model can exhibit, I consider methods by which an ALV model can be estimated from discrete observations of the term structure and derive the conditional and unconditional moment expressions needed for a method of moments estimation procedure. I go on to develop a method of derivative pricing that is not much more complicated (if at all) than that commonly used for traditional term structure models. Finally, I examine several derivative pricing examples to determine when use of an accurate volatility specification is likely to be important.

Much remains to be done in this field. For example, the latent variables in an ALV model follow a joint affine diffusion. Although the general LV model allows nonlinear diffusions, I have not examined any such models in detail. Other generalizations might include allowing jumps in either the prices of bonds or the values of the latent variables (or both). However, issues regarding absence of arbitrage in the presence of infinitely many assets with discontinuous sample paths, as per Artzner and Heath (1995), must be addressed in this case. Finally, my model is purely econometric. It remains to discover the economic processes of production, consumption, investment, and inflation that generate an LV-like process.

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