Optimal Portfolio Selection with Transaction Costs and "Event Risk" *

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Abstract

In this paper we consider the optimal trading strategy for an investor with an exponentially distributed horizon who invests in a riskless asset and a risky asset. The risky asset is subject to proportional transaction costs and its price follows a jump diffusion. In this situation, the optimal trading strategy is to maintain the fraction of the dollar amount invested in the riskless asset to the dollar amount invested in the risky asset in between two bounds. In contrast to the pure diffusion setting where the investor faces no jump risk, this fraction can jump discontinuously outside the bounds which is optimally followed by a transaction to the boundary. We characterize the value function and provide bounds on the trading boundaries. Our numerical results show the introduction of jumps ("event risk") dramatically affect the optimal transaction strategy. In particular jumps tend to reduce the amount of stock the investor holds and increase the width of the no transaction region. We also show that the boundaries are affected not only by the size of the jump but can be very sensitive to the uncertainty in the jump size. We also examine how the optimal transaction boundaries vary through time for investors with deterministic horizons by looking at the optimal policies for investors with Erlang distributed horizons, which has been shown to provide good approximations to the deterministic horizon optimal policies.

1. Introduction

In this paper we consider the optimal trading strategy for an investor who faces proportional transaction costs in a model with a riskless asset and a risky asset for which the price follows a jump diffusion. It is commonly accepted that investors face "event risk," the risk of a large change in asset prices which cannot be offset by the type of continuous trade postulated in a diffusion model. It is also a fact that investors face costs of transacting. While the former has been studied (for example Liu, Longstaff and Pan(2003)) in a setting with no transactions costs and the latter has been studied in a pure diffusion setting (for example Constantinides(1986), Davis and Norman(1990), Dumas and Luciano(1991), Shreve and Soner(1994) and Liu and Loewenstein(2002)) little is known about how an investor should optimally transact in the presence of both transactions costs and event risk. Framstad, Oksendal, and Sulem (2001) consider an infinite horizon consumption investment problem in a jump diffusion setting. They provide useful properties of the value function and and optimal trading strategies but do not provide numerical results.

We work in a setting where the investor maximizes expected terminal utility of wealth when his horizon is given at the first jump of a poission process as in Liu and Loewenstein(2002). This has the advantage of making the optimal trading strategy independent of time which greatly simplifies the analysis. We show in a later section that our results can give a reasonably accurate description of the optimal trading behavior for an investor with a deterministic horizon as in Liu and Loewenstein(2002). In contrast to Liu and Loewenstein(2002), the Hamilton Jacobi Bellman equation includes a jump term which leads to a function differential equation which must be solved to satisfy two free boundary conditions. Remarkably, we are able to solve for the value function as a sequence of solutions to ordinary differential equations. As in the pure diffusion case with no jumps, the no-transaction region is characterized by two boundaries in which the investor always transacts so as to maintain the ratio of the dollar amount in the riskless asset to the dollar amount in the risky asset in these boundaries. In contrast to the pure diffusion setting, however, this ratio can jump outside these boundaries which is optimally followed by a transaction to the boundary. We characterize the value function in the jump diffusion setting and provide comparative statics and analytical bounds on the value function and optimal trading boundaries which are new in this setting.

Our numerical results indicate that the presence of jumps can dramatically affect the optimal transaction strategy. First, jumps tend to reduce the amount of the risky asset held by the investor. Second, because of the possibility that the optimal fraction of riskless asset to risky asset value can jump discontinuously, the investor optimally widens the no-transaction region to offset increased transactions costs resulting from this possibility. We also show that the introduction of jumps with uncertain jump sizes makes the optimal policy more sensitive to the jump arrival rate. Finally, we show how one can generalize our results to understand the optimal trading behavior for an investor with a deterministic horizon.

Many of our theoretical results should be easily generalized to the infinite horizon consumption investment problem in a jump diffusion model. In particular, the bounds on the transaction boundaries should generalize to this setting.

2. The Basic Model

2.1 The Asset Market

Throughout this paper we are assuming a probability space $(\Omega, \mathcal{F}, P)$. Uncertainty and the filtration $\{\mathcal{F}_t\}$ in the model are generated by a standard one dimensional Brownian motion w and two Poisson processes defined below. We will assume that these processes are adapted.

There are two assets our investor can trade. The first asset ("the bond") is a money market account growing at a continuously compounded, constant rate r. The second asset ("the stock") is a risky investment. The investor can buy the stock at the ask price $S_t^A = (1+\theta)S_t$ and sell the stock at the bid price $S_t^B = (1-\alpha)S_t$, where $\theta \geq 0$ and $0 \leq \alpha < 1$ represent the proportional transaction cost rates and S_t follows the process

$$dS_t = (\mu - \eta E[J_t])S_{t-}dt + \sigma S_{t-}dw_t + J_t S_{t-}dN_t, \tag{1}$$

where w is a one-dimensional Brownian motion and N is an independent Poisson process with intensity $\eta > 0$. The parameters μ, r and σ are strictly positive constants with $\mu > r$. J_t is the time t realization of the random jump size J with $J > -1$ and for all $\varepsilon > 0$, $Prob\{J \leq \varepsilon - 1\} > 0$.

When $\alpha + \theta > 0$, the above model gives rise to equations governing the evolution of the amount invested in the bond, x_t, and the amount invested in the stock, y_t:

$$dx_t = rx_{t-}dt - (1+\theta)dI_t + (1-\alpha)dD_t, \tag{2}$$

$$dy_t = (\mu - \eta E[J_t])y_{t-}dt + \sigma y_{t-}dw_t + J_t y_{t-}dN_t + dI_t - dD_t, \tag{3}$$

where the processes D and I represent the cumulative dollar amount of sales and purchases of the stock, respectively. These processes are nondecreasing, right continuous adapted processes with $D(0) = I(0) = 0$. Let x_0 and y_0 be the given initial positions in the bond and the stock respectively. We let $\Theta(x_0, y_0)$ denote the set of admissible trading strategies (D, I) such that (2) and (3) are satisfied and the investor is always solvent, i.e.,[1]

$$x_t + (1-\alpha)y_t \geq 0, \forall t \geq 0. \tag{4}$$

The solvency requirement in the presence of jumps also requires

$$x_t + (1-\alpha)y_t(1+J_t) \geq 0, \forall t \geq 0. \tag{5}$$

which, as in Liu, Longstaff, and Pan(2003), restricts the fraction $\frac{x}{y}$.

2.2 The Investor's Problem

The investor's problem is to choose admissible trading strategies D and I so as to maximize $E[u(x_\tau + (1-\alpha)y_\tau)]$ for an event which occurs at the first jump time τ of a standard, independent Poisson process with intensity λ. τ is thus exponentially distributed with parameter λ, i.e.,

$$P\{\tau \in dt\} = \lambda e^{-\lambda t} dt.$$

This model captures bequest, accidents, retirement, and many other events that happen on uncertain dates.

If τ is interpreted to represent the investor's uncertain lifetime (as in Merton (1971) and Richard (1975)), the investor's average lifetime is then $1/\lambda$ and the variance of his lifetime is accordingly $1/\lambda^2$.

Assuming a constant relative risk averse preference (CRRA), we can then write the value function as

$$v(x,y) = \sup_{(D,I)\in\Theta(x,y)} E[\frac{\left(x_\tau + (1-\alpha)y_\tau\right)^{1-\gamma}}{1-\gamma}]. \tag{6}$$

[1] Since $\mu > r$ the investor optimally does not short the stock so $y \geq 0$.

In light of our assumptions on τ and the asset market, this can be rewritten as (see Merton (1971), Liu and Loewenstein (2002))

$$v(x,y) = \sup_{(D,I)\in\Theta(x,y)} \lambda E[\int_0^\infty e^{-\lambda t}\frac{(x_t + (1-\alpha)y_t)^{1-\gamma}}{1-\gamma}dt]. \tag{7}$$

2.3 Optimal Policies with No Transaction Costs

For purpose of comparison, let us first consider the case without transaction costs (i.e., $\alpha = \theta = 0$). Define the total wealth $W_t = x_t + y_t$ and let π be the fraction of wealth invested in the stock. In this case, the investor's problem becomes

$$v(x,y) = \sup_{\{\pi_t : t\geq 0\}} \lambda E[\int_0^\infty e^{-\lambda t}\frac{(x_t + y_t)^{1-\gamma}}{1-\gamma}dt],$$

subject to the self financing condition

$$dW_t = (r + \pi_t(\mu - r - \eta E[J]))W_{t-}dt + \pi_t \sigma W_{t-}dw_t + \pi_t J_t W_{t-} dN_t. \tag{8}$$

The above problem is formally similar to the one studied by Merton (1971) and Liu, Longstaff, and Pan (2003). As in these papers, conditions on the parameters and the jump distribution are required for the existence of the optimal solution. Define

$$\rho(\pi) = r + \pi(\mu - r - \eta E[J]) - \frac{\gamma}{2}\pi^2\sigma^2 + \eta E\left[\frac{(1+\pi J)^{1-\gamma}}{1-\gamma}\right]. \tag{9}$$

It can be easily shown that $\rho(\pi)$ is strictly concave in π. Let π^* be the unique maximizer of $\rho(\pi)$ subject to $\pi \leq 1$, i.e.,

$$\pi^* = \arg\max_{\pi\leq 1} \rho(\pi). \tag{10}$$

Assumption 1 $\lambda + \eta > (1-\gamma)\rho(\pi^*)$.

Assumption 1 is necessary to rule out the case where the investor can achieve bliss levels of utility and assures the existence of an optimal portfolio. We summarize the main result for this case of no transaction costs without proof in the following lemma. Notice the optimal portfolio in this case is independent of the investor horizon.

Lemma 1 *Suppose that $\alpha = \theta = 0$. Then under Assumption 1, for $0 \leq t < \tau$ the optimal stock investment policy π_t^* is equal to π^*. Moreover, the lifetime expected utility is*

$$v(x, y) = \frac{\lambda}{\lambda + \eta - \rho(\pi^*)} \frac{(x+y)^{1-\gamma}}{1-\gamma},$$

where $\rho(\pi)$ is defined in Equation (9).

Remark 1 *Since for all $\varepsilon > 0$, $Prob\{J \leq \varepsilon - 1\} > 0$, the investor never leverages (i.e., $\pi_t^* \leq 1$). It is then straightforward to show that $\pi_t^* = \pi^* = 1$ if and only if*

$$\frac{\mu - r}{\gamma\sigma^2} \geq 1 + \frac{\eta(E[J] - E\left[(1+J)^{-\gamma} J\right])}{\gamma\sigma^2}. \tag{11}$$

In our later analysis of the case with transaction costs, the case with $\pi^* = 1$ leads to interesting conclusions.

2.4 Optimal Policies with Transaction Costs

Suppose now that $\alpha + \theta > 0$. As in Liu and Lowenstein (2002), the value function is homogeneous of degree $1 - \gamma$ in (x, y). This implies that

$$v(x, y) = y^{1-\gamma} \psi\left(\frac{x}{y}\right) \tag{12}$$

for some concave function $\psi : (\alpha - 1, \infty] \to \mathbb{R}$.

In addition, the *solvency region* splits into three regions: Buy region, Sell region and No-Transaction (NT) region. Because of the time homogeneity of the value function, these regions can be identified by two critical numbers (instead of functions of time) r_s and r_b. The Buy region corresponds to $z \geq r_b$, the Sell region to $z \leq r_s$, and the No-Transaction region to $r_s < z < r_b$, where $z = \frac{x}{y}$. However, in contrast to the pure diffusion cases previously studied, the fraction $\frac{x}{y}$ can now jump outside the NT region, which is followed by an immediate transaction to the closest boundary of the NT region.

Under regularity conditions on v, we have the following Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE):

$$\frac{1}{2}\sigma^2 y^2 v_{yy} + rxv_x + (\mu - \eta E[J])yv_y - (\lambda+\eta)v + \eta E[v(x, y(1+J))] + \frac{\lambda(x+(1-\alpha)y)^{1-\gamma}}{1-\gamma} = 0, \tag{13}$$

in the NT region, with the associated conditions

$$(1+\theta)v_x = v_y \tag{14}$$

in the Buy region, and

$$(1-\alpha)v_x = v_y \tag{15}$$

in the Sell region.

Using (12), we can simplify the PDE in (13) to get the following not-so-ordinary differential equation in the NT region:

$$z^2\psi_{zz}(z) + \beta_2 z\psi_z(z) + \beta_1\psi(z) + \frac{2\eta}{\sigma^2}E\left[\psi(\frac{z}{1+J})(1+J)^{1-\gamma}\right] + \beta_0\frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} = 0, \tag{16}$$

where $\beta_2 = 2(\gamma\sigma^2 - (\mu - \eta E[J] - r))/\sigma^2$, $\beta_1 = -2(\lambda + (1-\gamma)(\gamma\sigma^2/2 - \mu + \eta E[J]))/\sigma^2 - 2\eta/\sigma^2$ and $\beta_0 = 2\lambda/\sigma^2$. The associated boundary conditions (14)–(15) are transformed into

$$(z+1+\theta)\psi_z(z) = (1-\gamma)\psi(z)$$

for all $z \geq r_b$ and

$$(z+1-\alpha)\psi_z(z) = (1-\gamma)\psi(z)$$

for all $z \leq r_s$.

One can prove

Theorem 1 *There exist constants A, B, r_s, r_b, and a function ψ such that*

1. *$\psi(z)$ is a C^2 function on $(0, \infty)$;*

2. *$\psi(z)$ satisfies the following: if $r_b = \infty$, then*

$$\lim_{y \to 0, x > 0} y^{1-\gamma} \psi(\frac{x}{y}) = \frac{\lambda}{\lambda - (1-\gamma)r} \frac{x^{1-\gamma}}{1-\gamma} \tag{17}$$

and if $r_s = 0$, then

$$\lim_{x \to 0} y^{1-\gamma} \psi(\frac{x}{y}) = \frac{\lambda}{\eta + \lambda - \rho(1)} \frac{((1-\alpha)y)^{1-\gamma}}{1-\gamma}; \tag{18}$$

3. *$\forall z \geq r_b$, $\psi(z) = \frac{A}{1-\gamma}(z + (1+\theta))^{1-\gamma}$;*

4. *$\forall z \leq r_s$, $\psi(z) = \frac{B}{1-\gamma}(z + (1-\alpha))^{1-\gamma}$;*

5. *$\psi(z)$ solves equation 16 for $r_s \wedge 0 \leq z \leq r_b \wedge 0$ and $r_s \vee 0 \leq z \leq r_b \vee 0$;*

6. *$v(x, y) = y^{1-\gamma} \psi(\frac{x}{y})$ is the value function.*

Moreover, the optimal transaction policy is to transact the minimal amount in order to maintain z between r_s and r_b.

Proof of Theorem 1. This follows from Theorem 2. Note that the value function may not be C^2 at $z = 0$ if $r_s = 0$. This does not create any conceptual difficulty since it is impossible to reach the axis from any point with $z > 0$. Moreover, if $r_s = 0$ and we start with a portfolio consisting of only the risky asset, we will never transact until the terminal event happens. $\square$

Given the existence of an optimal trading strategy, we now examine bounds on the transaction boundaries and value function. We also derive some useful comparative statics.

3. Boundary Behavior

Our first result identifies conditions under which the investor never buys stock. In fact this is the same as the condition given in Liu and Loewenstein (2002). The reason for this is intuitive. The jump, as we have modeled it, serves to increase the variance and other moments, but not the mean of the return. Since the condition in Liu and Loewenstein (2002) only concerns the expected return, but not the other moments it still holds.

Proposition 1 *Suppose* $0 < \alpha < 1$*. A necessary and sufficient condition for* r_b *to be infinite is*

$$\mu - r \leq \left(1 - \frac{1-\alpha}{1+\theta}\right)(\lambda - (1-\gamma)r) \tag{19}$$

Proof. Same as Liu and Loewenstein (2002). □

In the no transaction cost case, or in the infinite horizon transaction cost analysis of Davis and Norman (1990), Shreve and Soner (1994), and Dumas and Luciano (1991), the investor always optimally buys some of the risky asset if and only if $\mu - r > 0$. In contrast, with transactions costs, the above result says that if the investor does not expect to live long (i.e. λ is large), or the transaction cost rate α is high, or the investor is highly risk averse (i.e., γ is large), or the risk premium is low, then the investor will never buy the stock, even when the risk premium is positive. Thus, as opposed to the frictionless case, the trading strategy is now clearly horizon dependent.

While the above condition involves the expected lifetime $1/\lambda$, the following propositions provide bounds on the transaction boundaries which are independent of λ.

Proposition 2 *For* $\eta > 0$*, we have*

$$r_s \geq \max\left(0, \frac{\gamma\sigma^2(1-\alpha)}{2(\mu - r)} - (1-\alpha)\right). \tag{20}$$

PROOF OF PROPOSITION 2. See Appendix. □

Proposition 3 *We have the following bounds on the boundaries of the NT region, r_s and r_b:*

$$0 \leq r_s \leq (1-\alpha)r^*, \tag{21}$$

where r^* is the optimal ratio of $\frac{x}{y}$ in the no transaction cost case given by $r^* = \frac{1}{\pi^*} - 1$ and π^* is determined by the solution to (10). Moreover,

$$r_b \geq (1+\theta)r^*. \tag{22}$$

PROOF OF PROPOSITION 3. See Appendix. □

The previous propositions give useful information on the location of the transaction boundaries. In addition, if the optimal portfolio π^* without transaction costs is equal to 1, then the sell boundary $r_s = 0$ in the presence of transaction costs. As a result, if the investor starts with a leveraged portfolio then the optimal strategy is to immediately reduce leverage so the entire portfolio consists only of stock and hold until the terminal event. On the other hand, if the investor starts with all cash, the optimal strategy is to buy some stock and also hold some cash. Thereafter, the sell boundary is never hit and the investor only transacts at the buy boundary until the terminal event when he sells the stock. This result is interesting especially in the case where $\pi^* = 1$. [2] In this case, an increase of transaction cost rate may *increase* the trading frequency. For example, if there is no transaction cost and the investor starts with some cash, then the investor trades once time 0 to achieve 100% stock holding and once at the terminal event. However, when there is transaction cost, the

[2]For example, we consider later the case where the jump size satisfies $J_t = e^{(\mu_J + \sigma_J \nu_t)} - 1$, where ν_t is an independent standard normal random variable. As noted in Section 2.3, if for example $\frac{\mu - r - \eta E[J]}{\gamma\sigma^2} > 1$ then the optimal portfolio is $\pi^* = 1$.

investor will also buy stock from time to time. While a similar result obtains in the pure diffusion case, it only occurs when $\mu - r$ is exactly equal to $\gamma\sigma^2$, i.e., a measure zero event. In contrast, in the presence of jumps, this occurs as long as $\pi^* = 1$, i.e., as long as condition (11) holds, which is a positive measure event.

We also note some bounds on the value function which will are used in the proof of Proposition 3.

Lemma 2 *1. For all ζ with $1 - \alpha \leq \zeta \leq 1 + \theta$, we have the following bounds.*

$$\frac{\lambda}{\lambda - (1-\gamma)r} \frac{(x + (1-\alpha)y)^{1-\gamma}}{1-\gamma} \leq v(x, y) \leq \frac{\lambda}{\lambda + \eta - \rho(\pi^*)} \frac{(x + \zeta y)^{1-\gamma}}{1-\gamma}. \quad (23)$$

The right hand inequality is strict if $\pi^* \neq 1$.

2. For $r_s < \frac{x}{y} < r_b$

$$v(x, y) \geq \frac{B}{1-\gamma}(x + (1-\alpha)y)^{1-\gamma}, \quad (24)$$

and

$$v(x, y) \geq \frac{A}{1-\gamma}(x + (1+\theta)y)^{1-\gamma}. \quad (25)$$

Proof of Lemma 2. The left hand inequality in (23) follows from the fact that the investor must be at least as well off as liquidating and investing only in the riskless asset. The right hand inequality follows from Lemma 3 in the Appendix (see also Shreve and Soner (1994) Proposition 9.9). Statement 2 follows from optimality. □

In proving the above results, we employed Lemma 3 which is given in the Appendix. The same Lemma also yields interesting comparative statics.

Perhaps the most interesting comparative statics concern those related to the jump parameters. In particular, an investor has lower utility when more frequent jumps are introduced into the model. Moreover, for a deterministic jump size, the investor has lower utility for both positive and negative jumps. This follows from the fact that the jumps as we have modeled them do not affect the expected return but do affect the higher moments of the stock return distribution. While one might expect that positive jumps to benefit the investor by giving a positive skew to the return distribution, they also increase the variance of returns and this effect makes the investor worse off. In addition, the investor is worse off in a model when the jump distribution is subjected to a mean preserving spread. These ideas are useful to fix intuition for our later numerical results.

Proposition 4 *We have the following comparative statics (holding all else equal with the slight abuse of notation).*

1. *If* $\alpha_1 < \alpha_2$ *then* $v(x, y, \alpha_1) \geq v(x, y, \alpha_2)$.
2. *If* $\sigma_1 > \sigma_2$ *then* $v(x, y, \sigma_1) \leq v(x, y, \sigma_2)$.
3. *If* $\mu_1 > \mu_2$ *then* $v(x, y, \mu_1) \geq v(x, y, \mu_2)$.
4. *If* $\eta_1 > \eta_2$ *then* $v(x, y, \eta_1) \leq v(x, y, \eta_2)$.
5. *If* $J_1 \succ_{SSD} J_2$ *then* $v(x, y, J_1) \geq v(x, y, J_2)$
6. *If* J *is deterministic,* $v(x, y, J)$ *is increasing in* J *for* $J < 0$ *and decreasing in* J *for* $J > 0$.

Proof of Proposition 4. See Appendix. □

4. An Iterative Procedure to Find Optimal Trading Strategy

The fact that the ratio z can jump out of the NT region, reflected by the presence of the term $\frac{2\eta}{\sigma^2}E[\psi(\frac{z}{1+J})(1+J)^{1-\gamma}]$ in Equation 16, complicates the problem significantly. To solve this problem, we introduce an iterative technique.

Suppose we write a similar model but after the stock price jumps i times, the investor receives a continuation utility given by $v^0(x,y)$. Our choice of v^0 is quite arbitrary, but for concreteness and ease of boundary conditions, we will assume that

$$v^0(x,y) = \frac{\lambda}{\lambda + \rho - \rho(\pi^*)} \frac{(x + (1-\alpha)y)^{1-\gamma}}{1-\gamma} \tag{26}$$

Under this assumption, the investor liquidates the risky asset holdings after i jumps and reinvests all liquidated wealth without transaction costs. Other assumptions will work, however, and we stress that this choice of v^0 is for convenience. The important properties of v^0 are that it is finite, concave, homogeneous, and $v^0(x,y) \geq v(x,y)$.

Let $v^i(x,y)$ be the value function when there are i jumps left:

$$v^i(x,y) = \sup_{(D,I)\in\Theta(x,y)} E\left[\frac{\Big(x(\tau) + (1-\alpha)y(\tau)\Big)^{1-\gamma}}{1-\gamma}\right]$$

where $v^0(x,y)$ given above. Then to compute $v^i(x,y)$, for $i = 1, ..., n$, we can solve the following recursive structure

$$v^i(x,y) = \sup_{(D,I)\in\Theta(x,y)} E\left[\int_0^\infty e^{-(\eta+\lambda)t}\left(\eta v^{i-1}(x_t, y_t(1+J_t)) + \lambda\frac{(x_t + (1-\alpha)y_t)^{1-\gamma}}{1-\gamma}\right) dt\right]. \tag{27}$$

As before, because of the homogeneity of $v^i(x,y)$, there exists some function ψ^i such that

$$v^i(x,y) = y^{1-\gamma}\psi^i(\frac{x}{y}).$$

Solving (27) reduces to finding functions $\psi^i(z)$ such that

$$z^2\psi^i_{zz} + \beta_2 z\psi^i_z + \beta_1\psi^i + \beta_0\frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} + \frac{2\eta}{\sigma^2}E[\psi^{i-1}(\frac{z}{1+J})(1+J)^{1-\gamma}] = 0, \quad i = 1, ..., n \tag{28}$$

with the associated boundary conditions

$$(z + 1 + \theta)\psi^i_z(z) = (1-\gamma)\psi^i(z), \tag{29}$$

for all $z \geq r_b^i$ and

$$(z+1-\alpha)\psi_z^i(z) = (1-\gamma)\psi^i(z), \tag{30}$$

for all $z \leq r_s^i$, where β_2, β_1 and β_0 are the same as in (16) and r_s^i and r_b^i represent the Sell and Buy boundaries respectively when there are i jumps left. Moreover, the homogeneous solutions to (28) are given by $\psi_1^i(z) = |z|^{n_1}$ and $\psi_2^i(z) = |z|^{n_2}$ where

$$n_{1,2} = \frac{(1-\beta_2) \pm \sqrt{(1-\beta_2)^2 - 4\beta_1}}{2}. \tag{31}$$

This leads to the general solution to (28)

$$C_1^i\psi_1(z) + C_2^i\psi_2(z) + \psi_p^i(z), \tag{32}$$

where C_1^i and C_2^i are integration constants and the particular solution

$$\psi_p^i(z) = \int_{r^*}^{z} \frac{\psi_1(\xi)\psi_2(z) - \psi_1(z)\psi_2(\xi)}{\psi_1'(\xi)\psi_2(\xi) - \psi_1(\xi)\psi_2'(\xi)} \frac{\beta_0 \frac{(\xi+1-\alpha)^{1-\gamma}}{1-\gamma} + \frac{2\eta}{\sigma^2} E[\psi^{i-1}(\frac{\xi}{1+J})(1+J)^{1-\gamma}]}{\xi^2} d\xi.$$

or,

$$\psi_p^i(z) = \psi_p^{i-1}(z) + \int_{r^*}^{z} \frac{\psi_1(\xi)\psi_2(z) - \psi_1(z)\psi_2(\xi)}{\psi_1'(\xi)\psi_2(\xi) - \psi_1(\xi)\psi_2'(\xi)} \frac{\frac{2\eta}{\sigma^2} E[(\psi^{i-1}(\frac{\xi}{1+J}) - \psi^{i-2}(\frac{\xi}{1+J}))(1+J)^{1-\gamma}]}{\xi^2} d\xi.$$

Equations (28), (29), (30) and (32) imply that

$$\psi^i(z) = \begin{cases} A^i \frac{(z+1+\theta)^{1-\gamma}}{1-\gamma} & \text{if } z \geq r_b^i \\ C_1^i\psi_1(z) + C_2^i\psi_2(z) + \psi_p^i(z) & \text{if } r_s^i < z < r_b^i \\ B^i \frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} & \text{if } \alpha - 1 < z \leq r_s^i, \end{cases}$$

for some constants A^i,B^i,C_1^i,C_2^i, and the boundaries r_s^i and r_b^i.

We will now assume we can find coefficients which make $\psi^i(z)$ a C^2 function on $(\alpha-1, 0)$ and $(0, \infty)$ and satisfy the appropriate limiting conditions when r_b^i is infinite or $r_s^i = 0$ [3]. Notice that in this case, the coefficients A^i,B^i,C_1^i,C_2^i, and the boundaries r_s^i and r_b^i change each time the Poisson jump occurs.

[3] One can formally show this assumption is valid by performing the fairly lengthy type of convex analysis employed in Shreve and Soner(1994) or Framstad, Oksendal, and Sulem(2001).

To compute the optimal boundaries when there are $i > 0$ jumps left, we first compute r_s^0 and r_b^0 using the approach described previously. We then iterate $i-1$ times using the same approach to obtain r_s^i and r_b^i. It is important to realize that $r_s^i + (1-\alpha)(1+\underline{J}) > 0$ and for z in the solvency region such that $z+(1-\alpha)(1+\underline{J}) \leq 0$ an immediate transaction to r_s^i is optimal.

We then have the following result:

Theorem 2 *As $i \to \infty$, the functions $v^i(x,y) = y^{1-\gamma}\psi^i(\frac{x}{y})$ converge to $v(x,y)$.*

Proof of Theorem 2. Since many of the details of the proof follow fairly standard procedures, we will give a sketch and refer the reader to other sources which provide similar results. It is straightforward to prove that $\psi^i(z)$ is concave. Moreover, it is straightfoward to show that $\frac{\lambda}{\lambda-(1-\gamma)r}\frac{(z+(1-\alpha))^{1-\gamma}}{1-\gamma} \leq \psi^i(z) \leq \psi^{i-1}(z)$ using a variant of the proof in Lemma 2 and Proposition 4 item 4. Theorem 10.8 in Rockafellar(1970) then implies $\lim_{i\to\infty}\psi^i(z) \equiv \hat{\psi}(z)$ exists and is a continuous concave function on the solvency region and moreover, this convergence is uniform on each compact subset of the solvency region. Theorem 25.5 and Corollary 25.5.1 in Rockafellar(1970) implies $\hat{\psi}(z)$ is differentiable on a dense subset of the solvency region and this set can be written as a union of open intervals and in fact $\hat{\psi}$ is continuously differentiable on these intervals. On every closed bounded subset of each of the intervals, the derivatives of $\psi^i(z)$ converge uniformly to those of $\hat{\psi}(z)$ according to Theorem 25.7 in Rockafellar(1970). In particular, $(z+1+\theta)\hat{\psi}_z \geq (1-\gamma)\hat{\psi} \geq (z+1-\alpha)\hat{\psi}_z$ and there exist regions where these inequalities are in fact equalities. Thus, we have $\lim_{i\to\infty} A^i$ and $\lim_{i\to\infty} B^i$ exist and moreover there must exist some open interval (z_1, z_2) such that for i large enough

$$\lim_{i\to\infty} C_1^i\psi_1(z) + C_2^i\psi_2(z) + \psi_p^i(z) = \hat{\psi}(z) \tag{33}$$

for all $z \in (z_1, z_2)$.

Now recall the construction of the function $\psi_p^i(z)$ using the method of variation of parameters as described in Boyce and DiPrima (1969). We have

$$\psi_p^i(z) = u_1^i(z)\psi_1(z) + u_2^i(z)\psi_2(z) \tag{34}$$

$$\psi_p^{i\prime}(z) = u_1^i(z)\psi_1'(z) + u_2^i(z)\psi_2'(z) \tag{35}$$

$$\psi_p^{i\prime\prime}(z) = u_1^{i\prime}(z)\psi_1'(z) + u_2^{i\prime}(z)\psi_2'(z) + u_1^i(z)\psi_1''(z) + u_2^i(z)\psi_2''(z) \tag{36}$$

$$u_1^{i\prime}(z) = -\frac{\psi_2(z)}{\psi_1(z)\psi_2'(z) - \psi_1'(z)\psi_2(z)}\left(\beta_0\frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} + \frac{2\eta}{\sigma^2}E[\psi^{i-1}(\frac{z}{1+J})(1+J)^{1-\gamma}]\right) \tag{37}$$

$$u_2^{i\prime}(z) = \frac{\psi_1(z)}{\psi_1(z)\psi_2'(z) - \psi_1'(z)\psi_2(z)}\left(\beta_0\frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} + \frac{2\eta}{\sigma^2}E[\psi^{i-1}(\frac{z}{1+J})(1+J)^{1-\gamma}]\right) \tag{38}$$

The functions $E[\psi^{i-1}(\frac{z}{1+J})(1+J)^{1-\gamma}]$ are concave and hence continuous, satisfy $E[\psi^{i-1}(\frac{z}{1+J})(1+J)^{1-\gamma}] \geq E[\psi^{i}(\frac{z}{1+J})(1+J)^{1-\gamma}] \geq \frac{\lambda}{\lambda-(1-\gamma)r}E[\frac{(z+(1-\alpha)(1+J))^{1-\gamma}}{1-\gamma}]$ and thus converge uniformly to a concave continuous function on closed and bounded subsets of the solvency region where $z + (1-\alpha)(1+\underline{J}) > 0$ from Rockafellar (1970) Theorem 10.8. Monotone convergence Theorem implies $\lim_{i\to\infty} E[\psi^i(\frac{z}{1+J})(1+J)^{1-\gamma}] = E[\hat{\psi}(\frac{z}{1+J})(1+J)^{1-\gamma}]$ for $z + (1-\alpha)(1+\underline{J}) > 0$. Thus, $u_1^{i\prime}$ and $u_2^{i\prime}$ converge uniformly on closed and bounded subsets of the solvency region which do not contain 0 or the set $\{z | z + (1-\alpha)(1+\underline{J}) \leq 0\}$. This in turn implies $\psi_p^{i\prime\prime}$, $\psi_p^{i\prime}$ and ψ_p^i also converge uniformly on closed and bounded subsets of the solvency region which do not contain 0 or the set $\{z | z + (1-\alpha)(1+\underline{J}) \leq 0\}$.

From (33), the uniform convergence of ψ_p^i and ψ^i we can now conclude that $\lim_{i\to\infty} C_1^i$ and $\lim_{i\to\infty} C_2^i$ exist. It follows that the limiting function $\hat{\psi}$ can be written

$$\hat{\psi}(z) = \begin{cases} A\frac{(z+1+\theta)^{1-\gamma}}{1-\gamma} & \text{if } z \geq r_b \\ C_1\psi_1(z) + C_2\psi_2(z) + \psi_p(z) & \text{if } r_s < z < r_b \\ B\frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} & \text{if } \alpha - 1 < z \leq r_s, \end{cases}$$

for some constants A, B, C_1, C_2, and the boundaries r_s and r_b. Moreover, taking the pointwise limits (which is valid here due to the uniform convergence of the derivatives) in (28) shows that the limiting function satisfies the HJB equation on the interior of the solvency region except possibly at $z = 0$ if $r_s = 0$. However, in this case a fairly tedious calculation leads to the fact that $\hat{\psi}$ must have the form given in Theorem 1 at $z = 0$. To conclude the proof, we must prove a verification theorem to show that the limiting function $\hat{\psi}$ is indeed the value function and its implied trading policy is optimal. The proof of this is fairly long and involved, but follows a similar path as in Jang, Koo, Liu, and Loewenstein(2006), Shreve and Soner (1994), and Framstad, Oksendal, and Sulem (2001). $\square$

5. Numerical Results

For our analysis of the optimal trading strategy in the jump diffusion model we use as our default parameters $\alpha = 0.01$, $\theta = 0$, $\mu = 0.12$, $r = 0.05$, $\gamma = 5$, $\lambda = 0.04$ and $\eta = 0.5$ or $\eta = 0.1$. These parameters represent an investor with an expected horizon of 25 years in a model where jumps occur on average once every 2 years or 10 years. For the jump size, following Merton (1976) we assume $1 + J_t = e^{\mu_J + \sigma_J \nu_t}$, where ν_t are independent standard normal random variables. To determine the remaining baseline parameters for each value of jump frequency η, we calibrate the model to match the variance (.0082), skew (-1.33) and excess kurtosis (34.92) reported in Campbell, Lo, and MacKinlay (1996) page 21 for daily log returns. When $\eta = 0.5$ this procedure leads to $\sigma = 0.1190$, $\mu_J = -0.0259$, and $\sigma_J = 0.0666$. For $\eta = 0.1$, we obtain estimates $\sigma = 0.1239$, $\mu_J = -0.0675$, and $\sigma_J = 0.0853$. We then calculate comparative statics around these parameter estimates. Using these parameter estimates, the optimal fraction of bond to stock in the portfolio without transaction costs when $\eta = 0.5$ is

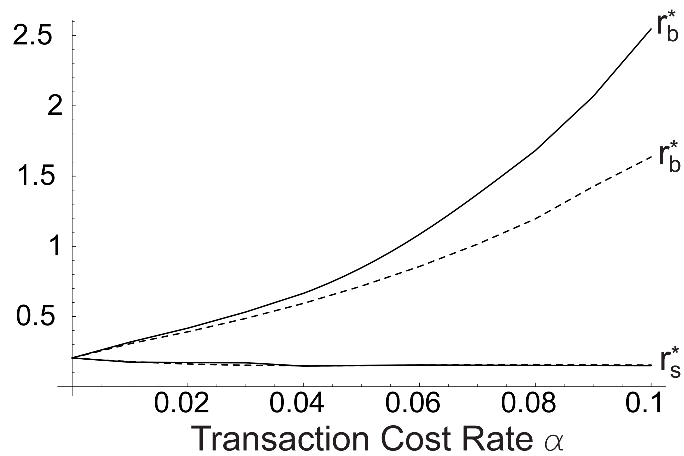

Figure 1: Optimal Trading Boundaries as a Function of the Transaction Cost Rate α.
This figure shows how the optimal trading boundaries vary with α. The dotted lines represent the function with $\lambda = 0.1$. The solid lines represent the function with $\lambda = 0.04$. Other parameters are given by $\eta = 0.1$, $\theta = 0$, $\mu = 0.12$, $r = 0.05$, $\gamma = 5$, $\sigma = 0.1239$, $\mu_J = -0.0675$, $\sigma_J = 0.0853$, and $\lambda = 0.04$.

$r^* = 0.2176$ and when $\eta = 0.1$, $r^* = 0.205$.

Figure 1 shows how the transaction boundaries change as the transaction costs vary for two different expected horizons. A shorter expected horizon leads to a significant widening of the no-transaction region. Remarkably, the sell boundary is insensitive to horizon, so as the expected horizon decreases, the widening occurs primarily from the buy boundary increasing. Of course we know as the expected horizon gets very short, the buy boundary will tend to infinity and the investor will never buy stock.

Figure 2 shows how the NT region varies as a function of the expected return μ for expected jump frequencies of once every two years and once every ten years. As one might expect, as the expected return decreases, the investor tends to hold less stock and the no-transaction region widens as μ decreases. As the jump frequency

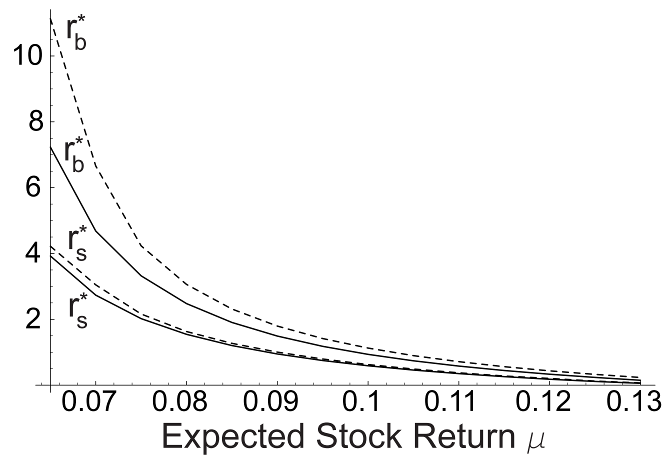

Figure 2: Optimal Trading Boundaries as a Function of Expected Return. This figure shows how the optimal trading boundaries vary with the expected return μ. The dashed lines represent $\eta = 0.5$, $\sigma = 0.1190$, $\mu_J = -0.0259$, and $\sigma_J = 0.0666$. The solid lines represent $\eta = 0.1$, $\sigma = 0.1239$, $\mu_J = -0.0675$, and $\sigma_J = 0.0853$. Other parameters are given by $\alpha = 0.01$, $\theta = 0$, $r = 0.05$, $\gamma = 5$, and $\lambda = 0.04$.

increases, the investor also tends to hold less stock. Once again, we observe the sell boundary is less sensitive to the parameter changes than the buy boundary.

Figure 3 displays the behavior of the optimal trading boundaries as a function of the coefficient of relative risk aversion γ for expected jump frequencies of once every two years and once every ten years. Once again the no-transaction region behaves as one might expect with more risk averse investors holding less stock than less risk averse investors. This is more pronounced when jumps are expected to occur more frequently.

The previous numerical results are similar to those one would find using a pure diffusion model. The next set of results address the sensitivity to jump size distribution. In contrast to the pure diffusion case, the parameters of the jump distribution have a fairly large effect on the optimal trading strategy.

Figure 4 shows how the optimal transaction boundaries vary against the expected

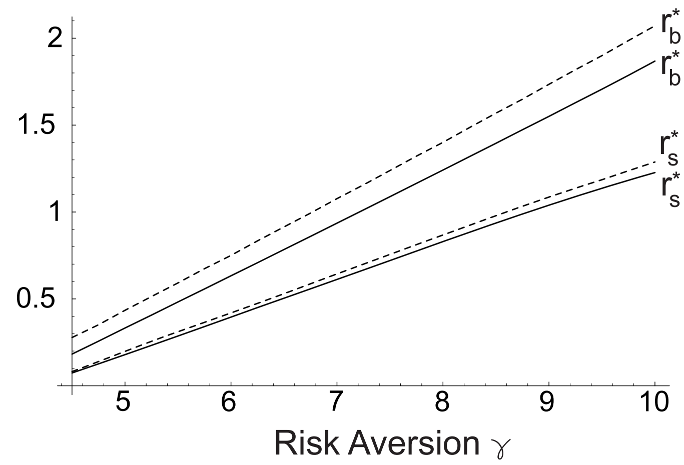

Figure 3: Optimal Trading Boundaries as a Function of Risk Aversion. This figure shows how the optimal trading boundaries vary with risk aversion γ. The dashed lines represent $\eta = 0.5$, $\sigma = 0.1190$, $\mu_J = -0.0259$, and $\sigma_J = 0.0666$. The solid lines represent $\eta = 0.1$, $\sigma = 0.1239$, $\mu_J = -0.0675$, and $\sigma_J = 0.0853$. Other parameters are given by $\alpha = 0.01$, $\theta = 0$, $r = 0.05$, $\mu = 0.12$, and $\lambda = 0.04$.

jump size $\exp(\mu_J - \frac{1}{2}\sigma_J^2)$ as we vary μ_J for two values of η corresponding to expected frequencies of jumps of once every two years and once every ten years. This figure reveals, similar to the findings in Longstaff, Liu, and Pan (2003), that the optimal trading boundaries are "U" shaped with some asymmetry. To understand this, recall that the jumps as we have modeled them do not affect the expected stock return. When the expected value of the jump is positive, the jump does help the investor by introducing a positive skew to returns, however, the inclusion of jumps in the model also increases variance of returns which is the dominant effect . The asymmetry occurs due to the fact that downward jumps tend introduce a negative skew, in other words a jump tends to bring the investor closer to the solvency line and the associated higher marginal utility. In addition the NT region widens as the expected magnitude of the jumps becomes larger. This is because for when jumps tend to be large on

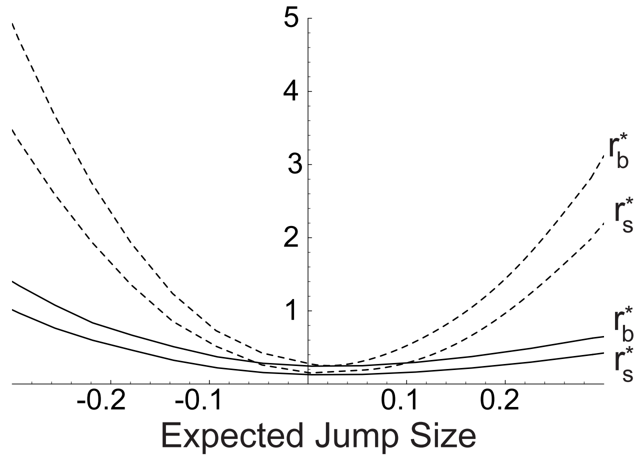

Figure 4: Optimal Trading Boundaries as a Function of Expected Jump Size.
This figure shows how the optimal trading boundaries vary with the expected jump size $\exp(\mu_J - \frac{1}{2}\sigma_J^2)$. The dashed lines represent $\eta = 0.5$, $\sigma = 0.1190$, and $\sigma_J = 0.0666$. The solid lines represent $\eta = 0.1$, $\sigma = 0.1239$, and $\sigma_J = 0.0853$. Other parameters are given by $\alpha = 0.01$, $\theta = 0$, $r = 0.05$, $\mu = 0.12$, and $\lambda = 0.04$.

average, if the NT region is too narrow, then the fraction $\frac{x}{y}$ will tend to jump out of the NT region, leading to large transaction costs.

Figure 5 shows how the optimal transaction region varies with the volatility of the jumps for expected jump frequencies of once every two years and once every ten years. As σ_J gets large the transaction boundaries generally go up due to the increase in volatility. The transaction boundaries also widen significantly since for a narrow NT region, a large σ_J leads to a higher probability of jumping outside the NT region followed by a transaction. It is interesting to note that the inclusion of uncertain jump sizes in the model leads to a much larger sensitivity to the frequency of jumps.

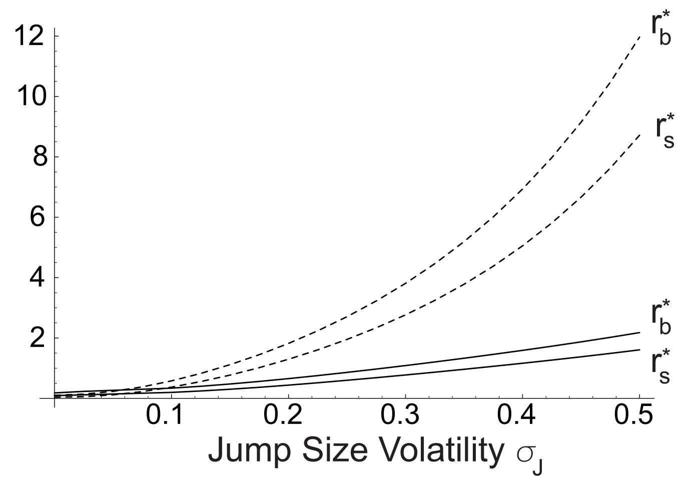

Figure 5: Optimal Trading Boundaries as a Function of Jump Volatility. This figure shows how the optimal trading boundaries vary with the jump volatility σ_J. The dashed lines represent $\eta = 0.5$, $\sigma = 0.1190$, and $\mu_J = -0.0259$. The solid lines represent $\eta = 0.1$, $\sigma = 0.1239$, and $\mu_J = -0.0675$. Other parameters are given by $\alpha = 0.01$, $\theta = 0$, $r = 0.05$, $\mu = 0.12$, $\gamma = 5$, and $\lambda = 0.04$.

6. Horizon

In this section, we explore the effects of the investor's horizon. One way to do this is to vary λ maintaining the assumption of an exponentially distributed horizon. However, it is natural to ask how the exponentially distributed horizon corresponds to a problem in which the investor has a deterministic horizon. For concreteness, we define the value function when the investor has a deterministic horizon of T as

$$v(x, y, T) = \sup_{(D,I)\in\Theta(x,y)} E[\frac{\Big(x_T + (1-\alpha)y_T\Big)^{1-\gamma}}{1-\gamma}]. \tag{39}$$

Directly attacking this problem is difficult. It involves solving for transaction boundaries which vary through time. However, in Liu and Loewenstein(2002) it is shown that an investor with an erlang distributed horizon can closely approximate the value function and optimal trading boundaries for the deterministic horizon. The erlang horizon corresonds to the case where the investor's horizon is equal to the time at which the nth jump of a standard poisson process occurs. The probability density of the ith jump of a standard Poisson process with intensity λ is

$$P\{\tau^i \in dt\} = \frac{\lambda^i}{(i-1)!} t^{i-1} e^{-\lambda t} dt,$$

If we set the investor's horizon to be τ^n, the expected horizon is i/λ when there are i jumps left. The variance of τ^n is n/λ. To help solve the problem with finite deterministic horizon T, we consider the case where τ always has expected value $E[\tau] = T$. Thus we set the intensity $\lambda = n/T$. The variance of τ is then T^2/n, which approaches 0 as n increases. For the existence of a solution, we still maintain Assumption 1, which will be satisfied if n is large enough for a fixed T. We then define

$$v(x, y, i) = \sup_{(D,I) \in \Theta(x,y)} E[\frac{\left(x_{\tau^i} + (1-\alpha) y_{\tau^i}\right)^{1-\gamma}}{1-\gamma}]. \tag{40}$$

We will state the following result without proof. The proof is quite similar to that in Liu and Loewenstein(2002)

Theorem 3 *Let* $\lambda = n/T$. *Then* $v(x, y, n) \to v(x, y, T)$ *as* $n \to \infty$.

To solve the Erlang case we essentially need to find solutions to

$$z^2 \psi^i_{zz}(z) + \beta_2 z \psi^i_z(z) + \beta_1 \psi^i(z) + \frac{2\eta}{\sigma^2} E[\psi^i(\frac{z}{1+J})(1+J)^{1-\gamma}] + \beta_0 \psi^{i-1} = 0, \tag{41}$$

where $\beta_2 = 2(\gamma\sigma^2 - (\mu - \eta E[J] - r))/\sigma^2$, $\beta_1 = -2(\lambda + (1-\gamma)(\gamma\sigma^2/2 - \mu + \eta E[J]))/\sigma^2 - 2\eta/\sigma^2$ and $\beta_0 = 2\lambda/\sigma^2$ and $\psi^0 = \frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma}$. The associated boundary conditions are

$$(z + 1 + \theta)\psi^i_z(z) = (1-\gamma)\psi^i(z)$$

for all $z \geq r_b$ and

$$(z + 1 - \alpha)\psi^i_z(z) = (1-\gamma)\psi^i(z)$$

for all $z \leq r_s$.

To solve this we can proceed for each i as we did in the exponentially distributed horizon. In particular we first iterate over the number of jumps to find ψ^i. We then increase to $i + 1$ and repeat the process.

Figure 6 shows how the optimal transaction boundaries vary with horizon for three cases. The first case is simply the transaction boundaries for an investor with an exponentially distributed horizon as the expected horizon $1/\lambda$ varies. The second case and third cases are when the investor has Erlang distributed horizons with expected value 25 years and jump intensities of 1 and 2. In other words the investors horizon is at the 25th jump of a Poisson process with intensity of 1 and the fiftieth jump of a Poisson process with intensity 2 for these cases. For all of the cases considered, the optimal fraction of riskless asset to risky asset without transactions costs is 0.205 and is independent of the horizon. We see that the that the investor with exponentially distributed horizon holds less stock than the erlang distributed horizon investors. However, the transaction boundaries in all three cases are reasonably close and thus our analysis of the exponentially distributed horizon will produce results quite similar to the deterministic horizon case.

7. Conclusion

We study a model where an investor can invest in a risk free and a riskless asset. The risky asset is subject to proportional transactions costs and, in contrast with previous models, the risky asset price can jump discontinuously. We show that the introduction of event risk can have large effects on the optimal trading strategy.

We envision future research could stem from our results. First, our analytic bounds should extend to cover the infinite horizon consumption investment problems. Second,

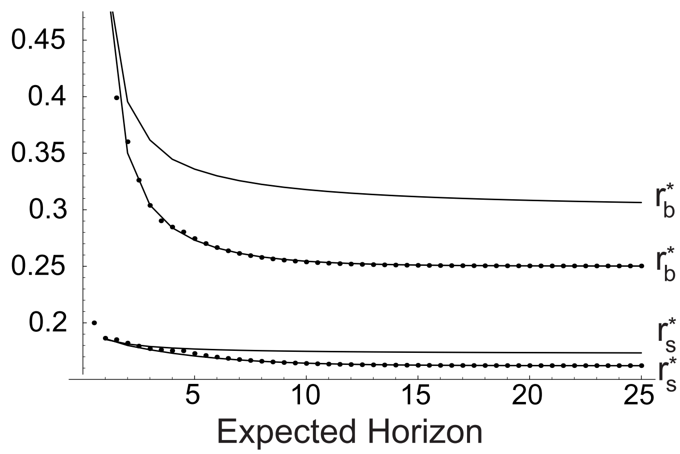

Figure 6: Optimal Trading Boundaries as a Function of Expected Horizon. This figure shows how the optimal trading boundaries vary with the remianing expected horizon. The dark solid lines represent the optimal transaction boundaries for an investor with an exponentially distributed horizon with expected horizon $1/\lambda$. The light solid lines represent the optimal transaction boundaries for an investor with an Erlang distributed horizon with $\lambda = 1$. The dotted lines represent the optimal transaction boundaries an investor with an Erlang distributed horizon with $\lambda = 2$. Other parameters are given by $\eta = 0.1$, $\theta = 0$, $r = 0.05$, $\sigma = 0.1239$, $\mu = 0.12$, $\gamma = 5$, $\mu_J = -0.0675$, and $\sigma_J = 0.0853$.

one might also incorporate regime shifting as in Jang, Koo, Liu, and Loewenstein (2006) in combination with jumps to model shifting model parameters.

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Appendix: Proofs

PROOF OF PROPOSITION 2. The fact that $r_s > (\alpha - 1)(1 + \underline{J})$ follows from the fact that to be solvent with probability 1, $x + (1 - \alpha)y(1 + J) > 0$. To show the other lower bound, if $r_s \neq 0$, ψ is C^2 at r_s and

$$\psi(z) = B\frac{(z + 1 - \alpha)^{1-\gamma}}{1 - \gamma}, \quad \forall z \leq r_s.$$

Notice $B > 0$ since $v(x, y)$ is strictly increasing in x and y. Putting this expression evaluated at r_s into (13), we get (after simplification)

$$\begin{aligned} & -\frac{1}{2}\gamma\sigma^2(1-\alpha)^2 B + (\mu - r)(1-\alpha)(r_s + 1 - \alpha)B \\ & +[-(\lambda - (1-\gamma)r)\frac{B}{1-\gamma} + \frac{\lambda}{1-\gamma}](r_s + 1 - \alpha)^2 \\ & +\eta\frac{E[\psi(\frac{r_s}{1+J})(1+J)^{1-\gamma}]}{(r_s + 1 - \alpha)^{-\gamma-1}} - \eta\frac{B}{1-\gamma}(r_s + 1 - \alpha)^2 - \eta E[J](1-\alpha)(r_s + 1 - \alpha)B = 0 \end{aligned} \tag{42}$$

If $r_s = 0$ one can verify by direct substitution using (18) that (42) still holds. The second line of (42) is less than or equal to zero since

$$\frac{\lambda}{\lambda - (1-\gamma)}(x + (1-\alpha)y)^{1-\gamma} \leq \frac{B}{1-\gamma}(x + (1-\alpha)y)^{1-\gamma}, \tag{43}$$

in other words the investor must be at least as well off as liquidating the position and investing the proceeds in the riskless asset. The third line of (42) is also less than or equal to zero since from the well known inequality for concave functions

$$v(x + \Delta x, y + \Delta y) - v(x, y) - \leq v_x(x, y)\Delta x + v_y(x, y)\Delta y \tag{44}$$

Evaluating this for x, y in the sell region and $\Delta x = 0$ and $\Delta y = Jy$, gives

$$v(x, y(1+J)) - \frac{B}{1-\gamma}(x + (1-\alpha)y)^{1-\gamma} - (1-\alpha)B(x + (1-\alpha)y)^{-\gamma}yJ \leq 0, \tag{45}$$

which after taking expectations with respect to J and dividing by $y^{1-\gamma}(r_s+1-\alpha)^{-\gamma-1}$ leads to the third line is less than or equal to zero. Thus

$$-\frac{1}{2}\gamma\sigma^2(1-\alpha)^2 B + (\mu - r)(1-\alpha)(r_s+1-\alpha)B \geq 0 \tag{46}$$

which gives the lower bound. $\square$

Proof of Proposition 3. We prove the proposition for the upper bound on r_s. The lower bound on r_b follows from virtually identical manipulation. If $r_s \neq 0$, ψ is C^2 at r_s and

$$\psi(z) = B\frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma}, \quad \forall z \leq r_s.$$

Notice $B > 0$ since $v(x,y)$ is strictly increasing in x and y. Putting this expression evaluated at $z \leq r_s$ into (13), we get (after simplification)

$$\begin{aligned}
&-\frac{1}{2}\gamma\sigma^2(1-\alpha)^2 B + (\mu - r - \eta E[J])(1-\alpha)(z+1-\alpha)B \\
&+[-(\lambda+\eta-(1-\gamma)r)\frac{B}{1-\gamma} + \frac{\lambda}{1-\gamma}](z+1-\alpha)^2 + \eta\frac{E[\psi(\frac{z}{1+J})(1+J)^{1-\gamma}]}{(z+1-\alpha)^{-\gamma-1}} \leq 0,
\end{aligned} \tag{47}$$

with equality at $z = r_s$. Recalling the definition of $\rho(\pi)$ in Assumption 1 and further manipulation gives

$$\begin{aligned}
&-\frac{1}{2}\gamma\sigma^2(1-\alpha)^2 B + (\mu - r - \eta E[J])(1-\alpha)(z+1-\alpha)B \\
&+[-(\lambda+\eta-\rho(\pi^*))\frac{B}{1-\gamma} + \frac{\lambda}{1-\gamma}](z+1-\alpha)^2 + \eta\frac{E[\psi(\frac{z}{1+J})(1+J)^{1-\gamma}]}{(z+1-\alpha)^{-\gamma-1}} \\
&-B\left[\pi^*(\mu - r - \eta E[J]) - \frac{\gamma}{2}\pi^{*2}\sigma^2 + \frac{\eta}{1-\gamma}E[(1+\pi^* J)^{1-\gamma}\right](z+1-\alpha)^2 \leq 0.
\end{aligned} \tag{48}$$

Dividing by $(z+1-\alpha)^2$ and rearranging gives

$$\begin{aligned}
&-\frac{1}{2}\gamma\sigma^2(\frac{(1-\alpha)^2}{(z+1-\alpha)^2} - \pi^{*2})B + (\mu - r - \eta E[J])(\frac{1-\alpha}{z+1-\alpha} - \pi^*)B \\
&+\eta\frac{E[\psi(\frac{z}{1+J})(1+J)^{1-\gamma}]}{(z+1-\alpha)^{1-\gamma}} - \frac{B}{1-\gamma}\eta E[(1+\pi^* J)^{1-\gamma}] \\
&+[-(\lambda+\eta-\rho(\pi^*))\frac{B}{1-\gamma} + \frac{\lambda}{1-\gamma}] \leq 0,
\end{aligned} \tag{49}$$

which can be written

$$\frac{\rho(\frac{1-\alpha}{z+1-\alpha})-\rho(\pi^*)}{1-\gamma}B+[-(\lambda+\eta-\rho(\pi^*))\frac{B}{1-\gamma}+\frac{\lambda}{1-\gamma}]$$
$$+\eta(\frac{E[\psi(\frac{z}{1+J})(1+J)^{1-\gamma}]}{(z+1-\alpha)^{1-\gamma}}-\frac{B}{1-\gamma}E[(1+\frac{1-\alpha}{z+1-\alpha}J)^{1-\gamma}])\leq 0, \tag{50}$$

for al1 $z \leq r_s$ with equality at $z = r_s$. From Lemma 2 if $\pi^* \neq 1$, we have

$$\frac{B}{1-\gamma}<\frac{1}{\lambda+\eta-\rho(\pi^*)}\frac{\lambda}{1-\gamma} \tag{51}$$

from equation (23). Moreover, we have $\frac{\rho(\frac{1-\alpha}{z+1-\alpha})-\rho(\pi^*)}{1-\gamma} \leq 0$ with equality when $\frac{1-\alpha}{z+1-\alpha} = \pi^*$ or $z = (1-\alpha)r^*$, because π^* maximizes $\frac{\rho(\pi)}{1-\gamma}$. From Lemma 2 Statement 2 we have

$$\psi(\frac{z}{1+J}) \geq \frac{B}{1-\gamma}(\frac{z}{1+J}+1-\alpha)^{1-\gamma} \tag{52}$$

and a bit of algebra gives

$$\frac{\psi(\frac{z}{1+J})(1+J)^{1-\gamma}}{(z+1-\alpha)^{1-\gamma}} \geq \frac{B}{1-\gamma}(1+\frac{1-\alpha}{z+1-\alpha}J)^{1-\gamma} \tag{53}$$

Thus,

$$\eta\frac{E[\psi(\frac{r^*}{1+J})(1+J)^{1-\gamma}]}{(r^*+1-\alpha)^{1-\gamma}}-\frac{B}{1-\gamma}\eta E[(1+\pi^*J)^{1-\gamma}]) \geq 0 \tag{54}$$

It thus develops that if $\pi^* \neq 1$ and $r_s \neq 0$, (50) cannot hold for $r_s = (1-\alpha)r^*$ and thus $r_s < (1-\alpha)r^*$. If $r_s = 0$, then (50) evaluated at $r_s = 0$ indicates (51) must hold with equality. This gives the bound in the proposition. □

Lemma 3 *suppose $\varphi(x, y)$ satisfies*

$$(1-\alpha)\varphi_x \leq \varphi_y \leq (1+\theta)\varphi_x \tag{55}$$

and if (x_t, y_t) correspond to the stock and bond accounts using the optimal c, I, D

$$E[\int_0^T e^{-\lambda t}\sigma y\varphi_y dw_t + \int_0^T e^{-\lambda t}(E[\varphi(x_t, y_t(1+J)) - \varphi(x_t, y_t))dM_t] = 0 \tag{56}$$

where $M_t = N_t - \eta t$ *and*

$$\lim_{T\to\infty} E[e^{-\lambda T}\varphi(x_T, y_T)] = 0 \tag{57}$$

Let

$$\begin{aligned}\mathcal{L}\varphi \quad &\equiv \quad \frac{1}{2}\sigma^2 y^2 \varphi_{yy} + rx\varphi_x + (\mu - \eta E[J])y\varphi_y - \lambda\varphi + \lambda\frac{(x+(1-\alpha)y)^{1-\gamma}}{1-\gamma} \\ &\quad\quad +\eta(E[\varphi(x, y(1+J))] - \varphi(x,y))\end{aligned} \tag{58}$$

Then if $\mathcal{L}\varphi \le 0$, $\varphi(x,y) \ge v(x,y)$ *for all* $(x,y) \in \mathcal{S}$. *If* $E[\int_0^\infty e^{-\lambda t}\mathcal{L}\varphi(x_t,y_t)dt] < 0$, *then* $\varphi(x,y) > v(x,y)$.

Proof of Lemma 3. Let (x_t, y_t) correspond to the optimal bond and stock account using the optimal c, D, I. By Ito's lemma we have,

$$\begin{aligned} e^{-\lambda T}\varphi(x_T, y_T) + \int_0^T e^{-\lambda t}\lambda\frac{(x_t+(1-\alpha)y_t)^{1-\gamma}}{1-\gamma}dt \\ = \varphi(x_0,y_0) + \int_0^T e^{-\lambda t}\mathcal{L}\varphi(x_t,y_t)dt \\ + \int_0^T e^{-\lambda t}(\varphi_y - (1+\theta)\varphi_x)dI_t + \int_0^T e^{-\lambda t}((1-\alpha)\varphi_x - \varphi_y)dD_t \\ + \int_0^T e^{-\lambda t}\sigma y\varphi_y dw_t + \int_0^T e^{-\lambda t}(E[\varphi(x_t, y_t(1+J)) - \varphi(x_t,y_t))dM_t, \end{aligned} \tag{59}$$

where $M_t = N_t - \eta t$. Taking expectations

$$\varphi(x,y) \ge E[\int_0^T e^{-\lambda t}\lambda\frac{(x_t+(1-\alpha)y_t)^{1-\gamma}}{1-\gamma}dt + e^{-\lambda T}\varphi(x_T,y_T)]. \tag{60}$$

Letting $T\to\infty$ and using the monotone convergence theorem gives

$$\varphi(x,y) \ge E[\int_0^\infty e^{-\lambda t}\lambda\frac{(x_t+(1-\alpha)y_t)^{1-\gamma}}{1-\gamma}dt] = v(x,y) \tag{61}$$

□

Proof of Proposition 4. Each of the claimed comparative statics follows from verifying the conditions of Lemma 3 hold for the value function which is claimed to be greatest. For example to prove statement 5, it is obvious that (55) holds for $v(x,y,J_1)$ which we shall rename v^1. Then we know

$$\frac{1}{2}\sigma^2y^2v^1_{yy} + rxv^1_x + (\mu - \eta E[J_1])yv^1_y - \lambda v^1 + \lambda\frac{(x+(1-\alpha)y)^{1-\gamma}}{1-\gamma}$$
$$+\eta(E[v^1(x,y(1+J_1))] - v^1(x,y)) \le 0 \tag{62}$$

which since $J_1 \succ_{SSD} J_2$ and its implication $E[J_1] = E[J_2]$ leads to

$$\frac{1}{2}\sigma^2y^2v^1_{yy} + rxv^1_x + (\mu - \eta E[J_2])yv^1_y - \lambda v^1 + \lambda\frac{(x+(1-\alpha)y)^{1-\gamma}}{1-\gamma}$$
$$+\eta(E[v^1(x,y(1+J_2))] - v^1(x,y)) \le 0 \tag{63}$$

which verifies that (58) holds. A fairly long and tedious calculation also leads to the conclusion that the other conditions hold and thus $v(x,y,J_1) \ge v(x,y,J_2)$.

We remark that to prove statement 4 the reader might find Equation (44) of use in showing (letting $v(x,y,\eta_2) = v^2(x,y)$), $(E[v^2(x,y(1+J))] - v^2(x,y) - E[J]yv^2_y) \le 0$ so if $\eta_1 > \eta_2$, $\eta_1(E[v^2(x,y(1+J))] - v^2(x,y) - E[J]yv^2_y) \le \eta_2(E[v^2(x,y(1+J))] - v^2(x,y) - E[J]yv^2_y)$.

We further remark that to show statement 6, denote the function $v(x,y,J)$ by $v^J(x,y)$. Now consider the function $v^J(x,y(b+1)) - ybv^J_y(x,y)$. This is a concave function of b. Differentiating with respect to b implies this function is maximized at $b=0$ and if $b \le J \le 0$ we have $v^J(x,y(b+1)) - ybv^J_y(x,y) \le v^J(x,y(J+1)) - yJv^J_y(x,y)$ and if $b \ge J \ge 0$, then $v^J(x,y(b+1)) - ybv^J_y(x,y) \le v^J(x,y(J+1)) - yJv^J_y(x,y)$. Statement 6 follows from the observation that since v^J satisfies the HJB equation

$$\frac{1}{2}\sigma^2y^2v^J_{yy} + rxv^J_x + (\mu - \eta J)yv^J_y - \lambda v^J + \lambda\frac{(x+(1-\alpha)y)^{1-\gamma}}{1-\gamma}$$
$$+\eta(E[v^J(x,y(1+J))] - v^J(x,y)) \le 0 \tag{64}$$

then if $b \ge J \ge 0$ or $b \le J \le 0$ we have

$$\frac{1}{2}\sigma^2y^2v^J_{yy} + rxv^J_x + (\mu - \eta b)yv^J_y - \lambda v^J + \lambda\frac{(x+(1-\alpha)y)^{1-\gamma}}{1-\gamma}$$

$$+\eta(E[v^J(x,y(1+b))]-v^J(x,y))\leq 0 \tag{65}$$

$\square$