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An analysis of VaR-based capital requirements

Domenico Cuoco [a,*], Hong Liu [b]

[a] *Finance Department, The Wharton School, University of Pennsylvania, 3620 Locust Walk, Philadelphia, PA 19104-6367*
[b] *John M. Olin School of Business, Washington University in St. Louis, St. Louis, MO 63130*

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Abstract

We study the behavior of a financial institution subject to capital requirements based on self-reported VaR measures, as in the Basel Committee's Internal Models Approach. We view these capital requirements and the associated backtesting procedure as a mechanism designed to induce financial institutions to reveal the risk of their investments and to support this risk with adequate levels of capital. Accordingly, we consider the simultaneous choice of an optimal dynamic reporting and investment strategy. Overall, we find that VaR-based capital requirements can be very effective not only in curbing portfolio risk but also in inducing revelation of this risk.

JEL classification: D91; D92; G11; C61

Keywords: Banking regulation; Capital requirements; Basel Capital Accord; Internal Models Approach; Value-at-Risk; Portfolio constraints

1. Introduction

Financial institutions are required by regulators to maintain minimum levels of capital. This regulation is normally justified as a response to the negative externalities arising from

* Corresponding author. Fax: +1 (215) 898 6200.
E-mail addresses: cuoco@wharton.upenn.edu (D. Cuoco), liuh@wustl.edu (H. Liu).

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bank failures and to the risk-shifting incentives created by deposit insurance.[1] The 1988 Basel Capital Accord imposed uniform capital requirements based on risk-adjusted assets, defined as the sum of asset positions multiplied by asset-specific risk weights. These risk weights were intended to reflect primarily the asset's credit risk.[2] In 1996 the Accord was amended to include additional minimum capital reserves to cover market risk, defined as the risk arising from movements in the market prices of trading positions (Basel Committee on Banking Supervision, 1996a).

The 1996 Amendment's Internal Models Approach (IMA) determines capital requirements on the basis of the output of the financial institutions' internal risk measurement systems. Financial institutions are required to report daily their Value-at-Risk (VaR) at the 99% confidence level over a one-day horizon and over a two-week horizon (ten trading days).[3] The minimum capital requirement on a given day is then equal to the sum of a charge to cover "credit risk" (or idiosyncratic risk) and a charge to cover "general market risk," where the credit-risk charge is equal to 8% of risk-adjusted assets and the market-risk charge is equal to a multiple of the average reported two-week VaRs in the last 60 trading days.[4] US-regulated banks and OTC derivatives dealers are subject to capital requirements determined on the basis of the IMA.

The reliance on the financial institution's self-reported VaRs to determine capital requirements creates an adverse selection problem, since the institution has an incentive to underreport its true VaR in order to reduce capital requirements. The procedure suggested by the Basel Committee to address this problem relies on "backtesting" (Basel Committee on Banking Supervision, 1996c): regulators should evaluate on a quarterly basis the frequency of "exceptions" (that is, the frequency of daily losses exceeding the reported VaRs) in the most recent twelve-month period and the multiplicative factor used to determine the market risk charge should be increased (according to a given scale varying between 3 and 4) if the frequency of exceptions is high.[5] Additional corrective actions in response to a high

[1] See Berger et al. (1995), Freixas and Santomero (2002) or Santos (2002) for a review of the theoretical justifications for bank capital requirements.

[2] Gordy (2003) shows how the credit risk weights might be determined in the context of a single-factor credit risk model.

[3] Simply stated, VaR is the maximum loss of a trading portfolio over a given horizon, at a given confidence level (i.e., a quantile of the projected profit/loss distribution at the given horizon). To avoid a duplication of risk-measurement systems, financial institutions are allowed to derive their two-week VaR measure by scaling up the daily VaR by the square root of ten (see: Basel Committee on Banking Supervision, 1996b, p. 4).

[4] More precisely, the market-risk charge is equal to the larger of: (i) the average reported two-week VaRs in the last 60 trading days times a multiplicative factor and (ii) the last-reported two-week VaR. However, since the multiplicative factor is not less than 3 (see below), the average of the reported VaRs in the last 60 trading days times the factor typically exceeds the last-reported VaR.

[5] The reason backtesting is based on a daily VaR measure in spite of the fact that the market risk charge is based on a two-week VaR measure is that VaR measures are typically computed ignoring portfolio revisions over the VaR horizon. According to the Basel Committee, "it is often argued that value-at-risk measures cannot be compared against actual trading outcomes, since the actual outcomes will inevitably be 'contaminated' by changes in portfolio composition during the holding period. [...] This argument is persuasive with regard to the use of value-at-risk measures based on price shocks calibrated to longer holding periods. That is, comparing the ten-day, 99th percentile risk measures from the internal models capital requirement with actual ten-day trading outcomes would probably not be a meaningful exercise. In particular, in any given ten day period, significant

number of exceptions are left to the discretion of regulators. This leaves open the question of how financial institutions will behave under the new regime of capital requirements.

This paper studies the optimal behavior of a financial institution subject to capital requirements determined according to the IMA. We view the system of capital requirements put in place by the 1996 Amendment as a revelation mechanism designed to induce financial institutions to truthfully reveal the risk (VaR) of their trading portfolios and to support this risk with adequate levels of capital, a view consistent with Rochet (1999) and Jorion (2001, p. 65). Accordingly, we consider the simultaneous choice of an optimal reporting and investment strategy. Since the incentives to truthful revelation arise in part from the threat of increased capital requirements in the future (through an increase in the multiplicative factor), we consider a fully dynamic model with discrete reporting and continuous trading.

Specifically, we consider a financial institution with preferences represented by a risk-averse utility function defined over the market value of its equity capital at the end of the planning horizon.[6] We assume that the institution has fully-insured deposits and limited liability: thus, our model allows for the risk-shifting incentives created by these institutional features. The institution is allowed to continuously rebalance its portfolio across a number of risky investment opportunities and a riskless asset. In addition, the financial institution is exposed to background risk. This risk takes the form of unhedgeable random capital shocks that reduce the market value of the institution's capital and can result in the institution's default (if the value of the institution's capital is below the realized shock at the time such a shock occurs). The probability of default is thus endogenous in our model, since it depends on the stochastic process followed by the institution's capital, and hence by the chosen investment strategy.

The institution is required by regulators to continuously maintain its capital above a minimum level, which depends on the risk of the chosen investment strategy. Specifically, regulatory minimum capital equals the sum of a charge to cover market risk and a charge to cover credit risk.

For the purpose of determining capital requirements, the institution's planning horizon is divided into non-overlapping "backtesting periods," each of which is in turn divided into non-overlapping "reporting periods." At the beginning of each reporting period, the institution must report to regulators its claimed VaR as well as the actual loss over the previous period. The market-risk charge for the current reporting period is then equal to a multiple k of the reported VaR,[7] while the credit-risk charge is equal to the sum of asset

changes in portfolio composition relative to the initial positions are common at major trading institutions. For this reason, *the backtesting framework described here involves the use of risk measures calibrated to a one-day holding period*" (Basel Committee on Banking Supervision, 1996c, p. 3).

[6] The assumption that the preferences of a financial institution can be represented by a risk-averse utility function has been widely used in the existing literature on capital regulation and can be justified by, among other things, managerial risk aversion and the value of the institution's charter, whose loss represents a significant default cost (see Keeley, 1990). Of course, when coupled with limited liability and deposit insurance, this assumption does not imply a globally risk-averse behavior. This will become clear in the subsequent analysis.

[7] Consistently with empirical evidence, we find that in our model reported VaRs display little variation from one reporting period to the next. Thus, making capital requirements proportional to an average of recently-reported VaRs (rather than proportional to the last-reported VaR) would make little difference in our results.

positions multiplied by asset-specific credit-risk weights. At the end of each backtesting period, the number of exceptions (i.e., the number of reporting periods in which the actual loss exceeded the reported VaR) is computed and this determines the multiple k over the next backtesting period, according to a given increasing scale.[8] To capture the effect of any additional regulatory actions or reputation losses that might be triggered by exceptions, we also assume that non-financial costs are incurred at the end of any reporting period in which a loss exceeding the reported VaR is observed. These non-financial costs are assumed to be proportional to the amount by which the actual loss exceeds the reported VaR.[9]

Therefore, the financial institution chooses the level of VaR to report in each period by trading off the cost of higher capital requirements in the current period resulting from a higher reported VaR against the benefit of a lower probability of non-financial costs and a lower probability of higher capital requirements in the future as a result of a loss exceeding the reported VaR. In addition, the institution simultaneously chooses a continuously-rebalanced trading strategy for its portfolio, subject to the applicable capital requirements. We stress that the problem we consider differs from a standard investment problem with portfolio constraints, since capital requirements are not exogenously fixed, but vary endogenously as a result of the institution's optimal reporting strategy.

We explicitly characterize the solution of the problem described above using martingale duality (as in Cuoco, 1997) and parametric quadratic programming. Even with constant price coefficients, optimal portfolios in the presence of capital requirements do not display two-fund separation: as capital requirements become progressively more binding following losses, financial institutions find it optimal to rebalance their portfolios in favor of assets characterized by high risk-weight-adjusted expected returns (high systematic risks). However, we show that optimal portfolios satisfy a local three-fund separation property, with the three funds being the riskless asset, the mean-variance efficient portfolio of risky assets and a risk-weight-constrained minimum-variance portfolio of risky assets. We find that VaR-based capital requirements are effective in offsetting the risk-taking incentives generated by deposit insurance and limited liability, with the risk-taking in the presence of capital requirements always being lower than that of a similar unregulated institution.

In general, financial institutions may optimally underreport or overreport their true VaRs, depending on their risk aversion, the current reserve multiple, the number of exceptions recorded in the current backtesting period, the time remaining to the end of the

[8] Thus, we assume that the VaR measure used for backtesting coincides with that used to determine the capital charge for market risk. This is without loss of generality, as any difference between the two VaR measures can be captured by rescaling the multiple k. For example, in our numerical calibration we take the reporting period to be one day, as suggested by the Basel Committee, and assume that the multiple is determined according to the scale suggested by the Basel Committee times the square root of ten: this adjustment captures the fact that the multiple should be applied to a two-week (rather than one-day) VaR (see footnote 3).

[9] Because reporting a zero VaR always results in the smallest possible capital requirement no matter how large the reserve multiple k is, the threat of higher reserve multiples in the future is by itself insufficient to induce institutions to report strictly positive VaRs unless is supplemented by additional penalties. While the Basel Capital Accord explicitly mentions the possibility of penalties in addition to the revision of the capital multiple, these penalties are—as already noted—left to the discretion of regulators. This margin of discretion leaves some ambiguity in the penalties associated with exceptions. We believe however that our assumption of proportionality to the size of the exception is a natural one.

current backtesting period and the level of the non-financial costs associated with an exception. Overall, capital requirements determined on the basis of the IMA appear to be quite effective not only in curbing portfolio risks, but also in inducing revelation of these risks, with reported VaRs being close to true VaRs.

To our knowledge, the optimal behavior of a financial institution subject to capital requirements determined in accordance to the IMA is an issue so far unexplored in the existing literature.

In a static mean-variance framework, Kahane (1977) and Kohen and Santomero (1980) showed that a more stringent capital requirement (in the form of a lower upper bound on feasible leverage ratios) may induce financial institutions to substitute riskier assets for less risky ones and thus may increase the risk of trading portfolios and the probability of default. Kim and Santomero (1988) established that the same result applies to capital requirements determined on the basis of risk-weighted assets, unless the risk weights happen to be proportional to the assets' betas. The conclusion that capital requirements could lead to an increase in risk taking and hence in the likelihood of bank failures has been the subject of extensive discussion in the subsequent theoretical literature.[10] Furlong and Keeley (1989) and Keeley and Furlong (1990) argued that the mean-variance framework is inappropriate to analyze the effect of capital requirements in the presence of deposit insurance and limited liability, because limited liability results in skewed portfolio return distributions. In addition, Furlong and Keeley (1989) considered a value-maximizing financial institution and showed that stricter leverage limits unambiguously reduce optimal risk-taking. This result derives from the fact that in their model an institution would always choose the portfolio having the maximum possible risk (i.e., a corner solution) in order to maximize the value of the deposit insurance (a put option). Gennotte and Pyle (1991) extended the analysis of Furlong and Keeley to allow for investment opportunities having non-zero net present value (NPV) and showed that in this setting tighter capital restrictions can lead financial institutions to increase asset risk.

Blum (1999) used a two-period model to show that the incentives to increase the risk of trading portfolios in response to tighter capital requirements are even higher in a dynamic setting. This is because capital requirements increase the marginal utility of a unit of capital in the next period and thus can lead to an increase in risk-taking in the current period in an effort to increase expected return. Calem and Rob (1999) considered an infinite-horizon discrete-time model with a riskless and a risky asset and minimum capital requirements given by a linear function of the allocation to the risky asset. Moreover, they assumed that capital requirements could be violated, but that such violation would result in a pecuniary penalty (e.g., in the form of a higher deposit insurance premium). They also concluded that in their model tighter capital requirements could result in increased risk-taking. Dangl and Lehar (2004) considered an infinite-horizon continuous-time model in which the financial institution could choose between two assets, each of which was assumed to follow a geometric Brownian motion with given drift and volatility. Switches were assumed to involve a fixed cost and no diversification across the two risky assets (or a riskless asset) was allowed. In addition, capital requirements were assumed to be monitored only at random

[10] See Jackson (1999) for a review of the related empirical evidence.

auditing times, at which times the financial institution would be liquidated if it were found in violation of the capital requirement. No other corrective actions (aside for liquidation) were assumed to be available to regulators and no incentives were assumed to be given to financial institutions to disclose their risk between auditing times. Dangl and Lehar showed that, in their model, capital requirements that depended on which of the two risky asset was selected (and hence on portfolio risk) dominated capital requirements that were independent of this choice, in the sense that, for a given monitoring frequency, financial institutions would find it optimal to switch less often to the riskier asset.

Differently from the dynamic models mentioned above, we examine the impact of capital requirement on the risk-shifting incentives created by deposit insurance and limited liability in a continuous-time trading model that allows for portfolio diversification and hence for portfolio return distributions that are not restricted to a simple parametric family (e.g., normal or lognormal). In addition, we allow for more general capital requirements that include a charge for market risk based on a self-reported risk measure in addition to that based on risk-weighted assets. To make our model tractable, we assume, differently from Dangl and Lehar (2004), that capital requirements can be continuously monitored. While asset substitution incentives are present in our model, we find that capital requirements never have perverse effects on risk-taking or on the probability of extreme losses and default. Moreover, differently from Furlong and Keeley (1989), this lack of perverse effects is not due to the fact that the financial institution acts globally as a risk-lover and hence is always at a corner solution.

Sentana (2001), Emmer et al. (2001), Vorst (2001), Basak and Shapiro (2001) and Cuoco et al. (2002) considered the investment problem of a trader subject to an exogenous limit on the VaR of the trading portfolio. None of these papers incorporated limited liability or a realistic model of capital requirements. The first four papers considered the case of a fixed VaR limit, which does not capture the constraint imposed by capital requirements on financial institutions. Cuoco et al. considered the case in which the limit varies as a function of the value of the trading portfolio. Their results for the proportional case imply that if the VaR of a financial institution's trading portfolio were perfectly and continuously observable by regulators and minimum capital requirements at any given point in time were simply equal to a fixed multiple of the contemporaneous VaR (with no penalties for observed exceptions), then, under the assumption of CRRA preferences and unlimited liability, the optimal portfolio for a financial institution subject to capital requirements would involve a constant proportional allocation to the mean-variance efficient portfolio. Moreover, the capital requirement would either always bind or never bind. Neither of these conclusions holds for the more realistic model of capital regulation considered in this paper.[11]

In a static setting, Chan et al. (1992) and Giammarino et al. (1993) studied the optimal design of a mechanism to induce truthful risk revelation in a setting in which regulators also provide deposit insurance. By contrast, our focus is not on mechanism design but on the analysis of the specific mechanism implemented by the 1996 Amendment. Ju and Pear-

[11] Leippold et al. (2003), using asymptotic approximation techniques, extended the analysis of an exogenous proportional VaR constraint in Cuoco et al. to incorporate stochastically-varying price coefficients and also examined the equilibrium implications of such a constraint.

son (1999) examined, also in a static setting, the bias that arises when the VaR of a portfolio is determined on the basis of the delta-normal method with variances and covariances estimated using past data: in this case, an institution (or a trader) subject to a binding VaR constraint and possessing private information about the relation between current variances and covariances and historical ones, is able to select portfolios whose true VaR exceeds the estimated VaR and hence to assume risks in excess of the stated limit. Ju and Pearson quantified the extent of this bias assuming that the regulator monitoring this limit provides no incentives for the institution to reveal its information and that the institution has one of three objectives: maximizing the portfolio VaR, maximizing the portfolio expected return, or minimizing the variance of the difference between the return of the chosen portfolio and the return of an exogenously-given reference portfolio. By contrast, because of the penalties associated with exceptions, the 1996 Amendment does provide incentives to financial institutions to reveal private information about risk: these incentives (in addition to a more realistic dynamic investment objective) are critical features of the model we consider.

The rest of the paper is organized as follows. Section 2 describes our model in detail. Section 3 explains our solution approach to the joint reporting and investment problem in the presence of capital requirements and provides some explicit characterization of optimal trading strategies in this section. Section 4 provides a numerical analysis. Section 5 concludes. Appendix A contains all the proofs.

2. The model

We consider a financial institution with a planning horizon equal to T backtesting periods, where T is a positive integer. Without loss of generality, we normalize the length of a backtesting period to 1. Each backtesting period comprises n non-overlapping reporting periods of equal length $\tau = 1/n$. At the beginning of each reporting period, the financial institution is required to report to a regulator its current VaR as well as the actual profit/loss over the previous reporting period. As explained later, the reported VaR determines the capital charge to cover market risk for the period.

The financial institution has liabilities represented by deposits and (equity) capital. For simplicity, we assume that the face value of deposits D is fixed over the planning horizon and that there are no equity issues or dividend payments over this period. Deposits are fully insured and earn the risk-free interest rate, which is paid out continuously to depositors. The market value of deposits is therefore constant and equal to D. The investment opportunities are represented by $m+1$ long-lived assets. The first asset is riskless and earns a constant continuously-compounded interest rate $r \geqslant 0$. The other m assets are risky and their price process S (inclusive of reinvested dividends) follows a geometric Brownian motion with drift vector $r\bar{1} + \mu$ and diffusion matrix σ, i.e.,

$$S(t) = S(0) + \int_0^t I^S(s)\big(r\bar{1} + \mu\big)\,\mathrm{d}s + \int_0^t I^S(s)\sigma\,\mathrm{d}w(s),$$

where $I^S(t)$ denotes the $m \times m$ diagonal matrix with elements $S(t)$, $\bar{1} = (1, \ldots, 1)^\top$ and w is an m-dimensional Brownian motion. We assume without loss of generality that σ

has rank m.[12] The financial institution can trade continuously and without frictions over $[0, T]$.[13,14]

Letting θ be the m-dimensional stochastic process representing the (dollar) investment in the risky assets, the evolution of the value A of the institution's asset portfolio over any reporting period is then given by

$$\mathrm{d}A(t) = \big(A(t)r + \theta(t)^\top \mu\big)\,\mathrm{d}t + \theta(t)^\top \sigma \,\mathrm{d}w(t) - rD\,\mathrm{d}t, \tag{1}$$

where the last term reflects interest payments to depositors.

We define the institution's *regulatory capital* $K = A - D$ as the difference between the value of the institution's asset portfolio and the value of the institution's deposits.[15] The financial institution is required to maintain this capital above a minimum level equal to the sum of the charge to cover general market risk plus a charge to cover credit (or idiosyncratic) risk. The charge to cover market risk equals the VaR reported at the beginning of the current reporting period times a multiple k. The charge to cover credit risk equals the sum of the institution's trading positions (long and short) multiplied by asset-specific risk weights. Thus, letting $\beta \in [0, 1]^m$ denote the vector of asset risk weights, the capital charge to cover credit risk at time t equals $\beta^\top(\theta(t)^+ + \theta(t)^-)$, where for any vector $x \in \mathbb{R}^m$ we denote by x^+ the vector with components $x_i^+ = \max[0, x_i]$ and by x^- the vector with components $x_i^- = \max[0, -x_i]$.[16] Hence, if $VaR \geqslant 0$ denotes the VaR reported to regulators at the beginning of the current reporting period and k is the currently-applicable multiple, the

12 If $d = \mathrm{rank}(\sigma) < m$, some stocks are redundant and can be omitted from the analysis. Moreover, w can be redefined in this case to be a d-dimensional Brownian motion.

13 The assumption of continuous frictionless trading is of course a simplification in the case of a financial institution for which loans constitute a significant portion of investments. However, incorporating illiquidity into the present model would significantly add to its complexity. We view the frictionless case as a reasonable starting point for a first analysis of VaR-based capital requirements, especially in consideration of the increasing use of loan securitization by financial institutions. In addition, the unhedgeable capital shocks in our model can be interpreted as capturing the risk associated with totally illiquid assets.

14 While we do not explicitly impose short-sale constraints, our results would be unchanged by these constraints. As will be shown in Proposition 4, capital requirements never induce financial institutions to short (long) assets that are held long (short) in the unconstrained mean-variance efficient (MVE) portfolio. Thus, assets that are held short in the unconstrained MVE portfolio would never be held in the presence of short-sale constraints (with or without capital requirements) and thus can simply be ignored. On the other hand, assets that are held long in the unconstrained MVE portfolio are also held in non-negative amounts in the presence of capital requirements and thus are unaffected by short-sale constraints. As noted in footnote 17, margin requirements could also be easily included in our model and would leave all qualitative results unchanged.

15 The above definition of regulatory capital differs from the market value of the institution's equity because the value of the institution's default option is not included in the value of the institution's assets. Thus, at the terminal date T, the value of the institution's equity is equal to $K(T)^+ = \max[0, K(T)]$. As it will become clear from Eqs. (2) and (4), we assume that capital requirements are defined in terms of regulatory capital, but that the institution has preferences defined over the terminal equity value.

16 This is consistent with the Basel Capital Accord, which sets the charge to cover credit risk equal to 0.08 times the sum of asset positions multiplied by asset-specific weights ranging from 0 to 1.5 (see: Basel Committee on Banking Supervision, 2001). This corresponds to risk weights between 0 and $0.08 \times 1.5 = 0.12$ in our definition. Unrated corporate claims (including equity) are assigned a weight of 100% (0.08 in our definition). Consistently with existing regulation, we assume a zero risk weight for investment in the money market account.

institution must satisfy the constraint

$$K(t) \geqslant kVaR + \beta^\top\big(\theta(t)^+ + \theta(t)^-\big) \tag{2}$$

at all times during the reporting period.[17] We assume that this constraint can be enforced. However, the institution's future trading strategy, and hence the institution's true VaR, are unobservable by regulators: therefore, the reported VaR can differ from the true VaR.

The institution incurs a non-financial cost c at the end of each reporting period in which the actual loss exceeds the reported VaR. This cost is meant to capture additional regulatory actions that can be undertaken in response to exceptions (besides the increase in the reserve multiple k) or reputation losses. For simplicity, we refer to these costs simply as reputation costs and assume that they are proportional to the amount by which the actual loss exceeds the reported VaR, that is, $c = \lambda(K_- - K - VaR)^+$, where $\lambda \geqslant 0$ is the proportional cost and K_- (respectively, K) is the value of the institution's capital at the beginning (respectively, at the end) of the reporting period. At the end of each backtesting period, the number i ($i = 0, 1, \ldots, n$) of reporting periods in which the actual loss exceeded the reported VaR is computed, and the capital reserve multiple k for the next backtesting period is set equal to $k(i)$, for some given positive numbers $k(0) \leqslant k(1) \leqslant \cdots \leqslant k(n)$. Clearly, reputation costs and the revision of the capital reserve multiple k at the end of each backtesting period represent incentives to not underreport the true VaR, while capital requirements provide an incentive to not overreport.

Aside for the market risk represented by the normal fluctuations in the value of its assets (as given in Eq. (1)), the institution is subject to unhedgeable idiosyncratic risk: at the end of every reporting period, there is a small probability p that a rare event will occur resulting in the loss of an amount equal to qK_-, where $q \in [0, 1]$. These rare events can force the institution into default if they result in the value of the institution's capital becoming negative (implying that the institution will be unable to repay its deposits). While these shocks are unhedgeable, the institution can control the probability of default by controlling the probability of losses in the market value of its assets exceeding $(1 - q)K_-$ in any given period (that is, by avoiding very risky investment strategies).[18]

Since the market value of deposits is fixed and there are no new equity issues, it follows from Eq. (1) that the institution's regulatory capital satisfies

$$\mathrm{d}K(t) = \big(K(t)r + \theta(t)^\top\mu\big)\,\mathrm{d}t + \theta(t)^\top\sigma\,\mathrm{d}w(t) \tag{3}$$

in the absence of rare events.

The financial institution has limited liability and preferences represented by an expected utility function over the market value of its equity at the end of the planning horizon. Thus,

[17] The constraint $K(t) \geqslant \beta^\top(\theta(t)^+ + \theta(t)^-)$ is identical to the one that would arise in the presence of margin requirements if the trader were allowed to earn market interest on the margin: see Cuoco and Liu (2000). Thus, while we assume that trading is frictionless, margin requirements could be easily accommodated and would amount to an increase in the vector of risk weights β by an amount equal to the proportional margin requirement. It would be a straightforward extension to allow β to be different across long and short positions.

[18] If the distribution of the random capital shocks has bounded support, as we assume, the institution's default could be avoided entirely by forcing it to maintain capital reserves in excess of the maximum possible realization of the shock. However, we implicitly assume that such a high level of capital requirements is suboptimal or that the distribution of the random shocks is unknown to regulators.

it chooses a reporting and trading strategy over $[0, T]$ so as to maximize

$$\mathrm{E}\big[u\big(K(T)^+\big)\big], \tag{4}$$

where u is an increasing and strictly concave utility function with $u(0) > -\infty$.[19]

3. Characterization results

3.1. Recursion for the value function

Let $V(K, K_-, VaR, i, k, t)$ denote the institution's value function at time t conditional on current capital being K, capital at the beginning of the current reporting period being K_-, the VaR reported at the beginning of the current reporting period being *VaR*, the number of exceptions in the current backtesting period being i and the current capital reserve multiple being k. Without loss of generality, suppose that t is in the hth reporting period, i.e., that $t \in [(h-1)\tau, h\tau)$. Finally, let $\mathcal{T} = \{1, 2, \dots, T\}$ denote the set of backtesting dates. Then it follows from the principle of dynamic programming that

$$\begin{aligned} V(K, K_-, VaR, i, k, t) &= \max_\theta \mathrm{E}\big[v\big(K(h\tau), K_-, VaR, i, k, h\tau\big) \bigm| K(t) = K\big] \\ \text{s.t.} \quad & \mathrm{d}K(s) = \big(K(s)r + \theta(s)^\top \mu\big)\,\mathrm{d}s + \theta(s)^\top \sigma \,\mathrm{d}w(s), \\ & K(s) \geqslant kVaR + \beta^\top\big(\theta(s)^+ + \theta(s)^-\big), \quad \text{for all } s \in [t, h\tau), \end{aligned} \tag{5}$$

for $K \geqslant kVaR$, where $v(K, K_-, VaR, i, k, h\tau)$ represents the value of having capital K at the end of the current period, before the capital shock is realized. In turn,

$$\begin{aligned} & v(K, K_-, VaR, i, k, h\tau) \\ & \quad = (1-p)\hat{v}(K, K_-, VaR, i, k, h\tau) + p\hat{v}(K - qK_-, K_-, VaR, i, k, h\tau), \end{aligned} \tag{6}$$

where

$$\begin{aligned} & \hat{v}(K, K_-, VaR, i, k, h\tau) \\ & \quad = \max_{VaR_1 \geqslant 0} V(K_1, K_1, VaR_1, i_1, k_1, h\tau) 1_{\{K \geqslant K_- - VaR\}} \\ & \qquad + \max_{VaR_2 \geqslant 0} V(K_2, K_2, VaR_2, i_2, k_2, h\tau) 1_{\{K < K_- - VaR\}} \end{aligned} \tag{7}$$

and

$$K_1 = K^+, \qquad K_2 = \big(K - \lambda(K_- - K - VaR)\big)^+,$$

$$i_1 = \begin{cases} 0 & \text{if } h\tau \in \mathcal{T}, \\ i & \text{otherwise}, \end{cases} \qquad i_2 = \begin{cases} 0 & \text{if } h\tau \in \mathcal{T}, \\ i+1 & \text{otherwise}, \end{cases}$$

$$k_1 = \begin{cases} k(i) & \text{if } h\tau \in \mathcal{T}, \\ k & \text{otherwise}, \end{cases} \qquad k_2 = \begin{cases} k(i+1) & \text{if } h\tau \in \mathcal{T}, \\ k & \text{otherwise}. \end{cases}$$

[19] The case $u(0) = -\infty$ would be less interesting, as it would rule out the possibility of default.

The function $\hat{v}$ in Eq. (7) represents the value of having capital K at the end of the current period after the shock is realized, This value depends on: (i) whether or not a loss exceeding the reported VaR is recorded at the end of the current reporting period (that is, whether or not $K_- - K(h\tau) > VaR$), and (ii) whether or not the end of the current reporting period coincides with the end of the current backtesting period (that is, whether or not $h\tau \in \mathcal{T}$). If the end of the current reporting period does not coincide with the end of the current backtesting period, the institution enters the next reporting period with the same reserve multiple k. Moreover, in this case the number of exceptions i going into the next period is incremented by one if a loss exceeding the reported VaR is recorded in the current period. On the other hand, if the end of the current reporting period coincides with the end of the current backtesting period, the reserve multiple going into the next period is set to the new value $k(i)$ or $k(i+1)$ and the number of exceptions is reset to zero. In all cases, the VaR reported at the beginning of the new period is determined optimally so as to maximize the continuation value. Thus, if no exception is recorded in the current period and capital after the shock realization is K, then, given limited liability, the continuation value associated with this capital is $V(K^+, K^+, VaR_1, i_1, k_1, h\tau)$. If, on the other hand, an exception is recorded, the institution incurs a reputation cost $c = \lambda(K_- - K - VaR)^+$, which lowers the continuation value associated with a given capital K from $V(K^+, K^+, VaR_2, i_2, k_2, h\tau)$ to $V((K-c)^+, (K-c)^+, VaR_2, i_2, k_2, h\tau)$.[20]

Equations (5)–(7) make it possible to compute the value function V recursively using the terminal condition at T

$$V(K, K_-, VaR, i, k, T) = u(K^+)$$

and solving the constrained continuous-time optimal investment problem in Eq. (5) backward one reporting period at a time. We therefore focus on this problem.

Remark 1. The capital requirement constraint in (5) implies

$$K(t) \geqslant kVaR \geqslant 0 \quad \text{for all } t \in \big[(h-1)\tau, h\tau\big). \tag{8}$$

Hence, given limited liability, the maximum possible loss $K_- - K(h\tau)$ over the reporting period cannot exceed $K_- - (kVaR - qK_-)^+$ after the capital shock. Clearly, reporting a VaR equal to the maximum possible loss over the reporting period is sufficient to avoid all the penalties, while reporting a VaR larger than this level is never optimal, since it provides no additional benefit and increases the capital reserve requirement. This implies

$$VaR \leqslant K_- - (kVaR - qK_-)^+,$$

or

$$VaR \leqslant \min\left[1, \frac{1+q}{1+k}\right] K_-. \tag{9}$$

[20] The cost c is non-financial in the sense that, while it reduces the continuation value, it does not affect the actual amount of capital with which the institutions enters the next period or the default boundary.

3.2. PDE characterization of the value function

We next derive a PDE characterization for the value function using a martingale duality approach as in Cvitanić and Karatzas (1992) and Cuoco (1997).[21,22] We start with a preliminary result. Let

$$\tilde{A} = \{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^m\colon\ \nu_0 \geqslant 0,\ \nu_0(\bar{1} - \beta) \leqslant \nu_- \leqslant \nu_0(\bar{1} + \beta)\},$$

and let $\mathcal{N}$ denote the set of $\tilde{A}$-valued bounded processes on $[0, \tau)$. Finally, for $\nu \in \mathcal{N}$ let

$$\xi_\nu(t) = \exp\left(-\int_0^t \left(r + \nu_0(s) + \frac{|\kappa_\nu(s)|^2}{2}\right) \mathrm{d}s - \int_0^t \kappa_\nu(s)^\top \mathrm{d}w(s)\right), \tag{10}$$

where

$$\kappa_\nu = \sigma^{-1}(\mu + \nu_- - \nu_0 \bar{1}).$$

The following result is then easily derived from Proposition 1 in Cuoco (1997).

Proposition 1. *Let*

$$\tilde{v}(z, K_-, VaR, i, k, h\tau) = \max_{K \geqslant kVaR}\left[v(K, K_-, VaR, i, k, h\tau) - zK\right], \tag{11}$$

where v is the function in Eq. (6) and consider the problem

$$\min_{\nu \in \mathcal{N}} \mathrm{E}\left[\tilde{v}(\psi \xi_\nu(h\tau), K_-, VaR, i, k, h\tau) - \psi\left(kVaR \int_{(h-1)\tau}^{h\tau} \xi_\nu(s)\nu_0(s)\,\mathrm{d}s - K_-\right)\right].$$

If the above problem has a solution for all $\psi > 0$, then

$$V(K, K_-, VaR, i, k, t) = \min_{\psi > 0}\left[\widetilde{V}(\psi, K_-, VaR, i, k, t) + \psi K\right] \tag{12}$$

for all $K \geqslant kVaR$ and all $t \in [(h-1)\tau, h\tau)$, where

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[21] Cvitanić and Karatzas (1992) develop the martingale duality technique for a class of investment problems involving convex constraints on the portfolio weights. The portfolio constraint in Eq. (5) involves the total value of capital K and thus is not included in the setup considered by Cvitanić and Karatzas (1992). Cuoco (1997) provides an extension of the martingale duality technique to a more general class of convex constraints on the portfolio amounts. This class of constraints includes the one in Eq. (5).

[22] The main reason to use the duality approach in this context is that the primal stochastic control problem has a discontinuous objective function (the function v defined in Eq. (6)). This fact prevents the application of results guaranteeing that the corresponding value function is the unique solution of the associated HJB equation and that finite-difference numerical approximations of this equation converge. On the other hand, as it will become clear in the sequel, the martingale duality approach leads to a stochastic control problem that has a continuous objective function (the function $\tilde{v}$ in Eq. (11)).

$$\widetilde{V}(z, K_-, VaR, i, k, t) = \min_{\nu\in\mathcal{N}} \mathrm{E}\left[\tilde{v}\big(Z_\nu(\tau), K_-, VaR, i, k, h\tau\big) - kVaR\int_t^{h\tau} Z_\nu(s)\nu_0(s)\,ds \;\Big|\; Z_\nu(t) = z\right]$$
$$\text{s.t.}\quad \mathrm{d}Z_\nu(t) = -Z_\nu(t)\big(\big(r+\nu_0(t)\big)\,\mathrm{d}t + \kappa_\nu(t)^\top\,\mathrm{d}w(t)\big). \tag{13}$$

Because the dual value function $\widetilde{V}$ solves the dynamic programming problem in Proposition 1, it must solve the Hamilton–Jacobi–Bellman (HJB) equation associated with this problem. Below we denote by ι_i the ith column of the $m \times m$ identity matrix.

Proposition 2. *If $\beta \in \mathbb{R}^m_{++}$, the dual value function $\widetilde{V}$ in Eq. (13) is strictly decreasing and strictly convex in z for all $t \in [(h-1)\tau, h\tau)$ and it solves the HJB equation*

$$0 = \widetilde{V}_t - rz\widetilde{V}_z + z^2\widetilde{V}_{zz}\min_{\nu\in\tilde{A}}\left[\frac{1}{2}\left|\sigma^{-1}\big(\mu+\nu_- - \nu_0\bar{1}\big)\right|^2 - \frac{\widetilde{V}_z + kVaR}{z\widetilde{V}_{zz}}\nu_0\right] \tag{14}$$

with terminal condition

$$\widetilde{V}(z, K_-, VaR, i, k, h\tau) = \tilde{v}(z, K_-, VaR, i, k, h\tau).$$

Moreover, the process ν^* attaining the minimum in Eq. (14) satisfies

$$0 \leqslant \nu_0^* \leqslant M$$

and

$$M\big(\bar{1} - \beta\big) \leqslant \nu_- \leqslant M\big(\bar{1} + \beta\big),$$

where $M = \max\{|\iota_i^\top\mu|/\iota_i^\top\beta\colon\ i = 1, \ldots, m\}$. Hence, $\nu^* \in \mathcal{N}$.

Remark 2. The dual value function for the institution's optimization problem in the absence of capital requirements (that is, with $k = \beta = \lambda = 0$) satisfies the PDE in Eq. (14) with $\tilde{A} = \{0\}$ (and hence $\nu^* = 0$).

3.3. Optimal investment strategy

Once the dual value function $\widetilde{V}$ is known, the optimal trading strategy θ and the process for the institution's capital K can be easily recovered. To prevent excessively cumbersome notation, we suppress from now on the dependence of the dual value function on the variables (K_-, VaR, i, k) which are constant within each reporting period.

Proposition 3. *If $\beta \in \mathbb{R}^m_{++}$, the optimal trading strategy θ for the constrained problem in (5) is given by $\theta(t) = \theta(Z_{\nu^*}(t), t)$, where*

$$\theta(z,t) = z\widetilde{V}_{zz}(z,t)(\sigma\sigma^\top)^{-1}\big(\mu + \nu_-^*(z,t) - \nu_0^*(z,t)\bar{1}\big). \tag{15}$$

Moreover, under the optimal trading strategy, the institution's capital at time t is given by $K(t) = K(Z_{\nu^*}(t), t)$, where

$$K(z,t) = -\widetilde{V}_z(z,t). \tag{16}$$

In particular,

$$K(z, h\tau) = \arg\max_K \big[v(K, h\tau) - zK\big].$$

Remark 3. The above proposition with $\nu^* = 0$ also characterizes the optimal trading strategy for the unconstrained problem.

The next result provides an explicit characterization of the optimal trading strategy in the presence of capital requirements. We denote by I_i the $i \times m$ matrix consisting of the first i rows of the $m \times m$ identity matrix.

Proposition 4. *Suppose that $\beta \in \mathbb{R}^m_{++}$ and that all the components of the unconstrained mean-variance efficient portfolio $(\sigma\sigma^\top)^{-1}\mu$ are different from zero.*[23] *For $i, j \in \{1, 2, \ldots, m\}$, $j \leqslant i$, let*

$$\eta_{i,j} = \frac{\iota_j^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i \mu}{\iota_j^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i H\beta},$$

where

$$H = \operatorname{diag}\big(\operatorname{sign}\big((\sigma\sigma^\top)^{-1}\mu\big)\big) \tag{17}$$

and suppose without loss of generality that the assets are sorted so that

$$\eta_{i,i} = \min\{\eta_{i,j}\colon\ \eta_{i,j} > 0,\ j = 1, \ldots, i\}.$$

For $i = 1, 2, \ldots, m$, let

$$h_i = \beta^\top H I_i^\top \big(I_i \sigma\sigma^\top I_i^\top\big)^{-1} I_i (\mu - \eta_{i,i} H\beta)$$

and let

$$h_{m+1} = \beta^\top H (\sigma\sigma^\top)^{-1}\mu.$$

Then

$$0 = h_1 \leqslant h_2 \leqslant \cdots \leqslant h_{m+1}.$$

If

$$-\frac{\widetilde{V}_z(z,t) + kVaR}{z\widetilde{V}_{zz}(z,t)} \geqslant h_{m+1},$$

then $\nu_0^*(z,t) = 0$ and

$$\theta(z,t) = z\widetilde{V}_{zz}(z,t)(\sigma\sigma^\top)^{-1}\mu. \tag{18}$$

[23] If an asset is not included in the unconstrained mean-variance efficient portfolio, it would not be included in the constrained portfolio and thus can be ignored.

If

$$h_i \leqslant -\frac{\widetilde{V}_z(z,t) + kVaR}{z\widetilde{V}_{zz}(z,t)} < h_{i+1},$$

for $i = 1, 2, \ldots, m$ *then*

$$\nu_0^*(z,t) = \frac{\beta^\top H I_i^\top (I_i \sigma \sigma^\top I_i^\top)^{-1} I_i \mu + \frac{\widetilde{V}_z(z,t)+kVaR}{z\widetilde{V}_{zz}(z,t)}}{\beta^\top H I_i^\top (I_i \sigma \sigma^\top I_i^\top)^{-1} I_i H\beta} \tag{19}$$

and

$$\theta(z,t) = z\widetilde{V}_{zz}(z,t) I_i^\top \big(I_i \sigma \sigma^\top I_i^\top\big)^{-1} I_i \big(\mu - \nu_0^*(z,t) H\beta\big). \tag{20}$$

In particular, $H\theta \geqslant 0$*, that is, the components of the constrained optimal portfolio never have the opposite sign of the corresponding components of the mean-variance efficient portfolio.*

The result in Proposition 4 that the components of the constrained optimal portfolio never have the opposite sign of the corresponding components of the mean-variance efficient (MVE) portfolio implies that the nonlinear portfolio constraint in Eq. (5) is equivalent to the pair of linear constraints

$$H\theta(s) \geqslant 0, \tag{21}$$

$$K(s) \geqslant kVaR + \beta^\top H\theta(s), \tag{22}$$

where H, as defined in Eq. (17), is the diagonal matrix whose ith diagonal element is the sign of the ith component of the MVE portfolio. The characterization of the optimal portfolio strategy in the previous proposition is then quite intuitive. As long as the non-negativity constraint in Eq. (21) is not binding, constrained optimal portfolios are combinations of a long position in the portfolio that maximizes expected return for a given variance (the MVE portfolio) and a short position in the portfolio that maximizes the charge for credit risk $\beta^\top H\theta$ for a given variance. We refer to the latter portfolio as the constrained minimum-variance (CMV) portfolio, since it is also the portfolio that minimizes variance subject to a constraint on the charge for credit risk. More generally, for $h_i < -(\widetilde{V}_z(z,t) + kVaR)/(z\widetilde{V}_{zz}(z,t)) \leqslant h_{i+1}$ the non-negativity constraint in Eq. (21) binds for the last $m-i$ assets, so that $\iota_j^\top \theta(z,t) = 0$ for $j = m-i, \ldots, m$ and (as shown in Eq. (20))

$$\theta(z,t) = z\widetilde{V}_{zz}(z,t)\pi_i^{\mathrm{MVE}} - z\widetilde{V}_{zz}(z,t)\nu_0^*(z,t)\pi_i^{\mathrm{CMV}},$$

where

$$\pi_i^{\mathrm{MVE}} = \big(I_i \sigma \sigma^\top I_i^\top\big)^{-1} I_i \mu$$

denotes the MVE portfolio of the first i risky assets and

$$\pi_i^{\mathrm{CMV}} = \big(I_i \sigma \sigma^\top I_i^\top\big)^{-1} I_i H\beta$$

denotes the CMV portfolio of the first i risky assets.

Both $h(z,t) = -(\widetilde{V}_z(z,t) + kVaR)/(z\widetilde{V}_{zz}(z,t))$ and $K(z,t) = -\widetilde{V}_z(z,t)$ are monotonically decreasing functions of z.[24] Thus, Proposition 4 shows that when capital $K(Z_{\nu^*}(t),t)$ is large (that is, when $Z_{\nu^*}(t)$ is small and $h(Z_{\nu^*}(t),t) \geqslant h_{m+1}$), the capital constraint does not bind (that is, $\nu_0^*(Z_{\nu^*}(t),t) = 0$) and the financial institution holds the mean-variance efficient portfolio of risky assets π_m^{MVE}. For lower levels of capital (that is, when $h(Z_{\nu^*}(t),t) < h_{m+1}$), the constraint starts to bind (that is, $\nu_0^*(Z_{\nu^*}(t),t)$ becomes positive) and the institution is forced to alter its leverage to satisfy the constraint. At the same time, it finds it optimal to rebalance its portfolio of risky assets: this rebalancing is done by shorting the constrained minimum-variance portfolio π_m^{CMV}.

For even lower levels of capital (higher values of $Z_{\nu^*}(t)$), shorting of the corrective portfolio progressively increases, until the institution reaches a point where its investment in the mth asset is zero (this happens when $h(Z_{\nu^*}(t),t) = h_m$). Beyond this point, the institution simply drops the mth asset from its portfolio, since it is never optimal to short (respectively, long) an asset that is held in positive (respectively, negative) amounts in the unconstrained mean-variance efficient portfolio. If the assets are uncorrelated (σ is diagonal) the mth asset is the one with the lowest ratio of absolute risk premium $|\iota_j^\top \mu|$ to risk weight $\iota_j^\top \beta$. If the assets are correlated, then correlations are also taken into account in deciding which asset is dropped first from the portfolio, and the mth asset is the one with the lowest ratio $\eta_{m,j}$. Thus, in either case, as the institution is forced to reduce its allocation to risky assets to satisfy the capital constraint, it finds it optimal to tilt its portfolio toward assets with high absolute risk premia and low risk weights.[25]

If capital decreases ($Z_{\nu^*}(t)$ increases) even further and the constraint becomes even more severe, the institution sequentially drops other risky assets from its portfolio, concentrating on those characterized by progressively higher absolute risk premia and lower risk weights. This happens each time that $h(Z_{\nu^*}(t),t)$ exceeds a new value h_j. Eventually, if $h(Z_{\nu^*}(t),t) = h_1 = 0$ (that is, if $K(Z_{\nu^*}(t),t) = kVaR$), the institution is forced to invest its entire portfolio in the riskless asset. In general, whenever $h(Z_{\nu^*}(t),t)$ is between h_i and h_{i+1}, the institution only holds the first i risky assets and its portfolio is a combination of the riskless asset and two funds of risky assets: the mean-variance efficient portfolio of the first i assets, π_i^{MVE} and the constrained minimum-variance portfolio of the first i assets, π_i^{CMV}. Thus, locally (that is, between any pair h_i and h_{i+1}), optimal portfolios satisfy three-fund separation.

3.4. The one-dimensional case

Not surprisingly, the results in the previous two subsections take a very simple form in the case of a single risky asset, as shown in the following corollary.

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[24] The term ν_0^* can be interpreted as the Lagrangian multiplier on the constraint in the primal problem in Eq. (5): thus ν_0^* is inversely related to capital K. Since h is a decreasing function of ν_0^* (as shown in Eq. (19)), h must be an increasing function of K (that is, a decreasing function of z).

[25] This substitution effect is similar to the one described by Kohen and Santomero (1980) and Kim and Santomero (1988) in a static setting and by Blum (1999) in a two-period setting.

Corollary 1. *In the case of a single risky asset, a positive risk premium and a positive risk weight* ($m = 1$, $\mu > 0$, $\beta > 0$) *the HJB equation* (14) *reduces to*

$$0 = \widetilde{V}_t - rz\widetilde{V}_z + z^2\widetilde{V}_{zz} \min\left[\left(\frac{\mu}{\sigma}\right)^2, -\frac{\mu}{\beta}\left(\frac{\widetilde{V}_z + kVaR}{z\widetilde{V}_{zz}(z,t)}\right)\right]$$
$$-\frac{1}{2}z^2\widetilde{V}_{zz} \min\left[\left(\frac{\mu}{\sigma}\right)^2, \left(\frac{\sigma}{\beta}\right)^2\left(\frac{\widetilde{V}_z + kVaR}{z\widetilde{V}_{zz}(z,t)}\right)^2\right].$$

Moreover, the optimal investment strategy in (15) *reduces to*

$$\theta(z,t) = \min\left[z\widetilde{V}_{zz}(z,t)\frac{\mu}{\sigma^2}, \frac{1}{\beta}\big(K(z,t) - kVaR\big)\right]$$
$$= \min\left[\frac{1}{\Gamma(z,t)}\frac{\mu}{\sigma^2}K(z,t), \frac{1}{\beta}\big(K(z,t) - kVaR\big)\right], \tag{23}$$

where

$$\Gamma(z,t) = -\frac{\widetilde{V}_z(z,t)}{z\widetilde{V}_{zz}(z,t)}$$

is the relative risk aversion coefficient of the primal value function.[26]

Because, as noted in Proposition 4, the optimal portfolio in the presence of capital requirements always has the same sign as the MVE portfolio, in the case of a single risky asset the assumption of a positive risk premium implies a non-negative allocation to the risky asset. The constraint in Eq. (5) then implies

$$\theta(z,t) \leqslant \frac{1}{\beta}\big(K(z,t) - kVaR\big). \tag{24}$$

Equation (23) shows that, under the stated assumptions, the optimal portfolio allocation is equal to the minimum of $\frac{1}{\Gamma(z,t)}\frac{\mu}{\sigma^2}K(z,t)$, which can be interpreted as the optimal portfolio allocation in the absence of currently-binding capital requirements, and the upper bound in Eq. (24), which represents the maximum feasible portfolio allocation under capital requirements.

[26] By duality,

$$z = V_K\big(K(z,t),t\big) = V_K\big(-\widetilde{V}_z(z,t),t\big)$$

and hence (differentiating with respect to z and rearranging)

$$-\frac{\widetilde{V}_z(z,t)}{z\widetilde{V}_{zz}(z,t)} = -\frac{K(z,t)V_{KK}(K(z,t),t)}{V_K(K(z,t),t)}.$$

4. Analysis of optimal policies

For our numerical analysis, we fix throughout the backtesting period to one year, the investment horizon to two years ($T = 2$) and the reporting period to one business day ($n = 250$, $\tau = 1/250$). We assume that the risk weight used to determine the capital charge to cover credit risk is $\beta = 0.08$,[27] and that the reserve multiple used to determine the capital charge to cover market risk is determined according to the schedule proposed by the Basel Committee, that is,[28]

$$k(i) = \begin{cases} 3.00\sqrt{10} = 9.49 & \text{if } i \leqslant 4, \\ 3.40\sqrt{10} = 10.75 & \text{if } i = 5, \\ 3.50\sqrt{10} = 11.07 & \text{if } i = 6, \\ 3.65\sqrt{10} = 11.54 & \text{if } i = 7, \\ 3.75\sqrt{10} = 11.86 & \text{if } i = 8, \\ 3.85\sqrt{10} = 12.17 & \text{if } i = 9, \\ 4.00\sqrt{10} = 12.65 & \text{if } i \geqslant 10. \end{cases}$$

We consider the cases in which the number m of risky assets equals 1 or 2: since, as noted in the previous section, optimal investment policies satisfy local three-fund separation, considering additional risky assets would not affect the analysis. In either case, we set $r = 0$ and choose the vector of risk premia μ and the volatility matrix σ so that the risk premium (respectively, the volatility) of the mean-variance efficient portfolio of risky assets equals 0.059 (respectively, 0.22).[29] In the case of two risky assets, we assume in addition that the volatility of the first asset (respectively, the second asset) is 25% higher (respectively, 25% lower) than the volatility of the mean-variance efficient portfolio and that the correlation coefficients between the returns on the two assets is 0.50.[30] We assume the probability p of the rare capital shock at the end of each reporting period is equal to 0.001 and the size q of the shock is 0.9. In addition, we assume a CRRA utility function, i.e., $u(K) = K^{1-\gamma}/(1-\gamma)$ for some $\gamma \in (0, 1)$. Under this assumption,

$$V(K, K_-, VaR, i, k, t) = K_-^{1-\gamma} V(K/K_-, VaR/K_-, i, k, t)$$

and hence it follows from Proposition 3 and Eq. (3) that the proportional portfolio allocation

$$\pi(t) = \frac{\theta(t)}{K(t)} = -\frac{Z_{\nu^*}(t)\widetilde{V}_{zz}(Z_{\nu^*}, t)}{\widetilde{V}_z(Z_{\nu^*}, t)}(\sigma\sigma^\top)^{-1}\big(\mu + \nu_-^*(Z_{\nu^*}, t) - \nu_0^*(Z_{\nu^*}, t)\bar{1}\big)$$

and the distribution of $K(t)/K_-$ are independent of K_-.

We solve for the dual value function recursively as explained in the previous section by numerically integrating the PDE (14) using a finite-difference approximation.[31] This is

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[27] See footnote 16.

[28] See footnote 8.

[29] These values correspond to the mean risk premium and the return standard deviation of the market portfolio as estimated by Ibbotson and Sinquefeld (1982).

[30] These assumptions imply that $\mu = \binom{0.07225}{0.02937}$ and $\sigma = \binom{0.27500\ 0.00000}{0.08250\ 0.14289}$ in our simulations with $m = 2$.

[31] Convergence of this approximation in our problem can be verified by rewriting the problem in terms of the state variable $z_\nu = \log(Z_\nu)$ and then using Theorem IX.4.1 in Fleming and Soner (1992).

done for each value of i and k and for a grid of values for the reported VaR between zero and the upper bound in Eq. (9). We also compute the distribution function P of the state variable Z_{ν^*} at the end of each reporting period by using a finite-difference approximation to solve the PDE

$$\frac{\partial}{\partial t}P(z,t)=\frac{1}{2}\frac{\partial}{\partial z}\big(\big|\kappa_{\nu^*}(z,t)z\big|^2 P_z(z,t)\big)+\big(r+\nu_0^*(z,t)\big)zP_z(z,t),$$
$$P\big(z,(h-1)\tau\big)=1_{\{z\geqslant\psi^*\}},\tag{25}$$

for $t\in[(h-1)\tau,h\tau)$, where ψ^* is the value of ψ solving the minimization in (12) with $t=(h-1)\tau$.[32] This allows us to compute the distribution of the financial institution's capital using Eq. (16), and hence the true VaR of the portfolio, which we compare to the reported VaR. For comparison purposes, we also compute the optimal policies and the true VaR for the unconstrained problem with no capital requirements using the results in Remarks 2 and 3. Finally, we compute and report the probabilities of default in both the constrained and the unconstrained problem. Given the assumption of a CRRA utility function (and hence of infinite marginal utility at zero), the probability of the institution defaulting conditional on there being no capital shock is zero. On the other hand, the probability of the institution defaulting conditional on there being a capital shock is equal to the probability that end-of-period capital before the shock is insufficient to cover the loss associated with the shock, that is to the probability that end-of-period capital is below qK_-.

4.1. One risky asset

Table 1 shows the optimal reporting and investment strategy at the beginning of the first and of the last reporting period in the first year ($t=(h-1)\tau$, where $h=1$ and $h=250$, respectively) for three different values of the number of violations in the current backtesting period ($i=0$, $i=5$ and $i=9$) and two different values of the current reserve multiple ($k=9.49$ and $k=12.65$) for the case $\gamma=0.25$, $\lambda=0.01$ and $m=1$. For each combination of (h,i,k), the table shows the reported VaR normalized by the beginning-of-period capital, $v=VaR/K_-$, the beginning-of-period maximum possible proportional allocation to the risky asset under the capital requirement constraint, $\bar{\pi}((h-1)\tau)=\frac{1}{\beta}(1-kv)$,[33] the beginning-of-period proportional allocation to risky assets, $\pi((h-1)\tau)$, the true 1-day 90 and 99% VaRs normalized by the initial capital value, $v_{0.90}$ and $v_{0.99}$, and the default probability conditional on a capital shock, p_D. For comparison purposes, the table also shows the beginning-of-period proportional allocation to risky assets, $\pi^U((h-1)\tau)$, the

[32] The PDE in (25) is obtained by integrating the forward Kolmogorov equation

$$\frac{\partial}{\partial t}p(z,t)=\frac{1}{2}\frac{\partial^2}{\partial z^2}\big(\big|\kappa_{\nu^*}(z,t)z\big|^2 p(z,t)\big)+\frac{\partial}{\partial z}\big(\big(r+\nu_0^*(z,t)\big)zp(z,t)\big)$$

solved by the density function $p(z,t)=P_z(z,t)$.

[33] See Eq. (24).

Table 1
Results for the parameter values: $\gamma = 0.25$, $\lambda = 0.01$, $m = 1$

h	i	k	v	$\bar{\pi}$	π	$v_{0.90}$	$v_{0.99}$	p_D	π^U	$v^U_{0.90}$	$v^U_{0.99}$	p^U_D
1	0	9.49	.0699	4.212	4.212	.0699	.0699	.0092	4.861	.0811	.1529	.0584
1	0	12.65	.0542	3.930	3.872	.0542	.0542	.0074	4.861	.0811	.1529	.0584
1	5	9.49	.0703	4.164	4.141	.0703	.0703	.0016	4.861	.0811	.1529	.0584
1	5	12.65	.0532	4.093	4.093	.0532	.1033	.0130	4.861	.0811	.1529	.0584
1	10	9.49	.0692	4.291	4.291	.0692	.1074	.0165	4.861	.0811	.1529	.0584
1	10	12.65	.0531	4.097	4.097	.0531	.1033	.0130	4.861	.0811	.1529	.0584
250	0	9.49	.0692	4.293	4.293	.0692	.1077	.0164	4.861	.0811	.1529	.0584
250	0	12.65	.0531	4.096	4.096	.0531	.0824	.0063	4.861	.0811	.1529	.0584
250	5	9.49	.0702	4.170	4.151	.0702	.0702	.0053	4.861	.0811	.1529	.0584
250	5	12.65	.0544	3.903	3.822	.0544	.0544	.0033	4.861	.0811	.1529	.0584
250	10	9.49	.0692	4.293	4.293	.0692	.1174	.0310	4.861	.0811	.1529	.0584
250	10	12.65	.0531	4.097	4.097	.0531	.1034	.0131	4.861	.0811	.1529	.0584

Note. The table shows the values of the reported normalized daily VaR (v), the upper bound on the portfolio allocation to the risky asset ($\bar{\pi}$), the portfolio allocation to the risky asset (π), the true normalized 90 and 99% VaRs ($v_{0.90}$ and $v_{0.99}$) and the conditional probability of default (p_D) in the presence of capital requirements, as well as the portfolio allocation to the risky asset (π^U), the true normalized 90 and 99% VaRs ($v^U_{0.90}$ and $v^U_{0.99}$) and the conditional probability of default (p^U_D) in the absence of capital requirements at the beginning of two different reporting periods.

true normalized 1-day 90 and 99% VaRs, $v^U_{0.90}$ and $v^U_{0.99}$, and the conditional default probability, p^U, in the unconstrained case.[34]

For the set of parameters considered in Table 1, a financial institution not subject to capital shocks or capital requirements would choose a proportional allocation to risky assets of $\kappa/(\gamma\sigma) = 487.6\%$. As shown in Table 1, an institution not subject to capital requirements but subject to capital shocks would lower the beginning-of-period portfolio weight slightly to 486.1% in response to the additional risk represented by the capital shocks. The lighter line in Fig. 1 shows how this beginning-of-period portfolio weight varies as a function of the normalized capital K/K_-: when capital is large, the background risk becomes relatively less relevant and the portfolio allocation converges to $\kappa/(\gamma\sigma)$. On the other hand, when capital is below the conditional default point (that is, when $K/K_- < q = 0.9$), the institution has incentives to exploit its limited liability and "bet for resurrection" by increasing its risk taking. In fact, as the end of the period approaches, the portfolio weight becomes unboundedly large if the institution is just below the default point. The lighter line in Fig. 2 shows the corresponding probability distribution of end-of-period capital. This probability distribution assigns no mass to values of end-of-period capital around the conditional default point: in other words, given limited liability, the institution will find it optimal not to default for infinitesimal amounts, choosing instead a trading strategy as described above to avoid such an outcome. As shown in Table 1, the probability mass to the left of the conditional default point in Fig. 2 is $p^U_D = 5.84\%$, which corresponds to an unconditional default probability $pp^U_D = 0.00584\%$.

[34] We report the portfolio allocation and VaRs normalized by capital K (rather than by total assets $A = D + K$), since these values are independent of the institution's leverage.

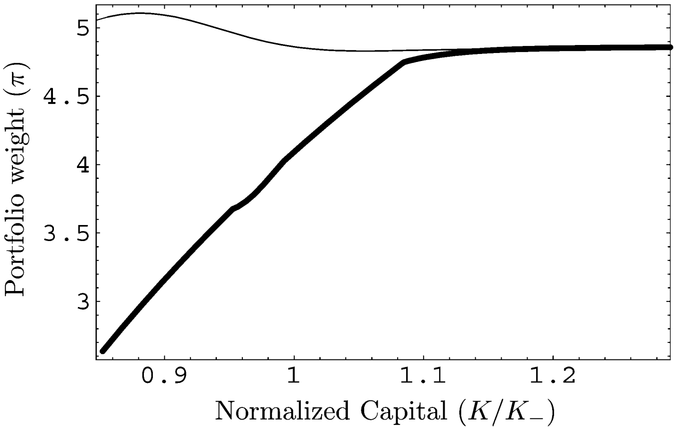

Fig. 1. The graph plots the optimal portfolio allocation to risky assets at the beginning of the first reporting period in the presence of capital requirements (heavier solid line) and in the absence of capital requirements (lighter solid line), for the case $\gamma = 0.25$, $\lambda = 0.01$, $m = 1$, $i = 5$, $k = 12.65$.

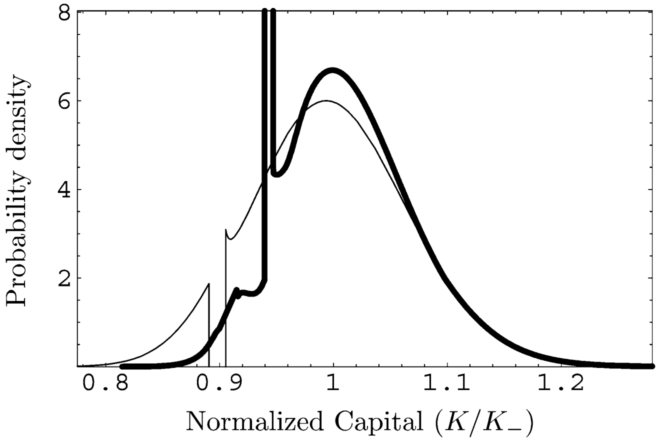

Fig. 2. The graph plots the probability distribution of normalized capital at the end of the first reporting period in the presence of capital requirements (heavier solid line) and in the absence of capital requirements (lighter solid line), for the case $\gamma = 0.25$, $\lambda = 0.01$, $m = 1$, $i = 5$, $k = 12.65$.

The heavier line in Fig. 1 (respectively, Fig. 2) shows the beginning-of-period portfolio weight (respectively, the probability distribution of end-of-period capital) for an institution subject to capital requirements, assuming the number of exceptions recorded in the current backtesting period is $i = 5$ and that the applicable reserve multiple is $k = 12.65$. As shown in Table 1, such an institution would choose to report a normalized VaR $v = 5.32\%$ at the beginning of the period. Two critical levels of capital then become relevant in Figs. 1 and 2. The first is the point below which, conditional on a capital shock occurring, the continuation value equals zero. This is the point at which

$$K - qK_- - \lambda\big(K_- - VaR - (K - qK_-)\big)^+ = 0,$$

or $K/K_- = q + \lambda(1-v)/(1+\lambda) = 0.909375$. The second is the point below which an exception is recorded for the current period. This is the point at which $K = K_- - VaR$, or $K/K_- = 1 - v = 0.9468$. For the same reasons described above in the unconstrained case, the institution has incentives to increase its risk taking as the end of the period approaches if capital is just below either of these two points. This behavior is however restricted by capital requirements. In fact, Eq. (23) implies that, with a single risky asset, the portfolio weight equals the minimum of $\frac{1}{\Gamma(t)}\frac{\kappa}{\sigma}$, which represents the portfolio weight that would be chosen in the absence of currently-binding capital requirements and $\bar{\pi}(t) = \frac{1}{\beta}(1 - \frac{kv}{K(t)/K_-})$, which represents the maximum feasible portfolio weight under capital requirements. Figure 3 plots the optimal portfolio weight at the beginning of the first reporting period (heavier solid line), together with the portfolio weight that would be chosen in the absence of currently-binding capital requirements and the maximum feasible portfolio weight (dotted line). Figure 4 plots the same curves at time $t = 0.003$. As shown in both figures, when normalized capital is large, capital requirements become not binding and the portfolio weights in both the constrained and the unconstrained model converge to the same value, $\kappa/(\gamma\sigma)$. However, when normalized capital is low, capital requirements bind and the institution is forced to reduce its allocation to risky assets. In particular, as shown in Table 1, the institution would have to lower its allocation to 409.3% at the beginning of the period (when $K/K_- = 1$) in order to meet capital requirements. Moreover, as shown in the figures, any increase in risk taking for values of capital just below the two critical points mentioned above is also prevented by binding capital requirements. As a result of the reduction in risk taking induced by capital requirements when normalized capital is low, Fig. 2 shows that the institution is unable to drive the probability of end-of-period levels of capital just below the two critical points to zero.

Figures 3 and 4 also show that, due to the costs associated with a violation of the reported VaR, the institution has incentives to decrease its risky position as the end of the period approaches and capital is just above the point below which an exception is recorded

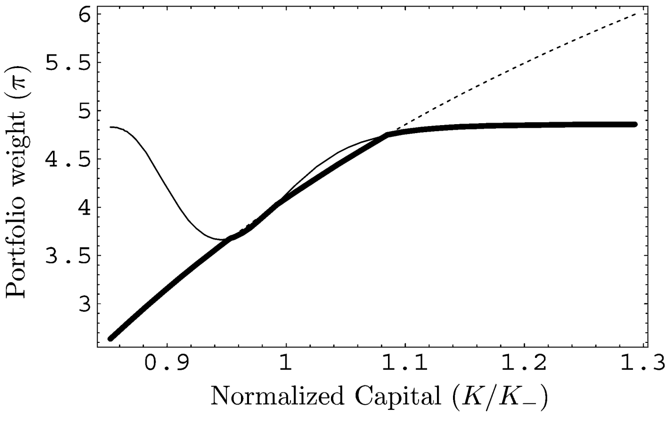

Fig. 3. The graph plots the optimal portfolio allocation to risky assets at the beginning of the first reporting period (heavier solid line), the allocation that would be chosen in the absence of currently-binding capital requirements (lighter solid line) and the maximum feasible portfolio allocation (dotted line), for the case $\gamma = 0.25$, $\lambda = 0.01$, $m = 1$, $i = 5$, $k = 12.65$.

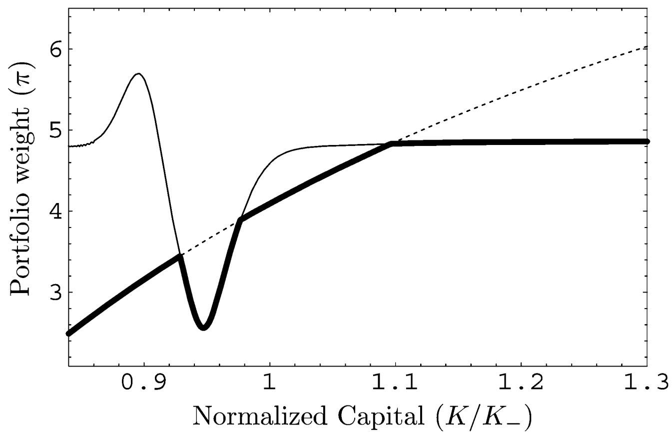

Fig. 4. The graph plots the optimal portfolio allocation to risky assets at time $t = 0.003$ (heavier solid line), the allocation that would be chosen in the absence of currently-binding capital requirements (lighter solid line) and the maximum feasible portfolio allocation (dotted line), for the case $\gamma = 0.25$, $\lambda = 0.01$, $m = 1$, $i = 5$, $k = 12.65$.

(that is, as K/K_- is just above $1 - v = 0.9468$), so as to reduce the probability of recording an exception. This reduction in risk taking is consistent with capital requirements and induces a singularity in the probability density in Fig. 2 at the point $K/K_- = 1 - v = 0.9468$: in other words, there is a positive probability that end-of-period capital will exactly equal this critical level.

Because capital requirements result in a decrease in risk taking, the end-of-period capital distribution in the presence of capital requirements has a thinner left tail than that in the absence of capital requirements. This is reflected in Table 1 in a reduction of both the 90 and the 99% true VaRs compared to the case of no capital requirements. More importantly, the conditional default probability is significantly reduced, from 5.84 to 1.30%. Since, as noted in Fig. 2, there is a positive probability that end-of-period capital will be exactly $K_- - VaR$, several different VaR values are concentrated at exactly the reported VaR. As shown in Table 1, for the case under consideration the 90% VaR happens to coincide with the reported VaR.

A general examination of Table 1 shows that, for all values of h, i and k, VaR-based capital requirements are effective in curbing the risk of trading portfolios and corresponding default probabilities. It is also interesting to note from the table that, given the number of exceptions in the current period, a larger value of the reserve multiplicative factor k does not necessarily result in more stringent capital requirements and lower default probabilities, since the institution has stronger incentives to underreport its true 99% VaR the higher the value of the multiplicative factor. Nevertheless, the penalties associated with backtesting appear to be effective in mitigating the incentives to underreport and, for the set of parameters considered in Table 1, reported VaRs are never below the true 90% VaR.

The incentives to underreport also depend critically on the institution's risk aversion. Since the utility loss of capital requirements is larger the lower the risk aversion, institutions with low risk aversion have a stronger incentive to underreport their true VaR, while

Table 2
Results for the parameter values: $\gamma = 0.50$, $\lambda = 0.01$, $m = 1$

h	i	k	v	$\bar{\pi}$	π	$v_{0.90}$	$v_{0.99}$	p_D	π^U	$v^U_{0.90}$	$v^U_{0.99}$	p^U_D
1	0	9.49	.0835	2.596	2.332	.0364	.0650	.0000	2.355	.0411	.0714	.0003
1	0	12.65	.0634	2.476	2.306	.0358	.0634	.0000	2.355	.0411	.0714	.0003
1	5	9.49	.0835	2.596	2.332	.0362	.0649	.0000	2.355	.0411	.0714	.0003
1	5	12.65	.0634	2.474	2.303	.0358	.0634	.0000	2.355	.0411	.0714	.0003
1	10	9.49	.0835	2.597	2.332	.0362	.0651	.0000	2.355	.0411	.0714	.0003
1	10	12.65	.0634	2.476	2.306	.0358	.0634	.0000	2.355	.0411	.0714	.0003
250	0	9.49	.0835	2.597	2.332	.0363	.0650	.0000	2.355	.0411	.0714	.0003
250	0	12.65	.0634	2.478	2.306	.0356	.0634	.0000	2.355	.0411	.0714	.0003
250	5	9.49	.0835	2.596	2.332	.0364	.0651	.0000	2.355	.0411	.0714	.0003
250	5	12.65	.0634	2.474	2.303	.0356	.0634	.0000	2.355	.0411	.0714	.0003
250	10	9.49	.0835	2.597	2.332	.0367	.0653	.0000	2.355	.0411	.0714	.0003
250	10	12.65	.0634	2.478	2.306	.0360	.0634	.0000	2.355	.0411	.0714	.0003

Note. See Table 1.

institutions with high risk aversion have a stronger incentive to overreport. As shown in Table 2, an institution with a risk aversion coefficient of 0.50 (or larger) would never underreport its true VaR at $t = 0$ and possibly over-report.[35]

Higher costs associated with exceptions (higher values of λ) lead to a decrease in risk-taking (as measured by the initial portfolio allocation π) and an increase in reported VaRs. With a value of λ equal to 5%, Table 3 shows that an institution with risk aversion $\gamma = 0.25$ would only underreport its true VaR in the extreme case in which the number of exceptions recorded in the current period has already reached $i = 10$, so that there is no opportunity cost of higher capital requirements in the future associated with additional exceptions (since no additional increases in the multiplicative factor k are imposed once the number of exceptions exceeds 10). As a consequence of the reduction in risk-taking, higher reputation costs also result in a reduction in the conditional default probabilities, so that these probabilities become essentially zero for the set of parameters considered in Table 3.

4.2. *Two risky assets*

To check how the results reported in the previous subsection are affected by the assumption of a single risky asset, we also compute the optimal reporting and investment strategies for the case of two risky assets. As mentioned at the beginning of this section, in order to allow a comparison with the case of a single risky asset, we choose the price coefficient μ and σ so that the risk premium and volatility of the mean-variance efficient portfolio of risky assets are the same as in previous examples. Finally, we assume that the two assets

[35] Berkowitz and O'Brien (2002) compare daily VaR reports with historical daily profit and losses for a sample of 6 large US commercial banks and conclude that the banks in their sample tend to overestimate (or to overreport) their true VaR. Our results suggest that it should be possible to relate the extent of under- or over-reporting in a cross section of financial institutions to the volatility of the institution's asset value (which proxies for the institution's risk aversion).

Table 3
Results for the parameter values: $\gamma = 0.25$, $\lambda = 0.05$, $m = 1$

h	i	k	v	$\bar{\pi}$	π	$v_{0.90}$	$v_{0.99}$	p_D	π^U	$v^U_{0.90}$	$v^U_{0.99}$	p^U_D
1	0	9.49	.0710	4.076	4.076	.0710	.0710	.0000	4.861	.0811	.1529	.0584
1	0	12.65	.0557	3.689	3.658	.0557	.0557	.0000	4.861	.0811	.1529	.0584
1	5	9.49	.0710	4.078	4.078	.0710	.0710	.0000	4.861	.0811	.1529	.0584
1	5	12.65	.0557	3.688	3.657	.0557	.0557	.0000	4.861	.0811	.1529	.0584
1	10	9.49	.0710	4.075	4.075	.0710	.0710	.0000	4.861	.0811	.1529	.0584
1	10	12.65	.0557	3.688	3.658	.0557	.0557	.0000	4.861	.0811	.1529	.0584
250	0	9.49	.0711	4.074	4.074	.0711	.0711	.0000	4.861	.0811	.1529	.0584
250	0	12.65	.0557	3.688	3.658	.0335	.0557	.0000	4.861	.0811	.1529	.0584
250	5	9.49	.0711	4.074	4.074	.0711	.0711	.0000	4.861	.0811	.1529	.0584
250	5	12.65	.0557	3.688	3.657	.0466	.0577	.0000	4.861	.0811	.1529	.0584
250	10	9.49	.0710	4.077	4.077	.0710	.1174	.0000	4.861	.0811	.1529	.0584
250	10	12.65	.0557	3.688	3.658	.0557	.1034	.0000	4.861	.0811	.1529	.0584

Note. See Table 1.

Table 4
Results for the parameter values: $\gamma = 0.25$, $\lambda = 0.01$, $m = 2$

h	i	k	v	$\overline{\pi_1 + \pi_2}$	$\pi_1 + \pi_2$	$v_{0.90}$	$v_{0.99}$	p_D	$\pi^U_1 + \pi^U_2$	$v^U_{0.90}$	$v^U_{0.99}$	p^U_D
1	0	9.49	.0711	4.071	4.071	.0711	.0711	.0079	4.861	.0811	.1529	.0584
1	0	12.65	.0555	3.721	3.721	.0555	.0555	.0060	4.861	.0811	.1529	.0584
1	5	9.49	.0717	4.002	4.002	.0717	.0717	.0016	4.861	.0811	.1529	.0584
1	5	12.65	.0542	3.923	3.923	.0542	.1046	.0131	4.861	.0811	.1529	.0584
1	10	9.49	.0702	4.169	4.169	.0702	.1089	.0168	4.861	.0811	.1529	.0584
1	10	12.65	.0541	3.942	3.942	.0541	.1053	.0137	4.861	.0811	.1529	.0584
250	0	9.49	.0703	4.168	4.168	.0703	.1101	.0133	4.861	.0811	.1529	.0584
250	0	12.65	.0541	3.941	3.941	.0541	.0823	.0067	4.861	.0811	.1529	.0584
250	5	9.49	.0716	4.010	4.010	.0716	.0716	.0049	4.861	.0811	.1529	.0584
250	5	12.65	.0558	3.684	3.684	.0558	.0558	.0028	4.861	.0811	.1529	.0584
250	10	9.49	.0703	4.167	4.167	.0703	.1203	.0309	4.861	.0811	.1529	.0584
250	10	12.65	.0541	3.941	3.941	.0541	.1051	.0145	4.861	.0811	.1529	.0584

Note. The table shows the values of the reported normalized daily VaR (v), the upper bound on the total portfolio allocation to risky assets ($\overline{\pi_1 + \pi_2}$), the total portfolio allocation to risky assets ($\pi_1 + \pi_2$), the true normalized 90 and 99% VaRs ($v_{0.90}$ and $v_{0.99}$) and the conditional probability of default (p_D) in the presence of capital requirements, as well as the portfolio allocation to risky assets ($\pi^U_1 + \pi^U_2$), the true normalized 90 and 99% VaRs ($v^U_{0.90}$ and $v^U_{0.99}$) and the conditional probability of default (p^U_D) in the absence of capital requirements at the beginning of two different reporting periods.

have the same risk weights, so that the risk weight for the market portfolio is also the same as in the previous examples.

Table 4 reports the results for the same set of parameters as in Table 1, but with $m = 2$. The ability to engage in asset substitution to limit the impact of capital requirements in the presence of multiple risky assets results in higher VaRs than in the case of a single risky asset. However, a comparison of the results in Table 4 with those in Table 1 shows that this effect is quite small. Moreover, all the qualitative features of the optimal reporting and trading strategy described in the previous subsection also apply in the presence of multiple risky assets.

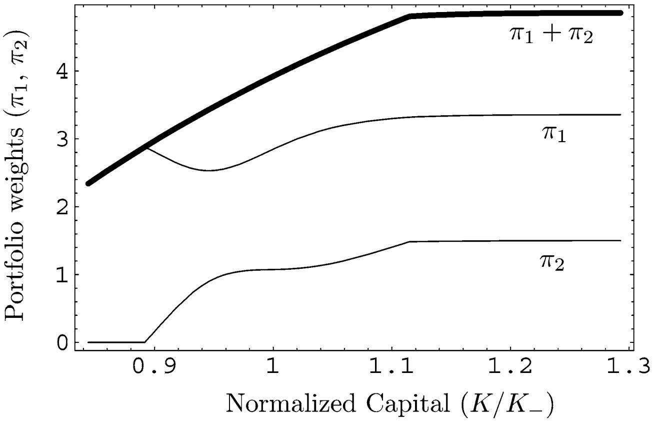

Fig. 5. The graph plots the optimal total portfolio allocation to risky assets (heavier solid line) and to each of the two risky assets (lighter solid lines) at the beginning of the first reporting period in the presence of capital requirements, for the case $\gamma = 0.25$, $\lambda = 0.01$, $m = 2$, $i = 5$, $k = 12.65$.

This is confirmed by Fig. 5, which plots the total allocation to risky assets as a function of normalized capital at the beginning of the first reporting period, for the case $i = 5$ and $k = 12.65$, as well as the contribution of the two risky assets to the total portfolio allocation. As implied by Proposition 4, the optimal portfolio policy significantly diverges from two-fund separation. As the constraint increasingly binds (that is, as normalized capital decreases), the weight of the second asset (the one with the lower ratio of risk premium to risk coefficient) in the optimal risky asset portfolio is progressively reduced. If normalized capital is sufficiently low, the second asset is entirely omitted from the optimal portfolio. However, in spite of this lack of portfolio separation, the total allocation to stocks $\pi_1 + \pi_2$ almost exactly matches that in Figs. 1 and 3 for the case of a single risky asset.

5. Concluding remarks

We study the dynamic investment and reporting problem of a financial institution subject to capital requirements based on self-reported VaR estimates, as in the Basel Committee's Internal Models Approach (IMA). We characterize the solution of this problem using martingale duality and parametric quadratic programming techniques. We find that capital requirements based on the IMA can be very effective in curbing portfolio risk and inducing revelation of this risk. Even with constant price coefficients, optimal portfolios in the presence of capital requirements do not display two-fund separation: we show that as capital requirements become progressively more binding following losses, financial institutions find it optimal to rebalance their portfolios in favor of assets characterized by high expected returns (high systematic risks) relative to the regulatory risk weights. However, optimal portfolios satisfy a local three-fund separation property, the three funds being the riskless asset, the mean-variance efficient portfolio of risky assets and a risk-weight-constrained minimum-variance portfolio of risky assets. For no choice of the parameters we find IMA-based capital requirements leading to an increased probability of default or of extreme losses.

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Appendix A. Proofs

Proof of Proposition 1. Letting $\alpha = K - \theta^\top \bar{1}$, the constraint in Eq. (5) is equivalent to $(\alpha_s, \theta_s) \in A$, where

$$A = \left\{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^m \colon \alpha \geqslant kVaR - \left(\bar{1} - \beta\right)^\top \theta^+ + \left(\bar{1} + \beta\right)^\top \theta^-\right\}.$$

For $\nu = (\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^m$, let $\delta_A(\nu) = \sup_{(\alpha,\theta)\in A}[-(\alpha\nu_0 + \theta\nu_-)]$ denote the support function of $-A$ and let $\tilde{A} = \{\nu \in \mathbb{R}^{m+1} \colon \delta_A(\nu) < +\infty\}$ denote its dual cone. It is easily verified that

$$\tilde{A} = \left\{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^m \colon \nu_0 \geqslant 0,\ \nu_0\left(\bar{1} - \beta\right) \leqslant \nu_- \leqslant \nu_0\left(\bar{1} + \beta\right)\right\},$$

and

$$\delta_A(\nu) = -kVaR\nu_0 \quad \text{for } \nu \in \tilde{A}.$$

The claim then follows from Proposition 1 in Cuoco (1997).[36] □

Proof of Proposition 2. The monotonicity and convexity of $\widetilde{V}$ follow from the fact that Eq. (13) implies

$$\begin{aligned}\widetilde{V}_z(z,t) = \mathrm{E}\Bigg[& \mathrm{e}^{-\int_t^{h\tau}(r+\nu^*(u)+\frac{1}{2}\kappa_{\nu^*}(u)^2)\,\mathrm{d}u - \int_t^{h\tau}\kappa_{\nu^*}(u)\,\mathrm{d}w(u)} \\ & \times \tilde{v}_z\Big(z\,\mathrm{e}^{-\int_t^{h\tau}(r+\nu^*(u)+\frac{1}{2}\kappa_{\nu^*}(u)^2)\,\mathrm{d}u - \int_t^{h\tau}\kappa_{\nu^*}(u)\,\mathrm{d}w(u)}\Big) \\ & - kVaR\int_t^{h\tau}\mathrm{e}^{-\int_t^{u}(r+\nu^*(s)+\frac{1}{2}\kappa_{\nu^*}(s)^2)\,\mathrm{d}s - \int_t^{u}\kappa_{\nu^*}(s)\,\mathrm{d}w(s)}\nu^*(u)\,\mathrm{d}u\Bigg] \end{aligned} \tag{26}$$

[36] It can be easily verified that the proof of Proposition 1 in Cuoco (1997) remains valid for the problem in Eq. (5) in spite of the fact that $v(K, K_-, VaR, i, k, h\tau)$ is discontinuous at $K = K_- - VaR$.

and the fact that $\tilde{v}$ is strictly decreasing and convex in z.

The PDE (14) is the HJB equation associated with the minimization problem in Eq. (13). The fact that the dual value function $\widetilde{V}$ is a (viscosity) solution of this PDE can be verified by rewriting the dual stochastic control problem in terms of the state variable $z_\nu = \log(Z_\nu)$ and then using Theorem 3 in Vargiolu (2001).

The bounds on ν^* follow from the fact that, letting $\zeta = \nu_- - \nu_0 \bar{1}$, the minimization problem in Eq. (14) is equivalent to

$$\min_{\substack{\nu_0 \geqslant 0 \\ -\nu_0\beta \leqslant \zeta \leqslant \nu_0\beta}} \left[\frac{1}{2}\left|\sigma^{-1}(\mu+\zeta)\right|^2 - \frac{\widetilde{V}_z + kVaR}{z\widetilde{V}_{zz}}\nu_0\right]. \tag{27}$$

The unconstrained solution to Eq. (27) is given by $\zeta^* = -\mu$, which satisfies the constraints in Eq. (27) as long as $\nu_0 \geqslant M = \max\{|\iota_i^\top \mu|/\iota_i^\top \beta\colon\ i = 1, \ldots, m\}$. Since $K = -\widetilde{V}_z$, the constraint in Eq. (5) implies

$$\widetilde{V}_z + kVaR \leqslant -\beta^\top(\theta^+ + \theta^-) \leqslant 0.$$

Hence, the term multiplying ν_0 in Eq. (27) is nonnegative. Thus, taking $\nu_0 > M$ in Eq. (27) increases the second term while leaving the first unchanged. Hence, $\nu_0^* \leqslant M$. The bounds on ν_-^* then follow immediately from the definition of $\tilde{A}$. □

Proof of Proposition 3. The proof of Theorem 1 in Cuoco (1997) shows that

$$K(t) = \mathrm{E}\left[-\frac{Z_{\nu^*}(h\tau)}{Z_{\nu^*}(t)}\tilde{v}_z\big(Z_{\nu^*}(h\tau)\big) + kVaR\int_t^{h\tau}\frac{Z_{\nu^*}(u)}{Z_{\nu^*}(t)}\nu^*(u)\,\mathrm{d}u \,\Big|\, \mathcal{F}_t\right].$$

Equation (16) then follows from Eq. (26), while Eq. (15) follows from an application of Itô's lemma to Eq. (16). □

Proof of Proposition 4. Letting $\zeta = \nu_- - \nu_0\bar{1}$, the minimization in Eq. (14) can be rewritten as

$$\min_{\substack{\nu_0 \geqslant 0 \\ -\nu_0\beta \leqslant \zeta \leqslant \nu_0\beta}} \left[\frac{1}{2}\left|\sigma^{-1}(\mu+\zeta)\right|^2 + h\nu_0\right], \tag{28}$$

where

$$h = -\frac{\widetilde{V}_z + kVaR}{z^2\widetilde{V}_{zz}}.$$

Letting (ν_0^*, ζ^*) denote the solution to Eq. (28), it will be shown below that ν_0^* is a monotonically decreasing function of h. Since the constraints on ζ in Eq. (28) become less binding as ν_0^* increases, a constraint that does not bind for a given value of h will never bind for lower values of h. It is easy to see that when h is large ν_0^* is close to zero and exactly one constraint binds for each component of ζ. The above argument then implies that the other constraint will never bind. Let H be the $m \times m$ diagonal matrix whose ith diagonal element is equal to $+1$ (respectively, -1) if the lower bound (respectively, the upper bound) binds on the ith component of ζ for h sufficiently large. The constraints on ζ in Eq. (28) is then equivalent to

$$H\zeta + \nu_0\beta \geqslant 0.$$

The Lagrangian for this problem is

$$L = \frac{1}{2}(\mu+\zeta)^\top(\sigma\sigma^\top)^{-1}(\mu+\zeta) + h\nu_0 - \lambda^\top(H\zeta+\nu_0\beta),$$

where λ is a vector of Lagrangian multipliers.

Now suppose without loss of generality that, after resorting the assets if needed, for a given value of h only the first $i \leqslant m$ constraints on ζ bind. Then the last $m-i$ components of λ equal 0 (that is, $\lambda = I_i^\top I_i\lambda$) and the first-order conditions for a minimum are

$$(\sigma\sigma^\top)^{-1}(\mu+\zeta^*) - H\lambda = 0, \tag{29}$$

$$I_i\bigl(H\zeta^* + \nu_0^*\beta\bigr) = 0, \tag{30}$$

$$h - \lambda^\top\beta \geqslant 0, \tag{31}$$

$$(h-\lambda^\top\beta)\nu_0^* = 0. \tag{32}$$

From Eq. (29)

$$\zeta^* = -\mu + \sigma\sigma^\top H\lambda \tag{33}$$

and hence, from Eq. (30) and the fact that $\lambda = I_i^\top I_i\lambda$

$$\begin{aligned}
\lambda &= I_i^\top\bigl(I_iH\sigma\sigma^\top HI_i^\top\bigr)^{-1}I_i\bigl(H\mu - \nu_0^*\beta\bigr)\\
&= I_i^\top I_iHHI_i^\top\bigl(I_iHI_i^\top I_i\sigma\sigma^\top I_i^\top I_iHI_i^\top\bigr)^{-1}I_iHH\bigl(H\mu-\nu_0^*\beta\bigr)\\
&= I_i^\top I_iHI_i^\top I_iHI_i^\top\bigl(I_iHI_i^\top\bigr)^{-1}\bigl(I_i\sigma\sigma^\top I_i^\top\bigr)^{-1}\bigl(I_iHI_i^\top\bigr)^{-1}I_iHI_i^\top I_iH\bigl(H\mu-\nu_0^*\beta\bigr)\\
&= HI_i^\top\bigl(I_i\sigma\sigma^\top I_i^\top\bigr)^{-1}I_i\bigl(\mu-\nu_0^*H\beta\bigr).
\end{aligned} \tag{34}$$

Substituting the above expression for λ in Eq. (33) gives

$$\zeta^* = -\mu + \sigma\sigma^\top I_i^\top\bigl(I_i\sigma\sigma^\top I_i^\top\bigr)^{-1}I_i\bigl(\mu-\nu_0^*H\beta\bigr), \tag{35}$$

while substituting the same expression in Eq. (31) and using Eq. (32) gives

$$\nu_0^* = \left(\frac{\beta^\top HI_i^\top(I_i\sigma\sigma^\top I_i^\top)^{-1}I_i\mu - h}{\beta^\top HI_i^\top(I_i\sigma\sigma^\top I_i^\top)^{-1}I_iH\beta}\right)^+. \tag{36}$$

As already noted, it is clear from Eq. (28) that there must exist some constant h_m such that ν_0^* is close to zero when $h > h_m$ and hence the constraint on each component of ζ binds (that is, all the components of λ are strictly positive). The above analysis (with $i = m$) then implies

$$\lambda = H(\sigma\sigma^\top)^{-1}\bigl(\mu-\nu_0^*H\beta\bigr), \qquad \zeta^* = -\nu_0^*H\beta,$$

$$\nu_0^* = \left(\frac{\beta^\top H(\sigma\sigma^\top)^{-1}\mu - h}{\beta^\top H(\sigma\sigma^\top)^{-1}H\beta}\right)^+$$

for $h > h_m$. Let

$$h_{m+1} = \beta^\top H(\sigma\sigma^\top)^{-1}\mu > h_m.$$

For $h \geqslant h_{m+1}$, $\nu_0^* = 0$, $\zeta^* = 0$ and $\lambda = H(\sigma\sigma^\top)^{-1}\mu$. For all the components of λ to be strictly positive, we must have

$$H = \operatorname{diag}\big(\operatorname{sign}\big((\sigma\sigma^\top)^{-1}\mu\big)\big).$$

On the other hand, for $h_m < h < h_{m+1}$ the strict positivity of λ amounts to

$$\nu_0^* < \min\{\eta_{m,j}\colon\ \eta_{m,j} > 0,\ j = 1, 2, \ldots, m\}, \tag{37}$$

where

$$\eta_{m,j} = \frac{\iota_j^\top H(\sigma\sigma^\top)^{-1}\mu}{\iota_j^\top H(\sigma\sigma^\top)^{-1}H\beta}.$$

The fact that $H(\sigma\sigma^\top)^{-1}H$ is positive definite and $\beta \in \mathbb{R}^m_{++}$ implies that there must be at least one $\eta_{m,j} > 0$, so that the minimum in (37) is well defined. By resorting the assets if needed, we can assume without loss of generality that the minimum is attained by $\eta_{m,m}$. The condition on ν_0^* is then equivalent to

$$h > \beta^\top H(\sigma\sigma^\top)^{-1}(\mu - \eta_{m,m}H\beta).$$

Thus,

$$h_m = \beta^\top H(\sigma\sigma^\top)^{-1}(\mu - \eta_{m,m}H\beta).$$

When $h \leqslant h_m$, the constraint on the last component of ζ no longer binds and there must exist some constant $h_{m-1} \leqslant h_m$ such that for $h_{m-1} < h \leqslant h_m$ the constraint on each of the first $m-1$ components of ζ binds. Since ν_0^* must be continuous at h_m (see, e.g., Theorem 1 in Tøndel et al., 2003) and hence strictly positive, for these values of h we must have (from Eqs. (34)–(36) with $i = m-1$)

$$\nu_0^* = \frac{\beta^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}\mu - h}{\beta^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}H\beta},$$

$$\zeta^* = -\mu + \sigma\sigma^\top I_{m-1}^\top \big(I_{m-1}\sigma\sigma^\top I_{m-1}^\top\big)^{-1} I_{m-1}\big(\mu - \nu_0^* H\beta\big),$$

$$\lambda = H I_{m-1}^\top \big(I_{m-1}\sigma\sigma^\top I_{m-1}^\top\big)^{-1} I_{m-1}\big(\mu - \nu_0^* H\beta\big).$$

The strict positivity of the first $m-1$ components of λ amounts to

$$\nu_0^* < \min\{\eta_{m-1,j}\colon\ \eta_{m-1,j} > 0,\ j = 1, 2, \ldots, m-1\}, \tag{38}$$

where

$$\eta_{m-1,j} = \frac{\iota_j^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}\mu}{\iota_j^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}H\beta}.$$

The fact that $I_{m-1}\lambda$ is continuous (see, e.g., Theorem 1 in Tøndel et al., 2003) and hence strictly positive at h_m, so that

$$I_{m-1}HI_{m-1}^\top\big(I_{m-1}\sigma\sigma^\top I_{m-1}^\top\big)^{-1}I_{m-1}\mu \\ > \eta_{m,m}I_{m-1}HI_{m-1}^\top\big(I_{m-1}\sigma\sigma^\top I_{m-1}^\top\big)^{-1}I_{m-1}H\beta,$$

$HI_{m-1}^\top(I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1}I_{m-1}H$ is positive definite and $\beta \in \mathbb{R}^m_{++}$ implies that there must be at least one $\eta_{m-1,j} > 0$. Without loss of generality, we can assume that the minimum in (38) is attained by $\eta_{m-1,m-1}$. The condition on ν_0^* is then equivalent to

$$h > \beta^\top HI_{m-1}^\top\big(I_{m-1}\sigma\sigma^\top I_{m-1}^\top\big)^{-1}I_{m-1}(\mu - \eta_{m-1,m-1}H\beta).$$

Thus,

$$h_{m-1} = \beta^\top HI_{m-1}^\top\big(I_{m-1}\sigma\sigma^\top I_{m-1}^\top\big)^{-1}I_{m-1}(\mu - \eta_{m-1,m-1}H\beta).$$

Continuing this way we obtain additional values $\{h_1, h_2, \ldots, h_{m-2}\}$ with

$$h_i = \beta^\top HI_i^\top\big(I_i\sigma\sigma^\top I_i^\top\big)^{-1}I_i(\mu - \eta_{i,i}H\beta)$$

such that for $h_i < h \leqslant h_{i+1}$ only the constraint on the first i components of ζ bind, in which case

$$\nu_0^* = \frac{\beta^\top HI_i^\top(I_i\sigma\sigma^\top I_i^\top)^{-1}I_i\mu - h}{\beta^\top HI_i^\top(I_i\sigma\sigma^\top I_i^\top)^{-1}I_iH\beta},$$

ζ^* is given by Eq. (35) and λ is given by Eq. (34). Since $h \geqslant 0$ (see the proof of Proposition 2) and $h_1 = 0$, this characterization exhausts all possible cases.

The expressions for the optimal trading strategy immediately follow from the fact that for $h_i < h < h_{i+1}$

$$\theta(z,t) = z\widetilde{V}_{zz}(z,t)(\sigma\sigma^\top)^{-1}(\mu + \zeta^*) \\ = \begin{cases} z\widetilde{V}_{zz}(z,t)(\sigma\sigma^\top)^{-1}\mu & \text{if } h \geqslant h_{m+1}, \\ z\widetilde{V}_{zz}(z,t)I_i^\top(I_i\sigma\sigma^\top I_i^\top)^{-1}I_i(\mu - \nu_0^*H\beta) & \text{if } h_i < h \leqslant h_{i+1}. \end{cases} \tag{39}$$

The fact that $\operatorname{sign}(H\theta) \geqslant 0$ follows from the fact that Eq. (29) implies

$$H\theta = z\widetilde{V}_{zz}\lambda \geqslant 0. \quad \square$$

Proof of Corollary 1. Letting $\zeta = \nu_- - \nu_0$, it follows from Eqs. (35) and (36) in the proof of Proposition 4 that, when $m = 1$ and $\mu > 0$,

$$\zeta^* = -\nu_0^*\beta \tag{40}$$

and

$$\nu_0^* = \left(\frac{\mu}{\beta} + \left(\frac{\sigma}{\beta}\right)^2\frac{\widetilde{V}_z + kVaR}{z\widetilde{V}_{zz}}\right)^+. \tag{41}$$

Substituting Eqs. (40) and (41) into Eq. (14) yields the PDE in the corollary. The expression for the optimal trading strategy follows directly from Eqs. (39)–(41). $\square$

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