Optimal Consumption and Investment with Asymmetric Long-term/Short-term Capital Gains Taxes*

Min Dai Hong Liu Yifei Zhong

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Abstract

We propose an optimal consumption and investment model with asymmetric long-term/short-term capital gains tax rates for both lower income and wealthy investors. We characterize and develop an iterative algorithm to compute the optimal policy. Opposite to the existing literature, we show that it may be optimal to defer even large long-term gains and losses. In addition, the optimal policy for lower income investors is qualitatively different from that for wealthy ones. Furthermore, raising capital gains tax rates for lower income investors can significantly increase their consumption, stock investment, and welfare, due to negative effective tax rates.

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1. Introduction

As shown by the existing literature (e.g., Constantinides (1983, 1984), Dybvig and Koo (1996) , Dammon, Spatt, and Zhang (2001)), capital gains tax can significantly affect the optimal trading strategies of investors. Two important features of the current tax code are that the tax rate for long term investment can be much lower than the short term tax rate for many investors and capital losses that are allowed to offset ordinary income are limited to $3,000 per year, with the rest carried over indefinitely for offsetting future gains and/or income. However, to our knowledge, research on how these important features jointly affect the optimal consumption and investment is still absent.

Analysis of optimal investment in the presence of capital gains tax is in general difficult, because of the path dependency of the optimal policy due to the need to keep track of the tax basis. Most existing literature approximate the exact tax basis using the average tax basis to reduce the dimensionality of the optimization problem (e.g., Dammon, Spatt, and Zhang (2001, 2004), Gallmeyer, Kaniel, and Tompaidis (2006)). Unfortunately, this approximation method is not suitable for studying the impact of asymmetric long-term/short-term tax rates. This is because with average tax basis, it may be optimal to hold shares that are purchased at different times and thus the optimal investment once again becomes path dependent due to the need to keep track of the time of each purchase to apply the corresponding tax rate. Using an exact basis model similar to DeMiguel and Uppal (2005) (see Appendix), we show that even with asymmetric tax rates, an investor rarely has more than one tax basis and the equivalent wealth loss from following a single-basis strategy is almost negligible (less than 0.1% of the initial wealth for reasonable parameter values). Based on these findings, we develop a continuous-time optimal consumption and investment model with a single tax basis at each point in time to study the impact of asymmetric long-term and short-term tax rates and the capital loss carry over provision.

More specifically, we consider the optimal consumption and investment problem of a small, constant relative risk aversion (CRRA) investor who can continuously trade a risk free asset and a risky stock to maximize his expected utility from intertemporal consumption and bequest.[1] There is no transaction cost, but the investor must pay capital gains tax with asymmetric long-term/short-term rates. Different from the existing literature but consistent with the U.S. tax law, the tax rate for both longer-term and short-term capital losses is set to be the short-term rate. We assume that the investor always maintains a single tax basis at each point in time. We consider both the case where an investor can get full tax rebate for all capital losses (i.e., there is no limit on the amount of losses that can offset ordinary income) and the case where the investor can only carry over capital losses (i.e., all losses can only be carried over to offset future gains). The full rebate case is a better model for lower income investors whose capital losses are likely below $3,000 a year, while the full carry over case is more suitable for wealthy investors whose potential capital losses can far exceed $3,000 a year.[2] In addition, we assume capital gains and losses can

[1] While an extension to a multi-stock case would help understand cross-stock tax management strategy, it would unlikely be important for our main purpose. First, for lower income investors it is widely documented that they typically directly hold at most 2 stocks and therefore any tax management strategy would unlikely make a significant difference especially when transaction costs are present. Second, for wealthy investors, they hold a well diversified portfolio which can be well approximated by ETF shares like SPDR. The tax treatment of an ETF share is almost the same as that of an individual stock and thus the stock in our model can be interpreted as an ETF share. In addition, as Gallmeyer, Kaniel and Tompaidis (2006) show, when shortsales are costly, cross-stock tax management is largely ineffective.

[2] To compare the impact of the single-basis assumption and the commonly adopted average-basis assumption, we also solve a corresponding average-basis model with full capital loss rebate and full carry over when the long term and short term tax rates are the same. The optimal trading strategies are quite similar, which is consistent with the finding of DeMiguel and Uppal (2005) that most of

be realized immediately, there is no wash sale restriction and shorting against the box is prohibited. The optimal trading strategy is characterized by a time-varying no-realization region outside which it is optimal to realize at least some capital gains or losses to achieve the optimal stock market exposure. We develop an iterative procedure to numerically compute the optimal consumption and investment policy and conduct an extensive analysis of the effect of asymmetric tax rates and capital loss treatment rules.

The existing literature (e.g., Constantinides (1983, 1984), Dammon and Spatt (1996), and Dammon, Spatt, and Zhang (2001) who assume full tax rebate for losses, Marekwica (2009) who considers limited rebate but assumes symmetric long-term/short-term rates) show that that it is optimal to realize all losses before it turns long-term and realize all gains right after it turns long-term. The main intuition behind this result is that (1) realizing losses before it turns long-term entitles the investor to the higher short-term rate; (2) realizing gains right after it turns long-term enjoys the lower long-term rate and reestablishes the short-term status so that subsequent losses can be realized at the higher short-term rate while subsequent gains can be deferred. In contrast, we find that it may be optimal for investors to not only have long-term losses, but also further defer even large long-term gains and long-term losses. This is because (1) for lower income investors (for whom tax rebate for capital losses is significant), unlike in the existing literature that assume long-term rates for long-term losses, under the current law, short-term rates apply to all losses (short-term or long-term) and thus the long-term status strictly dominates the short-term status. Therefore, for lower income investors, it is optimal for them to defer all small long-term gains and losses to keep the dominant tax status; (2) For wealthy investors

the time investors only carry a single tax basis.

(for whom tax rebate for capital losses is almost negligible), on the other hand, it is optimal for them to defer even large long-term gains and when the long-term rate is much lower than the short-term rate it is also optimal for them to defer even large long-term losses. Intuitively, there is an additional benefit of deferring realization of gains for these investors: It makes incremental losses effectively rebatable. As a simple example, suppose an investor has a gain of \$1, realizes the gain, reestablishes a stock position, and consequently loses \$1. If she did not realize the gain, then she would have no tax liability. In addition, the benefit of reestablishing the short-term status is small, because they cannot get much tax rebate which the higher short-term rate would help. As for long-term losses, even though wealthy investors can only get almost negligible tax rebate for losses, keeping the long-term status can provide significant benefit from the much lower long-term rate when current losses turn into gains.

We find that lower income investors are willing to pay a significant fraction of their initial wealth to gain the same capital gains tax treatment as wealthy investors have, because of the option of realizing capital losses short-term and realizing capital gains long-term. For example, suppose the interest rate is zero, the expected stock return is 4%, the volatility is 20%, the dividend yield is 2%, and investors have an expected remaining life time of 80 years and a relative risk aversion coefficient of 3. Consider a lower income investor with a marginal ordinary income tax rate of 10%, and a long-term capital gains tax rate of 0%. This investor would be willing to pay as much as 24.5% (about 24.5%/80 = 0.31% per year) of her initial wealth to gain the same capital gains tax treatment as that of a rich investor who has a marginal ordinary income tax rate of 39.6% and a long-term capital gains tax rate of 23.8%. Even when an investor is forced to realize short-term gains more often, e.g., due to liquidity shocks, higher rates would still make her significantly better off. For example, if a large liquidity shock that requires the liquidation of the entire stock position occurs once a year on average and thus the expected investment horizon is only one year, the lower income investor would still be willing to pay 3.0%, which is even higher than the case without liquidity shock on a per year basis, of the initial wealth to gain the same capital gains tax treatment as that of the rich investor. Consistent with this finding, we show that the effective tax rate for a lower income investor can be significantly negative. The higher tax (rebate) rates on capital losses and the ability to defer short-term capital gains effectively make stock investment much less risky. As a result, a lower income investor with higher tax rates would also invest and consume significantly more (as a fraction of their wealth). As Wilson and Liddell (2010) reported, in 2007, tax returns with an adjusted gross income of $100,000 or less have short-term net losses on average. These returns, totaling about 6 million, account for more than half of the total returns that have short-term gains or losses. This implies that many lower income investors could benefit from higher short-term capital gains tax rates. Our finding implies that raising capital gains tax short-term rates for lower income investors can significantly increase their consumption and stock market participation. In contrast, a wealthy investor for whom the carry-over case fits better is always worse off with capital gains tax and her stock investment and consumption is almost insensitive to changes in short-term tax rate (because of deferring short-term gains), but decreasing in the long-term tax rate.[3] In addition, she generally invests more than the no-tax case when she has tax losses carried over because some capital gains can be offset.

[3]Since marginal investors are likely wealthy ones, this is consistent with Sialm (2009) who find an economically and statistically significant negative relation between equity valuations and effective tax rates, contrary to some tax irrelevance theories (e.g., Miller and Sholes (1978)).

Because tax is only paid when it is realized, an investor has the option to defer capital gains tax. When long-term/short-term tax rates are the same, the value of this deferring comes from earning the interest on the capital gains tax by not paying it sooner and from effectively making capital loss rebatable while deferring for the full carry over case. When short-term rates are higher than long-term rates, the value of this deferring also comes from the benefit of realizing gains at a lower (long-term) rate in the future. We show that the value of deferring from realizing gains at a lower rate in the future is much greater than that from saving interest for lower income investors. For example, for a lower income investor with a short-term rate of 20% and a long-term rate of 15%, the value of deferring from realizing gain at the long-term rate (instead of the short-term rate) can be as high as 12.2% of the initial wealth, in contrast to a mere 1.2% for the value from interest saving. Therefore, by ignoring the difference between the long-term and short term rates, most of the existing literature significantly underestimates the value of deferring capital gains tax and largely overestimates the effective tax rates for lower income investors. In contrast, for wealthy investors, the value of deferring mainly comes from effectively making capital loss rebatable while deferring, about 6.1% of the initial wealth. The effective tax rate for wealthy investors is always positive, but still quite low, only about 2.82% of their initial wealth.

While we have several simplifying assumptions, their impact on our main results is likely limited. The impact of immediate rebate assumption is probably small, especially when interest rate is low. This is because if interest rate is zero, then the investor can borrow against the tax rebate to be received later without interest cost. Accordingly, to address the concern, we have set the interest rate to be zero in all the relevant analysis. The assumption of no wash sale restriction and no transaction costs also unlikely changes our main results because investors can purchase similar stocks for 30 days without much loss and transaction cost rates are typically very small, especially for liquid stocks and in recent years.

As far as we know, this is the first paper to examine the optimal consumption and investment problem with both asymmetric long-term/short-term tax rates and limited rebate for capital losses as stipulated by the current tax law. This paper is closely related to Constantinides (1984), Dammon and Spatt (1996), Cadenillas and Pliska (1999), Dammon, Spatt, and Zhang (2001, 2004), Gallmeyer, Kaniel, and Tompaidis (2006), Marekwica (2009), and Ehling, Gallmeyer, Srivastava, Tompaidis, and Yang (2010). Constantinides (1984) and Dammon and Spatt (1996) consider the impact of asymmetric long-term/short-term tax rates on the capital gains/losses realization timing of one share of stock in a discrete-time setting where an investor sells and repurchases the stock only for tax reasons (e.g., not for consumption or portfolio rebalancing). Cadenillas and Pliska (1999) study the portfolio choice problem of an investor who maximizes expected long run growth rate and faces symmetric capital gains tax rates. In a discrete time setting, Dammon, Spatt, and Zhang (2001) consider the optimal consumption and portfolio decisions with capital gains tax and short-sale constraints, assuming a binomial stock price process and average tax basis approximation. They show that contrary to standard financial advice, optimal equity holding can increase until late in lifetime because of the forgiveness of capital gains tax at death. While their main analysis focuses on the symmetric long-term and short-term tax rates case, they do consider the impact of asymmetric tax rates on optimal life cycle equity holding in Subsection 3.2. In contrast to our model, however, they assume an investor can only trade once a year and Constantinides (1984)'s condition for realizing long-term gains each year is always satisfied, therefore, it is always optimal to realize all short-term losses and all long-term gains in their model, even with asymmetric tax rates. In addition, as Constantinides (1984) and Dammon and Spatt (1996), Dammon, Spatt, and Zhang (2001) also assume that the tax rate for long-term losses is the same as that for long-term gains. Dammon, Spatt, and Zhang (2004) examine optimal asset allocation and location decisions for investors making taxable and tax-deferred investments. They show that it is significantly advantageous to hold bonds in the tax-deferred account and equity in the taxable account. Gallmeyer, Kaniel and Tompaidis (2006) consider the optimal consumption-portfolio problem with symmetric capital gains tax rates and multiple stocks to understand how short selling influences portfolio choice when shorting against the box is prohibited. They find that shorting one stock even when no stock has embedded gain may be optimal. In addition, when short selling is costly, the benefit of trading separately in multiple stocks is not economically significant. Both Marekwica (2009) and Ehling et. al. (2010) consider the case with with limited use of capital losses. They assume average basis and symmetric tax rates. Ehling et. al. (2010) find that stock holdings are similar between investors with full tax rebate and investors with full carry over if investors have large embedded capital gains. Marekwica (2009) shows that it is still optimal to realize losses immediately. In contrast, we show that stock holdings of investors with full carry over can be also significantly different from those with full tax rebate even with large embedded gains. In addition, with asymmetric tax rates (arguably more appropriate for investors for whom tax loss rebate is likely negligible), it is optimal to defer even large losses when the long-term rate is significantly lower than the short-term rate.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 provides some analytical results, a verification theorem, and the numerical solution procedure. In Section 4, we conduct numerical analysis on the optimal consumption and trading strategies, the bias against lower income investors and the value of deferring realization. Section 5 concludes and all the proofs are relegated to the Appendix.

2. The Model

Throughout this paper we are assuming a probability space $(\Omega, \mathcal{F}, P)$. Uncertainty and the filtration $\{\mathcal{F}_t\}$ in the model are generated by a standard one dimensional Brownian motion w and a Poisson process defined below. We will assume that all stochastic processes are adapted.

There are two assets an investor can trade without any transaction costs. The first asset is a money market account growing at a continuously compounded, constant rate r. The second asset ("the stock") is a risky investment.[4] The ex-dividend stock price S_t follows the process

$$dS_t = \mu S_t dt + \sigma S_t dw_t, \tag{1}$$

where μ and σ are constants with $\mu > r$.

The investor is subject to capital gains tax. We assume that the tax on dividend and interest is due when they are paid. Optimal investment in the presence of capital gains tax is in general extremely difficult because of the path dependency of tax basis, as pointed out by Dybvig and Koo (1996). Most existing literature approximate the exact tax basis using the average tax basis to reduce the dimensionality of the opti-

[4]The risky asset can be interpreted as an ETF that represents a diversified portfolio. As an individual stock and unlike a mutual fund, an ETF's capital gain tax can be deferred until the sale of the ETF share. As shown in Gallmeyer, Kaniel and Tompaidis (2006), when short sale is costly, the effect of cross-stock tax management strategy is small. In addition, most lower income investors directly hold not more than two stocks outside their retirement accounts (e.g., Campbell (2006)).

mization problem (e.g., Dammon, Spatt, and Zhang (2001, 2004), Gallmeyer, Kaniel and Tompaidis (2006)). Unfortunately, this approximation method is not suitable for studying the impact of asymmetric long-term/short-term tax rates, because with average tax basis, it may be optimal to hold shares that are purchased at different times and the optimal investment problem again becomes path dependent due to the need to keep track of the time of each purchase. Using an exact tax basis similar to DeMiguel and Uppal (2005) but with asymmetric tax rates, we find that as in DeMiguel and Uppal (2005), the investor almost always holds shares with a single tax basis (see Section 4.7). Based on this finding, we assume that an investor always keeps a single tax basis.[5] As a consequence, the investor liquidates all stock holdings before purchases. In addition, we assume that (1) capital gains and capital losses can be realized immediately after sale; (2) no wash sale restriction; (3) shorting against the box is prohibited. The investor is endowed with x_{0-} dollars in the risk free account and y_{0-} dollars in the stock account. This initial endowment includes the present value of all future after-tax ordinary income (i.e., all except capital gains/losses).[6] Let x_t denote the dollar amount invested in the riskless asset, y_t denote the current value of the stock holding, B_t be the tax basis at time t, and s_t be the length of stock holding period since last purchase.

[5] To compare the impact of the single-basis assumption and the commonly adopted average-basis assumption, we also solve a corresponding average-basis model when the long-term and short-term tax rates are the same. The optimal trading strategies are quite similar, which is consistent with the finding of DeMiguel and Uppal (2005) that most of the time investors only carry a single tax basis.

[6] More specifically, suppose the investor has a constant after-tax income rate of L until death which occurs at the first jump time of a Poisson process with intensity λ. Then with a complete market, the present value of the after-tax income is equal to $L/(r+\lambda)$, which can be simply added to her initial wealth to arrive at the initial endowment we consider, and all our analysis would remain valid. We do not explicitly model the ordinary income tax payment to simplify the exposition and focus on the main analysis.

Between purchases, we then have

$$\begin{aligned} dx_t &= (r(1-\tau_i)x_{t-} + (1-\tau_d)\delta y_{t-} - c_t)dt + [y_{t-} - \tau(s_t)\,(y_{t-} - B_{t-}) \\ &\quad + (1-\omega)\,\kappa\,(\tau_S - \tau(s_t))\,(B_{t-} - y_{t-})^+ - \omega\tau\,(s_t)\,(B_{t-} - y_{t-})^+\big]\, dM_t, \end{aligned} \tag{2}$$

$$dy_t = \mu y_{t-}dt + \sigma y_{t-}dw_t - y_{t-}dM_t, \tag{3}$$

$$dB_t = \left[-B_{t-} + \omega\,(B_{t-} - y_{t-})^+\right] dM_t, \tag{4}$$

$$ds_t = dt, \tag{5}$$

where r and δ are before-tax interest rate and dividend yield with tax rates τ_i and τ_d respectively, dM_t represents the fraction of the current stock position that is sold at t, $\omega = 0$ or 1 corresponds to the full-rebate case and the full carry-over case, respectively, $\kappa = 0$ corresponds to the case of applying the long-term tax rate to long-term losses and $\kappa = 1$ to the case of applying the short-term tax rate to all losses, and τ is the tax rate function for capital gains with

$$\tau(s) = \begin{cases} \tau_S & if \quad s < H \\ \tau_L & if \quad s \geq H, \end{cases} \tag{6}$$

where H is the shortest holding time for qualifying a long term tax status. The bracketed term in (2) denotes the after-tax dollar amount if the entire stock position is sold at t. Because only a fraction dM_t is sold, the after-tax dollar amount and the basis are both proportionally reduced. Because $\tau_S \geq \tau(s)$, compared to the case with $\kappa = 0$, an investor with a long-term status in the $\kappa = 1$ case has an additional put option represented by the term $(B - y)^+$, which enables the investor to realize long-term losses at the (higher) short-term rate.

At a purchase time η (a stopping time) with a holding period of s since last purchase, we have

$$x_\eta = x_{\eta-} + y_{\eta-} - \tau(s_{\eta-})\,(y_{\eta-} - B_{\eta-}) + (1-\omega)\,\kappa\,(\tau_S - \tau(s_{\eta-}))\,(B_{\eta-} - y_{\eta-})^+ \\ -\omega\tau(s_{\eta-})(B_{\eta-} - y_{\eta-})^+ - I_\eta, \tag{7}$$

$$y_\eta = I_\eta, \tag{8}$$

$$B_\eta = I_\eta + \omega(B_{\eta-} - y_{\eta-})^+, \tag{9}$$

$$s_\eta = 0, \tag{10}$$

where I is the dollar amount of the stock bought immediately after liquidating the existing position.

The investor maximizes expected utility from intertemporal consumption and the final after-tax wealth at the first jump time $\mathcal{T}$ of an independent Poisson process with intensity λ. This Poisson process can represent the time of a liquidity shock upon which one must liquidate the entire portfolio or the death time of the investor or the performance evaluation time of a fund.[7] If it represents a death time, then capital gains tax may be forgiven (e.g., in USA) or may be not (e.g., in Canada). Let $V(x_0, y_0, B_0, 0)$ be the time 0 value function, which is equal to

$$\sup_{\{M_t,\eta,I_\eta,c_t\}} E\left[\alpha \int_0^{\mathcal{T}} e^{-\beta t} u(c_t)dt + (1-\alpha)e^{-\beta\mathcal{T}} u\left((1-\iota)(x_{\mathcal{T}} + y_{\mathcal{T}}) + \iota f(x_{\mathcal{T}}, y_{\mathcal{T}}, B_{\mathcal{T}}, s_{\mathcal{T}})\right)\right], \tag{11}$$

subject to (2)-(10) and the solvency constraint

$$f(x_t, y_t, B_t, s_t) \geq 0,\ (1-\iota)(x_t + y_t) \geq 0,\ \forall t \geq 0, \tag{12}$$

where $\beta > 0$ is the subjective discount rate, $\alpha \in [0,1]$ is the weight on intertemporal

[7]As shown by Carr (1998) and Liu and Loewenstein (2002), one can use a series of random times to approximate a fixed time (e.g., of performance evaluation).

consumption, $\iota \in \{0,1\}$ indicating if tax is due or not at $\mathcal{T}$,

$$\begin{aligned} f(x,y,B,s) \;&\equiv\; x + y - \tau(s)(y-B) + (1-\omega)\,\kappa\,(\tau_S - \tau(s))\,(B-y)^{+} \\ &\quad -\omega\tau\,(s)\,(B-y)^{+} \end{aligned}$$

is the after-tax wealth, and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

with the relative risk aversion coefficient γ. If $\alpha = 0$, the problem can be also interpreted as an investment problem of a fund whose manager's compensation is proportional to the before-tax ($\iota = 0$) or after-tax ($\iota = 1$) asset under management.

Using the dynamic programming principle and integrating out the Poisson jump, we can rewrite the investor's problem in a recursive form as

$$\begin{aligned} & V(x_0, y_0, B_0, 0) \\ = \;& \sup_{\{M_t,\eta,I_\eta,c_t\}} E\Big[\int_0^{\eta} e^{-(\beta+\lambda)t}\,(\alpha u(c_t) + (1-\alpha)\lambda u((1-\iota)(x_t+y_t) + \iota f(x_t,y_t,B_t,s_t)))\,dt \\ & + e^{-(\beta+\lambda)\eta} V\left(f(x_{\eta-}, y_{\eta-}, B_{\eta-}, \eta-) - I_\eta, I_\eta, I_\eta, 0\right)\Big], \end{aligned} \tag{13}$$

subject to (2)-(10) and the solvency constraint (12).

The associated HJB equation is

$$\begin{aligned} \max\Big\{ & V_s + \mathcal{L}_0 V, \sup_{I\geq 0} V\left(f(x,y,B,s) - I, I, I + \omega\,(B-y)^{+}, 0\right) - V(x,y,B,s), \\ & f\,(0,y,B,s)\,V_x - yV_y - \left(B - \omega\,(B-y)^{+}\right)V_B \Big\} = 0 \end{aligned} \tag{14}$$

in $s > 0$, $B > 0$, $y > 0$, $f\,(x,y,B,s) > 0$, and $x + y > 0$ (if $\iota = 0$), where

$$\begin{aligned} \mathcal{L}_0 V \;=\;& \frac{1}{2}\sigma^2 y^2 V_{yy} + \mu y V_y + ((1-\tau_i)rx + (1-\tau_d)\delta y)\,V_x - (\beta+\lambda)V \\ & + \alpha^{1/\gamma}\frac{\gamma}{1-\gamma}\,(V_x)^{-\frac{1-\gamma}{\gamma}} + \frac{(1-\alpha)\lambda}{1-\gamma}((1-\iota)(x+y) + \iota f(x,y,B,s))^{1-\gamma}. \end{aligned}$$

Using the homogeneity property of the value function, we can reduce the dimensionality of the problem by the following transformation:

$$
\begin{aligned}
z &= \frac{x}{y}, \\
b &= \frac{B}{y}, \\
V(x, y, B, s) &= y^{1-\gamma}\Phi(z, b, s),
\end{aligned}
$$

for some functions Φ, where b is equal to the basis per share divided by the stock price, and so will be simply referred to as the basis to price ratio. Then (14) can be reduced to

$$
\begin{aligned}
&\max\left\{\Phi_s + \mathcal{L}_1\Phi, \quad G(z, b, s; \Phi) - \Phi(z, b, s),\right. \\
&\left. -(1-\gamma)\Phi + f(z, 1, b, s)\Phi_z + \omega(b-1)^+\Phi_b\right\} = 0
\end{aligned} \tag{15}
$$

in $s > 0$, $b > 0$, $f(z, 1, b, s) > 0$, and $z + 1 > 0$ (if $\iota = 0$), where

$$
\begin{aligned}
\mathcal{L}_1\Phi &= \frac{1}{2}\sigma^2 z^2\Phi_{zz} + \frac{1}{2}\sigma^2 b^2\Phi_{bb} + \sigma^2 zb\Phi_{zb} - \left(\mu - \gamma\sigma^2\right)b\Phi_b \\
&\quad - \left[(\mu - (1-\tau_i)r - \gamma\sigma^2)z - (1-\tau_d)\delta\right]\Phi_z + \left[(1-\gamma)(\mu - \frac{1}{2}\gamma\sigma^2) - \beta - \lambda\right]\Phi \\
&\quad + \frac{\gamma\alpha^{1/\gamma}}{1-\gamma}(\Phi_z)^{-\frac{1-\gamma}{\gamma}} + \frac{(1-\alpha)\lambda}{1-\gamma}((1-\iota)(z+1) + \iota f(z, 1, b, s))^{1-\gamma}
\end{aligned}
$$

and

$$
G(z, b, s; \Phi) = f(z, 1, b, s)^{1-\gamma} \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1}\Phi\left(k, 1 + \omega(k+1)\frac{(b-1)^+}{f(z, 1, b, s)}, 0\right).
$$

In terms of the fraction of wealth $\pi \equiv y/f(x, y, B, s) = 1/f(z, 1, b, s)$, the optimal trading strategy of the investor can be characterized by a sell boundary $\bar{\pi}(b, s)$ and a buy boundary $\underline{\pi}(b, s)$ in the π-b plane: transact to the sell boundary $(\bar{\pi}(b, s), b)$ if $\pi(z, b, s) > \bar{\pi}(b, s)$; no transaction if $\underline{\pi}(b, s) < \pi(z, b, s) < \bar{\pi}(b, s)$; otherwise liquidate the entire stock position and transact to $\hat{\pi}(b)$, where if $b < 1$ or $\omega = 0$,

$\hat{\pi} \equiv 1/f(z^*, 1, 1, 0)$ with z^* solving $\mathcal{L}_1 \Phi(z^*, 1, 0) = 0$, otherwise $\hat{\pi}(b) = \pi^*$ as defined in (A-5) in the Appendix.

3. Theoretical Analysis and Numerical Procedure

Before presenting numerical procedure to solve the HJB equation (15), we conduct some theoretical analysis.

The following proposition provides a Merton type strategy that keeps a constant fraction of wealth in stock, which is optimal in the full rebate case if the risk free rate is 0 and tax rates are constant. This result provides a baseline for computing the value of deferring tax realization. Define

$$\rho \equiv \beta + \lambda - (1-\gamma)\left(\frac{\left((1-\tau_S)\,\mu + (1-\tau_d)\,\delta - (1-\tau_i)\,r\right)^2}{2\,(1-\tau_S)^2\,\gamma\sigma^2} + (1-\tau_i)\,r\right).$$

Proposition 1 *Suppose $\omega = 0$, $\iota = 1$ and $\rho > 0$.*

1. *Within the class of strategies with a constant fraction of wealth in stock, the optimal fraction is*

$$\begin{aligned} \frac{y_t}{x_t + y_t - \tau_S\,(y_t - B_t)} &= \frac{(1-\tau)\,\mu + (1-\tau_d)\,\delta - (1-\tau_i)\,r}{(1-\tau_S)^2\,\gamma\sigma^2}, \\ \frac{c_t}{x_t + y_t - \tau_S\,(y_t - B_t)} &= \alpha^{1/\gamma}\nu^{-(1-\gamma)/\gamma}, \end{aligned}$$

   and the associated value function is

$$\frac{\left[\nu\,(x + y - \tau_S\,(y - B))\right]^{1-\gamma}}{1-\gamma},$$

   where ν is the unique positive root of

$$-\rho\nu^{1-\gamma} + \gamma\alpha^{1/\gamma}\nu^{-(1-\gamma)^2/\gamma} + (1-\alpha)\,\lambda = 0. \tag{16}$$

2. *If in addition,* $r = 0$ *and* $\tau_S = \tau_L$*, then the above strategy is indeed optimal among all feasible strategies.*

Proof: see Appendix.

Proposition 1 implies that if $r = 0$, all tax rates are equal, and an investor starts with all cash and gets the full tax rebate, then the investor can achieve the same expected utility as in the case without tax. However, the investor invests a greater fraction of the after-tax wealth in the stock. Intuitively, with zero interest rate and same tax rates, there is no benefit of deferring the realization of capital gains or capital losses. Thus the investor trades the stock continuously and the after-tax expected return becomes $(1-\tau_S)\mu + (1-\tau_d)\,\delta$, while the after-tax volatility becomes $(1-\tau_S)\sigma$. Therefore, if $\tau_d = \tau_S$, the stock investment increases because it is proportional to the ratio of the after-tax expected return to the after-tax variance. However, the expected utility does not change because it is determined by the Sharpe ratio, which remains the same when tax rates are all the same.

The following proposition indicates that in the carry over case, continuous trading is optimal at $s = 0$ when there is tax loss .

Proposition 2 *Suppose* $\omega = 1$ *and in addition,* $s = 0$ *or* $\tau_L = \tau_S$*. Let* $\Phi(z, b, s)$ *be a solution to the HJB equation (15). Then,*

1.

$$\Phi(z, b, 0) = (1+z)^{1-\gamma}\zeta(\theta)\,, \textit{for } b > 1,$$

where $\theta = \frac{z+1}{z+b} \in (0, 1]$ *and* $\zeta(\theta)$ *satisfies (A-3) in the Appendix.*

2. *The optimal trading strategy when there is capital loss is to trade continuously to keep the fraction* π^* *(as defined in (A-5)) of wealth in stock.*

Proof: See Appendix.

We now present the verification theorem.

Proposition 3 *(verification theorem). Let $\Phi(z,b,s)$ be a solution to the HJB equation (15) satisfying certain regularity conditions. Define the no-trading region NT, the buy region BR, and the sell region SR as follows:*

$$
\begin{aligned}
NT &= \left\{ (z,b,s) : \begin{array}{l} \Phi(z,b,s) > G(z,b,s;\Phi), \\ -(1-\gamma)\,\Phi + f\,(z,1,b,s)\,\Phi_z + \omega(b-1)^+\Phi_b < 0 \end{array} \right\}, \\
BR &= \{(z,b,s) : \Phi(z,b,s) = G(z,b,s;\Phi)\}, \\
SR &= \left\{(z,b,s) : -(1-\gamma)\,\Phi + f\,(z,1,b,s)\,\Phi_z + \omega(b-1)^+\Phi_b = 0\right\}.
\end{aligned}
$$

Denote $\partial B = \overline{NT} \cap BR$ and $\partial S = \overline{NT} \cap SR$. Define

$$V\,(x,y,B,s) = y^{1-\gamma}\Phi\left(\frac{x}{y},\frac{B}{y},s\right).$$

Then $V\,(x,y,B,s)$ is the value function, and the optimal policy is given as follows:

1. *optimal consumption: $c^*(x_t,y_t,B_t,t) = y_t\left(\Phi_z(\frac{x_t}{y_t},\frac{B_t}{y_t},t)\right)^{-1/\gamma}$;*
2. *optimal trading: if $\omega = 1$, $b > 1$ and $s = 0$, then continuous trading to keep the fraction π^* (as in (A-5)) of wealth in stock; Otherwise,*
   (a) *sell strategy M_t^*:*
   $$M_t^* = \int_0^t 1_{\left\{\left(\frac{x_\xi^*}{y_\xi^*},\frac{B_\xi^*}{y_\xi^*},s_\xi^*\right)\in\partial S\right\}} dM_\xi^*;$$
   (b) *buy strategy $(\eta_1^*,\eta_2^*,...;I_1^*,I_2^*,...)$:*
   $$
   \begin{aligned}
   \eta_{n+1}^* &= \inf\left\{t > \eta_n^* : \left(\frac{x_t^*}{y_t^*},\frac{B_t^*}{y_t^*},s_t^*\right) \in BR\right\}, \\
   I_{n+1}^* &= \pi_{n+1} f\left(x_{\eta_{n+1}^*-}^*, y_{\eta_{n+1}^*-}^*, B_{\eta_{n+1}^*-}^*, \eta_{n+1}^* - \eta_n^*\right),
   \end{aligned}
   $$

where $\eta_0^* = 0$ *and* $(x_t^*, y_t^*, B_t^*, s_t^*)$ *is the induced process with the combined control* $(c_t^*, M_t^*, (\eta_1^*, \eta_2^*, ...; I_1^*, I_2^*, ...))$, *and*

$$\pi_{n+1} = \begin{cases} \dfrac{1}{1+z^*} & \text{if } \omega = 0 \text{ or } \omega = 1 \text{ with } B^*_{\eta^*_{n+1}-} \leq y^*_{\eta^*_{n+1}-}, \\ \pi^* \left(\dfrac{x^*_{\eta^*_{n+1}-} + y^*_{\eta^*_{n+1}-}}{x^*_{\eta^*_{n+1}-} + B^*_{\eta^*_{n+1}-}} \right) & \text{if } \omega = 1 \text{ with } B^*_{\eta^*_{n+1}-} > y^*_{\eta^*_{n+1}-}, \end{cases}$$

where z^* *solves* $\mathcal{L}_1 \Phi(z^*, 1, 0) = 0$.

Proof. See Appendix.

Because $G(z, b, H; \Phi)$ in (15) depends on Φ, we need to provide an iterative procedure to find the solution satisfying the requirements of Proposition 3. Noting that the tax rates remain constant after the holding period exceeds H, we must have $\Phi_s = 0$ and $f(x, y, B, s) = f(x, y, B, H)$ for any $s > H$. We denote

$$\varphi(z, b) \equiv \Phi(z, b, s) \text{ for } s \geq H.$$

Then we have

$$\max \Big\{ \mathcal{L}_1 \varphi, \ G(z, b, H; \varphi) - \varphi(z, b), \\ -(1-\gamma)\varphi + f(z, 1, b, H)\varphi_z + \omega(b-1)^+ \varphi_b = 0 \Big\}, \quad s > H. \tag{17}$$

Hence, we can instead solve the system: (17) in $s > H$, (15) in $s < H$ with the terminal condition at $s = H : \Phi(z, b, H) \equiv \varphi(z, b)$. This motivates us to propose the following algorithm.

The algorithm of finding the solution numerically

1. Set

$$M_0 = \text{initial guess}.$$

2. Given M_i, for $\omega = 0$ we denote

$$G_i(z, b, s) = \frac{(M_i f(z, 1, b, s))^{1-\gamma}}{1-\gamma}$$

and for $\omega = 1$,

$$G_i(z, b, s) = \begin{cases} \dfrac{(M_i f(z, 1, b, s))^{1-\gamma}}{1-\gamma}, & b \leq 1 \\ (z+1)^{1-\gamma} \zeta\left(\dfrac{z+1}{z+b}\right), & b > 1, \end{cases}$$

where ζ is obtained by solving (A-3) with (A-4) and $\zeta(1) = \frac{M_i^{1-\gamma}}{1-\gamma}$.[8]

Using $G_i(z, b, s)$, solve

$$\begin{aligned} &\max\{\mathcal{L}_1\varphi, G_i(z, b, H) - \varphi(z, b), \\ &\quad -(1-\gamma)\varphi + f(z, 1, b, H)\varphi_z + \omega(b-1)^+\varphi_b\} = 0, \quad s > H, \qquad (18) \\ &\max\{\Phi_s + \mathcal{L}_1\Phi, \quad G_i(z, b, s) - \Phi(z, b, s), \\ &\quad -(1-\gamma)\Phi + f(z, 1, b, s)\Phi_z + \omega(b-1)^+\Phi_b\} = 0, \quad s < H \qquad (19) \end{aligned}$$

with the terminal condition at $s = H$:

$$\Phi(z, b, H) = \varphi(z, b). \qquad (20)$$

3. Set

$$M_{i+1} = \left((1-\gamma) \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1}\Phi(k, 1, 0)\right)^{1/(1-\gamma)};$$

4. If $|M_{i+1} - M_i| <$ tolerance, then stop, otherwise set $M_i = M_{i+1}$ and go to Step 2.

As long as the initial guess M_0 is small enough,[9] the above iterative procedure yields a monotonically increasing sequence $\{M_i\}_{i=1,2,\ldots}$ and is thus convergent (see the proof in the Appendix).

[8] See the proof of Proposition 2 for the justification.

[9] For example, one can set $M_0 = \nu$, where ν solves (16) with $\rho = \beta + \lambda - (1-\gamma)(1-\tau_i)r$, corresponding to the feasible strategy of only investing in the risk free asset.

4. Numerical Analysis

In this section, we provide some numerical analysis on the solution of the investor's problem. Similar to Dammon, Spatt, and Zhang (2001), we set the default parameter values as follows: the relative risk aversion $\gamma = 3$, $\lambda = 0.0125$ (an average of 80 years to investment horizon), subjective discount rate $\beta = 0.01$, threshold for long term status $H = 1$, interest rate $r = 0.01$, expected stock return $\mu = 0.05$, dividend yield $\delta = 0.02$, stock return volatility $\sigma = 0.2$, consumption utility weight $\alpha = 0.9$, short term tax rate $\tau_S = 0.35$, long term tax rate $\tau_L = 0.15$, interest and dividend tax rates $\tau_i = \tau_d = \tau_S$, and $\iota = 0$ (i.e., tax is forgiven at death).

4.1 Optimal trading boundaries

Figure 1 plots the optimal trading boundaries against the basis-price ratio b for three tax rates when the short term and long term rates are the same, for both the full rebate and the full carry over cases. When tax rate is zero, we have the standard Merton solution where the investor invests a constant fraction 50% of wealth in the stock. When the interest rate is zero, it is also optimal to keep a constant fraction of wealth in stock in the full rebate case, as shown in Proposition 1. However, because the capital gains tax and capital losses credits effectively reduce the variance of the stock return more than the expected return, the fractions become higher for higher tax rates, e.g., 58.8% if tax rates are 15% and 76.9% if tax rates are 35%, as shown by the horizontal lines.

With positive tax rates and a positive interest rate, Figure 1 shows that it is optimal to have a no-realization region when there are capital gains (i.e., $b < 1$) in both the full rebate and the full carry over cases. More specifically, if the fraction of wealth in the stock is (vertically) above the sell boundary, then the investor sells a minimum amount (and thus realizes some capital gains) to stay below the sell boundary. The trading direction is vertically downward in the figure because the basis-price ratio b does not change as the investor sells. If the fraction of wealth in the stock is (vertically) below the buy boundary, then the investor liquidates the entire stock position and rebalances to the corresponding dotted position at $b = 1$. Because the no-realization region is bounded above and below, Figure 1 shows that it can be optimal to realize capital gains even when the interest rate is positive. Intuitively, the no-realization region is a reflection of the tradeoff between the benefit of deferring tax payment and the cost of suboptimal risk exposure. When the fraction of wealth in stock is too low or too high relative to the optimal fraction in the absence of tax, the cost of suboptimal risk exposure is greater than the benefit of deferring capital gains tax. Therefore, the investor sells the stock and pays the tax. As tax rate increases, the no-realization region shifts up and the investor on average invests more in the stock in the rebate case, while the buy boundary goes down and the sell boundary goes up in the carry over case. This difference is because as tax rate increases the effective volatility of return is reduced in the rebate case, while it becomes more costly to realize capital gains without any benefit of a higher rebate in the carry over case. Since the long term and short term rates are the same, the optimal trading boundaries are independent of holding duration.

As predicted by the standard literature, Figure 1(a) shows that in the full rebate case, the entire region with capital losses (i.e., $b > 1$) belongs to the transaction region, which implies that it is always optimal to immediately realize any capital loss (by trading to the corresponding dotted positions at $b = 1$). Intuitively, immediately realizing losses can not only earn interest on the tax rebate earlier but also reduce the duration of a sub-optimal position. As shown later, always realizing losses immediately is no longer true when the long-term and short-term tax rates differ.

Figure 1(b) shows that the no-realization region in $b < 1$ is much wider in the carry over case than in the rebate case. This is because deferring tax realization has one additional benefit in the carry over case: It effectively makes potential tax losses rebatable. To understand this, suppose the investor holds one share with capital gain. If she realizes the gain, pays the tax, buys back some shares, but stock price drops, then she can only carry over the loss. If instead she did not realize the gain, then the drop in the stock price would just offset his gain, thus reducing the present value of tax liability and effectively making the loss rebatable. As we will show later, this additional benefit can be the main source of benefits for deferring tax in the carry over case even with asymmetric tax rates.

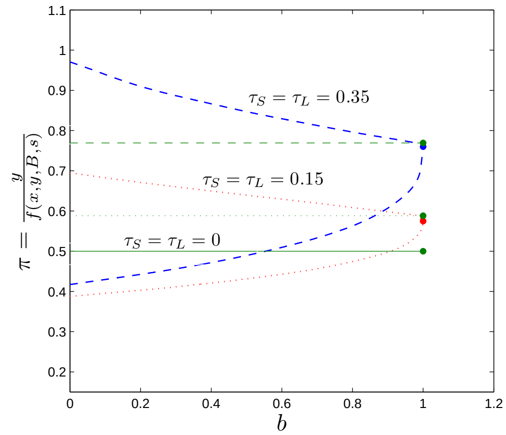
Even though there is no tax rebate for capital losses, the investor still prefers to realize capital losses immediately, because of the benefit of achieving the optimal risk exposure sooner. Indeed, Figure 1(b) confirms if there is a capital loss (i.e., $b > 1$), then it is optimal to continuously realize losses to stay at the corresponding lines for different tax rates, as shown in Proposition 2. In contrast to the full rebate case, the distance between optimal fraction at $b = 1$ (denoted by a dot) and the dotted line for $b > 1$ suggests that the optimal fraction of wealth invested in the stock is discontinuous at $b = 1$. This is because the investor needs to pay tax for capital gains but can only carry over capital losses. Because of this discrepancy, the investor tends to invest less when she has net gains and invest more when she has capital losses that can offset some potential capital gains.

Figure 1(b) shows that also different from the full rebate case, it is also optimal to defer very small capital gains, i.e., even if b is very close to 1 and the fraction of wealth in stock is far from the optimal targets (the dots), the investor still prefers to defer the realization. Intuitively, the benefit of achieving the optimal target sooner is smaller than the cost from the asymmetric treatment of gains and losses. In addition, as the tax rates increases, the optimal fraction at $b = 1$ (represented by the red or blue dot) always decreases. This is because for the carry over case, the after-tax stock return is always smaller than the no-tax return and tax does not reduce negative return fluctuation. Ehling et. al. (2010) find that stock holdings are similar between investors with full tax rebate and investors with full carry over if investors have large embedded capital gains. In contrast, Figure 1(b) suggests that stock holdings of investors with full carry over can be also significantly different from those with full tax rebate even with large embedded gains, because the after tax stock volatility is much lower and thus the buy boundary for the full rebate case is significantly higher than that for the carry over case.

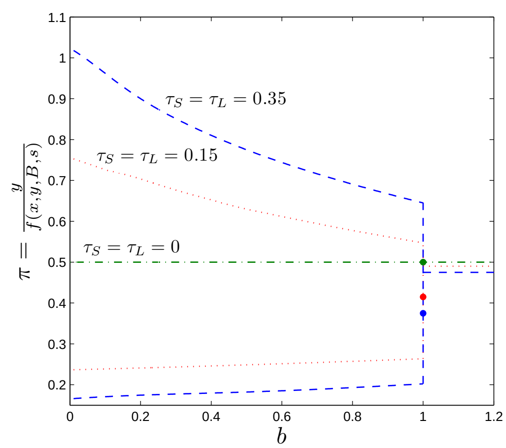
Figure 2 plots the optimal trading boundaries against the basis-price ratio b for four different holding periods when the short-term and long-term rates differ, for $\kappa = 0$ and $\kappa = 1$ and both the full rebate and the full carry over cases. Compared to the symmetric tax case with $\tau_L = \tau_S = 0.35$, the optimal target level at $b = 1$ and $s = 0$ is significantly higher and the no-realization region is much wider, a reflection of the greater benefit of deferring capital gains tax from realizing gains at the lower long-term tax rate. Still, in contrast to Constantinides (1984) and Dammon and Spatt (1996), Figure 2 implies that it can be optimal to realize short-term gains even when the long-term rate is much lower than the short-term rate, as long as the fraction of wealth in the stock becomes too high or too low relative to the optimal risk exposure in the absence of tax.

As in the symmetric rate case, Figure 2(b) shows that with full carry over, it is

Figure 1: Optimal trading boundaries against basis-price ratio b.



(a) Full Rebate



(b) Full Carry Over

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 1$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = \tau_d = \tau_S$, and $\kappa = 1$.

still optimal to immediately realize any losses. This is because the change of tax rates does not affect the benefit of carrying over the tax loss or adjusting the risk exposure. In contrast, Figure 2(a) shows that with full rebate, when the investor has already held the stock for some time (e.g., $s > 0.5$), there can be a no-realization region even when there is a capital loss (i.e., $b > 1$). Therefore, in contrast to the prediction by standard models and as pointed out by Dammon and Spatt (1996), it may be optimal to also defer tax losses realization even when there is no transaction cost or wash sale restriction. This is because there is a benefit of paying a lower (long-term) tax rate in case of an eventual gain if the investor does not realize the small tax loss and keeps the long-term status.

As far as we know, all of the existing literature on optimal consumption and investment assume a long-term loss is taxed at the long-term capital gains tax rate (i.e., $\kappa = 0$) instead of the marginal ordinary income rate (i.e., $\kappa = 1$). Next we examine how the optimal trading strategies differ across these two models. Figure 2(a) shows that when the holding duration is short, the boundaries for these two models are virtually indistinguishable (e.g., $s = 0$, $s = 0.5$). However, as the holding period approaches or exceeds the short-term threshold of 1 year, the optimal trading strategies become significantly different. If $\kappa = 0$, because the option value of realizing capital losses at the higher short-term rate after deferring decreases, the total benefit of not immediately realizing capital losses declines. Therefore the capital loss threshold beyond which it is optimal to realize the loss immediately becomes smaller when the holding time gets close to the short-term threshold of 1 year. The boundary at $s = 1^-$ shows that it is always optimal to realize all losses at the end of the year. Right after the status becomes long-term (i.e., $s = 1^+$), the entire region (for any $b \geq 0$) becomes the transaction region if $\kappa = 0$, i.e., it is optimal for the investor to sell all the stock and rebalance to the dotted position and thus realize all capital gains or capital losses immediately. These findings confirm the predictions of the existing literature (e.g., Constantinides (1984)). Intuitively, immediately realizing all the capital gains or losses converts the status to short-term, which entitles the investor to realize future losses at the higher short-term rate. Because $\kappa = 0$, the long-term tax status has costs and benefits, compared to the short-term status. The benefit is the capability of realizing capital gains at a lower rate. The cost is that the tax (rebate) rate for capital losses is also lower. When the short-term tax rate is sufficiently higher than the long-term rate (as it is in the figure), the benefit of deferring tax is smaller than the option value of realizing losses at the higher short-term rate, therefore, it is optimal for the investor to realize all the long-term gains and losses.[10]

In contrast to the predictions of the existing literature, the boundaries for $s = 1^-$ and $s = 1^+$ in Figure 2 (a) imply that if $\kappa = 1$, it can be optimal to defer some short-term losses beyond the end of a year and to defer all small long-term gains and losses in the rebate case.[11] This is because if $\kappa = 1$, then the long-term status strictly dominates the short-term status since the investor can realize losses at the higher rate and gains at the lower rate when she has the long-term status.[12]

Figure 2(a) also suggests that in the rebate case if $\kappa = 1$, the optimal trading strategies for short-term status and long-term status are qualitatively different. An investor tends to defer the realization of large short-term capital gains, as reflected by the wider no-realization region when b is small and $s < 1$. In contrast, it is always optimal to immediately realize large long-term capital gains, as reflected by the fact all positions with a large capital gain are in the transaction region if $s \geq 1$. The main intuition for always immediately realizing a large long-term capital gain is that if it is not realized and stock price goes down, then effectively the investor realizes the incremental loss at the lower long-term rate because the investor still has a cumulative

[10]Our additional numerical results unreported in the paper show that for $\tau_S = 0.35$, as long as τ_L is less than 0.34, it is optimal for the investor to realize all long-term gains and losses.

[11]Since the tax rates no longer change for holding period beyond 1 year, the trading boundaries are the same for all $s \geq 1$.

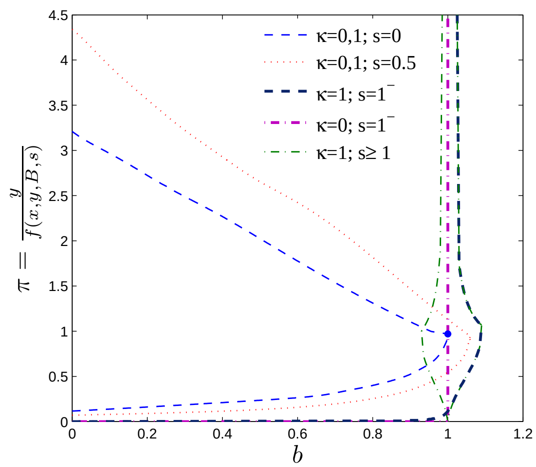
[12]We also show that the equivalent wealth loss from facing long-term rates for long-term losses can be as high as 1% of an investor's initial wealth for the set of default parameter values.

capital gain, whereas if it is realized and stock price goes down, then the investor can realize the incremental loss at the higher short-term rate.

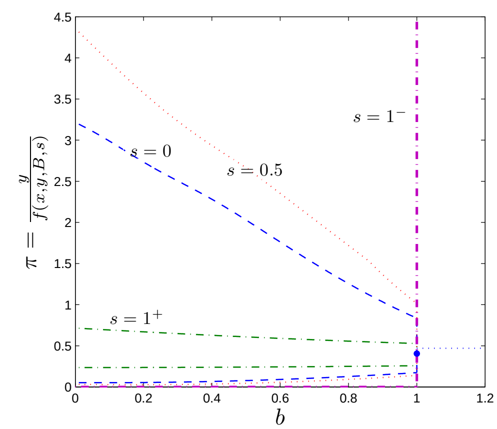
Different from the full rebate case with asymmetric tax rates, Figure 2(b) shows that it can be optimal to defer even large long-term capital gains ($s = 1^+$) if $\kappa = 1$, although the no-realization region shrinks significantly for $s > 1$ because of the lower long-term capital gain tax rate. As discussed earlier under Figure 2(a), one benefit of realizing large long-term capital gains is that in case the stock price declines, the investor can realize the loss at a higher short-term rate in the full rebate case. In the full carry over case, however, this benefit is absent and in addition, by deferring the investor can effectively make incremental losses rebatable as discussed for Figure 1, thus the investor prefers to defer capital gains, long-term or short-term. Figure 2 then shows that in contrast to the existing literature, it can be optimal to defer long-term capital gains for both the full rebate and the full carry over cases.

While Figure 2(b) may suggest that it is always optimal to realize capital losses immediately even with asymmetric tax rates, this is not true in general. Indeed, Figure 2(c), with $\tau_S = 0.396$ and $\tau_L = 0$, shows that if the difference between the long-term rate and short-term rate is large enough, then it is optimal to defer even large capital losses. This is because of the possibility that current losses can eventually turn into gain and then the investor would pay a much lower capital gains tax. This is in sharp contrast with Marekwica (2009) who shows that it is still optimal to realize losses immediately in the full carry over case with symmetric tax rates.

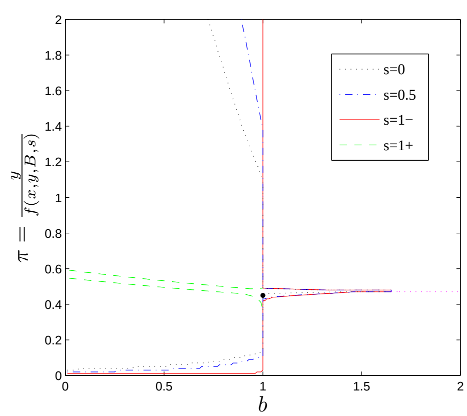
Figure 2: Optimal trading boundaries against basis-price ratio b with asymmetric long-term/short-term rates.



(a) Full Rebate



(b) Full Carry Over



(c) Full Carry Over, $\tau_i = \tau_d = \tau_S = 0.396$, $\tau_L = 0$

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 1$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = \tau_d = 0.35$, $\tau_L = 0.15$, and $\tau_S = 0.35$.

4.2 Would higher tax rates make lower income investors worse off?

The short-term tax rate is set to be equal to the marginal ordinary income tax rate applicable to the investor, while the long-term rate is independent of the income level as long as the ordinary tax rate is higher than the long-term rate. This implies that short-term rates applied to lower income investors are lower than those for higher income investors. Because investors can choose to realize gains at the long-term rate and losses at the higher short-term rate, this implies that lower income investors may be better off with higher tax rates such as those for high income investors. We next examine whether higher tax rates can indeed make lower income investors better off. To this end, we compute the equivalent wealth loss of a lower income investor for facing the lower short-term rate. More specifically, let $V_L(x, y, B, s)$ and $V_H(x, y, B, s)$ denote the value functions for a lower income investor with lower tax rates and the same lower income investor with higher tax rates. Let Δ be the equivalent wealth loss (EWL, in terms of the fraction of the initial wealth) at time 0 of the lower income investor from the lower tax rates, with the initial wealth of W_0 all in the riskless asset, i.e.,

$$V_L(W_0, 0, 0, 0) = V_H((1 - \Delta)W_0, 0, 0, 0).$$

Because of the homogeneity of the value functions, Δ is independent of W_0 which can thus be set to 1.

We use the tax rates in Table 1 for three ordinary income tax bracket levels.

In Figure 3, we plot the EWLs of a lower income investor with the low rate relative to with the high rate against the short-term tax rate τ_S for expected investment horizons of 80 years and 10 years. Figure 3 shows that not only a lower income investor

Table 1: Default Tax Rates for Figure 3 and Table 2

Ordinary Income Level	τ_i	τ_d	τ_S	τ_L
Low	0.1	0	0.1	0
Medium	0.25	0.15	0.25	0.15
High	0.396	0.238	0.396	0.238

can be better off from facing a higher short-term tax rate, but also the benefit from a higher short-term tax rate can be significant. For example, the investor is willing to pay as much as 24.5% of his initial wealth to have the higher short-term tax rate of 39.6%. As the ordinary income tax rate increases, the benefit from facing a higher short-term capital gains tax rate increases dramatically. Figure 3 implies that indeed raising capital gains tax rates can make lower income investors significantly better off. Furthermore, Figure 3 also shows that even elder lower income investors can also be significantly better off with higher tax rates. For example, when the expected time to death is only 10 years ($\lambda = 0.1$), the equivalent wealth loss of investors from the low tax rates relative to a high rate of 39.6% is as high as 12.6% of their initial wealth. On the other hand, the difference between the short-term and long-term rates needs to be large for a lower income investor to be better off with the higher rates, because the option value of realizing losses at the short-term rate would be smaller than the cost of paying at a higher rate for short-term gains when the difference is small.

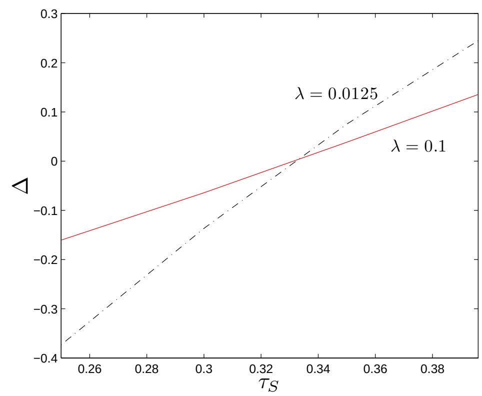
To help evaluate the robustness of the results shown in Figure 3, we report in Table 2 how optimal initial target fractions of wealth invested in stock, initial consumption wealth ratios and EWLs of a lower income investor with low, medium, and high tax rates (as defined in Table 1) change as we change a set of parameter values (the case in Figure 3 corresponds to the case with $\mu = 0.05$ and $r = 0.01$). Table 2 suggests

Table 2: **Comparisons among Low Rates, Medium Rates and High Rates**

	High Rates		Medium Rates		Low Rates		EWL	EWL
Cases	π^*	c^*	π^*	c^*	π^*	c^*	H-M	H-L
Base case	1.065	0.0300	0.755	0.0236	0.615	0.0225	0.3010	0.3480
$r = 0.01$	0.955	0.0299	0.655	0.0249	0.530	0.0249	0.2375	0.2385
$\delta = 0$	0.775	0.0202	0.530	0.0159	0.425	0.0152	0.3026	0.3487
$\delta = 0.01$	0.920	0.0247	0.645	0.0194	0.515	0.0185	0.3035	0.3505
$\mu = 0.06$	1.325	0.0411	0.975	0.0332	0.795	0.0318	0.2739	0.3181
$\sigma = 0.3$	0.495	0.0203	0.335	0.0159	0.270	0.0152	0.3084	0.3550
$\alpha = 0.5$	1.065	0.0300	0.755	0.0236	0.615	0.0225	0.3009	0.3480
$\lambda = 0.1$	1.055	0.0591	0.750	0.0528	0.605	0.0517	0.1567	0.1823
$\lambda = 0.2$	1.040	0.0924	0.735	0.0860	0.605	0.0850	0.1021	0.1177
$\gamma = 5$	0.625	0.0206	0.445	0.0161	0.355	0.0153	0.2685	0.3121
$\iota = 1$	1.065	0.0300	0.755	0.0236	0.615	0.0225	0.3010	0.3480
$\kappa = 0$	1.060	0.0298	0.755	0.0235	0.615	0.0225	0.2984	0.3453
$r = 0.01$ $\mu = 0.05$	1.080	0.0347	0.765	0.0289	0.615	0.0288	0.2377	0.2448
$\delta = 0.01$ $\mu = 0.05$	1.045	0.0294	0.745	0.0233	0.605	0.0223	0.2916	0.3376
$\delta = 0$ $\mu = 0.06$	1.020	0.0288	0.735	0.0231	0.600	0.0221	0.2824	0.3274
$\iota = 1$ $\lambda = 0.5$	0.995	0.1921	0.715	0.1859	0.585	0.1848	0.0485	0.0567
$\iota = 1$ $\lambda = 1$	0.945	0.3570	0.695	0.3509	0.570	0.3498	0.0257	0.0301

Parameter default values: $\omega = 0$, $\gamma = 3$, $\beta = 0.01$, $H = 1$, $r = 0$, $\mu = 0.04$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\iota = 0$, $\kappa = 1$, $\lambda = 0.0125$, $x_0 = 1$, and $y_0 = 0$.

Figure 3: Equivalent Wealth Loss with the Low Tax Rate



Parameter default values: $\omega = 0$, $\gamma = 3$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, and $\kappa = 1$.

that the significant welfare gain from higher tax rates is robust to these parameter value changes. For example, even when an elder investor with low rates (medium rates, respectively) has only an expected 5-year remaining lifetime ($\lambda = 0.2$), he is still willing to pay as much as 12% (column H-L) (10% (column H-M), respectively) of his current wealth to qualify for the higher 39.6% short-term rate. As the stock market volatility increases, the gain becomes even greater. For example, when the volatility increases to 30%, the EWL of an investor from low rates relative relative to high rates increases to 35.5%. As the expected return μ of the stock increases, the EWLs decrease because the chance of realizing losses decreases. However, the decrease in EWLs is small. For example, a 50% increase in the expected return (from 0.04 to 0.06) only results in a 3.0% reduction in the EWL of an investor from low rates relative to high rates. As the dividend yield decreases, the EWLs increase because the disadvantage of higher income investors from paying higher tax rates on dividend payment decreases. This suggests that lower income investors prefer stocks with high dividend yield and higher income investors prefer stocks with low dividend yield. As risk aversion increases from 3 to 5, the optimal stock investment decreases, the optimal consumption increases, and the EWLs also decrease, but only slightly. With an expected time to death of 80 years, the tax forgiveness at death has almost no impact on the EWLs and optimal consumption and investment, as shown by the row with $\iota = 1$. So our main results also apply to investors in Canada where capital gains tax is not forgiven at death.

An investor with a higher short-term rate pays more tax when she realizes a short-term gain. This is why as shown in Figures 1 and 2, the no-realization region for an investor with a short-term gain and a higher short-term rate is wider than an investor with a lower short-term rate. However, investors may be forced to liquidate due to liquidity reasons. To examine the impact of this forced liquidation on the EWLs of an investor from low rates relative to high rates, we report the results when an investor is forced to liquidate the entire stock position due to a large liquidity shock that occurs at the intensity of $\lambda = 0.5, 1$ with $\iota = 1$ (so that tax needs to be paid at liquidation). Indeed the EWLs are much smaller because the investor has to realize short-term gains more often and the expected investment horizon is much shorter. For example, when the large liquidity shock occurs once a year on average ($\lambda = 1$), the EWL of an investor from low rates relative to high rates decreases to 3.0% of the initial wealth. On the other hand, since the expected investment horizon is now only 1 year, the 3.0% loss is still quite economically significant. In addition, to bias against us, we assume that the large liquidity shock occurs once a year and when the liquidity shock occurs, the investor must liquidate the entire position and thus likely realize more gains than necessary for most liquidity needs in practice. These findings suggest that liquidity induced capital gain realization can unlikely eliminate or reverse the result that lower income investors can be better off with higher tax rates.

Table 2 also shows that an investor with higher tax rates invests more and consumes more. In addition, investment decreases with volatility, risk aversion, interest rate, and increases with expected return and expected investment horizon.

To summarize, we show that lower income investors can be significantly better off with higher tax rates like those faced by high income investors and raising their tax rates would make them invest more and consume more.

4.3 The value of deferring capital gains realization

Because tax is only paid when it is realized, an investor has the option to defer capital gains tax and immediately realize capital losses. When long-term/short-term tax rates are the same, the value of this deferring comes from earning the interest on the capital gains tax (for both the rebate and the carry over cases) and from making losses effectively rebatable (the carry over case). When long-term rates are lower than short-term rates, the value of this deferring can also come from the benefit of realizing gains at the lower long-term rate. We next decompose the value of deferring into these two sources to compare their relative magnitude. More specifically, let $\underline{V}(x, y, B, s; \tau_S, \tau_L)$ be the value function when the investor cannot defer capital gains realization, i.e., is forced to realize both gains and losses continuously (and thus short-term). Let $V(x, y, B, s; \tau_S, \tau_S)$ be the value function when the investor can defer capital gains realization, but long-term rates are equal to the short-term rates, and $V(x, y, B, s; \tau_S, \tau_L)$ be the value function when the investor can defer capital gains realization and long-term rates are lower than the short-term rates. We use the time 0 EWLs (in terms of the fraction of the initial wealth as before) Δ_0 and Δ_1 to measure the values of deferring from these two sources respectively, assuming all the initial wealth is in the risk free asset, i.e.,

$$\underline{V}(1,0,0,0;\tau_S,\tau_L) = V(1-\Delta_0,0,0,0;\tau_S,\tau_S)$$

and

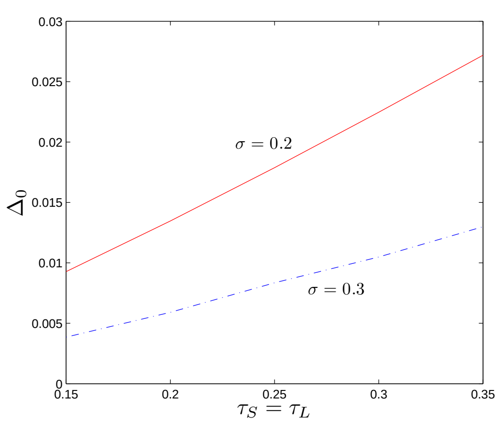
$$V(1,0,0,0;\tau_S,\tau_S) = V(1-\Delta_1,0,0,0;\tau_S,\tau_L).$$

Figure 4 plots the equivalent wealth loss Δ_0 for two volatility levels $\sigma = 0.2$ and $\sigma = 0.3$ for both the full rebate and the full carry over cases. As expected, this figure shows that as the single tax rate increases, the value of deferring from saving the interest on tax increases, because of the increase in the capital gains tax. The EWL magnitude varies from 0.81% to 2.7% of the initial wealth for the full rebate case and from 4.6% to 12.7% of the initial wealth for the full carry over case. The significantly greater value of deferring for the carry over case comes from the additional benefit of making losses effectively rebatable, while the part from interest saving (the difference between blue lines and red lines) is smaller than 1%. Figure 4(b) also shows that as the stock volatility increases, this value of deferring decreases, because the investor invests less in the stock and thus the dollar amount of the capital gains tax deferred decreases.

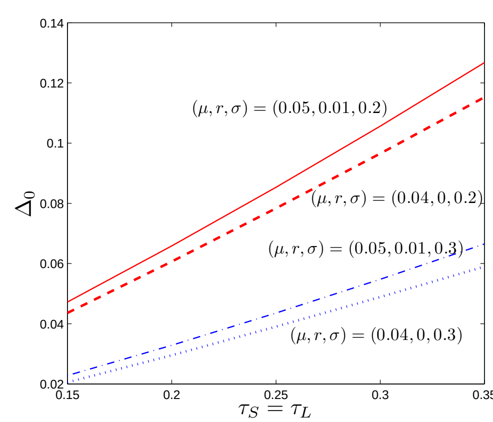
Figure 5 plots the equivalent wealth loss Δ_1 against the short-term rate for two volatility levels $\sigma = 0.2$ and $\sigma = 0.3$ when the long-term rate $\tau_L = 15\%$. This figure shows that as the short-term tax rate increases, the value of deferring from realizing gains at a lower rate increases significantly, because of the increase in the difference between the long-term rate and the short-term rate. The value of deferring from realizing gains at a lower rate is much greater than that from saving interest for lower income investors. For the full rebate case, the EWL magnitude can be as high as 48% of the initial wealth, in contrast to 2.6% for the benefit of interest saving. Therefore, by ignoring the difference between the long-term and short term rates, most of the existing literature significantly underestimates the value of deferring capital gains tax and largely overestimates the effective tax rates. Similar to Figure 4, the value of deferring from realizing gains at a lower rate also decreases with volatility because of the reduced investment in the stock.

In contrast, for the full carry over case, the EWL Δ_1 is much smaller because without rebate, realizing losses at a higher short-term rate does not provide any additional benefit compared to realizing them long-term. Consistent with this result, we show that in the carry over case, both investment and consumption are insensitive to changes in tax rates, as shown in Figure 6. Also different from the full carry over case, an increase in the volatility increases the EWL because the option value of realizing gains at the lower long-term rate is greater and more than offsets the effect from the decrease in the stock investment.

Figure 4: The value of deferring capital gains tax from saving the interest.



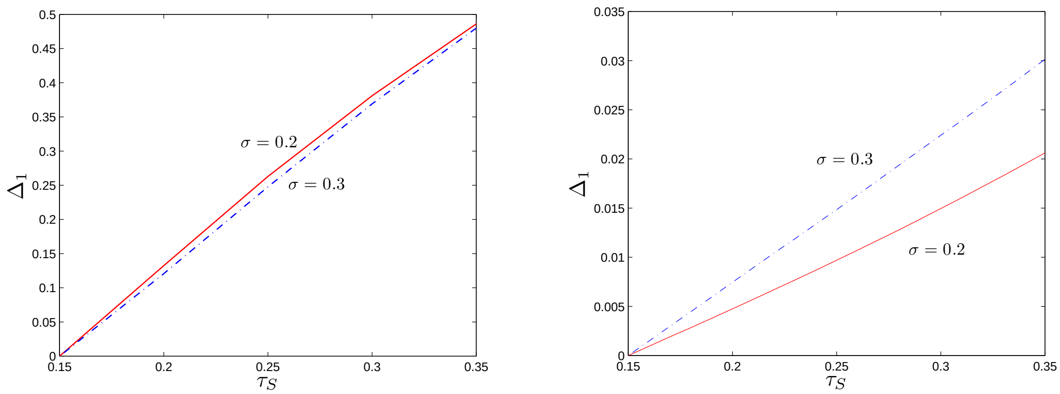
(a) Full Rebate



(b) Full Carry Over

Parameter default values: $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = \tau_d = \tau_S$, and $\kappa = 1$.

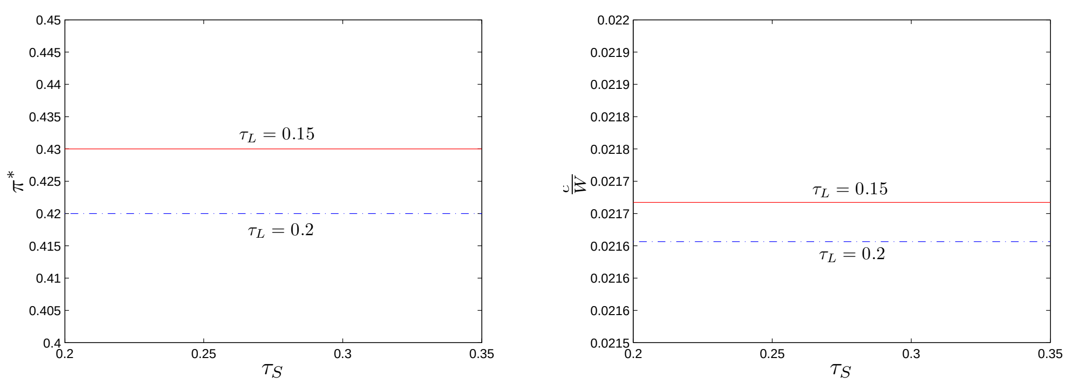
Figure 5: The value of deferring capital gains tax from realizing gains at the lower long-term rate.



(a) Full Rebate (b) Full Carry Over

Parameter default values: $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 1$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = \tau_d = \tau_S$, $\tau_L = 0.15$, and $\kappa = 1$.

Figure 6: Optimal initial fraction of wealth in stock and consumption against the short-term tax rate.



(a) Full Carry Over (b) Full Carry Over

Parameter default values: $\omega = 1$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.15$, and $\kappa = 1$.

4.4 Tax payment and holding period

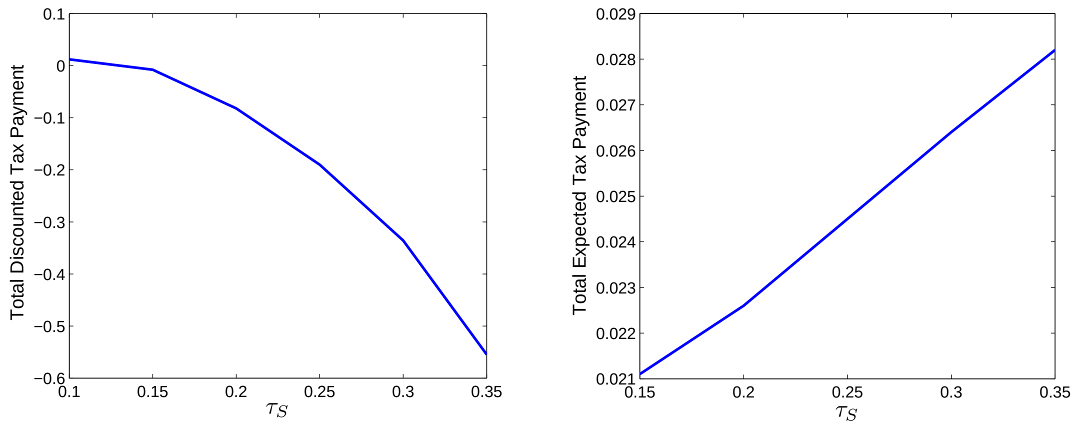
In this subsection, we use simulation to study how tax revenue and average holding period change in response to changes in tax rates. More specifically, we simulate

10,000 paths and compute the average present value of tax paid as a fraction of the initial wealth and the time between a purchase and the next sale (i.e., holding period) across all these paths.

Figure 7 plots how the expected tax payment change as the shorter-term rate changes for the full rebate and the full carry over cases. Figure 7 shows that as the short-term rate increases, the effective tax rate in the full rebate case can become negative because of the greater tax rebate for capital losses. This is consistent with the finding that less wealthy investors may benefit from an increase in the short-term tax rate. In contrast, for wealthy investors for whom tax rebate is insignificant, expected tax payment increases with the short-term rate. However, even with short-term rate of 35% and long-term rate of 15%, the effective tax rate is low, only 2.82%. This is because the loss carry-over provision and capital gains deferring.

Figure 8 plots how the expected time between purchase and the next sale changes as the long-term rate changes. The conventional wisdom holds that as long-term capital gains tax rate increases, the average holding period should increase because of the higher tax cost of realization. This is indeed true for wealthy investors, as shown in Figure 8(b). This is because the no realization region for long-term gains widens (see Figure 2(b)), i.e., they defer long-term gain longer to save on interest and make potential incremental losses rebatable. However, for less wealthy investors, the opposite holds. As shown in Figure 8(a), the average holding period decreases as the long-term rate increases because the option value of realization at long-term rate decreases and thus the no realization region narrows and the investors defer less. Still, the average tax payment increases as the long-term rate increases in both cases, as shown in Figure 9.

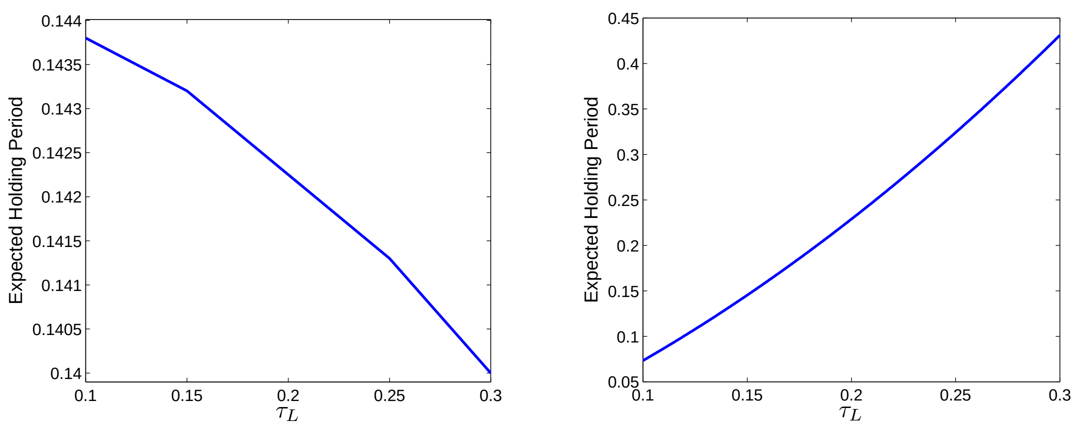
Figure 7: The expected tax payment as a fraction of initial wealth against τ_S.



(a) Full Rebate (b) Full Carry Over

Parameter default values: $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.15$, $\tau_L = 0.15$, and $\kappa = 1$.

Figure 8: The expected time from purchase to sale against τ_L



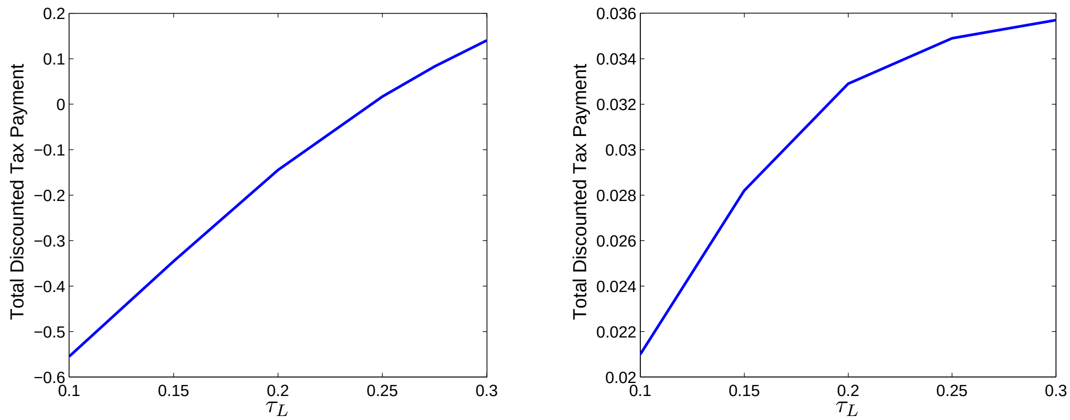
(a) Full Rebate (b) Full Carry Over

Parameter default values: $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.15$, $\tau_S = 0.35$, and $\kappa = 1$.

5. Conclusions

The optimal trading strategy with asymmetric long-term/short-term tax rates can be significantly different from that with a single tax rate. In addition, the impact of

Figure 9: The expected tax payment as a fraction of initial wealth against τ_L



(a) Full Rebate (b) Full Carry Over

Parameter default values: $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.15$, $\tau_S = 0.35$, and $\kappa = 1$.

capital gain tax on a less wealthy investor can be qualitatively different from on a wealthy investor. For example, in contrast to the standard literature, we show that for a less wealthy investor it can be optimal to defer capital losses beyond one year even in the absence of transaction costs and wash sale restriction. In addition, lower income investors can be significantly better off with higher tax rates such as those for wealthy investors. Moreover, raising the short-term tax rate can increase both consumption and stock investment for a lower income investor.

The existing literature assumes that the tax rate for long-term capital losses is the same as that for long-term capital gains, instead of the same as the marginal ordinary income tax rate as the law stipulates. Under this assumption, it is always optimal to realize all short-term capital losses before they turn into long-term. In contrast, for lower income investors, we show that it can be optimal to defer long-term capital losses, no matter how high the short-term tax rate is compared to the long-term rate.

In contrast, for an investor whose majority of capital losses can only be carried over (e.g., wealthy investors), capital gain tax always makes her worse off and she is willing to hold a tax-exempt security with a much lower expected return. In addition, an increase in tax rates tends to decrease the optimal target stock holdings of a wealthy investor and it can be optimal for him to defer even large long-term capital gains.

To conclude, our paper shows that the impact of capital gain tax on the optimal trading strategy critically depends on the difference between long-term and short-term tax rates and whether capital loss rebate or carry-over is more representative of the capital loss treatment.

Appendix

A.1 Proof of Proposition 1

Let

$$W_t = x_t + y_t - \tau_S (y_t - B_t) .$$

It is easy to verify

$$dW_t = [(1-\tau_i) r x_t - c_t + ((1-\tau_S)\mu + (1-\tau_d)\delta) y_t] dt + (1-\tau_S)\sigma y_t dw_t. \quad \text{(A-1)}$$

Because the investor always liquidates the entire stock holdings to maintain a constant fraction of wealth in stock, it follows $y_t = B_t$, $x_t = W_t - y_t$. So, problem (11) reduces to a classical Merton's consumption-investment problem with interest rate $(1-\tau_i) r$ and wealth process following (A-1) and stock prices following

$$\frac{dS_t}{S_t} = [(1-\tau_S)\mu + (1-\tau_d)\delta] dt + (1-\tau_S)\sigma dw_t.$$

It is well-known that Merton problem's optimal consumption and investment strategy is

$$\begin{cases} \frac{c_t^*}{W_t} = \alpha^{1/\gamma} v^{-(1-\gamma)/\gamma}, \\ \frac{y_t^*}{W_t} = \frac{(1-\tau_S)\mu + (1-\tau_d)\delta - (1-\tau_i) r}{\gamma (1-\tau_S)^2 \sigma^2}, \end{cases}$$

which gives a value function $\frac{(vW)^{1-\gamma}}{1-\gamma}$. Notice that Merton's optimal strategy is admissible to problem (11), then these two value functions coincide. It is easy to show that (16) has a unique positive root.

If $r = 0$ and $\tau_S = \tau_L = \tau$, then for any allowable strategy, we always have

$$dW_t = -c_t dt + y_t [((1-\tau)\mu + (1-\tau_d)\delta) dt + (1-\tau)\sigma dw_t] , \quad \text{(A-2)}$$

which is independent of the tax basis B_t. Since the dynamics of y_t in (3) is independent of B_t, we again obtain a Merton-like problem, and the resulting Merton's strategy must be optimal. The proof is complete.

A.2 Proof of Proposition 2

Let us start from the HJB equation (14) for $V(x,y,B,t)$. Note that for $B>y$ when $s=0$ or $\tau_S=\tau_L$, we have

$$\begin{aligned} V(x,y,B,0) &\geq V\left(f(x,y,B,0)-I,I,I+(B-y)^{+},0\right) \\ &= V(x+y-I,I,I+B-y,0). \end{aligned}$$

Setting $I=y+\delta,\ \delta>0$, we have

$$V(x,y,B,0)\geq V(x-\delta,y+\delta,B+\delta,0),$$

from which we deduce

$$V_x-V_y-V_B\geq 0,\ \text{for } B>y.$$

On the other hand, by (14), we always have

$$yV_x-yV_y-yV_B\leq 0,\ \text{for } B>y.$$

So,

$$V_x-V_y-V_B=0 \text{ for } B>y,$$

which implies continuous trading for $B>y$ when $s=0$ or $\tau_S=\tau_L$.

We will restrict attention to $B_t>y_t$. In this region, we have

$$V(x,y,B,0)=(x+y)^{1-\gamma}\zeta(\theta),\ \theta=\frac{x+y}{x+B},$$

for some function $\zeta(\cdot)$. Plugging into the HJB equation (14), we have

$$L\zeta=0 \quad \text{in} \quad 0<\theta<1, \tag{A-3}$$

with

$$\zeta(0)=\frac{\nu^{1-\gamma}}{1-\gamma},\quad \zeta(1)=\frac{M^{1-\gamma}}{1-\gamma}, \tag{A-4}$$

where ν solves (16), M is such that $\Phi(z,1,0) = \frac{(M(z+1))^{1-\gamma}}{1-\gamma}$,

$$
\begin{aligned}
L\zeta &= \frac{\alpha(\varkappa^*)^{1-\gamma}}{1-\gamma} + \frac{(1-\alpha)\lambda}{1-\gamma} + \frac{1}{2}\sigma^2\pi^{*2}\theta^2\zeta_{\theta\theta} \\
+ & \Big\{ \left[(1-\tau_i)\, r\,(1-\pi^*) + (1-\tau_d)\,\delta\pi^* - \varkappa^*\right](1-\theta) + \mu\pi^* + \sigma^2\pi^{*2}(1-\gamma) \Big\}\theta\zeta_\theta \\
+ & \left\{ \left[(1-\tau_i)\, r\,(1-\pi^*) + (\mu + (1-\tau_d)\,\delta)\,\pi^* - \frac{1}{2}\gamma\sigma^2\pi^{*2} - \varkappa^* \right](1-\gamma) - \beta - \lambda \right\}\zeta,
\end{aligned}
$$

and

$$
\varkappa^* = \left[(1-\theta)\,\theta\zeta_\theta + (1-\gamma)\,\zeta\right]^{-1/\gamma},
$$

$$
\pi^* = \frac{\left[(1-\tau_d)\,\delta - (1-\tau_i)\, r\right]\theta^2\zeta_\theta - \left[\mu + (1-\tau_d)\,\delta - (1-\tau_i)\, r\right]\left[\theta\zeta_\theta + (1-\gamma)\,\zeta\right]}{\sigma^2\theta^2\zeta_{\theta\theta} + 2\,(1-\gamma)\,\theta\zeta_\theta - \gamma\,(1-\gamma)\,\zeta}. \tag{A-5}
$$

A.3 Proof of Proposition 3

The proof is similar to that in Davis and Norman (1990) and Øksendal and Sulem (2002). Here we only provide the main steps for the proof.

Define

$$
\begin{aligned}
N_t &= \int_0^t e^{-(\beta+\lambda)h}\Big(\alpha u(c_h) + (1-\alpha)\lambda u((1-\iota)(x_h+y_h) + \iota f(x_h, y_h, B_h, s_h))\Big)dh \\
&\quad + e^{-(\beta+\lambda)t} V(x_t, y_t, B_t, s_t).
\end{aligned} \tag{A-6}
$$

Let M_t^c be the continuous part of M_t and $\Delta M_h = M_h - M_{h-} \in (0,1]$ be the discrete jump at time h. An application of the generalized version of Ito's lemma implies that

$$
\begin{aligned}
N_t &= N_0 + \int_0^t e^{-(\beta+\lambda)h}(V_s + \mathcal{L}_0 V)dh \\
&\quad + \int_0^t e^{-(\beta+\lambda)h}(f\,(0, y_h, B_h, s)\, V_x - y_h V_y - (B_h - \omega(B_h - y_h)^+)V_B)dM_h^c \\
&\quad + \sum_{0\le h\le t} e^{-(\beta+\lambda)h}\Big(V(x_{h-} + f\,(0, y_{h-}, B_{h-}, s_h)\,\Delta M_h, y_{h-} - y_{h-}\Delta M_h, \\
&\qquad B_{h-} - (B_{h-} - \omega(B_{h-} - y_{h-})^+)\Delta M_h, s_h) - V(x_{h-}, y_{h-}, B_{h-}, s_h)\Big) \\
&\quad + \int_0^t e^{-(\beta+\lambda)h}\sigma y_h V_y dw_h.
\end{aligned} \tag{A-7}
$$

First, we show that the fourth term is non-positive for any feasible trading strategy. By the mean-value theorem there exists $0 \leq m \leq \Delta M_h \leq 1$ such that the fourth term is equal to[13]

$$
\begin{aligned}
& \sum_{0 \leq h \leq t} e^{-(\beta+\lambda)h} \Big(f\left(0, y_{h-}, B_{h-}, s_h\right) V_x - y_{h-} V_y - (B_{h-} - \omega(B_{h-} - y_{h-})^+) V_B \Big) \Delta M_h \\
= \; & \sum_{0 \leq h \leq t} e^{-(\beta+\lambda)h} \frac{f\left(0, y_m, B_m, s_h\right) V_x - y_m V_y - (B_m - \omega(B_m - y_m)^+) V_B}{1-m} \Delta M_h \\
\leq \; & 0, \qquad\qquad \text{(A-8)}
\end{aligned}
$$

where the inequality follows from the HJB equation, V_x, V_y, and V_B are evaluated at (x_m, y_m, B_m, s_h) with

$$
\begin{aligned}
& x_m = x_{h-} + f\left(0, y_{h-}, B_{h-}, s_h\right) m, \quad y_m = (1-m) y_{h-}, \\
& B_m = B_{h-} - (B_{h-} - \omega(B_{h-} - y_{h-})^+) m.
\end{aligned}
$$

Then N_t is a martingale under the proposed strategy and a supermartingale for any feasible strategy, because V satisfies the HJB equation (14), $dM_t^c \geq 0$ for any feasible M_t, and thus both the second and the third term are non-positive for any feasible trading strategy and equal to 0 for the proposed one, the next-to-last term is 0 for the proposed strategy and non-positive for any feasible trading strategy, and the last term is a martingale for any feasible strategy due to the lognormal distributed stock price and the boundedness of V_y.

Let

$$
\eta^* := \inf\{\eta \geq t : V(x_\eta, y_\eta, B_\eta, s_\eta) = \sup_{I>0} V\left(f(x_\eta, y_\eta, B_\eta, s_\eta) - I, I, I + \omega(B_\eta - y_\eta)^+, 0 \right).
$$

[13] If $m = 1$, then the inequality follows from (A-10).

We then have

$$
\begin{aligned}
& V(x_0, y_0, B_0, s_0) \\
= \;& N_0 \geq E[N_\eta] \\
= \;& E\Big[\int_0^\eta e^{-(\beta+\lambda)h}\big(\alpha u(c_h) + (1-\alpha)\lambda u((1-\iota)(x_h+y_h) + \iota f(x_h, y_h, B_h, s_h))\big)dh \\
& + e^{-(\beta+\lambda)\eta} V(x_\eta, y_\eta, B_\eta, s_\eta)\Big] \\
\geq \;& E\Big[\int_0^\eta e^{-(\beta+\lambda)h}\big(\alpha u(c_h) + (1-\alpha)\lambda u((1-\iota)(x_h+y_h) + \iota f(x_h, y_h, B_h, s_h))\big)dh \\
& + e^{-(\beta+\lambda)\eta} V(f(x_{\eta-}, y_{\eta-}, B_{\eta-}, s_{\eta-}) - I_\eta, I_\eta, I_\eta + \omega(B_{\eta-} - y_{\eta-})^+, 0)\Big] \qquad \text{(A-9)}
\end{aligned}
$$

for any feasible strategy (c_t, M_t, η, I_η), with equality for $(c_t^*, M_t^*, \eta^*, I_{\eta^*}^*)$, where the second inequality follows from the fact that

$$V(x, y, B, s) \geq \sup_I V\left(f(x, y, B, s) - I, I, I + \omega(B_\eta - y_\eta)^+, 0\right) \qquad \text{(A-10)}$$

as implied by the HJB equation (14). The proof is complete.

A.4 Convergence of the algorithm

By step 3 of the algorithm,

$$\Phi(k, 1, 0) \leq \frac{(M_{i+1}(k+1))^{1-\gamma}}{1-\gamma} \qquad \text{(A-11)}$$

for all $k \in (-1, +\infty)$. From (19), we infer

$$\Phi(k, 1, 0) \geq G_i(k, 1, 0) = \frac{(M_i f(k, 1, b, 0))^{1-\gamma}}{1-\gamma} = \frac{(M_i(k+1))^{1-\gamma}}{1-\gamma} \qquad \text{(A-12)}$$

for all $k \in (-1, +\infty)$. Combination of (A-11) and (A-12) gives

$$M_i \leq M_{i+1}.$$

Hence, $\{M_i\}_{i=1,2,\ldots}$ is a monotonically increasing sequence. It remains to find an upper bound of the sequence.

Let $\Phi_0(z,b,s)$ be the solution to the HJB equation (15), as given in the verification theorem. We will prove

$$M_i \leq \left((1-\gamma) \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1} \Phi_0(k,1,0) \right)^{1/(1-\gamma)}$$

or equivalently,

$$\frac{M_i^{1-\gamma}}{1-\gamma} \leq \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1} \Phi_0(k,1,0) \tag{A-13}$$

for all i. Let us prove it using the method of induction. Clearly it is true for $i=0$ because Φ_0 corresponds to the value function while M_0 is only associated with a suboptimal strategy. Suppose (A-13) is true for i. Then

$$G_i(z,b,s) \leq f(z,1,b,s)^{1-\gamma} \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1} \Phi_0(k,1,0).$$

Applying the maximum principle (cf. [18]) to the problem (18)-(20), we have

$$\Phi(z,b,s) \leq \Phi_0(z,b,s) \text{ for all } z,b,s.$$

Therefore,

$$\begin{aligned} \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1} \Phi_0(k,1,0) &\geq \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1} \Phi(k,1,0) \\ &= \frac{M_{i+1}^{1-\gamma}}{1-\gamma}, \end{aligned}$$

which is desired. The proof is complete.

A.5 Is the assumption of single-basis a good approximation?

To keep a single tax basis, we assume that if an investor needs to buy more stock, she must first liquidate the entire stock position before buying. This assumption is not as restrictive as it may appear. Intuitively, if there is a capital loss, then it is optimal to sell the entire stock position to realize the capital loss earlier. The only case this assumption might be restrictive is when there is a capital gain and the investor wants to buy more of the stock. This happens when the fraction of wealth invested in the stock moves downward and reaches the buy boundary. However, since the stock has a higher expected return than the risk free asset and intertemporal consumption withdrawn from the risk free asset account also tends to move the fraction of wealth invested in the stock upward, the likelihood of buying when there is a capital gain is low.

To verify this intuition, we first use simulation to show the infrequency of hitting the buy boundary and then solve a discrete-time exact basis model to compare the results.

A.5.1 Simulations

we simulated 1000 sample paths of the π and b for a 20 year horizon (2500 trading dates per path) and report the average (across sample paths) number of transactions in these 20 sample paths in Tables 3 and 4. We use nbuy and nsell to denote respectively the average number of transactions involving first liquidating the entire position and then rebalancing to the optimal position and those involving selling a fraction of the current position. We also note if a transaction is a realization of a loss ($b \geq 1$) or a gain ($b < 1$). For these 1000 sample paths and the full rebate case, Table 3 implies that there are 137.43 average number of transactions for a 20 year horizon, 107.11 of which involve first liquidating the entire position and then rebalancing occur to realize *capital losses* (short-term 106.03, long-term 1.08), which is likely optimal even without the assumption of a single tax basis. Out of 2500 possible trading dates for each path, the investor liquidates her entire position to realize capital gains only on an average of 10.11 dates. More importantly, these capital gains are all long-term. Realizing all large long-term capital gains to reestablish the short-term status is also likely optimal even without the assumption of a single tax basis, because of the benefit of higher short-term rate for incremental losses, as explained below Figure 2. Therefore the impact of the assumption of a single tax basis is small and unlikely affects our results significantly. For the full carry over case, Table 4 shows similar patterns.

Table 3: The average number of transactions for 1000 sample paths, Full Rebate

holding period	nbuy ($b \geq 1$)	nbuy ($b < 1$)	nsell ($b \geq 1$)	nsell ($b < 1$)
$s < H$	106.03	0	19.17	1.04
$s \geq H$	1.08	10.11	0	0

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.3$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.35$, $\tau_S = 0.35$, $\tau_L = 0.15$ and $\kappa = 1$.

Table 4: The average number of transactions for 1000 sample paths, Full Carry Over

holding period	nbuy ($b \geq 1$)	nbuy ($b < 1$)	nsell ($b \geq 1$)	nsell ($b < 1$)
$s < H$	462.95	0	447.10	3.52
$s \geq H$	1.05	0.07	0.17	41.39

Parameter default values: $\omega = 1$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.3$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.35$, $\tau_S = 0.35$, $\tau_L = 0.15$ and $\kappa = 1$.

A.5.2 The exact tax basis model versus the single tax basis model

First let us give a brief introduction to the discrete-time exact tax basis model proposed by DeMiguel and Uppal (2005).

If m is the number of discrete time points, we have time points $t_i = i\Delta t$, $i = 0, 1, ..., m$ with $\Delta t = T/m$. Let C_i, c_i, S_i denote respectively cash in bank, consumption, and stock price at time t_i. At each time point, the investor needs to determine stock holdings, in addition to consumption. To keep track of the exact tax basis, we introduce the variable $N_{j,i}$ to represent the number of shares bought at time t_j and held at time t_i, where $j = 0, 1, ..., m$ and $i = j, j+1, ..., m$. When short sales are not optimal, e.g., when the risk premium is positive, we have

$$N_{j,i} \geq N_{j,i+1} \geq ... \geq N_{j,m} \geq 0 \tag{A-14}$$

for any $j \in \{0, 1, ..., m\}$.

Given the trading and consumption strategy $\{N_{j,i}, c_i\}_{j=0,...,i;i=1,...,m}$, the cash amount at time t_i becomes

$$C_i = C_{i-1}e^{r\Delta t} - c_i - N_{i,i}S_i + \sum_{j=0}^{i-1} \left(N_{j,i-1} - N_{j,i}\right) \left[S_i - \left(S_i - S_j\right) \tau \left(t_i - t_j\right)\right] \tag{A-15}$$

for any $i \geq 1$, where $\tau(\cdot)$ is the tax rate as given in Section 2. We aim to choose $\{N_{j,i}, c_i\}$ to maximize

$$E\left[\sum_{i=1}^{m} e^{-\beta t_i} \frac{c_i^{1-\gamma}}{1-\gamma}\right] \tag{A-16}$$

subject to (A-14), (A-15), and the solvency constraint

$$C_i + \sum_{j=0}^{i} N_{j,i} \left[S_i - \left(S_i - S_j\right) \tau \left(t_i - t_j\right)\right] \geq 0 \quad \text{for any } i.$$

For further discretization, we assume that the stock price follows a binomial tree process with probability p of going up by u, and $1-p$ of going down by d. Without loss of generality, we set $p = 1/2$. To be consistent with the continuous time model, we determine u and d through

$$\begin{aligned} \mu \Delta t &= pu + (1-p)\, d, \\ \sigma^2 \Delta t &= pu^2 + (1-p)\, d^2 - \mu^2 \Delta t^2. \end{aligned}$$

Then the problem can be formulated as a constrained optimization problem which can be solved by the Sparse Nonlinear Optimizer algorithm (SNOPT) [see DeMiguel and

Uppal (2005)]. It should be emphasized that the problem is strongly path-dependent and leads to 2^{i+1} states at time t_i. As a consequence, it is very time consuming and can work only for a very limited number of time steps.

The single tax basis model assumes that all stock holdings should be sold out before purchase. Hence we need add more constraints:

$$\forall i, N_{i,i} \sum_{j=0}^{i-1} N_{j,i} = 0$$

to deal with the discrete-time single tax basis model. Unfortunately the solution to the optimization problem with the additional nonlinear constraints is not accurate enough to examine the wealth loss of the single tax basis model from the exact tax basis model. Hence, we use the following suboptimal single-basis strategy that clearly cannot be better than the optimal single-basis strategy: 1) if the second tax basis occurs in the optimal strategy with exact tax basis, we first sell all stock holdings to realize capital gains or losses, then buy back stock to reach the same stock holding as in the strategy with exact tax basis; 2) the consumption remains unchanged. Using a suboptimal single-basis strategy biases against us.

Table A.5.2 reports the equivalent wealth loss from following the suboptimal single-basis strategy in terms of the percentage of the initial wealth. It shows that the loss relative to the optimal exact basis strategy is small and seems almost negligible even for the the suboptimal single-basis strategy. This seems to suggest that single-basis is a good approximation.

Parameter Values	$\Delta t = 1/2, T = 4$ Single-basis EWL (%)	$\Delta t = 1/3, T = 3$ Single-basis EWL (%)
Default	0.00017	0.033
$\mu = 0.04$	0.00000	0.0096
$\mu = 0.06$	0.00079	0.39
$\sigma = 0.15$	0.0013	0.55
$\sigma = 0.25$	0.000001	0.0092
$\tau_S = 0.30$	0.00059	0.026
$\tau_S = 0.25$	0.0010	0.020
$\tau_L = 0.20$	0.00060	0.027
$\tau_L = 0.25$	0.0011	0.021

Table 5: Equivalent wealth loss of the suboptimal single-basis strategy relative to the optimal exact-basis strategy. Default parameters: $\mu = 0.05, \sigma = 0.20, r = 0.01, \tau_S = 0.35, \tau_L = 0.15, \gamma = 3, \beta = 0.01$

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