Optimal Portfolio Selection with Transaction Costs and Finite Horizons

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We examine the optimal trading strategy for a CRRA investor who maximizes the expected utility of wealth on a finite date and faces transaction costs. Closed-form solutions are obtained when this date is uncertain. We then show a sequence of analytical solutions converge to the solution to the problem with a deterministic finite horizon. Consistent with the common life-cycle investment advice, the optimal trading strategy is found to be horizon dependent and largely buy and hold. Moreover, it might be optimal for the investor in our model not to buy any stock, even when the risk premium is positive. Further analysis of the optimal policy is also provided.

Financial advisers typically recommend that younger investors should allocate a greater share of wealth to stocks than older investors and all investors should follow a largely buy-and-hold strategy. Representative of this conventional wisdom, Malkiel (2000), in his popular book *A Random Walk Down Wall Street*, states that "The longer period over which you can hold on to your investments, the greater should be the share of common stocks in your portfolio. . . . [M]oreover, these returns are gained by the steady strategy of *buying* and *holding* your diversified portfolio." To be consistent with this clearly horizon-dependent portfolio rule, a model must be of finite horizon by definition. Moreover, when an investor invests for a specific event, such as bequest or retirement, his horizon is also clearly finite. However, the finiteness of the horizon alone is not sufficient to justify the horizon-dependent investment strategy. For example, in Samuelson (1969) and Merton (1971), even though the investor has a finite horizon, his optimal fraction of wealth invested in the stock is still horizon independent.

Assuming time-varying investment opportunities or other time-varying parameters would imply horizon-dependent portfolio rules [Kim and Omberg (1996), Brennan, Schwartz, and Lagnado (1997), Liu (1999)]. However, these models do not produce buy-and-hold rules and it is still hard

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to explain in general why, as an investor gets older, he should invest a smaller fraction of wealth in stock. Introduction of labor income can potentially explain the inverse relationship between age and the fraction of wealth invested in stocks [Bodie, Merton, and Samuelson (1992), Jagannathan and Kocherlakota (1996), Campbell and Viceira (1999)]. However, these models generally do not produce buy-and-hold strategies either.

Jagannathan and Kocherlakota (1996) examine several possible explanations of the above life-cycle investment advice. In particular, they show that if investors with constant relative risk aversion (CRRA) preferences over terminal wealth are restricted to buy-and-hold strategies due to transaction costs, the optimal portfolio choice is largely horizon independent. However, they do not account for the transaction costs incurred by an investor, and in this case whether an investor optimally chooses to engage in a buy-and-hold strategy should depend on the time interval over which the portfolio is held.

In this article, instead of assuming time-varying parameters or labor income or buy and hold as in the above articles, we show that the presence of transaction costs (which are certainly present in most financial markets) together with a finite horizon would imply a time-varying and largely buy-and-hold trading strategy which is consistent with the above life-cycle investment advice.[1] In particular, we examine the optimal transaction policy for a CRRA investor who has a finite horizon and is subject to proportional transaction costs in stock trading. Our analysis reveals that an investor with a longer horizon would tend to hold more stock in his portfolio. Thus the horizon becomes an important element of the investor's optimal decision process. Moreover, even small transaction costs lead to dramatic changes in the optimal behavior for an investor: from continuous trading to virtually buy-and-hold strategies. For example, an investor whose horizon is 10 years may expect to hold a position in the asset subject to transaction costs for 5 years. In addition, for the first time in the literature (as far as we know), we derive explicit bounds on the transaction boundaries. Our analysis also shows that an investor might optimally never buy the stock subject to transaction costs, even when there is a positive risk premium. We provide explicit necessary and sufficient conditions for this to happen in all the cases we analyze. Intuitively, an investor who does not expect to live long enough for the excess return on the asset to overcome the transaction costs would optimally never buy the asset.

There are a large number of articles studying the optimal transaction policy for an agent facing transaction costs in the financial markets. Constantinides (1979, 1986), Davis and Norman (1990), and Shreve and Soner (1994) study an infinite horizon problem where the investor maximizes discounted utility of intermediate consumption. Dumas and Luciano (1991) study the problem

[1] Adding labor income to our model would clearly reinforce the main results in the article in the same manner as in Jagannathan and Kocherlakota (1996).

of maximizing terminal utility of wealth in the limit as the horizon gets very large. For these analyses, the investor's horizon is infinite and, as a result, if the risk premium is positive the investor always optimally invests in the stock, even with transaction costs. Davis, Panas, and Zariphopoulou (1993) show the existence and uniqueness of the solution to a deterministic finite horizon problem and provide a discretization scheme to numerically solve the problem. Cvitanić and Karatzas (1996) and Loewenstein (2000) also study a deterministic finite horizon problem but do not provide specific solutions. Gennotte and Jung (1994) and Balduzzi and Lynch (1999) use binomial or discrete approximations to numerically compute the optimal trading strategy for an investor with a finite horizon. While a numerical approach may allow more flexible specifications of the form of the asset market, it provides little insight into the global properties of the optimal solutions. Moreover, the optimal solutions in these approaches can be sensitive to the choice of discretization. In contrast, this article proposes a methodology to analytically approximate the optimal strategy. It shows that this approach is indeed valid and provides approximations that are less prone to approximation error.

In order to focus on the effect of the horizon on an investor's investment decision in the presence of transaction costs, we restrict our attention to the case where the investor wishes to maximize the utility of wealth on a finite date, although much of our analysis is applicable to the case with intermediate consumption. This choice of objective function is appropriate for an individual who is investing for a specific event in the future. A direct attack on solving the transaction cost problem with a deterministic horizon involves solving a partial differential equation with two free boundaries; this is difficult because these two free boundaries also change through time.[2] Here we propose a different approach, based on an idea in Carr (1998).

We first examine the optimization problem for an investor maximizing expected CRRA utility of wealth at an uncertain time, which is assumed to be the first jump time of an independent Poisson process (thus the horizon is exponentially distributed). This analysis is of independent interest since many lifetime events such as disability or retirement occur at an uncertain time.[3] This case bears some resemblance, although with important economic differences, to the analysis in Dumas and Luciano (1991), where they examine the limiting strategy as the investor's horizon becomes large. In particular, the optimal trading strategy is also time independent in this case. However, in contrast to the asymptotic analysis in Dumas and Luciano (1991), who found no bias in favor of cash in the optimal portfolio, a finite horizon can induce a bias in favor of cash; our investor may optimally never buy the asset subject to transaction costs if the expected horizon is short.

[2] Recall that there is only one moving boundary for an American put option with finite maturity.

[3] Previous asset pricing literature has explored portfolio optimization in frictionless markets with uncertain lifetime such as Cass and Yaari (1967), Merton (1971), and Richard (1975).

As in most of the literature on optimal investment with transaction costs [e.g., Davis and Norman (1990), Grossman and Laroque (1990), Cuoco and Liu (2000)], the optimization problem in this case amounts to a singular stochastic control problem. We obtain analytic expressions for the value function as the closed form solution to an ordinary differential equation subject to certain free boundary conditions. We find that the optimal transaction policy is to maintain the ratio of the dollar amount in the risk-free asset to the amount in the risky asset within a wedge, represented by the buy boundary and the sell boundary.

We also derive explicit, horizon-independent bounds on the boundaries. We show that the ratio at the buy boundary is always greater than the ratio in the absence of transaction costs (the Merton line). However, the ratio at the sell boundary could also be greater than the Merton line, which means that the entire no-transaction region could be above the Merton line.

We then extend the above analysis to the case where the terminal date occurs at the time of the nth ($n > 1$) jump of an independent Poisson process (thus the horizon is Erlang distributed). As expected, the optimal transaction boundaries become state dependent and jump each time the Poisson process jumps. We also demonstrate that the bounds on the transaction boundaries obtained in the previous case still apply.

Finally, we show that the value function and transaction boundaries in the Erlang distributed case converge to the value function and transaction boundaries, respectively, for an investor with a deterministic finite horizon. This implies that the bounds on boundaries derived in the previous cases are also valid for the deterministic horizon case. We also show that the trading boundaries for the exponentially distributed horizon case can be regarded as approximations to the trading boundaries for the deterministic finite horizon case.

Since the optimal trading boundaries for investors with exponentially distributed horizons approximate those for the Erlang distributed and deterministic horizons, we provide detailed analysis of the trading behavior of investors with exponentially distributed horizons. In particular, we examine how the optimal transaction boundaries change as the coefficients of the model change. In general, the comparative statics follow those known in the frictionless case. However, we find that the buy boundary is more sensitive to parameter changes than the sell boundary. Furthermore, the sensitivity of the buy boundary increases as the horizon decreases. We also examine the expected time to sale after purchase and find that the optimal trading strategy is indeed largely buy and hold, consistent with much of the conventional wisdom.

The remainder of the article is organized as follows. In Section 1 we describe the basic model where an investor has a finite deterministic horizon. In Section 2 we solve the problem for an investor with an exponentially distributed horizon. In Section 3 we extend the problem to the case where the investor's horizon is Erlang distributed, and in Section 4 we show that the solutions in Section 3 converge to those for the basic model specified in Section 1. Section 5 provides comparative statics and further analysis of optimal trading policies. Section 6 concludes with some possible extensions of the model and applications of the methodology.

1. The Basic Model

1.1 The asset market

Throughout this article we are assuming a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t\}$. Uncertainty in the model is generated by a standard one-dimensional Brownian motion w. We will assume that w_t is adapted.

There are two assets our investor can trade. The first asset ("the bond") is a money market account growing at a continuously compounded, constant rate r. The second asset ("the stock") is a risky investment. The investor can buy the stock at the ask price, $S_t^A = S_t$, and sell the stock at the bid price, $S_t^B = (1-\alpha)S_t$,[4] where $0 \leq \alpha < 1$ represents the proportional transaction cost rate[5] and S_t is given by

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma w_t}, \tag{1}$$

where we assume all parameters are positive constants and $\mu > r$.

When $\alpha > 0$, the above model gives rise to equations governing the evolution of the amount invested in the bond, x_t, and the amount invested in the stock, y_t:

$$dx_t = rx_t\,dt - dI_t + (1-\alpha)dD_t, \tag{2}$$

$$dy_t = \mu y_t\,dt + \sigma y_t\,dw_t + dI_t - dD_t, \tag{3}$$

where the processes D and I represent the cumulative dollar amount of sales and purchases of the stock, respectively. These processes are nondecreasing, right continuous adapted processes with $D(0) = I(0) = 0$. Let x_0 and y_0 be the given initial positions in the bond and the stock, respectively. We let $\Theta(x_0, y_0)$ denote the set of admissible trading strategies (D, I) such that Equations (2) and (3) are satisfied and the investor is always solvent, that is,

$$x_t + (1-\alpha)y_t \geq 0, \forall\, t \geq 0. \tag{4}$$

[4] We choose the ask price S_t^A instead of the midpoint [as in Davis and Norman (1990)] as the numeraire for notational simplicity without any loss of generality. This does not imply no transaction costs when purchasing the stock. In fact, S_t^A should be interpreted as the stock price inclusive of the transaction cost for purchasing.

[5] In our model, the case where $\alpha = 1$ is a trivial case since no investor with monotonic preferences would ever buy the stock.

1.2 The investor's problem

In order to highlight the role of the horizon, we assume the utility of an investor only depends on the market value of his portfolio at a deterministic time T. This is consistent with earlier models in the literature [e.g., Dumas and Luciano (1991), Brennan, Schwartz, and Lagnado (1997)]. The investor's problem is to choose trading strategies D and I so as to maximize $E[u(x_T + (1-\alpha)y_T)]$ subject to Equations (2), (3), and (4). We assume that the investor has CRRA preference, that is, $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$ for $\gamma > 0$, $\gamma \neq 1$.[6] To solve this problem, we define the value function at time t as

$$V(x, y, t) = \sup_{(D,I)\in\Theta(x,y)} E\left[\frac{(x_T + (1-\alpha)y_T)^{1-\gamma}}{1-\gamma} \,\middle|\, \mathcal{F}_t\right]. \tag{5}$$

1.3 Optimal policies with no transaction costs

For the purpose of comparison we present results, due to Merton (1971), for the case when there are no transaction costs ($\alpha = 0$) without proof. In this case, the cumulative purchases and sales of the stock can be of infinite variation. The investor's problem can be written as

$$V(x, y, 0) = \sup_{\{y_t : t\geq 0\}} E\left[\frac{(x_T + y_T)^{1-\gamma}}{1-\gamma}\right],$$

subject to

$$d(x_t + y_t) = r(x_t + y_t)\,dt + (\mu - r)y_t\,dt + \sigma y_t\,dw_t. \tag{6}$$

In this case, it is well known that the optimal policy involves investing a constant fraction of wealth in the stock and the fraction is independent of the investor's horizon. It is important to note that as long as $\mu > r$, the investor always optimally holds some of the risky asset. Here we will sketch the solution and define parameters which will be used in the sequel. Without transaction costs, the optimal stock investment policy can be shown to be

$$y_t^* = \frac{1}{r^* + 1}(x_t^* + y_t^*) \tag{7}$$

for all $0 < t < T$, where the "Merton line" r^* is given by

$$r^* = \frac{\gamma\sigma^2}{\mu - r} - 1. \tag{8}$$

The lifetime expected utility is

$$V(x, y, 0) = e^{\rho T}\frac{(x+y)^{1-\gamma}}{1-\gamma}, \tag{9}$$

[6] Similar results for $\gamma = 1$ (i.e., log utility) can be derived.

where

$$\rho = (1-\gamma)\left(r + \frac{\kappa}{\gamma}\right) \tag{10}$$

and

$$\kappa = \frac{(\mu - r)^2}{2\sigma^2}. \tag{11}$$

1.4 The transaction cost case

In the case where $\alpha > 0$, the problem is considerably more complicated. Here we outline a direct approach: first, we postulate that the region where the investor has positive wealth, the *solvency region*,

$$\mathcal{S} = \{(y, x) : x + (1-\alpha)y > 0\},$$

at each point in time splits into a "buy" region, a "no-transaction" region, and a "sell" region, as in Davis and Norman (1990). Under regularity conditions on the value function, we have the following Hamilton–Bellman–Jacobi (HJB) equation,

$$\frac{1}{2}\sigma^2 y^2 V_{yy} + rxV_x + \mu y V_y + V_t = 0,$$

in the no-transaction region. In the buy region, the marginal cost of decreasing the amount in the bond is equal to the marginal benefit of increasing the amount in the stock, that is,

$$V_x = V_y.$$

Similarly, in the sell region, the marginal benefit of increasing the amount in the bond must be equal to the marginal cost of decreasing the amount in the stock, that is,

$$(1-\alpha)V_x = V_y.$$

In addition, we must have the terminal condition

$$\lim_{t \to T} V(x, y, t) = \frac{(x + (1-\alpha)y)^{1-\gamma}}{1-\gamma}.$$

It follows immediately from the homogeneity of the utility function u, the convexity of the set of admissible strategies, and the fact that $\Theta(\beta x, \beta y) = \beta\Theta(x, y)$ for all $\beta > 0$ that the value function V is concave and homogeneous of degree $1-\gamma$ in (x, y) [cf. Fleming and Soner (1993), Lemma VIII.3.2]. This homogeneity implies

$$V(x, y, t) = y^{1-\gamma}\varphi\left(\frac{x}{y}, t\right),$$

for some function φ: $(\alpha - 1, \infty) \times [0, T] \to \mathbb{R}$. Let

$$z = \frac{x}{y} \tag{12}$$

denote the ratio of the amount invested in the bond to the amount invested in the stock. The homogeneity property then implies that the buy, no-transaction, and sell regions can be described by two functions of time $r_1(t)$ and $r_2(t)$. The buy region corresponds to $z \geq r_2(t)$, the sell region to $z \leq r_1(t)$, and the no-transaction region to $r_1(t) < z < r_2(t)$, as depicted in Figure 1.

Using these properties, we obtain a partial differential equation for φ in the no-transaction region $(r_1(t) < z < r_2(t))$:

$$\frac{1}{2}\sigma^2 z^2 \varphi_{zz} + (\gamma\sigma^2 - (\mu - r))z\varphi_z - (1-\gamma)(\gamma\sigma^2/2 - \mu)\varphi + \varphi_t = 0.$$

In the buy region $(z \geq r_2(t))$, we have

$$(z+1)\varphi_z(z,t) = (1-\gamma)\varphi(z,t).$$

Similarly, in the sell region $(z \leq r_1(t))$, we have

$$(z+1-\alpha)\varphi_z(z,t) = (1-\gamma)\varphi(z,t).$$

In addition, φ must also satisfy the terminal condition

$$\lim_{t\to T} \varphi(z,t) = \frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma}.$$

This system of equations involves finding a pair of moving boundaries $r_1(t)$ and $r_2(t)$ and is difficult to solve. In the subsequent sections we develop an

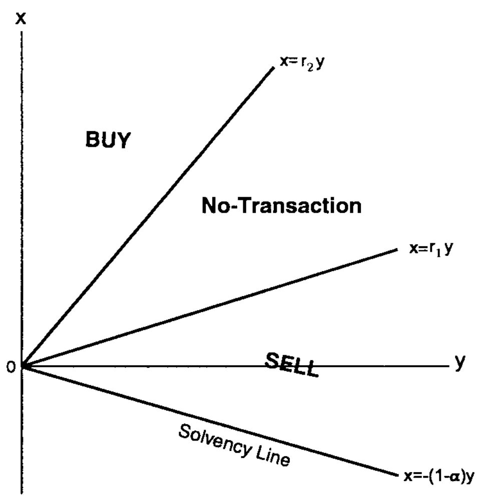

Figure 1
The solvency region

alternative methodology which can circumvent this difficulty and lead to a solution for the optimal trading policy.

2. Exponentially Distributed Horizon

In this section, as a first step toward solving the problem, we modify the optimization problem so that our investor has an uncertain horizon. In particular, the investor's problem is now to choose admissible trading strategies D and I so as to maximize $E[u(x_\tau + (1-\alpha)y_\tau)]$ for an event which occurs at the first jump time τ of a standard, independent Poisson process with intensity λ. τ is thus exponentially distributed with parameter λ, that is,

$$P\{\tau \in dt\} = \lambda e^{-\lambda t}\, dt.$$

This modified model yields a closed-form solution for the value function and serves as a foundation for solving the basic model specified in the previous section. Moreover, it can also be of independent interest. For example, bequest, accidents, retirement, and many other events happen on uncertain dates.

If τ is interpreted to represent the investor's uncertain lifetime [as in Merton (1971) and Richard (1975)], the investor's average lifetime is then $1/\lambda$ and the variance of his lifetime is accordingly $1/\lambda^2$.

We can then write the value function as

$$v(x, y) = \sup_{(D,I)\in\Theta(x,y)} E\left[\frac{(x_\tau + (1-\alpha)y_\tau)^{1-\gamma}}{1-\gamma}\right]. \tag{13}$$

In light of our assumptions on τ and the asset market, this can be rewritten as [see Merton (1971), Carr (1998)]

$$v(x, y) = \sup_{(D,I)\in\Theta(x,y)} \lambda E\left[\int_0^\infty e^{-\lambda t}\frac{(x_t + (1-\alpha)y_t)^{1-\gamma}}{1-\gamma}dt\right]. \tag{14}$$

Thus the investor's problem [Equation (13)] can be solved by solving the transformed problem [Equation (14)]. The critical difference from the basic model is the absence of the time dimension, which significantly simplifies the problem.

2.1 Optimal policies with no transaction costs

Again, for purpose of comparison, let us first consider the case without transaction costs ($\alpha = 0$). In this case, the investor's problem becomes

$$v(x, y) = \sup_{\{y_t:t\geq 0\}} \lambda E\left[\int_0^\infty e^{-\lambda t}\frac{(x_t + y_t)^{1-\gamma}}{1-\gamma}dt\right],$$

subject to the self-financing condition [Equation (6)].

The above problem is formally similar to the one studied by Merton (1971). As in Merton (1971), a condition on the parameters is required for the existence of the optimal solution.[7]

Assumption 1. *The investor's expected horizon parameter λ satisfies*

$$\lambda > (1-\gamma)\left(r+\frac{\kappa}{\gamma}\right),$$

where κ is as defined in Equation (11).

Assumption 1 is necessary because if the investor expects to live a long time (λ is small) then the risk-free rate must be low enough and the stock cannot deliver too high a risk premium ($\mu - r$) with low risk (σ), otherwise the investor can obtain bliss levels of utility by investing in either the stock (if κ is too high) or the bond (if r is too high). We summarize the main result for this case of no transaction costs without proof in the following lemma.

Lemma 1. *Suppose that $\alpha = 0$. Then the optimal stock investment policy is Equations* (7) *and* (8) *for $0 \le t \le \tau$. Moreover, the lifetime expected utility is*

$$v(x, y) = \frac{\lambda}{\lambda-\rho}\frac{(x+y)^{1-\gamma}}{1-\gamma},$$

where ρ is as defined in Equation (10).

Thus, without transaction costs, the optimal policy involves investing the same horizon-independent, constant fraction of total wealth in the stock and the bond as in the deterministic horizon case in Section 1.3. This is similar in spirit to the observation in Samuelson (1969) that the optimal portfolio does not depend on the investor's horizon. Moreover, it is always optimal to invest some in the stock if the expected return in the stock is greater than the interest rate. We will see later that all these features disappear in the presence of even small transaction costs.

2.2 Optimal policies with transaction costs

Suppose now that $\alpha > 0$. As in Section 1.4, the value function is homogeneous of degree $1-\gamma$ in (x, y). This implies that

$$v(x, y) = y^{1-\gamma}\psi\left(\frac{x}{y}\right) \tag{15}$$

for some concave function ψ: $(\alpha - 1, \infty) \to \mathbb{R}$.

[7] Introducing time discounting in the preference or restricting to the case with $\gamma > 1$ would make this assumption unnecessary or automatically satisfied and all the subsequent results still hold with, at most, minor modifications.

Similar to Section 1.4, the *solvency region* splits into three regions: buy region, sell region, and no-transaction region. In contrast to Section 1.4, however, because of the time homogeneity of the value function, these regions can be identified by two critical numbers (instead of functions of time) r_1 and r_2. The buy region corresponds to $z \geq r_2$, the sell region to $z \leq r_1$, and the no-transaction region to $r_1 < z < r_2$, where z is as defined in Equation (12).

Under regularity conditions on v, we have the following HJB equation:

$$\frac{1}{2}\sigma^2 y^2 v_{yy} + rxv_x + \mu y v_y - \lambda v + \frac{\lambda(x+(1-\alpha)y)^{1-\gamma}}{1-\gamma} = 0 \tag{16}$$

in the no-transaction region, with the associated conditions

$$v_x = v_y$$

in the buy region, and

$$(1-\alpha)v_x = v_y$$

in the sell region.

Using Equation (15), we can simplify the PDE in Equation (16) to get the following ordinary differential equation in the no-transaction region:

$$z^2\psi_{zz} + \beta_2 z\psi_z + \beta_1\psi + \beta_0\frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} = 0, \tag{17}$$

where $\beta_2 = 2(\gamma\sigma^2 - (\mu - r))/\sigma^2$, $\beta_1 = -2(\lambda + (1-\gamma)(\gamma\sigma^2/2 - \mu))/\sigma^2$, and $\beta_0 = 2\lambda/\sigma^2$. The associated boundary conditions are transformed into

$$(z+1)\psi_z(z) = (1-\gamma)\psi(z)$$

for all $z \geq r_2$ and

$$(z+1-\alpha)\psi_z(z) = (1-\gamma)\psi(z)$$

for all $z \leq r_1$. Define

$$n_{1,2} = \frac{(1-\beta_2) \pm \sqrt{(1-\beta_2)^2 - 4\beta_1}}{2}.$$

Assumption 1 implies that $(1-\beta_2)^2 - 4\beta_1 > 0$. The solutions to the homogeneous part of Equation (17) can therefore be characterized by the fundamental solutions ψ_1 and ψ_2, where

$$\psi_1(z) = |z|^{n_1}, \qquad \psi_2(z) = |z|^{n_2}.$$

The general solution to Equation (17) can thus be written as

$$C_1\psi_1(z) + C_2\psi_2(z) + \psi_p(z),$$

where C_1 and C_2 are integration constants and the particular solution [see Boyce and DiPrima (1969)]

$$\psi_p(z) = \beta_0 \int_{r^*}^{z} \frac{\psi_1(\xi)\psi_2(z) - \psi_1(z)\psi_2(\xi)}{\psi_1'(\xi)\psi_2(\xi) - \psi_1(\xi)\psi_2'(\xi)} \frac{(\xi + 1 - \alpha)^{1-\gamma}}{(1-\gamma)\xi^2}\, d\xi.$$

The above discussions imply that

$$\psi(z) = \begin{cases} A\frac{(z+1)^{1-\gamma}}{1-\gamma} & \text{if } z \geq r_2 \\ C_1\psi_1(z) + C_2\psi_2(z) + \psi_p(z) & \text{if } r_1 \vee 0 < z < r_2 \vee 0 \\ \overline{C}_1\psi_1(z) + \overline{C}_2\psi_2(z) + \psi_p(z) & \text{if } r_1 \wedge 0 < z < r_2 \wedge 0 \\ B\frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} & \text{if } \alpha - 1 < z \leq r_1, \end{cases} \tag{18}$$

for some constants $A, B, C_1, C_2, \overline{C}_1, \overline{C}_2, r_1$, and r_2.

We have the following result on the existence of the value function and the optimal trading strategy for the modified model.

Theorem 1. *There exist constants A, B, C_1, C_2, $\overline{C}_1$, $\overline{C}_2$, r_1, and r_2 such that*

1. *$\psi(z)$ is a C^2 function on $(\alpha - 1, 0)$ and $(0, \infty)$,*
2. *$\psi(z)$ satisfies the following: if $r_2 = \infty$, then*

$$\lim_{y \to 0, x > 0} y^{1-\gamma}\psi\left(\frac{x}{y}\right) = \frac{\lambda}{\lambda - (1-\gamma)r} \frac{x^{1-\gamma}}{1-\gamma} \tag{19}$$

and if $r_1 \leq 0 \leq r_2$, then

$$\lim_{x \to 0} y^{1-\gamma}\psi\left(\frac{x}{y}\right) = \frac{\lambda}{(1-\gamma)(\lambda - (1-\gamma)\mu + \gamma(1-\gamma)\frac{\sigma^2}{2})}((1-\alpha)y)^{1-\gamma}, \tag{20}$$

3. *$v(x, y) = y^{1-\gamma}\psi\left(\frac{x}{y}\right)$ is the value function.*

Moreover, the optimal transaction policy is to transact the minimal amount in order to maintain z between r_1 and r_2.

Proof. The proof is similar to the references below. Since this is not the focus of our analysis, we refer the interested reader to these sources for details. The basic idea is to note that the value function in Equation (14) is a piecewise C^2 solution of Equation (16).[8] This, combined with convex analysis

[8] Here we need to allow for the fact that the value function may not be a C^2 function at $z = 0$ (and thus not a classic solution) if the x axis is contained in the no-transaction region. This really causes no problem, since the optimal policy will never cross the axis in this case. Proposition 5 formally demonstrates this fact.

of the value function along the lines of Shreve and Soner (1994), reveals that the value function itself will satisfy the conditions stated in the theorem. Thus we are assured of the existence of the function above. The existence of an optimal transaction policy then follows directly from Fleming and Soner (1993), Theorem VIII.4.1. ■

In order to solve for the value function and the optimal trading strategy, we need to consider three cases. If r_2 is finite and the no-transaction region does not contain 0, then we need to determine six constants A, B, C_1, C_2, r_1, and r_2 using Equation (18) and the C^2 property (the "smooth pasting conditions") of ψ across r_1 and r_2. If r_2 is finite but the no-transaction region contains 0, then we need to determine eight constants $A, B, C_1, C_2, \overline{C}_1, \overline{C}_2, r_1$, and r_2 using Equation (18), the C^2 property of ψ across r_1 and r_2 and the condition [Equation (20)] at $z = 0$. If r_2 is infinite, then the only boundary condition at r_2 would be Equation (19), but other boundary conditions apply as in the previous two cases. For the first case, algebraic manipulation can reduce the six equations to two nonlinear equations for r_1 and r_2, which can be solved numerically. Of course this search is easier if we can find bounds on r_1 and r_2 and conditions which tell us when r_2 is infinite. The next section provides this information.

2.3 The behavior of the no-transaction region

Before we prove general properties of the transaction boundaries, we first turn our attention to Figure 2, which plots optimal boundaries for the bond-

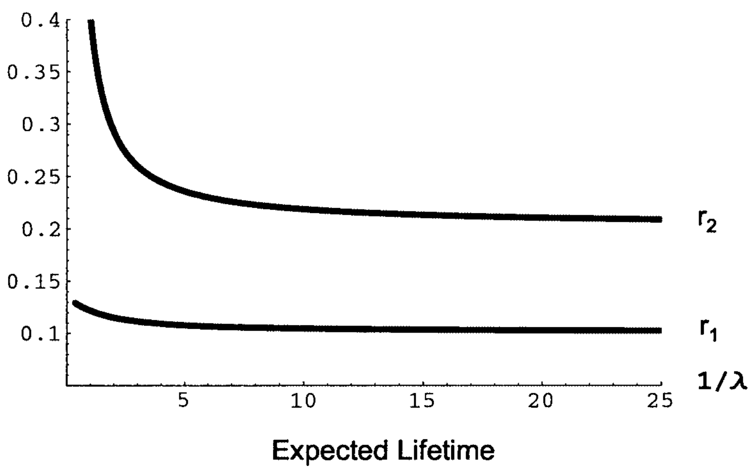

Figure 2
Optimal boundaries for the bond-to-stock ratio as functions of the expected lifetime
The graph plots r_1 and r_2 against $1/\lambda$ for parameters $r = 0.05$, $\mu = 0.12$, $\sigma = 0.20$, $\alpha = 0.01$, and $\gamma = 2$.

to-stock ratio as functions of the expected lifetime.[9] It shows that both the buy boundary r_2 and the sell boundary r_1 increase as the expected lifetime $1/\lambda$ decreases. This implies that starting with all the wealth in the bond, the optimal fraction of wealth invested in the stock decreases as the horizon decreases, consistent with the typical life-cycle investment advice. In fact, the speed of decrease in the optimal fraction of wealth invested in the stock increases as the horizon shortens. In addition, the buy boundary is much more sensitive to the lifetime than the sell boundary. Section 5 contains more analysis and comparative statics for this modified model.

In contrast to many of the previous studies, our investor has a finite horizon and is subject to transaction costs. Due to the finiteness of the horizon and the presence of transaction costs, it may be suboptimal to buy additional stock even when the risk premium is positive. In this case, the optimal buy boundary is vertical, or in other words, $r_2 = \infty$. Intuitively, if the investor does not have a long enough expected horizon to recover at least the transaction costs, then it does not pay to buy any additional stock. The following result confirms this intuition and provides a necessary and sufficient condition under which this occurs.

Proposition 1. *Suppose* $0 < \alpha < 1$. *A necessary and sufficient condition for* r_2 *to be infinite is*

$$\mu - r \leq \alpha(\lambda - (1-\gamma)r). \tag{21}$$

Proof. See the appendix. ■

Recall that in the notransaction cost case, or in the infinite horizon transaction cost analysis of Davis and Norman (1990), Shreve and Soner (1994), and Dumas and Luciano (1991), the investor always optimally buys some of the risky asset if and only if $\mu - r > 0$.[10] In contrast, in our model, the above result says that if the investor does not expect to live long (i.e., λ is large), or the transaction cost rate α is high, or the investor is highly risk averse (i.e., γ is large), or the risk premium is low, then the investor will never buy the stock, even when the risk premium is positive. Thus, as opposed to the frictionless case, the trading strategy is now clearly horizon dependent.

While the above condition involves the expected lifetime $1/\lambda$, the following lemma provides bounds on the transaction boundaries which are independent of λ.

[9] The optimal boundaries are computed using the smooth pasting conditions at r_1 and r_2 for the first case described in the last paragraph in the previous subsection.

[10] As in Equation (21), this condition is independent of σ.

Proposition 2. *We have the following bounds on the boundaries of the no-transaction region, r_1 and r_2:*

$$(1-\alpha)r^* \geq r_1 \geq \frac{\gamma\sigma^2(1-\alpha)}{2(\mu-r)} - (1-\alpha) \tag{22}$$

and

$$r_2 \geq r^*, \tag{23}$$

where r^* is the Merton line as defined in Equation (8). *Moreover, if $\gamma\sigma^2 = \mu - r$, then $r_1 = 0$.*

Proof. See appendix. ■

Proposition 2 shows that the buy boundary is always above the Merton line r^*. It also shows that the sell boundary is always below $(1-\alpha)r^*$. In particular, if $r^* > 0$, then the sell boundary is always below the Merton line. However, it is not the case that the sell boundary is always a decreasing function of transaction cost rate. In fact, if $r^* < 0$, then the sell boundary can be above the Merton line (which implies that the entire no-transaction region is above r^*) for large enough transaction cost rate. This is because when $\alpha = 1$, the sell boundary has to be on or above 0 (which is the solvency line in this case).

3. Erlang Distributed Horizon

As a second step toward solving the basic model, we generalize the model described in the previous section. In this section we assume the investor's horizon τ occurs after the nth i.i.d. Poisson jump, which means that τ will be Erlang distributed as shown in Carr (1998). Therefore, if we let

$$P\{\tau^i \in dt\} = \frac{\lambda^i}{(i-1)!} t^{i-1} e^{-\lambda t}\, dt,$$

then $\tau = \tau^n$.

Under this assumption the expected horizon is i/λ when there are i jumps left. The variance of τ is n/λ^2. To help solve the problem with finite deterministic horizon T, we consider the case where τ always has expected value $E[\tau] = T$. Thus we set the intensity $\lambda = n/T$ in this and the next section. The variance of τ is then T^2/n, which approaches 0 as n increases. For the existence of a solution, we still maintain Assumption 1, which will be satisfied if n is large enough for a fixed T. The investor's problem is once again to choose admissible D and I so as to maximize $E[u(x_\tau + (1-\alpha)y_\tau)]$ subject to Equations (2), (3), and (4).

3.1 Optimal policies with no transaction costs

In this subsection we present results without proof for the case when there are no transaction costs ($\alpha = 0$) and the investor's horizon τ is Erlang distributed. The investor's problem can be written as

$$v^n(x, y) = \sup_{\{y_t : t \geq 0\}} E\left[\frac{(x_\tau + y_\tau)^{1-\gamma}}{1-\gamma}\right],$$

subject to the budget constraint in Equation (6).

Lemma 2. *Suppose that $\alpha = 0$. The optimal stock investment policy is then*

$$y_t^* = \frac{1}{r^* + 1}(x_t^* + y_t^*).$$

Moreover, the lifetime expected utility is

$$v^n(x, y) = \frac{\lambda^n}{(\lambda - \rho)^n} \frac{(x + y)^{1-\gamma}}{1-\gamma},$$

where ρ is as defined in Equation (10). *Moreover, with $\lambda = n/T$, we have*

$$\lim_{n \to \infty} v^n(x, y) = e^{\rho T} \frac{(x + y)^{1-\gamma}}{1-\gamma} = V(x, y, 0),$$

where $V(x, y, 0)$ is as defined in Equation (9).

Once again, without transaction costs, the optimal policy involves investing a constant fraction of total wealth in the stock and the bond, and this is independent of the investor's horizon. Also, notice that as we make n very large, the value function converges to the value function for the deterministic horizon case. We will show in Section 4 that this convergence result also holds in the presence of transaction costs.

3.2 Optimal policies with transaction costs

Let $v^i(x, y)$ be the value function when there are i jumps left until the horizon,

$$v^i(x, y) = \sup_{(D, I) \in \Theta(x, y)} E\left[\frac{(x(\tau^i) + (1-\alpha)y(\tau^i))^{1-\gamma}}{1-\gamma}\right],$$

and thus $v^0(x, y) = \frac{(x+(1-\alpha)y)^{1-\gamma}}{1-\gamma}$. Then to compute $v^i(x, y)$, we can solve the following recursive structure:

$$v^i(x, y) = \lambda E\left[\int_0^\infty e^{-\lambda t} v^{i-1}(x_t, y_t)\, dt\right], \quad i = 1, \ldots, n. \tag{24}$$

As before, because of the homogeneity of $v^i(x, y)$, there exists some function ψ^i such that

$$v^i(x, y) = y^{1-\gamma}\psi^i\left(\frac{x}{y}\right).$$

Solving Equation (24) reduces to finding functions $\psi^i(z)$ such that

$$z^2\psi^i_{zz} + \beta_2 z\psi^i_z + \beta_1\psi^i + \beta_0\psi^{i-1} = 0, \quad i = 1, \ldots, n \tag{25}$$

with the associated boundary conditions

$$(z+1)\psi^i_z(z) = (1-\gamma)\psi^i(z), \tag{26}$$

for all $z \geq r^i_2$ and

$$(z+1-\alpha)\psi^i_z(z) = (1-\gamma)\psi^i(z), \tag{27}$$

for all $z \leq r^i_1$, where β_2, β_1, and β_0 are the same as in Equation (17) and r^i_1 and r^i_2 represent the sell and buy boundaries, respectively, when there are i jumps left. Moreover, the homogeneous solutions to Equation (25) are also the same as those for Equation (17). This leads to the general solution to Equation (25),

$$C^i_1\psi_1(z) + C^i_2\psi_2(z) + \psi^i_p(z), \tag{28}$$

where C^i_1 and C^i_2 are integration constants and the particular solution

$$\psi^i_p(z) = \beta_0 \int_{r^*}^{z} \frac{\psi_1(\xi)\psi_2(z) - \psi_1(z)\psi_2(\xi)}{\psi_1'(\xi)\psi_2(\xi) - \psi_1(\xi)\psi_2'(\xi)} \frac{\psi^{i-1}(\xi)}{\xi^2}\, d\xi.$$

Equations (25)–(28) imply that

$$\psi^i(z) = \begin{cases} A^i \frac{(z+1)^{1-\gamma}}{1-\gamma} & \text{if } z \geq r^i_2 \\ C^i_1\psi_1(z) + C^i_2\psi_2(z) + \psi^i_p(z) & \text{if } r^i_1 \vee 0 < z < r^i_2 \vee 0 \\ \overline{C}^i_1\psi_1(z) + \overline{C}^i_2\psi_2(z) + \psi^i_p(z) & \text{if } r^i_1 \wedge 0 < z < r^i_2 \wedge 0 \\ B^i \frac{(z+1-\alpha)^{1-\gamma}}{1-\gamma} & \text{if } \alpha - 1 < z \leq r^i_1, \end{cases}$$

for some constants A^i,B^i,C^i_1,C^i_2, $\overline{C}^i_1$, $\overline{C}^i_2$ and the boundaries r^i_1 and r^i_2.

As in the previous section, we need to find coefficients that make $\psi^i(z)$ a C^2 function on $(\infty, 0)$ and $(0, \infty)$ and satisfy the appropriate limiting conditions when r^i_2 is infinite or $r^i_2 \geq 0 \geq r^i_1$. Notice that in this case the coefficients A^i,B^i,C^i_1,C^i_2, $\overline{C}^i_1$, $\overline{C}^i_2$, and the boundaries r^i_1 and r^i_2 change each time the Poisson jump occurs. To save space, we omit the analogue of Theorem 1 for the Erlang distributed horizon, but we are assured of the existence of a solution to these equations by such a result.

To compute the optimal boundaries when there are $i > 1$ jumps left, we first compute r_1^1 and r_2^1 using the approach described in the last paragraph of Section 2.2. We then iterate $i-1$ times using the same approach to obtain r_1^i and r_2^i.

3.3 Behavior of the no-transaction region

Figure 3 plots r_1^i and r_2^i for i from 1 to 25 when $T = 25$ and $n = 25$ (i.e., $\lambda = 1$). For example, when there remain 25 jumps until the event time, the sell boundary is 0.0980 and the buy boundary is 0.1903. As is clear from the graph, as the number of remaining jumps decreases, the changes in the boundaries become larger. This feature is similar to that in Figure 2. It is also consistent with the intuition that as an investor gets closer to the terminal date, buying stock becomes less attractive and eventually the investor never buys any additional stock.

Some other important features are also worth noting. Let us compare the case in Figure 2, where there is only one jump to the terminal date, but with expected time of 25 years, and the case in Figure 3, where there are 25 jumps to the terminal date, but with expected time of 1 year to the next jump. The optimal no-transaction range in the first case is $(0.1026, 0.2091)$ versus $(0.0980, 0.1903)$ in the second case. First, we note that even if the two cases have the same expected time to the terminal date, the investor in the first case will be less willing to buy stock. The intuition is that with only one jump to go (even if the expected time to the jump is longer), the

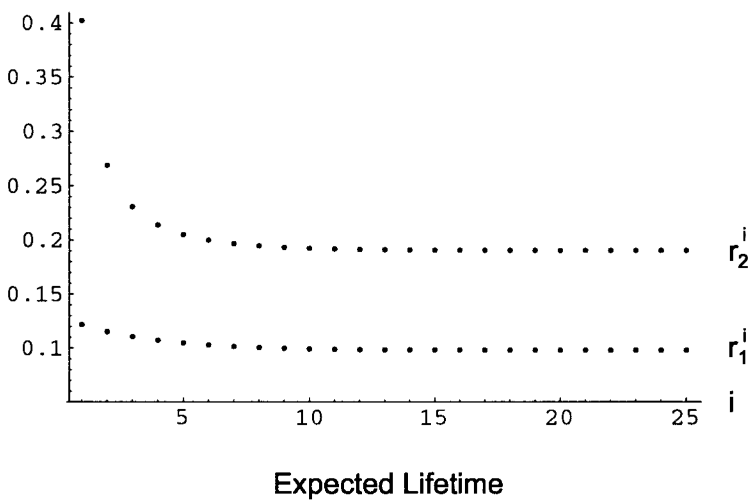

Figure 3
Optimal boundaries for the bond-to-stock ratio corresponding to the remaining number of jumps
The graph plots r_1^i and r_2^i for i from 1 to 25 when $T = 25$ and $n = 25$ for parameters $\lambda = 1$, $r = 0.05$, $\mu = 0.12$, $\sigma = 0.20$, $\alpha = 0.01$, and $\gamma = 2$.

uncertainty in the first case is much greater. In the second case, the investor has a better idea along the way about how much time is left. It is this higher uncertainty that makes the investor in the first case less willing to buy stock. We also note that even though the first case has a much coarser grid than the second case, the differences between the initial trading boundaries in these two cases are small, only 0.0046 and 0.0188, respectively, for the sell and buy boundary. Thus we conjecture that the analysis in Section 2 will produce a fairly accurate description of the initial trading boundaries for reasonable parameter values.

Next we provide the analogues to Propositions 1 and 2 for the investor who has an Erlang distributed horizon:

Proposition 3. *Suppose* $0 < \alpha < 1$. *A necessary and sufficient condition for* r_2^i *to be infinite is*

$$(1-\alpha)^{\frac{1}{i}}(\lambda-(1-\gamma)r) \leq (\lambda-\mu+\gamma r).$$

We also have the following bounds:

$$(1-\alpha)r^* \geq r_1^i \geq \frac{\gamma\sigma^2(1-\alpha)}{2(\mu-r)} - (1-\alpha)$$

and

$$r_2^i \geq r^*.$$

Moreover, if $\gamma\sigma^2 = \mu - r$, *then* $r_1^i = 0$.

Proof. Similar to Propositions 1 and 2. ■

4. Deterministic Horizon

The deterministic finite horizon case in Section 1 can now be dealt with by using the model of the previous section. Suppose that in the previous model we make n very large and always maintain $E[\tau] = T$ (i.e., $\lambda = n/T$). Intuitively we should expect that the limiting value function would converge to the value function of the case with a deterministic horizon T, since the variance of τ goes to zero as n gets large. This is indeed the case in the no transaction cost case as shown in Lemma 2 of Section 3.1. The following theorem confirms that this is also the case with the presence of transaction costs.

Theorem 2. *Let* $V(x, y, t)$ *be as defined in Equation* (5). *Then*

$$\lim_{n\to\infty} v^n(x, y) = V(x, y, 0).$$

Proof. See the appendix. ■

This result shows that to approximate the value function for the problem with finite deterministic horizon, one can solve the case with Erlang distributed horizon with a large n.

4.1 Behavior of the no-transaction region

A natural question is whether the optimal transaction boundaries converge. Theorem 25.5 in Rockafellar (1970) implies that the value function will be differentiable on a dense set of the solvency region, and the homogeneity property of the value function implies this dense set can be written as the union of open convex cones. Theorem 25.7 in Rockafellar (1970) then implies that the derivatives of the value functions for the Erlang distributed horizon will converge to those of the value function for the deterministic horizon on this union. Thus the transaction boundaries (which are defined by the ratio of the derivatives) must also converge. In addition, the necessary and sufficient condition for not buying any additional stock will converge to the condition for the deterministic finite horizon case. Moreover, since the bounds in Proposition 3 are independent of λ, they are still valid for the deterministic horizon case. In particular, this shows that in the deterministic, finite horizon case, the time-varying buy boundary will always be above the Merton line r^*, and if $r^* > 0$, the time-varying sell boundary will always be below the Merton line. We summarize the preceding remarks in the following proposition.

Proposition 4. *Let $r_1(t)$ and $r_2(t)$ be the optimal no-transaction boundaries at $t \in [0, T]$ for the deterministic horizon problem as defined in Equation* (5). *A necessary and sufficient condition for $r_2(t)$ to be infinite is*

$$\mu - r \leq -\frac{1}{T-t}\log(1-\alpha).$$

We also have the following bounds: $\forall\, t \in [0, T]$,

$$(1-\alpha)r^* \geq r_1(t) \geq \frac{\gamma\sigma^2(1-\alpha)}{2(\mu - r)} - (1-\alpha)$$

and

$$r_2(t) \geq r^*.$$

Moreover, if $\gamma\sigma^2 = \mu - r$, then $\forall\, t \in [0, T]$, $r_1(t) = 0$.

Proof. This is summarized in the discussion preceding the proposition. ■

As far as we know, this is the first time explicit bounds for the optimal no-transaction boundaries are derived in the deterministic finite horizon case. Some articles, for example, Gennotte and Jung (1994), have used discrete approximations to solve the deterministic horizon investor problem with transaction costs. However, these approximations can produce no trade regions which violate the above bounds due to discretization.

Theorem 2 and Figure 3 suggest that (0.0980, 0.1903) is a good approximation to the initial optimal range for the case with a deterministic horizon of 25 years. According to Theorem 2, Figure 3 also approximates the behavior of the transaction boundaries as a function of remaining time in the case with a deterministic horizon of 25 years.

Also, if we compare Figure 2 with Figure 3, we see that we can closely approximate the initial trading boundaries for an investor with a deterministic horizon T by the boundaries for an investor with an exponentially distributed horizon with mean T. In addition, Propositions 4 and 2 imply that the buy boundary in the deterministic finite horizon case is infinite for a remaining lifetime of less than 0.1436, while the buy boundary is infinite in the exponentially distributed horizon case for an expected lifetime of less than 0.1439, which is only 0.0003 away. We thus conjecture that the trading boundaries for exponentially distributed horizons produce a reasonable approximation of the trading boundaries for the deterministic horizon for reasonable parameter values.

5. Further Analysis of the Exponential Horizon Case

In the previous section we showed that we can regard the optimal trading boundaries for the exponentially distributed horizon case in Section 2 as a good approximation of the initial trading boundaries in the deterministic horizon case. In this section we provide further analysis of the optimal policies for an investor with an exponentially distributed horizon. This analysis should provide a fairly accurate description of the trading behavior of an investor with a deterministic horizon.

5.1 Changes in risk aversion

Figure 4 shows the effect of the coefficient of relative risk aversion, γ, on the optimal trading boundaries for expected horizons of 1 year and 25 years. As γ increases, both r_1 and r_2 increase and the width of the no-transaction region increases. Essentially a more risk-averse investor holds more of the risk-free asset. Of interest is that the sell boundary r_1 is not sensitive to horizon, even as risk aversion increases, but the buy boundary r_2 increases at a faster rate as the horizon decreases.

5.2 Changes in risk

Figure 5 shows how the optimal transaction boundaries change as we change the riskiness of the stock σ for investors with expected horizons of 1 year and 25 years. As stock return volatility increases, we see that not only r_1 and r_2, but also the width of the no-transaction region increase. Intuitively the risk-averse investor tends to invest less in the stock on average as the risk rises and he needs to widen the no-transaction region in order to avoid transacting too frequently as the volatility increases. Also note that the sell boundary is not sensitive to the horizon as σ increases. The buy boundary, on the other hand, increases at a significantly higher rate for a shorter-horizon investor as the risk increases.

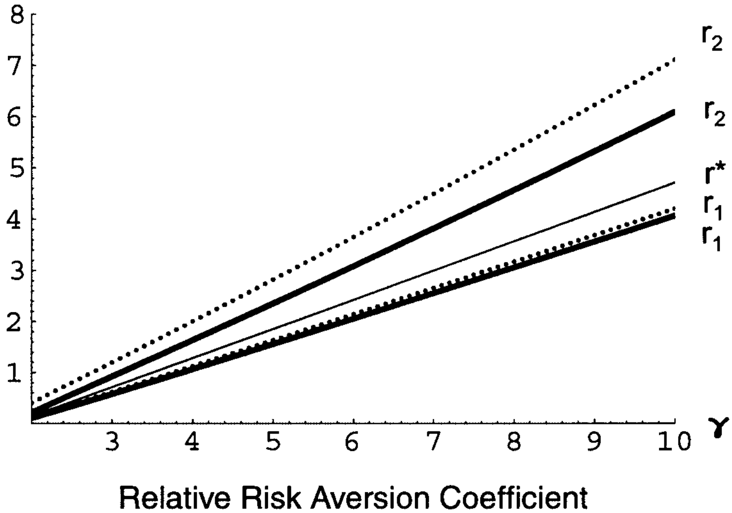

Figure 4
Changes in risk aversion
The graph plots r_1 and r_2 against γ. The dotted lines correspond to $\lambda = 1$ and the solid lines correspond to $\lambda = 0.04$. The thin line in the middle is the Merton line. Other parameters are $r = 0.05$, $\mu = 0.12$, $\sigma = 0.20$, and $\alpha = 0.01$.

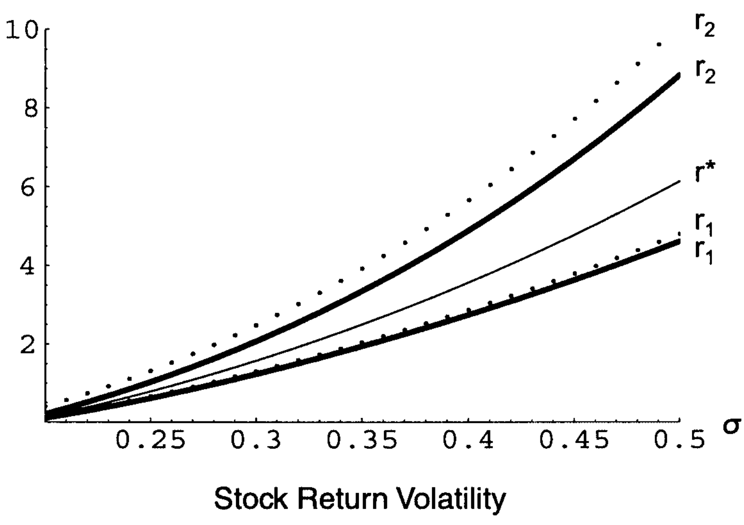

Figure 5
Changes in risk
The graph plots r_1 and r_2 against σ. The dotted lines correspond to $\lambda = 1$ and the solid lines correspond to $\lambda = 0.04$. The thin line in the middle is the Merton line. Other parameters are $r = 0.05$, $\mu = 0.12$, $\alpha = 0.01$, and $\gamma = 2$.

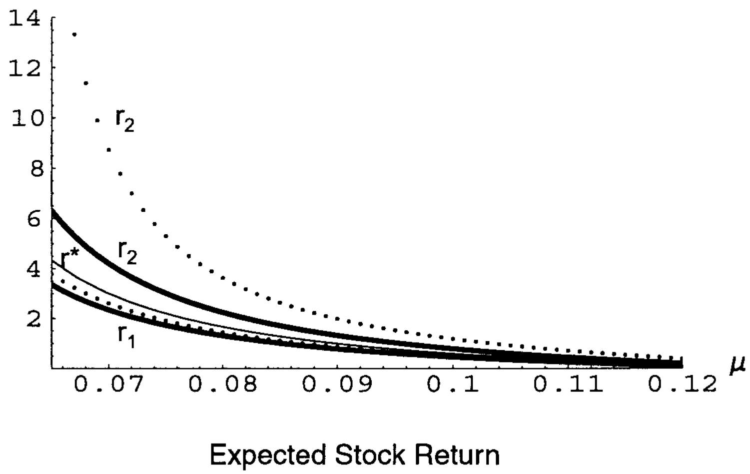

Figure 6
Changes in expected return of the stock
The graph plots r_1 and r_2 against μ. The dotted lines correspond to $\lambda = 1$ and the solid lines correspond to $\lambda = 0.04$. The thin line in the middle is the Merton line. Other parameters are $r = 0.05$, $\sigma = 0.2$, $\alpha = 0.01$, and $\gamma = 2$.

5.3 Changes in the expected return of the stock

Figure 6 shows how the buy and sell boundaries change as the expected stock return changes for expected horizons of 1 year and 25 years. We see that as the expected return increases both r_1 and r_2 decrease. Naturally, as the expected return on the stock becomes more attractive, an investor would want to hold more stock, all else being equal. However, this relationship is also affected by the horizon of the investor. We see that for investors with shorter horizons the buy boundary moves much farther up for smaller expected returns than for investors with longer horizons. Once again, the sell boundary is much less sensitive to the horizon than the buy boundary.

5.4 Changes in transaction costs

Figure 7 shows how the transaction boundaries change as the transaction cost rate α changes. The sell boundary is much less sensitive to the change in the transaction cost rate than the buy boundary. When $\alpha = 1\%$, the investor would let the ratio of the bond to the stock fluctuate between 0.1032 and 0.2037 before adjusting. In the absence of transaction costs, the investor would constantly keep the ratio at 0.1428. In contrast to Dumas and Luciano (1991), who found no bias toward cash in the optimal portfolio, here we see that as transaction cost rate increases, there is a clear bias toward cash in the optimal portfolio caused by the finiteness of the horizon. Notice the convexity of the buy boundary when α gets large enough (about 2%). In fact, this has to be the case because by Proposition 1, as the transaction cost rate α increases to 77.8%, the buy boundary has to approach infinity.

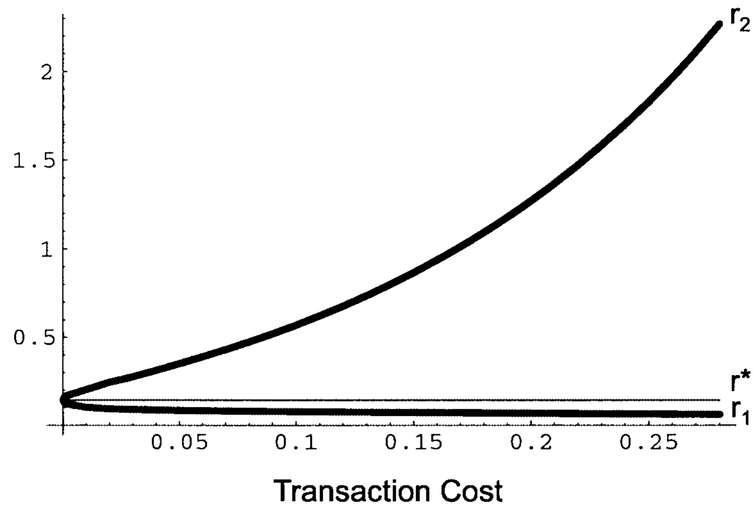

Figure 7
Optimal boundaries for the bond-to-stock ratio as a function of the transaction cost rate
The graph plots r_1 and r_2 against α for parameters $\lambda = 0.04$, $r = 0.05$, $\mu = 0.12$, $\sigma = 0.20$, and $\gamma = 2$. The thin line in the middle is the Merton line.

5.5 Frequency of trading

In order to analyze in more detail the stochastic behavior of the investment in the stock, we note that within the no-transaction region

$$dz_t = (\sigma^2 - (\mu - r))z_t\,dt - \sigma z_t\,dw_t.$$

Now fix $z_0 = z \in (r_1, r_2)$ and define

$$\tau_e = \inf\{t \geq 0 : z_t \notin (r_1, r_2)\}$$

to be the time of the next transaction. Let

$$P_z(\tau_e < \infty) = P(\tau_e < \infty \mid z_0 = z)$$

denote the conditional probability that τ_e is finite and

$$E_z[\tau_e] = E[\tau_e \mid z_0 = z]$$

denote the conditional expectation of τ_e.

The following proposition confirms that the investor will never buy the stock if and only if the buy boundary r_2 is ∞ and that if $r_1 \leq 0 \leq r_2$, then 0 can never be reached.

Proposition 5. *If $0 \notin [r_1, r_2]$ and $r_2 < \infty$, then $P_z(\tau_e < \infty) = 1$ and $E_z[\tau_e] < \infty$ for all $z \in (r_1, r_2)$. Moreover, either boundary of the no-transaction region* *can be reached with positive probability. On the other hand, if* $0 \in [r_1, r_2]$ ($r_2 = \infty$, *respectively*), 0 (∞, *respectively*) *is never reached from within the interior of the first and the fourth orthants in the* (y, x) *plane.*

Proof. This follows immediately from propositions in Section 5.5 of Karatzas and Shreve (1988). ■

The previous analysis implies that if $0 \in [r_1, r_2]$, then z_t can never cross the x-axis, which shows that the value function might not be C^2 at 0 in this case as pointed out in note 8. However, if $0 \notin [r_1, r_2]$ and $r_2 < \infty$, then both boundaries can be reached in finite expected time and we can compute a set of measures of trading frequency, for example, expected time to the next trade, expected time to the next sale after a purchase, etc. In this section we are going to focus on the expected time to the next sale after a purchase to measure the average turnover time. To do this we utilize the following result.

Proposition 6. *Suppose* $0 \notin [r_1, r_2]$ *and* $r_2 < \infty$. *Then*

$$E[\tau_s \wedge \tau | x_0 = x, y_0 = y] = F(x/y),$$

where

$$\tau_s \equiv \inf\{t \geq 0 : z_t = r_1\},$$

$$F(z) = \frac{k_1|r_2|^{k_1}|z|^{k_2} - k_2|r_2|^{k_2}|z|^{k_1}}{\lambda(k_2|r_1|^{k_1}|r_2 vert^{k_2} - k_1|r_1|^{k_2}|r_2|^{k_1})} + \frac{1}{\lambda},$$

and

$$k_{1,2} = \frac{-\left(\frac{1}{2}\sigma^2 - (\mu - r)\right) \pm \sqrt{\left(\frac{1}{2}\sigma^2 - (\mu - r)\right)^2 + 2\lambda\sigma^2}}{\sigma^2},$$

Proof. See the appendix. ■

Figure 8 displays the expected time to the next sale after a purchase as a function of the transaction cost α. Even for a small transaction cost rate of 1%, it would take about 5 years to sell after a purchase for an investor with an expected lifetime of 25 years. For a transaction cost rate of $\alpha = 2\%$, it takes about 10 years to sell after a purchase. This is consistent with the buy-and-hold advice in Malkiel (2000) and illustrates the dramatic impact of even small transaction costs; recall that without transaction costs, the investor would transact continuously.

To further illustrate the buy-and-hold strategy implied by the model and the trading frequency as a function of the expected horizon, we plot the expected time to next sale after purchase versus expected horizon in Figure 9. For investors with short horizons, on the order of three years or less, the expected time to sale after a purchase is roughly equal to the investor's expected horizon. Therefore this model indeed implies a largely buy-and-hold trading strategy. The figure also shows that as the expected horizon increases, the expected turnover time increases.

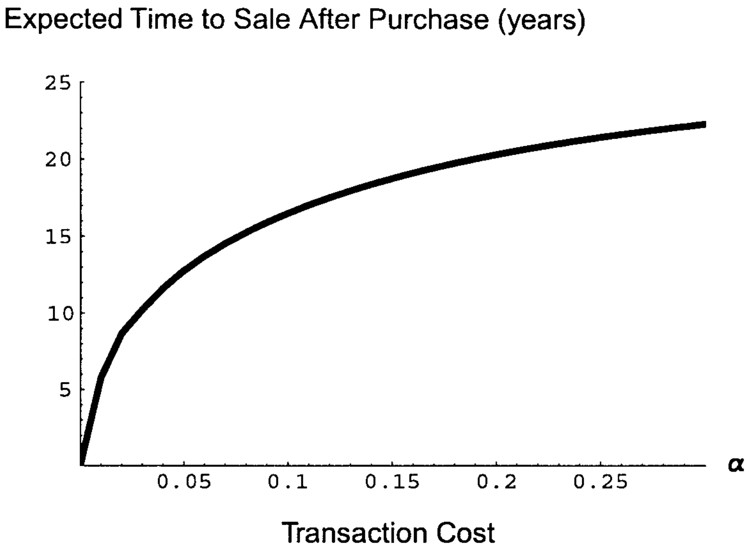

Figure 8
Expected time to the next sale after purchase as a function of transaction cost
The graph plots $E[\tau_s \wedge \tau]$ against α for parameters $\lambda = 0.04$, $r = 0.05$, $\mu = 0.12$, $\sigma = 0.20$, and $\gamma = 2$.

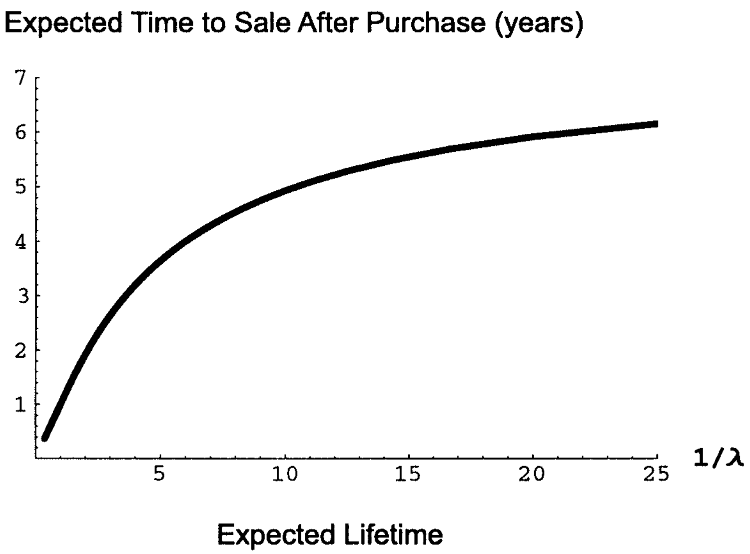

Figure 9
Expected time to the next sale after purchase as a function of expected horizon
The graph plots $E[\tau_s \wedge \tau]$ against $1/\lambda$ for parameters $r = 0.05$, $\mu = 0.12$, $\sigma = 0.20$, $\alpha = 0.01$, and $\gamma = 2$.

6. Conclusion

In this article we propose a methodology to study the optimal transaction policy for an investor with a finite horizon who is also subject to transaction costs. In particular, we show that, in contrast to the frictionless case, there is a clear link between the investor's horizon and the optimal portfolio trading strategy: investors with shorter horizons will buy relatively less of the risky asset and basically follow a buy-and-hold strategy. This is consistent with the conventional wisdom on life-cycle investing. Our analysis derives explicit solutions for investors whose horizons are exponential and Erlang distributed. We then show that the solution to the case with an Erlang distributed horizon converges to the solution to the deterministic finite horizon problem. We demonstrate that the optimal trading boundaries for the case with an exponentially distributed horizon can be a good approximation for those for the case with a deterministic horizon. In addition, we derive explicit bounds on the transaction boundaries for all the cases we consider. Moreover, we obtain the necessary and sufficient conditions under which it is not optimal to buy any stock, even when the risk premium is positive.

Our approach could be exploited to give approximate solutions for a greater range of transaction cost problems with finite horizons. First, we can generalize our model to allow for a dividend paying stock and still have closed-form solutions for the exponentially distributed horizon case. Second, the analysis in this article can be further generalized to cover the case where the coefficients (including λ) change at each jump of the Poisson process. From our analysis, we can conjecture that these value functions will converge to the value function for an economy where the coefficients are time varying. Such generalizations could lead to interesting market microstructure studies in the future.

Our approach should also find applications in a greater range of optimal consumption/investment problems with time-varying components. To employ this methodology, one could first derive a modified problem with time invariant solution and then solve a series of such problems whose solutions converge to the optimal solution to the original problem.

Appendix

In this appendix, we collect the proofs for Propositions 1, 2, 6, and Theorem 2.

Proof of Proposition 1. The proof relies on the inequality

$$u(x) - u(y) \geq u'(x)(x - y), \tag{A.1}$$

which is valid for differentiable concave functions [see Rockafellar (1970), Theorem 25.1]. Suppose our investor starts with initial endowments $x_0 > 0$ and $y_0 = 0$. For convenience, we take $S_0 = 1$. To show necessity, notice that if it is optimal not to buy stock, we must have for x, y corresponding to any feasible $(D, I) \in \Theta(x_0, y_0)$,

$$E[u'(x_\tau + (1-\alpha)y_\tau)(x_\tau + (1-\alpha)y_\tau - x_0 e^{r\tau})] \le 0.$$

In particular, letting $x_\tau = (1-a)x_0 e^{r\tau}$ and $y_\tau = a x_0 S_\tau$ for $0 < a < 1$, we have

$$E[u'((1-a)x_0 e^{r\tau} + a x_0 (1-\alpha) S_\tau)((1-\alpha)S_\tau - e^{r\tau})] \le 0.$$

Choose a sequence $0 < a_n < 1$ which goes down to 0. Taking limits as $a_n \downarrow 0$ and interchanging the limit and expectation, which is justified from using dominated convergence, since $u'(x_0(e^{r\tau} + S_\tau)) \le u'((1-a_n)x_0 e^{r\tau} + a_n x_0 (1-\alpha) S_\tau) \le u'(x_0(1-a_0)e^{r\tau})$, we then have

$$E[u'(x_0 e^{r\tau})((1-\alpha)S_\tau - e^{r\tau})] \le 0,$$

which leads to

$$\int_0^\infty (x_0 e^{rt})^{-\gamma}((1-\alpha)E[S_t] - e^{rt})\lambda e^{-\lambda t}\, dt \le 0.$$

Recalling $E[S_t] = e^{\mu t}$, we can integrate the above equations to get Equation (21). To prove sufficiency, if we can show

$$E[u'(x_0 e^{r\tau})(x_0 e^{r\tau} - x_\tau - (1-\alpha)y_\tau)] \ge 0 \tag{A.2}$$

for x, y corresponding to any $(D, I) \in \Theta(x_0, y_0)$, then from Equation (A.1), it is not optimal to buy stock. Note that Equation (A.2) is equivalent to

$$\int_0^\infty x_0 e^{(-\lambda + (1-\gamma)r)t}\, dt \ge \int_0^\infty e^{(-\lambda - \gamma r)t} E[(x_t + (1-\alpha)y_t)]\, dt. \tag{A.3}$$

By Equation (21), it is easy to check that

$$\int_0^s e^{(-\lambda-\gamma r)t}(1-\alpha)E[S_t]\, dt \le \int_0^s e^{(-\lambda-\gamma r)t} e^{rt}\, dt \tag{A.4}$$

for all $s \ge 0$. This implies that the right-hand side of Equation (A.3) is maximized by lending x_0 and not buying any stock. ■

Proof of Proposition 2. If $r_1 \neq 0$, ψ is C^2 at r_1 and

$$\psi(z) = B\frac{(z + 1 - \alpha)^{1-\gamma}}{1-\gamma}, \quad \forall z \le r_1.$$

Notice $B > 0$, since $v(x, y)$ is strictly increasing in x and y. Putting this expression evaluated at r_1 into Equation (16), we get (after simplification)

$$-\frac{1}{2}\gamma\sigma^2(1-\alpha)^2 B + (\mu - r)(1-\alpha)(r_1 + 1 - \alpha)B + \left[-(\lambda - (1-\gamma)r)\frac{B}{1-\gamma} + \frac{\lambda}{1-\gamma}\right](r_1 + 1 - \alpha)^2 = 0. \tag{A.5}$$

If $r_1 = 0$, then $B = \frac{\lambda}{\lambda - (1-\gamma)\mu + \gamma(1-\gamma)\frac{\sigma^2}{2}}$ by Equation (20) and direct substitution shows that Equation (A.5) still holds. In either case we know that $B\frac{(x+y(1-\alpha))^{1-\gamma}}{1-\gamma} \ge \frac{\lambda(x+y(1-\alpha))^{1-\gamma}}{(1-\gamma)(\lambda-(1-\gamma)r)}$, since the investor must do at least as well as liquidating his stock holdings and lending until the horizon.

Applying this inequality to the last term in Equation (A.5) leads to the lower bound in Equation (22). For the upper bound in Equation (22), let us first define [recalling the definition of κ in Equation (11)]

$$f(z) = -B\left(\sqrt{\frac{\kappa}{\gamma}}(z+1-\alpha) - \sigma(1-\alpha)\sqrt{\frac{\gamma}{2}}\right)^2$$

and

$$g(z) = \left(-\frac{(\lambda-(1-\gamma)(r+\frac{\kappa}{\gamma}))B}{1-\gamma} + \frac{\lambda}{1-\gamma}\right)(z+1-\alpha)^2.$$

Notice $f(z) \le 0$ and $y^{1-\gamma}(f(z)+g(z))$ is equal to the left-hand side of Equation (16) evaluated at $z \le r_1$. The supermartingale property of $v(x_t, y_t)$ then implies

$$f(z)+g(z) \le 0, \quad \forall z \le r_1$$

and Equation (A.5) implies

$$f(r_1)+g(r_1) = 0. \tag{A.6}$$

If $f(r_1) = 0$, then $r_1 = \frac{\gamma\sigma^2(1-\alpha)}{\mu - r} - (1-\alpha) = (1-\alpha)r^*$. If $f(r_1) < 0$, then $g(r_1) > 0$ by Equation (A.6). This implies $g(z) > 0$, $\forall z \le r_1$, which in turn implies that $f(z) < 0$, $\forall z \le r_1$. Since $f((1-\alpha)r^*) = 0$, we must have $r_1 < (1-\alpha)r^*$. The lower bound for r_2 can be similarly derived.

Finally, if $\mu - r = \gamma\sigma^2$ (i.e., $r^* = 0$), then from above we must have $r_1 \le 0$. Comparing the optimal strategy for some $z \le r_1$ to the strategy of immediately closing out the short bond position and holding stock until the horizon gives

$$\frac{B}{1-\gamma} \ge \frac{\lambda}{(1-\gamma)\big(\lambda-(1-\gamma)\mu+\gamma(1-\gamma)\frac{\sigma^2}{2}\big)} = \frac{\lambda}{(1-\gamma)\big(\lambda-(1-\gamma)\big(r+\frac{\kappa}{\gamma}\big)\big)},$$

or in other words,

$$-\frac{(\lambda-(1-\gamma)(r+\frac{\kappa}{\gamma}))B}{1-\gamma} + \frac{\lambda}{1-\gamma} \le 0.$$

This implies that $g(r_1) \le 0$. Equation (A.6) and the fact that $f(r_1) \le 0$ implies in fact that $g(r_1) = f(r_1) = 0$. As a result,

$$\left(\sqrt{\frac{\kappa}{\gamma}}(r_1+1-\alpha) - \sigma(1-\alpha)\sqrt{\frac{\gamma}{2}}\right)^2 = 0,$$

which implies $r_1 = 0$. ■

Proof of Theorem 2. Let x_t, y_t correspond to any feasible strategy $(D, I) \in \Theta(x, y)$ with the properties that $t \to E[u(x_t + (1-\alpha)y_t)]$ is continuous and for all t, $x_t + (1-\alpha)y_t > \varepsilon$ for some fixed constant $\varepsilon > 0$. Notice that this ensures $u(x_t + (1-\alpha)y_t)$ is bounded uniformly from below for all $t \le T$. We can modify each strategy in $\Theta(x, y)$ to be feasible for horizon τ by liquidating at τ if $\tau \le T$ and otherwise liquidating at T and leaving all wealth in the bond until τ. Let $V^t(x, y, 0)$ be the value function corresponding to the optimization problem of Equation (5), but with the deterministic horizon t. Then from the feasibility of the above strategy for horizon τ, we deduce

$$E\big[u\big((x_{T\wedge\tau} + (1-\alpha)y_{T\wedge\tau})e^{r(\tau-T)1_{\{\tau>T\}}}\big)\big] \le v^n(x, y) \le \int_0^\infty V^t(x, y, 0)\frac{\left(\frac{n}{T}\right)^n}{(n-1)!}t^{n-1}e^{-\frac{n}{T}t}\,dt.$$

Notice that $V^t(x,y,0)$ is bounded above by the Merton (1971) frictionless value function for a deterministic horizon t, as in Equation (9), and bounded below by the value function corresponding to the strategy which liquidates all wealth and lends until the horizon t. In other words,

$$u((x+(1-\alpha)y)e^{rt}) \leq V^t(x,y,0) \leq e^{\rho t}u(x+y),$$

where ρ is defined in Equation (10). Moreover, using Theorem 2 of Davis, Panas, and Zariphopoulou (1993), it is easy to verify that the conditions for a variant of the Helly–Bray theorem hold. As a result [see Chow and Teicher (1988), Theorem 8.1.2],

$$E[u(x_T+(1-\alpha)y_T)] \leq \lim_{n\to\infty} v^n(x,y) \leq V^T(x,y,0). \tag{A.7}$$

The theorem then follows from the fact that Equation (A.7) holds for any $\varepsilon > 0$ and by taking the supremum over all feasible strategies on the left-hand side. ■

Proof of Proposition 6. Let $\varphi(x,y) = E[(\tau_s \wedge \tau)|x_0 = x, y_0 = y]$. Since τ follows the exponential distribution, we have

$$\varphi(x,y) = E\left[\int_0^\infty \lambda e^{-\lambda t}(\tau_s \wedge t)\,dt \,\middle|\, x_0 = x, y_0 = y\right] = E\left[\int_0^{\tau_s} e^{-\lambda t}\,dt \,\middle|\, x_0 = x, y_0 = y\right].$$

It is easily verified that φ is homogeneous of degree 0 in (x,y), so that $\varphi(x,y) = F(x/y)$ for some function F. By Itô's lemma and the fact that the process

$$e^{-\lambda t}\varphi(x_t, y_t) + \int_0^t e^{-\lambda s}\,ds$$

is a martingale $\forall\, t < \tau_s$, F must be the unique solution to the ordinary differential equation

$$\frac{1}{2}\sigma^2 z^2 F_{zz} + (\sigma^2 - (\mu - r))zF_z - \lambda F + 1 = 0$$

on (r_1, r_2), with boundary conditions $F(r_1) = 0$ and $F_z(r_2) = 0$. It is straightforward to check that the function F defined in the proposition solves the above ODE. ■

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