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Optimal consumption of a divisible durable good☆

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Abstract

We examine the intertemporal optimal consumption and investment problem in a continuous-time economy with a divisible durable good. Consumption services are assumed to be proportional to the stock of the good held and adjustment of the stock is costly, in that it involves the payment of a proportional transaction cost. For the case in which the investor has an isoelastic utility function and asset prices follow a geometric Brownian motion, we establish the existence of an optimal policy and provide an explicit representation for the value function. We show that the investor acts so as to maintain the ratio of the stock of the durable to total wealth in a fixed (nonstochastic) range and that the optimal investment policy involves stochastic portfolio weights. The dependence of the optimal policies on the parameters of the model is also discussed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper studies, in a continuous-time economy with constant price coefficients, the intertemporal optimal consumption and investment problem of an infinitely lived investor with an isoelastic utility function for the services provided by a perfectly divisible durable good. The consumption services provided by the durable good are assumed to be proportional to the current holdings of the good (net of depreciation), and thus depend on past purchasing decisions. Investments in the durable good are reversible, but a proportional transaction cost has to be paid whenever the good is bought or sold. We allow for different transaction cost rates on purchases and sales.

In the absence of transaction costs, the solution to the problem we study can be easily obtained through a straightforward change of variables from the classical model with a perishable good (Merton, 1971). The investor would continuously adjust the stock of the durable good so as to maintain the marginal utility of consumption equal to the marginal utility of wealth. With the prices of risky assets following geometric Brownian motions, this amounts to keeping a constant fraction of wealth invested in the durable. Similarly, the optimal portfolio policy would involve constant weights.

In the presence of transaction costs, adjusting the stock of the durable continuously would lead to incurring infinitely large transaction costs. Therefore, the stock of the durable is adjusted only infrequently: transaction costs introduce a wedge between the marginal utility of consumption and the marginal utility of wealth, and the optimal consumption policy involves possibly a discrete change (jump) in the initial stock of the durable, followed by the minimal amount of transactions necessary to maintain the fraction of wealth invested in the durable in a given constant range. The optimal portfolio policy involves investing in the same portfolio of risky assets (the mean-variance efficient portfolio) as in the Merton case (no transaction costs), but the fraction of wealth allocated to stocks becomes stochastic.

As in the literature dealing with optimal consumption in the presence of proportional costs for transactions in the risky assets (e.g., Davis and Norman, 1990; Shreve and Soner, 1994; Akian et al., 1996) or with optimal investment in the presence of costly reversibility (e.g., Bertola and Caballero, 1994; Abel and Eberly, 1996), the problem we study amounts to a singular stochastic control problem. We show that the same condition on the parameters of the model that is necessary and sufficient to guarantee the existence of an optimal policy in the Merton case is also sufficient to guarantee the existence of an optimal policy in the presence of proportional transaction costs. Moreover, we show that the boundaries of the optimal range for the fraction of wealth invested in the durable good can be determined by solving a system of nonlinear equations, and we provide an explicit representation for the value function for the problem. For the case in which the transaction cost rate for durable sales is 100%, i.e., purchases are irreversible, we provide a closed-form expression for the boundaries of the optimal consumption range and for the value function.

As expected given the nature of the problem (cf. Øksendal, 1997), we find that small transaction costs can induce large deviations from the amount of durable consumption that would be optimal in the Merton case. The extent of these deviations depends critically on the depreciation rate of the durable good: the optimal policy for short-lived durables involves much more frequent purchases than the one for longer-lived durables. The boundaries of the optimal range for the fraction of wealth invested in the durable are not necessarily monotonic functions of the transaction cost rates, and the fraction of wealth invested in the durable good can be uniformly (i.e., throughout the optimal range) lower than what would be optimal in the Merton case, due to additional savings by the investor to meet future transaction costs. We provide a simple necessary and sufficient condition on the parameters of the model for this to happen.[1]

We also show that the optimal proportional investment in the durable good converges to a steady-state distribution, which we obtain in closed form. Numerical computations show that, even in cases where the size of the no-transaction region is monotonically increasing in the transaction cost rates, the steady-state average investment in the durable is always monotonically decreasing in the transaction cost rates.

While transaction costs in the market for the consumption good can induce large deviations of the holdings of the good from the Merton case, deviations of the fraction of wealth invested in risky assets are by comparison more limited. We show that this fraction is always higher than in the Merton case when the investor's wealth is high relative to the current stock of durable (i.e., immediately before or after a purchase), and is always lower when the investor's wealth is low (i.e., immediately before or after a sale). As a result, the investor can behave in either a more or a less risk-averse manner than in the absence of transaction costs, depending on his current wealth and durable holdings, even though the steady-state average investment in stocks is monotonically decreasing in the transaction cost rates. Since two-fund separation holds, the standard CAPM would characterize equilibrium prices in this economy. On the other hand, the consumption-based CAPM (CCAPM) would fail to hold due to the possible divergence between the marginal utility of consumption and the marginal utility of wealth.

Closely related models of optimal consumption of a durable good have been previously analyzed by Grossman and Laroque (1990) and Hindy and Huang (1993).

[1] In their analysis of optimal portfolio policies in the presence of proportional transaction costs, Shreve and Soner (1994, Remark 11.3) provided a sufficient (but not necessary) condition for the optimal fraction of wealth invested in the stock to be uniformly lower than what would be optimal in the absence of transaction costs.

Grossman and Laroque (1990) consider an economy similar to ours, but in which the durable good comes in stocks of various sizes and is indivisible once bought. Moreover, the consumer does not derive additional utility from holding multiple units of the good. Therefore, in order to change his durable consumption beyond what is caused by depreciation, the consumer must sell the existing stock and buy a new one. Accordingly, any adjustment in the stock of the durable held involves the payment of a transaction cost that is proportional to the existing stock. As Grossman and Laroque point out, this transaction cost acts as a *fixed* cost in an optimal stopping problem. The optimal consumption policy again involves only infrequent adjustments, but the corresponding durable holding process is discontinuous, as the investor makes discrete (rather than continuous) adjustments to the durable stock at the boundaries of the no-transaction region.

The analysis in Grossman and Laroque (1990) and the one in this paper are thus complementary, the first conforming closer to the case of an indivisible durable good such as a house or a car, and the second being a more natural modeling choice for divisible durable goods such as furniture or clothing. This appears to be consistent with the empirical evidence in Caballero (1993). In examining the extent to which models of durable consumption based on the presence of fixed costs can explain the behavior of aggregate expenditure on cars and furniture, Caballero reports that expenditure on furniture is much smoother than expenditure on cars: for this aggregate behavior to be consistent with the presence of fixed costs (which induce lumpy expenditure at the microeconomic level), one must assume that the optimal no-transaction region (and hence, the average time between purchases or sales) is much larger for furniture than for cars. In particular, Caballero reports an implied average time between individual car transactions of 4.35 yr, versus an implied average time between furniture transactions of 13.5 yr. Caballero points out that the latter estimate seems too large and that "allowing for other realistic features like habit formation (e.g., Constantinides, 1990; Heaton, 1993) and nonseparabilities across goods and time (e.g., Eichenbaum and Hansen, 1987; Heaton, 1993) should […] reduce the need for large inaction range estimates". We conjecture that an alternative explanation might lie in the presence of proportional (rather than fixed) transaction costs for furniture expenditure and in the ensuing continuity in the optimal holding process at the microeconomic level.[2] In addition, divisibility is a natural assumption in models with a single (i.e., composite) durable good.

[2] In comparing the model with proportional adjustment costs to the one with fixed costs, it may also be worth pointing out that the solution for the former model is much easier to compute numerically and, as we show in the paper, can be reduced to finding the zero of a continuous real-valued function that changes sign at the boundaries of a given finite interval. It is thus straightforward to implement numerical solution algorithms that always converge.

Hindy and Huang (1993) study, in a general continuous-time Markovian economy, the optimal consumption problem of an investor with preferences over the service flows from irreversible purchases of a durable good. They provide sufficient conditions for a consumption and portfolio policy to be optimal and derive a closed-form solution for the case in which the investor has isoelastic preferences and asset prices follow a geometric Brownian motion. Their closed-form solution is a special case of ours when the transaction cost rate for sales equals 100% (as in this case reselling the durable would clearly never be optimal) and there are no transaction costs for purchases.[3]

Also closely related to our analysis is the work of Dybvig (1995), who studies the optimal intertemporal consumption of a perishable good given extreme habit formation that prevents consumption from ever falling. His closed-form solution for this problem can also be obtained as a special case of ours when the transaction cost rate for sales is 100% and there are no transaction costs for purchases, by a straightforward change of variables that sets the instantaneous consumption rate of the perishable good in Dybvig's model equal to the instantaneous durable rental rate in our model.

The rest of the paper is organized as follows. Section 2 describes in more detail the economy we consider. Section 3 solves the investor's optimal consumption problem in the absence of transaction costs. This provides a benchmark for the subsequent analysis. Section 4 contains a heuristic derivation of the optimal policies in the presence of transaction costs and provides sufficient conditions under which the conjectured policies are indeed optimal. Section 5 shows that an optimal policy exists and derives an explicit representation for the value function. Section 6 contains an analysis of the optimal policies. Section 7 concludes the paper and points to some possible extensions.

2. The economy

We consider an infinite-horizon, continuous-time stochastic economy, with the uncertainty represented by a filtered probability space $(\Omega, \mathscr{F}, \boldsymbol{F}, P)$ on which is defined a d-dimensional Brownian motion w.

The investment opportunities are represented by $n+1$ long-lived securities. The first security (the "bond") is a money market account growing at a continuously compounded interest rate $r > 0$. The other n assets (the "stocks") are risky

[3] Detemple and Giannikos (1996) have recently considered a model with irreversible durable purchases in which the durable provides 'status' as well as consumption services. The latter are assumed as usual to be proportional to the stock of the durable held, while 'status' is related to contemporaneous purchases. As a result, in the model they consider the investor's preferences are affected by both the stock of the durable held and current purchases.

and their price process S (inclusive of reinvested dividends) is an n-dimensional geometric Brownian motion with drift vector μ and diffusion matrix σ, i.e.,

$$S_t = S_0 + \int_0^t I_s^S \mu \, \mathrm{d}s + \int_0^t I_s^S \sigma \, \mathrm{d}w_s,$$

where I_t^S denotes the $n \times n$ diagonal matrix with elements S_t. We assume without loss of generality that $1 \leq n \leq d$ and that $\mathrm{rank}(\sigma) = n$.[4] Also, letting

$$\kappa = \tfrac{1}{2}(\mu - r\mathbf{1})^{\mathrm{T}}(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1}), \tag{1}$$

where $\mathbf{1} = (1,1,\dots,1)^{\mathrm{T}} \in \mathbb{R}^n$, we assume that $\kappa > 0$.[5] Notice that if $n < d$ the market is dynamically incomplete. Trading in the bond and in the stocks takes place continuously and is frictionless (in particular, there are no transaction costs in the securities market). There is a single durable consumption good and holding a stock K of the good provides a consumption service flow $s(K)$ that is proportional to the stock, i.e., $s(K) = \alpha K$, where $\alpha > 0$. The good depreciates at a rate $\beta \geq 0$. Adjusting the stock of the durable is costly and involves the payment of a proportional transaction cost, at a rate $\iota \geq 0$ for purchases and $\delta \in [0,1]$ for sales.[6] A consumption and investment strategy is then characterized by a triple (I, D, θ), where θ is an n-dimensional adapted process with

$$\int_0^\infty |\theta_t|^2 \, \mathrm{d}t < \infty \quad \text{a.s.}$$

representing portfolio holdings of the risky assets, and I and D are nondecreasing, right-continuous adapted processes with $I_0 = D_0 = 0$ representing, respectively, cumulative purchases and sales of the durable good.

Now consider an investor who starts with an initial total wealth of $W_0 \geq 0$, of which an amount $K_0 \geq 0$ is invested in the durable good and the remainder $W_0 - K_0$ in financial assets. Given the choice of a consumption and investment strategy (I, D, θ), the investor's stock of the consumption good at time t equals the initial stock, plus purchases, minus sales and depreciation, i.e.,

$$K_t = K_0 - \int_0^t \beta K_s \, \mathrm{d}s + I_t - D_t, \tag{2}$$

[4] If $n > d$ or $\mathrm{rank}(\sigma) < n$, some stocks are redundant and can be omitted from the analysis.

[5] If $\kappa = 0$ (i.e., $\mu = r\mathbf{1}$), then the optimal investment policy involves no investment in the risky assets and the optimal consumption policy is deterministic.

[6] While assuming that $\iota = 0$ can be done without loss of generality, by taking as numeraire the purchase price of the good inclusive of any non-monetary search or adjustment costs, we prefer to capture these additional costs explicitly.

while his total wealth equals the initial wealth, plus the portfolio gains, minus depreciation and total transaction costs paid, i.e.,

$$W_t = W_0 + \int_0^t (r(W_s - K_s) + \theta_s^{\mathrm{T}}(\mu - r\mathbf{1}) - \beta K_s)\,\mathrm{d}s + \int_0^t \theta_s^{\mathrm{T}}\sigma\,\mathrm{d}w_s - \iota I_t - \delta D_t. \tag{3}$$

A consumption and investment strategy is *admissible* if it satisfies the solvency constraint

$$W_t - \delta K_t \geq 0 \quad \forall t \geq 0$$

(i.e., if total wealth after liquidating the stock of the durable good is nonnegative) and

$$K_t \geq 0 \quad \forall t \geq 0.$$

The investor's preferences are represented by a time-additive, isoelastic, von Neumann–Morgenstern utility function

$$U(K) = \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} u(s(K_t))\,\mathrm{d}t\right], \tag{4}$$

where $\rho > 0$ is a time-preference parameter and $u(c) = c^{1-\gamma}/(1-\gamma)$ for some $\gamma > 0$, $\gamma \neq 1$.[7]

The investor's consumption/investment problem is then that of choosing an admissible trading strategy (I^*, D^*, θ^*) so that the corresponding durable holding process K^* in (2) maximizes his lifetime expected utility (4).

Remark 1. The solution of the linear stochastic differential equation (2) is given by

$$K_t = K_0 \mathrm{e}^{-\beta t} + \int_0^t \mathrm{e}^{-\beta(t-s)}(\mathrm{d}I_s - \mathrm{d}D_s).$$

It is then immediate to see that the infinite-horizon model considered in Hindy and Huang (1993), in which the investor's preferences are defined over an exponentially-weighted average of past purchases (to capture durability and local substitution), is a special case of the model we consider with $\alpha = \beta$, $\iota = 0$ and $\delta = 1$ (in which case $D^* \equiv 0$).

[7] The case of logarithmic preferences ($\gamma = 1$) can be analyzed along similar lines. To avoid redundancies, we report all the results for this case (without proofs) in Appendix B.

Remark 2. Letting $\alpha = r$, $\beta = \iota = 0$, $\delta = 1$ and $c = rK$, the investor's problem can be rewritten as

$$\max_{(I,\theta)} \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} u(c_t)\mathrm{d}t\right],$$

subject to

$$c_t = c_0 + rI_t,$$

$$W_t = W_0 + \int_0^t (rW_s + \theta_s^{\mathrm{T}}(\mu - r\mathbf{1}) - c_s)\,\mathrm{d}s + \int_0^t \theta_s^{\mathrm{T}}\sigma\,\mathrm{d}w_s,$$

$$W_t \geq \frac{c_t}{r}.$$

This is the problem studied by Dybvig (1995), who considered optimal consumption of a perishable good under absolute intolerance for any decline in consumption.

Remark 3. The general consumption/investment problem we consider is feasible if and only if $W_0 - \delta K_0 \geq 0$, since it is always possible to liquidate the initial assets and invest all the proceeds in the durable (in which case $W_t = K_t \geq \delta K_t$ for all $t > 0$). Moreover, if along any feasible strategy $W(t,\omega) - \delta K(t,\omega) = 0$ for some $(t,\omega) \in [0,\infty) \times \Omega$, then $K(s,\omega) = 0$ for all $s \geq t$, unless $\delta = 1$. This can be seen by noticing that (2) and (3) imply

$$\mathrm{d}(W_t - \delta K_t) = (r(W_t - \delta K_t) + \theta_t^{\mathrm{T}}(\mu - r\mathbf{1}) - (1-\delta)(r+\beta)K_t)\,\mathrm{d}t + \theta_t^{\mathrm{T}}\sigma\,\mathrm{d}w_t - (\iota + \delta)\,\mathrm{d}I_t.$$

Therefore, if $W(t,\omega) - \delta K(t,\omega) = 0$ and $\delta < 1$, the only way to avoid a positive probability of violating the solvency constraint immediately afterward is to have $\theta(t,\omega) = 0$ and $K(t,\omega) = 0$. On the other hand, if $\delta = 1$, then the only rational continuation strategy would involve $\theta(s,\omega) = 0$ and $K(s,\omega) = K(t,\omega)\mathrm{e}^{-\beta(s-t)}$ for all $s \geq t$.

To rule out the special case noticed in Remark 3, we assume unless otherwise noted that $\delta < 1$.[8] Moreover, we assume that $W_0 - \delta K_0 > 0$, so that the investor can afford a strictly positive durable-holding process. Since $\lim_{c \downarrow 0} u_c(c) = \infty$, we can then restrict ourselves without loss of generality to admissible trading strategies (I, D, θ) for which the corresponding optimal wealth and durable-holding processes (W, K) are strictly positive and satisfy $W_t - \delta K_t > 0$ for all $t \geq 0$. We let $\Theta(W_0, K_0)$ denote this set of trading strategies.

[8] The optimal policies and the value function for the case $\delta = 1$ are given at the end of Section 5.

3. Optimal policies with no transaction costs

For purpose of comparison, let us consider first the case of no transaction costs (i.e., $\iota = \delta = 0$). In this case, letting $c = (r + \beta)K$ denote the instantaneous durable holding cost, we can rewrite the investor's problem as[9]

$$\max_{(c,\theta)} \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} u\left(\frac{\alpha}{r+\beta} c_t\right) \mathrm{d}t\right]$$

$$\text{s.t. } W_t = W_0 + \int_0^t (rW_s + \theta_s^{\mathrm{T}}(\mu - r\mathbf{1}) - c_s)\,\mathrm{d}s + \int_0^t \theta_s^{\mathrm{T}} \sigma\,\mathrm{d}w_s,$$

$$c_t \geq 0, \quad W_t \geq 0.$$

The above problem is formally similar to the one studied by Merton (1971). An optimal policy exists if and only if either $W_0 = 0$ or the following assumption, which will be maintained for the rest of the paper, is satisfied (otherwise arbitrarily large expected utility can be obtained by postponing consumption and investing in the stock market).

Assumption 1. The investor's impatience parameter ρ satisfies

$$\rho > (1 - \gamma)(r + \kappa/\gamma),$$

where κ is the constant in (1).

We summarize the main result for the case of no transaction costs in the following theorem.

Theorem 1. Suppose that $\delta = \iota = 0$ *and let*

$$r^* = \gamma(r + \beta)/\xi, \tag{5}$$

where

$$\xi = \rho - (1 - \gamma)(r + \kappa/\gamma) > 0. \tag{6}$$

Then the optimal policies are

$$K_t^* = \frac{1}{r^*} W_t^*$$

and

$$\theta_t^* = \frac{(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})}{\gamma} W_t^*$$

[9] We allow in this case durable holding processes K that are not necessarily of finite variation, i.e., that do not necessarily have representation (2) for some nondecreasing processes I and D. Durable holding processes of infinite variation are suboptimal in the presence of a proportional adjustment cost, as they involve an infinite cost.

for all $t > 0$. *The lifetime expected utility is*

$$v(W_0) = \frac{\alpha^{1-\gamma}(r^*)^\gamma}{(1-\gamma)(r+\beta)} W_0^{1-\gamma}.$$

Thus, with no transaction costs, the optimal policy involves investing a constant fraction of total wealth in the durable good and in each of the traded assets. Moreover, the value function v depends only on the investor's initial total wealth W_0.

4. Optimal policies with transaction costs

Suppose from now on that $\delta + \iota > 0$ and let

$$v(w,k) = \sup_{(I,D,\theta)\in\Theta(w,k)} U(K)$$

denote the value function for the investor's problem in this case (we will prove later that, under Assumption 1, the value function is finite for $w > \delta k > 0$).

It follows immediately from the concavity of the utility function u, the convexity of the set of admissible strategies $\Theta(w,k)$ and the fact that $\Theta(\lambda w, \lambda k) = \lambda\Theta(w,k)$ for all $\lambda > 0$ that the value function v is concave and homogeneous of degree $1-\gamma$ (cf. Fleming and Soner, 1993, Lemma VIII.3.2). This in turn implies that

$$v(w,k) = k^{1-\gamma}\psi\left(\frac{w}{k}\right) \tag{7}$$

for some concave function $\psi : (\delta, \infty) \to \mathbb{R}$.

To get some idea on the shape of the optimal policies, let us consider first, as in Davis and Norman (1990), a restricted class of policies in which I and D are required to be absolutely continuous with bounded derivatives, i.e.,

$$I_t = \int_0^t i_s \, ds$$

and

$$D_t = \int_0^t d_s \, ds$$

for some processes i, d with $0 \le i_t,\ d_t \le \eta$ for some $\eta < \infty$ and all $t > 0$.

The Hamilton–Jacobi–Bellman (HJB) equation for the investor's problem is then

$$0=\max_{(i,d,\theta)}\left[\frac{1}{2}|\theta^{\mathrm{T}}\sigma|^2 v_{ww}+[r(w-k)+\theta^{\mathrm{T}}(\mu-r\bar{\mathbf{1}})-\beta k]v_w\right.$$
$$\left.-\beta k v_k+(v_k-\iota v_w)i-(v_k+\delta v_w)d-\rho v+\frac{(\alpha k)^{1-\gamma}}{1-\gamma}\right].$$

The maximum is achieved by

$$\theta=-(\sigma\sigma^{\mathrm{T}})^{-1}(\mu-r\bar{\mathbf{1}})\frac{v_w}{v_{ww}},$$
$$i=\eta 1_{\{v_k\geq \iota v_w\}},$$
$$d=\eta 1_{\{v_k\leq -\delta v_w\}}.$$

Thus, the agent tries to adjust the stock of durable so as to keep the marginal utility of durable consumption between $(1-\delta)$ times the marginal utility of liquid wealth and $(1+\iota)$ times the marginal utility of liquid wealth.[10] As a result, the optimal durable adjustment policies are *bang-bang* (that is, adjustments in the stock of durable only take place at the maximum possible rate) and the *solvency region*

$$\mathscr{S}=\{(w,k)\colon k>0,\ w-\delta k>0\}$$

splits into three regions: 'buy' (B), 'sell' (S) and 'no transaction' (NT). At the boundary between S and NT $v_k=-\delta v_w$, while at the boundary between NT and B $v_k=\iota v_w$.

If the restriction that the optimal policies be absolutely continuous is removed, transactions in the durable will take place at infinite speed: that is, the investor will make an initial discrete transaction to the boundary of NT, and after that all subsequent transactions will take place at the boundary and involve the minimum amount necessary to maintain the durable stock in the NT region.

Also, it follows from the homogeneity of the value function that if v is continuously differentiable, then

$$v_w(\lambda w,\lambda k)=\lambda^{-\gamma}v_w(w,k)$$

and

$$v_k(\lambda w,\lambda k)=\lambda^{-\gamma}v_k(w,k)$$

for all $\lambda>0$, so that the boundaries between the B and NT regions and between

[10] Since we define the investor's wealth w to include investment in the durable, the marginal utility of durable consumption equals v_w+v_k.

the NT and S regions are straight lines through the origin in the (w,k) space. Call the slopes of these lines $1/r_1^*$ and $1/r_2^*$, respectively, with $1/r_1^* < 1/r_2^* < 1/\delta$.

Since the optimal policy in S or B is to immediately proceed to the boundary with NT by moving along a line of slope $1/\delta$ in S or $-1/\iota$ in B, the value function is constant along these lines. In terms of the function ψ in (7), this amounts to

$$\psi(x) = \begin{cases} \dfrac{A}{1-\gamma}(x-\delta)^{1-\gamma} & \text{for } \delta < x < r_2^*, \\ \dfrac{B}{1-\gamma}(x+\iota)^{1-\gamma} & \text{for } x > r_1^* \end{cases} \tag{8}$$

for some constants A, B.

On the other hand, in NT the value function satisfies the HJB equation

$$-\kappa\frac{v_w^2}{v_{ww}} + [r(w-k) - \beta k]v_w - \beta k v_k - \rho v + \frac{(\alpha k)^{1-\gamma}}{1-\gamma} = 0.$$

Equivalently, since

$$v_w = k^{-\gamma}\psi',$$

$$v_{ww} = k^{-(1+\gamma)}\psi''$$

and

$$v_k = (1-\gamma)k^{-\gamma}\psi - wk^{-(1+\gamma)}\psi',$$

the above HJB equation translates into an ordinary differential equation for ψ:

$$-\kappa\frac{(\psi')^2}{\psi''} + (r+\beta)(x-1)\psi' - (\rho + (1-\gamma)\beta)\psi + \frac{\alpha^{1-\gamma}}{1-\gamma} = 0 \quad \text{for } r_2^* \le x \le r_1^*. \tag{9}$$

The following verification theorem formalizes the previous heuristic discussion. For simplicity, we only consider investment policies involving bounded portfolio weights. We let

$$\hat{\Theta}(W_0, K_0) = \{(I, D, \theta) \in \Theta(W_0, K_0)\colon |\theta_t| \le \eta W_t \text{ for some } \eta < \infty \text{ and all } t \ge 0\}$$

denote this restricted class of policies.

Theorem 2. Suppose that there exists an increasing, strictly concave, twice continuously differentiable function $\psi\colon(\delta,\infty)\to\mathbb{R}$ *satisfying*

$$-\kappa\frac{(\psi')^2}{\psi''} + (r+\beta)(x-1)\psi' - (\rho + (1-\gamma)\beta)\psi + \frac{\alpha^{1-\gamma}}{1-\gamma} \le 0 \quad \textit{on } \mathscr{S}, \tag{10}$$

$$\frac{(\psi')^2}{\psi''} + \frac{1-\gamma}{\gamma}\psi \geq 0 \quad on\ \mathscr{S} \tag{11}$$

and (8)–(9) *for some constants* A, B, r_1^*, r_2^* *with* $r_1^* > r_2^* > \delta$. *Let*

$$NT = \{(w,k){:}\ k>0,\ r_2^* k \leq w \leq r_1^* k\}$$

and for $(w,k) \in NT$ *set*

$$\theta^*(w,k) = -(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})k\frac{\psi'(w/k)}{\psi''(w/k)}. \tag{12}$$

Then, for any initial endowment $(W_0, K_0) \in NT$, *there exist unique continuous processes* (W^*, K^*, I^*, D^*) *with* I^*, D^* *nondecreasing such that*

$$W_t^* = W_0 + \int_0^t (r(W_s^* - K_s^*) + \theta^*(W_s^*, K_s^*)^{\mathrm{T}}(\mu - r\bar{\mathbf{1}}) - \beta K_s^*)\,\mathrm{d}s$$

$$+ \int_0^t \theta^*(W_s^*, K_s^*)^{\mathrm{T}}\sigma\,\mathrm{d}w_s - \iota I_s^* - \delta D_s^*,$$

$$K_t^* = K_0 - \int_0^t \beta K_s^*\,\mathrm{d}s + I_t^* - D_t^*,$$

$$I_t^* = \int_0^t 1_{\{W_s^* = r_1^* K_s^*\}}\,\mathrm{d}I_s^*,$$

$$D_t^* = \int_0^t 1_{\{W_s^* = r_2^* K_s^*\}}\,\mathrm{d}D_s^*,$$

and the policy (I^*, D^*, θ^*) *is optimal in* $\hat{\Theta}(W_0, K_0)$. *Otherwise, as long as* $(W_0, K_0) \in \mathscr{S}$, *the optimal policy consists of an immediate transaction to the closest boundary of NT, followed by an application of the policy* (I^*, D^*, θ^*). *The maximal lifetime expected utility is*

$$v(W_0, K_0) = K_0^{1-\gamma}\psi\left(\frac{W_0}{K_0}\right). \tag{13}$$

5. Existence and explicit solution

In this section we use Theorem 2 to derive an explicit representation for the value function and to prove the existence of an optimal policy.

Differentiating the HJB equation (9) once with respect to x and dividing by ψ'' gives a first-order differential equation in $\varphi(x) = -\psi'(x)/\psi''(x) > 0$:

$$\kappa\varphi\,\varphi' - (\rho - r - \gamma\beta + \kappa)\varphi - (r + \beta)(x - 1) = 0. \tag{14}$$

Integrating the above ODE leads to the following result.

Lemma 1. For $v \geq 0$, $x \in \mathbb{R}$, *let* $\varphi_v(x) \geq \max[\alpha_1(x-1), \alpha_2(x-1)]$ *denote the unique solution of the equation*

$$[\varphi_v(x) - \alpha_1(x-1)]^{\beta_1}[\varphi_v(x) - \alpha_2(x-1)]^{\beta_2} = v, \tag{15}$$

where

$$\alpha_1 = \frac{\rho + \kappa - r - \gamma\beta - \sqrt{(\rho + \kappa - r - \gamma\beta)^2 + 4(r+\beta)\kappa}}{2\kappa} < 0,$$

$$\alpha_2 = \frac{\rho + \kappa - r - \gamma\beta + \sqrt{(\rho + \kappa - r - \gamma\beta)^2 + 4(r+\beta)\kappa}}{2\kappa} > 0,$$

$\beta_1 = \alpha_1/(\alpha_1 - \alpha_2) \in (0,1)$ *and* $\beta_2 = 1 - \beta_1 \in (0,1)$. *If* ψ *satisfies the assumptions of Theorem* 2, *then*

$$-\frac{\psi'(x)}{\psi''(x)} = \varphi_v(x) \tag{16}$$

for all $x \in (r_2^*, r_1^*)$ *and some* $v \geq 0$.

Proof. See Appendix A. □

Integrating Eq. (16) twice and recalling (8) gives an explicit representation for ψ (and hence for the value function v) up to the five constants A, B, v, r_1^*, r_2^*, plus two additional constants of integration. However, these can be determined using the HJB equation (9) and smooth-pasting of ψ and of its first two derivatives at the boundaries of the NT region.

Theorem 3. Suppose that $\delta < 1$ *and there exist constants* $r_1^* > r_2^*$ *and* $v^* > 0$ *solving*

$$\left[\frac{1}{\gamma}(r_1^* + \imath) - \alpha_1(r_1^* - 1)\right]^{\beta_1}\left[\frac{1}{\gamma}(r_1^* + \imath) - \alpha_2(r_1^* - 1)\right]^{\beta_2} = v^*, \tag{17}$$

$$\left[\frac{1}{\gamma}(r_2^* - \delta) - \alpha_1(r_2^* - 1)\right]^{\beta_1}\left[\frac{1}{\gamma}(r_2^* - \delta) - \alpha_2(r_2^* - 1)\right]^{\beta_2} = v^*, \tag{18}$$

and

$$r_2^* - \delta + (1-\gamma)\int_{r_2^*}^{r_1^*} \exp\left(-\int_{r_2^*}^{y} \frac{\mathrm{d}z}{\varphi_{v^*}(z)}\right)\mathrm{d}y$$
$$= (r_1^* + \imath)\exp\left(-\int_{r_2^*}^{r_1^*} \frac{\mathrm{d}z}{\varphi_{v^*}(z)}\right). \tag{19}$$

Then the function

$$\psi(x) = \begin{cases} \dfrac{A}{1-\gamma}(x-\delta)^{1-\gamma} & \text{if } \delta < x < r_2^*, \\ C_1 - C_2 \int_x^{r_1^*} \exp\left(\int_y^{r_1^*} \dfrac{\mathrm{d}z}{\varphi_{v^*}(z)}\right) \mathrm{d}y & \text{if } r_2^* \le x \le r_1^*, \\ \dfrac{B}{1-\gamma}(x+\imath)^{1-\gamma}, & \text{if } x > r_1^*, \end{cases} \tag{20}$$

where

$$A = \frac{\alpha^{1-\gamma}}{\eta}(r_2^* - \delta)^{\gamma} \exp\left(\int_{r_2^*}^{r_1^*} \frac{\mathrm{d}z}{\varphi_{v^*}(z)}\right),$$

$$B = \frac{\alpha^{1-\gamma}}{\eta}(r_1^* + \imath)^{\gamma},$$

$$C_1 = \frac{\alpha^{1-\gamma}}{\eta(1-\gamma)}(r_1^* + \imath),$$

$$C_2 = \frac{\alpha^{1-\gamma}}{\eta}$$

and

$$\eta = \xi(r_1^* + \imath) + (1-\gamma)(r+\beta)(1+\imath), \tag{21}$$

satisfies the conditions of Theorem 2.

Proof. See Appendix A. □

Before establishing the existence of a solution to (17)–(19), we record some useful inequalities.

Proposition 1. If r_1^* and r_2^* satisfy the conditions of Theorem 3, *then*

$$\frac{\gamma(r+\beta)(1+\imath)}{\xi} - \imath < r_1^* < 1 - \frac{1+\imath}{1-\gamma\alpha_2} \tag{22}$$

and

$$1 - \frac{1-\delta}{1-\gamma\alpha_1} < r_2^* \le \frac{\gamma(r+\beta)(1-\delta)}{\xi} + \delta. \tag{23}$$

Moreover, the constant η in (21) *is strictly positive.*

We are now ready to show that, under Assumption 1, the nonlinear system (17)–(19) always has a solution.[11] This establishes the existence of an optimal policy.

Theorem 4. If $\delta < 1$, *there exist constants* $r_1^* > r_2^*$ *and* $v^* > 0$ *solving* (17)–(19).

Proof. See Appendix A. □

Remark 4. Since the function ψ in (20) solves the HJB equation (9), we have

$$
\begin{aligned}
0 = & -\kappa\frac{\psi'(r_2^*)^2}{\psi''(r_2^*)} + (r+\beta)(r_2^* - 1)\psi'(r_2^*) - [\rho + (1-\gamma)\beta]\psi(r_2^*) + \frac{\alpha^{1-\gamma}}{1-\gamma}\\
= & \frac{\alpha^{1-\gamma}}{(1-\gamma)\eta}\exp\left(\int_{r_2^*}^{r_1^*}\frac{dz}{\varphi_{v^*}(z)}\right)\left[\eta\exp\left(-\int_{r_2^*}^{r_1^*}\frac{dz}{\varphi_{v^*}(z)}\right)\right.\\
& -\xi(r_2^*-\delta) - (1-\gamma)(r+\beta)(1-\delta)\\
& -(\rho+(1-\gamma)\beta)\left(r_2^* - \delta + (1-\gamma)\int_{r_2^*}^{r_1^*}\exp\left(-\int_{r_2^*}^{y}\frac{dz}{\varphi_{v^*}(z)}\right)dy\right.\\
& \left.\left.-(r_1^*+\iota)\exp\left(-\int_{r_2^*}^{r_1^*}\frac{dz}{\varphi_{v^*}(z)}\right)\right)\right],
\end{aligned}
$$

so that Eq. (19) is equivalent to

$$
\xi(r_2^*-\delta) + (1-\gamma)(r+\beta)(1-\delta) = \eta\exp\left(-\int_{r_2^*}^{r_1^*}\frac{dz}{\varphi_{v^*}(z)}\right) \tag{24}
$$

provided that $\rho \neq (\gamma-1)\beta$. If the latter condition is satisfied, Eq. (24) is more convenient to use than Eq. (19) in numerical search algorithms for (r_1^*, r_2^*, v^*), since it involves a single, rather than a double, integration.

We conclude this section by providing an explicit solution for the case $\delta = 1$. Clearly, in this case it is never optimal to sell the durable, so that the solvency region contains only a 'no transaction' (NT) and a 'buy' (B) region (i.e, $r_2^* = \delta = 1$). Also, if $W^*(t,\omega) = K^*(t,\omega)$ for some $(t,\omega)\in[0,\infty)\times\Omega$, then, as

[11] The proof of the theorem also reveals that the solution of the system is easily computed once the zero of a real-valued continuous function h (defined in Eq. (A.17)) has been found. Moreover, it is shown that

$$
h\left(1-\frac{1-\delta}{1-\gamma\alpha_1}\right) > 0 > h\left(\frac{\gamma(r+\beta)(1-\delta)}{\xi}+\delta\right).
$$

Thus, it is trivial to implement numerical search procedures that always converge to the constants r_1^*, r_2^* and v^* identifying the optimal policies.

already observed in Remark 3, the only rational continuation strategy involves $\theta^*(s,\omega) = 0$ and $K^*(s,\omega) = K^*(t,\omega)e^{-\beta(s-t)}$ for all $s \geq t$. Thus, ψ satisfies the ODE (9) in NT, with the boundary condition

$$\psi(1) = \int_0^\infty e^{-\rho t} \frac{(\alpha e^{-\beta t})^{1-\gamma}}{1-\gamma}\, dt.$$

To ensure that the above value is finite, assume that $\rho > (\gamma - 1)\beta$. We can then obtain the value function for this case from Theorem 3. Since $r_2^* = \delta = 1$, we have from (18) that $v^* = 0$. Eq. (17) then gives

$$r_1^* = \frac{\gamma\alpha_2 + \iota}{\gamma\alpha_2 - 1} > 1.$$

Also, since $\varphi_0(x) = \alpha_2(x-1)$ for $x \geq 1$, we have from (20)

$$\psi(x) = \begin{cases} \dfrac{\alpha^{1-\gamma}}{(1-\gamma)(\rho + (1-\gamma)\beta)} + \dfrac{\alpha^{1-\gamma}}{\eta}(r_1^* - 1)^{1/\alpha_2} \dfrac{(x-1)^{1-1/\alpha_2}}{1 - 1/\alpha_2} & \text{if } 1 \leq x \leq r_1^*, \\ \dfrac{\alpha^{1-\gamma}}{\eta}(r_1^* + \iota)^\gamma \dfrac{(x+\iota)^{1-\gamma}}{1-\gamma} & \text{if } x > r_1^*, \end{cases}$$

where η is the constant in (21). While our verification result (Theorem 2) does not apply directly to this case (since $r_2^* = \delta$), it is straightforward to use a similar argument to show that the value function for this case is indeed given by

$$v(w,k) = k^{1-\gamma}\psi\left(\frac{w}{k}\right),$$

with ψ as above. Moreover, the optimal investment policy is given by

$$\theta_t^* = -(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})K_t^* \frac{\psi'(W_t^*/K_t^*)}{\psi''(W_t^*/K_t^*)} 1_{\{W_t^* > K_t^*\}},$$

and the optimal consumption policy involves only purchasing the durable when $K_t^* = (1/r_1^*)W_t^*$ (and never selling it).

6. Analysis of optimal policies

Recall from Theorem 2 that the optimal consumption policy consists of maintaining the fraction K_t^*/W_t^* of total wealth invested in the durable in the range $[1/r_1^*, 1/r_2^*]$, while the optimal portfolio weights are given by

$$\frac{\theta_t^*}{W_t^*} = -(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1}) \frac{K_t^*}{W_t^*} \frac{\psi'(W_t^*/K_t^*)}{\psi''(W_t^*/K_t^*)} = \frac{(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})}{\Gamma(W_t^*/K_t^*)}, \tag{25}$$

where

$$\Gamma(x) = -\frac{x\psi''(x)}{\psi'(x)}$$

denotes the Arrow–Pratt relative risk aversion coefficient of the indirect utility function ψ. In the case of no transaction costs ($\delta = \iota = 0$),

$$\frac{1}{r_1^*} = \frac{1}{r_2^*} = \frac{1}{r^*} = \frac{\rho - (1-\gamma)(r + \kappa/\gamma)}{\gamma(r+\beta)}$$

and

$$\frac{\theta_t^*}{W_t^*} = \frac{(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})}{\gamma}.$$

6.1. The no-transaction region

Table 1 shows the optimal ranges for the fraction of wealth invested in the durable good for different levels of the relative risk aversion coefficient γ and of the transaction cost rates δ and ι. As in Grossman and Laroque (1990), we assume that $n = 1$, $\mu = 0.069$, $\sigma = 0.22$ and $r = 0.01$. Moreover, we take $\rho = 0.01$ and $\beta = 0$ (no depreciation), so as to allow a direct comparison between the values in the table and those reported in Table I in Grossman and Laroque (1990).

As expected, a small percentage transaction cost can induce large deviations of optimal consumption from the Merton line. For example, in the absence of transaction costs an investor with logarithmic utility[12] would keep the investment in the durable good equal to total wealth (any stock investment would be financed entirely by borrowing). The same investor would let the ratio of durable investment to wealth fluctuate between 0.672 and 1.416 with a transaction cost rate of 0.5% in either direction, and between 0.221 and 1.899 with a transaction cost rate of 25%. Not surprisingly, the optimal ranges for the fraction of wealth invested in the durable reported in Table I of Grossman and Laroque (1990), computed under the assumption that the durable good is indivisible, are considerably larger and always strictly contain the corresponding ranges for the model we study.

In the examples reported in Table 1, the lower bound of the optimal range for the fraction of wealth invested in the durable appears to be strictly decreasing in

[12] While the analysis so far has focused on the case $\gamma \neq 1$, the case of logarithmic utilities ($\gamma = 1$) can be analyzed in a similar fashion. The relevant results are reported in Appendix B.

the percentage transaction costs δ and ι, while the upper bound appears to be strictly increasing in ι, but not necessarily a monotonic function of δ.[13] Indeed, it is easy to see that the upper bound $1/r_2^*$ cannot be monotonically increasing in

Table 1
Optimal range for the fraction of wealth invested in the durable

	$\gamma = 0.9$					
	$\iota = 0$	$\iota = 0.005$	$\iota = 0.05$	$\iota = 0.10$	$\iota = 0.25$	$\iota = 1$
$\delta = 0$	(0.556,0.556)	(0.394,0.765)	(0.259,1.057)	(0.210,1.210)	(0.145,1.481)	(0.065,2.045)
$\delta = 0.005$	(0.395,0.767)	(0.360,0.828)	(0.253,1.076)	(0.207,1.222)	(0.144,1.486)	(0.065,2.039)
$\delta = 0.05$	(0.265,1.063)	(0.258,1.081)	(0.215,1.210)	(0.185,1.313)	(0.136,1.520)	(0.064,1.984)
$\delta = 0.10$	(0.220,1.209)	(0.216,1.220)	(0.189,1.305)	(0.167,1.380)	(0.128,1.542)	(0.063,1.920)
$\delta = 0.25$	(0.161,1.382)	(0.160,1.386)	(0.147,1.421)	(0.136,1.454)	(0.110,1.532)	(0.059,1.731)
$\delta = 1$	(0.094,1.000)	(0.094,1.000)	(0.090,1.000)	(0.086,1.000)	(0.077,1.000)	(0.049,1.000)
	$\gamma = 1$					
	$\iota = 0$	$\iota = 0.005$	$\iota = 0.05$	$\iota = 0.10$	$\iota = 0.25$	$\iota = 1$
$\delta = 0$	(1.000,1.000)	(0.731,1.326)	(0.497,1.744)	(0.409,1.947)	(0.290,2.284)	(0.136,2.901)
$\delta = 0.005$	(0.732,1.325)	(0.672,1.416)	(0.486,1.764)	(0.403,1.956)	(0.288,2.279)	(0.135,2.879)
$\delta = 0.05$	(0.503,1.693)	(0.492,1.716)	(0.415,1.874)	(0.361,1.995)	(0.271,2.227)	(0.133,2.686)
$\delta = 0.10$	(0.421,1.805)	(0.414,1.817)	(0.366,1.911)	(0.328,1.990)	(0.255,2.153)	(0.130,2.497)
$\delta = 0.25$	(0.312,1.779)	(0.310,1.783)	(0.287,1.811)	(0.267,1.838)	(0.221,1.899)	(0.123,2.045)
$\delta = 1$	(0.185,1.000)	(0.184,1.000)	(0.178,1.000)	(0.171,1.000)	(0.154,1.000)	(0.102,1.000)
	$\gamma = 2$					
	$\iota = 0$	$\iota = 0.005$	$\iota = 0.05$	$\iota = 0.10$	$\iota = 0.25$	$\iota = 1$
$\delta = 0$	(1.899,1.899)	(1.611,2.165)	(1.300,2.414)	(1.163,2.509)	(0.950,2.641)	(0.578,2.824)
$\delta = 0.005$	(1.605,2.153)	(1.533,2.214)	(1.279,2.410)	(1.151,2.498)	(0.944,2.623)	(0.577,2.800)
$\delta = 0.05$	(1.272,2.260)	(1.255,2.271)	(1.143,2.339)	(1.058,2.387)	(0.895,2.468)	(0.564,2.597)
$\delta = 0.10$	(1.125,2.191)	(1.115,2.195)	(1.044,2.230)	(0.981,2.258)	(0.849,2.310)	(0.551,2.403)
$\delta = 0.25$	(0.904,1.888)	(0.900,1.889)	(0.865,1.897)	(0.831,1.905)	(0.747,1.922)	(0.517,1.959)
$\delta = 1$	(0.592,1.000)	(0.591,1.000)	(0.581,1.000)	(0.569,1.000)	(0.538,1.000)	(0.421,1.000)

Note: The table shows numerical values of the interval $(1/r_1^*, 1/r_2^*)$ for different values of the investor's risk aversion and of the proportional transaction cost rates. The other parameters are set as follows: $r = 0.01, \mu = 0.069, \sigma = 0.22, \beta = 0$ and $\rho = 0.01$.

[13] A similar behavior can be detected in Table I of Grossman and Laroque (1990). The non-monotonicity of the no-transaction region is also a feature of models with proportional costs for transaction in the risky asset: cf. Shreve and Soner (1994, Remark 11.3).

δ if $\iota = 0$ and $r^* < 1$, i.e., if

$$\rho > (1-\gamma)(r+\kappa/\gamma) + \gamma(r+\beta).$$

This is so because we must have $1/r_2^* = 1/r^* > 1$ for $\delta = 0$ and $1/r_2^* = 1$ for $\delta = 1$. The following proposition confirms the above analysis.

Proposition 2. Let r_1^*, r_2^* satisfy the conditions of Theorem 3. *Then r_1^* is strictly increasing in ι and δ, while r_2^* is strictly decreasing in ι.*

Proof. It is easy to see from Eqs. (17)–(19) that $r_1^* = r_1(\delta, \iota)$ and $r_2^* = r_2(\delta, \iota)$ for some continuously differentiable functions r_1, r_2. Also, letting

$$\psi_1(x, r_1, \iota) = \frac{\alpha^{1-\gamma}(r_1+\iota)^\gamma}{\xi(r_1+\iota) + (1-\gamma)(r+\beta)(1+\iota)} \frac{(x+\iota)^{1-\gamma}}{1-\gamma}$$

and

$$\psi_2(x, r_2, \delta) = \frac{\alpha^{1-\gamma}(r_2-\delta)^\gamma}{\xi(r_2-\delta) + (1-\gamma)(r+\beta)(1-\delta)} \frac{(x-\delta)^{1-\gamma}}{1-\gamma},$$

it follows from Eqs. (20) and (19) that $\psi(x) = \psi_1(x, r_1^*, \iota)$ for $x \geq r_1^*$ and $\psi(x) = \psi_2(x, r_2^*, \delta)$ for $\delta < x \leq r_2^*$.

Since $\psi(x)$ represents the value function for the investor's problem when $K_0 = 1$ and the maximum expected utility is strictly decreasing in the transaction cost rates δ and ι (this follows from the fact that the optimal policy always involves a positive probability of hitting either boundary: see Proposition 4), we must have $\partial\psi_1(x, r_1(\delta, \iota), \iota)/\partial\delta < 0$ and $\partial\psi_1(x, r_1(\delta, \iota), \iota)\,\iota/\partial\iota < 0$ for all $x \geq r_1(\delta, \iota)$ and $\partial\psi_2(x, r_2(\delta, \iota), \delta)/\partial\iota < 0$ for all $\delta < x \leq r_2(\delta, \iota)$.

The first inequality amounts to

$$0 > \frac{\partial\psi_1(x, r_1^*, \iota)}{\partial r_1} \frac{\partial r_1(\delta, \iota)}{\partial\delta}$$

$$= -\frac{\alpha^{1-\gamma}(r_1^*+\iota)^{\gamma-1}[\xi(r_1^*+\iota) - \gamma(r+\beta)(1+\iota)]}{[\xi(r_1+\iota) + (1-\gamma)(r+\beta)(1+\iota)]^2} (x+\iota)^{1-\gamma} \frac{\partial r_1(\delta, \iota)}{\partial\delta}.$$

Since the first fraction in the above expression is strictly positive (by Proposition 1), we conclude that $\partial r_1(\delta, \iota)/\partial\delta > 0$. The proof that $\partial r_1(\delta, \iota)/\partial\iota > 0$ and $\partial r_2(\delta, \iota)/\partial\iota < 0$ is similar. □

The fact that the upper bound $1/r_2^*$ of the optimal range for the fraction of wealth invested in the durable is not necessarily monotonically increasing in δ may at first appear counterintuitive. However, it can be rationalized as follows. An increase in the transaction cost rates δ and ι affects the location of the optimal range for K/W in two different ways. First, since adjusting the stock of durable becomes more expensive, the investor will tend to widen the no-transaction region by decreasing $1/r_1^*$ and increasing $1/r_2^*$ (the 'transaction-avoidance effect'). On the other hand, an increase in δ or ι also induces the investor to save more to compensate for lower future consumption due to higher future transaction costs: this is obtained by decreasing the upper bound $1/r_2^*$ (and possibly the lower bound $1/r_1^*$) of the optimal range for the fraction of wealth invested in the durable good (the 'saving' effect). While both effects will tend to unambiguously decrease $1/r_1^*$ as the transaction cost rates increase, the net impact on $1/r_2^*$ depends on which factor dominates.

Since $1/r_1^*$ is decreasing in δ and ι and $1/r_1^* = 1/r^*$ for $\delta = \iota = 0$, we always have $1/r_1^* \leq 1/r^*$. On the other hand, since $1/r_2^*$ is not necessarily increasing in δ, it is possible that $1/r_2^* \leq 1/r^*$, i.e., that the no-transaction region does not contain the Merton line. In this case, the investor always consumes less, for any given level of wealth, than what he would consume in the absence of adjustment costs. The following proposition provides a simple necessary and sufficient condition for this to happen.

Proposition 3. Let r^* be as in Theorem 1 *and let r_2^* be as in Theorem* 3. *There exists a $\delta \in (0,1)$ such that $r_2^* > r^*$ if and only if $r^* < 1$, i.e., if and only if*

$$\rho > (1-\gamma)(r+\kappa/\gamma) + \gamma(r+\beta).$$

Proof. The condition on ρ is equivalent to

$$\frac{\gamma[\alpha_1 \xi + (1-\gamma\alpha_1)(r+\beta)]}{\xi} < 1.$$

Fix δ with

$$\frac{\gamma[\alpha_1 \xi + (1-\gamma\alpha_1)(r+\beta)]}{\xi} < \delta < 1.$$

Then it follows from Proposition 1 that

$$r_2^* > 1 - \frac{1-\delta}{1-\gamma\alpha_1} > \frac{\gamma(r+\beta)}{\xi} = r^*.$$

Conversely, suppose that $r_2^* > r^*$ for some $\delta \in (0,1)$. Then it follows from Proposition 1 that

$$0 < r_2^* - r^* \leq \frac{\gamma(r+\beta)(1-\delta)}{\xi} + \delta - r^*$$

$$= \delta \frac{\rho - (1-\gamma)(r+\kappa/\gamma) - \gamma(r+\beta)}{\xi}. \quad \square$$

Assuming a durable's depreciation rate of 5% and no transaction costs for purchases ($\iota = 0$), Fig. 1 shows how the no-transaction region changes as

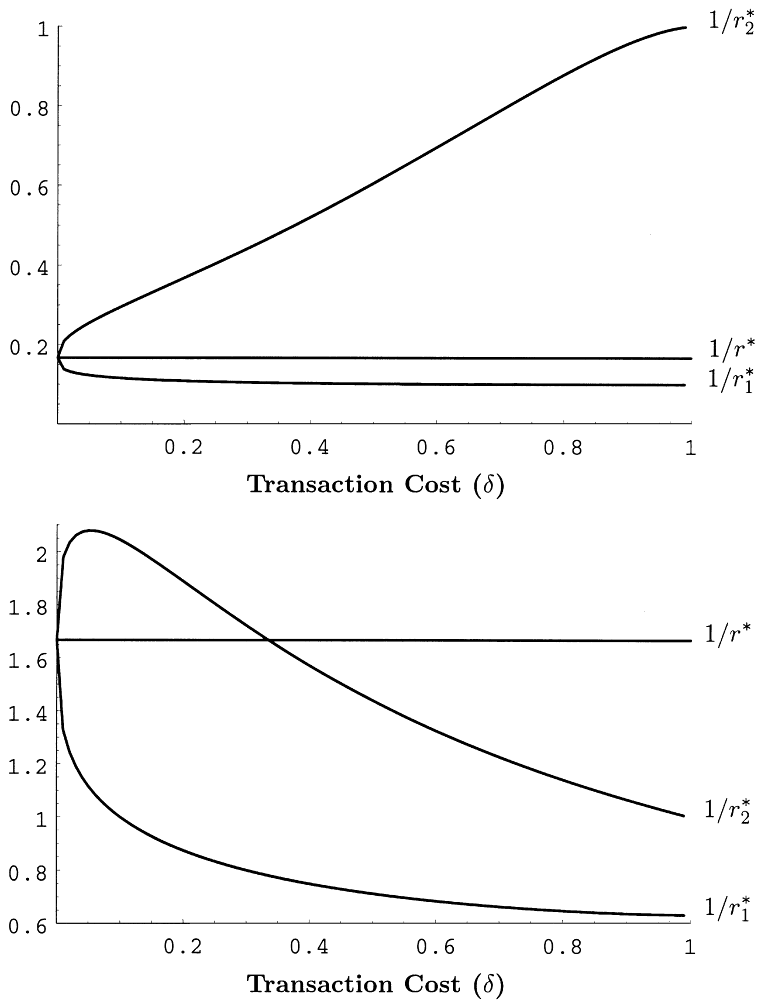

Fig. 1. Boundaries of the optimal range for the fraction of wealth invested in the durable as a function of the transaction cost rate. The graph plots $1/r_1^*$ and $1/r_2^*$ against δ for two different values of the investor's time preference parameter: $\rho = 0.01$ (top graph) and $\rho = 0.10$ (bottom graph). The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\iota = 0$, $\beta = 0.05$ and $\gamma = 1$.

a function of the transaction cost rate δ for a log investor ($\gamma = 1$) and for two different values of the time-preference parameter ($\rho = 0.01$ and $\rho = 0.10$). In the first case (in which $r^* > 1$) the no-transaction region becomes monotonically larger as the transaction cost rate increases, and it always includes the Merton line, while in the second case (in which $r^* < 1$) the no-transaction region is not monotonically increasing and it fails to include the Merton line when the transaction cost rate is large enough. This reflects the fact that in the latter case the investor is more impatient and thus initially less willing to save. Since this implies a higher marginal utility for future consumption, a reduction in future consumption due to an increase in transaction costs induces him to increase his savings proportionally more than a less impatient investor. Thus, the 'saving' effect dominates the 'transaction-avoidance' effect.

Figs. 2 and 3 show, respectively, how the boundaries of the no-transaction region change as a function of the investor's relative risk aversion γ and of the durable's depreciation rate β. As would be the case in the absence of transaction costs, the region's boundaries are a nonmonotonic function of the investor's risk-aversion coefficient and a decreasing function of the durable's depreciation rate.[14] Moreover, the size of the no-transaction region is a nonmonotonic

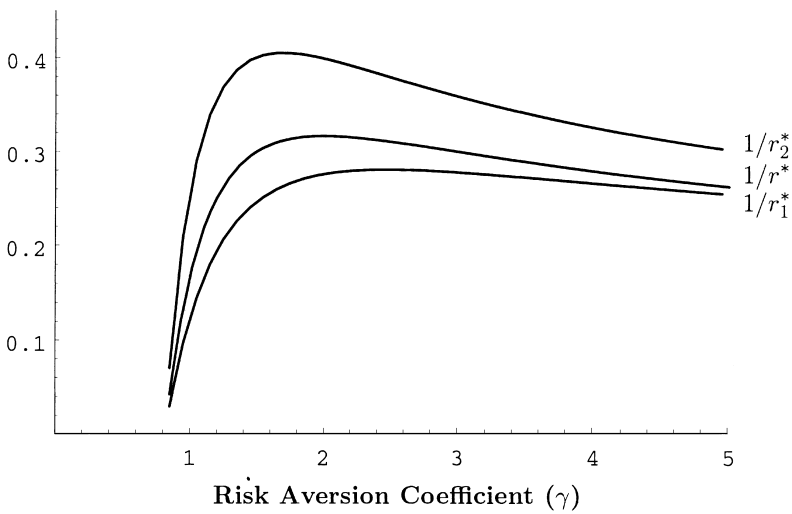

Fig. 2. Boundaries of the optimal range for the fraction of wealth invested in the durable as a function of the investor's relative risk aversion coefficient. The graph plots $1/r_1^*$, $1/r_2^*$ and $1/r^*$ against γ. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\delta = 0.05$, $\iota = 0$, $\beta = 0.05$ and $\rho = 0.01$.

[14] An optimal policy does not exist in Fig. 2 for

$$\gamma < \gamma^* = \frac{\sqrt{(\rho - r + \kappa)^2 + 4r\kappa} - (\rho - r + \kappa)}{2r} = 0.815,$$

as Assumption 1 is violated in this case. Moreover, as $\gamma \downarrow \gamma^*$, the optimal policy involves postponing consumption to increase investment in the stock market. Thus, both boundaries of the no-transaction region converge to zero. In Fig. 3, as $\beta \uparrow \infty$, the durable behaves increasingly as a perishable good, so that $\lim(1/r_1^*) = \lim(1/r_2^*) = \lim(1/r^*) = 0$.

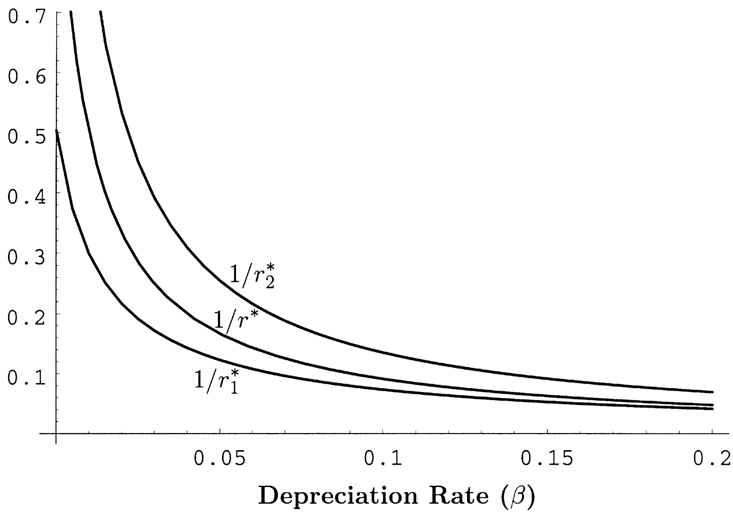

Fig. 3. Boundaries of the optimal range for the fraction of wealth invested in the durable as a function of the durable's depreciation rate. The graph plots $1/r_1^*$, $1/r_2^*$ and $1/r^*$ against β. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\delta = 0.05$, $\iota = 0$, $\rho = 0.01$ and $\gamma = 1$.

function of the risk aversion and a monotonically decreasing function of the depreciation rate. Thus, the optimal consumption policy for short-lived durable goods is to purchase small quantities frequently, while the optimal policy for long-lived durables calls for more sporadic and larger purchases. This agrees with the finding of Hindy and Huang (1993) for the case $\delta = 1$.

6.2. Stochastic behavior of the investment in the durable

In order to analyze in more detail the stochastic behavior of the investment in the durable, let $x_t = W_t^*/K_t^*$ denote the inverse of the fraction of wealth invested in the durable at time t. An application of Itô's lemma shows that, within the no-transaction region,

$$\mathrm{d}x_t = a(x_t)\,\mathrm{d}t + b(x_t)^{\mathrm{T}}\,\mathrm{d}w_t,$$

where

$$a(x) = (r + \beta)(x - 1) + 2\kappa\varphi_{v^*}(x)$$

and

$$b(x) = [(\mu - r\mathbf{1})^{\mathrm{T}}(\sigma\sigma^{\mathrm{T}})^{-1}\sigma]^{\mathrm{T}}\varphi_{v^*}(x).$$

Now fix $x_0 = x \in (r_2^*, r_1^*)$, and let

$$\tau = \inf\{t \geq 0\colon x_t \notin (r_2^*, r_1^*)\}.$$

denote the time of the next transaction in the durable. Also, let

$$\mathrm{P}_x(\tau < \infty) = \mathrm{P}(\tau < \infty \mid x_0 = x)$$

denote the conditional probability that τ is finite and let

$$\mathrm{E}_x[\tau] = \mathrm{E}[\tau \mid x_0 = x]$$

denote the conditional expectation of τ. Finally, fix an arbitrary number $c \in (r_2^*, r_1^*)$ and define the *scale function*

$$s(x) = \int_c^x \exp\left(-2\int_c^y \frac{a(z)}{|b(z)|^2}\,\mathrm{d}z \right)\mathrm{d}y$$

as well as the *speed density*

$$m(x) = \frac{2}{s'(x)\,|b(x)|^2}.$$

Proposition 4. If $\delta < 1$, *then* $\mathrm{P}_x(\tau < \infty) = 1$ *and* $\mathrm{E}_x[\tau] < \infty$ *for all* $x \in (r_2^*, r_1^*)$. *Moreover, either boundary of the no-transaction region can be reached with positive probability. On the other hand, if* $\delta = 1$ *then* $\mathrm{P}_x(\tau < \infty) = 1$ *if* $\rho \leq r + \kappa + \gamma\beta$ *and* $0 < \mathrm{P}_x(\tau < \infty) < 1$ *otherwise. In either case, the lower boundary is never reached.*

Proof. Since $\varphi_{v^*}(x) > 0$ for all $x \in (r_2^*, r_1^*)$, we have (recalling the standing assumption that $\kappa > 0$)

$$|b(x)|^2 > 0 \quad \forall x \in (r_2^*, r_1^*).$$

Moreover, since both $a(x)$ and $b(x)$ are continuous on (r_2^*, r_1^*),

$$\forall x \in (r_2^*, r_1^*), \quad \exists \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |a(x)|}{|b(x)|^2}\,\mathrm{d}x < \infty.$$

Now let

$$q(x) = \int_c^x (s(x) - s(y))m(y)\,\mathrm{d}y = \int_c^x s'(y)\left(\int_c^y m(z)\,\mathrm{d}z\right)\mathrm{d}y.$$

If $\delta < 1$, then $\varphi_{v^*}(x) > 0$ for all $x \in [r_2^*, r_1^*]$, so that s and m are defined and continuous on $[r_2^*, r_1^*]$, and we have $q(r_2^*) < \infty$ and $q(r_1^*) < \infty$. The fact that $\mathrm{E}_x[\tau] < \infty$ (and hence that $\tau < \infty$ a.s.) then follows from Proposition 5.5.32(i) in Karatzas and Shreve (1988), while the fact that either boundary can be reached with positive probability follows from Proposition 5.5.22(d).

If $\delta = 1$, then the above argument is not necessarily true, since $\varphi_{v^*}(r_2^*) = \varphi_0(1) = 0$. On the other hand, since $\varphi_0(x) = \alpha_2(x-1)$, for any $x > 1$, we have in this case

$$s(x) = \int_c^x \exp\left(-2\int_c^y \frac{r+\beta+2\kappa\alpha_2}{2\kappa\alpha_2^2(z-1)}\,\mathrm{d}z\right)\mathrm{d}y$$

$$= \int_c^x \frac{(y-1)^{-\zeta}}{(c-1)^{-\zeta}}\,\mathrm{d}y$$

$$= \begin{cases} (c-1)^{\zeta}\left(\dfrac{(x-1)^{1-\zeta}}{1-\zeta} - \dfrac{(c-1)^{1-\zeta}}{1-\zeta}\right) & \text{if } \zeta \neq 1, \\ (c-1)\log\left(\dfrac{x-1}{c-1}\right) & \text{otherwise} \end{cases}$$

and

$$q(x) = \int_c^x \frac{(y-1)^{-\zeta}}{(c-1)^{-\zeta}}\left(\int_c^y \frac{2(c-1)^{-\zeta}}{2\kappa\alpha_2^2(z-1)^{2-\zeta}}\,\mathrm{d}z\right)\mathrm{d}y$$

$$= \begin{cases} \dfrac{(x-1)^{1-\zeta} - (c-1)^{1-\zeta}}{\kappa\alpha_2^2(c-1)^{1-\zeta}(1-\zeta)^2} - \dfrac{\ln(x-1) - \ln(c-1)}{\kappa\alpha_2^2(1-\zeta)} & \text{if } \zeta \neq 1, \\ \dfrac{1}{2\kappa\alpha_2^2}\left(\log\left(\dfrac{x-1}{c-1}\right)\right)^2 & \text{otherwise,} \end{cases}$$

where

$$\zeta = \frac{r+\beta+2\kappa\alpha_2}{\kappa\alpha_2^2}.$$

If $\rho \leq r + \kappa + \gamma\beta$, then $\zeta \geq 1$, and hence $s(r_1^*) < \infty$, $s(r_2^* +) = s(1+) = -\infty$, $q(r_1^*) < \infty$ and $q(r_2^* +) = q(1+) = +\infty$. The fact that $\mathrm{P}_x(\tau < \infty) = 1$ then follows from Proposition 5.5.32(ii) in Karatzas and Shreve (1988), while the fact that the lower boundary is never reached follows from Proposition 5.5.22(c). On the other hand, if $\rho > r + \kappa + \gamma\beta$, then $\zeta < 1$, and hence $s(r_1^*) < \infty$, $s(r_2^*) = s(1) > -\infty$, $q(r_1^*) < \infty$ and $q(r_2^* +) = q(1+) = +\infty$, so that it follows from Propositions 5.5.29 and 5.5.32 in Karatzas and Shreve (1988) that $0 < \mathrm{P}_x(\tau < \infty) < 1$. The fact that the lower boundary is never reached follows from the fact that $q(+\infty) = +\infty$ and Proposition 5.5.29 in Karatzas and Shreve (1988). □

As a consequence of the fact that, when $\delta < 1$, the expected time to reach either boundary is finite, it follows that x is a positively recurrent process and that

$$f(x) = \frac{m(x)}{\int_{r_2^*}^{r_1^*} m(y)\,\mathrm{d}y}$$

is a stationary (or steady-state) probability density (Borodin and Salminen, 1996, Section II.12). In addition, x is ergodic and the distribution of x_t converges to the stationary distribution, that is

$$\lim_{t\to\infty}\left(\sup_{A\in\mathscr{B}([r_2^*,r_1^*])}\left|\mathrm{P}_x(x_t\in A) - \int_A f(z)\,\mathrm{d}z\right|\right) = 0$$

where $\mathscr{B}([r_2^*, r_1^*])$ denotes the Borel sigma-field on $[r_2^*, r_1^*]$ (Borodin and Salminen, 1996, Section II.35–36).

Fig. 4 shows the steady-state average fraction of wealth invested in the durable as a function of the transaction cost rate δ. As it could be expected, even though the no-transaction region is monotonically increasing in this case (as shown in Fig. 2), the steady-state average proportional investment in the durable good declines monotonically as the transaction costs increase.

6.3. Frequency of transactions in the durable

For the case in which $\delta < 1$ (so that $\mathrm{E}_x[\tau] < \infty$), we can compute the expected time to the next transaction in the durable using the following result.

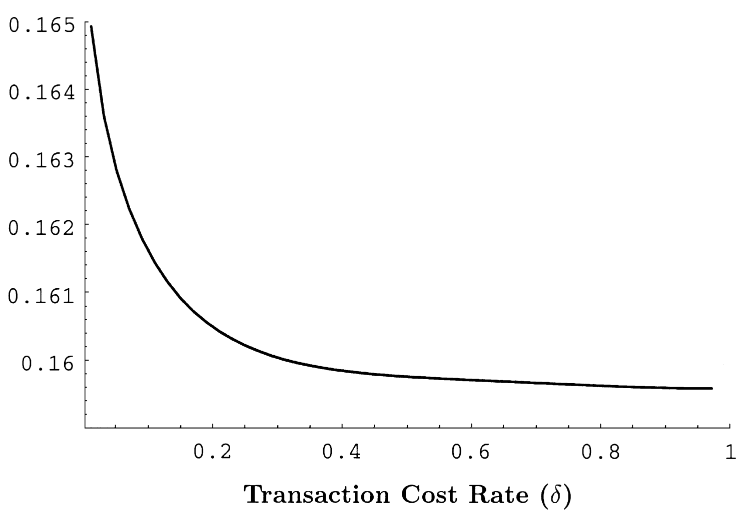

Fig. 4. Steady-state average fraction of wealth invested in the durable as a function of the transaction cost rate. The graph plots the average of K^*/W^* under the steady-state distribution against δ. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\iota = 0$, $\beta = 0.05$, $\rho = 0.01$ and $\gamma = 1$.

Proposition 5. Suppose that $\delta < 1$. *Then the function* $T(x) = \mathrm{E}_x[\tau]$ *solves the ordinary differential equation*

$$\tfrac{1}{2}|b(x)|^2 T''(x) + a(x)T'(x) + 1 = 0$$

on (r_2^*, r_1^*), *with boundary conditions* $T(r_2^*) = T(r_1^*) = 0$.

Proof. This follows immediately from Karlin and Taylor (1981, p. 192). □

Fig. 5 plots the expected length of time (in years) to the next adjustment in the stock of the durable $T(x) = \mathrm{E}_x[\tau]$ as a function of the current fraction of wealth invested in the durable $K^*/W^* = 1/x$ for different levels of the transaction cost rate δ. While the values in Fig. 5 are conditional expectations based on the current value of x, Figs. 6 and 7 plot the unconditional expectation of the time to the next transaction in the durable under the steady-state distribution for x, as a function of the transaction cost rate δ and of the depreciation rate β. The latter figure confirms our earlier statement that the optimal policy for longer-lived durables involves more sporadic adjustments. As expected, these figures also indicate that changes in durable consumption are much more frequent in the

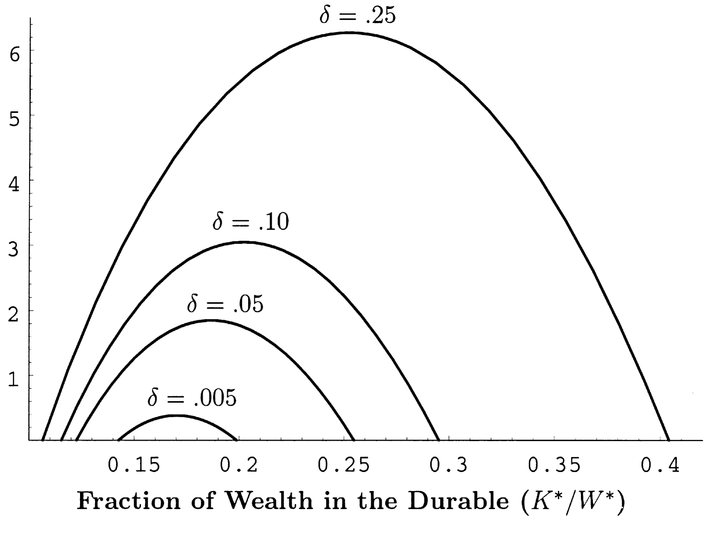

Fig. 5. Expected length of time (in years) to the next transaction in the durable as a function of the fraction of wealth invested in the durable. The graph plots $E_x[\tau]$ against $1/x = K^*/W^*$ for different values of δ. Each curve is plotted over the optimal range for K^*/W^*. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$,$\iota = 0$, $\beta = 0.05$, $\rho = 0.01$ and $\gamma = 1$.

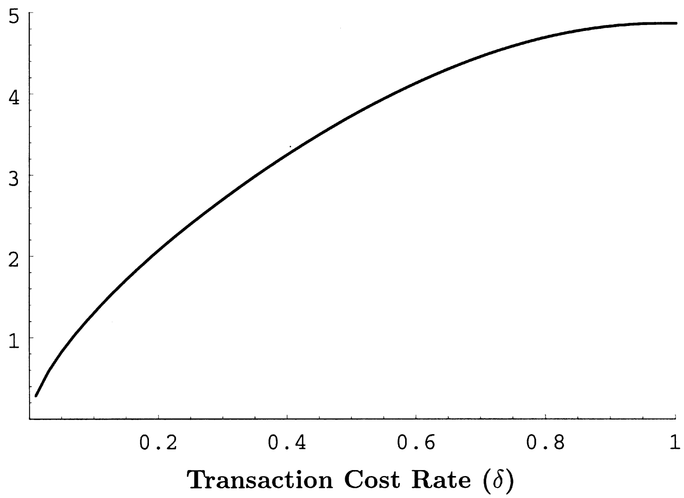

Fig. 6. Steady-state average time to the next transaction in the durable as a function of the transaction cost rate. The graph plots the unconditional mean of the time to the next transaction under the steady-state distribution of x as a function of δ. The other parameters are set as follows: $r=0.01$, $\mu=0.069$, $\sigma=0.22$, $\imath=0$, $\beta=0.05$, $\rho=0.01$ and $\gamma=1$.

case of a divisible durable good than in the case of an indivisible good studied by Grossman and Laroque (1990).

An alternative assessment of the frequency of transactions in the durable can be obtained by examining the expected discounted value of the lifetime purchases and sales of the durable.

Proposition 6. Let $\lambda, \lambda_1, \lambda_2$ be arbitrary constants with

$$\lambda > r + 2\kappa \frac{r_1^* + \imath}{\gamma r_1^*}.$$

Then

$$\mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\lambda t} |\lambda_1 \, \mathrm{d}I_t^* + \lambda_2 \, \mathrm{d}D_t^*| \middle| K_0^* = K, W_0^* = W\right] < \infty$$

for all (W, K) with $K>0$ and $W/K \in [r_2^*, r_1^*]$ if and only if

$$\mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\lambda t} (\lambda_1 \, \mathrm{d}I_t^* + \lambda_2 \, \mathrm{d}D_t^*) \middle| K_0^* = K, \ W_0^* = W\right] = Kg(W/K; \lambda, \lambda_1, \lambda_2),$$

where $g(x) = g(x; \lambda, \lambda_1, \lambda_2)$ solves the ordinary differential equation

$$\tfrac{1}{2} |b(x)|^2 g''(x) + a(x) g'(x) - (\lambda + \beta) g(x) = 0 \tag{26}$$

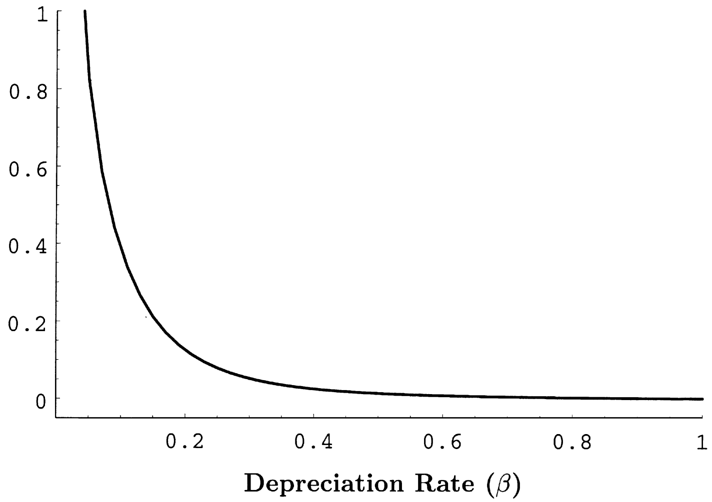

Fig. 7. Steady-state average time to the next transaction in the durable as a function of the depreciation rate. The graph plots the unconditional mean of the time to the next transaction under the steady-state distribution of x for different values of β. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\delta = 0.05$, $\imath = 0$, $\rho = 0.01$ and $\gamma = 1$.

on $[r_2^*, r_1^*]$ *with boundary conditions*

$$g(r_1^*) - (r_1^* + \imath)g'(r_1^*) + \lambda_1 = 0$$

and

$$g(r_2^*) - (r_2^* - \delta)g'(r_2^*) - \lambda_2 = 0.$$

Proof. Suppose first that there is a solution g to ODE (26) with the associated boundary conditions and let $C(W,K) = Kg(W/K)$. An application of Itô's lemma gives

$$\begin{aligned} C(W,K) = {} & \mathrm{e}^{-\lambda t}C(W_t^*, K_t^*) - \int_0^t \mathrm{e}^{-\lambda s} C_W(W_s^*, K_s^*)\theta_s^{*\mathrm{T}}\sigma \, \mathrm{d}w_s \\ & - \int_0^t \mathrm{e}^{-\lambda s}(\tfrac{1}{2} C_{WW}(W_s^*, K_s^*) \,|\, \theta_s^{*\mathrm{T}}\sigma \,|^2 \\ & + C_W(W_s^*, K_s^*)[rW_s^* + \theta_s^{*\mathrm{T}}(\mu - r) - (r + \beta)K_s^*] \\ & - C_K(W_s^*, K_s^*)\beta K_s^* - \lambda C(W_s^*, K_s^*))\mathrm{d}s \end{aligned}$$

$$+\int_0^t e^{-\lambda s}(\imath C_W(W_s^*,K_s^*) - C_K(W_s^*,K_s^*))\,dI_s^*$$

$$+\int_0^t e^{-\lambda s}(\delta C_W(W_s^*,K_s^*) + C_K(W_s^*,K_s^*))\,dD_s^*$$

$$= e^{-\lambda t}W_t^*\frac{g(x_t)}{x_t} - \int_0^t e^{-\lambda s}W_s^*\frac{g'(x_s)}{x_s}\varphi_{v^*}(x_s)(\mu - r\mathbf{1})^{\mathrm{T}}(\sigma\sigma^{\mathrm{T}})^{-1}\sigma\,dw_s$$

$$-\int_0^t e^{-\lambda s}K_s^*\left(\frac{1}{2}|b(x_s)|^2 g''(x_s) + a(x_s)g'(x_s) - (\lambda+\beta)g(x_s)\right)ds$$

$$-\int_0^t e^{-\lambda s}(g(r_1^*) - (r_1^* + \imath)g'(r_1^*))\,dI_s^*$$

$$+\int_0^t e^{-\lambda s}(g(r_2^*) - (r_2^* - \delta)g'(r_2^*))\,dD_s^*$$

$$= e^{-\lambda t}W_t^*\frac{g(x_t)}{x_t} - \int_0^t e^{-\lambda s}W_s^*\frac{g'(x_s)}{x_s}\varphi_{v^*}(x_s)(\mu - r\mathbf{1})^{\mathrm{T}}(\sigma\sigma^{\mathrm{T}})^{-1}\sigma\,dw_s$$

$$+\lambda_1\int_0^t e^{-\lambda s}\,dI_s^* + \lambda_2\int_0^t e^{-\lambda s}\,dD_s^*.$$

Since $x_t \in [r_2^*, r_1^*]$ for all t, the function $(g'(x)/x)\varphi_{v^*}(x)$ is continuous, and hence bounded on $[r_2^*, r_1^*]$ and W^* is square-integrable on $[0,t]$, the stochastic integral in the previous expression has zero expectation, so that

$$C(W,K) = \mathrm{E}\left[e^{-\lambda t}W_t^*\frac{g(x_t)}{x_t}\right] + \mathrm{E}\left[\int_0^t e^{-\lambda s}(\lambda_1\,dI_s^* + \lambda_2\,dD_s^*)\right].$$

Letting $\pi = \theta^*/W^*$ denote the portfolio weights process, we will show below that

$$0 < \pi_t^{\mathrm{T}}(\mu - r\mathbf{1}) = 2\kappa\frac{\varphi_{v^*}(x_t)}{x_t} \le 2\kappa\frac{r_1^* + \imath}{\gamma r_1^*}.$$

The claim then follows from the monotone convergence theorem, using the fact that $g(x)/x$ is bounded on $[r_2^*, r_1^*]$ and that the process

$$N_t = W\exp\left(-\frac{1}{2}\int_0^t |\pi_s^{\mathrm{T}}\sigma|^2\,ds + \int_0^t \pi_s^{\mathrm{T}}\sigma\,dw_s\right)$$

is a martingale, so that

$$\begin{aligned}\mathrm{E}[\mathrm{e}^{-\lambda t}W_t^*] &\le \mathrm{E}\left[\exp\left(\int_0^t (r+\pi_s^{\mathrm{T}}(\mu - r\mathbf{1}) - \lambda)\,\mathrm{d}s\right)N_t\right]\\ &\le \exp\left(\left(r + 2\kappa\frac{r_1^*+\iota}{\gamma r_1^*} - \lambda\right)t\right)\mathrm{E}[N_t]\\ &= \exp\left(\left(r + 2\kappa\frac{r_1^*+\iota}{\gamma r_1^*} - \lambda\right)t\right)W \to 0 \quad \text{as } t\to\infty.\end{aligned}$$

Conversely, letting

$$C(W,K) = \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\lambda t}(\lambda_1\,\mathrm{d}I_t^* + \lambda_2\,\mathrm{d}D_t^*)\middle| K_0^* = K, W_0^* = W\right],$$

it is easily verified that C is homogeneous of degree one in (W,K), so that $C(W,K) = Kg(W/K)$ for some function g. The ODE for g then follows from Itô's lemma and the fact that the process

$$\mathrm{e}^{-\lambda t}C(W_t,K_t) + \int_0^t \mathrm{e}^{-\lambda s}(\lambda_1\,\mathrm{d}I_s^* + \lambda_2\,\mathrm{d}D_s^*)$$

is a martingale. Finally, to show that

$$\frac{\varphi_{v^*}(x_t)}{x_t} \le \frac{r_1^*+\iota}{\gamma r_1^*},$$

let

$$v(x,y) = \beta_1\log[yx - \alpha_1(x-1)] + \beta_2\log[yx-\alpha_2(x-1)].$$

Since v is strictly increasing in y and

$$\frac{r_1^*+\iota}{\gamma r_1^*}r_2^* > \frac{r_2^*-\delta}{\gamma},$$

it follows from (17) and (18) that

$$v\left(r_1^*, \frac{r_1^*+\iota}{\gamma r_1^*}\right) = \log(v^*) < v\left(r_2^*, \frac{r_1^*+\iota}{\gamma r_1^*}\right).$$

The concavity of v in x then implies

$$v\left(x, \frac{r_1^*+\iota}{\gamma r_1^*}\right) \ge \log(v^*) \quad \text{for all } x\in[r_2^*, r_1^*].$$

The claim now follows from the fact that

$$v\left(x, \frac{\varphi_{v^*}(x)}{x}\right) = \log(v^*)$$

by (15) and v is increasing in y. □

The above proposition allows to compute the expected discounted value of the lifetime purchases (respectively, sales) of the durable good, conditional on the current values of W^* and K^*, by solving ODE (26) for g with $\lambda_1 = 1$ and $\lambda_2 = 0$ (respectively, $\lambda_1 = 0$ and $\lambda_2 = 1$). Figs. 8 and 9 report the unconditional expected discounted values of the lifetime purchases and sales of the durable, as a fraction of the initial stock of the durable, for different levels of the transaction cost rate δ. The unconditional expected discounted values are computed under the steady-state distribution of x and the discount rate λ is set at 0.1. While the expected discounted purchases and sales are both monotonically decreasing in δ, the latter are more responsive than the former to changes in selling costs.[15]

6.4. The portfolio policy

Looking next at the optimal portfolio policies, (25) shows that investors still hold the same portfolio of risky assets they would hold in the absence of

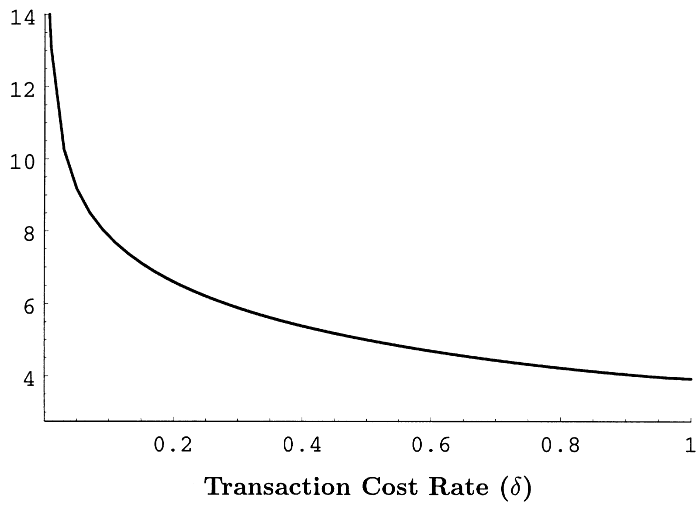

Fig. 8. Expected discounted lifetime purchases of the durable over existing stock as a function of the transaction cost rate. The graph plots the unconditional mean of $g(x; 0.10, 1, 0)$ under the steady-state distribution of x for different values of δ. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\iota = 0$, $\beta = 0.05$, $\rho = 0.01$ and $\gamma = 1$.

[15] The same monotonic pattern prevails in the case in which $\rho = 0.10$, even though (as shown in Fig. 1) the boundaries of the no-transaction region are non-monotonic in this case.

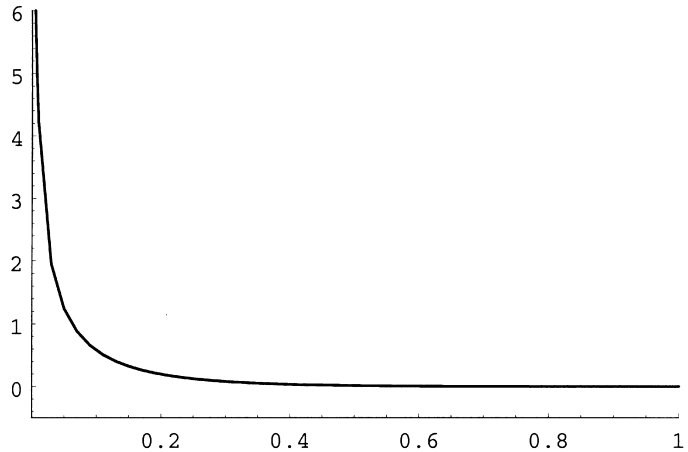

Fig. 9. Expected discounted lifetime sales of the durable over existing stock as a function of the transaction cost rate. The graph plots the unconditional mean of $g(x; 0.10, 0, 1)$ under the steady-state distribution of x for different values of δ. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\iota = 0$, $\beta = 0.05$, $\rho = 0.01$ and $\gamma = 1$.

transaction costs. However, their risk aversions, and hence the fraction of their wealth invested in stocks, are changed as a result of the presence of transaction costs. More precisely, the following proposition shows that investors are less risk averse than they would be in the absence of transaction costs when their wealth is large relative to the stock of durable (i.e., immediately before or after a purchase), and more risk averse when their wealth is small (i.e., immediately before or after a sale).

Proposition 7. Let r_1^*, r_2^* satisfy the conditions of Theorem 3. *Then*

$$\Gamma(r_1^*) \le \gamma \le \Gamma(r_2^*).$$

Proof. Let v^* be the constant in (17) and (18). Since $\varphi_{v^*}(r_1^*) = (r_1^* + \iota)/\gamma$ and $\varphi_{v^*}(r_2^*) = (r_2^* - \delta)/\gamma$ by (15), (17) and (18), we have from (16):

$$\Gamma(r_1^*) = \frac{r_1^*}{\varphi_{v^*}(r_1^*)} = \frac{\gamma r_1^*}{r_1^* + \iota} \le \gamma \le \frac{\gamma r_2^*}{r_2^* - \delta} = \frac{r_2^*}{\varphi_{v^*}(r_2^*)} = \Gamma(r_2^*). \qquad \square$$

Table 2 shows the optimal ranges for the fraction of wealth invested in stocks for different levels of the relative risk aversion coefficient γ and of the transaction cost rates δ and ι.[16] Transaction costs on the durable good appear to have a smaller impact on the portfolio weights than on the fraction of wealth invested in the durable. For the parameters we are considering, a logarithmic investor would keep the ratio of stock investment to wealth equal to 1.219 in the absence

Table 2
Optimal range for the fraction of wealth invested in stocks

	$\gamma = 0.9$					
	$\iota = 0$	$\iota = 0.005$	$\iota = 0.05$	$\iota = 0.10$	$\iota = 0.25$	$\iota = 1$
$\delta = 0$	(1.354,1.354)	(1.354,1.357)	(1.354,1.372)	(1.354,1.383)	(1.354,1.404)	(1.354,1.443)
$\delta = 0.005$	(1.349,1.354)	(1.349,1.357)	(1.347,1.372)	(1.346,1.382)	(1.344,1.403)	(1.341,1.443)
$\delta = 0.05$	(1.282,1.354)	(1.281,1.356)	(1.273,1.369)	(1.266,1.379)	(1.251,1.400)	(1.220,1.441)
$\delta = 0.10$	(1.191,1.354)	(1.189,1.356)	(1.178,1.367)	(1.167,1.377)	(1.146,1.398)	(1.094,1.439)
$\delta = 0.25$	(0.887,1.354)	(0.885,1.356)	(0.873,1.364)	(0.862,1.373)	(0.836,1.392)	(0.768,1.435)
$\delta = 1$	(0.000,1.354)	(0.000,1.355)	(0.000,1.361)	(0.000,1.366)	(0.000,1.380)	(0.000,1.421)
	$\gamma = 1$					
	$\iota = 0$	$\iota = 0.005$	$\iota = 0.05$	$\iota = 0.10$	$\iota = 0.25$	$\iota = 1$
$\delta = 0$	(1.219,1.219)	(1.219,1.223)	(1.219,1.249)	(1.219,1.269)	(1.219,1.307)	(1.219,1.384)
$\delta = 0.005$	(1.211,1.219)	(1.210,1.223)	(1.208,1.249)	(1.207,1.268)	(1.205,1.307)	(1.201,1.384)
$\delta = 0.05$	(1.116,1.219)	(1.114,1.222)	(1.105,1.244)	(1.097,1.263)	(1.083,1.302)	(1.055,1.381)
$\delta = 0.10$	(0.999,1.219)	(0.997,1.222)	(0.986,1.241)	(0.976,1.259)	(0.957,1.297)	(0.915,1.378)
$\delta = 0.25$	(0.677,1.219)	(0.676,1.221)	(0.667,1.237)	(0.659,1.252)	(0.640,1.286)	(0.596,1.369)
$\delta = 1$	(0.000,1.219)	(0.000,1.220)	(0.000,1.230)	(0.000,1.240)	(0.000,1.266)	(0.000,1.343)
	$\gamma = 2$					
	$\iota = 0$	$\iota = 0.005$	$\iota = 0.05$	$\iota = 0.10$	$\iota = 0.25$	$\iota = 1$
$\delta = 0$	(0.610,0.610)	(0.610,0.614)	(0.610,0.649)	(0.610,0.680)	(0.610,0.754)	(0.610,0.962)
$\delta = 0.005$	(0.603,0.610)	(0.603,0.614)	(0.602,0.648)	(0.602,0.680)	(0.602,0.753)	(0.601,0.961)
$\delta = 0.05$	(0.541,0.610)	(0.540,0.613)	(0.538,0.644)	(0.537,0.674)	(0.534,0.746)	(0.530,0.953)
$\delta = 0.10$	(0.476,0.610)	(0.476,0.613)	(0.474,0.641)	(0.472,0.669)	(0.469,0.739)	(0.463,0.945)
$\delta = 0.25$	(0.322,0.610)	(0.322,0.612)	(0.320,0.636)	(0.319,0.660)	(0.317,0.723)	(0.311,0.924)
$\delta = 1$	(0.000,0.610)	(0.000,0.611)	(0.000,0.627)	(0.000,0.644)	(0.000,0.691)	(0.000,0.866)

Note: The table shows numerical values of the interval $((\mu - r)/(\Gamma(r_2^*)\sigma^2),(\mu - r)/(\Gamma(r_1^*)\sigma^2))$ for different values of the investor's risk aversion and of the proportional transaction cost rates. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\beta = 0$ and $\rho = 0.01$.

[16] As in Table 1, we set $\beta = 0$ to allow direct comparison with the values in Table 1 of Grossman and Laroque (1990).

of transaction costs. This ratio would fluctuate between 1.210 and 1.223 with a transaction cost of 0.5% in either direction, and between 0.640 and 1.286 with a transaction cost of 25%.

Fig. 10 shows, for the logarithmic case ($\gamma = 1$) and for different levels of the transaction cost rates, how the fraction of wealth invested in stocks, θ^*/W^*, varies as a function of the fraction of wealth invested in the durable, K^*/W^*. While the relationship between θ^*/W^* and K^*/W^* is non-monotonic, an increase in the transaction cost rates seems to have the unambiguous result of reducing the fraction of wealth invested in stocks, for any given level of the investor's current consumption and wealth within the no-transaction region. The next proposition confirms that this is indeed the case.

Proposition 8. Let r_1^*, r_2^* satisfy the conditions of Theorem 3 and let $x \in (r_2^*, r_1^*)$. Then $\Gamma(x)$ increases as δ or ι increase, as long as x remains in the no-transaction region.

Proof. If $x \in (r_2^*, r_1^*)$, then

$$\Gamma(x) = -\frac{x\psi''(x)}{\psi'(x)} = \frac{x}{\varphi_{v^*}(x)},$$

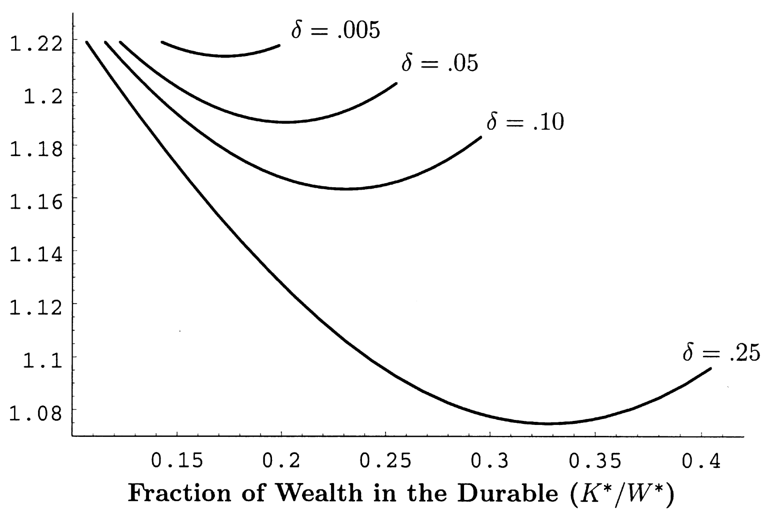

Fig. 10. Fraction of wealth invested in stocks as a function of the fraction of wealth invested in the durable. The graph plots θ^*/W^* against K^*/W^* for different values of δ. Each curve is plotted over the optimal range for K^*/W^*. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22, \iota = 0$, $\beta = 0.05$, $\rho = 0.01$ and $\gamma = 1$.

where v^* is the constant in (17) and (18). Since it follows immediately from the definition that $\varphi_{v^*}(x)$ is increasing in v^*, it is enough to show that $\partial v^*/\partial\delta < 0$ and $\partial v^*/\partial\imath < 0$.

Let

$$v_1(x) = \left[\frac{1}{\gamma}(x+\imath) - \alpha_1(x-1)\right]^{\beta_1}\left[\frac{1}{\gamma}(x+\imath) - \alpha_2(x-1)\right]^{\beta_2}.$$

Then $v^* = v_1(r_1^*)$ and $v_1'(x) < 0$ for $x > \gamma(r+\beta)(1+\imath)/\xi - \imath$. The fact that $\partial v^*/\partial\delta < 0$ then follows immediately from Propositions 1 and 2. The proof that $\partial v^*/\partial\imath < 0$ is similar. □

In the special case in which $\delta = 1$, we have $1 = r_2^* < r_1^*$ and $v^* = 0$, so that

$$\Gamma(x) = -\frac{x\psi''(x)}{\psi'(x)} = \frac{x}{\varphi_0(x)} = \frac{x}{\alpha_2(x-1)}$$

and

$$\frac{\theta_t^*}{W_t^*} = \alpha_2(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})\left(1 - \frac{K_t^*}{W_t^*}\right).$$

Hence, the optimal portfolio weights are a linearly decreasing function of the fraction of wealth invested in the durable. Alternatively,

$$\frac{\theta_t^*}{W_t^* - K_t^*} = \alpha_2(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1}),$$

so that the optimal portfolio policy involves investing a constant fraction of liquid wealth $W_t^* - K_t^*$ in stocks. Moreover, it can be shown that $\alpha_2 > 1/\gamma$ and $\alpha_2 \to 1/\gamma$ as $\beta \to \infty$. Both of these results are consistent with the findings of Hindy and Huang (1993), who considered irreversible purchases of the consumption good.

Fig. 11 plots the steady-state average fraction of wealth invested in stocks as a function of the transaction costs rate δ. Even though within the no-transaction region proportional investment in the stock can be higher or lower than in the Merton case, the average proportional investment is monotonically decreasing in the transaction cost rate δ, and thus always lower than in the Merton case.

6.5. Welfare impact of transaction costs

Fig. 12 plots the unconditional expected discounted values of the lifetime transaction costs, as a fraction of the initial stock of the durable, for different levels of the transaction cost rate δ. The expected values are computed under the steady-state distribution for x and the discount rate is set to 0.1. An increase in the transaction cost rate has a non-monotonic impact on expected lifetime costs, as beyond a certain level an increase in δ is more than compensated by

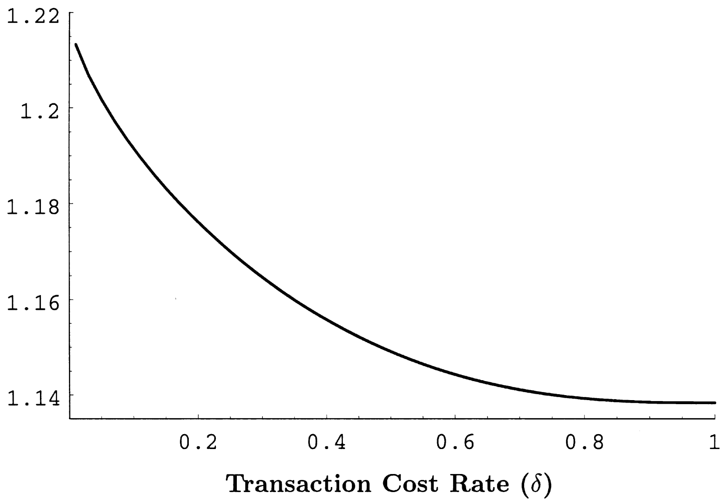

Fig. 11. Steady-state average fraction of wealth invested in stocks as a function of the transaction cost rate. The graph plots the steady-state average of θ^*/W^* against δ. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$,$\iota = 0$, $\beta = 0.05$, $\rho = 0.01$ and $\gamma = 1$.

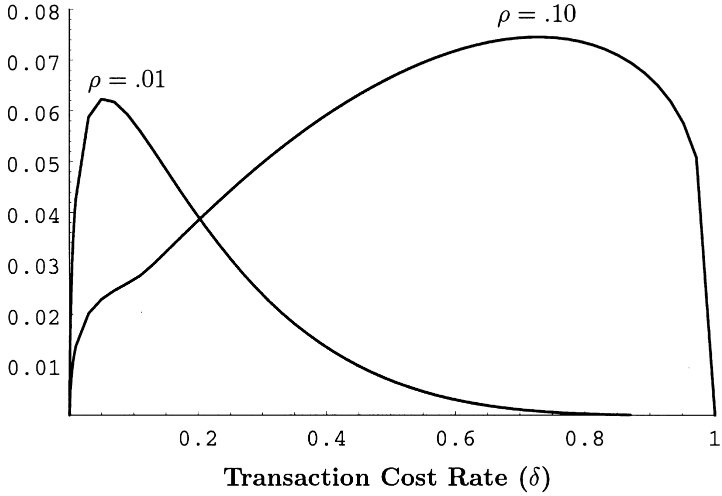

Fig. 12. Expected discounted lifetime transaction costs over existing stock of the durable as a function of the transaction cost rate. The graph plots the unconditional mean of $\delta g(x; 0.10, 0, 1)$ under the steady-state distribution of x as a function of δ for two different values of ρ. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\iota = 0$, $\beta = 0.05$ and $\gamma = 1$.

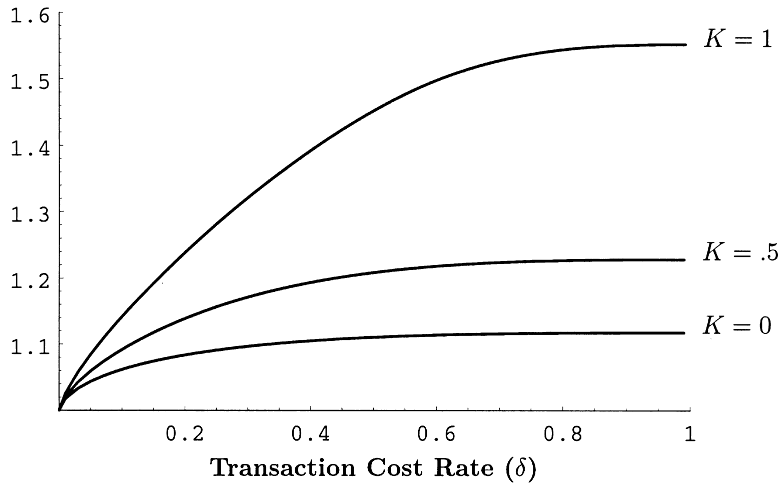

Fig. 13. Welfare impact of transaction costs. The graph plots the combinations of initial wealth and transaction cost rate δ that would give a logarithmic investor a constant lifetime expected utility. The other parameters are set as follows: $r = 0.01$, $\mu = 0.069$, $\sigma = 0.22$, $\beta = 0.05, \imath = 0$, $\rho = 0.01$ and $\gamma = 1$.

a corresponding decrease in the expected level of sales (as shown in Fig. 10). In fact, it follows from Proposition 4 that when $\delta = 1$ the lower boundary of the no-transaction region is never reached, and hence that expected sales and costs equal zero.

While Fig. 12 illustrates the direct cost associated with an illiquid market for the durable good, it does not capture the additional utility loss due to a suboptimal investment in the durable. In order to assess the welfare impact of transaction costs, Fig. 13 plots, for different levels of the initial holdings of the durable, the combinations of initial wealth and transaction cost rates that would give a logarithmic investor the same lifetime expected utility that he would be able to obtain with no transaction costs and unit wealth. For example, in the case in which transaction costs are only paid on sales of the durable, Fig. 13 shows that an investor who starts with all of his endowment in liquid securities, would be willing to give up about 8.3% of his wealth to avoid paying transaction costs if $\delta = 0.25$, about 10% if $\delta = 0.5$ and about 10.5% if $\delta = 1$. Additional increases in the transaction cost rates have a progressively smaller impact on the investor's welfare and can be compensated by progressively smaller increases in wealth. Moreover, since Fig. 13 is plotted for the case of no transaction costs on purchases, the welfare impact of transaction costs is higher the higher the initial fraction of wealth invested in the durable.

7. Conclusions and extensions

We have examined a continuous-time model in which an investor derives utility from the service flow provided by a durable consumption good. Adjustment of the stock of the durable is costly and entails a proportional transaction cost. Our analysis thus complements that of Grossman and Laroque (1990), who considered the case in which adjustment in the stock of durable involves payment of transaction cost proportional to the existing stock (rather than to the amount bought or sold). We show that an optimal consumption policy exists under the same set of conditions that are necessary and sufficient for existence in the absence of transaction costs. Moreover, we provide a closed-form expression for the value function in terms of three constants solving a system of nonlinear equations.

For the case of no-transaction costs, a change of variables reduces this problem to the one studied in Merton (1971). The optimal policies consist of maintaining a constant fraction of wealth invested in the durable and constant portfolio weights. In the presence of transaction costs, the optimal consumption policy consists of maintaining the fraction of total wealth invested in the durable good in a non-stochastic interval, which is easily computed. This interval may or may not include the ratio of durable to wealth that would be optimal in the no-transaction case. The optimal portfolio strategy involves investing in the same portfolio of risky assets that would be optimal in the absence of transaction costs, but the fraction of wealth allocated to risky assets is stochastic and depends on the current level of wealth relative to the stock of durable. Since the fraction of wealth invested in the durable is within a deterministic interval, the same is true for the fraction of wealth invested in stocks. We show that this interval always brackets the proportion that would be optimal in the absence of transaction costs. Moreover, numerical simulation reveals that this interval is typically small, so that the optimal investment strategy is not very sensitive to the presence of transaction costs for adjusting durable consumption. We also provide an explicit solution for the case in which the transaction cost rate for selling the durable is 100%. Clearly, the optimal consumption policy in this case involves never selling the durable.

Since the investor's optimal consumption policy does not satisfy the usual first-order condition due to the presence of transaction costs, the Consumption-based Capital Asset Pricing Model (CCAPM) would not hold in equilibrium in the economy we study. On the other hand, since investors still hold the same portfolio of risky assets (the mean-variance efficient portfolio) the standard Capital Asset Pricing Model (CAPM) would hold in equilibrium. This is analogous to what Grossman and Laroque (1990) reported for the case of an indivisible durable good.

Finally, we point out that it is easy to extend our analysis to the case in which the investor derives utility from both a durable good and a perishable consumption good, as long as the utility function is additive and the relative price of the two goods is constant. The value function for the extended problem is given by

$$v(W_0, K_0) = \max_{W_{10}+W_{20}=W_0} v_1(W_{10}, K_0) + v_2(W_{20}),$$

where v_1 is the value function for the problem with only the durable good (as studied in this paper) and v_2 is the value function for the problem with only the perishable good (as in Merton, 1971). The optimal consumption and investment policies can also be immediately retrieved. Clearly, the CAPM would still characterize the equilibrium in this economy, while the CCAPM would hold relative to aggregate nondurable consumption, but not relative to aggregate total consumption.

8. For further reading

The following references are also of interest to the reader: Cvitanić and Karatzas, 1996; Harrison, 1985.

Appendix A

Before embarking on the proof of Theorem 2, whose argument is adapted from Davis and Norman (1990), we start with a preliminary result.

Lemma A.1. Under the assumptions of Theorem 2, *the function v defined in* (13) *is concave and satisfies*:

$$\max_{\theta}\left[\frac{1}{2}|\theta^{\mathrm{T}}\sigma|^2 v_{ww} + [r(w-k) + \theta^{\mathrm{T}}(\mu - r\mathbf{1}) - \beta k]v_w - \beta k v_k - \rho v + \frac{(\alpha k)^{1-\gamma}}{1-\gamma}\right] \leq 0 \tag{A.1}$$

on $\mathscr{S}$, with equality on NT. Moreover,

$$\iota v_w - v_k \geq 0 \text{ on } \mathscr{S}, \text{ with equality on } B \tag{A.2}$$

and

$$\delta v_w + v_k \geq 0 \text{ on } \mathscr{S}, \text{ with equality on } S. \tag{A.3}$$

Proof. Recalling the definition of v and letting $x = w/k$, we have

$$v_{ww}(w,k) = k^{-(1+\gamma)}\psi''(x) < 0$$

and

$$v_{ww}(w,k)v_{kk}(w,k) - v_{wk}(w,k)^2$$
$$= -\gamma k^{-2(1+\gamma)}((1-\gamma)\psi(x)\psi''(x) + \gamma\psi'(x)^2) \geq 0,$$

where the first inequality follows from the strict concavity of ψ and the second inequality follows from (11). This establishes the concavity of v. On the other hand,

$$\imath v_w(w,k) - v_k(w,k) = k^{-\gamma}((x+\imath)\psi'(x) - (1-\gamma)\psi(x)) = k^{-\gamma}\psi'(x)g(x),$$

where

$$g(x) = x + \imath - (1-\gamma)\frac{\psi(x)}{\psi'(x)}.$$

By (8), $g(x) = 0$ for $x > r_1^*$, and, by (11),

$$g'(x) = \gamma + (1-\gamma)\frac{\psi(x)\psi''(x)}{\psi'(x)^2} \leq 0.$$

This establishes (A.2). The proof of (A.3) is similar, while (A.1) follows immediately from (9) and (10). □

Proof of Theorem 2. Since

$$\theta^*(w,k) = -(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})\frac{\psi'(w/k)}{(w/k)\psi''(w/k)}w$$

and the function $\psi'(x)/(x\psi''(x))$ is continuous and hence bounded on $[r_2^*, r_1^*]$, we conclude that $|\theta^*(w,k)| \leq \eta w$ holds for some $\eta < \infty$ and all $(w,k) \in NT$. Also, since the definition of θ^* implies that $\theta^*(\lambda w, \lambda k) = \lambda\theta^*(w,k)$ for all $\lambda > 0$, θ^* is Lipschitz continuous on NT. The existence and uniqueness of processes (W^*, K^*, I^*, D^*) satisfying the conditions of the theorem for all $t < \tau = \inf\{t \geq 0\colon W_t^* = K_t^* = 0\}$ then follows from the construction of diffusions with oblique reflections in Lions and Sznitman (1984) or Dupuis and Ishii (1993) (see the proof of Lemma 9.3 in Shreve and Soner (1994) for details).

We will start by showing that $\tau = \infty$ a.s., so that $(I^*, D^*, \theta^*) \in \hat{\Theta}(W_0, K_0)$. Recalling the definition of v, an application of Itô's lemma shows that

$$\mathrm{e}^{-\rho t}v(W_t^*, K_t^*) = v(W_0, K_0)\exp\left(-\int_0^t \frac{\alpha^{1-\gamma}}{(1-\gamma)\psi(W_s^*/K_s^*)}\mathrm{d}s\right)N_t \tag{A.4}$$

for all $t < \tau$, where

$$N_t = \exp\left(-\int_0^t \frac{(\psi'(W_s^*/K_s^*))^2}{\psi(W_s^*/K_s^*)\psi''(W_s^*/K_s^*)}(\mu - r\mathbf{1})^{\mathrm{T}}(\sigma\sigma^{\mathrm{T}})^{-1}\sigma\,\mathrm{d}w_s\right.$$
$$\left. - \frac{1}{2}\int_0^t \left|\frac{(\psi'(W_s^*/K_s^*))^2}{\psi(W_s^*/K_s^*)\psi''(W_s^*/K_s^*)}(\mu - r\mathbf{1})^{\mathrm{T}}(\sigma\sigma^{\mathrm{T}})^{-1}\sigma\right|^2 \mathrm{d}s\right). \tag{A.5}$$

Since the function $(\psi')^2/(\psi\,\psi'')$ is continuous, and hence bounded, on $[r_2^*, r_1^*]$ and $(1-\gamma)\psi$ is bounded below away from zero (because of (11)), the above implies that

$$0 < \lim_{t \uparrow \tau} |v(W_t, K_t)| < \infty \quad \text{on } \{\tau < \infty\}.$$

On the other hand, (8) implies that

$$\lim_{t \uparrow \tau} v(W_t, K_t) = \begin{cases} 0 & \text{if } \gamma < 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Thus $\tau = \infty$, almost surely. Next, let

$$M_t = \int_0^t e^{-\rho s} \frac{(\alpha K_s^*)^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} v(W_t^*, K_t^*).$$

An application of Itô's lemma shows that

$$\begin{aligned} M(t) = v(W_0, K_0) &+ \int_0^t e^{-\rho s}(\tfrac{1}{2}|\theta^*(W_s^*, K_s^*)^{\mathrm{T}}\sigma|^2 v_{ww}(W_s^*, K_s^*) \\ &+ [r(W_s^* - K_s^*) + \theta^*(W_s^*, K_s^*)^{\mathrm{T}}(\mu - r\mathbf{1}) - \beta K_s^*] v_w(W_s^*, K_s^*) \\ &- \beta K_s^* v_k(W_s^*, K_s^*) - \rho v(W_s^*, K_s^*) + u(\alpha K_s^*)) ds \\ &- \int_0^t e^{-\rho s}[\iota v_w(W_s^*, K_s^*) - v_k(W_s^*, K_s^*)]\, dI_s^* \\ &- \int_0^t e^{-\rho s}[\delta v_w(W_s^*, K_s^*) + v_k(W_s^*, K_s^*)]\, dD_s^* \\ &+ \int_0^t e^{-\rho s} v_w(W_s^*, K_s^*)\theta^*(W_s^*, K_s^*)^{\mathrm{T}}\sigma\, dw_s \end{aligned} \tag{A.6}$$

It then follows from Lemma A.1 that the first three integrals in the previous expression are identically zero. Turning next to the stochastic integral, we have from the continuity of $x^\gamma \psi'(x)$ that

$$|v_w(w,k)\theta^*(w,k)| = w^{1-\gamma}|(w/k)^\gamma \psi'(w/k)|\,|\theta^*(w,k)/w| \le \eta w^{1-\gamma}$$

for some $\eta < \infty$ and all $(w,k) \in NT$. Also, we have from (3) that

$$0 \le W_t \le W_0 \exp\left(\int_0^t \left[r + \pi_s^{\mathrm{T}}(\mu - r\mathbf{1}) - \frac{1}{2}|\pi_s^{\mathrm{T}}\sigma|^2 \right] ds + \int_0^t \pi_s^{\mathrm{T}}\sigma\, dw_s \right) \tag{A.7}$$

for any $(I, D, \theta) \in \Theta(W_0, K_0)$, where $\pi = \theta/W$ denotes the vector of portfolio weights. It then follows from the fact that π is uniformly bounded for

$(I, D, \theta) \in \hat{\Theta}(W_0, K_0)$ that the stochastic integral has zero expectation. Therefore,

$$\begin{aligned} v(W_0, K_0) &= \lim_{t \uparrow \infty} \mathrm{E}[M_t] \\ &= \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} \frac{(\alpha K_t^*)^{1-\gamma}}{1-\gamma} \mathrm{d}t\right] + \lim_{t \uparrow \infty} \mathrm{E}[\mathrm{e}^{-\rho t} v(W_t^*, K_t^*)] \\ &= \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} \frac{(\alpha K_t^*)^{1-\gamma}}{1-\gamma} \mathrm{d}t\right], \end{aligned}$$

where the second equality follows from the monotone convergence theorem and the third from (A.4), using the fact that $1/[(1-\gamma)\psi]$ is bounded below away from zero on NT and that the process N in (A.5) is a martingale. Therefore, v is indeed the lifetime expected utility from following the proposed optimal policy.

To conclude the proof, we only need to show that

$$v(W_0, K_0) \geq \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} \frac{(\alpha K_t)^{1-\gamma}}{1-\gamma} \mathrm{d}t\right] \tag{A.8}$$

for any investment policy $(I, D, \theta) \in \hat{\Theta}(W_0, K_0)$. Accordingly, fix from now on an arbitrary $(I, D, \theta) \in \hat{\Theta}(W_0, K_0)$ and let (W, K) denote the corresponding wealth and durable-holding processes.

Suppose at first that $\gamma < 1$. It then follows easily from the definitions that there exists a constant $\eta^\varepsilon > 0$ such that

$$|v(w,k)| + |w v_w(w,k)| \leq \eta^\varepsilon w^{1-\gamma} \quad \text{for all } (w,k) \in \mathscr{S} \text{ with } w - \delta k > \varepsilon > 0. \tag{A.9}$$

Let $v^\varepsilon(w, k) = v(w + \varepsilon, k)$ for $\varepsilon > 0$, and consider the process

$$M_t^\varepsilon = \int_0^t \mathrm{e}^{-\rho s} \frac{(\alpha K_s)^{1-\gamma}}{1-\gamma} \mathrm{d}s + \mathrm{e}^{-\rho t} v^\varepsilon(W_t, K_t). \tag{A.10}$$

It then follows from the generalized Itô's lemma that

$$\begin{aligned} M^\varepsilon(t) = v^\varepsilon(W_0, K_0) &+ \int_0^t \mathrm{e}^{-\rho s}(\tfrac{1}{2}|\theta_s^{\mathrm{T}}\sigma|^2 v_{ww}^\varepsilon(W_s, K_s) \\ &+ [r(W_s - K_s) + \theta_s^{\mathrm{T}}(\mu - r\mathbf{1}) - \beta K_s] v_w^\varepsilon(W_s, K_s) \\ &- \beta K_s v_k^\varepsilon(W_s, K_s) - \rho v^\varepsilon(W_s, K_s) + u(\alpha K_s))\mathrm{d}s \\ &- \int_0^t \mathrm{e}^{-\rho s}[\imath v_w^\varepsilon(W_s, K_s) - v_k^\varepsilon(W_s, K_s)]\, \mathrm{d}I(s) \\ &- \int_0^t \mathrm{e}^{-\rho s}[\delta v_w^\varepsilon(W_s, K_s) + v_k^\varepsilon(W_s, K_s)]\, \mathrm{d}D(s) \end{aligned}$$

$$+\int_0^t \mathrm{e}^{-\rho s} v_w^\varepsilon(W_s, K_s)\theta_s^{\mathrm{T}}\sigma\,\mathrm{d}w_s + \sum_{0\le s\le t} \mathrm{e}^{-\rho t}[\Delta v^\varepsilon(W_s, K_s)$$

$$- v_w^\varepsilon(W_{s-}, K_{s-})\Delta W_s - v_k^\varepsilon(W_{s-}, K_{s-})\Delta K_s]. \tag{A.11}$$

Since

$$\tfrac{1}{2}|\theta^{\mathrm{T}}\sigma|^2 v_{ww}^\varepsilon(w,k) + [r(w-k) + \theta^{\mathrm{T}}(\mu - r\mathbf{1}) - \beta k]v_w^\varepsilon(w,k)$$
$$- \beta k v_k^\varepsilon(w,k) - \rho v^\varepsilon(w,k) + u(\alpha k)$$
$$= \tfrac{1}{2}|\theta^{\mathrm{T}}\sigma|^2 v_{ww}(w+\varepsilon, k)$$
$$+ [r(w+\varepsilon-k) + \theta^{\mathrm{T}}(\mu - r\mathbf{1}) - \beta k]v_w(w+\varepsilon, k)$$
$$- \beta k v_k(w+\varepsilon,k) - \rho v(w+\varepsilon,k) + u(\alpha k) - r\varepsilon v_w(w+\varepsilon,k),$$

it follows from Lemma A.1 and the fact that $v_w > 0$ that the first three integrals in the above expression are nonpositive, while the term in the summation is nonpositive by the concavity of v^ε. We then conclude from (A.9) that M^ε is a supermartingale. Hence,

$$v^\varepsilon(W_0, K_0) \ge \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\rho t}\frac{(\alpha K_t)^{1-\gamma}}{1-\gamma}\,\mathrm{d}t\right] + \lim_{t\uparrow\infty}\mathrm{E}[\mathrm{e}^{-\rho t}v^\varepsilon(W_t, K_t)]$$
$$= \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\rho t}\frac{(\alpha K_t)^{1-\gamma}}{1-\gamma}\,\mathrm{d}t\right].$$

Since $v^\varepsilon \downarrow v$ as $\varepsilon\downarrow 0$, we obtain the desired inequality.

Finally, suppose that $\gamma > 1$. Fix an arbitrary λ with $0 < \lambda < r/(r + (1+\imath)\beta)$ and for any $\varepsilon > 0$ let $v^\varepsilon(w,k) = v(w+\varepsilon, k+\lambda\varepsilon)$. It can be immediately verified from the definitions that v^ε and v_w^ε are bounded on $\mathscr{S}$. Moreover,

$$\tfrac{1}{2}|\theta^{\mathrm{T}}\sigma|^2 v_{ww}^\varepsilon(w,k) + [r(w-k) + \theta^{\mathrm{T}}(\mu - r\mathbf{1}) - \beta k]v_w^\varepsilon(w,k)$$
$$- \beta k v_k^\varepsilon(w,k) - \rho v^\varepsilon(w,k) + u(\alpha k)$$
$$= \tfrac{1}{2}|\theta^{\mathrm{T}}\sigma|^2 v_{ww}(w+\varepsilon, k+\lambda\varepsilon)$$
$$+ [r(w+\varepsilon-(k+\lambda\varepsilon)) + \theta^{\mathrm{T}}(\mu - r\mathbf{1}) - \beta(k+\lambda\varepsilon)]v_w(w+\varepsilon, k+\lambda\varepsilon)$$
$$- \beta(k+\lambda\varepsilon)v_k(w+\varepsilon, k+\lambda\varepsilon) - \rho v(w+\varepsilon, k+\lambda\varepsilon) + u(\alpha(k+\lambda\varepsilon))$$
$$- \varepsilon[r - \lambda(r+\beta)]v_w(w+\varepsilon, k+\lambda\varepsilon) + \beta\lambda\varepsilon v_k(w+\varepsilon, k+\lambda\varepsilon)$$
$$- [u(\alpha(k+\lambda\varepsilon)) - u(\alpha k)]$$
$$\le -\varepsilon[r - \lambda(r + (1+\imath)\beta)]v_w(w+\varepsilon, k+\lambda\varepsilon)$$
$$- [u(\alpha(k+\lambda\varepsilon)) - u(\alpha k)] < 0,$$

where the first inequality follows from (A.1) and(A.2). We then have from (A.11) that the process M^ε in (A.10) is a supermartingale. Hence,

$$v^{\varepsilon}(W_0, K_0) \geq \mathrm{E}\left[\int_0^{\infty} \mathrm{e}^{-\rho t}\frac{(\alpha K_t)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] + \lim_{t\uparrow\infty} \mathrm{E}[\mathrm{e}^{-\rho t}v^{\varepsilon}(W_t, K_t)]$$
$$= \mathrm{E}\left[\int_0^{\infty} \mathrm{e}^{-\rho t}\frac{(\alpha K_t)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right],$$

where the last equality follows from the boundedness of v^{ε}. Since $v^{\varepsilon} \to v$ as $\varepsilon\downarrow 0$, we conclude that the policy (I^*, D^*, θ^*) is optimal for all $\gamma \neq 1$. □

Proof of Lemma 1. It is easy to verify that any solution of the ODE (14) satisfies

$$|\varphi_v(x) - \alpha_1(x-1)|^{\beta_1}|\varphi_v(x) - \alpha_2(x-1)|^{\beta_2} = v$$

for some v and that any nonnegative solution has one of three possible shapes on the positive orthant:

1. $0 \leq \varphi_v(x) \leq \alpha_1(x-1)$, $\varphi_v'(x) < 0$ and $\varphi_v''(x) \leq 0$, defined for x in a subset of $[0,1]$,
2. $0 \leq \varphi_v(x) \leq \alpha_2(x-1)$, $\varphi_v'(x) > 0$ and $\varphi_v''(x) \leq 0$, defined for x in a subset of $[1,\infty)$,
3. $\varphi_v(x) \geq \max[\alpha_1(x-1), \alpha_2(x-1)]$, $\varphi_v''(x) \geq 0$, defined for $x \in [0,\infty)$

(see Lemma 3 in Grossman and Laroque (1987) for details). We can rule out the first two solutions as follows. Let

$$f(x) = \frac{\psi'(x)^2}{\psi''(x)} + \frac{1-\gamma}{\gamma}\psi(x). \tag{A.12}$$

Then (11) implies $f(x) \geq 0$, while (8) and the continuity of ψ give $f(r_2^*) = f(r_1^*) = 0$. Hence, $f'(r_2^*) \geq 0$ and $f'(r_1^*) \leq 0$. Since

$$f'(x) = \psi'(x)\left(\frac{1}{\gamma} - \varphi_v'(x)\right), \tag{A.13}$$

we cannot have $\varphi_v''(x) \leq 0$ on (r_2^*, r_1^*), unless $\varphi_v''(x) = 0$ and $\varphi_v'(x) = 1/\gamma$ for all x. This rules out the first two solutions. Thus, $\varphi_v(x) \geq \max[\alpha_1(x-1), \alpha_2(x-1)]$. □

Proof of Proposition 1. Let

$$v_1(x) = \left[\frac{1}{\gamma}(x+\imath) - \alpha_1(x-1)\right]^{\beta_1}\left[\frac{1}{\gamma}(x+\imath) - \alpha_2(x-1)\right]^{\beta_2} \tag{A.14}$$

and

$$v_2(x) = \left[\frac{1}{\gamma}(x-\delta) - \alpha_1(x-1)\right]^{\beta_1}\left[\frac{1}{\gamma}(x-\delta) - \alpha_2(x-1)\right]^{\beta_2}, \tag{A.15}$$

which are defined for $x \in [\underline{r}_1, \bar{r}_1]$ and $x \in [\underline{r}_2, \bar{r}_2]$, respectively, where $\underline{r}_1 = 1 - (1+\imath)/(1-\gamma\alpha_1)$, $\bar{r}_1 = 1-(1+\imath)/(1-\gamma\alpha_2)$, $\underline{r}_2 = 1-(1-\delta)/(1-\gamma\alpha_1)$ and $\bar{r}_2 = 1-(1-\delta)/(1-\gamma\alpha_2)$. Eqs. (17) and (18) then amount to $v_1(r_1^*) = v_2(r_2^*) = v^* > 0$. Since

$$v_1(\underline{r}_1) = v_1(\bar{r}_1) = v_2(\underline{r}_2) = v_2(\bar{r}_2) = 0, \tag{A.16}$$

the above implies $r_1^* \le \bar{r}_1$ and $r_2^* \ge \underline{r}_2$. Letting

$$\hat{r}_1 = \frac{\gamma(r+\beta)(1+\imath)}{\xi} - \imath$$

and

$$\hat{r}_2 = \frac{\gamma(r+\beta)(1-\delta)}{\xi} + \delta,$$

we are then left to show that $r_1^* > \hat{r}_1$, $r_2^* \le \hat{r}_2$ and $\eta > 0$. It can be easily verified that

$$\underline{r}_1 < \underline{r}_2 < \bar{r}_2 < \bar{r}_1$$

and

$$v_1(x) > v_2(x) \quad \text{for } x \in [\underline{r}_2, \bar{r}_2].$$

Moreover, we have $v_1'(x) > 0$ for $x < \hat{r}_1$ and $v_1'(x) < 0$ for $x > \hat{r}_1$. Recalling (A.16), this implies that, for any given $r_2 \in [\underline{r}_2, \bar{r}_2]$, the equation $v_1(r_1) = v_2(r_2)$ has two different solutions: the first with $r_1 < \min[\hat{r}_1, r_2]$, and the second with $r_1 > \max[\hat{r}_1, r_2]$. Since by assumption $r_1^* > r_2^*$, we must also have $r_1^* > \hat{r}_1$. The latter inequality implies $\eta > 0$.

Finally, letting f be the function in (A.12), (20) and the continuity of ψ imply that $f(r_2^*) = f(r_1^*) = 0$. Since φ_{v^*} is convex, (A.13) then implies that $f'(r_2^*) \ge 0$, or

$$\gamma\varphi_{v^*}'(r_2^*) \le 1.$$

Since φ_{v^*} solves (14), we have

$$\varphi_{v^*}'(x) = \frac{(\rho - r - \gamma\beta + \kappa)\varphi_{v^*}(x) + (r+\beta)(x-1)}{\kappa\varphi_{v^*}(x)}.$$

Moreover, (15) and (18) imply

$$\varphi_{v^*}(r_2^*) = (r_2^* - \delta)/\gamma.$$

Thus,

$$\frac{(\rho - r - \gamma\beta + \kappa)(r_2^* - \delta) + \gamma(r+\beta)(r_2^* - 1)}{\kappa/\gamma\,(r_2^* - \delta)} \le 1.$$

Rearranging the latter inequality gives $r_2^* \le \hat{r}_2$. □

Proof of Theorem 3. The fact that ψ is twice continuously differentiable and satisfies ODE (9) can be easily verified. The fact that ψ is increasing and strictly concave follows from the fact that the constants A,B and C_2 are all strictly positive (because $\eta > 0$).

Next, letting $f(x)$ be the function in (A.12), it follows from the definition of ψ that $f(x) = 0$ for $x \in (\delta, r_2^*] \cup [r_1^*, \infty)$. Eq. (A.13), the convexity of φ_{v^*} and the continuity of f then imply $f(x) \geq 0$ on (r_2^*, r_1^*), and hence condition (11) of Theorem 2 is satisfied.

Also, letting

$$g(x) = -\kappa \frac{\psi'(x)^2}{\psi''(x)} + (r+\beta)(x-1)\psi'(x) - (\rho + (1-\gamma)\beta)\psi(x) + \frac{\alpha^{1-\gamma}}{1-\gamma},$$

Eq. (9) implies that $g(x) = 0$ for $x \in [r_2^*, r_1^*]$. Since

$$\frac{\psi'(x)^2}{\psi''(x)} = -\frac{1-\gamma}{\gamma}\psi(x) \quad \text{for } x \in (\delta, r_2^*] \cup [r_1^*, \infty),$$

we have

$$g(x) = (r+\beta)(x-1)\psi'(x) - (\rho + (1-\gamma)(\beta - \kappa/\gamma))\psi(x) + \frac{\alpha^{1-\gamma}}{1-\gamma}$$

$$\text{for } x \in (\delta, r_2^*] \cup [r_1^*, \infty).$$

Thus, for $x > r_1^*$,

$$g'(x) = (r+\beta)(x-1)\psi''(x) - (\rho - r - \gamma\beta - (1-\gamma)\kappa/\gamma)\psi'(x)$$

$$= \frac{\psi''(x)}{\gamma}(\xi(x+\iota) - \gamma(r+\beta)(1+\iota)) < 0,$$

where the inequality follows from the concavity of ψ and the fact that $x > r_1^* > \gamma(r+\beta)(1+\iota)/\xi - \iota$ (by (22)). Hence, $g(x) < 0$ for $x > r_1^*$. The proof that $g(x) < 0$ for $\delta < x < r_2^*$ is similar, using the fact that $r_2^* \leq \gamma(r+\beta)(1-\delta)/\xi + \delta$ (by (23)).

Finally, the fact that $r_2^* > \delta$ follows from the fact that $r_2^* \geq 1 - (1-\delta)/(1-\gamma\alpha_1)$ (by (23)) and $\alpha_1 < 0$. □

Proof of Theorem 4. Let $v_1, v_2, \underline{r}_1, \bar{r}_1, \underline{r}_2, \bar{r}_2, \hat{r}_1$ and $\hat{r}_2$ be as in the proof of Proposition 1. For $r_2 \in [\underline{r}_2, \hat{r}_2]$, let $r_1(r_2)$ denote the unique solution with $r_1 > r_2$ of the equation $v_1(r_1) = v_2(r_2)$. Since $v_2(x) > 0$ for $x \in (\underline{r}_2, \hat{r}_2]$, the existence of constants $r_1^* > r_2^*$ and $v^* > 0$ satisfying (17)–(19) amounts to the existence of a $r_2^* \in (\underline{r}_2, \hat{r}_2]$ such that $h(r_2^*) = 0$, where

$$h(x) = x - \delta + (1-\gamma)\int_x^{r_1(x)} \exp\left(-\int_x^y \frac{dz}{\varphi_{v_2(x)}(z)}\right) dy$$

$$- (r_1(x) + \imath)\exp\left(- \int_x^{r_1(x)} \frac{\mathrm{d}z}{\varphi_{v_2(x)}(z)}\right). \tag{A.17}$$

We will show below that $h(\underline{r}_2) > 0 > h(\hat{r}_2)$. The claim then follows from the continuity of h on $(\underline{r}_2, \hat{r}_2)$. Since $v_2(\underline{r}_2) = 0$, it follows from (15) that

$$\varphi_{v_2(\underline{r}_2)}(x) = \begin{cases} \alpha_1(x-1) & \text{if } x \leq 1, \\ \alpha_2(x-1) & \text{if } x \geq 1. \end{cases}$$

Since $\underline{r}_2 < 1$ and $1/\alpha_2 > 0 > 1/\alpha_1$, we then have, for all $y > 1$,

$$\begin{aligned}
&\exp\left(- \int_{\underline{r}_2}^{y} \frac{\mathrm{d}z}{\varphi_{v_2(\underline{r}_2)}(z)}\right) \\
&= \lim_{\varepsilon \downarrow 0} \left[\exp\left(- \int_{\underline{r}_2}^{1-\varepsilon} \frac{\mathrm{d}z}{\alpha_1(z-1)} - \int_{1+\varepsilon}^{y} \frac{\mathrm{d}z}{\alpha_2(z-1)}\right)\right] \\
&= \lim_{\epsilon \downarrow 0} \left[\exp\left(\left(\frac{1}{\alpha_2} - \frac{1}{\alpha_1}\right)\log(\varepsilon) + \frac{\log(1-\underline{r}_2)}{\alpha_1} - \frac{\log(y-1)}{\alpha_2}\right)\right] \\
&= 0
\end{aligned}$$

and hence (since $r_1(\underline{r}_2) = \bar{r}_1 > 1$)

$$\begin{aligned}
h(\underline{r}_2) &= \underline{r}_2 - \delta + (1-\gamma)\int_{\underline{r}_2}^{\bar{r}_1} \exp\left(- \int_{\underline{r}_2}^{y} \frac{\mathrm{d}z}{\varphi_{v_2(\underline{r}_2)}(z)}\right)\mathrm{d}y \\
&= \underline{r}_2 - \delta + (1-\gamma)\int_{\underline{r}_2}^{1} \exp\left(- \int_{\underline{r}_2}^{y} \frac{\mathrm{d}z}{\alpha_1(z-1)}\right)\mathrm{d}y \\
&= \underline{r}_2 - \delta + (1-\gamma)\frac{1-\underline{r}_2}{1-\dfrac{1}{\alpha_1}} = \frac{1-\delta}{1-\dfrac{1}{\alpha_1}} > 0.
\end{aligned}$$

Next, it follows immediately from the definitions of v_2 and φ_v that

$$\varphi_{v_2(\hat{r}_2)}(\hat{r}_2) = (\hat{r}_2 - \delta)/\gamma.$$

Also, since $\varphi_{v_2(\hat{r}_2)}$ solves ODE (14), we have

$$\varphi'_{v_2(\hat{r}_2)}(\hat{r}_2) = \frac{\rho - r - \gamma\beta + \kappa}{\kappa} + \frac{(r+\beta)(\hat{r}_2 - 1)}{\kappa\varphi_{v_2(\hat{r}_2)}(\hat{r}_2)} = \frac{1}{\gamma}.$$

It then follows from the convexity of $\varphi_{v_2(\hat{r}_2)}$ that

$$\varphi_{v_2(\hat{r}_2)}(x) > \frac{x-\delta}{\gamma}$$

for all $x > \hat{r}_2$. Now, suppose first that $\gamma > 1$. Then the above implies

$$\begin{aligned} h(\hat{r}_2) &< \hat{r}_2 - \delta + (1-\gamma)\int_{\hat{r}_2}^{r_1(\hat{r}_2)} \exp\left(- \int_{\hat{r}_2}^{y} \frac{\gamma\,\mathrm{d}z}{z-\delta}\right)\mathrm{d}y \\ &\quad - (r_1(\hat{r}_2) + \imath)\exp\left(- \int_{\hat{r}_2}^{r_1(\hat{r}_2)} \frac{\gamma\,\mathrm{d}z}{z-\delta}\right) \\ &= \hat{r}_2 - \delta + (1-\gamma)\int_{\hat{r}_2}^{r_1(\hat{r}_2)} \left(\frac{\hat{r}_2-\delta}{y-\delta}\right)^{\gamma}\mathrm{d}y - (r_1(\hat{r}_2)+\imath)\left(\frac{\hat{r}_2-\delta}{r_1(\hat{r}_2)-\delta}\right)^{\gamma} \\ &= -(\delta+\imath)\left(\frac{\hat{r}_2-\delta}{r_1(\hat{r}_2)-\delta}\right)^{\gamma} < 0. \end{aligned}$$

On the other hand, if $\gamma < 1$,

$$\begin{aligned} h(\hat{r}_2) &= \exp\left(- \int_{\hat{r}_2}^{r_1(\hat{r}_2)} \frac{\mathrm{d}z}{\varphi_{v_2(\hat{r}_2)}(z)}\right)\Bigg[(\hat{r}_2-\delta)\exp\left(\int_{\hat{r}_2}^{r_1(\hat{r}_2)} \frac{\mathrm{d}z}{\varphi_{v_2(\hat{r}_2)}(z)}\right) \\ &\quad + (1-\gamma)\int_{\hat{r}_2}^{r_1(\hat{r}_2)} \exp\left(\int_{y}^{r_1(\hat{r}_2)} \frac{\mathrm{d}z}{\varphi_{v_2(\hat{r}_2)}(z)}\right)\mathrm{d}y - (r_1(\hat{r}_2)+\imath)\Bigg] \\ &< \exp\left(- \int_{\hat{r}_2}^{r_1(\hat{r}_2)} \frac{\mathrm{d}z}{\varphi_{v_2(\hat{r}_2)}(z)}\right)\Bigg[(\hat{r}_2-\delta)\exp\left(\int_{\hat{r}_2}^{r_1(\hat{r}_2)} \frac{\gamma\,\mathrm{d}z}{z-\delta}\right) \\ &\quad + (1-\gamma)\int_{\hat{r}_2}^{r_1(\hat{r}_2)} \exp\left(\int_{y}^{r_1(\hat{r}_2)} \frac{\gamma \mathrm{d}z}{z-\delta}\right)\mathrm{d}y - (r_1(\hat{r}_2)+\imath)\Bigg] \\ &= \exp\left(- \int_{\hat{r}_2}^{r_1(\hat{r}_2)} \frac{\mathrm{d}z}{\varphi_{v_2(\hat{r}_2)}(z)}\right)\Bigg[(\hat{r}_2-\delta)\left(\frac{r_1(\hat{r}_2)-\delta}{\hat{r}_2-\delta}\right)^{\gamma} \\ &\quad + (1-\gamma)\int_{\hat{r}_2}^{r_1(\hat{r}_2)} \left(\frac{r_1(\hat{r}_2)-\delta}{y-\delta}\right)^{\gamma}\mathrm{d}y - (r_1(\hat{r}_2)+\imath)\Bigg] \\ &= -(\delta+\imath)\exp\left(- \int_{\hat{r}_2}^{r_1(\hat{r}_2)} \frac{\mathrm{d}z}{\varphi_{v_2(\hat{r}_2)}(z)}\right) < 0. \qquad \square \end{aligned}$$

Appendix B

We collect in this appendix the results for the logarithmic case ($\gamma = 1$). Since the derivation is similar to that for the power case ($\gamma \neq 1$), all the results are stated without proof.

First, if $\delta = \imath = 0$, the optimal policies can be obtained directly from the analysis in Merton (1971) and are given by

$$K_t^* = \frac{1}{r^*}W_t^*$$

and

$$\theta_t^* = (\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})W_t^*,$$

where

$$r^* = (r + \beta)/\rho.$$

The lifetime expected utility is

$$v(W_0) = \frac{r - \rho + \kappa}{\rho^2} + \frac{1}{\rho}\log\left(\frac{\alpha W_0}{r^*}\right).$$

Next, if $\delta + \iota > 0$ and $\delta < 1$, the lifetime expected utility is given by

$$v(W_0, K_0) = \frac{1}{\rho}\log(K_0) + \psi\left(\frac{W_0}{K_0}\right),$$

where

$$\psi(x) = \begin{cases} A + \dfrac{1}{\rho}\log(x - \delta) & \text{if } \delta < x < r_2^*, \\ C_1 - C_2\int_x^{r_1^*}\exp\left(\int_y^{r_1^*}\dfrac{\mathrm{d}z}{\varphi_{v^*}(z)}\right)\mathrm{d}y & \text{if } r_2^* \le x \le r_1^*, \\ B + \dfrac{1}{\rho}\log(x + \iota) & \text{if } x > r_1^*, \end{cases}$$

and

$$C_1 = \frac{\log(\alpha)}{\rho} + \frac{\kappa + r}{\rho^2} - \frac{(r + \beta)(1 + \iota)}{\rho^2(r_1^* + \iota)},$$

$$C_2 = \frac{1}{\rho(r_1^* + \iota)},$$

$$A = C_1 - \frac{\log(r_2^* - \delta)}{\rho} - \frac{\int_{r_2^*}^{r_1^*}\exp\left(\int_y^{r_1^*}\dfrac{\mathrm{d}z}{\varphi_{v^*}(z)}\right)\mathrm{d}y}{\rho(r_1^* + \iota)},$$

$$B = C_1 - \frac{\log(r_1^* + \iota)}{\rho},$$

and r_1^*, r_2^*, v^* solve (17)–(19) with $\gamma = 1$. The optimal policies are as in Theorem 2.

Finally, if $\delta = 1$, the lifetime expected utility is given by

$$v(W_0, K_0) = \frac{1}{\rho}\log(K_0) + \psi\left(\frac{W_0}{K_0}\right),$$

where

$$\psi(x) = \begin{cases} \dfrac{1}{\rho}\left(\log(\alpha) - \dfrac{\beta}{\rho}\right) + \dfrac{(r_1^* - 1)^{1/\alpha_2}}{\eta} \dfrac{(x-1)^{1-1/\alpha_2}}{1 - \dfrac{1}{\alpha_2}} & \text{if } 1 \le x \le 1/r_1^* \\ \dfrac{1}{\rho}\left(\log(\alpha) - \dfrac{\beta}{\rho} + \dfrac{1}{\alpha_2 - 1} - \log(r_1^* + \imath)\right) + \dfrac{1}{\rho}\log(x + \imath) & \text{if } x > 1/r_1^*, \end{cases}$$

$$r_1^* = \frac{\alpha_2 + \imath}{\alpha_2 - 1},$$

and $\eta = \rho(r_1^* + \imath)$. The optimal investment policy is given by

$$\theta_t^* = -(\sigma\sigma^{\mathrm{T}})^{-1}(\mu - r\mathbf{1})K_t^* \frac{\psi'(W_t^*/K_t^*)}{\psi''(W_t^*/K_t^*)} 1_{\{W_t^* > K_t^*\}},$$

and the optimal consumption policy involves only purchasing the durable when $K_t^* = W_t^*/r_1^*$ (and never selling it).

References

Abel, A.B., Eberly, J.C., 1996. Optimal investment with costly reversibility. Review of Economic Studies 63, 581–593.

Akian, M., Menaldi, J.L., Sulem, A., 1996. On an investment-consumption model with transaction costs. SIAM Journal of Control and Optimization 34, 329–364.

Bertola, G., Caballero, R.J., 1994. Irreversibility and aggregate investment. Review of Economic Studies 61, 223–246.

Borodin, A.N., Salminen, P., 1996. Handbook of Brownian Motion: Facts and Formulae. Birkhäuser, Boston.

Caballero, R.J., 1993. Durable goods: an explanation for their slow adjustment. Journal of Political Economy 101, 351–384.

Constantinides, G.M., 1990. Habit formation: a resolution of the equity premium puzzle. Journal of Political Economics 98, 519–543.

Cvitanić, J., Karatzas, I., 1996. Hedging and portfolio optimization under transaction costs: a martingale approach. Mathematical Finance 6, 133–165.

Davis, M.H.A., Norman, A.R., 1990. Portfolio selection with transaction costs. Mathematics of Operations Research 15, 676–713.

Detemple, J.B., Giannikos, C.I., 1996. Asset and commodity prices with multi-attribute durable good. Journal of Economic Dynamics and Control 20, 1451–1504.

Dupuis, P., Ishii, H., 1993. Stochastic differential equations with oblique reflections on nonsmooth domains. Annals of Probability 21, 554–580.

Dybvig, P.H., 1995. Duesenberry's ratcheting of consumption and investment given intolerance for any decline in standard of living. Review of Economic Studies 62, 287–313.

Eichenbaum, M.S., Hansen, L.P., 1987. Estimating models with intertemporal substitution using aggregate time-series data. NBER Working Paper No. 2181.

Fleming, W.H., Soner, H.M., 1993. Controlled Markov Processes and Viscosity Solutions. Springer-Verlag, New York.

Grossman, S.J., Laroque, G., 1987. Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods. NBER Working Paper No. 2369.

Grossman, S.J., Laroque, G., 1990. Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods. Econometrica 58, 25–51.

Harrison, M., 1985. Brownian Motion and Stochastic Flow Systems. Wiley, New York.

Heaton, J., 1993. The interaction between time-nonseparable preferences and time aggregation. Econometrica 61, 353–385.

Hindy, A., Huang, C.-f., 1993. Optimal consumption and portfolio rules with durability and local substitution. Econometrica 61, 85–121.

Karatzas, I., Shreve, S.E., 1988. Brownian Motion and Stochastic Calculus. Springer-Verlag, New York.

Karlin, S., Taylor, H., 1981. A Second Course in Stochastic Processes. Academic Press, New York.

Lions, P.L., Sznitman, A.S., 1984. Stochastic differential equations with reflecting boundary conditions. Communications on Pure and Applied Mathematics 37, 511–537.

Merton, R.C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 373–413.

Øksendal, B., 1997. Stochastic Control Problems where Small Intervention Costs Have Dramatic Effects. University of Oslo, mimeo.

Shreve, S.E., Soner, H.M., 1994. Optimal investment and consumption with transaction costs. Annals of Applied Probability 4, 909–962.