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A MARTINGALE CHARACTERIZATION OF CONSUMPTION CHOICES AND HEDGING COSTS WITH MARGIN REQUIREMENTS

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This paper examines optimal consumption and investment choices and the cost of hedging contingent claims in the presence of margin requirements or, more generally, of nonlinear wealth dynamics and constraints on the portfolio policies. Existence of optimal policies is established using martingale and duality techniques under general assumptions on the securities' price process and the investor's preferences. As an illustration, explicit solutions are provided for an agent with "logarithmic" utility. A PDE characterization of the cost of hedging a nonnegative path-independent European contingent claim is also provided.

Key Words: optimal consumption, portfolio optimization, option pricing, margin requirements, borrowing constraints, martingales, convex duality

1. INTRODUCTION

Margin requirements oblige investors who short securities or buy on margin (that is, borrow to buy securities) to deposit and maintain a minimum amount of cash or securities with their broker-dealer in order to reduce the potential losses in case of default. In particular, initial margin requirements set the minimum margin deposit with which a position can be opened, and maintenance requirements set a floor below which the margin deposit is not allowed to fall as long as the position remains open. Regulation T of the Federal Reserve Board determines the initial margin requirement for stock positions undertaken through broker-dealers, and the NYSE determines the maintenance margin requirement applicable to its members' customers. The initial margin requirement for long or short stock positions is currently 50 percent of the value of the stock position. In the case of a long position, this means that the investor cannot borrow more than 50 percent of the market value of the stock. In the case of a short position, this means that not only does the investor not have availability of the cash proceeds from the short sale, but that an additional amount equal to 50 percent of the value of the stock must be deposited with the broker-dealer. This additional deposit need not be in cash: securities can be used at the broker-dealer's discretion. The maintenance margin requirement is currently 25 percent for long positions and 30 percent for short positions.[1]

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[1] A more detailed overview of margin regulations can be found in Sofianos (1988).

This paper examines optimal consumption and investment choices and the cost of hedging European contingent claims in the presence of margin requirements. Margin requirements introduce two types of deviations from the standard frictionless setting studied by Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989). First, margin requirements limit the size of long or short positions that can be undertaken by an investor, and hence introduce a constraint on the feasible portfolio policies. Second, the loss of interest on the proceeds from short sales (and, typically, on the associated margin) affect the usual dynamic budget constraint, making it nonlinear in the investor's portfolio policy. Models with constraints on the portfolio policies were studied by Cvitanić and Karatzas (1992, 1993), building on previous work by Karatzas et al. (1991), He and Pearson (1991), and Xu and Shreve (1992). Models with nonlinear wealth dynamics similar to the one that arises in the presence of margin requirements (but no constraints on the portfolio policies) were studied by El Karoui, Peng, and Quenez (1997) and Cuoco and Cvitanić (1998).

We show that martingale techniques similar to those employed in the above-mentioned papers can be employed to establish the existence of optimal policies in the presence of margin requirements and to provide some characterization by duality. As an illustration, an explicit solution is provided for an agent with "logarithmic" utility. More generally, we show that the characterization and existence results obtained in this paper apply to a class of consumption problems with nonlinear wealth dynamics and constraints on the portfolio policies which generalizes those considered by Cvitanić and Karatzas (1992, 1993) and Cuoco and Cvitanić (1998).

Finally, under the assumption that the stock prices in the economy follow a Markov process, we provide a PDE characterization of the cost of hedging a nonnegative path-independent European contingent claim. This characterization builds on recent work by Broadie, Cvitanić, and Soner (1998), who have shown that, in the presence of constraints on the portfolio weights, the minimum cost of hedging (that is, superreplicating) a contingent claim equals the unconstrained price of a related dominating claim, and thus solves the Black–Scholes PDE, but with a different terminal condition. In the presence of margin requirements, the minimum hedging cost of a European contingent claim is shown to equal the price of a related dominating claim in an economy with no constraints on the portfolio policies, but with nonlinear wealth dynamics. As a consequence, the minimum hedging cost is shown to be the solution of a quasi-linear PDE with an appropriate terminal condition. Once again, this result is shown to generalize to a class of problems with nonlinear wealth dynamics and constraints on the portfolio policies.

As far as we are aware, the only related work to examine the impact of margin requirements on consumption choices and/or the cost of hedging contingent claims is Heath and Jarrow (1987). Heath and Jarrow show that margin requirements are sufficient to rule out arbitrage opportunities such as the doubling strategy discussed by Harrison and Kreps (1979). Moreover, they argue that, although margin requirements rule out the standard replication strategy used to derive the Black–Scholes formula, this formula should still correctly price a call option in equilibrium *as long as some investor has sufficient excess equity in his margin account at all times*—that is, as long as the constraint imposed by margin requirements is never binding. This argument would obviously not apply to a put option once the loss of interest on the proceeds from a short sale is taken into account. Differently from Heath and Jarrow (and in the spirit of Cvitanić and Karatzas (1993)), we characterize the minimal cost of superreplicating a given contingent claim in isolation of the investor's residual portfolio. This cost can thus be interpreted as an upper bound on the price that could be charged by a financial intermediary selling a nontraded contingent claim to a client. A lower bound could be similarly obtained (see Karatzas and Kou 1996).

The rest of the paper is organized as follows. Section 2 describes the economic setting. Section 3 derives a martingale characterization of the minimum cost of replicating a contingent claim (consumption plan). Section 4 establishes the existence of an optimal consumption/investment policy and provides a dual characterization of such a policy. Section 5 provides explicit solutions for some special cases. Section 6 derives the PDE characterization of the minimum hedging cost in a Markovian setting. Section 7 generalizes the results of the paper to a related class of problems and Section 8 contains some concluding remarks.

2. THE ECONOMIC SETTING

We consider a continuous-time economy on the finite time span $[0, T]$.

Information Structure. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$, on which is defined an n-dimensional Brownian motion

$$w = \left\{(w_1(t), \dots, w_n(t))^\top : t \in [0, T]\right\}.$$

The filtration $\mathbf{F} = \{\mathcal{F}_t\}$ is the augmentation under P of the filtration generated by w. We assume that $\mathcal{F} = \mathcal{F}_T$, or that the true state of nature is completely determined by the sample paths of w on $[0, T]$. We interpret the sigma-field $\mathcal{F}_t$ as representing the information of the individual at time t and the probability measure P as representing his beliefs. All the stochastic processes to appear in the sequel are progressively measurable with respect to $\mathbf{F}$ and all the (in)equalities involving random variables (respectively, random processes) are understood to hold P-a.s. (respectively, $(\lambda \times P)$-a.e., where λ denotes the Lebesgue measure on $[0, T]$).

Securities Market. We assume that there are n long-lived risky securities (stocks) traded. The investors in the economy are allowed to hold the numéraire (cash), as well as to lend (respectively, borrow) at an instantaneous interest rate r (respectively, R), where r and R are assumed to be bounded nonnegative processes with $r \le R$. Also, letting $S = (S_1, \dots, S_n)$ denote the price process for the traded stocks and $D = (D_1, \dots, D_n)$ their cumulative dividend process, we assume that $S + D$ is an Itô process:

$$S(t) + D(t) = S(0) + \int_0^t I_S(s)\mu(s)\,ds + \int_0^t I_S(s)\sigma(s)\,dw(s), \tag{1}$$

where $I_S(t)$ denotes the $n \times n$ diagonal matrix with elements $S(t)$, μ is a bounded n-dimensional process, and σ is a bounded $(n \times n)$-dimensional process.

ASSUMPTION 1. *The diffusion matrix $\sigma(t)$ satisfies the nondegeneracy condition*

$$x^\top \sigma(t)\sigma(t)^\top x \ge \varepsilon |x|^2 \tag{2}$$

almost surely for all $(x, t) \in \mathbb{R}^n \times [0, T]$ and some $\varepsilon > 0$.

Condition (2) implies in particular that $\sigma(t)$ has full rank almost surely for all $t \in [0, T]$, and that $\sigma(t, \omega)^{-1}$ has essentially bounded matrix norm, uniformly in $(t, \omega) \in [0, T] \times \Omega$ (Karatzas and Shreve 1988, Prob. 5.8.1). Together with the boundedness of the price coefficients, this is sufficient to guarantee the existence of an equivalent martingale measure and rule out arbitrage opportunities.

Consumption Space. There is a single consumption good (the numéraire). A *consumption plan* is a pair (c, W), where c is a nonnegative consumption rate process with $\int_0^T c(t)\,dt < +\infty$ and W a nonnegative random variable representing terminal wealth.

Trading Strategies and Margin Requirements. Trading takes place continuously. A trading strategy is an $(n+2)$-dimensional process $(\alpha_C, \alpha_B, \theta)$—where α_C denotes the amount held in cash, α_B the amount loaned (if $\alpha_B > 0$) or borrowed (if $\alpha_B < 0$), and θ the amount invested in each of the n stocks—satisfying

$$\int_0^T \left|\alpha_B(t)^+ r(t) - \alpha_B(t)^- R(t) + \theta(t)^\top \mu(t)\right| dt + \int_0^T \left|\theta(t)^\top \sigma(t)\right|^2 dt < +\infty, \tag{3}$$

where $x^+ = \max[0, x]$ denotes the positive part of the real number x and $x^- = \max[0, -x]$ the negative part. A trading strategy $(\alpha_C, \alpha_B, \theta)$ is said to *finance* a consumption plan $(c, W) \in \mathcal{C}$ if there exists a nondecreasing process C with $C(0) = 0$ such that the wealth process

$$W(t) = \alpha_C(t) + \alpha_B(t) + \sum_{i=1}^n \theta_i(t)$$

satisfies the dynamic budget constraint

$$\begin{aligned} W(t) = W(0) &+ \int_0^t \left(\alpha_B(s)^+ r(s) - \alpha_B(s)^- R(s) + \theta(s)^\top \mu(s)\right) ds \\ &- \int_0^t c(s)\,ds - C(t) + \int_0^t \theta(s)^\top \sigma(s)\,dw(s) \end{aligned} \tag{4}$$

and

$$W(T) \geq W.$$

The process C in (4) captures the possibility of free disposal of wealth: in other words, the agent is allowed not to reinvest some of his wealth if he chooses to do so. The total amount of wealth "wasted" up to time t is given by $C(t)$.

To model margin requirements, we assume that there is no difference between initial and maintenance margins, and we denote by $\lambda_- \geq 0$ the proportional margin requirement on short positions and by $\lambda_+ \in [0, 1]$ the proportional margin requirement on long positions.[2] Moreover, we assume that a fraction $\iota \in [0, 1]$ of the margin requirement on short positions must be met with cash, and that the rest can be met with bonds.[3] Margin

[2] Different initial and maintenance margins would make the balance in the margin account a function of the sample path of the trading strategy and thus considerably complicate the analysis.

[3] As mentioned in the Introduction, the brokerage firm can accept interest-bearing securities in lieu of cash to margin short-sales.

requirements then impose the following constraints on an admissible trading strategy $(\alpha_C, \alpha_B, \theta)$:

$$\alpha_C \geq (1 + \iota\lambda_-) \sum_{i=1}^{n} \theta_i^- \tag{5}$$

and

$$\alpha_B \geq -\left(\alpha_C - (1 + \lambda_-) \sum_{i=1}^{n} \theta_i^- + (1 - \lambda_+) \sum_{i=1}^{n} \theta_i^+\right). \tag{6}$$

Equation (5) states that the investor must hold an amount of cash at least equal to the value of his short stock positions plus a fraction ι of the required margin, and equation (6) states that the investor can only borrow using as collateral cash or stocks in excess of the required margin.

Clearly, since cash is a dominated asset, the constraint in (5) will always be satisfied as an equality, so that the size of an investor's cash position is completely determined by the size of his short stock position. Moreover, (6) is equivalent to the constraint

$$W \geq \lambda_- \sum_{i=1}^{n} \theta_i^- + \lambda_+ \sum_{i=1}^{n} \theta_i^+,$$

which implies in particular that the wealth process corresponding to any admissible trading strategy under margin requirements is nonnegative. Since (5) and (6) also imply that $W = 0$ only if $\alpha_C = \alpha_B = 0$ and $\theta_i = 0$ for $i = 1, \ldots, n$, we can then equivalently represent a trading strategy in terms of the portfolio weights $\pi = (\theta/W)1_{\{W>0\}}$ and equations (3)–(5) are then equivalent to

$$\int_0^T \left|r(t) + \pi(t)^\top(\mu(t) - r(t)\bar{1}) + g(\pi(t), t)\right| dt + \int_0^T \left|\pi(t)^\top \sigma(t)\right|^2 dt < +\infty, \tag{7}$$

$$\begin{aligned} W(t) = W(0) &+ \int_0^t W(s)\Big(r(s) + \pi(s)^\top\big(\mu(s) - r(s)\bar{1}\big) + g\big(\pi(s), s\big)\Big)\, ds \\ &+ \int_0^t W(s)\pi(s)^\top \sigma(s)\, dw(s) - \int_0^t c(s)\, ds - C(t), \end{aligned} \tag{8}$$

and

$$\pi(t, \omega) \in K \qquad (\lambda \times P)\text{-a.e.}, \tag{9}$$

where $\bar{1} = (1, \ldots, 1)^\top$,

$$\begin{aligned} g(\pi, t, \omega) = -r(t, \omega)(1 + \iota\lambda_-) \sum_{i=1}^{n} \pi_i^- - \big(R(t, \omega) - r(t, \omega)\big) \\ \times \left(1 - \sum_{i=1}^{n} \pi_i^+ - \iota\lambda_- \sum_{i=1}^{n} \pi_i^-\right)^-, \end{aligned} \tag{10}$$

and

$$K = \left\{ \pi \in \mathbb{R}^n : \lambda_- \sum_{i=1}^{n} \pi_i^- + \lambda_+ \sum_{i=1}^{n} \pi_i^+ \leq 1 \right\}.$$

Note that because of the term involving g, the dynamic budget constraint in (8) is nonlinear in the portfolio policy (although it is piecewise linear). We will denote by Π the set of portfolio weight processes satisfying (7) and (9).

3. COST OF HEDGING CONTINGENT CLAIMS

To be able to approach the investor's consumption problem, we start by deriving a static (martingale) characterization of the minimal cost of financing a consumption plan $(c, W) \in \mathcal{C}$, similar to the one obtained by Karatzas et al. (1987) and Cox and Huang (1989) for the standard frictionless case. In our setting, such a characterization is complicated by the presence of the constraint (9) on the admissible portfolio policies and by the fact that the stochastic differential equation (8) is nonlinear in the portfolio policy (because of the presence of the function g). Constraints on the admissible portfolio policies of the type in equation (9) were dealt with by Cvitanić and Karatzas (1992, 1993), and nonlinear wealth dynamics of the type in equation (8) were dealt with by El Karoui et al. (1997) and Cuoco and Cvitanić (1998). The new feature of the problem at hand is that it includes both a constraint on the portfolio policies and nonlinear wealth dynamics. As we show in Section 7, all the main results to be obtained below apply to a class of problems with constraints and nonlinear wealth dynamics which include those considered in the above-mentioned papers as special cases.

Let

$$\delta_K(\pi) = \begin{cases} 0 & \text{if } \pi \in K \\ +\infty & \text{if } \pi \notin K \end{cases}$$

denote the indicator function (in the sense of convex analysis) of K, and let

$$g_K(\pi, t, \omega) = g(\pi, t, \omega) - \delta_K(\pi).$$

It is easily verified that $g_K(\cdot, t, \omega)$ is a nonpositive upper semicontinuous concave function with $g_K(0, t, \omega) = 0$ for all $(t, \omega) \in [0, T] \times \Omega$. We denote by

$$\tilde{g}_K(\nu, t, \omega) = \sup_{\pi \in \mathbb{R}^n} \left[g_K(-\pi, t, \omega) + \pi^\top \nu \right] = \sup_{\pi \in K} \left[g(\pi, t, \omega) - \pi^\top \nu \right]$$

the convex conjugate of $-g_K(-\pi, t, \omega)$. It is clear from its definition that $\tilde{g}_K$ is nonnegative (since $0 \in K$). Moreover, it follows from Theorem 12.2 in Rockafellar (1970) that $\tilde{g}_K(\cdot, t, \omega)$ is a lower semicontinuous convex function on $\mathbb{R}^n$ and that

$$g_K(\pi, t, \omega) = \inf_{\nu \in \mathbb{R}^n} \left[\tilde{g}_K(\nu, t, \omega) + \pi^\top \nu \right]. \tag{11}$$

Letting

$$\mathcal{N}_K(t, \omega) = \left\{ \nu \in \mathbb{R}^n : \tilde{g}_K(\nu, t, \omega) < +\infty \right\}$$

denote the effective domain of $\tilde{g}_K(\cdot, t, \omega)$, it can be easily verified that

$$\mathcal{N}_K(t,\omega) = \begin{cases} [-(R(t,\omega)-r(t,\omega)), r(t,\omega)]^n & \text{if } \lambda_- = 0,\ \lambda_+ = 0 \\ [-(R(t,\omega)-r(t,\omega)), +\infty)^n & \text{if } \lambda_- > 0,\ \lambda_+ = 0 \\ (-\infty, r(t,\omega)]^n & \text{if } \lambda_- = 0,\ \lambda_+ > 0 \\ (-\infty,+\infty)^n & \text{if } \lambda_- > 0,\ \lambda_+ > 0. \end{cases} \tag{12}$$

Also, for $p \in [0,+\infty]$ let $\mathcal{N}_K^p$ denote the set of n-dimensional processes ν such that

$$\int_0^T |\nu(t)|^2\,dt + \int_0^T \tilde{g}_K(\nu(t),t)\,dt < +\infty$$

and

$$\nu \in L^p(\lambda \times P),$$

where $L^0(\lambda \times P)$ denotes in particular the set of all (progressively measurable) processes. Clearly, $0 \in \mathcal{N}_K^p$ for all $p \in [0,+\infty]$. Moreover, if $\nu \in \mathcal{N}_K^p$ then

$$\nu(t,\omega) \in \mathcal{N}_K(t,\omega) \qquad (\lambda \times P)\text{-a.e.}$$

PROPOSITION 1. *For any $\pi \in \Pi$ there exists a $\nu \in \mathcal{N}_K^\infty$ such that*

$$g(\pi(t),t) = g_K(\pi(t),t) = \tilde{g}_K(\nu(t),t) + \pi(t)^\top \nu(t) \qquad \forall t \in [0,T].$$

Proof. This can be verified directly by considering the process ν with

$$\nu(t) = \begin{cases} r_t(1+\iota\lambda_-)1_{\{\pi_t \le 0\}} \\ \quad \text{if } \sum_{i=1}^n \pi_i(t)^+ + \iota\lambda_- \sum_{i=1}^n \pi_i(t)^- \le 1 \\ (r_t + \iota\lambda_- R_t)1_{\{\pi_t \le 0\}} - (R_t - r_t)1_{\{\pi_t > 0\}} \\ \quad \text{if } \sum_{i=1}^n \pi_i(t)^+ + \iota\lambda_- \sum_{i=1}^n \pi_i(t)^- > 1, \end{cases}$$

where $1_{\{\pi(t)\le 0\}} = (1_{\{\pi_1(t)\le 0\}}, \ldots, 1_{\{\pi_n(t)\le 0\}})^\top$, and noticing that

$$\tilde{g}_K\left(r_t(1+\iota\lambda_-)1_{\{\pi_t\le 0\}}, t\right) = 0$$

and

$$\tilde{g}_K\left((r_t+\iota\lambda_- R_t)1_{\{\pi_t\le 0\}} - (R_t - r_t)1_{\{\pi_t>0\}}, t\right) = R_t - r_t. \qquad \square$$

For $\nu \in \mathcal{N}_K^0$, define the processes

$$\beta_\nu(t) = \exp\left(-\int_0^t (r_s + \tilde{g}_K(\nu_s,s))\,ds\right)$$

$$\kappa_\nu(t) = -\sigma_t^{-1}(\mu_t + \nu_t - r_t\bar{1})$$

$$Z_\nu(t) = \exp\left(\int_0^t \kappa_\nu(s)^\top dw(s) - \frac{1}{2}\int_0^t |\kappa_\nu(s)|^2\,ds\right)$$

and

$$\xi_\nu(t) = \beta_\nu(t) Z_\nu(t).$$

Finally, for $\nu \in \mathcal{N}_K^\infty$ let Q_ν denote the probability measure with

$$\frac{dQ_\nu}{dP} = Z_\nu(T).$$

We remark that each ξ_ν corresponds to the unique state-price density that would prevail in a fictitious frictionless economy with interest rate $r_t + \tilde{g}_K(\nu_t, t)$ and stock drift $\mu_t + \nu_t + \tilde{g}_K(\nu_t, t)$.

Now consider an arbitrary consumption plan $(c, W) \in \mathcal{C}$ and let $P(0)$ denote the minimal cost at time 0 of financing (superreplicating) (c, W); that is,

$$P(0) = \inf\{x \geq 0 : \exists \text{ a nondecreasing process } C \text{ with } C(0) = 0 \text{ s.t. } W^{x,c,C,\pi}(T) \geq W\},$$

where $W^{x,c,C,\pi}$ denotes the solution to the stochastic differential equation (8) with $W(0) = x$. The following result provides a martingale characterization of $P(0)$.

THEOREM 1. *We have*

$$\begin{aligned} P(0) &= \sup_{\nu \in \mathcal{N}_K^0} \mathrm{E}\left[\int_0^T \xi_\nu(s) c(s)\, ds + \xi_\nu(T) W\right] \\ &= \sup_{\nu \in \mathcal{N}_K^\infty} \mathrm{E}^{Q_\nu}\left[\int_0^T \beta_\nu(s) c(s)\, ds + \beta_\nu(T) W\right]. \end{aligned}$$

Proof. See the Appendix. □

4. OPTIMAL CONSUMPTION POLICIES

We now turn to a characterization of optimal consumption policies in the presence of margin requirements. We consider an investor endowed with some initial wealth $W_0 > 0$ and whose preferences are represented by a time-additive utility function for intertemporal consumption[4]

$$U(c) = \mathrm{E}\left[\int_0^T u(c(t), t)\, dt\right]. \tag{13}$$

ASSUMPTION 2. *The function $u(\cdot, t)$ is increasing, strictly concave and continuously differentiable on $(0, +\infty)$ for all $t \in [0, T]$. Moreover, it satisfies the Inada conditions*

$$\lim_{c \downarrow 0} u_c(c, t) = +\infty \quad \text{and} \quad \lim_{c \uparrow +\infty} u_c(c, t) = 0, \tag{14}$$

and there exist constants $\delta \in (0, 1)$ and $\gamma \in (0, +\infty)$ such that

$$\delta u_c(c, t) \geq u_c(\gamma c, t) \qquad \forall (c, t) \in (0, +\infty) \times [0, T]. \tag{15}$$

Finally, $u(c, \cdot)$ is continuous on $[0, T]$ for all $c > 0$.

[4] The case in which preferences include a bequest function $V(W)$ for final wealth is not substantially different and the results in this paper apply, provided that $V(\cdot)$ satisfies the same conditions placed on $u(\cdot, t)$.

REMARK 1. Condition (14) is well understood and it implies in particular that the derivative function $u_c(\cdot, t)$ has a continuous and strictly decreasing inverse $f(\cdot, t)$ mapping $(0, +\infty)$ onto itself. Condition (15) has the purpose of guaranteeing that certain functionals to be introduced in the sequel can be differentiated under the integral sign. It is easily verified that this condition holds for the utility functions $u(c, t) = \rho(t) \log c$ or $u(c, t) = \rho(t)c^{1-b}/(1 - b)$, for some $\rho : [0, T] \to (0, +\infty)$, $b > 0$, $b \neq 1$. Also, taking $c = f(y, t)$ in (15), applying $f(\cdot, t)$ to both sides, and iterating shows that the following property holds:

$$(16) \qquad \begin{aligned} &\forall \delta \in (0, +\infty), \exists \gamma \in (0, +\infty) \quad \text{such that} \\ &f(\delta y, t) \le \gamma f(y, t), \forall (y, t) \in (0, +\infty) \times [0, T]. \end{aligned}$$

It follows immediately from Theorem 1 that the problem of choosing an optimal consumption plan in the investor's budget set can be written as the static problem

$$(P) \qquad \begin{aligned} &\max_{c \in \mathcal{C}} \; U(c) \\ &\text{s.t.} \; \sup_{\nu \in \mathcal{N}_K^p} \mathrm{E}\left[\int_0^T \xi_\nu(t) c(t)\, dt\right] >\leq> W_0 \end{aligned}$$

for some $p \in [0, +\infty]$. Letting c^* denote the optimal consumption policy this suggests that there should exist a Lagrangian multiplier $\psi^* > 0$ and a process $\nu^* \in \mathcal{N}_K^p$ such that (c^*, ψ^*, ν^*) is a saddle point of the map

$$(17) \qquad Ł(c, \psi, \nu) = U(c) - \psi\left(\mathrm{E}\left[\int_0^T \xi_\nu(t) c(t)\, dt\right] - W_0\right),$$

where we maximize with respect to $c \in \mathcal{C}^*$ and minimize with respect to $(\psi, \nu) \in (0, +\infty) \times \mathcal{N}_K^p$.

Let

$$(18) \qquad \tilde{u}(y, t) \equiv \max_{c \ge 0}\left[u(c, t) - yc\right] = u(f(y, t), t) - y f(y, t)$$

denote the convex conjugate of $-u(-c, t)$. The following lemma collects some properties of the function $\tilde{u}$ that will be useful in the sequel.

LEMMA 1. *The function* $\tilde{u}(\cdot, t) : (0, +\infty) \to \mathbb{R}$ *is strictly decreasing and strictly convex for all* $t \in [0, T]$, *with*

$$(19) \qquad \frac{\partial}{\partial y} \tilde{u}(y, t) = -f(y, t).$$

Moreover

$$(20) \qquad \tilde{u}(0+, t) = u(+\infty, t), \qquad \tilde{u}(+\infty, t) = u(0+, t).$$

Proof. See, for example, Karatzas et al. (1991, p. 707). □

Maximization of equation (17) with respect to c gives

$$J(\psi,\nu) = \mathrm{E}\left[\int_0^T \tilde{u}(\psi\xi_\nu(t),t)\,dt\right] + \psi W_0, \tag{21}$$

where we remark that the above expectation is well defined for all $(\psi,\nu) \in (0,+\infty) \times \mathcal{N}_K^0$, since from the inequality

$$u(1,t) - y \le \max_{c>0}\big[u(c,t) - yc\big] = u(f(y,t),t) - yf(y,t), \tag{22}$$

the nonnegativity of r and $\tilde{g}_K$ and the fact that Z_ν is a supermartingale for all $\nu \in \mathcal{N}_K^0$, we have

$$\begin{aligned} \mathrm{E}\left[\int_0^T \tilde{u}(\psi\xi_\nu(t),t)^-\,dt\right] &\le \int_0^T u(1,t)^-\,dt + \psi\,\mathrm{E}\left[\int_0^T \xi_\nu(t)\,dt\right] \\ &\le \int_0^T u(1,t)^-\,dt + \psi T < +\infty. \end{aligned}$$

Therefore $J : (0,+\infty) \times \mathcal{N}_K^0 \to \mathbb{R} \cup \{+\infty\}$ and we are left with the shadow state-price problem

$$\min_{(\psi,\nu)\in(0,+\infty)\times\mathcal{N}_K^p} J(\psi,\nu). \tag{P^*}$$

The following theorem establishes the duality between the individual's optimization problem and (P^*).

THEOREM 2. *Assume that $(\psi^*,\nu^*) \in (0,+\infty) \times \mathcal{N}_K^p$ solves the dual state-price problem (P^*) for some $p \in [0,+\infty]$ and*

$$\mathrm{E}\left[\int_0^T \xi_{\nu^*}(t) f(\psi^*\xi_{\nu^*}(t),t)\,dt\right] < +\infty. \tag{23}$$

Then the policy

$$c^*(t) = f(\psi^*\xi_{\nu^*}(t),t) \tag{24}$$

is optimal and the optimal wealth process is given by

$$W(t) = \xi_{\nu^*}(t)^{-1}\,\mathrm{E}\left[\int_t^T \xi_{\nu^*}(s)c(s)\,ds \,\middle|\, \mathcal{F}_t\right]. \tag{25}$$

This wealth process satisfies the budget constraint (8) *with $C = 0$. Moreover, the optimal investment strategy π satisfies*

$$g(\pi(t),t) = \tilde{g}_K(\nu^*(t),t) + \pi(t)^\top \nu^*(t), \qquad (\lambda \times P)\text{-a.e.} \tag{26}$$

Proof. See the Appendix. □

The next theorem provides sufficient conditions for the existence of a solution $(\psi^*,\nu^*) \in (0,+\infty) \times \mathcal{N}_K^2$ to the dual problem and of an optimal consumption/investment policy.

THEOREM 3. *Assume that*

(a) $u(0+, t) > -\infty$, $u(+\infty, t) = +\infty$ *and* $-cu''(c, t)/u'(c, t) \leq 1$ *on* $(0, +\infty) \times [0, T]$*;*
(b) *for all* $\psi \in (0, +\infty)$*, there exists a* $\nu \in \mathcal{N}_K^2$ *such that* $J(\psi, \nu) < +\infty$.

Then the minimum in (P^*) *is attained by some* $\nu^* \in \mathcal{N}_K^2$*. If, additionally,*

(c) $cu_c(c, t) \leq a + (1 - b)u(c, t)$ *on* $(0, +\infty) \times [0, T]$ *for some* $a \geq 0$, $b > 0$,

then condition (23) *of Theorem* 2 *is also satisfied, and hence there exists an optimal consumption/investment policy.*

Proof. See the Appendix. □

REMARK 2. Conditions (a) and (c) of Theorem 3 are in particular satisfied if $u(c, t) = \rho(t)\frac{c^{1-b}}{1-b}$ for some bounded measurable function $\rho : [0, T] \mapsto (0, \bar{\rho}]$ and some $b \in (0, 1)$. In this case, condition (b) will also hold with $\nu = 0$ (cf. Karatzas et al. 1991, Remark 11.9).

REMARK 3. Proceeding as in Cuoco (1997), it would have also been possible to establish the existence of an optimal consumption policy for more general utility functions and without resorting to duality. However, we will see in the next section that the dual problem offers computational advantages.

5. EXPLICIT SOLUTIONS

5.1. Logarithmic Utility

Suppose that $u(c, t) = e^{-\rho t} \log(c)$ for some $\rho \in \mathbb{R}$. Then we have

$$\tilde{u}(y, t) = \max_{c \geq 0}[e^{-\rho t} \log(c) - yc] = -e^{-\rho t}\big(1 + \rho t + \log(y)\big),$$

and the dual problem becomes

$$\min_{(\psi, \nu) \in (0, +\infty) \times \mathcal{N}_K^2} \mathrm{E}\left[-\int_0^T e^{-\rho t}\big(1 + \rho t + \log(\psi \xi_\nu(t))\big)\, dt + \psi W_0\right]$$
$$= Te^{-\rho T} - 2\,\frac{1 - e^{-\rho T}}{\rho} + \mathrm{E}\left[\int_0^T e^{-\rho t}\left(\int_0^t r(s)\, ds\right) dt\right]$$
$$+ \min_{\psi > 0}\left[\psi W_0 - \frac{1 - e^{-\rho T}}{\rho} \log(\psi)\right]$$
$$+ \min_{\nu \in \mathcal{N}_K^2} \mathrm{E}\left[\int_0^T e^{-\rho t}\left(\int_0^t \left(\tilde{g}_K(\nu(s), s) + \frac{1}{2}\left|\kappa_0(s) - \sigma(s)^{-1}\nu(s)\right|^2\right) ds\right) dt\right].$$

The above implies

$$\psi^* = \frac{1 - e^{-\rho T}}{\rho W_0}$$

and

$$(27)\qquad \nu^*(t,\omega) = \arg \min_{\nu \in \mathcal{N}_K(t,\omega)} \left[\tilde{g}_K(\nu,t) + \frac{1}{2}\left|\kappa_0(t,\omega) - \sigma(t,\omega)^{-1}\nu\right|^2 \right].$$

The optimal consumption, investment, and wealth policies are then given by

$$\begin{aligned} c_{\nu^*}(t) &= W_0 \frac{\rho e^{-\rho t}}{1-e^{-\rho T}} \xi_{\nu^*}(t)^{-1} = \frac{\rho}{1-e^{-\rho(T-t)}} W(t) \\ \pi(t) &= \left(\sigma(t)\sigma(t)^\top\right)^{-1}\left(\mu(t) + \nu^*(t) - r(t)\bar{1}\right) \\ W(t) &= W_0 \frac{e^{-\rho t} - e^{-\rho T}}{1-e^{-\rho T}} \xi_{\nu^*}(t)^{-1}. \end{aligned}$$

In particular, if $n = 1$, $\lambda_- > 0$ and $\lambda_+ > 0$, then it can be easily verified that

$$\tilde{g}_K(\nu,t) = (\lambda_+^{-1} - 1)(R_t - r_t + \nu)^- + \nu^- + \lambda_-^{-1}(\nu - (1+\iota\lambda_-)r_t)^+$$

and

$$\nu_t^* = \begin{cases} \sigma_t(\kappa_{0t} + \lambda_+^{-1}\sigma_t) & \text{if } \sigma_t\kappa_{0t} \le -(R_t - r_t) - \lambda_+^{-1}\sigma_t^2 \\ -(R_t - r_t) & \text{if } -(R_t - r_t) - \lambda_+^{-1}\sigma_t^2 \le \sigma_t\kappa_{0t} \le -(R_t - r_t) - \sigma_t^2 \\ \sigma_t(\kappa_{0t} + \sigma_t) & \text{if } -(R_t - r_t) - \sigma_t^2 \le \sigma_t\kappa_{0t} \le -\sigma_t^2 \\ 0 & \text{if } -\sigma_t^2 \le \sigma_t\kappa_{0t} \le 0 \\ \sigma_t\kappa_{0t} & \text{if } 0 \le \sigma_t\kappa_{0t} \le (1+\iota\lambda_-)r_t \\ (1+\iota\lambda_-)r_t & \text{if } (1+\iota\lambda_-)r_t \le \sigma_t\kappa_{0t} \le (1+\iota\lambda_-)r_t + \lambda_-^{-1}\sigma_t^2 \\ \sigma_t(\kappa_{0t} - \lambda_-^{-1}\sigma_t) & \text{if } \sigma_t\kappa_{0t} \ge (1+\iota\lambda_-)r_t + \lambda_-^{-1}\sigma_t^2, \end{cases}$$

which implies

$$\pi_t = \begin{cases} \min[\hat{\pi}_t, \lambda_+^{-1}] & \text{if } \mu_t \ge r_t \\ 0 & \text{if } -\iota\lambda_- r_t \le \mu_t \le r_t \\ \max\left[\sigma_t^{-2}(\mu_t + \iota\lambda_- r_t), -\lambda_-^{-1}\right] & \text{if } \mu_t \le -\iota\lambda_- r_t, \end{cases}$$

where

$$\hat{\pi}_t = \begin{cases} \sigma_t^{-2}(\mu_t - R_t) & \text{if } \sigma_t^{-2}(\mu_t - R_t) \ge 1 \\ 1 & \text{if } \sigma_t^{-2}(\mu_t - R_t) \le 1 \le \sigma_t^{-2}(\mu_t - r_t) \\ \sigma_t^{-2}(\mu_t - r_t) & \text{if } \sigma_t^{-2}(\mu_t - r_t) \le 1 \end{cases}$$

denotes the optimal investment policy in the absence of margin requirements (see Cvitanić and Karatzas 1992 or Cuoco and Cvitanić 1998).

As long as $\mu_t > r_t$ (i.e., as long as $\hat{\pi}_t > 0$), the optimal policy at time t consists in holding a long position in the stock (and lending if $\mu_t < r_t + \sigma_t^2$ or borrowing if $\mu > R_t + \sigma_t^2$). The fraction of wealth invested in the stock equals the one that would prevail with different borrowing or lending rates but no margin requirements until the latter reaches λ_+^{-1} (the maximum allowable percentage long position under margin requirements).

When $\mu_t < -\iota\lambda_- r_t$, the optimal policy consists in holding a short position in the stock (and lending). The proportional short position in the stock equals the one that would prevail with a lending rate equal to $-\iota\lambda_- r_t$ and no margin requirements until the latter reaches $-\lambda_-^{-1}$ (the maximum allowable percentage short position under margin requirements). The reason for using a "shadow" interest rate equal to $-\iota\lambda_- r_t$ is of course that shorting \$1 worth of stock involves not only no interest on the proceeds, but a loss of $\$\iota\lambda_- r_t$ in interest on the portion of the required margin that must be met with cash. The induced disincentive to shorting can be significant. In particular, the above implies that

$$\pi(t)^- \leq \left(\hat{\pi}(t) + (1+\iota\lambda_-)\frac{r(t)}{\sigma(t)^2}\right)^- .$$

For example, with $r(t) = 0.06$, $\sigma(t) = 0.2$, $\lambda_- = 0.3$ and $\iota = 1$, one would need to have

$$\hat{\pi}(t) < -(1+\iota\lambda_-)\frac{r(t)}{\sigma(t)^2} = -195\%$$

(or $\mu(t) < -1.8\%$) before any shorting at all is done in the presence of margin requirements.

Finally, when $-\iota\lambda_- r_t \leq \mu_t \leq r_t$, the agent does not undertake any stock position and invests all of his wealth in the bond.

5.2. CRRA Utility and Deterministic Price Coefficients

Next, suppose that $u(c,t) = e^{-\rho t}c^{1-b}/(1-b)$ for some $\rho \in \mathbb{R}$, $b \in (0,1)$, and that the price coefficients r, μ, and σ are deterministic continuous functions of time. Then the proof of Theorem 3 shows that the function

$$V(\psi) = \min_{\nu\in\mathcal{N}_K^2} J(\psi,\nu) = \min_{\nu\in\mathcal{N}_K^2} \mathrm{E}\left[\int_0^T \tilde{u}(\psi\xi_\nu(t),t)\,dt\right] + \psi W_0$$

is well defined for all $\psi \in (0,\infty)$, so that

$$\min_{(\psi,\nu)\in(0,\infty)\times\mathcal{N}_K^2} J(\psi,\nu) = \min_{\psi\in(0,\infty)} V(\psi).$$

Moreover, it follows from standard stochastic control theory that $V(\psi) = v(\psi,0)+\psi W_0$, where v solves the Hamilton–Jacobi–Bellman (HJB) equation

$$\min_{\nu\in\mathcal{N}_K(t)}\left[\frac{1}{2}x^2 v_{xx}(x,t)|\kappa_0(t)-\sigma(t)^{-1}\nu|^2 - xv_x(x,t)\tilde{g}_K(\nu,t)\right]$$
$$- xv_x(x,t)r(t) + v_t(x,t) + \tilde{u}(x,t) = 0$$

with terminal condition

$$v(x,T) = 0$$

(where we write $\mathcal{N}_K(t)$ instead of $\mathcal{N}_K(t,\omega)$ for the sets in (12), as they do not depend on ω in this case). Since

$$\tilde{u}(y,t) = \max_{c\geq 0}\left[e^{-\rho t}\frac{c^{1-b}}{1-b} - yc\right] = e^{-\rho t/b}\frac{by^{-(1-b)/b}}{1-b},$$

it can be easily verified that the solution of the above HJB equation is given by

$$v(x,t) = f(t)\frac{bx^{-(1-b)/b}}{1-b},$$

where

$$f(t) = \int_t^T \exp\left(-\frac{\rho s}{b} + \int_t^s h(u)\,du\right) ds$$

and

$$h(t) = \frac{1-b}{b}\left(r(t) + \min_{\nu(t)\in\mathcal{N}_K(t)}\left[\tilde{g}_K(\nu(t),t) + \frac{1}{2b}\left|\kappa_0(t) - \sigma(t)^{-1}\nu(t)\right|^2\right]\right).$$

The above implies

$$\psi^* = \left(\frac{f(0)}{W_0}\right)^b$$

and

$$\nu^*(t) = \arg\min_{\nu\in\mathcal{N}_K(t)}\left[\tilde{g}_K(\nu,t) + \frac{1}{2b}\left|\kappa_0(t,\omega) - \sigma(t,\omega)^{-1}\nu\right|^2\right].$$

The optimal consumption, investment, and wealth policies are then given by

$$\begin{aligned}
c_{\nu^*}(t) &= \frac{W_0}{f(0)}\left(e^{\rho t}\xi_{\nu^*}(t)\right)^{-1/b} = \frac{e^{-\rho t/b}}{f(t)}\,W(t)\\
\pi(t) &= \frac{1}{b}\left(\sigma(t)\sigma(t)^\top\right)^{-1}\left(\mu(t) + \nu^*(t) - r(t)\bar{1}\right)\\
W(t) &= W_0\frac{f(t)}{f(0)}\,\xi_{\nu^*}(t)^{-1/b}.
\end{aligned}$$

Thus, with deterministic price coefficients, future investment opportunities affect the investor's propensity to consume $c_{\nu^*}(t)/W(t)$ through the "forward-looking" term $f(t)$, but not his investment policy. Moreover, unlike the logarithmic case, in which margin requirements affect the investment policy but not the propensity to consume, in the power case margin requirements also have an impact on the propensity to consume (again, through the term $f(t)$).

In particular, if $n = 1$, $\lambda_- > 0$ and $\lambda_+ > 0$, then

$$\nu_t^* = \begin{cases}
\sigma_t(\kappa_{0t} + b\lambda_+^{-1}\sigma_t) & \text{if } \sigma_t\kappa_{0t} \le -(R_t - r_t) - b\lambda_+^{-1}\sigma_t^2\\
-(R_t - r_t) & \text{if } -(R_t - r_t) - b\lambda_+^{-1}\sigma_t^2 \le \sigma_t\kappa_{0t} \le -(R_t - r_t) - b\sigma_t^2\\
\sigma_t(\kappa_{0t} + b\sigma_t) & \text{if } -(R_t - r_t) - b\sigma_t^2 \le \sigma_t\kappa_{0t} \le -b\sigma_t^2\\
0 & \text{if } -b\sigma_t^2 \le \sigma_t\kappa_{0t} \le 0\\
\sigma_t\kappa_{0t} & \text{if } 0 \le \sigma_t\kappa_{0t} \le (1+\iota\lambda_-)r_t\\
(1+\iota\lambda_-)r_t & \text{if } (1+\iota\lambda_-)r_t \le \sigma_t\kappa_{0t} \le (1+\iota\lambda_-)r_t + b\lambda_-^{-1}\sigma_t^2\\
\sigma_t(\kappa_{0t} - b\lambda_-^{-1}\sigma_t) & \text{if } \sigma_t\kappa_{0t} \ge (1+\iota\lambda_-)r_t + b\lambda_-^{-1}\sigma_t^2,
\end{cases}$$

which implies

$$\pi_t = \begin{cases} \min[\hat{\pi}_t, \lambda_+^{-1}] & \text{if } \mu_t \geq r_t \\ 0 & \text{if } -\iota\lambda_- r_t \leq \mu_t \leq r_t \\ \max[b^{-1}\sigma_t^{-2}(\mu_t + \iota\lambda_- r_t), -\lambda_-^{-1}] & \text{if } \mu_t \leq -\iota\lambda_- r_t, \end{cases}$$

where

$$\hat{\pi}_t = \begin{cases} b^{-1}\sigma_t^{-2}(\mu_t - R_t) & \text{if } b^{-1}\sigma_t^{-2}(\mu_t - R_t) \geq 1 \\ 1 & \text{if } b^{-1}\sigma_t^{-2}(\mu_t - R_t) \leq 1 \leq b^{-1}\sigma_t^{-2}(\mu_t - r_t) \\ b^{-1}\sigma_t^{-2}(\mu_t - r_t) & \text{if } b^{-1}\sigma_t^{-2}(\mu_t - r_t) \leq 1 \end{cases}$$

denotes the optimal investment policy in the absence of margin requirements.

6. MINIMAL HEDGING COST IN THE MARKOVIAN CASE

We now analyze in more detail the computation of the minimal cost of hedging (superreplicating) a claim to a payoff $b(S_T)$ at time T, where $b : \mathbb{R}^n \mapsto \mathbb{R}_+$ is lower semicontinuous.[5] We focus on the Markovian case in which the price coefficients (r, R, μ, σ) are deterministic functions of the vector S of contemporaneous stock prices and time and denote by $P(S_t, t)$ the minimal cost at time t of superreplicating the contingent claim. It then follows from Theorem 1 and the fact that S is an Itô process with drift $I_S(r - \nu)$ and diffusion $I_S\sigma$ under any probability measure Q_ν with $\nu \in \mathcal{N}_K^\infty$ that

$$P(S,t) = \sup_{\nu \in \mathcal{N}_K^\infty} \mathrm{E}\left[b(S_T^\nu) e^{-\int_t^T (r_s + \tilde{g}_K(\nu_s, s))\, ds} \;\middle|\; S_t^\nu = S \right], \tag{28}$$

where S^ν denotes the solution to the stochastic differential equation

$$dS^\nu(t) = I_S(t)(r(t)\bar{1} - \nu(t))\, dt + I_S(t)\sigma(t)\, dw(t). \tag{29}$$

The difficulty with the characterization in (28) is that the sets $\mathcal{N}_K(t, \omega)$ are not uniformly bounded, so that P is not necessarily a classical solution of the HJB equation corresponding to the stochastic control problem in (28). One possible computational approach would be to consider a sequence of approximate solutions obtained by constraining ν to satisfy a bound of the form $|\nu(t, \omega| \leq n$ $(\lambda \times P)$-a.e., each of which would be a solution to the HJB equation, as in El Karoui and Quenez (1995) and Cvitanić et al. (1997). On the other hand, Broadie et al. (1998) have shown that, in the case of constraints on the portfolio policies and linear wealth dynamics, the solution to (28) satisfies the Black–Scholes (linear) partial differential equation, but with a different terminal condition. Unfortunately, this result does not extend to the present case with nonlinear wealth dynamics. On the other hand, we will show that it is possible to characterize the solution to (28) in terms of a *quasi-linear* partial differential equation with a different terminal condition.

[5] Naik and Uppal (1994) provide related results in a binomial setting assuming no loss of interest on the proceeds from short sales.

Let

$$\tilde{g}(\nu, t, \omega) = \sup_{\pi \in \mathbb{R}^n} \left[g(-\pi, t, \omega) + \pi^\top \nu \right] = \sup_{\pi \in \mathbb{R}^n} \left[g(\pi, t, \omega) - \pi^\top \nu \right]$$

denote the convex conjugate of $-g(-\pi, t, \omega)$ and let $\mathcal{N}(t, \omega)$ denote its effective domain. Similarly, let

$$\tilde{\delta}_K(\nu) = \sup_{\pi \in \mathbb{R}^n} \left[-\delta_K(-\pi) + \pi^\top \nu \right] = \sup_{\pi \in K} [-\pi^\top \nu] \tag{30}$$

denote the convex conjugate of $\delta_K(-\pi)$ and let $\tilde{K}$ denote its effective domain (the barrier cone of $-K$). Clearly, $\tilde{\delta}_K$ is a positively homogeneous function (of degree 1). Moreover, it is easily verified that

$$\mathcal{N}(t, \omega) = \left[-(R(t, \omega) - r(t, \omega)), r(t, \omega) + \iota\lambda_- R(t, \omega) \right]^n, \tag{31}$$

a uniformly bounded set, and

$$\tilde{K} = \begin{cases} \{0\} & \text{if } \lambda_- = 0,\ \lambda_+ = 0 \\ [0, +\infty)^n & \text{if } \lambda_- > 0,\ \lambda_+ = 0 \\ (-\infty, 0]^n & \text{if } \lambda_- = 0,\ \lambda_+ > 0 \\ (-\infty, +\infty)^n & \text{if } \lambda_- > 0,\ \lambda_+ > 0. \end{cases} \tag{32}$$

Let $\mathcal{N}$ denote the set of n-dimensional processes ν such that $\nu(t, \omega) \in \mathcal{N}(t, \omega)$ $(\lambda \times P)$-a.e., and let $\mathcal{K}^\infty$ denote the set of $\tilde{K}$-valued essentially bounded processes.

REMARK 4. Since $g_K(\pi, t) = g(\pi, t) - \delta_K(\pi) \le \min[g(\pi, t), -\delta_K(\pi)]$, we have

$$\tilde{g}_K(\nu, t) \le \min[\tilde{g}(\nu, t), \tilde{\delta}_K(\nu)]. \tag{33}$$

Moreover, it follows from Theorem 16.4 in Rockafellar (1970) that

$$\tilde{g}_K(\nu, t) = \min_{\nu_1 + \nu_2 = \nu} [\tilde{g}(\nu_1, t) + \tilde{\delta}_K(\nu_2)]. \tag{34}$$

Equations (33)–(34) (or direct verification from (12), (31), and (32)) imply that

$$\mathcal{N}_K(t, \omega) = \mathcal{N}(t, \omega) + \tilde{K} \tag{35}$$

and hence that

$$\mathcal{N}_K^\infty = \mathcal{N} + \mathcal{K}^\infty. \tag{36}$$

As in Broadie et al. (1998), define the dominating payoff $\hat{b}$ by

$$\hat{b}(S) = \sup_{\nu \in \tilde{K}} \left[b\left(Se^{-\nu} \right) e^{-\tilde{\delta}_K(\nu)} \right], \tag{37}$$

where $Se^{-\nu} = (S_1 e^{-\nu_1}, \dots, S_n e^{-\nu_n})^\top$.

THEOREM 4. *We have*

$$P(S,t) = \sup_{\nu\in\mathcal{N}} \mathrm{E}\big[\hat{b}(S_T^\nu)e^{-\int_t^T (r_s+\tilde{g}(\nu_s,s))\,ds} \mid S_t^\nu = S\big].$$

Proof. We start by showing that in order to hedge $b(S_T)$ it is necessary to hedge at least $\hat{b}(S_T)$; that is, $\lim_{t\uparrow T} P(S,t) \geq \hat{b}(S)$. For this, fix $S > 0$ and $t \in [0,T)$. Also, let $\{\mu^n\} \subset \tilde{K}$ be such that $b(Se^{-\mu^n})e^{-\delta_K(\mu^n)} \to \hat{b}(S)$ as $n \to +\infty$ and consider the (deterministic) processes $\nu^n \in \mathcal{N}_K^\infty$ with $\nu^n(s) = \mu^n/(T-t)$ for all $s \in [0,T)$. Then (28), (33), the positive homogeneity of $\tilde{\delta}_K$, and the fact that (29) implies

$$S^\nu(T) = S^0(T)e^{-\int_t^T \nu_s\,ds} \tag{38}$$

give

$$\begin{aligned}
P(S,t) &\geq \mathrm{E}\left[b\left(S_T^0 e^{-\int_t^T \nu_s^n\,ds}\right) e^{-\int_t^T (r_s+\tilde{g}_K(\nu_s^n,s))\,ds} \mid S_t^0 = S\right]\\
&\geq \mathrm{E}\left[b\left(S_T^0 e^{-\int_t^T \nu_s^n\,ds}\right) e^{-\int_t^T (r_s+\tilde{\delta}_K(\nu_s^n,s))\,ds} \mid S_t^0 = S\right]\\
&= \mathrm{E}\left[b\left(S_T^0 e^{-\mu^n}\right) e^{-\tilde{\delta}_K(\mu^n)} e^{-\int_t^T r_s\,ds} \;\Big|\; S_t^0 = S\right]
\end{aligned}$$

and hence

$$\lim_{t\uparrow T} P(S,t) \geq b\left(S_T^0 e^{-\mu^n}\right) e^{-\tilde{\delta}_K(\mu^n)}.$$

Letting $n \to +\infty$ shows that $\lim_{t\uparrow T} P(S,t) \geq \hat{b}(S)$. Since in order to hedge $b(S_T)$ it is necessary to hedge at least $\hat{b}(S_T)$, we have from (28), (33), and the fact that $\mathcal{N} \subset \mathcal{N}_K^\infty$ (because of (36))

$$\begin{aligned}
P(S,t) &\geq \sup_{\nu\in\mathcal{N}_K^\infty} \mathrm{E}\left[\hat{b}(S_T^\nu)e^{-\int_t^T (r_s+\tilde{g}_K(\nu_s,s))\,ds} \mid S_t^\nu = S\right]\\
&\geq \sup_{\nu\in\mathcal{N}} \mathrm{E}\left[\hat{b}\left(S_T^\nu\right) e^{-\int_t^T (r_s+\tilde{g}_K(\nu_s,s))\,ds} \mid S_t^\nu = S\right].
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
&\mathrm{E}\left[\hat{b}(S_T^\nu)e^{-\int_t^T (r_s+\tilde{g}(\nu_s,s))\,ds} \;\Big|\; S_t^\nu = S\right]\\
&\quad\geq \mathrm{E}\left[b\left(S_T^\nu e^{-\int_t^T \mu_s\,ds}\right) e^{-\tilde{\delta}_K(\int_t^T \mu_s\,ds)-\int_t^T (r_s+\tilde{g}(\nu_s,s))\,ds} \;\Big|\; S_t^\nu = S\right] \qquad \forall \mu \in \mathcal{K}^\infty
\end{aligned}$$

(because $\tilde{K}$ is a convex cone and hence $\{\int_t^T \mu(s,\omega)\,ds : \mu \in \mathcal{K}^\infty\} \subset \tilde{K}$), we also have

$$\begin{aligned}
&\sup_{\nu\in\mathcal{N}} \mathrm{E}\left[\hat{b}(S_T^\nu)e^{-\int_t^T(r_s+\tilde{g}(\nu_s,s))\,ds} \;\Big|\; S_t^\nu = S\right] \\
&\quad\geq \sup_{\substack{\nu\in\mathcal{N}\\ \mu\in\mathcal{K}^\infty}} \mathrm{E}\left[b\left(S_T^0 e^{-\int_t^T(\mu_s+\nu_s)\,ds}\right) e^{-\tilde{\delta}_K(\int_t^T \mu_s\,ds)-\int_t^T(r_s+\tilde{g}(\nu_s,s))\,ds} \;\Big|\; S_t^0 = S\right] \\
&\quad\geq \sup_{\substack{\nu\in\mathcal{N}\\ \mu\in\mathcal{K}^\infty}} \mathrm{E}\left[b\left(S_T^0 e^{-\int_t^T(\mu_s+\nu_s)\,ds}\right) e^{-\int_t^T(r_s+\tilde{g}(\nu_s,s)+\tilde{\delta}_K(\mu_s)\,ds)\,ds} \;\Big|\; S_t^0 = S\right] \\
&\quad= \sup_{\substack{\nu\in\mathcal{N}\\ \mu\in\mathcal{K}^\infty}} \mathrm{E}\left[b\left(S_T^0 e^{-\int_t^T(\mu_s+\nu_s)\,ds}\right) e^{-\int_t^T(r_s+\tilde{g}_K(\mu_s+\nu_s,s))\,ds)\,ds} \;\Big|\; S_t^0 = S\right] \\
&\quad= \sup_{\nu\in\mathcal{N}_K^\infty} \mathrm{E}\left[b\left(S_T^0 e^{-\int_t^T \nu_s\,ds}\right) e^{-\int_t^T(r_s+\tilde{g}_K(\nu_s,s))\,ds)\,ds} \;\Big|\; S_t^0 = S\right] \\
&\quad= P(S,t),
\end{aligned}$$

where the second (in)equality follows from the fact that $\tilde{\delta}_K(\int_t^T \mu_s\,ds) \leq \int_t^T \tilde{\delta}_K(\mu_s)\,ds$ because δ is a positively homogeneous convex function, the third follows from (34), and the last from (36). □

The above theorem allows us to obtain a PDE characterization of P.

COROLLARY 1. *Suppose that the function $\hat{b}$ in (37) satisfies the polynomial growth condition*

$$\hat{b}(S) \leq k_1(1+S^{\alpha_1})$$

for some $k_1, \alpha_1 > 0$. If P solves the HJB equation

$$\begin{aligned}
(39)\qquad 0 = {}&\frac{1}{2}\,\mathrm{tr}\big(P_{SS}(S,t)I_S\sigma(S,t)\sigma(S,t)^\top I_S\big) + r(S,t)\big(P_S(S,t)^\top S - P(S,t)\big) \\
&+ \max_{\nu\in\mathcal{N}(t)}\Big(-P_S(S,t)^\top S\nu - \tilde{g}(\nu,t)P\Big) + P_t(S,t)
\end{aligned}$$

with terminal condition

$$P(S,T) = \hat{b}(S),$$

and

$$P(S,t) \leq k_2(1+S^{\alpha_2})$$

for some $k_2, \alpha_2 > 0$, then $P(S_t,t)$ is the minimal cost of hedging b at time t.

Proof. This is an immediate consequence of Theorem 4 and of Theorem IV.3.1 in Fleming and Soner (1993). □

In particular, with constant coefficients (r, R, σ) and $n = 1$, it can be easily verified that

$$\tilde{g}(\nu, t) = \begin{cases} \nu^- + (\iota\lambda_-)^{-1}(\nu - (1 + \iota\lambda_-)r)^+ & \text{if } \iota\lambda > 0; \\ \nu^- & \text{otherwise,} \end{cases}$$

so that the HJB equation (39) reduces to

$$0 = \frac{1}{2}\sigma^2 S^2 P_{SS}(S, t) + h(SP_S(S, t), P(S, t)) + P_t(S, t), \tag{40}$$

where

$$h(x, y) = \begin{cases} R(x - y)^+ - r(x - y)^- & \text{if } x \geq 0 \\ R(\iota\lambda_- x + y)^- - r(\iota\lambda_- x + y)^+ & \text{if } x < 0. \end{cases}$$

Also, in this case

$$\tilde{\delta}_K(\nu) = \lambda_-^{-1}\nu^+ + \lambda_+^{-1}\nu^-.$$

As an illustration, for a call option, $b(S) = (S - K)^+$ and equation (37) gives $\hat{b}(S) = b(S)$ if $\lambda_+ = 0$, $\hat{b}(S) = S$ if $\lambda_+ = 1$, and

$$\hat{b}(S) = \begin{cases} S - K & \text{if } S \geq \dfrac{K}{1 - \lambda_+} \\ \dfrac{\lambda_+ K}{1 - \lambda_+}\left(\dfrac{(1 - \lambda_+)S}{K}\right)^{\lambda_+^{-1}} & \text{if } S < \dfrac{K}{1 - \lambda_+} \end{cases}$$

if $0 < \lambda_+ < 1$. In this case, the margin requirement on short-sales is irrelevant. Figure 1 plots the hedging cost as a function of the proportional margin requirement for long positions (λ_+), under the assumption that $S = K = 50$, $R = 0.08$, $\sigma = 0.40$, and $T = 0.75$. The hedging cost equals the Black–Scholes price for the call option when $\lambda_+ = 0$ and converges to the underlying stock price as λ_+ converges to 1.

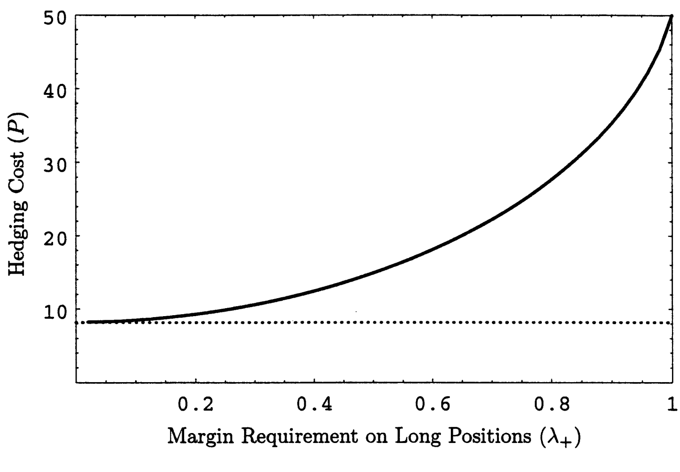

FIGURE 1. The cost of hedging a call option as a function of the proportional margin requirement on long positions (λ_+), assuming $S = K = 50$, $R = 0.08$, $\sigma = 0.40$, and $T = 0.75$. The dotted line corresponds to the Black–Scholes price computed using the borrowing rate.

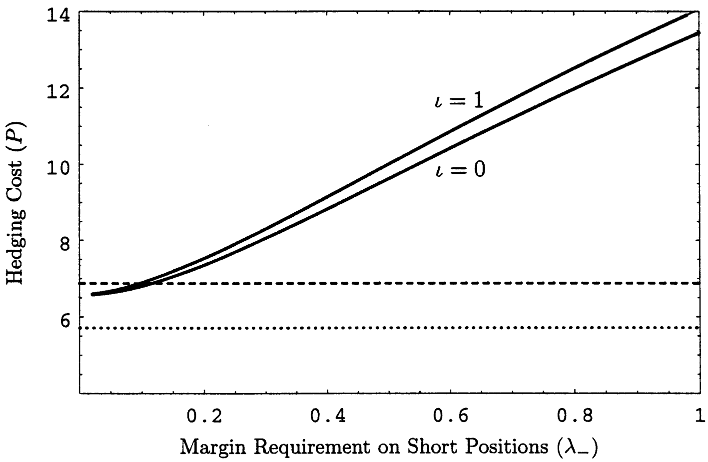

FIGURE 2. The cost of hedging a put option as a function of the proportional margin requirement on short positions (λ_-), assuming $S = K = 50$, $r = 0.06$, $\sigma = 0.40$, and $T = 0.75$. The dotted line corresponds to the Black–Scholes price computed using the lending rate; the dashed line corresponds to the Black–Scholes price computed using a zero interest rate.

For a put option, $b(S) = (K - S)^+$. Moreover $\hat{b}(S) = b(S)$ if $\lambda_- = 0$ and

$$\hat{b}(S) = \begin{cases} K - S & \text{if } S \leq \dfrac{K}{1+\lambda_-} \\ \dfrac{\lambda_- K}{1+\lambda_-}\left(\dfrac{K}{(1+\lambda_-)S}\right)^{\lambda_-^{-1}} & \text{if } S > \dfrac{K}{1+\lambda_-} \end{cases}$$

if $\lambda_- > 0$. In this case, the margin requirement on long stock positions is irrelevant. Figure 2 plots the hedging cost as a function of the proportional margin requirement on short positions (λ_-), under the assumption that $S = K = 50$, $r = 0.06$, $\sigma = 0.40$, and $T = 0.75$. In this case, even with no margin requirements ($\lambda_- = 0$), the hedging cost is higher than the Black–Scholes price for the put option computed using the lending rate, because of the loss of interest on the proceeds from short-sales. Moreover, the hedging cost converges to Ke^{-rT} as λ_- becomes unboundedly large.

7. EXTENSIONS AND GENERALIZATIONS

The results in the paper can be easily extended to cover the case in which the investor is allowed to earn interest on the cash proceeds from short-sales at a rate r_C with $0 \leq r_C \leq r$. Theorems 1–4 and Corollary 1 still hold, provided that r and R are replaced with $r - r_C$ and $R - r_C$, respectively, in the definition of the function g and of the sets $\mathcal{N}_K(t, \omega)$ and $\mathcal{N}(t, \omega)$. Moreover, the PDE (40) still characterizes the hedging cost of contingent claims in the case of constant price coefficients (r_C, r, R, σ) and $n = 1$ provided that the function h in (40) is redefined as

$$h(x, y) = \begin{cases} R(x - y)^+ - r(x - y)^- & \text{if } x \geq 0 \\ R(\iota\lambda_- x + y)^- - r(\iota\lambda_- x + y)^+ + r_C(1 + \iota\lambda_-) & \text{if } x < 0. \end{cases}$$

More generally, all the main results of this paper extend to a class of problems with nonlinear wealth dynamics and constraints on the portfolio policies which includes as special cases those considered in Cvitanić and Karatzas (1992, 1993), El Karoui et al. (1997), and Cuoco and Cvitanić (1998).

Indeed, suppose that the admissible portfolio policies are subject to the constraint (9) for some set $K \subset \mathbb{R}^n$ containing the origin and that the investor's wealth evolves according to the stochastic differential equation (8) for some given function $g : \mathbb{R}^n \times [0, T] \times \Omega \to \mathbb{R}$. We assume as in Cvitanić and Karatzas (1992, 1993) that K is closed and convex. Moreover, we assume as in El Karoui et al. (1997) and Cuoco and Cvitanić (1998) that $g(\cdot, t, \omega)$ is concave and upper semicontinuous on $\mathbb{R}^n$ with $g(0, t, \omega) = 0$ for all $(t, \omega) \in [0, T] \times \Omega$ and that the sets $\mathcal{N}(t, \omega)$ are uniformly bounded (a sufficient condition is that $g(\cdot, t, \omega)$ is uniformly Lipschitz continuous).

Defining the functions g_K and $\tilde{g}_K$ as in Section 3, the result in Proposition 1 still holds—that is, the minimum in (11) is attained. To see this, notice that, for any $\pi \in K$, we have from (11), (34), and (35)

$$
\begin{aligned}
(41) \qquad g(\pi, t, \omega) &= \inf_{\nu \in \mathcal{N}_K(t,\omega)} \left[\tilde{g}_K(\nu, t, \omega) + \pi^\top \nu \right] \\
&= \inf_{\nu \in \mathcal{N}_K(t,\omega)} \left[\min_{\nu_1 + \nu_2 = \nu} [\tilde{g}(\nu_1, t, \omega) + \tilde{\delta}_K(\nu_2)] + \pi^\top \nu \right] \\
&\equiv \inf_{\substack{\nu_1 \in \mathcal{N}(t,\omega) \\ \nu_2 \in \tilde{K}}} \left[\tilde{g}(\nu_1, t, \omega) + \tilde{\delta}_K(\nu_2) + \pi^\top (\nu_1 + \nu_2) \right] \\
&= \inf_{\nu_1 \in \mathcal{N}(t,\omega)} \left[\tilde{g}(\nu_1, t, \omega) + \pi^\top \nu_1 \right] + \inf_{\nu_2 \in \tilde{K}} \left[\tilde{\delta}_K(\nu_2) + \pi^\top \nu_2 \right].
\end{aligned}
$$

Since each set $\mathcal{N}(t, \omega)$ is compact, for any $\pi \in \Pi$ there is a $\nu^* \in \mathcal{N}_K^\infty$ such that

$$
\nu^*(t, \omega) \in \arg \min_{\nu_1 \in \mathcal{N}(t,\omega)} \left[\tilde{g}(\nu_1, t, \omega) + \pi(t, \omega)^\top \nu_1 \right].
$$

Moreover, it follows from (30) that

$$
0 \in \arg \min_{\nu_2 \in \tilde{K}} \left[\tilde{\delta}_K(\nu_2) + \pi(t, \omega)^\top \nu_2 \right].
$$

Equation (41) then shows that

$$
g(\pi(t, \omega), t, \omega) = \tilde{g}_K(\nu^*(t, \omega), t, \omega) + \pi(t, \omega)^\top \nu^*(t, \omega).
$$

Thus, Proposition 1 still holds. In turn, it can be easily checked that this result, together with the assumed properties of g and K, is all that is needed to derive Theorems 1–4.

8. CONCLUDING REMARKS

This paper has examined optimal consumption and investment choices and the cost of hedging contingent claims in the presence of margin requirements. The main results are related to the existence and characterization of optimal policies under fairly general assumptions on the security price coefficients and on the income process and to a simple PDE characterization of the hedging cost in the Markovian case. These results generalize to a class of problems with nonlinear wealth dynamics and constraints on the portfolio policies.

APPENDIX

This Appendix contains the proofs of Theorems 1–3. The techniques used are largely adapted from those in Cvitanić and Karatzas (1992, 1993) and Cuoco and Cvitanić (1998).

Proof of Theorem 1. We start by showing that

$$P(0) \geq \sup_{\nu \in \mathcal{N}_K^0} \mathrm{E}\left[\int_0^T \xi_\nu(s)c(s)\,ds + \xi_\nu(T)W\right].$$

Let $\pi \in \Pi$ be any trading strategy financing (c, W) with initial wealth x and let $W(t) = W^{x,c,C,\pi}(t)$ denote the corresponding wealth process. For any $\nu \in \mathcal{N}_K^0$, we then have by Itô's lemma

$$\begin{aligned}(42)\qquad & \xi_\nu(t)W(t) + \int_0^t \xi_\nu(s)c(s)\,ds + \int_0^t \xi_\nu(s)\,dC(s) \\ &\quad + \int_0^t \xi_\nu(s)W(s)\Big(\tilde{g}_K(\nu(s), s) + \pi(s)^\top \nu(s) - g(\pi(s), s)\Big)\,ds \\ &= x + \int_0^t \xi_\nu(s)W(s)\Big(\pi(s)^\top\sigma(s) + \kappa_\nu(s)^\top\Big)\,dw(s).\end{aligned}$$

Since

$$g(\pi(t), t) = g_K(\pi(t), t) \leq \tilde{g}_K(\nu(t), t) + \pi(t)^\top \nu(t)$$

(because of (11)), the process on the left-hand side of (42) is a nonnegative local martingale, and hence a supermartingale. Thus,

$$\begin{aligned} x &\geq \mathrm{E}\bigg[\xi_\nu(T)W(T) + \int_0^T \xi_\nu(t)c(t)\,dt + \int_0^T \xi_\nu(t)\,dC(t) \\ &\qquad + \int_0^T \xi_\nu(t)\Big(\tilde{g}_K(\nu(t), s) + \pi(t)^\top\nu(t) - g(\pi(t), t)\Big)\,dt\bigg] \\ &\geq \mathrm{E}\left[\xi_\nu(T)W(T) + \int_0^T \xi_\nu(t)c(t)\,dt\right].\end{aligned}$$

This shows that

$$P(0) \geq \sup_{\nu \in \mathcal{N}_K^0} \mathrm{E}\left[\int_0^T \xi_\nu(t)c(t)\,dt + \xi_\nu(T)W\right].$$

Next, we show that

$$P(0) \leq \sup_{\nu \in \mathcal{N}_K^\infty} \mathrm{E}\left[\int_0^T \xi_\nu(t)c(t)\,dt + \xi_\nu(T)W\right].$$

Clearly, we may assume that the above supremum is finite. Let

$$W(t) = \operatorname{ess\,sup}_{\nu \in \mathcal{N}_K^\infty} \beta_\nu(t)^{-1}\,\mathrm{E}^{Q_\nu}\left[\int_t^T \beta_\nu(s)c(s)\,ds + \beta_\nu(T)W \,\middle|\, \mathcal{F}_t\right]$$

and

$$V_\nu(t) = \int_0^t \beta_\nu(s)c(s)\,ds + \beta_\nu(t)W(t).$$

It then follows from the principle of dynamic programming that V_ν is a Q_ν-supermartingale for all $\nu \in \mathcal{N}_K^\infty$ (see Prop. 6.3 in Cvitanić and Karatzas (1993) for details). By the Doob–Meyer decomposition and the martingale representation theorems, this implies that for all $\nu \in \mathcal{N}_K^\infty$ there exists an n-dimensional process φ_ν with $\int_0^T |\varphi(t)|^2\,dt < +\infty$ a.s. and an increasing process C_ν with $C_\nu(0) = 0$ such that

$$V_\nu(t) = W(0) + \int_0^t \varphi_\nu(s)^\top dw_\nu(s) - C_\nu(t). \tag{43}$$

Since

$$\begin{aligned}\beta_\nu(t)^{-1}&\left[V_\nu(t) - \int_0^t \beta_\nu(s)c(s)\,ds\right]\\ &= W(t) = \beta_0(t)^{-1}\left[V_0(t) - \int_0^t \beta_0(s)c(s)\,ds\right]\end{aligned} \tag{44}$$

for all $\nu \in \mathcal{N}_K^\infty$, we have (applying Itô's lemma to the above equality)

$$\beta_\nu(t)^{-1}\varphi_\nu(t) = \beta_0(t)^{-1}\varphi_0(t)$$

and

$$\begin{aligned}\int_0^t \beta_\nu(s)^{-1}\,dC_\nu(s) &- \int_0^t \Big(W(s)\tilde{g}_K(\nu(s), s) + \beta_\nu(s)^{-1}\varphi_\nu(s)^\top\sigma(s)^{-1}\nu(s)\Big)\,ds\\ &= \int_0^t \beta_0(s)^{-1}\,dC_0(s).\end{aligned}$$

Thus, there exists a process π such that

$$\beta_\nu(t)^{-1}\varphi_\nu(t)^\top = W(t)\pi(t)^\top\sigma(t) \tag{45}$$

and

$$\begin{aligned}\int_0^T W(t)\Big(\tilde{g}_K(\nu(t), t) + \pi(t)^\top\nu(t)\Big)\,dt &+ \int_0^T \beta_0(t)^{-1}\,dC_0(t)\\ &= \int_0^T \beta_\nu(t)^{-1}\,dC_\nu(t) \geq 0\end{aligned} \tag{46}$$

for all $\nu \in \mathcal{N}_K^\infty$. Since $W > 0$ a.e., equation (46) implies that $\pi \in \Pi$, as otherwise it would follow from (11) that

$$\inf_{\nu\in\mathcal{N}_K^\infty} \int_0^t W(s)\Big(\tilde{g}_K(\nu(s), s) + \pi(s)^\top\nu(s)\Big)\,ds = -\infty$$

and the above inequality would be violated.

Next, taking $\nu \in \mathcal{N}_K^\infty$ to be the process of Proposition 1, an application of Itô's lemma to (44) (using (43) and (45)) gives

$$\begin{aligned} W(t) &= W(0) + \int_0^t W(s)\Big(r(s) + \pi(s)^\top(\mu(s) - r(s)\bar{1}) + \tilde{g}_K(\nu(s), s) + \pi(s)^\top \nu(s)\Big)\, ds \\ &\quad - \int_0^t c(s)\, ds - \int_0^t \beta_\nu(s)^{-1}\, dC_\nu(s) + \int_0^t W(s)\pi(s)^\top \sigma(s)\, dw(s) \\ &= W(0) + \int_0^t W(s)\Big(r(s) + \pi(s)^\top(\mu(s) - r(s)\bar{1}) + g(\pi(s), s)\Big)\, ds \\ &\quad - \int_0^t c(s)\, ds - C(t) + \int_0^t W(s)\pi(s)^\top \sigma(s)\, dw(s), \end{aligned}$$

where

$$C(t) = \int_0^t \beta_\nu(s)^{-1}\, dC_\nu(s).$$

Thus π finances (c, W) with initial cost $W(0)$, and hence

$$P(0) \le W(0) = \sup_{\nu \in \mathcal{N}_K^\infty} \mathrm{E}^{Q_\nu}\left[\int_0^T \beta_\nu(t)c(t)\, dt + \beta_\nu(T)W\right].$$

Since

$$\begin{aligned} &\sup_{\nu \in \mathcal{N}_K^\infty} \mathrm{E}^{Q_\nu}\left[\int_0^T \beta_\nu(t)c(t)\, dt + \beta_\nu(T)W\right] \\ &\qquad = \sup_{\nu \in \mathcal{N}_K^\infty} \mathrm{E}\left[\int_0^T \xi_\nu(t)c(t)\, dt + \xi_\nu(T)W\right] \\ &\qquad \le \sup_{\nu \in \mathcal{N}_K^0} \mathrm{E}\left[\int_0^T \xi_\nu(t)c(t)\, dt + \xi_\nu(T)W\right], \end{aligned}$$

the above shows that

$$\begin{aligned} P(0) &= \sup_{\nu \in \mathcal{N}_K^0} \mathrm{E}\left[\int_0^T \xi_\nu(s)c(s)\, ds + \xi_\nu(T)W\right] \\ &= \sup_{\nu \in \mathcal{N}_K^\infty} \mathrm{E}^{Q_\nu}\left[\int_0^T \beta_\nu(s)c(s)\, ds + \beta_\nu(T)W\right]. \end{aligned}$$ □

Proof of Theorem 2. Define the consumption policy c^* and the wealth process W as in equations (24) and (25), respectively (the latter is finite because of (23)). In order to prove that c^* is optimal we will proceed in two steps: first we will show that $U(c^*) \ge U(c)$ holds for all feasible consumption processes $c \in \mathcal{C}$, and then that c^* is feasible.

Step 1: By (16), equation (23) implies that

$$\mathrm{E}\left[\int_0^T \xi_{\nu^*}(t) f(\psi \xi_{\nu^*}(t), t)\, dt\right] < +\infty \qquad \forall \psi \in (0, +\infty). \tag{47}$$

By the optimality of ψ^*, we then have

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \frac{J(\psi^* + \varepsilon, \nu^*) - J(\psi^*, \nu^*)}{\varepsilon} \\ &= \mathrm{E}\left[\int_0^T \lim_{\varepsilon \to 0} \frac{\tilde{u}((\psi^* + \varepsilon)\xi_{\nu^*}(t), t) - \tilde{u}(\psi^* \xi_{\nu^*}(t), t)}{\varepsilon}\, dt + W_0\right] \\ &= W_0 - \mathrm{E}\left[\int_0^T \xi_{\nu^*}(t) c^*(t)\, dt\right], \end{aligned}$$

where the second equality follows from Lebesgue's dominated convergence theorem and (47), using the fact that

$$\begin{aligned} &\left| \frac{\tilde{u}\big((\psi^* + \varepsilon)\xi_{\nu^*}(t), t\big) - \tilde{u}\big(\psi^* \xi_{\nu^*}(t), t\big)}{\varepsilon} \right| \\ &\qquad \le \frac{\tilde{u}\big((\psi^* - |\varepsilon|)\xi_{\nu^*}(t), t\big) - \tilde{u}\big(\psi^* \xi_{\nu^*}(t), t\big)}{|\varepsilon|} \\ &\qquad \le \xi_{\nu^*}(t) f\big((\psi^* - |\varepsilon|)\xi_{\nu^*}(t), t\big) \le \xi_{\nu^*}(t) f\Big((\psi^*/2)\xi_{\nu^*}(t), t\Big) \end{aligned}$$

for $|\varepsilon| < \psi^*/2$, because $\tilde{u}(\cdot, t)$ is decreasing and convex, $(\partial/\partial y)\tilde{u}(y, t) = -f(y, t)$, and $f(\cdot, t)$ is decreasing. Therefore,

$$\mathrm{E}\left[\int_0^T \xi_{\nu^*}(t) c^*(t)\, dt\right] = W_0. \tag{48}$$

Next, let $c \in \mathcal{C}$ be any feasible consumption process in (P). Since by concavity

$$u(f(y, t), t) - u(c, t) \ge y[f(y, t) - c] \qquad \forall c > 0, y > 0, \tag{49}$$

we have

$$\begin{aligned} U(c^*) - U(c) &= \mathrm{E}\left[\int_0^T \big(u(c^*(t), t) - u(c(t), t)\big)\, dt\right] \\ &\ge \psi^* \,\mathrm{E}\left[\int_0^T \xi_{\nu^*}(t)(c^*(t) - c(t))\, dt\right] \ge 0. \end{aligned}$$

Hence, c^* must be optimal provided it is feasible.

Step 2: Since the process

$$M(t) = \xi_{\nu^*}(t)W(t) + \int_0^t \xi_{\nu^*}(\tau)c(\tau)\,d\tau$$

is a P-martingale with $M(0) = W_0$, it follows from the martingale representation theorem that there exists an adapted process φ with $\int_0^T |\varphi(t)|^2\,dt < +\infty$ a.s. such that

$$\xi_{\nu^*}(t)W(t) + \int_0^t \xi_{\nu^*}(s)c(s)\,ds = M(t) = W_0 + \int_0^t \varphi(s)^\top dw(s). \tag{50}$$

Define a process π by

$$\xi_{\nu^*}(t)W(t)\Big(\sigma(t)^\top\pi(t) + \kappa_{\nu^*}(t)\Big) = \varphi(t).$$

An application of Itô's lemma to (50) then shows that

$$\begin{aligned} W(t) = W_0 &+ \int_0^t W(s)\big(r(s) + \pi(s)^\top(\mu(s) + \nu^*(s) - r(s)\bar{1}) + \tilde{g}_K(\nu^*(s), s)\big)\,ds \\ &- \int_0^t c(s)\,ds + \int_0^t W(s)\pi(s)^\top\sigma(s)\,dw(s), \end{aligned}$$

so that in order to prove that c is feasible we only need to show that $\pi \in \Pi$ and

$$g(\pi(t), t) = \tilde{g}_K(\nu^*(t), t) + \pi(t)^\top\nu^*(t), \qquad (\lambda \times P)\text{-a.e.} \tag{51}$$

Let $\nu \in \mathcal{N}_K^p$ be arbitrary and define the processes

$$N_\nu(t) = \int_t^T \big(\sigma(s)^{-1}(\nu(s) - \nu^*(s))\big)^\top\big(dw(s) - \kappa_{\nu^*}(s)\,ds\big)$$

and

$$G_\nu(t) = \int_0^t \big(\tilde{g}_K(\nu(s), s) - \tilde{g}_K(\nu^*(s), s)\big)\,ds$$

as well as the stopping times

$$\begin{aligned} \tau_n = T \wedge \inf\Big\{ t \in [0, T] : |N_\nu(t)| \geq n, \text{ or } |G_\nu(t)| \geq n, \\ \text{or} \int_0^t |\sigma(s)^\top(\nu(s) - \nu^*(s))\,ds \geq n, \\ \text{or} \int_0^t \xi_{\nu^*}(s)^2 W(s)^2 |\sigma(s)^{-1}(\nu(s) - \nu^*(s))|^2\,ds \geq n, \\ \text{or} \int_0^t \xi_{\nu^*}(s)^2 W(s)^2 |\sigma(s)^\top\pi(s) + \kappa_{\nu^*}(s)|^2\,ds \geq n \Big\}. \end{aligned}$$

Then $\tau_n \uparrow T$ almost surely. Moreover, letting

$$\nu_{\varepsilon,n}(t) = \nu^*(t) + \varepsilon[\nu(t) - \nu^*(t)]1_{\{t \leq \tau_n\}},$$

we have $\nu_{\varepsilon,n} \in \mathcal{N}_K^p$ for all $\varepsilon \in (0,1)$ (because of the convexity of the sets $\mathcal{N}_K(t,\omega)$) and

$$(52)\quad \xi_{\nu_{\varepsilon,n}}(t) = \xi_{\nu^*}(t)\exp\biggl(-\varepsilon N_\nu(t\wedge\tau_n) - \frac{\varepsilon^2}{2}\int_0^{t\wedge\tau_n} |\sigma(s)^{-1}(\nu(s)-\nu^*(s))|^2\,ds - \int_0^{t\wedge\tau_n}\Bigl(\tilde{g}_K(\nu^*(s)+\varepsilon(\nu(s)-\nu^*(s)),s) - \tilde{g}_K(\nu^*(s),s)\Bigr)\,ds\biggr) \ge \xi_{\nu^*}(t)e^{-3n},$$

where the inequality follows from the definition of τ_n and the fact that

$$(53)\quad \tilde{g}_K(\nu^*(t)+\varepsilon(\nu(t)-\nu^*(t)),t) - \tilde{g}_K(\nu^*(t),t) \le \varepsilon(\tilde{g}_K(\nu(t),t) - \tilde{g}_K(\nu^*(t),t))$$

(because of the convexity of $\tilde{g}_K(\cdot,t)$). This implies

$$(54)\quad \frac{\tilde{u}(\psi^*\xi_{\nu_{\varepsilon,n}}(t),t) - \tilde{u}(\psi^*\xi_{\nu^*}(t),t)}{\varepsilon} \le \psi^* K_n \xi_{\nu^*}(t) f(\psi^* e^{-3n}\xi_{\nu^*}(t),t),$$

where $K_n = \sup_{\varepsilon\in(0,1)}(e^{3\varepsilon n}-1)/\varepsilon$

We then have

$$\begin{aligned}
(55)\quad 0 &\le \lim_{\varepsilon\downarrow 0}\frac{J(\psi^*,\nu_{\varepsilon,n}) - J(\psi^*,\nu^*)}{\varepsilon}\\
&\le \mathrm{E}\biggl[\int_0^T \lim_{\varepsilon\downarrow 0}\frac{\tilde{u}(\psi^*\xi_{\nu_{\varepsilon,n}}(t),t) - \tilde{u}(\psi^*\xi_{\nu^*}(t),t)}{\varepsilon}\,dt\biggr]\\
&= \psi^*\,\mathrm{E}\biggl[\int_0^T \xi_{\nu^*}(t)c^*(t) \times\biggl(N_\nu(t\wedge\tau_n) + \lim_{\varepsilon\downarrow 0} \times\int_0^{t\wedge\tau_n}\frac{\tilde{g}_K(\nu^*(s)+\varepsilon(\nu(s)-\nu^*(s)),s) - \tilde{g}_K(\nu^*(s),s)}{\varepsilon}\,ds\biggr)\,dt\biggr]\\
&\le \psi^*\,\mathrm{E}\biggl[\int_0^T \xi_{\nu^*}(t)c^*(t)(N_\nu(t\wedge\tau_n) + G_\nu(t\wedge\tau_n))\,dt\biggr]\\
&= \psi^*\,\mathrm{E}\biggl[\int_0^{\tau_n} \xi_{\nu^*}(t)c^*(t)(N_\nu(t)+G_\nu(t))\,dt + \xi_{\nu^*}(\tau_n)W(\tau_n)(N_\nu(\tau_n)+G_\nu(\tau_n))\biggr],
\end{aligned}$$

where the first (in)equality follows from the optimality of ν^*, the second from Fatou's lemma (using (54) and (47)), the third from (19) and (52), the fourth from (53), and the last from (25), and the law of iterated expectations.

On the other hand, by Itô's lemma

$$\begin{aligned}
&\xi_{\nu^*}(\tau_n)W(\tau_n)\big(N_\nu(\tau_n)+G_\nu(\tau_n)\big)\\
&\quad=\int_0^{\tau_n}\xi_{\nu^*}(t)W(t)\big(\sigma(t)^{-1}(\nu(t)-\nu^*(t))\big)^\top\big(dw(t)-\kappa_{\nu^*}(t)\,dt\big)\\
&\quad+\int_0^{\tau_n}\xi_{\nu^*}(t)W(t)\big(\tilde g_K(\nu(s),s)-\tilde g_K(\nu^*(s),s)\big)\,ds\\
&\quad-\int_0^{\tau_n}\xi_{\nu^*}(t)c^*(t)\big(N_\nu(t)+G_\nu(t)\big)\,dt\\
&\quad+\int_0^{\tau_n}\xi_{\nu^*}(t)W(t)\big(N_\nu(t)+G_\nu(t)\big)\big(\sigma(t)^\top\pi(t)+\kappa_{\nu^*}(t)\big)^\top dw(t)\\
&\quad+\int_0^{\tau_n}\xi_{\nu^*}(t)W(t)\big(\sigma(t)^\top\pi(t)+\kappa_{\nu^*}(t)\big)^\top\sigma(t)^{-1}(\nu(t)-\nu^*(t))\,dt.
\end{aligned}$$

Since the stochastic integrals in the previous expression are martingales, taking expectations gives

$$\begin{aligned}
\text{(56)}\quad &\mathrm{E}\bigg[\int_0^{\tau_n}\xi_{\nu^*}(t)c^*(t)\Big(N_\nu(t)+G_\nu(t)\Big)\,dt+\xi_{\nu^*}(\tau_n)W(\tau_n)\Big(N_\nu(\tau_n)+G_\nu(\tau_n)\Big)\bigg]\\
&\quad=\mathrm{E}\bigg[\int_0^{\tau_n}\xi_{\nu^*}(t)W(t)\Big(\pi(t)^\top(\nu(t)-\nu^*(t))+\tilde g_K(\nu(t),t)-\tilde g_K(\nu^*(t),t)\Big)\,dt\bigg].
\end{aligned}$$

Substituting the last equality in (55) gives

$$\begin{aligned}
\text{(57)}\qquad &\mathrm{E}\bigg[\int_0^{\tau_n}\xi_{\nu^*}(t)W(t)\Big(\tilde g_K(\nu(t),t)+\pi(t)^\top\nu(t)\Big)\,dt\bigg]\\
&\qquad\ge\mathrm{E}\bigg[\int_0^{\tau_n}\xi_{\nu^*}(t)W(t)\Big(\tilde g_K(\nu^*(t),t)+\pi(t)^\top\nu^*(t)\Big)\,dt\bigg]>-\infty
\end{aligned}$$

for all $\nu\in\mathcal{N}_K^p$ and all n (where the second inequality follows from taking $\nu=0$ in (56)).

Since

$$\inf_{\nu\in\mathcal{N}_K(t,\omega)}[\tilde g_K(\nu,t,\omega)+\pi(t,\omega)^\top\nu]=g_K(\pi(t,\omega),t,\omega)$$

and the right-hand side of the above equation equals $-\infty$ if $\pi(t,\omega)\notin K$, equation (57) implies that $\pi\in\Pi$. Then, taking ν to be the process in Proposition 1, (57) gives

$$\mathrm{E}\bigg[\int_0^{\tau_n}\xi_{\nu^*}(t)W(t)\Big(g_K(\pi(t),t)-\tilde g_K(\nu^*(s),s)-\pi(t)^\top\nu^*(t)\Big)\,ds\bigg]\ge 0.$$

Since equation (11) implies that $g_K(\pi(t),t)-\tilde g_K(\nu^*(s),s)-\pi(t)^\top\nu^*(t)\le 0$, this shows that (51) holds and hence that π finances c^*. □

Proof of Theorem 3. It is easy to see that, under the assumptions of Theorem 3, a sufficient condition for the minimum in (P^*) to be attained is that for all $\psi \in (0, +\infty)$ there exists a solution to the problem

$$\min_{\nu \in \mathcal{N}_K^2} J(\psi, \nu). \tag{58}$$

In fact, letting $V(\psi)$ denote the value function in (58), it is easily verified that V is strictly convex and continuous on $(0, +\infty)$. Moreover, condition (a) and the definition of $\tilde{u}$ imply that

$$\tilde{u}(y, t) > u(0+, t) > -\infty. \tag{59}$$

It then follows from Jensen's inequality, the fact that $\mathrm{E}[\xi_\nu(t)] \leq 1$ for all $\nu \in \mathcal{N}_K^0$ and all $t \in [0, T]$ (because $\beta_\nu \leq 1$ and Z_ν is a supermartingale), and equation (20) that V satisfies the coercivity conditions

$$\lim_{\psi \uparrow +\infty} V(\psi) \geq \int_0^T \tilde{u}(\psi, t)\, dt + \psi W_0 \geq \int_0^T u(0+, t)\, dt + \psi W_0 = +\infty$$

and

$$\lim_{\psi \downarrow 0} V(\psi) \geq \int_0^T \tilde{u}(\psi, t)\, dt + \psi W_0 = +\infty.$$

Therefore, given condition (b), V must attain a (unique) minimum on $(0, +\infty)$, and hence (P^*) has a solution.

By Proposition 2.1.2 in Ekeland and Temam (1976), in order to prove that the problem in (58) has a solution it is sufficient to show that (i) $\mathcal{N}_K^2$ is convex and closed in $L^2(\lambda \times P)$, and (ii) $J(\psi, \cdot)$ is convex, coercive, and lower semicontinuous on $\mathcal{N}_K^2$.

The convexity and closedness of $\mathcal{N}_K^2$ follow from the fact that each of the sets $\mathcal{N}_K(t, \omega)$ is convex and closed in $\mathbb{R}^n$ and the fact that convergence in $L^2(\lambda \times P)$ implies convergence $(\lambda \times P)$-a.e. along a subsequence.

The convexity of $J(\psi, \cdot)$ follows from the fact that the map $\nu \mapsto \log \xi_\nu(t)$ is convex and the map $x \mapsto \tilde{u}(e^x, t)$ is decreasing (by Lemma 1) and convex (by condition (a)). Coercivity follows from the fact that

$$\begin{aligned} J(\psi, \nu) &\geq \mathrm{E}\left[\int_0^T \tilde{u}(\psi Z_\nu(t), t)\, dt\right] \\ &\geq \int_0^T \tilde{u}\Big(\psi \exp\Big(\mathrm{E}[\log Z_\nu(t)]\Big), t\Big)\, dt \\ &= \int_0^T \tilde{u}\left(\psi \exp\left(-\frac{1}{2}\mathrm{E}\int_0^T |\kappa_0(t) + \sigma(t)^{-1}\nu(t)|^2\, dt\right), t\right) dt \end{aligned}$$

(because of Jensen's inequality and the convexity of the map $x \mapsto \tilde{u}(e^x, t)$), together with (20) and condition (a). Lower semicontinuity follows from Fatou's lemma, using (59).

Thus, under conditions (a) and (b), (P^*) has a solution. Moreover, we have from condition (c) that

$$yf(y,t) \leq a + (1-b)u(f(y,t),t),$$

and hence

$$byf(y,t) \leq a + (1-b)[u(f(y,t),t) - yf(y,t)] = a + (1-b)\tilde{u}(y,t),$$

so that

$$\begin{aligned} \mathrm{E}\left[\int_0^T \xi_{\nu^*}(t) f(\psi^* \xi_{\nu^*}(t), t)\, dt\right] &\leq \frac{a}{b\psi^*} + \frac{1-b}{b\psi^*}\,\mathrm{E}\left[\int_0^T \tilde{u}(\psi^* \xi_{\nu^*}(t), t)\, dt\right] \\ &= \frac{a}{b\psi^*} + \frac{1-b}{b\psi^*}(J(\psi^*, \nu^*) - \psi^* W_0) \\ &< +\infty. \end{aligned}$$

Therefore, condition (23) of Theorem 2 is also satisfied, and hence there exists an optimal consumption/investment policy. □

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