Asset Pricing Implications of Short-sale Constraints in Imperfectly Competitive Markets *

Hong Liu† Yajun Wang ‡

December 7, 2015

Abstract

We study the impact of short-sale constraints on market prices and liquidity in imperfectly competitive markets where market makers have significant market power. We show that with or without information asymmetry, short-sale constraints decrease bid, bid depth and trading volume, but increase bid-ask spread and its volatility. If market makers are risk averse, short-sale constraints also increase ask and decrease ask depth. In addition, the impact of short-sale constraints can increase with market transparency. Our results are largely unaffected by endogenization of information acquisition and information revelation.

*JEL Classification Codes: G11, G12, G14, D82.

Keywords: Short-sale Constraints, Bid-Ask Spread, Market Liquidity, Imperfect Competition.

---

*We thank Matthew Ringgenberg, Haoxiang Zhu and seminar participants at 2015 EFA, 2014 CICF, Copenhagen Business School, NUS, University of Southern Denmark, INSEAD, University of Maryland, and Washington University in St. Louis for helpful comments.

†Olin Business School, Washington University in St. Louis and CAFR, liuh@wustl.edu.

‡Robert H. Smith School of Business, University of Maryland, ywang22@rhsmith.umd.edu.
Asset Pricing Implications of Short-sale Constraints in Imperfectly Competitive Markets

Abstract

We study the impact of short-sale constraints on market prices and liquidity in imperfectly competitive markets where market makers have significant market power. We show that with or without information asymmetry, short-sale constraints decrease bid, bid depth and trading volume, but increase bid-ask spread and its volatility. If market makers are risk averse, short-sale constraints also increase ask and decrease ask depth. In addition, the impact of short-sale constraints can increase with market transparency. Our results are largely unaffected by endogenization of information acquisition and information revelation.
1. Introduction

Competition among market makers in many financial markets is far from perfect, even after the introduction of electronic trading platforms (e.g., Christie and Schultz (1994) and Biais, Bisière and Spatt (2010)). Implicit and explicit short-sale constraints are prevalent in many financial markets and especially in those less liquid ones. One of the most robust findings in the existing empirical studies is that short-sale constraints can significantly increase bid-ask spreads.\(^1\) However, as far as we know, the existing theories on how short-sale constraints affect asset prices and market liquidity exclusively focus on perfectly competitive markets and cannot explain this widely documented impact of short-sale constraints on bid-ask spreads.\(^2\)

In this paper, we propose an equilibrium model to study the impact of short-sale constraints in imperfectly competitive markets where market makers have market power. Our analysis highlights that the impact of short-sale constraints on asset prices and market liquidity critically depends on buyers’ market power and risk aversion toward inventory holdings. Our model can also help explain why bid-ask spreads may be significantly wider with short-sale constraints.

We solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed-form in a market with a monopolistic, risk averse market maker in the presence of short-sale constraints with or without asymmetric information. We find if the market maker has significant market power, short-sale constraints decrease bid, bid depth and trading volume, but increase bid-ask spread and its volatility. In addition, as long as the imposition of short-sale constraints itself does not imply strongly negative information and the market maker is risk averse, short-sale constraints also increase ask and decrease ask depth. More public disclosure may increase the adverse impact of short-sale constraints on market liquidity. Our main results still hold for a large set of parameter values when the effect of short-sale constraints on information acquisition and information revelation is taken into account.

Specifically, we consider a one-period equilibrium model and three types of risk averse investors: hedgers, nonhedgers, and a market maker. On date 0, all investors optimally

\(^1\)See, for example, Beber and Pagano (2013), Boehmer, Jones and Zhang (2013), Ang, Shtauber, and Tetlock (2013).

choose how to trade a risk-free asset and a risky security (e.g., a stock in an imperfectly competitive market, a corporate bond) to maximize their expected constant absolute risk aversion utility from the terminal wealth on date 1. Hedgers are subject to a liquidity shock which we model as a random endowment of a nontradable asset whose payoff is correlated with that of the risky security. Hedgers may or may not observe a private signal about the date 1 payoff of the security before trading on date 0 and thus may or may not have trading demand motivated by private information. Both hedgers and nonhedgers trade through the market maker, possibly due to high search costs for non-market-maker counterparty. As in Goldstein, Li, and Yang (2013), because investors have different motives to trade, their reservation prices may differ, which causes trading in equilibrium.

Because short-sale constraints restrict non-market-makers’ selling at the bid, one may expect that their selling price increases in equilibrium, as shown in the existing theories (e.g., Yuan (2006), Wang (2015)), which is exactly the opposite to what we find. One key difference of our model from the existing literature is that competition among some buyers (i.e., market makers) is imperfect in our model. To pinpoint market makers’ market power as the cause of the opposite result, we show in Theorems 2 and 3 in Appendix B that keeping everything else the same as in our model, if market makers did not have market power, then short-sale constraints would indeed increase equilibrium bid prices. The intuition for our opposite result can be illustrated with a simple example. Suppose without short-sale constraints, the short-seller short-sells 10 shares (at the bid) in equilibrium and with short-sale constraints, the short-seller can only short 5 shares. Because the optimal number of shares the short-seller chooses to short decreases as the bid price decreases, a market maker with market power can lower the bid price to the level at which the constraints just start to bind (i.e., at this bid price, the short-seller shorts 5 shares when unconstrained). By doing this, the market maker pays a lower price for the shares without any adverse impact on the number of shares she can buy (still 5 shares). Therefore, because of the market power of the market maker, the equilibrium bid price is lower with short-sale constraints. More generally, when some investors are restricted from selling more, if buyers do not have market power, then they will compete for the reduced supply and thus drive up the equilibrium trading price, as found in the existing literature with competitive markets. On the other hand, if

---

3The reservation price is the critical price such that an investor buys (sells) the security if and only if the ask (bid) is lower (higher) than this critical price.
buyers have significant market power, then the equilibrium price goes down, as found in this paper, because a lower price makes buyers better off without affecting the number of shares they can buy. This shows how short-sale constraints affect the price at which short-sale occurs (bid price) critically depends on whether buyers have significant market power, but other factors such as whether buyers can make offsetting trades at the ask or whether there is information asymmetry or whether market makers are risk averse is not important.

Because the market maker buys less from short-sellers as a result of the short-sale constraints, she also charges a higher ask price and sells less at the ask to achieve optimal inventory risk exposure. The simplest example to explain the intuition is when the market maker is infinitely risk averse. In this case, the market maker does not carry any inventory (and makes profit only from the spread). Therefore, when her purchase at bid decreases as a result of short-sale constraints, she also charges a higher ask price to reduce the sale by the same amount. If the market maker is risk neutral, then the short-sale constraints do not affect ask price or ask depth, but still lower bid and bid depth and increase spread. This is because the change in the inventory risk due to the short-sale constraints is irrelevant for a risk neutral market maker, but the same intuition for the determination of the bid price and bid depth still applies. This shows how short-sale constraints affect ask price and ask depth critically depends on a market maker’s risk aversion. Therefore, while the market maker’s market power is the key driving force behind the result that short-sale constraints increase the bid price, the market maker’saversion to inventory risk is the one that causes the constraints to increase the ask price and decrease the ask depth. To the extent that for small stocks, markets are less competitive and inventories are riskier, our results are consistent with the empirical evidence that bid-ask spreads increase more for small-cap stocks as a result of short-sale bans (e.g., Beber and Pagno (2013)).

Because short-sale constraints increase spread and decrease trading volume, they reduce market liquidity. In addition, with short-sale constraints, bid and ask prices become more sensitive to shocks in the economy because risk sharing is reduced by the constraints. Thus short-sale constraints also make the bid-ask spread more volatile.4

4Although our main model focuses on short-sale constraints, our main results apply also to any constraints that restrict the amount of sales or purchases of non-market-makers by the same mechanism. For example, our model implies that long position limits drive up ask, drive down bid, reduce bid-ask depths, and increase bid-ask spread volatility. This is because with reduced demand at the ask due to the limits, the market maker increases the ask, decreases the ask depth, and if she is risk averse, she also decreases bid and bid
The presence of even significant information asymmetry does not change our main qualitative results. More public disclosure about asset payoff reduces overall risk, can increase investors' trading demand, and thus can make short-sale constraints bind more often and have a greater effect. As a result, the adverse impact of short-sale constraints on prices and market liquidity may be greater in more transparent markets.

Because the imposition of short-sale constraints may change the benefit of private information, we further study whether endogenizing information acquisition invalidates our main results. To this end, we assume that the cost for the private signal about the risky security payoff is an increasing and convex function of the signal’s precision. We find that for a large set of parameter values, our main results, such as the increase in the expected spread and the decrease in trading volume, remain valid. We also show that even when short-sale constraints prevent some negative news from being revealed, our main results still hold.

To our knowledge, Diamond and Verrecchia (1987) (hereafter DV) is the only theoretical paper in the existing literature that considers the effect of short-sale constraints on bid-ask spreads. They show that immediately after the imposition of short-selling prohibition for both the informed and the uninformed, neither the bid nor the ask changes, and thus the bid-ask spread stays the same (see Corollary 2 in DV). In contrast, the existing empirical literature finds that shortly after the short-sale constraint imposition, bid-ask spreads significantly increase (e.g., Beber and Pagano (2013), Boehmer, Jones and Zhang (2013), Ang, Shtauber, and Tetlock (2013)). Like most of the rational expectations models in market microstructure literature (e.g., Glosten and Milgrom (1985), Admati and Pfleiderer (1988)), DV consider a perfect competition market with risk neutral market makers, and thus impose a zero expected profit condition for each trade of a market maker. However, there are other markets where competition among market makers is imperfect (e.g., Christie and Schultz (1994) and Biais, Bisière and Spatt (2010)), market makers are risk-averse to carrying inventory (e.g., Garman (1976), Lyons (1995)), and market makers can make offsetting trades to avoid significant inventory position (e.g., Sofianos (1993), Shachar (2012)). Our model focuses on these markets and generates several empirically testable implications that differ from the existing literature. Table 1 summarizes the differences that may help empirical tests differentiate our model and DV. For example, Table 1 implies that if before a short-sale ban it is mostly the potentially informed hedgers who can short, then after imposing a short-sale ban the depth to buy less to control inventory risk.
Table 1: Comparison of Predictions on Average Bid, Ask and Spread

<table>
<thead>
<tr>
<th>Cases</th>
<th>This Paper</th>
<th>Diamond and Verrecchia (1987)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bid</td>
<td>Ask</td>
</tr>
<tr>
<td>Base case to Case 1</td>
<td>↓</td>
<td>↑</td>
</tr>
<tr>
<td>Base case to Case 2</td>
<td>↓</td>
<td>↑</td>
</tr>
<tr>
<td>Case 2 to Case 1</td>
<td>↓</td>
<td>↑</td>
</tr>
<tr>
<td>Case 2 to Case 3</td>
<td>↓</td>
<td>↓</td>
</tr>
</tbody>
</table>

Base case: Unconstrained; Case 1: Short-sale prohibition for both hedgers (the potentially informed) and nonhedgers (the uninformed); Case 2: Only hedgers can short; Case 3: Only hedgers can short, conditional on a positive short interest.

DV model implies that bid price goes up (because conditional on observing a sell order, the probability that the order comes from the potentially informed is lower), the ask stays the same, and thus the bid-ask spread decreases, while our model predicts that bid price goes down, but ask price and spread go up.

The remainder of the paper proceeds as follows. We first discuss the applicability of models in imperfectly competitive markets and additional literature review in the next section. In Section 3 we present the model. In Section 4 we derive the closed-form equilibrium results. In Section 5 we examine the effect of short-sale constraints on bid/ask prices, bid-ask spread and liquidity risk with and without endogenous information acquisition. We conclude in Section 6. All proofs are in Appendix A. In Appendix B, we provide related results for the competitive case to identify the source of difference of our results from those in the literature.

2. Applicable markets and additional related literature

Competition among market makers in many financial markets such as those for small stocks and corporate bonds is imperfect. For example, Christie and Schultz (1994) suggest that Nasdaq dealers may implicitly collude to maintain wide spread. Biais, Bisière and Spatt (2010) analyze trades and order placement on Nasdaq and a competing electronic order book, Island. They conclude that competition among market makers in these markets is still imperfect even after the introduction of electronic markets. The opaqueness and illiquidity of many dealers markets make them even less competitive (e.g., Ang, Shtauber, and Tetlock
Because there tend to be less liquidity and less trading volume in imperfectly competitive markets, implicit and explicit short-sale constraints are more prevalent in these markets. For example, short selling of small stocks and OTC stocks is difficult and rare, possibly due to low ownership by market makers and institutions (the main lenders of shares), which leads to high short-sale costs. Even though we model short-sale constraints in the form of explicit limit on short positions instead of in the form of short-sale costs, the qualitative results from these two alternative approaches are the same. This is because as short-sale costs increase, investors reduce the amount of short-sales, and thus yielding the same qualitative effect on the amount of short-sales as imposing explicit short-sale constraints. As an extreme example, if short-sale costs are infinity, then it is equivalent to imposing no-short-sale constraints. In addition, explicit short-sale constraints are also often imposed by market making firms in many imperfectly competitive markets (e.g., some dealers markets). For example, Ang, Shtauber, and Tetlock (2013) collect short selling data for a sample of 50 OTC stocks and 50 similarly-sized (small) listed stocks in June 2012 and find “A retail customer of Fidelity could buy all 100 of these stocks, but the broker would allow short selling in only one of the OTC stocks and eight of the listed stocks.”

Although explicit or implicit short-sale constraints are usually more stringent in imperfectly competitive markets, shorting activity is still significant in these markets. For example, Asquith, Au, Covert, and Pathak (2013) find that between 2004 and 2007, trading activity in corporate bond markets averaged $17.3 billion per day with shorting represents 19.1% of all the trades. The stringency of short-sale constraints varies significantly across securities, as suggested by the presence of explicit short-sale constraints for some securities and the cross sectional variation of security lending costs for other ones. This suggests that it is important to understand the impact of short-sale constraints in these markets.

There is a vast literature on the impact of short-sale constraints on asset prices such as Scheinkman and Xiong (2003), and Wang (2015). These models focus on competitive markets and find that short-sale constraints drive up trading prices. As our model, Hong and Stein (2003) and Bai et. al. (2006) show that short-sale constraints can cause trading prices to go down. However, the driving force in Hong and Stein (2003) and Bai et. al. (2006) is the assumption that short-sale constraints prevent some investors from revealing negative information. With less information revealed, uncertainty increases, demand for
assets decreases, and thus prices may go down when this uncertainty effect dominates the effect of reduced sales, as shown in Bai et. al. (2006). When the negative information initially prevented from revealing is revealed later, prices decrease, as shown in Hong and Stein (2003). In contrast, the driving force behind our result that short-sale constraints can lower trading prices is buyers’ market power and therefore our result holds even when there is no information asymmetry. In addition, neither Hong and Stein (2003) nor Bai et. al. (2006) consider the impact of short-sale constraints on ask price or bid-ask spreads. Duffie, Gărleanu, and Pedersen (2002) consider the effect of lending fee on asset prices in a random matching model and find that the prospect of lending fee can push the initial price higher than even the most optimistic evaluation. Although Liu and Wang (2015) use a similar setting to this paper, short-sale constraints are absent and thus they are silent on the impact of short-sale constraints on market prices and liquidity. In addition, while a market maker’s capability of making offsetting trades is critical for the results in Liu and Wang (2015), it is not important for our results. Nezafat, Schroder, and Wang (2014) consider a partial equilibrium model with endogenous information acquisition and short-sale constraints. In contrast to our model, they do not study the impact of short-sale constraints on equilibrium asset prices or bid-ask spreads, and there are no strategic traders in their model.

3. The model

We consider a one period setting with dates 0 and 1. There are a continuum of identical hedgers with mass $N_h$, a continuum of identical non-hedgers with mass $N_n$, and $N_m = 1$ designated market maker. They can trade one risk-free asset and one risky security on date 0 to maximize their constant absolute risk aversion (CARA) utility from the terminal wealth on date 1. There is a zero net supply of the risk-free asset, which also serves as the numeraire and thus the risk-free interest rate is normalized to 0. The total supply of the security is $N \times \bar{\theta} > 0$ shares where $N = N_h + N_n + N_m$ and the date 1 payoff of each share is $\tilde{V}$, where $\tilde{V} \sim N(\bar{V}, \sigma_V^2)$, $\bar{V}$ is a constant, $\sigma_V > 0$, and $N(\cdot)$ denotes the normal distribution. The aggregate risky asset endowment is $N_i \bar{\theta}$ shares for type $i \in \{h, n, m\}$ investors, but no investor is endowed with any risk-free asset.$^5$

$^5$Given the CARA preferences, having different cash endowment would not change any of the results. Throughout this paper, “bar” variables are constants, “tilde” random variables are realized on date 1 and
Hedgers are subject to a liquidity shock that is modeled as a random endowment of \( \hat{X}_h \sim N(0, \sigma_X^2) \) units of a non-tradable risky asset on date 0, with \( \hat{X}_h \) realized on date 0 and only known to hedgers.\(^6\) The non-tradable asset has a per-unit payoff of \( \tilde{L} \sim N(0, \sigma_L^2) \) that has a covariance of \( \sigma_{VL} \) with the the risky asset’s payoff \( \hat{V} \). The payoff of the nontradable asset is realized and becomes public on date 1. The correlation between the non-tradable asset and the security results in a liquidity demand for the risky asset to hedge the nontradable asset payoff.

All trades go through the designated market maker (dealer).\(^7\) The market maker posts her price schedules first. Then hedgers and nonhedgers decide how much to sell to the designated market maker at the bid \( B \) or buy from her at the ask \( A \) or do not trade at all. When deciding on what price schedules to post, the market maker takes into account the best response functions (i.e., the demand schedules) of other investors given the to-be-posted price schedules.\(^8\) In equilibrium, the risk-free asset market also clears.

We assume that both \( h \) and \( n \) investors are subject to short-sale constraints, i.e., the after trade position \( \theta_i + \bar{\theta} \geq -\kappa_i, \ i = h, n \), where \( \kappa_i \geq 0 \) can be different for the hedgers and the nonhedgers, a smaller \( \kappa_i \) means a more stringent short-sale constraint.\(^9\) If \( \kappa_i = 0 \), then type \( i \) investors are prohibited from short selling. If \( \kappa_i = \infty \), on the other hand, then it is equivalent to the absence of short-sale constraints. Heterogeneous short-sale constraint stringency for hedgers and nonhedgers captures the essence of possibly different short-sale costs across them (e.g., Kolasinki, Reed and Ringgenberg (2013)).

Because from equilibrium prices, nonhedgers can infer out the liquidity shock realized

---

\(^6\)The random endowment can represent any shock in the demand for the security, such as a liquidity shock or a change in the needs for rebalancing an existing portfolio or a change in a highly illiquid asset.

\(^7\)Searching for a direct, non-market-maker counterparty can be costly. We assume zero market making cost because a positive cost complicates analysis and does not change our main results.

\(^8\)This can be reinterpreted as a Stackelberg game between the market maker and other investors where the market maker moves first by posting bid and ask price schedules (that depend on order sizes), then other players move by trading the optimal amount given the price schedules. This is equivalent to a setting where other investors submit demand schedules to the market maker, similar to Kyle (1989), Glosten (1994), and Biais, Martimort, and Rochet (2000). See Biais, Martimort, and Rochet (2000) for the equivalence of order size dependent quotes and a sequence of limit orders posted by a market maker as in an order-driven market (e.g., the Paris Bourse, or the Tokyo Stock Exchange).

\(^9\)As in most markets, we assume that a market maker is exempted from short-sale constraints. An alternative way of modeling short-sale constraints is to impose short-sale costs. This alternative model would yield the same qualitative results, because as the costs increase, the amount and frequency of short-sale decrease, as in our model when we decrease \( \kappa_i, \ i = h, n \).
\( \hat{X}_h \), there is no information asymmetry in equilibrium. After observing liquidity shock \( \hat{X}_h \), each hedger chooses a demand schedule \( \Theta_h(\hat{X}_h, \cdot) \) and each nonhedger chooses a demand schedule \( \Theta_n(\hat{X}_h, \cdot) \). The schedules \( \Theta_h \) and \( \Theta_n \) are traders’ strategies. Given bid price \( B \) and ask price \( A \), the quantities demanded by hedgers and nonhedgers can be written as \( \theta_h = \Theta_h(\hat{X}_h, A, B) \) and \( \theta_n = \Theta_n(\hat{X}_h, A, B) \).

Given \( A \) and \( B \), for \( i \in \{h, n\} \), a type \( i \) investor’s problem is to choose \( \theta_i \) to solve

\[
\max E[-e^{-\delta \tilde{W}_i}],
\]

subject to the budget constraint

\[
\tilde{W}_i = \theta_i^- B - \theta_i^+ A + (\tilde{\theta} + \theta_i) \tilde{V} + \hat{X}_i \tilde{L},
\]

and the short-sale constraint

\[
\theta_i + \tilde{\theta} \geq -\kappa_i,
\]

where \( \hat{X}_n = 0 \), \( \delta > 0 \) is the absolute risk-aversion parameter, \( x^+ := \max(0, x) \), and \( x^- := \max(0, -x) \).\(^{10}\)

Since \( h \) and \( n \) investors buy from the designated market maker at ask and sell to them at bid, we can view these trades occur in two separate markets: the “ask” market and the “bid” market. In the ask market, the market maker is the supplier, other investors are demanders and the opposite is true in the bid market. The monopolistic market maker chooses bid and ask prices, taking into account other investors’ demand curve in the ask market and supply curve in the bid market.

Given liquidity shock \( \hat{X}_h \), bid price \( B \) and ask price \( A \), let the realized demand schedules of hedgers and nonhedgers be denoted as \( \Theta_h(\hat{X}_h, A, B) \) and \( \Theta_n(\hat{X}_h, A, B) \) respectively. By market clearing conditions, the equilibrium ask depth \( \alpha \) must be equal to the total amount bought by other investors and the equilibrium bid depth \( \beta \) must be equal to the total amount bought by other investors and the equilibrium bid depth \( \beta \) must be equal to the total amount

\(^{10}\) We have solved the more general case where investors have different liquidity shocks, different information, different initial endowment, and different risk aversions, the results of which are available from authors. Qualitative results on the impact of short-sale constraints are the same because the same intuition still applies. We focus on the current case where all investors have the same risk aversion and only \( h \) investors have liquidity shocks to make the main intuitions as clear as possible and to save space.
sold by other investors,\textsuperscript{11} i.e.,

$$\alpha = \sum_{i=h, n} N_i \Theta_i(\hat{X}_h, A, B)^+, \quad \beta = \sum_{i=h, n} N_i \Theta_i(\hat{X}_h, A, B)^-.$$  \hspace{1cm} (4)

Note that if an investor decides to buy (sell), then only the ask (bid) price affects how much he buys (sells), i.e., $\Theta_i(\hat{X}_h, A, B)^+$ only depends on $A$ and $\Theta_i(\hat{X}_h, A, B)^-$ only depends on $B$, in addition to the dependence on $\hat{X}_h$. Therefore, the bid depth $\beta$ only depends on $B$, henceforth referred as $\beta(B)$ and the ask depth $\alpha$ only depends on $A$, henceforth referred as $\alpha(A)$, suppressing the dependence on $\hat{X}_h$.

We denote market maker’s pricing strategies as $A(\cdot)$ and $B(\cdot)$. For any realized demand schedules $\Theta_h(\hat{X}_h, A, B)$ and $\Theta_n(\hat{X}_h, A, B)$, the designated market maker’s problem is to choose ask price level $A := A(\Theta_h, \Theta_n)$ and bid price level $B := B(\Theta_h, \Theta_n)$ to solve

$$\max E \left[ -e^{-\delta \hat{W}_m} \right],$$  \hspace{1cm} (5)

subject to

$$\hat{W}_m = \alpha(A)A - \beta(B)B + (\bar{\theta} + \beta(B) - \alpha(A)) \hat{V}.$$  \hspace{1cm} (6)

This leads to the definition of an equilibrium.

\textbf{Definition 1} Given any liquidity shock $\hat{X}_h$, an equilibrium $(\Theta_h^*(\hat{X}_h, A, B), \Theta_n^*(\hat{X}_h, A, B), (A^*, B^*))$ is such that\textsuperscript{12}

1. given any $A$ and $B$, $\Theta_i^*(\hat{X}_h, A, B)$ solves a type $i$ investor’s Problem (1) for $i \in \{h, n\}$;

2. given $\Theta_h^*(\hat{X}_h, A, B)$ and $\Theta_n^*(\hat{X}_h, A, B)$, $A^*$ and $B^*$ solve the market maker’s Problem (5).

3.A Discussions on the assumptions of the model

In this subsection, we discuss our main assumptions and discuss whether these assumptions are important for our main results.

\begin{footnotesize}
\textsuperscript{11}The risk-free asset market will be automatically cleared by the Walras’ law. To help remember, Alpha denotes Ask depth and Beta denotes Bid depth.

\textsuperscript{12}The market clearing conditions (Equation (4)) are implicitly enforced in the market maker’s problem.
\end{footnotesize}
The assumption that there is only one market maker is for expositional focus. A model with multiple market makers was solved in an earlier version of this paper, the results of which are available from authors. In this more general model with Cournot competition, we show that our main qualitative results still hold (e.g., short-sale constraints increase expected bid-ask spread). The assumption that the market maker is risk averse is not important for the main results that short-sale constraints decrease bid price and bid depth, but increase bid-ask spread. With a risk neutral market maker, the difference is that short-sale constraints no longer affect ask or ask depth. This is because although short-sale constraints reduce the amount that the market maker can buy at the bid and thus change her inventory level, the risk neutrality makes changed inventory risk irrelevant for her choice of ask and ask depth.

The existing empirical analyses of how short-sale constraints affect spreads focus on the spread difference shortly after the constraint imposition dates. Accordingly, we use a one-period setting to examine the immediate impact of short-sale constraints. This one period setting also helps highlight the main driving forces behind our results and simplify exposition. Extending to a dynamic model would not change the immediate impact of short-sale constraints. More importantly, the qualitative impact of market makers' market power and risk aversion on the effect of short-sale constraints on asset prices would remain the same even over time because the same intuition applies. In illiquid markets such as some dealers markets, it is costly for non-market-makers to find and directly trade with each other. Therefore, most trades are through a small number of market makers, as we assumed in the model. We also assume that the market maker can buy at the bid from some investors and sell at the ask to other investors at the same time. This assumption captures the fact that in many dealers markets, when a dealer receives an inquiry from a client, she often contacts other clients (or dealers) to see at which price and by how much she can unload the inquired trade before she trades with the initial client. However, this capability of the market maker to make offsetting trades simultaneously is not important for our main results. Even when the market maker cannot make an offsetting trade at the ask simultaneously, short-sale constraints still decrease bid, increase ask, and decrease trading volume. This is because

\[13\] In contrast to Bertrand competition, Cournot competition allows market makers to keep some market power, which is critical for our main results. As is well-known, it takes only two Bertrand competitors to reach a perfect competition equilibrium (and thus no market maker has any market power). However, market prices can be far from the perfect competition ones (e.g., Christie and Schultz (1994), Chen and Ritter (2000), and Biais, Bisière and Spatt (2003)).
the market maker’s market power would still result in a lower bid price, and her aversion to inventory risk would still make her raise the next sale price (ask) to achieve the optimal inventory position.

To keep the exposition as simple as possible to show the driving force behind our main results, we assume there is no information asymmetry. We relax this assumption in Section 6 to show the robustness of our results to the presence of information asymmetry. We assume that the market maker posts price schedules first and then other investors submit orders to the market maker. This is consistent with the common practice in dealer markets where a dealer making two-sided markets typically provides a take-it-or-leave-it pair of prices, a bid and an offer, to customers (e.g., Duffie (2012), Chapter 1).

As the existing literature on the impact of short-sale constraints, we do not take into account the possibility that imposition of short-sale constraints itself may convey negative information about the stock payoff. However, the effect of this negative signal is clear from our model: It decreases both bid and ask prices. Therefore, while the result that short-sale constraints increase ask price might be reversed if this negative information effect dominates, the result that short-sale constraints decrease bid price would be strengthened. In addition, if the negative information effect lowers bid and ask by a similar amount, the result that short-sale constraints increase spread would still hold. In the Appendix, we report the results for the other extreme case where market makers have no market power in the bid market to show that it is the market power that causes the bid price to go down with short-sale constraints.

4. The equilibrium

In this section, we solve the equilibrium bid and ask prices, bid and ask depth and trading volume in closed form.

Given \( A \) and \( B \), the optimal demand schedule for a type \( i \) investor for \( i \in \{h, n\} \) is

\[
\theta^*_i(A, B) = \begin{cases} 
\frac{P^R_i - A}{\delta \sigma^2} & A < P^R_i, \\
0 & B \leq P^R_i \leq A, \\
\max \left[ -(\kappa_i + \bar{\theta}) \frac{B - P^R_i}{\delta \sigma^2}, \frac{B - P^R_i}{\delta \sigma^2} \right] & B > P^R_i, 
\end{cases} 
\]  

(7)
where
\[ P_i^R = \bar{V} + \omega \hat{X}_i - \delta \sigma^2 \bar{\theta} \] (8)
is the reservation price of a type \( i \) investor (i.e., the critical price such that non-market-makers buy (sell, respectively) the security if and only if the ask price is lower (the bid price is higher, respectively) than this critical price) and \( \omega = -\delta \sigma_{VL} \) represents the hedging premium per unit of liquidity shock.

Let \( \Delta \) denote the difference in the reservation prices of \( h \) and \( n \) investors, i.e.,
\[ \Delta := P_h^R - P_n^R = \omega \hat{X}_h. \] (9)
The following theorem provides the equilibrium bid and ask prices and equilibrium security demand in closed form.

**Theorem 1** 1. If
\[ -\frac{2(N+1)\delta \sigma^2 (\kappa_h + \bar{\theta})}{N_n + 2} < \Delta < \frac{2(N+1)\delta \sigma^2 (\kappa_n + \bar{\theta})}{N_h}, \]
then short-sale constraints do not bind for any investors and

(a) the equilibrium bid and ask prices are
\[ A^* = P_n^R + \frac{N_h}{2(N+1)} \Delta + \frac{\Delta^+}{2}, \] (10)
\[ B^* = P_n^R + \frac{N_h}{2(N+1)} \Delta - \frac{\Delta^-}{2}, \] (11)

which implies that the bid-ask spread is
\[ A^* - B^* = \frac{|\Delta|}{2} = \frac{|\omega \hat{X}_h|}{2}. \] (12)

(b) the equilibrium quantities demanded are
\[ \theta^*_h = \frac{N_n + 2}{2(N+1)\delta \sigma^2} \Delta, \quad \theta^*_n = -\frac{N_h}{2(N+1)\delta \sigma^2} \Delta, \quad \theta^*_m = 2\theta^*_n; \] (13)

the equilibrium quote depths are
\[ \alpha^* = N_h(\theta^*_h)^+ + N_n(\theta^*_n)^+, \] (14)
\[ \beta^* = N_h(\theta_h^*) - N_n(\theta_n^*). \] (15)

2. If \[ \Delta \leq -\frac{2(N+1)\delta\sigma^2_V(\kappa_h+\bar{\theta})}{N_n+2}, \] then short-sale constraints bind for hedgers and

(a) the equilibrium bid and ask prices are

\[ A^*_{c1} = P^R_n - \frac{\delta N_h \sigma^2_V(\kappa_h + \bar{\theta})}{N_n + 2}, \] (16)

\[ B^*_{c1} = P^R_n + \delta\sigma^2_V(\kappa_h + \bar{\theta}), \] (17)

and bid-ask spread is

\[ A^*_{c1} - B^*_{c1} = \Delta - \frac{N+1}{N_n+2} \delta\sigma^2_V(\kappa_h + \bar{\theta}); \] (18)

(b) the equilibrium quantities demanded are

\[ \theta_{hc1}^* = -(\kappa_h + \bar{\theta}), \quad \theta_{nc1}^* = \frac{N_h(\kappa_h + \bar{\theta})}{N_n + 2}, \quad \theta_{mc1}^* = \frac{2N_h(\kappa_h + \bar{\theta})}{N_n + 2}. \] (19)

the equilibrium quote depths are

\[ \alpha^*_{c1} = \frac{N_h N_n(\kappa_h + \bar{\theta})}{N_n + 2}, \quad \beta^*_{c1} = N_h(\kappa_h + \bar{\theta}). \] (20)

3. If \[ \Delta \geq \frac{2(N+1)\delta\sigma^2_V(\kappa_n+\bar{\theta})}{N_h}, \] then short-sale constraints bind for nonhedgers and

(a) the equilibrium bid and ask prices are

\[ A^*_{c2} = P^R_h - \frac{\Delta + \delta N_n \sigma^2_V(\kappa_n + \bar{\theta})}{N_h + 2}, \] (21)

\[ B^*_{c2} = P^R_h + \delta\sigma^2_V(\kappa_n + \bar{\theta}), \] (22)

and bid-ask spread is

\[ A^*_{c2} - B^*_{c2} = \frac{N_h + 1}{N_h + 2} \Delta - \frac{N+1}{N_n+2} \delta\sigma^2_V(\kappa_n + \bar{\theta}); \] (23)
(b) the equilibrium quantities demanded are

\[
\theta_{hc}^* = \frac{\Delta + \delta N_n \sigma_V^2 (\kappa_n + \bar{\theta})}{(N_h + 2)\delta \sigma_V^2}, \quad \theta_{nc}^* = - (\kappa_n + \bar{\theta}),
\]

(24)

\[
\theta_{mc}^* = -N_h \Delta + 2\delta N_n \sigma_V^2 (\kappa_n + \bar{\theta}),
\]

(25)

the equilibrium quote depths are

\[
\alpha_{c2}^* = \frac{N_h \Delta + \delta N_h N_n \sigma_V^2 (\kappa_n + \bar{\theta})}{(N_h + 2)\delta \sigma_V^2}, \quad \beta_{c2}^* = N_n (\kappa_n + \bar{\theta}).
\]

(26)

Theorem 1 shows that whether short-sale constraints bind depends on whether the magnitude of the reservation price difference is large. If hedgers’ reservation price is close to that of nonhedgers, then no one trades a large amount in equilibrium, and thus short-sale constraints do not bind for any of the investors (Case 1). If hedgers’ reservation price is much smaller than that of nonhedgers, then the equilibrium bid price in the no-constraint case is much higher than the reservation price of hedgers, hedgers would like to sell a large amount, and thus short-sale constraints bind for hedgers (Case 2). Similarly, if hedgers’ reservation price is much greater than that of nonhedgers, on the other hand, then nonhedgers would like to sell a large amount and thus short-sale constraints bind for nonhedgers (Case 3). The thresholds for the reservation price difference such that short-sale constraints bind are determined by equalizing the unconstrained equilibrium short-sale quantities (\(\theta_{hc}^*\) or \(\theta_{nc}^*\)) to the short-sale bounds (\(- (\kappa_h + \bar{\theta})\) or \(- (\kappa_n + \bar{\theta})\) respectively).

Part 1 of Theorem 1 implies that when short-sale constraints do not bind, in equilibrium both bid and ask prices are nonlinear functions of the reservation prices of hedgers and nonhedgers. In addition, nonhedgers can indeed infer \(\hat{X}_h\) from observing the equilibrium trading price as we conjectured, because of the one-to-one mapping between the two. Furthermore, Part 1 shows that when short-sale constraints do not bind, the equilibrium bid-ask spread is equal to the absolute value of the reservation price difference between hedgers and nonhedgers, divided by 2 (more generally by \(N_m + 1\)).

When short-sale constraints bind for hedgers or nonhedgers, the maximum amount of purchase the market maker can make with the constrained investors is fixed, thus the market maker’s utility always decreases in the bid price in the region where the constraints bind.
As explained in the next section, the market power of the market maker then implies that
the optimal bid price when short-sale constraints bind in equilibrium must be such that
short-sale constraints just start to bind, which gives rise to the constrained equilibrium bid
prices as in (17) and (22), and bid depths as in (20) and (26). Given these bid prices and
depths, ask prices and depths are then determined optimally by the market maker to trade
off profit from the spread and inventory risk facing the demand schedules of the buyers.

5. The effect of short-sale constraints

In this section, we analyze the effect of short-sale constraints on bid prices, ask prices, bid-ask
spreads, and liquidity risk.

5.A Bid/ask prices, bid-ask spread, and trading volume

By Theorem 1, we have:

**Proposition 1**

1. As short-sale constraints become more stringent, equilibrium bid price
decreases, equilibrium ask price increases, and so does equilibrium bid-ask spread.

2. As short-sale constraints become more stringent, equilibrium bid depth, ask depth, and
   trading volume decrease.

Because short-sale constraints restrict sales at the bid, one might expect that short-
sale constraints increase the equilibrium bid price. In contrast, Proposition 1 implies that
prohibition of short-sales decreases the bid. We next provide the essential intuition for this
seemingly counterintuitive result and other implications of Proposition 1 through graphical
illustrations. Suppose $P^R_h < P^R_n$ and thus hedgers sell and nonhedgers buy. The market
clearing condition (4) implies that the inverse demand and supply functions faced by the
market maker are respectively

$$
A = P^R_n - \frac{\delta \sigma^2}{N_n} \alpha, \quad B = P^R_h + \frac{\delta \sigma^2}{N_h} \beta.
$$

To make the intuition as simple as possible, we first plot the above inverse demand
and supply functions and equilibrium spreads in Figure 1(a) for the extreme case where
Figure 1: Inverse Demand/Supply Functions and Bid/Ask Prices with and without Short-sale Constraints.
the market maker has infinite risk aversion and no initial endowment of the risky security. Then we illustrate in Figure 1(b) the case where the market maker has the same risk aversion and initial endowment as other investors. Figure 1 shows that as the market maker increases ask (decreases bid) other investors buy (sell) less. Facing the inverse demand and supply functions, a monopolistic market maker optimally trades off profit from the spread and inventory risk. Similar to the results of monopolistic competition models, the bid and ask spread is equal to the absolute value of the reservation price difference $|\Delta|$, divided by 2 (by $N_m + 1$ with multiple market makers engaging in Cournot competition). In Figure 1(a) because the market maker has infinite risk aversion and no initial endowment, the market maker buys and sells the same amount so that there is zero inventory carried to date 1. With short-sale constraints, a market maker can only buy from hedgers up to $N_h(\kappa_h + \bar{\theta})$. Because the market maker has market power and obtains a greater utility with a lower bid price when the amount of purchase at the bid is fixed, the market maker chooses a bid price such that the short-sale constraint never strictly binds. Therefore, if the unconstrained equilibrium sale amount from hedgers is larger than the upper bound $N_h(\kappa_h + \bar{\theta})$, the market maker lowers the bid price such that in the constrained equilibrium, hedgers sell less and the short-sale constraint just starts to bind. Because the market maker buys less from hedgers in equilibrium, the market maker must sell less to nonhedgers at the ask than in the unconstrained case to avoid inventory risk. Therefore, the market maker optimally increases the ask price to achieve the desired reduced amount of sale and as a result, the ask depth is lower. When the market maker has positive but finite risk aversion, the same motive of reducing inventory risk also drives up the ask price and drives down the ask depth, although the market maker may choose to carry some inventory. On the other hand, if the market maker is risk neutral, then because inventory risk is irrelevant for her, she maximizes $A\alpha(A) - B\beta(B) + (\bar{\theta} + \beta(B) - \alpha(A))\bar{V}$ and thus choices of $B$ and $A$ are independent. Therefore, with a risk neutral market maker, short-sale constraints only reduce bid price and bid depth, but do not affect ask price or ask depth. The above intuition suggests that position limits on long position or short position would have the same qualitative impact: increasing ask, decreasing bid, increasing bid-ask spread, and decreasing depths and trading.

---

14 Even though in the main model, we assume that the market maker has the same risk aversion as other investors, this extreme case can be easily solved to yield the results shown in this figure, because the ask depth is always equal to the bid depth and the market maker maximizes only the profit from the spread and carries no inventory.
To further identify the driving force behind the reduction of bid price due to short-sale constraints, in Theorem 5 in Appendix B we report the equilibrium results for an alternative model where the market maker is a price taker in the “bid” market as in most of the existing literature, but a monopolist in the “ask” market as in the main model. Theorem 5 shows the same qualitative results for the impact of short-sale constraints on ask price, depths, and trading volume. However, in contrast to the main model, Theorem 5 implies that short-sale constraints increase equilibrium bid price. Because this alternative model differs from our main model only in that the market maker is a price taker in the “bid” market, this shows that the driving force behind our result that short-sale constraints decrease bid price is indeed the market maker’s market power. If buyers do not have market power (i.e., are price takers), then they compete for the reduced supply and thus the constrained equilibrium price becomes higher. In this case, the constrained equilibrium price is determined by the unconstrained’s (i.e., buyers’) demand. On the other hand, if buyers have all the market power, then because their utility increases as the bid price decreases, they dictate a lower bid price. In this case, the constrained equilibrium price is determined by the constrained’s (i.e., short-sellers’) reduced supply. More generally, if both the constrained short-sellers and the buyers have market power, then the bid price can be higher or lower depending on whether the buyers’ market power is smaller or greater respectively than that of the short-sellers. As far as we know, this paper is the first to show that when buyers have market power, equilibrium selling price can go down when sellers are constrained.

Our model predicts that in markets where market makers have significant market power and are risk averse, imposing short-sale constraints will cause bid prices to go down and ask prices to go up. There is a caveat for this result: as in the existing literature, we do not model the information content of the imposition itself. The imposition of short-sale constraints by regulators may signal some negative information about the stocks being regulated. If this negative information content were taken into account (e.g., in terms of a lower unconditional expected payoff upon the imposition, i.e., smaller $\bar{V}$), then the joint impact of this negative signal and short-sale constraints would lower the bid price further, but might also lower the ask price in the net, because negative information drives both bid price and ask price down.

---

15 The results on the effect of position limits (on both long and short positions) on prices and depths are available from authors.
Figure 2: The percentage changes in expected bid price (dashed), expected ask price (solid), expected mid-price (dot-dashed), and expected bid-ask spread with short-sale constraints against $\sigma_X$. The parameter values are: $\delta = 1$, $\sigma_V = 0.9$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\bar{\theta} = 1/(N_h + N_n + N_m)$, and $\kappa_h = \kappa_n = 0$.

as implied by Theorem 1. On the other hand, because negative information drives both bid price and ask price down, the information content of the imposition of the constraints affects less the result that short-sale constraints increase bid-ask spread, as long as the magnitude of the impact is similar on bid and ask.\textsuperscript{16}

To illustrate the magnitude of the impact, we plot the percentage changes in expected bid, expected ask, and expected spread in Figure 2 and the expected percentage changes in trading volume, bid depth and ask depth in Figure 3 against the liquidity shock volatility $\sigma_X$. Figures 2 and 3 show that short-sale prohibition can have significant impact on expected prices, expected spread, expected depths, and expected trading volume.\textsuperscript{17} For example, the bid price can go down by more than 6% and the ask can go up by more than 4%, even though on average there is no liquidity shock for hedgers.\textsuperscript{18} Figure 3 shows that the impact of short-sale prohibition on depths and trading volume can be as high as more than 80%. In addition,

\textsuperscript{16}If one models the impact of the information content of short-sale constraints imposition as having a lower unconditional expected payoff $\bar{V}$ in the case with short-sale constraints than without, then our model implies that equilibrium bid and ask are lowered by the same amount and thus spread would be unaffected.

\textsuperscript{17}All the figures in the paper are for illustrations of qualitative results only and we do not attempt to calibrate to some imperfectly competitive markets because of the complexity and opaqueness of these markets. $\bar{\theta}$ is chosen to normalize the total supply of the security to 1 share.

\textsuperscript{18}Bid is lower and ask is higher with short-sale constraints. Figure 2 shows that the mid quote price can also go down with short-sale constraints.
the impact of short-sale constraints increases with the liquidity shock volatility. Intuitively, as the liquidity shock volatility increases, the probability that short-sale constraints bind increases. In addition, conditional on constraints binding, the average impact on bid and ask prices also increases. As a result, the unconditional average impact becomes greater with a greater liquidity shock volatility.

5.B Bid-ask spread volatility

One type of liquidity risk faced by investors just before time 0 is that time 0 bid-ask spread is stochastic, depending on realizations of liquidity shock $\hat{X}_h$. The more volatile the spread, the greater the risk.\textsuperscript{19} We next study how short-sale constraints affect time 0 volatility of bid-ask spread caused by the liquidity shock.\textsuperscript{20} To this end, we have

\textsuperscript{19}Note that time 1 bid-ask spread is zero, because payoff becomes publicly known at time 1. This implies that just before time 0, there is only uncertainty about time 0 bid-ask spread, but no uncertainty about time 1 spread. Thus time 0 spread volatility captures total risk due to random spread fluctuation.

\textsuperscript{20}Time 0 volatility of bid-ask spread can also be equivalent to volatility of spread change over time. For example, consider an extension of our model to the same time 0 and time 1 setup as in our main model but include an earlier time $-1$ when all investors are identical. Because all investors are identical at time $-1$, they all have the same reservation prices, which implies that the equilibrium spread at time $-1$ is zero (with and without short-sale constraints). This implies that time 0 volatility of bid-ask spread can also be interpreted as the volatility of the change in the spread between time $-1$ and time 0, i.e., time series...
Proposition 2 Short-sale constraints increase stock’s liquidity risk measured by time 0 volatility of bid-ask spread, i.e., \( \text{Vol}(A^*_c - B^*_c) \geq \text{Vol}(A^* - B^*) \).

The main intuition for Proposition 2 is as follows. When short-sale constraints bind, there is less risk sharing among investors and thus bid and ask prices change more in response to a random shock. For example, keeping everything else constant, we plot the bid-ask spreads as a function of reservation price difference \( \Delta \) for the case without constraints (dashed lines) and the case with short-sale constraints (solid lines) in Figure 4. This figure shows that indeed when the constraints bind, for the same change in the reservation price difference, spread changes more (i.e., steeper lines), which in turn implies that the volatility of spread goes up.

To examine the impact of liquidity shock volatility on the volatility of spread, we plot bid-ask spread volatilities against liquidity shock volatility \( \sigma_X \) for the case where both investors are subject to short-sale constraints (solid), the case without short-sale constraints (dashed), and the case when only nonhedgers are subject to short-sale constraints (dotted). As illustrated in Figure 5, spread volatility is the greatest when both hedgers and nonhedgers are subject to short-sale constraints, which is significantly greater than (almost double) the volatility.
Both $h$ and $n$ are subject to constraints
Only $n$ are subject to constraints
No constraints

Figure 5: The volatility of bid-ask spread with and without short-sale constraints against $\sigma_X$. The default parameters are: $\delta = 1$, $\sigma_V = 0.9$, $\sigma_L = 0.9$, $\sigma_VL = 0.3$, $\bar{V} = 3$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\bar{\theta} = 1/(N_h + N_n + N_m)$, and $\kappa_h = \kappa_n = 0$.

lowest volatility when no one is subject to short-sale constraints. Figure 5 suggests that the impact of short-sale constraints on volatility of spreads can be significant. As the liquidity shock volatility increases, the spread volatility increases because with higher liquidity shock volatility, short-sale constraints bind more often and are also more restrictive on average when they bind.

6. Robustness to information asymmetry and information acquisition

In this section, we examine whether our main results are robust when there is asymmetric information and when information acquisition is endogenous.

6.A Asymmetric Information

In this subsection, we extend our model to incorporate asymmetric information. To make sure private information about the risky asset’s payoff does not affect hedging demand, we decompose the date 1 payoff of each share to $\tilde{V} = \tilde{v} + \tilde{u}$, where $\tilde{v} \sim \mathcal{N}(\bar{V}, \sigma_v^2)$ and $\tilde{u} \sim \mathcal{N}(0, \sigma_u^2)$ are independent with $\sigma_v > 0$, $\sigma_u > 0$, and $\text{Cov}(\tilde{u}, \tilde{L}) = \sigma_{VL}$. 

23
We assume that on date 0, hedgers observe a private signal

\[ \hat{s} = \hat{\psi} - \hat{V} + \hat{\varepsilon} \]  

about the payoff \( \hat{\psi} \), where \( \hat{\varepsilon} \) is independently normally distributed with mean zero and variance \( \sigma^2_{\hat{\varepsilon}} \). We assume that the payoff of the nontraded asset to be independent of the first component \( \hat{\psi} \) (i.e., \( \text{Cov}(\hat{\psi}, \hat{L}) = 0 \)) so that private information about the security payoff does not affect the hedging demand and thus information motivated trades are separated from hedging motivated trades. This way, hedgers’ trades can also be viewed as pooled trades from pure information traders and pure liquidity traders. Assuming it is hedgers who observe the private signal is to preserve information asymmetry in equilibrium. If it were nonhedgers who observe the private signal, then because hedgers know their own liquidity shock, they would be able to infer the private signal precisely from equilibrium prices and thus there would be no information asymmetry in equilibrium.

In addition, we assume that there is a public signal

\[ \hat{S}_s = \hat{s} + \hat{\eta} \]  

about hedgers’ private signal \( \hat{s} \) that all investors (i.e., nonhedgers, the designated market maker, and hedgers) can observe, where \( \hat{\eta} \) is independently normally distributed with mean zero and volatility \( \sigma_{\hat{\eta}} > 0 \). This public signal represents public disclosure about the asset payoff determinants, such as macroeconomic conditions, cash flow news and regulation shocks, which is correlated with but less precise than hedgers’ private signal. While this additional signal \( \hat{S}_s \) is not critical for our main results, it has three main benefits. First, it allows us to examine the impact of public disclosure about the asset payoff on the effect of short-sale constraints. Second, it can serve as a good measure of information asymmetry that does not affect aggregate information quality in the economy (measured by the precision of security payoff distribution conditional on all information in the economy).\(^{22}\) Third, its introduction also makes our model nest models with different degrees of information asymmetry in one

\(^{21}\)Observing the private signal may also be reinterpreted as extracting more precise information from public news (e.g., Engelberg, Reed, and Ringgenberg (2012)).

\(^{22}\)For example, the precision of a private signal about asset payoff would not be a good measure, because a change in the precision also changes the quality of aggregate information about the payoff and both information asymmetry and information quality can affect economic variables of interest (e.g., prices, liquidity).
For each type $i \in \{h, n, m\}$, investors of type $i$ are ex ante identical. Accordingly, we restrict our analysis to symmetric equilibria where all investors of the same type adopt the same trading strategy. Investors’ problems are exactly the same as those in the main model, except that the different information sets. Let $I_i$ represent a type $i$ investor’s information set on date 0 for $i \in \{h, n, m\}$. Because hedgers know exactly $\{\hat{s}, \hat{X}_h\}$, we have

$$E[\tilde{V}|I_h] = \bar{V} + \rho_h \hat{s}, \quad \text{Var}[\tilde{V}|I_h] = (1 - \rho_h)\sigma^2_v + \sigma^2_u,$$

where

$$\rho_h := \frac{\sigma^2_v}{\sigma^2_v + \sigma^2_s}. \quad (30)$$

The hedgers’ reservation price becomes

$$P^R_h = \bar{V} + \hat{S} - \delta((1 - \rho_h)\sigma^2_v + \sigma^2_u)\bar{\theta}, \quad (31)$$

where $\hat{S} := \rho_h \hat{s} + \omega \hat{X}_h$.

Given that the joint impact of $\hat{s}$ and $\hat{X}_h$ on hedgers’ demand is through the composite signal $\hat{S}$, we restrict our analysis to equilibrium prices $A^*$ and $B^*$ that are piecewise linear in the composite signal $\hat{S}$ and the public signal $\hat{S}_s$ such that other investors can infer the value of $\hat{S}$ from market prices. Accordingly, the information sets for the unhedgers and the market maker are

$$I_n = I_m = \{\hat{S}, \hat{S}_s\}. \quad (32)$$

Then the conditional expectation and conditional variance of $\tilde{V}$ for nonhedgers and the market maker are respectively

$$E[\tilde{V}|I_n] = \bar{V} + \rho_n(1 - \rho_X)\hat{S} + \rho_n\rho_X\rho_h\hat{s}_s, \quad (33)$$

$$\text{Var}[\tilde{V}|I_n] = (1 - \rho_n\rho_h)\sigma^2_v + \sigma^2_u. \quad (34)$$

---

23For example, the case where $\sigma_\eta = 0$ implies that nonhedgers and the market maker can perfectly observe $\hat{s}$ from the public signal and thus represents the symmetric information case we considered in the main model. The case where $\sigma_\eta = \infty$, on the other hand, implies that the public signal is useless and thus corresponds to the asymmetric information case as modeled in the standard literature, i.e., there is no public signal about the private information.
where
\[ \rho_X := \frac{\omega^2 \sigma_X^2}{\omega^2 \sigma_X^2 + \rho_n^2 \sigma_n^2}, \quad \rho_n := \frac{\sigma_v^2}{\sigma_v^2 + \rho_X \rho_h \sigma_n^2}. \] (35)

It follows that the reservation price for nonhedgers and the market maker is
\[ P_R^n = P_R^m = \bar{V} + \rho_n (1 - \rho_X) \hat{S} + \rho_n \rho_X \rho_h \hat{S}_s - \delta ((1 - \rho_n \rho_h) \sigma_v^2 + \sigma_n^2) \bar{\theta}. \] (36)

We solve this model with information asymmetry and report the equilibrium results in Theorem 3 in Appendix A, of which Theorem 1 is a special case with \( \sigma_\epsilon = \infty \). Theorem 3 shows that information asymmetry quantitatively changes the prices and quantities, but qualitative results remain the same. For example, when short-sale constraints do not bind, the spread is still equal to half of the absolute value of reservation price difference between hedgers and nonhedgers. When short-sale constraints bind, the bid price is still such that the constraints just start to bind.

More importantly, we show in Appendix A that Proposition 1 holds with information asymmetry. Proposition 1 thus implies that with or without asymmetric information, prohibition of short-sales decreases bid, increases ask, and thus also increases spread. This is in sharp contrast with the conclusion of DV who show that right after a short-sale prohibition imposed on both informed investors and uninformed investors neither the bid nor the ask changes and thus the spread also stays the same (see Corollary 2 in DV).\(^{24}\) The main driving forces for this stark difference between the conclusions of these two models are the market power and the risk aversion of the market maker in our model. To facilitate future empirical analysis, next we compare the predictions of our model with information asymmetry to those of DV for several scenarios. For this purpose, we restrict to four main cases: Base case: No short-sale constraints; Case 1: Short-sale prohibition for both the informed (i.e., hedgers) and the uninformed (i.e., nonhedgers); Case 2: Only the informed can short; Case 3: Only the informed can short, conditional on shorting. We consider the impact of changing from one case to another. Cases 2 and 3 are motivated by empirical evidence that short-sale costs

\(^{24}\)The intuition in DV is that since short-sale prohibition restricts both the informed and the uninformed symmetrically, conditional on a sell order, the percentage of the informed trading does not change and thus the conditional expected payoff stays the same. Because for a risk neutral, competitive market maker, the bid price is equal to the conditional expected payoff, the equilibrium bid price also remains the same. In addition, since ask price is equal to the expected payoff conditional on a buy order and short-sale prohibition does not affect an investor’s purchasing decision in their model, ask price also remains the same.
are smaller for relatively informed investors (e.g., for some institutional traders) than for relatively uninformed investors (e.g., most retail investors). In our model, whenever additional investors, whether informed or uninformed, become effectively subject to short-sale constraints, expected bid price and depths go down, while expected ask price and spread go up. In contrast, in DV, whether the informed or the uninformed become constrained is critical for their prediction. First, consider a change from Base case to Case 2. In DV, because less uninformed investors submit sell orders, the expected payoff conditional on a sell order goes down and thus bid price goes down, which is consistent with our model’s prediction, although for different reasons. In addition, a key differentiation in this case is that our model predicts that ask price goes up, but DV predict that it stays the same. Second, consider a change from Case 2 to Case 1. Because the ban prohibits the informed from shorting and thus a sell order becomes less likely from the informed, the DV model implies that in these markets the ban increases the expected bid price. As explained above, in the DV model, short-sale constraints do not have any impact on ask price because market makers are risk neutral and short-sale constraints do not change the information content of a buy order. This implies that the DV model predicts that the expected spread goes down after the additional short-sale ban on the informed (moving from Case 2 to Case 1). In contrast, our model predicts expected bid goes down, expected ask goes up and thus expected spread goes up. Third, consider a change from Case 2 to Case 3. Because conditional on a short-sale by the informed, the news is on average negative, DV predict that while on average bid price is lower, the ask price does not change, and thus the spread goes up. In contrast, our model predicts both bid and ask go down, but the spread can increase or decrease. We summarize the main differences in predictions in Table 1 of the introduction. One can use these differences in predictions to test which theory applies better in which markets.

To illustrate the impact of information asymmetry, we plot the percentage changes in expected bid, expected ask and expected spread against the information asymmetry measure $\sigma_\eta$ in Figure 6.

Figure 6 shows that indeed short-sale constraints always decrease expected bid, increase expected ask even in the presence of asymmetric information. In addition, as information asymmetry increases, the magnitudes of the percentage changes in the expected ask, expected bid and the expected spread can all decrease. This is because as information asymmetry increases, both the adverse selection effect and the uncertainty faced by the uninformed
Figure 6: The percentage changes in expected bid price (dashed), expected ask price (solid), expected mid-price (dot-dashed), and expected bid-ask spread with short-sale constraints against $\sigma_\eta$. The parameter values are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_v = 0.9$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $\sigma_X = 0.8$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\bar{\theta} = 1/(N_h + N_n + N_m)$, and $\kappa_h = \kappa_n = 0$.

Figure 7: The percentage changes in expected bid depth (dashed), ask depth (solid), expected trading volume (dot-dashed) with short-sale constraints, and the volatility of bid-ask spread with and without short-sale constraints against $\sigma_\eta$. The default parameter values are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_v = 0.9$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $\sigma_X = 0.8$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\sigma_\eta = 0.4$, $\bar{\theta} = 1/(N_h + N_n + N_m)$, and $\kappa_h = \kappa_n = 0$. 

28
increase, consequently investors may trade less on average, which results in the constraints binding less.

In the left panel of Figure 7, we plot the expected percentage changes in trading volume, bid depth and ask depth against the information asymmetry measure $\sigma_\eta$. Figure 7 shows that short-sale constraints always decrease expected depths and trading volume. In addition, as information asymmetry increases, the impact of short-sale prohibition on depths and trading volume decreases due to less trading on average.

To examine the impact of information asymmetry on the volatility of spread, we plot bid-ask spread volatilities against the information asymmetry measure $\sigma_\eta$ for the case where both investors are subject to short-sale constraints (solid), the case without short-sale constraints (dashed), and the case when only uninformed are subject to short-sale constraints (dotted). As illustrated in the right panel of Figure 7, spread volatility always increases as a result of short-sale constraints even in the presence of asymmetric information. In addition, as information asymmetry increases, the volatility increase caused by the short-sale constraints decreases because with higher uncertainty and greater adverse selection effect investors tend to trade less and the short-sale constraints bind less often. Thus more public disclosure and greater transparency which reduces information asymmetry can increase the impact of short-sale constraints on the spread volatility.

6.B Robustness with information acquisition

So far, we have assumed that hedgers’ information quality is not affected by the imposition of short-sale constraints. We next examine whether our results can still hold when aggregate information quality is indeed affected by imposing short-sale constraints.

To this extent, we assume that, on date 0, hedgers can acquire a costly signal $\hat{s}$ as defined in (27) with precision of $\rho_\varepsilon = \frac{1}{\sigma^{2}_\varepsilon}$ at a cost of $c(\rho_\varepsilon) := k\rho^{2}_\varepsilon$, where $k$ is a positive constant.

As expected, for moderate information acquisition cost, there exists an optimal level of precision for hedgers who trade off the benefit from trading with more precise private information and the cost of acquiring more precise private information. We find that the optimal precision of private information for hedgers in the presence of short-sale constraints tend to be lower than that in the absence of short-sale constraints. Intuitively, the presence of short-sale constraints may reduce the incentive of investors to produce more precise information
Figure 8: The percentage changes of expected bid and ask prices, and the optimal precisions of information with and without short-sale constraints. The default parameters are: \( \delta = 1 \), \( \sigma_u = 0.4 \), \( \sigma_v = 0.9 \), \( \sigma_L = 0.9 \), \( \sigma_{VL} = 0.3 \), \( \bar{V} = 3 \), \( \sigma_X = 0.8 \), \( N_h = 1 \), \( N_m = 1 \), \( N_n = 10 \), \( \bar{\theta} = 0.01 \), \( \kappa_h = \kappa_n = 0 \), and \( k = 0.001 \).
because short-sale constraints prevent them from benefiting from the private information in some states.

Most importantly, Figure 8 shows that short-sale constraints may still increase the expected ask price and spread volatility, and decrease the expected bid price even after the impact of short-sale constraints on endogenous information acquisition is taken into account. For a large set of parameter values, we obtain similar patterns to those in Figure 8. This suggests in particular that the main result that short-sale constraints increase bid-ask spread is largely unaffected by endogenization of information acquisition. In addition, Figure 8 illustrates that more disclosure (i.e., smaller \( \sigma_\eta \)) might actually increase the incentive of hedgers to acquire more precise private information. This is because public disclosure reduces information asymmetry and the loss of hedgers from the adverse selection problem. Figure 8 also suggests that the optimal precision increases with liquidity shock. Intuitively, high liquidity shock volatility tends to increase hedgers’s trading volume and thus makes them benefit more from more precise information.

6.C Model extensions with heterogenous information and reduced information revelation

In the model with asymmetric information, all hedgers have the same information and all submit orders that reveal the composite signal \( \hat{S} \) in all states. In addition, there is no restriction on the width of the spread that the market maker can choose. We now extend our model to include multiple hedgers with different private information and their orders might not fully reveal the composite signal \( \hat{S} \). We also allow initial endowment of the risky asset, liquidity shocks, and risk aversions to differ across investors. To illustrate the impact of reduced market power and to keep tractability, we consider the extreme case where the spread must be set equal to zero, i.e., bid must be equal to ask, which approximates the practice of restricting bid-ask spread by regulators to improve market liquidity.\(^{25}\)

\(^{25}\)The market maker still has (reduced) market power with the spread restricted to 0 because she chooses the optimal price to maximize her expected utility taking into account the demand schedules of other investors.
6.C.1 Robustness with heterogenous information

Let $\bar{\theta}_i$, $\delta_i$, $\bar{X}_i$, $\bar{V}_i$ and $\mathcal{I}_i$ denote respectively the initial inventory, the risk aversion coefficient, the liquidity shock, the date 1 resale value of the security and the information set for a type $i$ investor for $i \in \{h, n, m\}$. Given price $P$, for $i \in \{h, n\}$, a type $i$ investor’s problem is to choose $\theta_i$ to solve

$$\max E[-e^{-\delta_i \bar{W}_i}|\mathcal{I}_i],$$

subject to the budget constraint

$$\bar{W}_i = -\theta_i P + (\bar{\theta}_i + \theta_i)\bar{V}_i + \bar{X}_i \bar{L},$$

and the short-sale constraint

$$\theta_i + \bar{\theta}_i \geq -\kappa_i,$$

where $\delta_i > 0$ is the absolute risk-aversion parameter. By the same argument as before, a type $i$ investor’s reservation price can be written as

$$P^R_i = E[\bar{V}_i|\mathcal{I}_i] - \delta_i \text{Cov}[\bar{V}_i, \bar{L}|\mathcal{I}_i] \bar{X}_i - \delta_i \text{Var}[\bar{V}_i|\mathcal{I}_i] \bar{\theta}_i, \ i \in \{h, n\}.$$

Then the designated market maker’s problem is to choose price $P$ to solve

$$\max E \left[ -e^{-\delta_m \bar{W}_m}|\mathcal{I}_m \right],$$

subject to

$$\bar{W}_m = -\sum_{i=h,n} \min \left[ \frac{P - P^R_i}{\delta_i \text{Var}[\bar{V}_i|\mathcal{I}_i]} \kappa_i + \bar{\theta}_i \right] P + \left( \theta_m + \sum_{i=h,n} \min \left[ \frac{P - P^R_i}{\delta_i \text{Var}[\bar{V}_i|\mathcal{I}_i]} \kappa_i + \bar{\theta}_i \right] \right) \bar{V}.$$  

Let $\Delta_{ij} := P^R_i - P^R_j$ denote the reservation price difference between type $i$ and type $j$ investors for $i,j \in \{h, n, m\}$. Define

$$P_h = P^R_h + \delta_h (\kappa_h + \bar{\theta}_h) \text{Var}[\bar{V}_h|\mathcal{I}_h], \ P_n = P^R_n + \delta_n (\kappa_n + \bar{\theta}_n) \text{Var}[\bar{V}_n|\mathcal{I}_n],$$

$$P^* = \varphi_m P^R_m + \varphi_n P^R_n + (1 - \varphi_m - \varphi_n) P^R_h,$$
\begin{align*}
P^h_c &= \lambda_m P^R_m + (1 - \lambda_m) P^R_h - \lambda_h \delta_h (\kappa_h + \bar{\theta}_h) \text{Var}[\hat{V}_h|I_h], \quad (45) \\
P^n_c &= \gamma_m P^R_m + (1 - \gamma_m) P^R_h - \gamma_n \delta_n (\kappa_n + \bar{\theta}_n) \text{Var}[\hat{V}_n|I_n], \quad (46)
\end{align*}

where \( \varphi_m, \varphi_n, \lambda_m, \lambda_h, \gamma_m, \) and \( \gamma_n \) are as defined in (A-30)-(A-33) in the Appendix. \( \bar{P}_h (\bar{P}_n) \) is the critical price above which short-sale constraints bind for \( h (n) \) investors, \( P^* \) is the equilibrium price in the absence of short-sale constraints, and \( P^h_c (P^n_c) \) is the equilibrium price when short-sale constraints bind only for \( h \) (only for \( n \)). In addition, let \( V(P) \) denote the market maker’s expected utility in the absence of short-sale constraints and \( V^h_c(P) \) be the market maker’s expected utility in the presence of short-sale constraints which bind only for hedgers. For the model with reduced market power, we report the results for the case with \( \bar{P}_h \leq \bar{P}_n \). The case \( \bar{P}_h \geq \bar{P}_n \) is symmetric (i.e., switching “h” and “n” in the notations in Theorem 2) and thus omitted to save space.

**Theorem 2** Suppose \( \bar{P}_h \leq \bar{P}_n \).

1. If \( P^* \leq \bar{P}_h \) and \( P^h_c \leq \bar{P}_h \), short-sale constraints do not bind for any investor, and the equilibrium price is \( P^*_c = P^* \);

2. If \( P^* \leq \bar{P}_h \) and \( \bar{P}_h < P^h_c \leq \bar{P}_n \), then the equilibrium price is

\[
P^*_c = \begin{cases} 
  P^* & \text{if } V(P^*) > V^h_c(P^h_c), \\
  P^* \text{ or } P^h_c & \text{if } V(P^*) = V^h_c(P^h_c), \\
  P^h_c (> P^*) & \text{if } V(P^*) < V^h_c(P^h_c), 
\end{cases}
\]

(47)

and if the equilibrium price is equal to \( P^h_c \) then short-sale constraints bind for hedgers;

3. If \( P^* > \bar{P}_h \geq P^h_c \), short-sale constraints bind for hedgers, and the equilibrium price is \( P^*_c = \bar{P}_h (< P^*) \);

4. If \( P^* > P^h_c > \bar{P}_h \) and \( P^h_c \leq \bar{P}_n \), short-sale constraints bind for hedgers, and the equilibrium price is \( P^*_c = P^h_c (< P^*) \);

5. If \( \bar{P}_n \geq P^h_c \geq P^* > \bar{P}_h \), then short-sale constraints bind for hedgers, and the equilibrium price is \( P^*_c = P^h_c (\geq P^*) \);

6. If \( P^h_c > \bar{P}_n > P^* > \bar{P}_h \), then short-sale constraints bind for both hedgers and nonhedgers, and the equilibrium price is \( P^*_c = \bar{P}_n (> P^*) \);

\[ \text{It can be shown that } P^* \leq \bar{P}_h \text{ and } P^h_c > \bar{P}_n \text{ cannot happen under the assumption that } \bar{P}_h \leq \bar{P}_n. \]
7. If $P^* \geq \bar{P}_n$ and $P^h_c > \bar{P}_n$, then short-sale constraints bind for both hedgers and nonhedgers, and the equilibrium price is $P^*_c = \bar{P}_n$ ($\leq P^*$).

It can be shown that as the reservation price of the market maker varies from low to high, all 7 cases in Theorem 2 can occur. It is clear from Equations (43) through (46) that $P^* - \bar{P}_h$ and $P^h_c - \bar{P}_h$ both increase with $P^R_m$. The intuition for Theorem 2 is that if the reservation price of the market maker is small enough relative to those of hedgers and nonhedgers, then the market maker will be the only seller in equilibrium and thus short-sale constraints do not bind for any investor and the equilibrium price is the same as the case without short-sale constraints (Case 1). In the other extreme, if the the reservation price of the market maker is large enough, then the market maker is the only buyer and short-sale constraints bind for both hedgers and nonhedgers. In addition, the equilibrium price is equal to $\bar{P}_n$, which is smaller than the optimal price in the absence of short-sale constraints (i.e., $P^*$, Case 7). This shows that even with reduced market power, the equilibrium price can also be lower with short-sale constraints (e.g., Cases 3, 4, and 7).

In contrast to the results in the main model, Theorem 2 shows that the equilibrium price with short-sale constraints can be greater than the equilibrium price without the constraints when buyers have insufficient market power. For example, in Case 5 short-sale constraints bind for hedgers. If nonhedgers’ reservation price is high enough, then nonhedgers are buyers and the market maker is a seller in equilibrium. In this case, short-sale constraints reduce supply, the equilibrium price becomes higher because buyers (i.e., nonhedgers) do not have any market power, as in the existing literature that focuses on a competitive setting. In addition, it is also possible that the equilibrium trade price becomes higher with the constraints than without, even when the market maker is a buyer in equilibrium (e.g., Case 6). This is due to the reduced market power of the market maker. These results highlight that the key driving force behind our main result that short-sale constraints drive down bid price is buyers’ market power.

6.C.2 Robustness with reduced information revelation

The above analysis still assumes that short-sale constraints do not affect information revealed by the orders submitted by the constrained. Next we show that even when short-sale constraints reduce information revelation, the average equilibrium sale (bid) price with constraints may still be lower than without. For this purpose, we consider the model in Subsubsection 6.C.1, specializing to the case where hedgers have zero initial endowment and nonhedgers have different amount of initial endowment from the market maker. As shown in Appendix A, hedgers’ demand increases with the composite signal $\hat{S}$ which combines the hedging demand and information motivated demand. When
the composite signal \( \hat{S} \) is smaller than a threshold \( S \), hedgers would like to short sell, but with a short-sale constraint that prevents any short-selling. In contrast to the model in Subsubsection 6.C.1, we assume that hedgers do not submit any order in this case and thus do not reveal the value of \( \hat{S} \). Nonhedgers and the market maker accordingly update their beliefs conditional on the composite signal \( \hat{S} < S \) and thus information revelation is reduced by the presence of short-sale constraints. We provide the analysis details of this case at the end of Appendix A.

The equilibrium price is a constant from all \( \hat{S} < S \) and there is a discontinuous jump downward at \( \hat{S} = S \), as shown in Figure 9. Loosely speaking, when nonhedgers and the market maker only know that \( \hat{S} < S \), they use some conditional average of \( \hat{S} \) for the estimation of \( \hat{S} \), and therefore they underestimate \( \hat{S} \) for \( S < \hat{S} < S \), which is reflected by the downward discontinuity at \( S \). On the other hand, they overestimate \( \hat{S} \) for \( \hat{S} < S \), as shown in Figure 9.

Figure 10, where the market maker sells in equilibrium, and Figure 11, where the market maker buys in equilibrium, show that even when short-sale constraints prevent some of the information of hedgers being revealed, conditional on \( \hat{S} < S \) (i.e., short-sale constraints bind), the expected trading price with short-sale constraints can still be lower than that without the constraints, and therefore our main results still hold. Intuitively, because nonhedgers and the market maker underestimate \( \hat{S} \) for \( S < \hat{S} < S \) but overestimate \( \hat{S} \) for \( \hat{S} < S \), this translates to a lower equilibrium price for \( S < \hat{S} < S \) but a higher equilibrium price for \( \hat{S} < S \) than the unconstrained equilibrium price, as shown in Figure 9. Therefore, as long as the probability of \( S < \hat{S} < S \) is significantly higher than the probability of \( \hat{S} < S \), the equilibrium price with short-sale constraints for \( \hat{S} < S \) is lower than the expected price without the constraints, conditioned on \( \hat{S} < S \). Because the equilibrium prices
Figure 10: The expected unconstrained equilibrium price $P^*$ and the constrained equilibrium price $P^*_c$ conditional on $\hat{S} < \underline{S}$ against $\sigma_X$ and $\bar{\theta}_n$ when the market maker sells in equilibrium. The default parameters are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_v = 0.4$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $\sigma_X = 0.3$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\bar{\theta}_n = 0.1$, $\bar{\theta}_m = 0.6$, and $\kappa_h = \kappa_n = 0$.

Figure 11: The expected unconstrained equilibrium price $P^*$ and the constrained equilibrium price $P^*_c$ conditional on $\hat{S} < \underline{S}$ against $\sigma_X$ and $\bar{\theta}_n$ when the market maker buys in equilibrium. The default parameters are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_v = 0.4$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $\sigma_X = 0.3$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\bar{\theta}_n = 2$, $\bar{\theta}_m = 0.2$, and $\kappa_h = \kappa_n = 0$.

are the same with and without constraints for $\hat{S} \geq \underline{S}$, the (unconditional) expected equilibrium price may also be lower with short-sale constraints.
7. Conclusions

In this paper, we develop an equilibrium model to help explain the empirical evidence that short-sale constraints tend to increase bid-ask spread. In contrast to the existing literature, our analysis suggests that if market makers have significant market power, then short-sale constraints drive bid price down and if in addition market makers are risk averse, then short-sale constraints drive ask price up. This implies a greater equilibrium bid-ask spread. Furthermore, short-sale constraints decrease market trading volume and increase liquidity risk measured by the volatility of bid-ask spreads. More public disclosure that reduces information asymmetry can further magnify the adverse impact of short-sale constraints on asset prices and market liquidity. Furthermore, the main result that short-sale constraints increase bid-ask spread is largely unaffected by endogenization of information acquisition and partial revelation of negative information that might be caused by the constraints.

Our model provides some new empirically testable implications. For example, in markets where market makers have significant market power, short-sale constraints decrease average bid and average trading volume, but increase average spread and spread volatility; and the impact of short-sale constraints is greater in more transparent markets.
References


Appendix A

We first state the main results for the extended model with asymmetric information. Let \( \Delta \) denote the difference in the reservation prices of \( h \) and \( n \) investors, i.e.,

\[
\Delta := P^R_h - P^R_n = (1 - \rho_n) \left( \left( \frac{\sigma^2 \nu}{\rho_h \hat{s}} \right) \hat{S} - \frac{\sigma^2}{\sigma^2 \nu} \hat{S}_s + \delta \rho_h \sigma^2 \theta \right)
\]  

(A-1)

Let

\[
\nu := \frac{\text{Var}[\hat{V}|I_h]}{\text{Var}[\hat{V}|I_n]} \geq 1
\]

be the ratio of the security payoff conditional variance of nonhedgers to that of hedgers, and

\[
\bar{N} := \nu N_h + N_n + 1 \geq N
\]

be the information weighted total population. The following theorem provides the equilibrium bid and ask prices and equilibrium security demand in closed form.

**Theorem 3** 1. If

\[
-2 \frac{(N+1)\delta \text{Var}[\hat{V}|I_h]|(\kappa_h + \bar{\theta})}{N_n + 2} < \Delta < 2 \frac{(N+1)\delta \text{Var}[\hat{V}|I_n]|(\kappa_n + \bar{\theta})}{\nu N_h},
\]

then short-sale constraints do not bind for any investors and

(a) the equilibrium bid and ask prices are

\[
A^* = P^R_n + \frac{\nu N_h}{2 (N + 1)} \Delta + \frac{\Delta^+}{2}, \quad (A-2)
\]

\[
B^* = P^R_n + \frac{\nu N_h}{2 (N + 1)} \Delta - \frac{\Delta^-}{2}, \quad (A-3)
\]

which implies that the bid-ask spread is

\[
A^* - B^* = \frac{\Delta}{2} = \frac{(1 - \rho_n) \left( \left( \frac{\sigma^2 \nu}{\rho_h \hat{s}} \right) \hat{S} - \frac{\sigma^2}{\sigma^2 \nu} \hat{S}_s + \delta \rho_h \sigma^2 \theta \right)}{2}. \quad (A-4)
\]

(b) the equilibrium quantities demanded are

\[
\theta^*_h = \frac{N_n + 2}{2 (N + 1) \delta \text{Var}[\hat{V}|I_h]} \Delta, \quad \theta^*_n = -\frac{\nu N_h}{2 (N + 1) \delta \text{Var}[\hat{V}|I_n]} \Delta, \quad \theta^*_m = 2 \theta^*_n, \quad (A-5)
\]
the equilibrium quote depths are

$$\alpha^* = N_h(\theta^*_h)^+ + N_n(\theta^*_n)^+, \quad (A-6)$$

$$\beta^* = N_h(\theta^*_h)^- + N_n(\theta^*_n)^-.$$  \hspace{1cm} (A-7)

2. If $\Delta \leq -\frac{2(N+1)\delta \text{Var}[\widetilde{V} | I_h](\kappa_h + \bar{\theta})}{N_n+2}$, then short-sale constraints bind for hedgers and

(a) the equilibrium bid and ask prices are

$$A^*_{c1} = P_n^R - \frac{\delta N_h \text{Var}[\widetilde{V} | I_h](\kappa_h + \bar{\theta})}{N_n+2}, \quad (A-8)$$

$$B^*_{c1} = P_h^R + \delta \text{Var}[\widetilde{V} | I_h](\kappa_h + \bar{\theta}), \quad (A-9)$$

and bid-ask spread is

$$A^*_{c1} - B^*_{c1} = -\Delta - \frac{\bar{N} + 1}{N_n+2} \delta \text{Var}[\widetilde{V} | I_h](\kappa_h + \bar{\theta}); \quad (A-10)$$

(b) the equilibrium quantities demanded are

$$\theta^*_{hc1} = -(\kappa_h + \bar{\theta}), \quad \theta^*_{nc1} = \frac{N_h(\kappa_h + \bar{\theta})}{N_n+2}, \quad \theta^*_{mc1} = \frac{2N_h(\kappa_h + \bar{\theta})}{N_n+2}, \quad (A-11)$$

the equilibrium quote depths are

$$\alpha^*_{c1} = \frac{N_h N_n(\kappa_h + \bar{\theta})}{N_n+2}, \quad \beta^*_{c1} = N_h(\kappa_h + \bar{\theta}). \quad (A-12)$$

3. If $\Delta \geq \frac{2(N+1)\delta \text{Var}[\widetilde{V} | I_n](\kappa_n + \bar{\theta})}{N_h}$, then short-sale constraints bind for nonhedgers and

(a) the equilibrium bid and ask prices are

$$A^*_{c2} = P_h^R - \frac{\Delta + \delta N_h \text{Var}[\widetilde{V} | I_n](\kappa_n + \bar{\theta})}{\nu N_h+2}, \quad (A-13)$$

$$B^*_{c2} = P_n^R + \delta \text{Var}[\widetilde{V} | I_n](\kappa_n + \bar{\theta}), \quad (A-14)$$

and bid-ask spread is

$$A^*_{c2} - B^*_{c2} = \frac{\nu N_h+1}{\nu N_h+2} \Delta - \frac{\bar{N} + 1}{\nu N_h+2} \delta \text{Var}[\widetilde{V} | I_n](\kappa_n + \bar{\theta}); \quad (A-15)$$
(b) the equilibrium quantities demanded are

\[ \theta_{hc}^* = \frac{\Delta + \delta N_n \text{Var}[\tilde{V}|\mathcal{I}_n](\kappa_n + \bar{\theta})}{(\nu N_h + 2)\delta \text{Var}[V|\mathcal{I}_h]}, \quad \theta_{mc}^* = -(\kappa_n + \bar{\theta}), \]

(A-16)

\[ \theta_{mc}^* = \frac{-\nu N_h \Delta + 2\delta N_n \text{Var}[\tilde{V}|\mathcal{I}_n](\kappa_n + \bar{\theta})}{(\nu N_h + 2)\delta \text{Var}[V|\mathcal{I}_h]}, \]

(A-17)

the equilibrium quote depths are

\[ \alpha_{c2}^* = \frac{N_h \Delta + \delta N_h N_n \text{Var}[\tilde{V}|\mathcal{I}_n](\kappa_n + \bar{\theta})}{(\nu N_h + 2)\delta \text{Var}[V|\mathcal{I}_h]}, \quad \beta_{c2}^* = N_n(\kappa_n + \bar{\theta}). \]

(A-18)

**Proof of Theorems 1 and 3:**

We only prove the generalized model with information asymmetry, because it nests the main model with symmetric information. We consider the case when \( \Delta < 0 \), the other case is similar. In this case, we conjecture that \( h \) investors sell and \( n \) investors buy. First, suppose no investors are constrained. Given bid price \( B \) and ask price \( A \), the optimal demand of \( h \) and \( n \) are respectively:

\[ \theta_{h}^* = \frac{P_R^h - B}{\delta \text{Var}[V|\mathcal{I}_h]} \quad \text{and} \quad \theta_{n}^* = \frac{P_R^n - A}{\delta \text{Var}[V|\mathcal{I}_n]}, \]

(A-19)

Substituting (A-19) into the market clearing condition (4), we get that the market clearing ask and bid depths are respectively:

\[ \beta = -N_h \theta_{h}^* = N_k \frac{B - P_R^h}{\delta \text{Var}[V|\mathcal{I}_h]}, \quad \alpha = N_n \theta_{n}^* = N_n \frac{P_R^n - A}{\delta \text{Var}[V|\mathcal{I}_n]} \]

(A-20)

Because of the CARA utility and normal distribution of the date 1 wealth, the market maker’s problem is equivalent to:

\[ \max_{A,B} \alpha A - \beta B + (\bar{\theta} + \beta - \alpha)E[\tilde{V}|\mathcal{I}_m] - \frac{1}{2} \delta \text{Var}[\tilde{V}|\mathcal{I}_m](\bar{\theta} + \beta - \alpha)^2, \]

(A-21)

subject to (A-20). The F.O.C with respect to \( B \) (noting that \( \beta \) is a function of \( B \)) gives us:

\[ -\beta - B \frac{N_h}{\delta \text{Var}[V|\mathcal{I}_h]} + E[\tilde{V}|\mathcal{I}_m] \frac{N_h}{\delta \text{Var}[V|\mathcal{I}_h]} - \delta \text{Var}[\tilde{V}|\mathcal{I}_m](\bar{\theta} + \beta - \alpha) \frac{N_h}{\delta \text{Var}[V|\mathcal{I}_h]} = 0, \]

which can be reduced to

\[ (\nu N_h + 2)\beta - \nu N_h \alpha = -\frac{N_h \Delta}{\delta \text{Var}[V|\mathcal{I}_h]}, \]

(A-22)
by using (36) and expressing $B$ in terms of $\beta$ using (A-20).

Similarly using the F.O.C with respect to $A$, we get:

$$\alpha + A \left( -\frac{N_n}{\delta \text{Var}[\tilde{V}|I_n]} \right) - E[\tilde{V}|I_m] \left( -\frac{N_n}{\delta \text{Var}[\tilde{V}|I_n]} \right) + \delta \text{Var}[\tilde{V}|I_m] \left( \theta + \beta - \alpha \right) \left( -\frac{N_n}{\delta \text{Var}[\tilde{V}|I_n]} \right) = 0,$$

(A-23)

which can be reduced to

$$(N_n + 2)\alpha - N_n \beta = 0,$$

(A-24)

by using (36), expressing $A$ in terms of $\alpha$ using (A-20), and noting that $I_m = I_n$.

Solving (A-24) and (A-22), we can get the equilibrium ask depth and bid depth $\alpha^*$ and $\beta^*$ as in (A-6) and (A-7). Substituting $\alpha^*$ and $\beta^*$ into (A-20), we can get the equilibrium ask and bid prices $A^*$ and $B^*$ as in (A-2) and (A-3). In addition, by the market clearing condition, we have $\theta^*_n = \alpha^*/N_n$, $\theta^*_h = -\beta^*/N_h$, $\theta^*_m = \beta^* - \alpha^*$, which can be simplified into equation (A-5).

The short-sale constraints bind for hedgers if and only if $\theta^*_h \leq -(\kappa_h + \bar{\theta})$, equivalently, if and only if $\Delta \leq -C_h \delta \text{Var}[\tilde{V}|I_h](\kappa_h + \bar{\theta})$. When short-sale constraints bind for hedgers, we have $\theta^*_{hc1} = -(\kappa_h + \bar{\theta})$ and $\beta^*_{c1} = N_h(\kappa_h + \bar{\theta})$. Because the first order condition (A-24) with respect to $\alpha$ remains the same, we have:

$$\alpha^*_{c1} = \frac{N_h N_n}{N_n + 2}(\kappa_h + \bar{\theta}).$$

Then from (A-20), we get the equilibrium bid price $B^*_{c1}$ and ask price $A^*_{c1}$ when short-sale constraints bind for hedgers. Other quantities can then be derived. Similarly, we can prove Theorem 3 for the other case where $h$ investors buy and $n$ investors sell.

**Q.E.D.**

**Proof of Proposition 1:** We prove this proposition for the case when $h$ investors sell, the proof of the other case is very similar and we thus skip it here. Conditional on the constraint binding for hedgers, it is clear from Theorem 3 that $A^*_{c1}$ decreases in $\kappa_h$, $B^*_{c1}$ increases in $\kappa_h$ and $(A^*_{c1} - B^*_{c1})$ decreases in $\kappa_h$. We next show that compared to the case without short-sale constraints, bid is lower and ask is higher with the constraints. By Theorem 3, we have

$$B^*_{c1} - B^* = \Delta/C_h + \delta \text{Var}[\tilde{V}|I_h](\kappa_h + \bar{\theta}),$$

(A-25)

and

$$A^*_{c1} - A^* = -\Delta/C_n - \frac{N_h}{N_n + 2} \delta \text{Var}[\tilde{V}|I_n](\kappa_h + \bar{\theta}).$$

(A-26)
The condition $\Delta < -C_h \delta \text{Var}[\tilde{V}[\mathcal{L}_h]](\kappa_h + \bar{\theta})$ implies that $B_{c1}^* \leq B^*$ and $A_{c1}^* \geq A^*$, which leads to $A_{c1}^* - B_{c1}^* \geq A^* - B^*$. Similarly, the results on depths and trading volume can be shown. Q.E.D.

The following lemma is used to prove Proposition 2.

**Lemma 1** Let $f(x) := |x|$ and

$$g(x) := \begin{cases} 
c_1(k_1x - h_1) & x > \frac{h_1}{k_1} \\
c_2(-k_2x - h_2) & x < -\frac{h_2}{k_2}, \\
0 & \text{otherwise},
\end{cases}$$

where $k_1$, $k_2$, $h_1$, $h_2$, $c_1$ and $c_2$ are positive constants and $x$ is randomly distributed in $(-\infty, +\infty)$ with probability density function $p(x)$ which is an even function. Then we have $\text{Cov}(f(x), g(x)) > 0$.

**Proof:**

$$\text{Cov}(f(x), g(x)) = E(f(x)g(x)) - E(f(x))E(g(x))$$

$$= \int_{-\infty}^{+\infty} f(x)g(x)p(x)dx - \int_{-\infty}^{+\infty} f(x)p(x)dx \int_{-\infty}^{+\infty} g(x)p(x)dx$$

$$= \int_{-\infty}^{+\infty} p(y)dy \int_{-\infty}^{+\infty} f(x)g(x)p(x)dx - \int_{-\infty}^{+\infty} f(y)p(y)dy \int_{-\infty}^{+\infty} g(x)p(x)dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x)g(x) - f(y)g(y)) p(x)p(y)dx dy$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))(g(x) - g(y)) p(x)p(y)dx dy. \quad (A-27)$$

Since $p(-x) = p(x)$ and $p(-y) = p(y)$, we have

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(-x) - f(-y))(g(-x) - g(-y)) p(x)p(y)dx dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))(g(x) - g(y)) p(x)p(y)dx dy. \quad (A-28)$$

From (A-27) and (A-28), we have $\text{Cov}(f(x), g(x)) =$

$$\frac{1}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))(g(x) + g(-x) - g(y) - g(-y)) p(x)p(y)dx dy. \quad (A-29)$$

45
(1) If \( x \) and \( y \) have the same sign, the term inside the integral can be written as 
\[
(f(x) - f(y))(g(x) - g(y)) + (f(-x) - f(-y))(g(-x) - g(-y)),
\]
which is \( \geq 0 \).

(2) If \( x < 0 \) and \( y > 0 \), the term inside the integral can be written as 
\[
(f(-x) - f(y))(g(-x) - g(y)) + (f(x) - f(-y))(g(x) - g(-y)),
\]
which is \( \geq 0 \).

(3) If \( x > 0 \) and \( y < 0 \), the term inside the integral can be written as 
\[
(f(x) - f(-y))(g(x) - g(-y)) + (f(-x) - f(y))(g(-x) - g(y)),
\]
which is \( \geq 0 \). In addition, at least for some \( x \) and \( y \), the term inside the integral is non-zero. Therefore, \( \text{Cov}(f(x), g(x)) > 0 \). \( \text{Q.E.D.} \)

**Proof of Proposition 2:** The spread with short-sale constraints \( A^*_c - B^*_c \) can be written as 
\[ f(\Delta) \pm (\Delta), \]
where 
\[
f(\Delta) = A^* - B^* = \frac{|\Delta|}{2}
\]
and
\[
g(\Delta) = \begin{cases} 
\frac{\nu N}{2(\nu N + 2)} \Delta - \frac{N + 1}{2N + 2} \delta \text{Var}[\tilde{V}_n \mid I_n](\kappa_n + \bar{\theta}) & \Delta \geq C_\delta \text{Var}[\tilde{V}_n \mid I_n](\kappa_n + \bar{\theta}) \\
-\frac{1}{2} \Delta + \delta \text{Var}[\tilde{V}_h \mid I_h](\kappa_h + \bar{\theta}) & \Delta \leq -C_\delta \text{Var}[\tilde{V}_h \mid I_h](\kappa_h + \bar{\theta}) \\
0 & \text{otherwise},
\end{cases}
\]
By Lemma 1, \( f(\Delta) \) and \( g(\Delta) \) are positively correlated. Then it follows that 
\( \text{Var}(A^*_c - B^*_c) > \text{Var}(A^* - B^*) \). \( \text{Q.E.D.} \)

**Proof of Theorem 2:** The proof is straightforward. Here we only outline the proof. Define 
\[
\nu_1 = \frac{\text{Var}[\tilde{V}_n \mid I_n]}{\text{Var}[\tilde{V}_h \mid I_h]}, \quad \nu_2 = \frac{\text{Var}[\tilde{V}_m \mid I_m]}{\text{Var}[\tilde{V}_h \mid I_h]},
\]
\[
\varphi_m = \frac{\delta_h \delta_n \nu_1}{\delta_m \delta_n \nu_2 N_h + 2 \delta_h \delta_n \nu_1 + \delta_m \delta_h \nu_2 N_n}, \quad (A-30)
\]
\[
\varphi_n = N_n \left( \frac{\delta_m \nu_2}{\delta_n \nu_1} + \frac{\delta_h}{N_n \delta_h + N_h \nu_1 \delta_n} \right) \varphi_m, \quad (A-31)
\]

46
\[ \lambda_m = \frac{\delta_n \nu_1}{N_n \delta_m \nu_2 + 2 \delta_n \nu_1}, \lambda_h = \frac{N_n \delta_m \nu_2 + \delta_n \nu_1 \bar{N}_h}{N_n \delta_m \nu_2 + 2 \delta_n \nu_1 \bar{N}_n}, \]
\[ \gamma_m = \frac{\delta_h}{N_h \delta_m \nu_2 + 2 \delta_h}, \gamma_n = \frac{N_h \delta_m \nu_2 + \delta_h \bar{N}_n}{N_h \delta_m \nu_2 + 2 \delta_h \delta_n \nu_1 \bar{N}_h}. \] (A-32) (A-33)

First, assuming there are no short-sale constraints, then the market maker’s problem is equivalent to

\[
\max_P N_h \frac{p_R - p - P}{\delta_n \text{Var} [V_h]} P + N_n \frac{p_R - p}{\delta_n \text{Var} [V_n | I_m]} P + \left( \bar{\theta}_m + N_h \frac{p - p_R}{\delta_h \text{Var} [V_h | I_m]} \right) P
\times \left( p_m + \delta_m \text{Var} [\hat{V}_m | I_m] \bar{\theta}_m - \frac{1}{2} \delta_m \text{Var} [\hat{V}_m | I_m]\bar{\theta}_m \right)
\times \left( \bar{\theta}_m + N_h \frac{p - p_R}{\delta_h \text{Var} [V_h | I_m]} \right) + N_n \frac{p - p_R}{\delta_n \text{Var} [V_n | I_m]} \right)^2.
\]

The first order condition then yields \( P^* \). Assuming only hedgers are constrained, then the market maker’s problem is equivalent to

\[
\max_P -N_h (\kappa_h + \bar{\theta}_h) P + N_n \frac{p_R - p - P}{\delta_n \text{Var} [V_n | I_m]} P + \left( \bar{\theta}_m + N_h (\kappa_h + \bar{\theta}_h) \right) P
\times \left( p_m + \delta_m \text{Var} [\hat{V}_m | I_m] \bar{\theta}_m - \frac{1}{2} \delta_m \text{Var} [\hat{V}_m | I_m]\bar{\theta}_m \right)
\times \left( \bar{\theta}_m + N_h (\kappa_h + \bar{\theta}_h) \right) + N_n \frac{p - p_R}{\delta_n \text{Var} [V_n | I_m]} \right)^2.
\]

The first order condition then yields \( P^c_h \). Similarly, assuming only nonhedgers are constrained, we can derive \( P^c_n \). Then the comparison of the maximum expected utility with the constraint that \( P \leq \bar{P}_h \) and the maximum expected utility with the constraint that \( \bar{P}_h \leq P \leq \bar{P}_n \), while noting that the maximum expected utility with the constraint that \( P \geq \bar{P}_n \) is equal to the expected utility at \( P = \bar{P}_n \), yields the equilibrium prices under different conditions. \( Q.E.D. \)

**Robustness with reduced information when hedgers are constrained**

We assume the informed (hedgers) are not endowed with any shares of the stock, the market maker is endowed with \( \bar{\theta}_m \) shares of the stock, and each uninformed trader (nonhedger) is endowed with \( \bar{\theta}_n \) shares of the stock. For tractability, we study the case when the market maker has to post \( A = B = P \). To simplify computations, we also assume that there is no public signal \( \hat{S}_m \), i.e., \( \sigma_n = \infty \).

It can be shown that hedgers are constrained by short-selling constraints and thus are not trading when

\[
\hat{S} \leq \hat{S} := -\frac{\delta \text{Var} [\hat{V} | I_m] \nu \left( (N_h \nu + N_n)(\bar{\theta}_m + N_n \bar{\theta}_n) + N_n \bar{\theta}_n \right)}{(1 - \rho_n)(N_h \nu(N_h + 1) + N_n(N_n + 2))}.
\]

When the informed are constrained by short-selling constraints, in equilibrium, there are two cases: (1) if \( \bar{\theta}_m \geq \bar{\theta}_n \), the uninformed buy from the market maker; (2) if \( \bar{\theta}_m \leq \bar{\theta}_n \), then (i) the
uninformed sell and they are not constrained by short-selling constraints when $S_n < \hat{S} < S$, (ii) the uninformed buy when $\hat{S} < S_n$, where

$$S_n = \frac{\delta \text{Var}[\tilde{V}|\mathcal{I}_h]}{(1 - \rho_n) N_h + N_n + 2)} \theta_m. $$

We present the details of case (1), i.e., $\bar{\theta}_m > \bar{\theta}_n$, the uninformed buy from the market maker. Case (2) can be solved similarly.

When the informed are constrained by short-selling constraints and they are not endowed with any shares of risky asset, therefore informed traders are not trading. The market maker and the uninformed only know that $\hat{S} \leq S$, they cannot observe $\hat{S}$. The uninformed’s problem becomes

$$\max_{\theta_n} E[-e^{-\delta(-\theta_n + (\bar{\theta}_m + \theta_n)(\tilde{v} + \tilde{u}))}|\hat{S} \leq S].$$

(A-34)

It can be shown that (A-34) is equivalent to

$$\min_{\theta_n} e^{\delta \theta_n + \frac{1}{2} \delta^2 (\tilde{\theta}_n + \theta_n)^2 (\sigma^2_u + \sigma^2_v) - \delta (\theta_n + \theta_n) \bar{V}} N \left( \frac{S + \delta(\tilde{\theta}_n + \theta_n) \rho_h \sigma^2_v}{\sqrt{\rho_h \sigma^2_v + \omega^2 \sigma^2_X}} \right) / N \left( \frac{S}{\sqrt{\rho_h \sigma^2_v + \omega^2 \sigma^2_X}} \right).$$

(A-35)

Taking first order condition with respect to $\theta_n$ in equation (A-35) yields

$$(P + \delta (\tilde{\theta}_n + \theta_n)(\sigma^2_u + \sigma^2_v) - \tilde{V}) N \left( \frac{S + \delta(\tilde{\theta}_n + \theta_n) \rho_h \sigma^2_v}{\sqrt{\rho_h \sigma^2_v + \omega^2 \sigma^2_X}} \right) + \frac{1}{\sqrt{2\pi}} \frac{\rho_h \sigma^2_v}{\sqrt{\rho_h \sigma^2_v + \omega^2 \sigma^2_X}} e^{-\frac{(S + \delta(\tilde{\theta}_n + \theta_n) \rho_h \sigma^2_v)^2}{2(\rho_h \sigma^2_v + \omega^2 \sigma^2_X)}} = 0.$$ 

(A-36)

Market maker’s problem becomes

$$\max_P E[-e^{-\delta(\alpha + (\theta_m - \alpha)(\tilde{v} + \tilde{u}))}|\hat{S} \leq S].$$

(A-37)

It can be shown that (A-37) is equivalent to

$$\min_P e^{-\delta P + \frac{1}{2} \delta^2 (\bar{\theta}_m - \alpha)^2 (\sigma^2_u + \sigma^2_v) - \delta (\bar{\theta}_m - \alpha) \bar{V}} N \left( \frac{S + \delta(\bar{\theta}_m - \alpha) \rho_h \sigma^2_v}{\sqrt{\rho_h \sigma^2_v + \omega^2 \sigma^2_X}} \right) / N \left( \frac{S}{\sqrt{\rho_h \sigma^2_v + \omega^2 \sigma^2_X}} \right).$$

(A-38)

Taking first order condition with respect to $P$ in equation (A-38) yields
\[
\left(\alpha + \frac{\partial \alpha}{\partial P}(P + \delta(\bar{\theta}_m - \alpha)(\sigma^2_u + \sigma^2_v) - \bar{V})\right)N\left(\frac{S + \delta(\bar{\theta}_m - \alpha)\rho_h\sigma^2_v}{\sqrt{\rho_h\sigma^2_v + \omega^2\sigma^2_X}}\right)
+ \frac{1}{\sqrt{2\pi}} \frac{\rho_h\sigma^2_v}{\sqrt{\rho_h\sigma^2_v + \omega^2\sigma^2_X}} \frac{\partial \alpha}{\partial P} e^{-\frac{(S + \delta(\bar{\theta}_m - \alpha)\rho_h\sigma^2_v)^2}{2(\rho_h\sigma^2_v + \omega^2\sigma^2_X)}} = 0.
\] (A-39)

From (A-36), we can express \(P\) and \(\partial \theta_n / \partial P\) as functions of \(\theta_n\). Substituting \(\alpha = N_n\theta_n\) and \(\partial \alpha / \partial P = N_n \partial \theta_n / \partial P\) into (A-39) yields a function of \(\theta_n\) which can be solved numerically and then we obtain the equilibrium price.

Appendix B

Equilibrium with a price taking market maker

To show the impact of the market maker’s market power on equilibrium results, in this Appendix, we consider the case where the market maker is also a price taker, which yields the same results as in a perfect competition market. Let \(P\) denote the stock price. Given \(P\), the optimal demand schedule for a type \(i\) investor for \(i \in \{h, n\}\) is

\[
\theta^*_i(P) = \max \left[ -\left(\kappa_i + \bar{\theta}\right), -\frac{P - P^R_i}{\delta \text{Var}[\bar{V}|\mathcal{I}_i]} \right].
\] (B-1)

Solving for the equilibrium, we have

**Theorem 4**

1. If \(\frac{N_n \delta \text{Var}[\bar{V}|\mathcal{I}_h](\kappa_h + \bar{\theta})}{N_n + 1} < \Delta < \frac{N_n \delta \text{Var}[\bar{V}|\mathcal{I}_n](\kappa_n + \bar{\theta})}{N_n + 1}\), then no one is constrained and the equilibrium price is

\[
P^* = \frac{\nu N_h}{\Delta} P^R_h + \frac{N_n}{\Delta} P^R_n + \frac{1}{\Delta} P^R_m
\] (B-2)

and the investors’ optimal stock demand are given by

\[
\theta^*_h = \frac{N_n + 1}{N} \frac{\Delta}{\delta \text{Var}[\bar{V}|\mathcal{I}_h]}, \quad \theta^*_n = \theta^*_m = -\frac{\nu N_h}{\Delta} \frac{\Delta}{\delta \text{Var}[\bar{V}|\mathcal{I}_n]},
\] (B-3)

2. If \(\Delta \geq \frac{N_n \delta \text{Var}[\bar{V}|\mathcal{I}_n](\kappa_n + \bar{\theta})}{N_n}\), then short-sale constraints bind for nonhedgers and the equilibrium price is

\[
P^*_{ch} = P^R_h - \frac{\Delta + N_n \delta \text{Var}[\bar{V}|\mathcal{I}_n](\kappa_n + \bar{\theta})}{\nu N_h + 1},
\] (B-4)
and the investors’ optimal stock demand are given by

\[
\theta_{hc1}^* = \frac{\Delta + N_n \delta \text{Var}[\tilde{V} | \mathcal{I}_n](\kappa_n + \bar{\theta})}{(\nu N_h + 1)\delta \text{Var}[V | \mathcal{I}_h]}, \quad \theta_{nc1}^* = -(\kappa_n + \bar{\theta}), \tag{B-5}
\]

\[
\theta_{mc1}^* = -\nu N_h \Delta + N_n \delta \text{Var}[\tilde{V} | \mathcal{I}_h](\kappa_n + \bar{\theta}), \tag{B-6}
\]

3. If \( \Delta \leq -\frac{\delta \text{Var}[\tilde{V} | \mathcal{I}_n](\kappa_n + \bar{\theta})}{N_n + 1} \), then short-sale constraints bind for hedgers and the equilibrium price is

\[
P_{c2}^* = P_n^R - \frac{\nu N_h \delta \text{Var}[\tilde{V} | \mathcal{I}_h](\kappa_n + \bar{\theta})}{N_n + 1}, \tag{B-7}
\]

and the investors’ optimal stock demand are given by

\[
\theta_{hc2}^* = -(\kappa_n + \bar{\theta}), \quad \theta_{nc2}^* = \theta_{mc2}^* = \frac{N_h(\kappa_h + \bar{\theta})}{N_n + 1}. \tag{B-8}
\]

As shown by Theorem 4, the equilibrium price is a weighted average of the reservation prices of the investors in the economy. In addition, it is easy to show when short-sale constraints bind, the equilibrium selling price goes up (i.e., \( P_{c1}^* > P^* \) and \( P_{c2}^* > P^* \)) and trading volume decreases, as in the existing literature. Theorem 4 assumes that the market maker is a price taker in both “bid” and “ask” markets. Next, to isolate the impact of the market power on how short-sale constraints affect bid price, we assume that the market maker is a monopolist in the “ask” market, as in our main model, but is a price taker in the “bid” market. Under this assumption, we have,

**Theorem 5**

1. If \( 0 < \Delta < \frac{\nu N_n + 2(N + 1)}{\nu N_n} \delta \text{Var}[\tilde{V} | \mathcal{I}_n](\kappa_n + \bar{\theta}) \), then no one is constrained and the equilibrium prices are

\[
A_1^* = P_n^R + \frac{\bar{N}}{2(N_n + 1) + \nu N_h} \Delta, \quad B_1^* = P_n^R + \frac{\nu N_h}{2(N_n + 1) + \nu N_h} \Delta, \tag{B-9}
\]

which implies the bid-ask spread is

\[
A_1^* - B_1^* = \frac{N_n + 1}{2(N_n + 1) + \nu N_h} \Delta.
\]

the equilibrium depths are

\[
\alpha_1^* = \frac{\nu N_h(N_n + 1)}{2(N_n + 1) + \nu N_h} \frac{\Delta}{\delta \text{Var}[V | \mathcal{I}_n]}, \quad \beta_1^* = \frac{N_n}{N_n + 1} \alpha_1^*.
\]
the investors’ optimal stock demand are given by

\[ \theta^*_h = \frac{\alpha^*_1}{N_h}, \quad \theta^*_n = -\frac{\beta^*_1}{N_n}, \quad \theta^*_m = -\frac{\alpha^*_1}{N_n + 1}. \]  

(B-10)

2. If \(-\frac{2(\nu N_h + 1) + N_n}{N_n + 2} \delta \text{Var}[\tilde{V} | I_n](\kappa_n + \bar{\theta}) < \Delta < 0\), then no one is constrained and the equilibrium prices are

\[ A^*_2 = P^R_n + \frac{\nu N_h}{2\nu N_h + N_n + 2} \Delta, \quad B^*_2 = P^R_n + \frac{2\nu N_h}{2\nu N_h + N_n + 2} \Delta \]  

(B-11)

which implies the bid-ask spread is

\[ A^*_2 - B^*_2 = -\frac{\nu N_h}{2\nu N_h + N_n + 2} \Delta \]

the equilibrium depths are

\[ \alpha^*_2 = -\frac{\nu N_h N_n}{N_n + 2 + 2\nu N_h \delta \text{Var}[\tilde{V} | I_n]}, \quad \beta^*_2 = \frac{N_n + 2}{N_n} \alpha^*_2, \]

the investors’ optimal stock demand are given by

\[ \theta^*_h_2 = -\frac{\beta^*_2}{N_h}, \quad \theta^*_n_2 = \frac{\alpha^*_2}{N_n}, \quad \theta^*_m_2 = \frac{2}{N_n + 2} \beta^*_2. \]  

(B-12)

3. If \(\Delta \geq \frac{\nu N_h + 2(\nu N_h + 1)}{\nu N_h} \delta \text{Var}[\tilde{V} | I_n](\kappa_n + \bar{\theta})\), then short-sale constraints bind for nonhedgers and the equilibrium prices are

\[ A^*_c_1 = P^R_n + \frac{(\nu N_h + 1) \Delta - N_n \delta \text{Var}[\tilde{V} | I_n](\kappa_n + \bar{\theta})}{\nu N_h + 2}, \]  

(B-13)

\[ B^*_c_1 = P^R_n + \frac{\nu N_h \Delta - 2N_n \delta \text{Var}[\tilde{V} | I_n](\kappa_n + \bar{\theta})}{\nu N_h + 2}, \]  

(B-14)

which implies the bid-ask spread is

\[ A^*_c_1 - B^*_c_1 = \frac{\Delta + N_n \delta \text{Var}[\tilde{V} | I_n](\kappa_n + \bar{\theta})}{\nu N_h + 2}, \]

the equilibrium depths are

\[ \alpha^*_c_1 = \frac{\nu N_h \Delta + \nu N_h N_n \delta \text{Var}[\tilde{V} | I_n](\kappa_n + \bar{\theta})}{(\nu N_h + 2) \delta \text{Var}[\tilde{V} | I_n]}, \quad \beta^*_c_1 = N_n(\kappa_n + \bar{\theta}). \]
the investors’ optimal stock demand are given by
\[ \theta_{hc1}^* = \frac{\alpha_{c1}^*}{N_h}, \quad \theta_{nc1}^* = -(\kappa_n + \bar{\theta}), \quad \theta_{mc1}^* = N_n(\kappa_n + \bar{\theta}) - \alpha_{c1}^*. \] (B-15)

4. If \( \Delta < -\frac{2(\nu N_h + 1) + N_n}{N_n + 2} \delta \text{Var}[\tilde{V}|\mathcal{I}_h](\kappa_h + \bar{\theta}) \), then short-sale constraints bind for hedgers and the equilibrium prices are
\[ A_{c2}^* = P_n^R - \frac{\nu N_h}{N_n + 2} \delta \text{Var}[\tilde{V}|\mathcal{I}_h](\kappa_h + \bar{\theta}), \quad B_{c2}^* = P_n^R - \frac{2\nu N_h}{N_n + 2} \delta \text{Var}[\tilde{V}|\mathcal{I}_h](\kappa_h + \bar{\theta}), \] (B-16)
which implies the bid-ask spread is
\[ A_{c2}^* - B_{c2}^* = \frac{\nu N_h}{N_n + 2} \delta \text{Var}[\tilde{V}|\mathcal{I}_h](\kappa_h + \bar{\theta}), \]
the equilibrium depths are
\[ \alpha_{c2}^* = \frac{N_n}{N_n + 2} N_h(\kappa_h + \bar{\theta}), \quad \beta_{c2}^* = N_h(\kappa_h + \bar{\theta}). \]
the investors’ optimal stock demand are given by
\[ \theta_{hc2}^* = -(\kappa_h + \bar{\theta}), \quad \theta_{nc2}^* = \frac{1}{N_n + 2} N_h(\kappa_h + \bar{\theta}), \quad \theta_{mc2}^* = \frac{2}{N_n + 2} N_h(\kappa_h + \bar{\theta}). \] (B-17)

Given the results stated in Theorem 5, it is easy to show that as in our main model, short-sale constraints increases equilibrium ask price, decreases bid/ask depths (and thus trading volume). In contrast to our main model, Theorem 5 implies short-sale constraints increase equilibrium bid price (i.e., \( B_{c1}^* > B_1^* \) and \( B_{c2}^* > B_2^* \)).