Event Risks, Illiquidity, and Portfolio Selection*

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Abstract

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Journal of Economic Literature Classification Numbers: D11, D91, G11, C61.

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1. Introduction

The recent financial crisis highlights several potentially important fundamental elements for optimal portfolio selection. First, event risks, such as a market crash, may be significant. Second, market liquidity may dry up after a crash. Third, the probability of another crash may increase after a crash. Fourth, the investment opportunity set (e.g., market volatility) may change after a crash. However, the combined effects of the correlated event risks, market illiquidity, and investment opportunity set are unexplored in the existing literature.

In this paper, we develop a flexible portfolio selection workhorse model that incorporates possibly correlated market crashes, illiquidity, and stochastic investment opportunity set. In particular, both liquidity and investment opportunity set may change after a crash and crashes themselves may be correlated. This model captures the essence of all the above mentioned important features, but still remains tractable. More specifically, we consider the optimal trading strategy of a constant relative risk averse (CRRA) investor who derives utility from terminal wealth and can trade continuously a riskless asset and a risky stock. Stock price crashes in a liquid regime can trigger switching into an illiquid regime where other parameters such as crash intensity, expected return, and volatility can also change. Similarly, large upward price jumps in the illiquid regime can trigger regime switching into the liquid regime.

Because of the possibility of price jumps, the coupled Hamilton-Jacobi-Bellman (HJB) equations become integro-differential equations, which makes our problem much more difficult to solve even numerically than that of Jang, Koo, Liu and Loewenstein (2007) who do not consider event risks. Remarkably, we are able to develop an iterative procedure that solves for the value function as a sequence of solutions to ordinary differential equations, which significantly reduces computation intensity. As in the pure diffusion case, the no-transaction region is characterized by two regime-dependent boundaries within which the investor always maintains the ratio of the dollar amount in the riskless asset to the dollar amount in the risky asset. In contrast to the pure diffusion setting in Liu and Loewenstein (2002) and Jang et al (2007), however, this ratio can jump outside these boundaries which requires an immediate discrete transaction back to the closest boundary. We prove the existence of the optimal trading strategy and characterize the value function. We also provide an extensive numerical analysis to illustrate how the various elements of our model affect the optimal trading strategy.

We quantify when an investor should sell stock after a crash even if the market is illiquid. This typically occurs when the investment opportunity set significantly worsens after a market crash (e.g., much lower expected return, much higher volatility, or much greater further crash probability) and the illiquidity may persist for a period of time. Intuitively, this is because a significantly worsened investment opportunity set makes the marginal benefit of selling the stock high and the expected duration of the illiquid regime justifies incurring the necessary transaction cost. This finding might be consistent with "flight to quality" after a crash, but in sharp contrast to the contrarian style prediction of the standard portfolio selection models with i.i.d. returns (e.g., Merton (1969)). On the other hand, we also find that if the illiquid regime is relatively short then the investor optimally tends to reduce trading frequency until market conditions improve. In general, to determine the optimal trading strategy after a crash, the investor trades off the benefit of rebalancing due to the change in the investment opportunity set and the cost of transaction. Loosely speaking, the greater the change in the investment opportunity set and the greater the expected duration of the illiquid regime, the greater the benefit of rebalancing. Depending on the relative magnitude of the benefit and cost, the investor may choose to sell, to buy, or to wait out the illiquid regime after a crash.

Holding the expected return constant, the total return volatility increases with the expected jump size. We show that an increase in the expected jump size may increase the optimal stock holding even when the expected returns are held constant and the total return volatility increases. This is because increasing the expected jump size (but keeping the expected return constant) helps an investor by making returns less negatively skewed. In the presence of event risks, within any regime the optimal strategy trades off the benefit of rebalancing after a large price move against the transaction costs of rebalancing. Interestingly, price jumps can help *reduce* rebalancing costs across regimes and increase stock holdings. For example, suppose a large price drop triggers the illiquid regime and the optimal fraction of wealth that should be invested in stock decreases. With a large price drop, the fraction of wealth invested in stock is already lower so a transaction may be unnecessary. Given this, the investor can hold more stock in the liquid regime since large price drops help achieve the optimal risk exposure with lower rebalancing costs.

In addition, we show that misestimating the correlation between market crashes and market illiquidity can be costly to investors. For example, if an investor underestimates the correlation between market crashes and market illiquidity and adopts the corresponding "optimal" trading strategy, the certainty equivalent wealth loss from this trading strategy can be as high as 3.5% of the investor's initial wealth in some reasonable scenarios.

Closely related work include the literature on portfolio selection with transaction costs but with no event risks (e.g., Constantinides (1986), Davis and Norman

(1990), Dumas and Luciano (1991), Shreve and Soner (1994) and Liu and Loewenstein (2002))), the literature on portfolio selection with event risks but with no transaction costs (e.g., Liu, Longstaff and Pan (2003)). The closest work to ours are Jang et al (2007) and Framstad, Oksendal, and Sulem (2001). Jang et al (2007) uses a regime switching model to show that transaction costs can have a first order effect when investment opportunity set varies through time. However, they do not consider the effect of event risks such as market crashes on trading strategies. Framstad, Oksendal, and Sulem (2001) consider the optimal consumption/investment problem with an infinite horizon in a jump diffusion setting with constant proportional transaction costs. However, they do not consider the effect of correlated market crashes and market illiquidity, or crash-triggered investment opportunity set changes, which are all important features for understanding the optimal trading strategy in a financial crisis. In addition, they do not offer a procedure to solve for the optimal strategy either.

The rest of the paper is organized as follows. In Section 2 we describe our portfolio selection model with "event risk" and liquidity risk in a two-regime framework. We also show the existence of the optimal trading strategy and provide characterization of the value function and the no-transaction region. Section 3 describes an iterative procedure to compute the optimal trading strategy. We conduct an extensive numerical analysis on the optimal trading strategy in Section 4. We conclude in Section 5 and provide proofs in the Appendix.

2. The Basic Model

2.1 The Asset Market

Throughout this paper we are assuming a probability space $(\Omega, \mathcal{F}, P)$. Uncertainty and the filtration $\{\mathcal{F}_t\}$ in the model are generated by a standard one dimensional Brownian motion w and the Poisson processes defined below. We will assume that all stochastic processes are adapted.

An investor can trade two assets in the financial market: One risk free ("the bond") and one risky ("the stock"). There are two regimes with different liquidity (Regime 0 (liquid) and Regime 1 (illiquid)) across which other parameter values may also change. We use $\iota_t \in \{0, 1\}$ as a state variable to indicate the regime at time t. The time t interest rate is $r(\iota_t)$. The investor can buy the stock at the ask price $S_t^A = (1 + \theta(\iota_t))S_t$ and sell it at the bid price $S_t^B = (1 - \alpha(\iota_t))S_t$, where $\theta(\iota) \geq 0$ and $0 \leq \alpha(\iota) < 1$ represent the proportional transaction cost rates in Regime ι. We assume that stock price S_t may jump. To capture the idea that large jumps may have larger impact than moderate jumps and large down jumps may have different impact compared to large up jumps, we sort a stock price jump into a large up jump ("U"), a moderate up/down jump ("M"), or a large down jump ("D"), occurring at the jump times of independent Poisson processes N^j with intensities $\eta^j(\iota)$ and random jump sizes $J^j \in (-1, \infty)$ for $j \in \{U, M, D\}$ respectively. The stock price process S_t then evolves as:

$$\begin{aligned} dS_t &= (\mu(\iota_t) - \nu(\iota_t))S_{t-}dt + \sigma(\iota_t)S_{t-}dw_t + J_t^U S_{t-}dN_t^U \\ &\quad + J_t^M S_{t-}dN_t^M + J_t^D S_{t-}dN_t^D, \end{aligned} \tag{1}$$

where

$$\nu(\iota) = \eta^U(\iota)E[J^U] + \eta^M(\iota)E[J^M] + \eta^D(\iota)E[J^D] \tag{2}$$

represents the expected return compensation for the presence of jumps so that the instantaneous stock expected return is $\mu(\iota)$ with $\mu(\iota) > r(\iota)$, w is a one-dimensional Brownian motion, $\sigma(\iota)$ is the stock return volatility, and $J_t^{U,M,D}$ are the time t realizations of $J^{U,M,D}$. We assume for simplicity that the jump sizes are drawn from identical independent distributions at each time.[1] Let $\underline{\mathrm{J}}$ be the greatest lower bound that satisfies $\mathrm{Prob}\{J^{U,M,D} \geq \underline{\mathrm{J}}\} = 1$.

To capture the idea that liquidity changes may be correlated with large price jumps (e.g., large down jumps may be positively correlated with switching into an illiquid regime), we decompose each of the large jump processes into two independent components. Specifically, let

$$N_t^U = N_{1t}^U + N_{2t}^U, \quad N_t^D = N_{1t}^D + N_{2t}^D,$$

where if the first components (N_{1t}^D and N_{1t}^U) jump then the current regime switches into the other regime, while the second components (N_{2t}^U and N_{2t}^D) are independent of regime switching. In addition, N_{it}^j has an intensity of $\eta_i^j(\iota)$ with $\eta_1^j(\iota) + \eta_2^j(\iota) = \eta^j(\iota)$ for $i = 1, 2$ and $j \in \{U, D\}$. Moreover, for simplicity we assume $\eta_1^U(0) = \eta_1^D(1) = 0$ so that a large upward (downward) price jump in the liquid regime (illiquid regime) never results in switching into the illiquid (liquid) regime. To model the possibility that regimes can also change due to other factors such as general macroeconomic conditions, we assume that regime also switches into a different regime at the jump times of another independent Poisson jump process N^R with intensity $\xi(\iota)$. With these assumptions, we have that the state variable ι evolves (almost surely) as

$$d\iota_t = \begin{cases} dN_{1t}^D + dN_t^R, & \text{if } \iota_{t-} = 0 \\ -(dN_{1t}^U + dN_t^R), & \text{if } \iota_{t-} = 1. \end{cases} \tag{3}$$

[1] Alternatively, maintaining the independence of jump sizes through time, we could let the jump distribution vary with the state variable ι.

Suppose the current regime is Liquid (i.e., $\iota_{t-} = 0$). Equation (3) implies that if N_1^D jumps then we have a large downward jump in the stock price *and* the regime switches into the illiquid regime ($\iota_t = 1$). On the other hand, if N^R jumps, then the regime also shifts but there is no jump in the stock price. Finally, if N_2^D jumps then there is a downward jump in the stock price but the market stays in the liquid regime. Similarly, suppose the current regime is Illiquid (i.e., $\iota_{t-} = 1$). Equation (3) implies that if N_1^U jumps then we have a large upward jump in the stock price *and* the regime switches into the liquid regime ($\iota_t = 0$). On the other hand, if N^R jumps, then the regime also shifts but stock price does not jump. Finally, if N_2^U jumps then there is a large upward jump in the stock price but the market stays in the illiquid regime. In contrast, moderate price jumps do not have any impact on regime switching.

Thus we have a fairly parsimonious model which nests many possible submodels and allows liquidity to be correlated with stock prices in many interesting ways. For example, a pure jump diffusion model with constant proportional transaction costs is obtained by setting $\eta_1^D = \eta_1^U = \xi = 0$. Our model allows for regime switching to be correlated with large stock price jumps. For example, large downward stock price jumps may be followed by switching into the illiquid regime. In addition, this model also allows an investor to revise parameter values after a large price jump. For example, after a large downward price jump, an investor may believe that the fundamentals of the economy have changed and the expected return may become lower or the volatility may become higher.

When $\alpha(\iota) + \theta(\iota) > 0$, the above model gives rise to equations governing the evolution of the amount invested in the bond, x_t, and the amount invested in the stock, y_t:

$$dx_t = r(\iota_t)x_{t-}dt - (1 + \theta(\iota_t))dI_t + (1 - \alpha(\iota_t))dD_t, \tag{4}$$

$$
\begin{aligned}
dy_t &= (\mu(\iota_t) - \nu(\iota_t))\, y_{t-}dt + \sigma(\iota_t)y_{t-}dw_t + J_t^U y_{t-}dN^U \\
&\quad + J_t^M y_{t-}dN_t^M + J_t^D y_{t-}dN_t^D + dI_t - dD_t,
\end{aligned} \tag{5}
$$

where the processes D and I represent the cumulative dollar amount of sales and purchases of the stock, respectively. These processes are nondecreasing, right continuous adapted processes with $D(0) = I(0) = 0$.

Let x_0 and y_0 be the given initial dollar amounts in the bond and the stock respectively. We let $\Theta(x_0, y_0)$ denote the set of admissible trading strategies (D, I) such that (4) and (5) are satisfied given (3) and the investor is always solvent, i.e.,[2]

$$
x_t + (1 - \alpha(\iota_t))y_t(1 + \underline{\mathrm{J}}) \geq 0, \forall t \geq 0. \tag{6}
$$

which, as in Liu, Longstaff, and Pan (2003), restricts the fraction $\frac{x}{y}$.

2.2 The Investor's Problem

The investor's problem is to choose admissible trading strategies D and I so as to maximize $E[u(x_\tau + (1 - \alpha(\iota_\tau))y_\tau)]$ for an event that occurs at the first jump time τ of a standard, independent Poisson process with intensity λ. τ is thus exponentially distributed with parameter λ, i.e.,

$$
P\{\tau \in dt\} = \lambda e^{-\lambda t} dt.
$$

This formulation captures bequest, accidents, retirement, and many other events that happen on uncertain dates.[3]

[2] Since $\mu(\iota) > r(\iota)$ the investor optimally does not short the stock and so $y \geq 0$.

[3] We also used the method proposed by Liu and Loewenstein (2002) to solve the case with a deterministic horizon. The solution shows that the exponentially distributed horizon case is a close approximation to the case with a long horizon (about 20 years). Since this finding is similar to that in Liu and Loewenstein (2002), we do not report it in the paper to save space.

If τ is interpreted to represent the investor's uncertain lifetime (as in Merton (1971) and Richard (1975)), the investor's average lifetime is then $1/\lambda$ and the variance of his lifetime is accordingly $1/\lambda^2$.

Assuming a constant relative risk averse preference (CRRA), we can then write the value function as

$$v(x,y,\iota)=\sup_{(D,I)\in\Theta(x,y)} E\left[\frac{\Big(x_\tau+(1-\alpha(\iota_\tau))y_\tau\Big)^{1-\gamma}}{1-\gamma}|\iota_0=\iota\right]. \tag{7}$$

In light of our assumptions on τ and the asset market, this can be rewritten as (see Merton (1971), Liu and Loewenstein (2002), Jang et al (2007))

$$\begin{aligned} v(x,y,\iota) &= \sup_{(D,I)\in\Theta(x,y)} E\Big[\int_0^\infty e^{-\delta(\iota)t}\Big(\eta_1^U(\iota)v(x_t,y_t(1+J_t^U),1-\iota) \\ &\quad +\eta_2^U(\iota)v(x_t,y_t(1+J_t^U),\iota)+\eta^M(\iota)v(x_t,y_t(1+J_t^M),\iota) \\ &\quad +\eta_1^D(\iota)v(x_t,y_t(1+J_t^D),1-\iota)+\eta_2^D(\iota)v(x_t,y_t(1+J_t^D),\iota) \\ &\quad +\xi(\iota)v(x_t,y_t,1-\iota)+\lambda\frac{(x_t+(1-\alpha(\iota))y_t)^{1-\gamma}}{1-\gamma}\Big)dt\Big], \end{aligned}$$

where

$$\delta(\iota)=\lambda+\xi(\iota)+\eta^U(\iota)+\eta^M(\iota)+\eta^D(\iota). \tag{8}$$

2.3 Optimal Policies with No Transaction Costs

For purpose of comparison, we first consider the case without transaction costs (i.e., $\alpha(\iota)=\theta(\iota)=0$). Define the total wealth $W_t=x_t+y_t$ and let π be the fraction of wealth invested in the stock. The investor's problem becomes

$$v(W,\iota)=\sup_{\{\pi_t:t\geq 0\}}\lambda E[\int_0^\infty e^{-\lambda t}\frac{W_t^{1-\gamma}}{1-\gamma}dt|\iota_0=\iota,W_0=W],$$

subject to (3) and the self financing condition

$$\begin{aligned} dW_t &= (r(\iota_t)+\pi_{t-}(\mu(\iota_t)-\nu(\iota_t)-r(\iota_t)))W_{t-}dt+\pi_{t-}\sigma(\iota_t)W_{t-}dw_t \\ &\quad +\pi_{t-}(J_t^U W_{t-}dN_t^U+J_t^M W_{t-}dN_t^M+J_t^D W_{t-}dN_t^D). \end{aligned} \tag{9}$$

Merton (1971) and Liu, Longstaff, and Pan (2003) considered similar problems. In the absence of transaction costs, the optimal trading strategy is to invest a constant fraction $\pi(\iota)$ of wealth in stock in Regime ι and the value function in Regime ι is of the form:

$$v(W,\iota) = M(\iota)\frac{W^{1-\gamma}}{1-\gamma}.$$

From the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE), it is straightforward to show that $(M(\iota),\pi(\iota))$ for $\iota = 0,1$ satisfies

$$a(\pi(\iota),\iota)M(\iota) + h(\pi(\iota),\iota)M(1-\iota) + \lambda = 0 \tag{10}$$

and

$$\pi(\iota) = \arg\max_{\pi}\left(a(\pi,\iota)M(\iota) + h(\pi,\iota)M(1-\iota) + \lambda\right), \tag{11}$$

where

$$\begin{aligned} a(\pi,\iota) = &\left(r(\iota) + \pi(\mu(\iota) - r(\iota) - \nu(\iota)) - \frac{1}{2}\gamma\pi^2\sigma(\iota)^2\right)(1-\gamma) - \delta(\iota) \\ &+\eta^M(\iota)E\left[(1+\pi J^M)^{1-\gamma}\right] + \eta_2^U(\iota)E\left[(1+\pi J^U)^{1-\gamma}\right] + \eta_2^D(\iota)E\left[(1+\pi J^D)^{1-\gamma}\right] \end{aligned} \tag{12}$$

and

$$h(\pi,\iota) = \eta_1^U(\iota)E\left[(1+\pi J^U)^{1-\gamma}\right] + \eta_1^D(\iota)E\left[(1+\pi J^D)^{1-\gamma}\right] + \xi(\iota). \tag{13}$$

Although appearing complicated, equations (10)-(11) yield four equations for four unknowns $(M(\iota),\pi(\iota))$ $(\iota = 0,1)$ and are easily solved numerically. As in Merton (1971) and Liu, Longstaff, and Pan (2003), conditions on the parameters and the jump distribution are required for the existence of the optimal solution.

Assumption 1 *The solution $(M(\iota),\pi(\iota))$ to (10)-(11) is such that $M(\iota) > 0$ for $\iota = 0,1$.*

The positivity of $M(\iota)$ rules out the case where the investor can achieve bliss levels of utility and assures the existence of an optimal portfolio.[4] We summarize the main result for this no-transaction-cost case without proof in the following theorem. Notice that the optimal portfolio in this case is independent of the investor horizon.

Theorem 1 *Suppose that $\alpha(\iota) = \theta(\iota) = 0$. Then under Assumption 1, for $0 \leq t < \tau$ the optimal stock investment policy π_t^* in Regime ι_t is equal to $\pi(\iota_t)$ and the lifetime expected utility is*

$$v(x, y, \iota) = M(\iota)\frac{(x+y)^{1-\gamma}}{1-\gamma},$$

where $(M(\iota), \pi(\iota))$ solve (10)-(11) for $\iota = 0, 1$.

Remark 1 *If $\underline{J} = -1$, then the investor never leverages (i.e., $\pi_t^* \leq 1$). In general, when $\underline{J} < 0$ leverage is limited since solvency requires $x + y(1 + \underline{J}) \geq 0$ or $\pi \leq -\frac{1}{\underline{J}}$. This is why Equation (11) is not written in terms of the first order conditions.*

3. Optimal Policies with Transaction Costs

Suppose now that $\alpha(\iota) + \theta(\iota) > 0$ for $\iota = 0, 1$. As in Liu and Loewenstein (2002), the value functions are homogeneous of degree $1 - \gamma$ in (x, y). This implies that for $\iota = 0, 1$,

$$v(x, y, \iota) = y^{1-\gamma}\psi\left(\frac{x}{y}, \iota\right) \tag{14}$$

for some concave function $\psi : (\alpha(\iota) - 1, \infty) \times \{0, 1\} \to \mathbb{R}$.

In the presence of transaction cost, the *solvency region* in each regime splits into three regions: Buy region, Sell region and No-Transaction (NT) region. Because of the time homogeneity of the value function, these regions can be identified by two

[4] It can be easily verified that Assumption 1 reduces to the well-known Merton condition in the absence of jumps and regime shifts.

critical numbers (instead of functions of time) $r_s(\iota)$ and $r_b(\iota)$ in Regime ι. The Buy region corresponds to $z \geq r_b(\iota)$, the Sell region to $z \leq r_s(\iota)$, and the No-Transaction region to $r_s(\iota) < z < r_b(\iota)$, where $z = \frac{x}{y}$. However, in contrast to the pure diffusion cases previously studied, the fraction $\frac{x}{y}$ can jump out of the NT region, which is followed by an immediate lump-sum transaction to the closest boundary of the NT region. Moreover, when the regime shifts, an investor might also need to make a lump-sum trade to the new boundary in the new regime.

Under regularity conditions on v we have the following coupled HJB equation:

$$\max\{\mathcal{L}v, (1-\alpha(\iota))v_x - v_y, -(1+\theta(\iota))v_x + v_y\} = 0, \tag{15}$$

where

$$\begin{aligned}\mathcal{L}v &= \frac{1}{2}\sigma(\iota)^2 y^2 v_{yy} + r(\iota)xv_x + (\mu(\iota)-\nu(\iota))yv_y - \delta(\iota)v \\ &\quad + h(x,y,\iota) + \lambda\frac{(x+(1-\alpha(\iota))y)^{1-\gamma}}{1-\gamma},\end{aligned} \tag{16}$$

$$\begin{aligned}h(x,y,\iota) = \quad & \xi(\iota)v(x,y,1-\iota) + \eta^M(\iota)E[v(x,y(1+J^M),\iota)] \\ & + \eta_1^U(\iota)E[v(x,y(1+J^U),1-\iota)] + \eta_2^U(\iota)E[v(x,y(1+J^U),\iota)] \\ & + \eta_1^D(\iota)E[v(x,y(1+J^D),1-\iota)] + \eta_2^D(\iota)E[v(x,y(1+J^D),\iota)].\end{aligned} \tag{17}$$

Using (14), we can simplify (15) to get the following not-so-ordinary differential-integral equations:

$$\max\{\mathcal{L}_1\psi, (z+1-\alpha(\iota))\psi_z(z,\iota)-(1-\gamma)\psi(z,\iota), -(z+1+\theta(\iota))\psi_z(z,\iota)+(1-\gamma)\psi(z,\iota)\} = 0, \tag{18}$$

$$\mathcal{L}_1\psi = \frac{1}{2}\sigma(\iota)^2 z^2 \psi_{zz}(z,\iota) + \beta_2(\iota)z\psi_z(z,\iota) + \beta_1(\iota)\psi(z,\iota) + g(z,\iota), \tag{19}$$

where

$$
\begin{aligned}
g(z,\iota) &= \eta_1^U(\iota)E\left[\psi(\frac{z}{1+J^U},1-\iota)(1+J^U)^{1-\gamma}\right] + \eta_2^U(\iota)E\left[\psi(\frac{z}{1+J^U},\iota)(1+J^U)^{1-\gamma}\right] \\
&+\eta_1^D(\iota)E\left[\psi(\frac{z}{1+J^D},1-\iota)(1+J^D)^{1-\gamma}\right] + \eta_2^D(\iota)E\left[\psi(\frac{z}{1+J^D},\iota)(1+J^D)^{1-\gamma}\right] \\
&+\xi(\iota)\psi(z,1-\iota) + \eta^M(\iota)E\left[\psi(\frac{z}{1+J^M},\iota)(1+J^M)^{1-\gamma})\right] + \lambda\frac{(z+1-\alpha(\iota))^{1-\gamma}}{1-\gamma},
\end{aligned}
$$

$$\beta_2(\iota) = \gamma\sigma(\iota)^2 - \mu(\iota) + r(\iota) + \nu(\iota),$$

$$\beta_1(\iota) = -\delta(\iota) - (1-\gamma)(\gamma\sigma(\iota)^2/2 - \mu(\iota) + \nu(\iota)).$$

4. An Iterative Procedure to Find Optimal Trading Strategy

The fact that the ratio z can jump out of the NT region (reflected by the presence of the term $g(z,\iota)$ in Equation (18)) and the regime can shift, , complicates the problem significantly. We next develop an iterative technique that solves the investor's problem using a sequence of closed form expressions..

First, we choose an initial function $v^0(x,y,\iota)$ that is finite, concave, increasing, and homogeneous such that $v^0(x,y,\iota) \geq v(x,y,\iota)$ for $\iota = 0,1$. For concreteness and ease of boundary conditions, we assume that

$$v^0(x,y,\iota) = M(\iota)\frac{(x+y)^{1-\gamma}}{1-\gamma}, \tag{20}$$

where $M(\iota)$ are the coefficients solved for the no-transaction-cost case.

Then to compute $v^{i+1}(x,y,\iota)$, for $i = 0,1,...,n$, we can solve the following recursive structure

$$v^{i+1}(x,y,\iota) \quad = \quad \sup_{(D,I)\in\Theta(x,y)} E\Big[\int_0^\infty e^{-\delta(\iota)t}\Big(f^i(x_t,y_t,\iota) + \lambda\frac{(x_t+(1-\alpha(\iota))y_t)^{1-\gamma}}{1-\gamma}\Big)dt\Big],$$

where

$$
\begin{aligned}
f^i(x_t, y_t, \iota) &= \eta_1^U(\iota) v^i(x_t, y_t(1+J_t^U), 1-\iota) + \eta_2^U(\iota) v^i(x_t, y_t(1+J_t^U), \iota) \\
&\quad + \eta^M(\iota) v^i(x_t, y_t(1+J_t^M), \iota) + \eta_1^D(\iota) v^i(x_t, y_t(1+J_t^D), 1-\iota) \\
&\quad + \eta_2^D(\iota) v^i(x_t, y_t(1+J_t^D), \iota) + \xi(\iota) v^i(x_t, y_t, 1-\iota). \qquad (21)
\end{aligned}
$$

We have the following useful result for proving the validity of the iterative approach.

Lemma 1 *For $\iota = 0, 1$ and $i = 0, 1, ..., n$, v^i is increasing, concave, and satisfies*

$$
v^i(x, y, \iota) \geq v^{i+1}(x, y, \iota) \geq \frac{\lambda}{\lambda - (1-\gamma) r(\iota)} \frac{(x + (1-\alpha(\iota))y)^{1-\gamma}}{1-\gamma}. \qquad (22)
$$

In addition, v^i satisfies $v^i(p\,x, p\,y) = p^{1-\gamma} v^i(x, y)$ for any $p > 0$. Thus these functions converge to concave functions $\hat{v}(x, y, \iota)$. The convergence is uniform on compact subsets of the interior of the solvency region.

Proof: See Appendix.

Lemma 1 guarantees the convergence of this iterative procedure and the concavity of the limit function $\hat{v}$. We now explicitly construct the functions $v^i(x, y, \iota)$ to facilitate the proof that $\hat{v}$ is indeed the value function. As before, for $\iota = 0, 1$, because of the homogeneity of $v^i(x, y, \iota)$, there exists a function ψ^i such that

$$
v^i(x, y, \iota) = y^{1-\gamma} \psi^i(\frac{x}{y}, \iota).
$$

Solving (21) reduces to finding functions $\psi^i(z, \iota)$ for $\iota = 0, 1$ such that

$$
\frac{1}{2}\sigma(\iota)^2 z^2 \psi^i_{zz} + \beta_2(\iota) z \psi^i_z + \beta_1(\iota) \psi^i + g^i(z, \iota) = 0, \qquad i = 1, ..., n, \qquad (23)
$$

where

$g^i(z, \iota) =$

$$\begin{aligned}
&\eta_1^U(\iota)E\left[\psi^{i-1}(\frac{z}{1+J^U},1-\iota)(1+J^U)^{1-\gamma}\right]+\eta_2^U(\iota)E\left[\psi^{i-1}(\frac{z}{1+J^U},\iota)(1+J^U)^{1-\gamma}\right]\\
&+\eta_1^D(\iota)E\left[\psi^{i-1}(\frac{z}{1+J^D},1-\iota)(1+J^D)^{1-\gamma}\right]+\eta_2^D(\iota)E\left[\psi^{i-1}(\frac{z}{1+J^D},\iota)(1+J^D)^{1-\gamma}\right]\\
&+\xi(\iota)\psi^{i-1}(z,1-\iota)+\eta^M(\iota)E\left[\psi^{i-1}(\frac{z}{1+J^M},\iota)(1+J^M)^{1-\gamma})\right]+\lambda\frac{(z+1-\alpha(\iota))^{1-\gamma}}{1-\gamma},
\end{aligned}$$

and $\beta_2(\iota)$ and $\beta_1(\iota)$ are the same as in (18).

Assumption 1 implies $\beta_1(\iota)<0$. Thus, the homogeneous solution to (23) is given by $\psi_1(z,\iota)=|z|^{n_1(\iota)}$ and $\psi_2(z,\iota)=|z|^{n_2(\iota)}$ where

$$n_{1,2}(\iota)=\frac{(1-\beta_2(\iota))\pm\sqrt{(1-\beta_2(\iota))^2-4\beta_1(\iota)}}{2} \tag{24}$$

with $n_1(\iota)>0$ and $n_2(\iota)<0$. This leads to the general solution to (23)

$$\psi^i(z,\iota)=C_1^i(\iota)\psi_1(z,\iota)+C_2^i(\iota)\psi_2(z,\iota)+\psi_p^i(z,\iota), \tag{25}$$

where $C_1^i(\iota)$ and $C_2^i(\iota)$ are integration constants and $\psi_p^i(z,\iota)$ is the particular solution:

$$\psi_p^i(z,\iota)=u_1^i(z,\iota)\psi_1(z,\iota)+u_2^i(z,\iota)\psi_2(z,\iota), \tag{26}$$

where ,

$$\begin{aligned}
u_1^i(z,\iota)&=\int^z\frac{\psi_2(s,\iota)}{\psi_1'(s,\iota)\psi_2(s,\iota)-\psi_1(s,\iota)\psi_2'(s,\iota)}\frac{2g^i(s,\iota)}{\sigma(\iota)^2s^2}ds,\\
u_2^i(z,\iota)&=-\int^z\frac{\psi_1(s,\iota)}{\psi_1'(s,\iota)\psi_2(s,\iota)-\psi_1(s,\iota)\psi_2'(s,\iota)}\frac{2g^i(s,\iota)}{\sigma(\iota)^2s^2}ds,
\end{aligned}$$

equivalently,

$$\psi_p^i(z,\iota)=\psi_p^{i-1}(z,\iota)+\int^z\frac{\psi_1(s,\iota)\psi_2(z,\iota)-\psi_1(z,\iota)\psi_2(s,\iota)}{\psi_1'(s,\iota)\psi_2(s,\iota)-\psi_1(s,\iota)\psi_2'(s,\iota)}\frac{2(g^i(s,\iota)-g^{i-1}(s,\iota))}{\sigma(\iota)^2s^2}ds.$$

For $\iota=0,1$, the HJB equations imply that

$$\psi^i(z,\iota)=\begin{cases}
A^i(\iota)\frac{(z+1+\theta(\iota))^{1-\gamma}}{1-\gamma} & \text{if } z\geq r_b^i(\iota)\\
C_1^i(\iota)\psi_1(z,\iota)+C_2^i(\iota)\psi_2(z,\iota))+\psi_p^i(z,\iota) & \text{if } \max(r_s^i(\iota),0)<z<r_b^i(\iota)\\
\hat{C}_1^i(\iota)\psi_1(z,\iota)+\hat{C}_2^i(\iota)\psi_2(z,\iota)+\psi_p^i(z,\iota) & \text{if } \min(r_s^i(\iota),0)<z<0\\
B^i(\iota)\frac{(z+1-\alpha(\iota))^{1-\gamma}}{1-\gamma} & \text{if } \alpha(\iota)-1<z\leq r_s^i(\iota),
\end{cases}$$

for some constants $A^i(\iota)$,$B^i(\iota)$,$C_1^i(\iota)$,$C_2^i(\iota)$, $\hat{C}_1^i(\iota)$, $\hat{C}_2^i(\iota)$ and the boundaries $r_s^i(\iota)$ and $r_b^i(\iota)$.

For $\iota = 0, 1$, since $\mu(\iota) > r(\iota)$, the buy boundary must lie in the region $y > 0$. If the buy and sell boundaries $r_b^i(\iota)$ and $r_s^i(\iota)$ are positive, then the third branch is vacuous and the value function is C^2 in the entire solvency region. However, the sell boundary $r_b^i(\iota)$ can be nonpositive while the buy boundary $r_s^i(\iota)$ is positive. In this case, the homogeneous solution suggests that $\hat{C}_2^i(\iota)$ must take the same value as $C_2^i(\iota)$, which must be equal to $-\lim_{z\to 0} u_2^i(z,\iota)$ to keep the value function finite. In addition, one can show by Hospital's rule that $\lim_{z\to 0} \psi^i(z,\iota) = -\frac{g^i(0,\iota)}{\beta_1(\iota)}$ which agrees with direct computation in (21).

The case where $r_b^i(\iota) = \infty$ only arises when it is optimal to never buy stock. Intuitively, this can happen when the transaction cost is large and the investor's expected lifetime is short as shown in Liu and Loewenstein (2002). A similar, albeit more complex, set of conditions will arise in our model. In this case, to keep the value function finite, we must have $C_1^i(\iota) = -\lim_{z\to\infty} u_1^i(z,\iota)$. One can show that

$$\lim_{y\to 0} y^{1-\gamma}\psi^i(\frac{x}{y},\iota) = \lim_{y\to 0} \frac{y^{1-\gamma}\frac{\sigma^2}{2}g^i(\frac{x}{y},\iota)}{\delta(\iota) - (1-\gamma)r}$$

which agrees with direct computation in (21) if it is optimal to never buy stock given an initial position 100% in cash.

Using a similar approach to those in Shreve and Soner (1994) or Framstad, Oksendal, and Sulem (2001), one can show that there exist constants $A^i(\iota)$, $B^i(\iota)$, $C_1^i(\iota)$, $C_2^i(\iota)$, $\hat{C}_1^i(\iota)$, $\hat{C}_2^i(\iota)$ and the boundaries $r_s^i(\iota)$ and $r_b^i(\iota)$ which make $\psi^i(z,\iota)$ a C^2 function in the solvency region except at $z = 0$ or $z = \infty$. We can thus iteratively compute the optimal boundaries and value functions for each i, by following the approach described in Liu and Loewenstein (2002).[5] We then have the following result:

Theorem 2 *As $i \to \infty$, for $\iota = 0, 1$, the functions $v^i(x, y, \iota) = y^{1-\gamma}\psi^i(\frac{x}{y}, \iota)$ converge to $v(x, y, \iota)$.*

Proof: See Appendix.

Theorem 2 shows that the iterative procedure can indeed closely approximate the value function and the corresponding optimal trading strategy.

The optimal trading strategy in Regime ι is no trading if $r_s(\iota) \leq \frac{x}{y} \leq r_b(\iota)$, selling stock to the boundary $r_s(\iota)$ if $\frac{x}{y} < r_s(\iota)$, and buying stock to the boundary $r_b(\iota)$ if $\frac{x}{y} > r_b(\iota)$. In contrast to a diffusion model, it is possible that $\frac{x_t}{y_t}$ jumps out of the NT region, which would be followed by an immediate transaction back to the closest boundary. When the regime shifts, the optimal strategy may or my not involve immediate transaction depending on how the boundaries change across regimes. It is helpful to discuss three possible types of NT regions across regimes we will encounter in our numerical work later.

Case 1. Separated. For example, $r_s(0) < r_b(0) < r_s(1) < r_b(1)$. In this case investors may sell some stock and buy more of the riskfree asset right after a stock price crash. This is consistent with the so-called "flight to quality" phenomenon, but in sharp contrast with the contrarian strategy predicted by a model with i.i.d returns. This case occurs if the regime shifts from the liquid regime ($\iota = 0$) to the illiquid regime ($\iota = 1$) after a large downward price jump *and* the new ratio $\frac{x}{y}$ right after the crash stays in the sell region of the illiquid regime. This case will typically obtain when the shift in the investment opportunity set across regimes is large and the expected time spent in the new regime is long so that the required transaction

[5]Note that $r_s^i(\iota) + (1 - \alpha(1))(1 + \underline{J}) > 0$ and for z in the solvency region such that $z + (1 - \alpha(1))(1 + \underline{J}) \leq 0$ an immediate transaction to $r_s^i(\iota)$ is optimal.

cost is justified.

Case 2. Nested. For example, $r_s(1) < r_s(0) < r_b(0) < r_b(1)$. In this case, if the regime shifts from the liquid regime and the jump magnitude at this time is not too large, then the investor will optimally not rebalance. However, if the regime shifts from the illiquid regime to the liquid regime, then the investor may buy or sell stock even without a price jump. In this case an investor optimally reduces transaction frequency until market conditions improve. Intuitively, this case will occur when the difference in the investment opportunity set is relatively small, the time spent in the illiquid regime is relatively short, and the transaction costs are relatively large.

Case 3. Overlapping but non-nested. For example, $r_s(0) < r_s(1) < r_b(0) < r_b(1)$. In contrast to Case 2, in the absence of large upward jumps, the investor never sells the stock when the regime shifts from the illiquid regime to the liquid regime.

5. Numerical Results

It should be apparent that the jump parameters and liquidity effects interact to determine the optimal transaction boundaries. To gain some understanding of how the various elements of our model are optimally traded off, we now present a baseline case and perform comparative statics to see how the optimal boundaries behave.

For our analysis of the optimal trading strategy, in the liquid regime we use as our default parameters $\alpha(0) = 0.5\%$, $\theta(0) = 0$, $\mu(0) = 7\%$, and in the illiquid regime $\alpha(1) = 2.5\%$, $\theta(1) = 0$, $\mu(1) = 7\%$. We set $r(0) = r(1) = 1\%$, $\gamma = 5$, and $\lambda = 0.04$. Our baseline case will also set $\sigma(0) = \sigma(1)$. These parameters represent an equity premium of 6% in both regimes and an expected horizon of 25 years. The round trip transaction cost is 0.5% in the liquid regime and 2.5% in the illiquid regime.

In our baseline case we assume that a jump arrives on average once every two years and jump intensities do not change across regimes, that is, $\eta^U + \eta^M + \eta^D = 0.5$ with $\eta^U(0) = \eta^U(1) = \eta^U$, $\eta^M(0) = \eta^M(1) = \eta^M$, and $\eta^D(0) = \eta^D(1) = \eta^D$.[6]

For $i \in \{U, M, D\}$, log jump size $\log(1 + J_t^i)$ is assumed to be truncated normal with parameters μ_J and σ_J and support interval $[a^i, b^i]$, where $a^U = \bar{R} > 0$, $b^U = \infty$, $a^M = \underline{R} < 0$, $b^M = \bar{R}$, $a^D = -\infty$, and $b^D = \underline{R}$.

To determine the remaining baseline parameters we calibrate the model to match the variance (.0082), skewness (-1.33) and excess kurtosis (34.92) reported on page 21 in Campbell, Lo, and MacKinlay (1996) for daily log returns. This procedure leads to $\sigma(0) = \sigma(1) = 0.1190$, $\mu_J = -0.0259$, and $\sigma_J = 0.0666$. As default parameter values, we set $\underline{R} = -0.03$ and $\bar{R} = 0.03$, which implies the average large up jump size is 7.0%, the average large down jump size is -7.8%, and the average moderate jump size is 0.0%. Using these parameter values, then we obtain $\eta^U = 0.1003$, $\eta^M = 0.2377$, $\eta^D = 0.1620$. These parameters indicate that the probability of a large down jump is greater than the probability of a large up jump, consistent with the negative skewness of the stock returns. Moderate jumps occur roughly once every 4 years, large up jumps once every 10 years, and large down jumps once every six years.

We assume Regime 0 switches to Regime 1 if and only if a large down jump occurs. To accomplish this, we set $\xi(0) = 0$, $\eta_1^U(0) = 0$, $\eta_2^U(0) = \eta^U$, $\eta_1^D(0) = \eta^D$, $\eta_2^D(0) = 0$. These choices capture the idea that worsened liquidity conditions are usually accompanied by large downward jumps in the stock price. Thus, the fraction $\frac{x}{y}$ will jump up whenever there is a shift from the liquid to the illiquid regime.

Our baseline assumption is that the expected duration of the illiquid regime is one year. To accomplish this we set $\xi(1) = 0.9552$ and $\eta_1^U(1) = 0.0448$, so that

[6] We also conducted analysis on different baseline cases with lower jump frequencies, which implies larger jump sizes on average. The qualitative results are the same.

the regime shifts from 1 to 0 with intensity $\xi(1) + \eta_1^U(1) = 1$. This choice implies the regime can shift from the illiquid regime to the liquid regime with or without large upward price jumps. Our remaining parameters are set to be consistent with our assumptions that the jump intensities do not vary across regimes. The relation that $\eta_1^U(1) + \eta_2^U(1) = \eta^U = .1003$ dictates that $\eta_2^U(1) = 0.0555$. The moderate jump intensity remains fixed at $\eta^M = 0.2377$. For the down jump intensities we set $\eta_1^D(1) = 0$ so that the regime does not shift back to the liquid regime coincident with a large down jump. Thus $\eta^D = \eta_2^D(1) = 0.1620$.

Figure 1 shows how the transaction boundaries vary as a function of the illiquid-regime expected return $\mu(1)$. In the baseline case, the transaction boundaries are nested. In other words, the investor does not transact when the regime shifts from the liquid regime to the illiquid regime unless the price drop is very large in magnitude. With less liquidity, the trading frequency in the illiquid regime is lower, as implied by the wider no-transaction boundaries. If an investor believes the expected return is lower after a large downward price jump, the investor tends to hold less stock in both regimes. This occurs because an investor accounts for the fact that when the regime shifts to a less favorable investment opportunity set, he will want to hold less stock and thus holds less even in the liquid regime to save transaction costs. The no transaction regions are separated when $\mu(1)$ is low enough. Thus, if the investor revises his belief about the expected return significantly downward upon a price crash, then he may sell some stock and buy more the riskfree asset right after the price crash, behaving like "flight to quality." For example, after a 5% price crash, if the investor believes that the price crash reflects that the fundamentals have significantly worsened (e.g., the expected payoff of the stock has dramatically dropped) and the reestimated expected return changes to 2%, then he will sell enough to reach the sell-boundary of the illiquid regime right after the crash. In contrast, standard portfolio choice models with i.i.d. returns (e.g., Merton (1971)) predict the opposite: After a price drop, investor should buy more to rebalance. On the right most side of Figure 1 the investment opportunity set is better in the illiquid regime and the transaction boundaries are again separated, although the no transaction region is lower in the illiquid regime than in the liquid regime (in other words the investor wants to hold more stock in the illiquid regime). In this case the investor will always buy more of the risky asset to take advantage of the higher expected return when the regime shifts from the liquid to illiquid regime and liquidate the position when the market becomes more liquid. It is interesting to note that when the expected return is high in the illiquid regime the transaction boundaries become quite narrow. This is driven by the risk return tradeoff from buying the risky asset in the illiquid regime and waiting a relatively short time for the liquidity to return to the market. Therefore, with heterogenous beliefs about the investment opportunity set after a crash, "flight to quality" might be consistent with an equilibrium where investors differ in their beliefs about the investment opportunity set such as expected return.

Our baseline case assumes that only liquidity changes after a market crash. However, it is possible that given a market crash an investor might perceive the probability of another crash to be also different. Figure 2 shows how the optimal trading boundaries vary as we vary the intensity of the large down jump in the illiquid regime ($\eta_2^D(1)$). For higher values of the intensity of another market crash in the illiquid regime, the investor is more likely to sell right after a market crash in the liquid regime, cetera paribus. Recall that varying the jump parameters does not affect the expected return but does affect variance, skewness, and kurtosis. A large negative jump accompanied by the transition into the illiquid regime where large downward

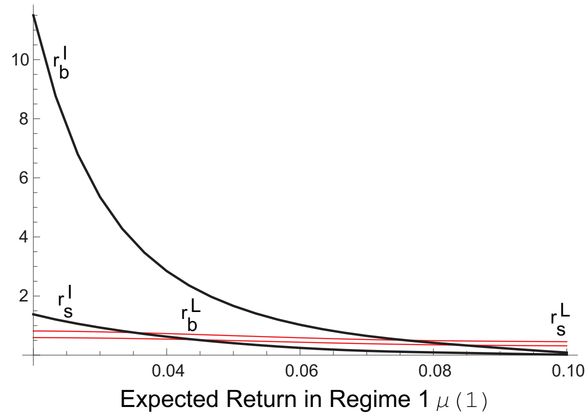

Figure 1: Optimal Trading Boundaries as a Function of $\mu(1)$.
This figure shows how the optimal trading boundaries vary with the expected return in the illiquid regime $\mu(1)$ for parameters: $\mu_J = -0.0259$, $\sigma_J = 0.0666$, $\sigma(0) = \sigma(1) = 0.1190$, $\mu(0) = 0.07$, $r = 0.01$, $\lambda = 0.04$, $\gamma = 5$, $\xi(0) = 0$, $\xi(1) = 0.9552$, $\theta(0) = \theta(1) = 0$, $\alpha(0) = 0.5\%$, $\alpha(1) = 2.5\%$, $\underline{R} = -0.03$, $\bar{R} = 0.03$, $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$, $\eta_2^U(0) = \eta^U = 0.1003$, $\eta_2^U(1) = 0.0555$, $\eta_1^U(1) = 0.0448$, $\eta^M = 0.2377$, and $\eta_1^D(0) = \eta_2^D(1) = \eta^D = 0.1620$.

jumps occur more frequently represents a significant deterioration of the investment opportunity set. Thus the investor might optimally incur the transaction cost to rebalance. In addition, the no transaction region widens significantly in the illiquid regime to reduce transaction costs as large downward jumps becomes more frequent.

Since expected return and jump intensity are not directly observable, one may argue that these parameter values might not have changed after a crash. To address this concern, next we examine the effect of the post-crash changes of market volatility and liquidity that are directly observable. One stylized fact is that markets tend to become more volatile after a market crash. Figure 3 shows how the optimal transaction policy varies when volatility rises and markets become less liquid after a crash. Even for modest increases in volatility in the illiquid regime the transaction boundaries become separated and as volatility in the illiquid regime rises this separation becomes more pronounced. Despite the higher transaction costs the investor

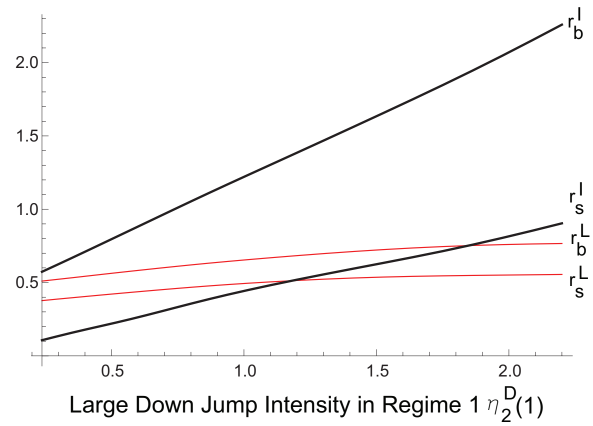

Figure 2: Optimal Trading Boundaries as a Function of $\eta_2^D(1)$.
This figure shows how the optimal trading boundaries vary with the intensity of the downward jump $\eta_2^D(1)$ for parameters: $\mu_J = -0.0259$, $\sigma_J = 0.0666$, $\sigma(0) = \sigma(1) = 0.1190$, $\mu(0) = \mu(1) = 0.07$, $r = 0.01$, $\lambda = 0.04$, $\gamma = 5$, $\xi(0) = 0$, $\xi(1) = 0.9552$, $\theta(0) = \theta(1) = 0$, $\alpha(0) = 0.5\%$, $\alpha(1) = 2.5\%$, $\underline{R} = -0.03$, $\bar{R} = 0.03$, $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$, $\eta_2^U(0) = \eta^U = 0.1003$, $\eta_2^U(1) = 0.0555$, $\eta_1^U(1) = 0.0448$, $\eta^M = 0.2377$, and $\eta_1^D(0) = \eta^D = 0.1620$.

still optimally sells stock and buys the riskless asset after a crash. In addition, the no transaction region in the illiquid regime widens significantly to reduce transaction frequency as the volatility goes up. Once the market becomes less volatile and liquidity improves, the investor then rebalances to buy more stock and hold less of the riskless asset.

Figure 4 shows how the transaction boundaries change as the transaction costs vary in the illiquid regime. The illiquid regime NT region nests the liquid regime NT region. For large transaction costs in the illiquid regime, the investor significantly widens the NT region in the illiquid regime to reduce trading frequency. Thus for large transaction costs, it is optimal to try to wait out the illiquid regime. As transaction costs in the illiquid regime increase, the investor also optimally holds less stock in the *liquid* regime.

The next set of results address the sensitivity to the jump size distribution. For

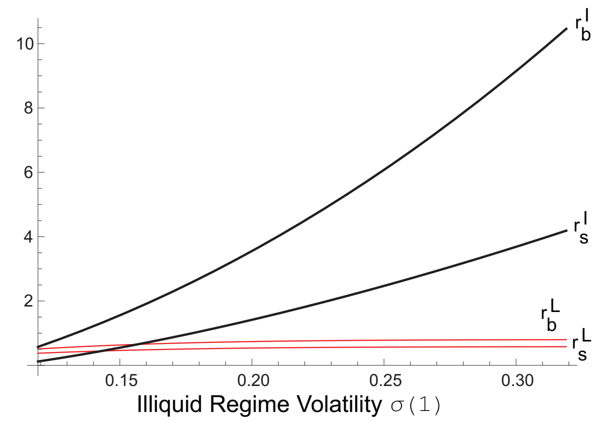

Figure 3: Optimal Trading Boundaries as a Function of $\sigma(1)$.
This figure shows how the optimal trading boundaries vary with the volatility in the illiquid regime $\sigma(1)$ for parameters: $\mu_J = -0.0259$, $\sigma_J = 0.0666$, $\sigma(0) = 0.1190$, $\mu(0) = \mu(1) = 0.07$, $r = 0.01$, $\lambda = 0.04$, $\gamma = 5$, $\xi(0) = 0$, $\xi(1) = 0.9552$, $\theta(0) = \theta(1) = 0$, $\alpha(0) = 0.5\%$, $\alpha(1) = 2.5\%$, $\underline{R} = -0.03$, $\bar{R} = 0.03$, $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$, $\eta_2^U(0) = \eta^U = 0.1003$, $\eta_2^U(1) = 0.0555$, $\eta_1^U(1) = 0.0448$, $\eta^M = 0.2377$, and $\eta_1^D(0) = \eta_2^D(1) = \eta^D = 0.1620$.

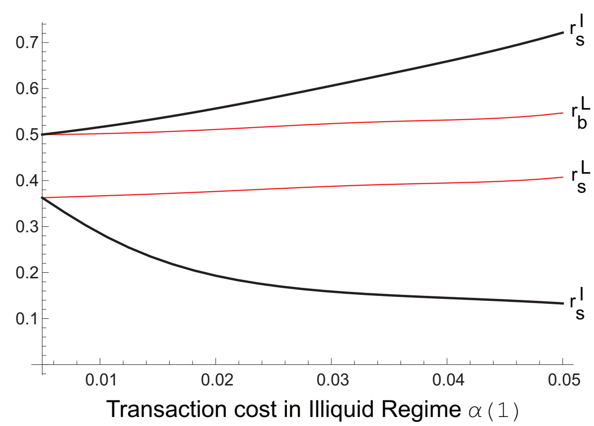

Figure 4: Optimal Trading Boundaries as a Function of $\alpha(1)$.
This figure shows how the optimal trading boundaries vary with the transactions cost in the illiquid regime $\alpha(1)$ for parameters: $\mu_J = -0.0259$, $\sigma_J = 0.0666$, $\sigma(0) = \sigma(1) = 0.1190$, $\mu(0) = \mu(1) = 0.07$, $r = 0.01$, $\lambda = 0.04$, $\gamma = 5$, $\xi(0) = 0$, $\xi(1) = 0.9552$, $\theta(0) = \theta(1) = 0$, $\alpha(0) = 0.5\%$, $\underline{R} = -0.03$, $\bar{R} = 0.03$, $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$, $\eta_2^U(0) = \eta^U = 0.1003$, $\eta_2^U(1) = 0.0555$, $\eta_1^U(1) = 0.0448$, $\eta^M = 0.2377$, and $\eta_1^D(0) = \eta_2^D(1) = \eta^D = 0.1620$.

this we return to our baseline model and maintain the assumption that the unconditional log jump size $\log(1 + J_t)$ is normally distributed with mean μ_J and volatility σ_J. As we vary μ_J or σ_J, we change the the values of η^U, η^M, and η^D so that large up jumps correspond to greater than 3% jump size, moderate jumps between -3% and 3%, and large down jumps less than -3% as before. We maintain all other assumptions such as jumps arrive on average every two years so that $\eta^U + \eta^M + \eta^D = 0.5$. Notice that as μ_J goes down the jumps tend to be more negatively skewed and, in addition, the possibility of a large down jump goes up while the possibility of a large up jump decreases. Thus as we decrease μ_J, in our baseline model, it becomes more likely that the liquid regime shifts to the illiquid regime.

Figure 5 shows how the optimal transaction boundaries vary against μ_J when we assume the expected return remains the same at 7% in both the liquid and the illiquid regimes. This figure reveals, similar to the findings in Longstaff, Liu, and Pan (2003), that the optimal trading boundaries are "U" shaped with some asymmetry. To understand this, recall that the jumps as we have modeled them do not affect the expected stock return. When the expected value of a jump is positive, the jump helps the investor by introducing a positive skew to returns. However, the jump also increases return volatility which can become the dominant force. The asymmetry occurs due to the fact that downward jumps tend to introduce a negative skew, which tends to bring the investor closer to the solvency line and the associated higher marginal utility. In addition, the width of the no transaction regions widen as the expected magnitude of the jumps becomes very large. This is because when jumps tend to be large on average, if the no transaction region is too narrow, then the fraction $\frac{x}{y}$ will tend to jump out of the no transaction region, leading to a large transaction cost payment. For the range of μ_J considered in Figure 5 the transaction

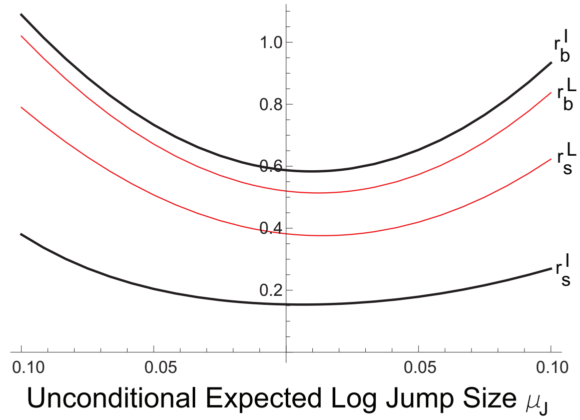

Figure 5: Optimal Trading Boundaries as a Function of μ_J.
This figure shows how the optimal trading boundaries vary with μ_J for parameters: $\sigma_J = 0.0666$, $\sigma(0) = \sigma(1) = 0.1190$, $\mu(0) = \mu(1) = 0.07$, $r = 0.01$, $\lambda = 0.04$, $\gamma = 5$, $\xi(0) = 0$, $\xi(1) = 0.9552$, $\theta(0) = \theta(1) = 0$, $\alpha(0) = 0.5\%$, $\alpha(1) = 2.5\%$, $\underline{R} = -0.03$, $\bar{R} = 0.03$, $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$, and $\eta_1^U(1) = 0.0448$.

boundaries are nested and the investor optimally reduces trading frequency in the illiquid regime.

Figure 6 shows how the optimal trading boundaries vary as we change μ_J as before, but instead we assume that the expected return is 4% in the illiquid regime. Again we see the "U" shaped transaction boundaries but now they are no longer always nested and the no transaction region is higher in the illiquid regime than in Figure 5. This is natural because the investment opportunity set is worse in the illiquid regime. Interestingly, in the liquid regime the no-transaction region in Figure 6 is quite similar to that in Figure 5. This occurs because even though the investor knows he will hold less of the risky asset in the illiquid regime following a market crash, he also knows that a market crash will already make him hold less of the risky asset even without any trading and thus the required transaction cost payment may be small when regime switches. Therefore it is less costly to hold more of the risky asset in the liquid regime with a larger expected jump size.

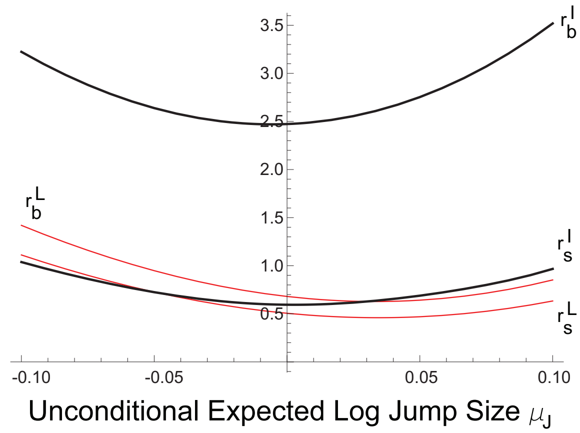

Figure 6: Optimal Trading Boundaries as a Function of μ_J.
This figure shows how the optimal trading boundaries vary with μ_J for parameters: $\sigma_J = 0.0666$, $\sigma(0) = \sigma(1) = 0.1190$, $\mu(0) = 0.07$, $\mu(1) = 0.04$, $r = 0.01$, $\lambda = 0.04$, $\gamma = 5$, $\xi(0) = 0$, $\xi(1) = 0.9552$, $\theta(0) = \theta(1) = 0$, $\alpha(0) = 0.5\%$, $\alpha(1) = 2.5\%$, $\underline{R} = -0.03$, $\bar{R} = 0.03$, $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$, and $\eta_1^U(1) = 0.0448$.

Figure 7 shows how the optimal transaction region varies with the unconditional log jump size volatility σ_J (with the same expected return in both regimes). When σ_J gets large the transaction boundaries generally go up because the increase in volatility makes the stock less attractive. The transaction boundaries also widen significantly since for a narrow NT region, higher σ_J would lead to a higher probability of jumping outside the NT region and thus a higher probability of transaction cost payment from a transaction back to the closest boundary. In addition, varying σ_J affects the probability of a regime shift. For example if σ_J equals zero, we have a model with deterministic jumps which are, given our assumptions, always medium sized. Thus in this case the probability of shifting from Regime 0 to Regime 1 is zero. As σ_J increases, the probability of large up and large down jumps increases and regime shifts become more frequent on average and the investment opportunity set deteriorates.

So far we have been focusing on the analysis of the optimal trading strategies. Next, as an example, we show the economic significance of correctly taking into

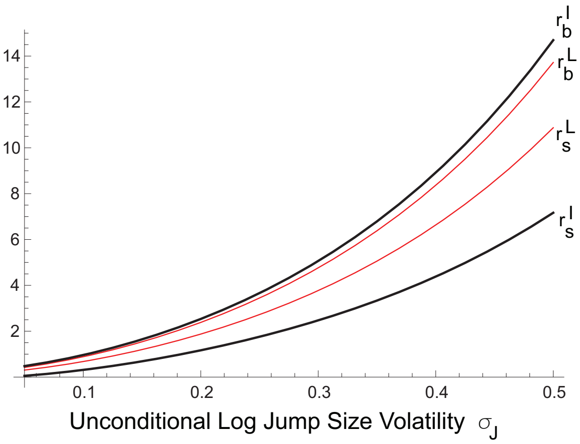

Figure 7: Optimal Trading Boundaries as a Function of Jump Volatility. This figure shows how the optimal trading boundaries vary with σ_J for parameters: $\mu_J = -0.0259$, $\sigma(0) = \sigma(1) = 0.1190$, $\mu(0) = \mu(1) = 0.07$, $r = 0.01$, $\lambda = 0.04$, $\gamma = 5$, $\xi(0) = 0$, $\xi(1) = 0.9552$, $\theta(0) = \theta(1) = 0$, $\alpha(0) = 0.5\%$, $\alpha(1) = 2.5\%$, $\underline{R} = -0.03$, $\bar{R} = 0.03$, $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$, and $\eta_1^U(1) = 0.0448$.

account the correlation between market crash and market illiquidity. Specifically, suppose an investor underestimates the correlation between market crashes and market illiquidity to be 0.1, while the true correlation is 1. The investor then adopts the optimal trading strategy that is based on the wrong estimate. We compute the certainty equivalent wealth loss as a fraction of his initial wealth from this misestimation. Figure 8 plots this loss against the volatility in the iliquid regime for two cases, one with expected illiquidity regime duration of 1 year and the other 2 years. Figure 8 shows that misestimation of the correlation is costly to the investor. For example, the the equivalent wealth loss can be as high as 2% for the 1 year case and about 3.6% for the two year case. This finding suggests the economic importance of correctly taking into account the correlation between market crashes and market illiquidity.

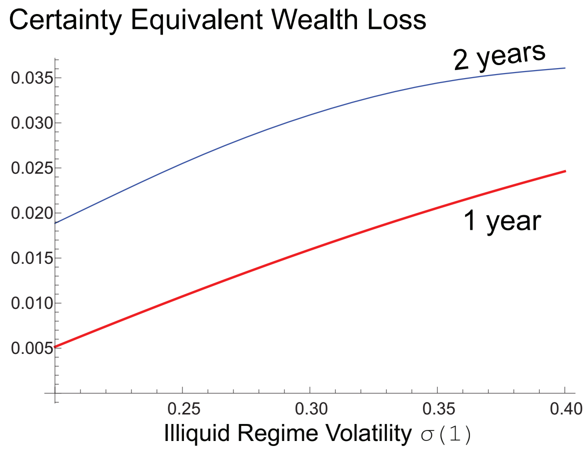

Figure 8: Certainty Equivalent Wealth Loss as a Function of $\sigma(1)$. This figure shows how the certainty equivalent wealth loss as a fraction of the initial wealth varies with $\sigma(1)$ for parameters: $\mu_J = -0.0259$, $\sigma_J = 0.0666$, $\sigma(0) = 0.1190$, $\mu(0) = \mu(1) = 0.07$, $r = 0.01$, $\lambda = 0.04$, $\gamma = 5$, $\xi(0) = 0$, $\theta(0) = \theta(1) = 0$, $\alpha(0) = 0.5\%$, $\alpha(1) = 5\%$, $\underline{R} = -0.03$, $\bar{R} = 0.03$, $\eta_1^U(0) = \eta_1^D(1) = 0$, $\eta_2^U(0) = \eta^U = 0.1003$, $\eta_2^U(1) = 0.0555$, $\eta^M = 0.2377$, and $\eta_2^D(1) = \eta^D = 0.1620$.

6. Conclusion

In this paper we develop a flexible, but tractable workhorse model of optimal portfolio choice with correlated event risks, illiquidity, and stochastic investment opportunity set. Our analysis demonstrates the presence of these risks can be an important factor in determining an optimal portfolio. We also provide an efficient iterative solution procedure that can be applied to a wide class of models with coupled integral-differential equations with free boundaries. Given its incorporation of many of the important determinants of portfolio selection and its tractability, our model provides an attractive framework for studying the joint qualitative and quantitative impact of event risks, liquidity risks, and time-varying return dynamics.

Several extensions to our analysis are immediate. For example, one can examine the effects of a deterministic horizon by using the methodology proposed in Liu and

Loewenstein (2002). While our model is formulated with two regimes, the extension to n regimes is conceptually straightforward.

Appendix

In this Appendix, we collect all the proofs.

PROOF of Lemma 2: Monotonicity, concavity, and homogeneity are fairly obvious for v^1 which inherits these properties from v^0 and subsequently v^{i+1} inherits these properties from v^i, see Shreve and Soner (1994) for example. The inequalities

$$v^i(x,y,\iota) \geq \frac{\lambda}{\lambda-(1-\gamma)r(\iota)}\frac{(x+(1-\alpha(\iota))y)^{1-\gamma}}{1-\gamma} \tag{27}$$

follow from the fact that an investor must have greater utility than that obtained from liquidating the risky asset investment and investing all wealth in the riskless asset and optimally consuming thereafter. The other inequalities $v^i(x,y,\iota) \geq v^{i+1}(x,y,\iota)$ are deduced as follows. Observe that for $\iota = 0, 1$, we have

$$\begin{aligned} v^1(x,y,\iota) &= \sup_{(D,I)\in\Theta(x,y)} E\Big[\int_0^\infty e^{-\delta(\iota)t}\Big(f^0(x_t,y_t,\iota)+\lambda\frac{(x_t+(1-\alpha(\iota))y_t)^{1-\gamma}}{1-\gamma}\Big)dt\Big] \\ &\leq \sup_\pi E\Big[\int_0^\infty e^{-\delta(\iota)t}\Big(f^0(x_t,y_t,\iota)+\lambda\frac{(x_t+y_t)^{1-\gamma}}{1-\gamma}\Big)dt\Big] \\ &= v^0(x,y,\iota), \end{aligned} \tag{28}$$

where the first equality follows from the dynamic programming principle for the no-transaction cost case and the inequality holds because feasible trading policies with transaction costs are also feasible in the no-transaction-cost case. Now assume $v^i(x,y,\iota) \leq v^{i-1}(x,y,\iota)$. Then this implies

$$\begin{aligned} v^{i+1}(x,y,\iota) &= \sup_{(D,I)\in\Theta(x,y)} E\Big[\int_0^\infty e^{-\delta(\iota)t}\Big(f^i(x_t,y_t,\iota)+\lambda\frac{(x_t+(1-\alpha(\iota))y_t)^{1-\gamma}}{1-\gamma}\Big)dt\Big] \\ &\leq \sup_{(D,I)\in\Theta(x,y)} E\Big[\int_0^\infty e^{-\delta(\iota)t}\Big(f^{i-1}(x_t,y_t,\iota)+\lambda\frac{(x_t+(1-\alpha(\iota))y_t)^{1-\gamma}}{1-\gamma}\Big)dt\Big] \\ &= v^i(x,y,\iota). \end{aligned} \tag{29}$$

The last statements of the lemma follow from Rockafellar, Theorem 10.8. □

Proof of Theorem 2. Lemma 1 implies that by passing to a subsequence if necessary we must have as $i \to \infty$, $A^i(\iota) \to A(\iota)$, $B^i(\iota) \to B(\iota)$, $C_1^i(\iota) \to C_1(\iota)$, $C_2^i(\iota) \to C_2(\iota)$, $\hat{C}_1^i(\iota) \to \hat{C}_1(\iota)$, $r_b^i(\iota) \to r_b(\iota)$ and $r_s^i(\iota) \to r_s(\iota)$, for some constants $A(\iota)$, $B(\iota)$, $C_1(\iota)$, $C_2(\iota)$, $\hat{C}_1(\iota)$, $r_s(\iota)$, and $r_b(\iota)$. Note that $r_s(\iota) > \alpha(\iota) - 1$ and $r_b(\iota) > r_s(\iota)$. For a complete proof one would need to provide verification theorems for the functions obtained in each iteration as well as for the limiting value function. Since this part is fairly long, involved, and very similar to those in Jang, Koo, Liu, and Loewenstein (2007), Shreve and Soner (1994), and Framstad, Oksendal, and Sulem (2001), we omit it to minimize repetition. We proceed to show that in all possible cases the limiting value function in Lemma 1 is a solution to the HJB equation with boundary conditions for the investor's problem and thus satisfy the conditions in the verification theorem for the limiting value function.[7]

First for a fixed ι , suppose $0 < r_s(\iota) < r_b(\iota) < \infty$. Define

$$\psi(z,\iota) = \begin{cases} A(\iota)\frac{(z+1+\theta(\iota))^{1-\gamma}}{1-\gamma} & \text{if } z \geq r_b(\iota) \\ C_1(\iota)\psi_1(z,\iota) + C_2(\iota)\psi_2(z,\iota) + \psi_p(z,\iota) & \text{if } r_s(\iota) \leq z \leq r_b(\iota) \\ B(\iota)\frac{(z+1-\alpha(\iota))^{1-\gamma}}{1-\gamma} & \text{if } \alpha(\iota) - 1 < z \leq r_s(\iota), \end{cases}$$

Then by the convergence of the constants, we have that ψ^i converges uniformly to ψ on any compact set of the solvency region, in particular in $[r_s(\iota), r_b(\iota)]$. The functions g^i are concave and converge uniformly on compact sets to a limiting concave function g as defined in (20), from Rockafellar Theorem 10.8 and Lemma 1. Observe that ψ_p^i and its first and second derivatives also converge uniformly on compact sets. Thus we see that for $z \in (r_s(\iota), r_b(\iota))$ the function $C_1(\iota)\psi_1(z,\iota)+C_2(\iota)\psi_2(z,\iota)+\psi_p(z,\iota)$ solves (18) in the NT region.

Observe from the C^2 property of the ψ^i we have (suppressing the fixed ι depen-

[7]By construction, corresponding conditions are satisfied for each iteration.

dence)

$$A^i(r_b^i+1+\theta)^{-\gamma} = C_1^i\psi_1'(r_b^i) + C_2^i\psi_2'(r_b^i) + \psi_p^{i\prime}(r_b^i) \tag{30}$$

$$B^i(r_s^i+1-\alpha)^{-\gamma} = C_1^i\psi_1'(r_s^i) + C_2^i\psi_2'(r_s^i) + \psi_p^{i\prime}(r_s^i) \tag{31}$$

$$-\gamma A^i(r_b^i+1+\theta)^{-\gamma-1} = C_1^i\psi_1''(r_b^i) + C_2^i\psi_2''(r_b^i) + \psi_p^{i\prime\prime}(r_b^i) \tag{32}$$

$$-\gamma B^i(r_s^i+1-\alpha)^{-\gamma-1} = C_1^i\psi_1''(r_s^i) + C_2^i\psi_2''(r_s^i) + \psi_p^{i\prime\prime}(r_s^i) \tag{33}$$

So thanks to the uniform convergence of ψ_p^i, in the limit we have

$$A(r_b+1+\theta)^{-\gamma} = C_1\psi_1'(r_b) + C_2\psi_2'(r_b) + \psi_p'(r_b) \tag{34}$$

$$B(r_s+1-\alpha)^{-\gamma} = C_1\psi_1'(r_s) + C_2\psi_2'(r_s) + \psi_p'(r_s) \tag{35}$$

$$-\gamma A(r_b+1+\theta)^{-\gamma-1} = C_1\psi_1''(r_b) + C_2\psi_2''(r_b) + \psi_p''(r_b) \tag{36}$$

$$-\gamma B(r_s+1-\alpha)^{-\gamma-1} = C_1\psi_1''(r_s) + C_2\psi_2''(r_s) + \psi_p^{j\prime\prime}(r_s). \tag{37}$$

So ψ is a solution to the HJB equation (18) with the boundary conditions.

Next, assume for a fixed ι that $0 = r_s(\iota) < r_b(\iota) < \infty$. The basic approach above still works. However, we must recognize since $\psi^{i,j}(z)$ converge to a finite valued concave function we must have $C_2(\iota) = -\lim_{z\to 0} u_2(z,\iota)$ to keep the value function finite. Otherwise the situation above still applies.

The case $r_s(\iota) < 0 < r_b(\iota) < \infty$ is also similar to the above. In this case, we can write the limiting function as

$$\psi(z,\iota) = \begin{cases} A(\iota)\frac{(z+1+\theta(\iota))^{1-\gamma}}{1-\gamma} & \text{if } z \geq r_b(\iota) \\ C_1(\iota)\psi_1(z,\iota) + C_2(\iota)\psi_2(z,\iota) + \psi_p(z,\iota) & \text{if } 0 \leq z \leq r_b(\iota) \\ \hat{C}_1(\iota)\psi_1(z,\iota) + C_2(\iota)\psi_2(z,\iota) + \psi_p(z,\iota) & \text{if } r_s(\iota) \leq z \leq 0 \\ B(\iota)\frac{(z+1-\alpha(\iota))^{1-\gamma}}{1-\gamma} & \text{if } \alpha(\iota)-1 < z \leq r_s(\iota), \end{cases}$$

where we must recognize that $C_2(\iota) = -\lim_{z\to 0} u_2(z,\iota)$ to keep the value function finite.

Finally, we must also consider the possibility that $r_b(\iota) = \infty$. Again the proof is similar to the above arguments once we recognize this requires restrictions on $C_1(\iota)$. We leave the details to the determined reader.

$\square$

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