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Verification Theorems for Models of Optimal Consumption and Investment with Retirement and Constrained Borrowing

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Proving verification theorems can be tricky for models with both optimal stopping and state constraints. We pose and solve two alternative models of optimal consumption and investment with an optimal retirement date (optimal stopping) and various wealth constraints (state constraints). The solutions are parametric in closed form up to at most a constant. We prove the verification theorem for the main case with a nonnegative wealth constraint by combining the dynamic programming and Slater condition approaches. One unique feature of the proof is the application of the comparison principle to the differential equation solved by the proposed value function. In addition, we also obtain analytical comparative statics.

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1. Introduction. Retirement is one of the most important economic events in a worker's life. This paper contains a rigorous formulation and analysis of several models of life cycle consumption and investment with voluntary or mandatory retirement and with or without a borrowing constraint against future labor income. In these models, optimal consumption jumps at retirement and, if retirement is voluntary, the optimal portfolio choice also jumps at retirement. If retirement is voluntary, the optimal retirement rule gives human capital a negative beta if wages are uncorrelated with the stock market because retirement comes later when the market is down. This leads to aggressive investment in the market, a result that is dampened when borrowing against future labor income is prohibited and may be reversed when wages are positively correlated with market returns. In the companion paper, Dybvig and Liu [1] focus on the economic intuitions for these results. In this paper, we provide rigorous proofs.

The main results in this paper are explicit parametric solutions (up to some constants) with verification theorems and analytical comparative statics. In particular, we combine the dual approach of Pliska [8], He and Pagès [3], Karatzas and Shreve [5] and Karatzas and Wang [6] with an analysis of the boundary to obtain a problem we can solve in a parametric form even if no known explicit solution exists in the primal problem. Having an explicit dual solution allows us to derive analytically the impact of parameter changes and, more importantly, allows us to prove a verification theorem showing that the first-order (Bellman equation) solution is a true solution to the choice problem. Compared to the existing literature (e.g., Pliska [8], Karatzas and Wang [6]), the no-borrowing constraint against future labor income significantly complicates the derivations and the proof of the verification theorem. The proof is subtle because of (1) the nonconvexity introduced by the retirement decision, (2) the market incompleteness (from the agent's view) caused by the nonnegative wealth constraint, and (3) the technical problems caused by utility unbounded above or below. Two common approaches to proving a verification theorem are the dynamic programming (Fleming-Richel) approach and the separating hyperplane (Slater condition) approach. Both approaches encounter difficulties in our setting so we use a hybrid of the two (a separating hyperplane after retirement and dynamic programming before retirement). The two are combined with optional sampling, where the continuation after retirement is replaced by the known value of the optimal continuation. One of the most challenging tasks for proving a verification theorem for optimal stopping problems is to show that the proposed value function satisfies certain inequality conditions so that it is indeed optimal to stop at the proposed boundaries. One unique feature of our proof is that this is shown indirectly by applying the comparison principle to the differential equations solved by the proposed value function. This indirect method makes the proof simple and elegant. So far as we know, we are the first to use this approach to prove a verification theorem for this type of control problem involving an optimal stopping time.

The rest of the paper is organized as follows. Section 2 presents the formal choice problems used in the paper. Section 3 presents analytical solutions, comparative statics, and proofs. Section 4 closes the paper.

2. Choice problems. We consider the optimal consumption and investment problem of an investor who can continuously trade a risk-free asset and n risky assets. The risk-free asset pays a constant interest rate of r. The risky asset price vector S_t evolves as

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma^\top dZ_t,$$

where Z_t is a standard n dimensional Wiener process; μ is an $n \times 1$ constant vector and σ is an $n \times n$ invertible constant matrix so that market is complete with no redundant assets; and the division is element by element.

The investor also earns labor income y_t:

$$y_t \equiv y_0 \exp\left[\left(\mu_y - \frac{\sigma_y{}^\top \sigma_y}{2}\right) t + \sigma_y{}^\top Z_t\right], \tag{1}$$

where y_0 is the initial income from working and μ_y and σ_y are constants of appropriate dimensions. The investor can choose to irreversibly retire at any point in time.

The arrival time τ_d of the investor's mortality follows an independent Poisson process with constant intensity δ. The investor can purchase insurance coverage of $B_t - W_t$ against mortality, where W_t is the financial wealth of the investor at time t so that, if death occurs at t, the investor has a bequest of $W_t + (B_t - W_t) = B_t$. To receive the insurance coverage $B_t - W_t$ at the time of mortality, the investor pays the insurer at a rate of $\delta(B_t - W_t)$, i.e., insurance is assumed to be fairly priced at the mortality rate δ per unit of coverage.

The investor derives utility from intertemporal consumption and bequest. The investor has a constant relative risk aversion (CRRA), time additive utility function (2) with a subjective time discount rate ρ:

$$E\left[\int_0^{\tau_d} e^{-\rho t}\left((1-R_t)\frac{c_t^{1-\gamma}}{1-\gamma} + R_t \frac{(Kc_t)^{1-\gamma}}{1-\gamma}\right) dt + e^{-\rho \tau_d} \frac{(kB_{\tau_d})^{1-\gamma}}{1-\gamma}\right], \tag{2}$$

where $\gamma > 0$ is the relative risk aversion coefficient and $\gamma \neq 1$,[1] the constant $K > 1$ indicates preferences for not working in the sense that the marginal utility of consumption is greater after retirement than before retirement, the constant $k > 0$ measures the intensity of preference for leaving a large bequest, the limit $k^{1-\gamma} \to 0$ implements the special case with no preference for bequest, and R_t is the right-continuous and nondecreasing indicator of the retirement status at time t (which is 1 after retirement and 0 before retirement). The state variable R_{0-} is the retirement status at the beginning of the investment horizon.

Define

$$\iota(\text{S}) = \begin{cases} 1 & \text{if statement S is true,} \\ 0 & \text{otherwise,} \end{cases}$$

$$g(t) \equiv \begin{cases} \left(\dfrac{1 - e^{-\beta_1 (T-t)}}{\beta_1}\right)^+ & \text{if } \beta_1 \neq 0, \\ (T-t)^+ & \text{if } \beta_1 = 0, \end{cases} \tag{3}$$

where

$$\beta_1 \equiv r + \delta - \mu_y + \sigma_y^\top \kappa > 0 \tag{4}$$

is the effective discount rate for labor income (assumed to be positive) and

$$\kappa \equiv (\sigma^\top \sigma)^{-1} \sigma^\top (\mu - r\mathbf{1}) \tag{5}$$

is the price of risk. Below are the two choice problems we focus on in this paper.

Problem 1. *Given initial wealth W_0, initial income from working y_0, and the deterministic time to retirement T with associated retirement indicator function $R_t = \iota(t \geq T)$, choose adapted nonnegative consumption $\{c_t\}$, adapted portfolio $\{\theta_t\}$, and adapted nonnegative bequest $\{B_t\}$ to maximize expected utility of lifetime consumption and bequest*

$$E\left[\int_0^{\tau_d} e^{-\rho t}\left((1-R_t)\frac{c_t^{1-\gamma}}{1-\gamma} + R_t \frac{(Kc_t)^{1-\gamma}}{1-\gamma}\right) dt + e^{-\rho \tau_d} \frac{(kB_{\tau_d})^{1-\gamma}}{1-\gamma}\right]$$

[1] $\gamma = 1$ corresponds to the log utility case, which can be examined similarly. Most of our results in the paper apply to the log case by taking $\gamma \to 1$.

subject to the budget constraint

$$W_t = W_0 + \int_0^t (rW_s\,ds + \theta_s^\top((\mu - r\mathbf{1})\,ds + \sigma^\top dZ_s) + \delta(W_s - B_s)\,ds - c_s\,ds + (1-R_s)y_s\,ds), \tag{6}$$

the labor income process (1), *and the limited borrowing constraint*

$$W_t \geq -g(t)y_t, \tag{7}$$

where $g(t)y_t$ *is the market value at* t *of the future labor income.*

Let $\mathscr{C} \in \{0, 1\}$ denote the type of borrowing constraints, with $\mathscr{C} = 1$ for the limited borrowing type and $\mathscr{C} = 0$ for the no-borrowing type. Problem 1 corresponds to Problem 1 of Dybvig and Liu [1], Problem 2 with $\mathscr{C} = 1$ corresponds to Problem 2 of Dybvig and Liu [1], and Problem 2 with $\mathscr{C} = 0$ corresponds to Problem 3 of Dybvig and Liu [1].

Problem 2. *Given initial wealth* W_0, *initial income from working* y_0, *initial retirement status* R_{0-}, *and borrowing constraint type* $\mathscr{C} \in \{0, 1\}$, *choose adapted nonnegative consumption* $\{c_t\}$, *adapted portfolio* $\{\theta_t\}$, *adapted nonnegative bequest* $\{B_t\}$, *and adapted nondecreasing retirement indicator* $\{R_t\}$ *(i.e., a right-continuous nondecreasing process taking values* 0 *and* 1) *to maximize the expected utility of lifetime consumption and bequest* (2) *subject to the budget constraint* (6), *the labor income process before retirement* (1), *and the borrowing constraint*

$$W_t \geq -\mathscr{C}(1-R_t)\frac{y_t}{\beta_1}, \tag{8}$$

where $(1-R_t)y_t/\beta_1$ *is the market value at* t *of the subsequent labor income.*

To summarize the differences across the problems, moving from Problem 1 to Problem 2, the fixed retirement date T ($R_t = \iota(t \geq T)$) is replaced by free choice of retirement date (R_t, a choice variable) along with a technical change in the calculation of the market value of future labor income $g(t)y_t$ to $(1-R_t)y_t/\beta_1$ when $\mathscr{C} = 1$. When $\mathscr{C} = 0$, then the investor faces a no-borrowing constraint $W_t \geq 0$.

Remark. Because the time of mortality τ_d is independent of the Brownian motion, we have that the objective function

$$\begin{aligned} E\bigg[\int_0^{\tau_d} e^{-\rho t}\bigg((1-R_t)\frac{c_t^{1-\gamma}}{1-\gamma} + R_t\frac{(Kc_t)^{1-\gamma}}{1-\gamma}\bigg)\,dt + e^{-\rho\tau_d}\frac{(kB_{\tau_d})^{1-\gamma}}{1-\gamma}\bigg] \\ = E\bigg[\int_0^\infty e^{-(\rho+\delta)t}\bigg((1-R_t)\frac{c_t^{1-\gamma}}{1-\gamma} + R_t\frac{(Kc_t)^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_t)^{1-\gamma}}{1-\gamma}\bigg)\,dt\bigg] \\ = E\bigg[\int_0^\infty e^{-(\rho+\delta)t}\bigg(\frac{(K^{R_t}c_t)^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_t)^{1-\gamma}}{1-\gamma}\bigg)\,dt\bigg]. \end{aligned}$$

We denote the value functions for Problems 1 and 2 by $v(W, y, t)$ and $V(W, y, R, \mathscr{C})$, respectively. Note that, because both problems become the same after retirement, we have

$$v(W, y, T) = V(W, y, 1, 1) = V(W, y, 1, 0). \tag{9}$$

3. The analytical solution and comparative statics. The general idea of solving Problems 1 and 2 is to perform a change of variables into a dual variable, the marginal utility of consumption. The general advantage of this dual approach (especially for Problem 2) is that it linearizes the nonlinear Hamilton-Jacobi-Bellman (HJB) equation in the primal problem. This is consistent with the method of Pliska [8] of converting a dynamic budget constraint into a static budget constraint.

Let

$$\nu \equiv \frac{\gamma}{\rho + \delta - (1-\gamma)(r + \delta + \kappa^\top\kappa/2\gamma)}. \tag{10}$$

For our solutions, we will assume $\nu > 0$, which is also the condition for the corresponding Merton problem (Merton [7]) to have a solution because, if $\nu < 0$, then an investor can achieve infinite utility by delaying consumption.

Define the state price density process ξ by

$$\xi_t \equiv e^{-(r+\delta+\frac{1}{2}\kappa^\top\kappa)t - \kappa^\top Z_t}. \tag{11}$$

This is the usual state price density adjusted to be conditional on living, given the mortality rate δ and fair pricing of long and short positions in term life insurance.

Also, define

$$b \equiv 1 - 1/\gamma,$$
$$f(t) \equiv (\hat{\eta} - \eta)\exp\left(-\frac{1+\delta k^{-b}}{\eta}(T-t)^+\right) + \eta,$$
$$\eta \equiv (1+\delta k^{-b})\nu,$$

and

$$\hat{\eta} \equiv (K^{-b} + \delta k^{-b})\nu.$$

The solution to Problem 1 can be stated as follows.

Theorem 3.1. *Suppose $\nu > 0$ and that the limited borrowing constraint is satisfied with strict inequality at the initial values*

$$W_0 > -g(0)y_0. \tag{12}$$

The solution to an investor's Problem 1 can be written in terms of a dual variable $\hat{x}_t$ (a normalized marginal utility of consumption), where

$$\hat{x}_t \equiv \left(\frac{W_0 + g(0)y_0}{f(0)}\right)^{-\gamma} e^{(\rho+\delta)t}\xi_t y_t^{\gamma}. \tag{13}$$

Then, the optimal wealth process is

$$W_t^* = f(t)y_t\hat{x}_t^{-1/\gamma} - g(t)y_t, \tag{14}$$

the optimal consumption policy is

$$c_t^* = K^{-bR_t} y_t \hat{x}_t^{-1/\gamma}, \tag{15}$$

the optimal trading strategy is

$$\theta_t^* = y_t\left[\frac{(\sigma^\top\sigma)^{-1}(\mu - r\mathbf{1})}{\gamma} f(t)\hat{x}_t^{-1/\gamma} - (\sigma^\top\sigma)^{-1}\sigma^\top\sigma_y g(t)\right],$$

and the optimal bequest policy is

$$B_t^* = k^{-b} y_t \hat{x}_t^{-1/\gamma}. \tag{16}$$

Furthermore, the value function for the problem is

$$v(W, y, t) = f(t)^\gamma \frac{(W + g(t)y)^{1-\gamma}}{1-\gamma}.$$

Proof. Because Problem 1 can be transformed into a standard dual problem with minor modifications (e.g., Pliska [8]), we omit the proof here. See Dybvig and Liu [1] for a sketch of the proof using a separating hyperplane to separate preferred consumptions from the feasible consumptions. □

Unlike Problem 1, Problem 2 also requires one to solve for the optimal retirement decision. We conjecture that it is optimal to retire when the wealth-to-income ratio is high enough, which corresponds to when a new dual variable x_t hits a lower bound $\underline{x}$. Then, as in a typical optimal stopping problem, one imposes C^1 condition for the dual value function across $\underline{x}$. In the presence of the no-borrowing constraint (i.e., $\mathscr{C} = 0$), one also imposes the no-risky-investment condition (i.e., the second derivative of the dual value function is 0) when wealth W_t hits 0 (or, equivalently, when x_t hits an upper bound $\bar{x}$). After obtaining the solutions, we verify that all of our conjectures are indeed correct.

Recall the definitions of ν in (10) and β_1 in (4). Define

$$\beta_2 \equiv \rho + \delta + \tfrac{1}{2}\gamma(1-\gamma)\sigma_y^\top\sigma_y - (1-\gamma)\mu_y,$$
$$\beta_3 \equiv (\gamma\sigma_y - \kappa)^\top(\gamma\sigma_y - \kappa),$$
$$\alpha_- \equiv \frac{\beta_1 - \beta_2 + \frac{1}{2}\beta_3 - \sqrt{(\beta_1 - \beta_2 + \frac{1}{2}\beta_3)^2 + 2\beta_2\beta_3}}{\beta_3}, \tag{17}$$

$$\alpha_+ \equiv \frac{\beta_1 - \beta_2 + \frac{1}{2}\beta_3 + \sqrt{(\beta_1 - \beta_2 + \frac{1}{2}\beta_3)^2 + 2\beta_2\beta_3}}{\beta_3}, \tag{18}$$

$$A_- \equiv (1-\mathscr{C})\left(\frac{\eta(b-\alpha_-)}{\alpha_+(\alpha_+-\alpha_-)}\bar{x}^{b-\alpha_+} - \frac{1-\alpha_-}{\alpha_+(\alpha_+-\alpha_-)\beta_1}\bar{x}^{1-\alpha_+}\right), \tag{19}$$

$$A_+ \equiv \mathscr{C}\frac{1}{\gamma(b-\alpha_-)\beta_1}\underline{x}^{1-\alpha_-} + (1-\mathscr{C})\left(\frac{\eta(\alpha_+-b)}{\alpha_-(\alpha_+-\alpha_-)}\bar{x}^{b-\alpha_-} - \frac{\alpha_+-1}{\alpha_-(\alpha_+-\alpha_-)\beta_1}\bar{x}^{1-\alpha_-}\right), \tag{20}$$

$$\bar{x} \equiv \left(\frac{(((\eta-\hat{\eta})/b)\zeta^{b-\alpha_-} - \eta/\alpha_-)(\alpha_+-b)\beta_1}{(\zeta^{1-\alpha_-} - 1/\alpha_-)(\alpha_+-1)}\right)^{\gamma}, \tag{21}$$

$$\underline{x} \equiv \mathscr{C}\left(\frac{(\eta-\hat{\eta})(b-\alpha_-)\beta_1}{b(1-\alpha_-)}\right)^{\gamma} + (1-\mathscr{C})\zeta\bar{x}, \tag{22}$$

where $\zeta \in (0, 1)$ is the unique solution to $q(\zeta) = 0$[2] and thus where

$$q(\zeta) \equiv \left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\zeta^{b-\alpha_-} - \frac{1}{\alpha_-}\right)\left(\zeta^{1-\alpha_+} - \frac{1}{\alpha_+}\right)(\alpha_+-b)(\alpha_--1)$$
$$- \left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\zeta^{b-\alpha_+} - \frac{1}{\alpha_+}\right)\left(\zeta^{1-\alpha_-} - \frac{1}{\alpha_-}\right)(\alpha_--b)(\alpha_+-1). \tag{23}$$

Then, the solution to Problem 2 can be stated as follows.

THEOREM 3.2. *Suppose $\nu > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$ and that the borrowing constraint holds with strict inequality at the initial condition:*[3]

$$W_0 > -\mathscr{C}(1-R_{0-})\frac{y_0}{\beta_1}. \tag{24}$$

The solution to an investor's Problem 2 can be written in terms of a dual variable x_t, where

$$x_t \equiv \mathscr{C}x_0 e^{(\rho+\delta)t}\xi_t\left(\frac{y_t}{y_0}\right)^{\gamma} + (1-\mathscr{C})\frac{x_0 e^{(\rho+\delta)t}\xi_t(y_t/y_0)^{\gamma}}{\max(1, \sup_{0\le s\le \min(t,\tau^*)} x_0 e^{(\rho+\delta)s}\xi_s(y_s/y_0)^{\gamma}/\bar{x})}$$

and where x_0 solves

$$-y_0\varphi_x(x_0, R_{0-}, \mathscr{C}) = W_0, \tag{25}$$

$$\tau^* = (1-R_{0-})\inf\left\{t \ge 0\colon x_0 e^{(\rho+\delta)t}\xi_t\left(\frac{y_t}{y_0}\right)^{\gamma} \le \underline{x}\right\}, \tag{26}$$

and

$$\varphi(x, R, \mathscr{C}) = \begin{cases} -\hat{\eta}\dfrac{x^b}{b} & \text{if } R = 1 \text{ or } x \le \underline{x}, \\ A_+x^{\alpha_-} + A_-x^{\alpha_+} - \eta\dfrac{x^b}{b} + \dfrac{1}{\beta_1}x & \text{otherwise.} \end{cases} \tag{27}$$

Then, the optimal consumption policy is

$$c_t^* = K^{-bR_t^*}y_t x_t^{-1/\gamma},$$

the optimal trading strategy is

$$\theta_t^* = y_t(\sigma^\top\sigma)^{-1}[(\mu - r\mathbf{1})x_t\varphi_{xx}(x_t, R_t^*, \mathscr{C}) - \sigma^\top\sigma_y(\gamma x_t\varphi_{xx}(x_t, R_t^*, \mathscr{C}) + \varphi_x(x_t, R_t^*, \mathscr{C}))],$$

the optimal bequest policy is

$$B_t^* = k^{-b}y_t x_t^{-1/\gamma},$$

[2] The existence and uniqueness of the solution to $q(\zeta) = 0$ is shown in Lemma 3.4.

[3] $\beta_1 > 0$ is to ensure the finiteness of the labor income; $\beta_2 > 0$ is to ensure the finiteness of the expected utility from working and consuming labor income forever; and $\beta_3 > 0$ is to avoid the degeneration of the dual process x_t to a deterministic function of time. The razor edge case where $\beta_3 = 0$ is less interesting and needs a separate treatment.

the optimal retirement policy is

$$R_t^* = \iota\{t \geq \tau^*\},$$

the corresponding retirement wealth threshold is

$$\bar{W}_t = -y_t \varphi_x(\underline{x}, 0, \mathscr{C}),$$

and the optimal wealth is

$$W_t^* = -y_t \varphi_x(x_t, R_t^*, \mathscr{C}). \tag{28}$$

Furthermore, the value function is

$$V(W, y, R, \mathscr{C}) = y^{1-\gamma}(\varphi(x, R, \mathscr{C}) - x\varphi_x(x, R, \mathscr{C})), \tag{29}$$

where x solves

$$-y\varphi_x(x, R, \mathscr{C}) = W. \tag{30}$$

The investor's problem can be associated with the dual optimal stopping problem, where the investor solves

$$\phi(x_t, y_t, \mathscr{C}) \equiv \max_\tau E_t\left[\int_t^\tau e^{-(\rho+\delta)(s-t)} y_s^{1-\gamma}\left(-(1+\delta k^{-b})\frac{x_s^b}{b} + x_s\right) ds + e^{-(\rho+\delta)(\tau-t)} y_\tau^{1-\gamma}\left(-\hat{\eta}\frac{x_\tau^b}{b}\right)\right]$$

subject to (1),

$$\frac{dx_t}{x_t} = \mu_x\, dt + \sigma_x^\top dZ_t,$$

and the borrowing constraint

$$-y_t^\gamma \phi_x(x_t, y_t) \geq -\mathscr{C}\frac{y_t}{\beta_1},$$

where

$$\mu_x \equiv -(r-\rho) - \tfrac{1}{2}\gamma(1-\gamma)\sigma_y^\top \sigma_y + \gamma\mu_y - \gamma\sigma_y^\top \kappa \tag{31}$$

and

$$\sigma_x \equiv \gamma\sigma_y - \kappa. \tag{32}$$

The transformed dual value function $\varphi(x, 0, \mathscr{C}) \equiv y^{\gamma-1}\phi(x/y^\gamma, y, \mathscr{C})$ then satisfies a variational inequality[4]

$$\mathscr{L}_0\varphi = 0, \qquad \underline{x} < x < \frac{\bar{x}}{1-\mathscr{C}},$$

$$\mathscr{L}_0\varphi < 0, \qquad 0 < x < \underline{x},$$

$$\varphi(x, 0, \mathscr{C}) > -\hat{\eta}\frac{x^b}{b}, \qquad \underline{x} < x < \frac{\bar{x}}{1-\mathscr{C}},$$

$$\varphi(x, 0, \mathscr{C}) = -\hat{\eta}\frac{x^b}{b}, \qquad 0 < x \leq \underline{x},$$

$$\varphi_x(x, 0, \mathscr{C}) < 0, \qquad 0 < x < \frac{\bar{x}}{1-\mathscr{C}},$$

$$\varphi_{xx}(x, 0, \mathscr{C}) > 0, \qquad 0 < x < \frac{\bar{x}}{1-\mathscr{C}}, \qquad \text{and} \qquad x \neq \underline{x}$$

with boundary conditions

$$\varphi_x(\bar{x}, 0, 0) = 0$$

and

$$\varphi_{xx}(\bar{x}, 0, 0) = 0,$$

where

$$\mathscr{L}_0\varphi \equiv \frac{1}{2}\beta_3 x^2\varphi_{xx} - (\beta_1 - \beta_2)x\varphi_x - \beta_2\varphi - (1+\delta k^{-b})\frac{x^b}{b} + x.$$

[4] We do not prove that the duality gap is zero or even that the first-order solution of the dual problem is an actual solution. However, we do not need these results because our verification theorem works with the primal objective function.

When $\mathscr{C}=1$, Problem 2 can be transformed into one that has been studied by Karatzas and Wang (2000) with minor modifications.[5] Therefore, we will only present the proof for the case with the no-borrowing constraint, i.e., $\mathscr{C}=0$. For notation simplicity, we use $V(W,y,R)$ and $\varphi(x,R)$ to represent $V(W,y,R,0)$ and $\varphi(x,R,0)$, respectively. Let

$$\psi(x)\equiv A_+x^{\alpha_-}+A_-x^{\alpha_+}-\eta\frac{x^b}{b}+\frac{1}{\beta_1}x \tag{33}$$

and

$$\hat{\psi}(x)\equiv -\hat{\eta}\frac{x^b}{b}. \tag{34}$$

Then, by (27), (33), and (34), we have

$$\varphi(x,R)=\begin{cases}\hat{\psi}(x) & \text{if } R=1 \text{ or } x\le \underline{x},\\ \psi(x) & \text{otherwise.}\end{cases}$$

The following lemmas are useful for the proof of Theorem 3.2.

Lemma 3.1. *Suppose $\mathscr{C}=0$, $\nu>0$, $\beta_1>0$, $\beta_2>0$, and $\beta_3>0$. Suppose there exists a solution $\zeta\in(0,1)$ to Equation (23) (to be shown in Lemma 3.4). Then,*

(i) *$\hat{\psi}(x)$ is strictly convex and strictly decreasing for $x\ge 0$.*

(ii) *$\forall\, x\le\bar{x}$, we have $\psi(x)\ge\hat{\psi}(x)$; $\forall\, x\in[\underline{x},\bar{x}]$, we have $\psi_x(x)\ge\hat{\psi}_x(x)$ and*

$$\underline{x}<\left(\frac{1-K^{-b}}{b}\right)^{\gamma}. \tag{35}$$

(iii)

$$A_-<0,\qquad A_+>0,\qquad \text{and}\qquad \bar{x}>\left(\frac{(1-\alpha_-)(1+\delta k^{-b})}{b-\alpha_-}\right)^{\gamma}.$$

(iv) *$\psi(x)$ is strictly convex and strictly decreasing for $x<\bar{x}$.*

(v) *Given $W_0>0$, there exists a unique solution $x_0>0$ to (25). In addition, W_t^* defined in (28) satisfies the borrowing constraint (8).*

Proof of Lemma 3.1. (i) $\gamma>0$ implies that $b=1-1/\gamma<1$. Then, because $\nu>0$, direct differentiation shows that $\hat{\psi}(x)$ is strictly convex and strictly decreasing for $x\ge 0$.

(ii) Let

$$h(x)\equiv\psi(x)-\hat{\psi}(x).$$

It can be easily verified that

$$\frac{1}{2}\beta_3x^2\hat{\psi}_{xx}(x)-(\beta_1-\beta_2)x\hat{\psi}_x(x)-\beta_2\hat{\psi}(x)-(K^{-b}+\delta k^{-b})\frac{x^b}{b}=0 \tag{36}$$

and

$$\frac{1}{2}\beta_3x^2\psi_{xx}(x)-(\beta_1-\beta_2)x\psi_x(x)-\beta_2\psi(x)-(1+\delta k^{-b})\frac{x^b}{b}+x=0, \tag{37}$$

with

$$\psi(\underline{x})=\hat{\psi}(\underline{x}), \tag{38}$$

$$\psi_x(\underline{x})=\hat{\psi}_x(\underline{x}), \tag{39}$$

$$\psi_x(\bar{x})=0, \tag{40}$$

and

$$\psi_{xx}(\bar{x})=0.$$

[5] One such transformation is

$$\tilde{W}_t\equiv\frac{W_t}{y_t},\qquad \tilde{\theta}_t\equiv\frac{\theta_t}{y_t}-\frac{W_t}{y_t}\sigma^{-1}\sigma_y,\qquad \tilde{B}_t\equiv\frac{B_t}{y_t},\qquad \tilde{c}_t\equiv\frac{c_t}{y_t},\qquad\text{and}\qquad \tilde{V}(\tilde{W},y,R,1)\equiv y^{\gamma-1}V(y\tilde{W},y,R,1).$$

We thank an anonymous referee for pointing this out.

Then, by (36) and (37), $h(x)$ must satisfy

$$\frac{1}{2}\beta_3 x^2 h'' - (\beta_1 - \beta_2)xh' - \beta_2 h = \frac{1-K^{-b}}{b}x^b - x. \tag{41}$$

By (38)–(40) and the fact that $\hat{\psi}(x)$ is monotonically decreasing for $x > 0$, we have

$$h(\underline{x}) = 0, \qquad h'(\underline{x}) = 0, \qquad h'(\bar{x}) > 0. \tag{42}$$

Differentiating (41) once, we obtain

$$\tfrac{1}{2}\beta_3 x^2 h''' + (\beta_3 - \beta_1 + \beta_2)xh'' - \beta_1 h' = (1-K^{-b})x^{b-1} - 1. \tag{43}$$

We consider two possible cases.

Case 1. $(1-K^{-b})\underline{x}^{b-1} - 1 < 0$. In this case, the right-hand side of Equation (43) is negative. Because $\beta_1 > 0$, $h'(x)$ cannot have any interior nonpositive minimum. To see this, suppose $\hat{x} \in (\underline{x}, \bar{x})$ achieves an interior minimum with $h'(\hat{x}) \le 0$. Then, we would have $h'''(\hat{x}) \ge 0$ and $h''(\hat{x}) = 0$, which implies that the left-hand side is positive. This is a contradiction. Because $h'(\underline{x}) = 0$, $h'(\bar{x}) > 0$, we must have $h'(x) > 0$ for any $x \in (\underline{x}, \bar{x}]$; otherwise, there would be an interior nonpositive minimum. Then, the fact that $h(\underline{x}) = 0$ implies that $h(x) > 0$ for any $x \in (\underline{x}, \bar{x}]$. Because $h'(x) > 0$ for any $x \in (\underline{x}, \bar{x}]$ and $h'(\underline{x}) = 0$, we must have $h''(\underline{x}) \ge 0$. In addition, if $h''(\underline{x})$ were equal to 0, then we would have $h'''(\underline{x}) < 0$ by (42) and (43) because $(1-K^{-b})\underline{x}^{b-1} - 1 < 0$. However, this would contradict the fact that $h'(x) > 0$ for any $x \in (\underline{x}, \bar{x}]$ and $h'(\underline{x}) = 0$. Therefore, we must have $h''(\underline{x}) > 0$. Then, (41), (42), and $h''(\underline{x}) > 0$ imply that

$$\underline{x} < \left(\frac{1-K^{-b}}{b}\right)^{\gamma}.$$

Case 2. $(1-K^{-b})\underline{x}^{b-1} - 1 \ge 0$. In this case, we must have $0 < b < 1$ because $K > 1$. Therefore, $\underline{x} \le (1-K^{-b})^{\gamma} < ((1-K^{-b})/b)^{\gamma}$. This implies that $h''(\underline{x}) > 0$ by (41) and (42). There exists $\epsilon > 0$ such that $h'(x) > 0$ for any $x \in (\underline{x}, \underline{x}+\epsilon]$ because $h'(\underline{x}) = 0$. The right-hand side of Equation (43) is monotonically decreasing in x. Let x^* be such that the right-hand side of (43) is 0. Then, for any $x \le x^*$, the right-hand side is nonnegative and thus $h'(x)$ cannot have any interior nonnegative (local) maximum in $[\underline{x}, x^*]$ for similar reasons to those in Case 1. There cannot exist any $\hat{x} \in (\underline{x}+\epsilon, x^*]$ such that $h'(\hat{x}) \le 0$. If $x^* < \bar{x}$, then, for any $x \in (x^*, \bar{x}]$, the right-hand side is nonpositive and thus $h'(x)$ cannot have any interior nonpositive (local) minimum in $(x^*, \bar{x}]$. There cannot exist any $\hat{x} \in (x^*, \bar{x}]$ such that $h'(\hat{x}) \le 0$. Therefore, there cannot exist any $\hat{x} \in (\underline{x}, \bar{x})$ such that $h'(\hat{x}) \le 0$ and thus we have $h'(x) > 0$ and $h(x) > 0$ for any $x \in (\underline{x}, \bar{x}]$.

Now, we show, for both cases, that $h(x) > 0$ for any $x < \underline{x}$. Equation (35) implies that the right-hand side of (41) is positive for $x < \underline{x}$ and h cannot achieve an interior positive maximum for $x < \underline{x}$. On the other hand, $h''(\underline{x}) > 0$, $h''(x)$ is continuous at $\underline{x}$ and $h'(\underline{x}) = 0$, which imply that there exists an $\epsilon > 0$ such that

$$\forall\, x \in [\underline{x} - \epsilon, \underline{x}], \quad h'(x) < 0.$$

Thus, $\forall\, x \in [\underline{x} - \epsilon, \underline{x})$, $h(x) > 0$ and, therefore, $\forall\, x < \underline{x}$, $h(x) > 0$; otherwise, h would achieve an interior positive maximum in $(0, \underline{x})$.

(iii) Recall that $\mathscr{C} = 0$. It can be shown that

$$A_+ = \frac{(\eta - \hat{\eta})(\alpha_+ - b)}{b(\alpha_+ - \alpha_-)}\underline{x}^{b-\alpha_-} - \frac{(\alpha_+ - 1)}{(\alpha_+ - \alpha_-)\beta_1}\underline{x}^{1-\alpha_-}$$

and

$$\eta = \frac{(\alpha_+ - 1)(1-\alpha_-)(1+\delta k^{-b})}{(\alpha_+ - b)(b - \alpha_-)\beta_1}.$$

Equation (35) then implies that $A_+ > 0$. Because we also have (20), $\bar{x}$ must satisfy

$$\bar{x} > \left(\frac{\eta(\alpha_+ - b)\beta_1}{\alpha_+ - 1}\right)^{\gamma}.$$

Because

$$\frac{\alpha_+ - b}{\alpha_+ - 1} > \frac{b - \alpha_-}{1 - \alpha_-},$$

we have

$$\bar{x} > \left(\frac{\eta(b - \alpha_-)\beta_1}{1 - \alpha_-}\right)^{\gamma}, \tag{44}$$

which (by the definition (19)) implies that $A_- < 0$.

(iv) Differentiating (33) twice, we have, for $x < \bar{x}$,

$$\begin{aligned}\psi_{xx}(x) &= \big(A_-\alpha_+(\alpha_+-1)x^{\alpha_+-b} + A_+\alpha_-(\alpha_--1)x^{\alpha_--b} - \eta(b-1)\big)x^{b-2}\\ &> \psi_{xx}(\bar{x})(x/\bar{x})^{b-2} = 0,\end{aligned} \tag{45}$$

where the inequality follows from the fact that

$$\frac{d}{dx}[A_-\alpha_+(\alpha_+-1)x^{\alpha_+-b} + A_+\alpha_-(\alpha_--1)x^{\alpha_--b}] < 0.$$

This is implied by $A_- < 0$, $A_+ > 0$, $\alpha_+ > 1 > b > \alpha_-$, and $\alpha_- < 0$, and the last equality in (45) follows from $\psi_{xx}(\bar{x}) = 0$. Thus, $\psi(x)$ is strictly convex $\forall\, x < \bar{x}$. Because $\psi_x(\bar{x}) = 0$ and $\forall\, x < \bar{x}$, $\psi_{xx}(x) > 0$, we must also have $\forall\, x < \bar{x}$, $\psi_x(x) < 0$.

(v) By part (i), part (iv), and $\psi_x(\underline{x}) = \hat{\psi}_x(\underline{x})$, $\varphi_x(x, R)$ is continuous and strictly increasing in $x \in (0, \bar{x}]$. By inspection of (33) and (34), $\varphi_x(x, R)$ takes on all nonpositive values. Because $y_0 > 0$, there exists a unique solution $x_0 > 0$ to (25) for each $W_0 > 0$. Also, because $\varphi_x(x, R) \le 0$, (28) implies that $W_t^* \ge 0$, $\forall\, t \ge 0$. □

Though the dual approach yields almost explicit solutions, it is simpler to show the optimality of the candidate policies in the primal for this combined optimal stopping and optimal control problem.

Define

$$\begin{aligned}M_t &= \int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + V(W_s, y_s, 1)\,dR_s\right]\\ &\quad + (1-R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0).\end{aligned} \tag{46}$$

The following lemma is a generalized dominated convergence theorem that is required for the proof of Lemma 3.3.

LEMMA 3.2. *Suppose that a.s. convergent sequences of random variables $X_n \to X$ and $Y_n \to Y$ satisfy $0 \le X_n \le Y_n$ and $E[Y_n] \to E[Y] < \infty$. Then, $E[X_n] \to E[X]$.*

PROOF OF LEMMA 3.2. Because $0 \le X_n \le Y_n$, by Fatou's lemma, $\liminf E[X_n] \ge E[X]$ and $\liminf E[Y_n - X_n] \ge E[Y - X]$. These inequalities imply that both $\limsup E[X_n] \ge E[X]$ and $\liminf E[X_n] \le E[X]$ because $E[Y_n] \to E[Y] < \infty$. Therefore, we must have $E[X_n] \to E[X]$. □

LEMMA 3.3. *Suppose $\mathscr{C} = 0$. Given the definitions for Theorem 3.2,*

(i) *M_t as defined in (46) is a supermartingale for any feasible policy and a martingale for the claimed optimal policy.*

(ii) *For any feasible policy,*

$$\lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0)] \ge 0 \tag{47}$$

with equality for the claimed optimal policy.

PROOF OF LEMMA 3.3. 1. Define $\bar{W} = -y\varphi_x(\underline{x}, 0)$. Then, for any $W \ge 0$,

$$V(W, y, 0) \ge V(W, y, 1) \tag{48}$$

with equality for $W \ge \bar{W}$. This can be shown as follows.

Let x and x^R be such that $-y\varphi_x(x, 0) = W$ and $-y\varphi_x(x^R, 1) = W$. Then, we have

$$\varphi(x,0) - \varphi(x^R,1) \ge \varphi(x,1) - \varphi(x^R,1) \ge \varphi_x(x^R,1)(x - x^R) = x\varphi_x(x,0) - x^R\varphi_x(x^R,1),$$

where the first inequality follows from $\varphi(x,0) \ge \varphi(x,1)$ by Lemma 3.1 and the second inequality follows from the convexity of $\varphi(x,1)$. After rearranging, we obtain (48).

Applying the generalized Itô's lemma to M_t (see, e.g., Harrison [2, §4.7]), we have

$$\begin{aligned}M_t &= M_0 + \int_0^t (1-R_s)\left\{e^{-(\rho+\delta)s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma} + \mathscr{L}V(W_s, y_s, 0)\right)\right\}ds\\ &\quad + \int_0^t e^{-(\rho+\delta)s}(V(W_s, y_s, 1) - V(W_s, y_s, 0))\,dR_s\\ &\quad + \int_0^t (1-R_s)e^{-(\rho+\delta)s}(V_W(W_s, y_s, 0)\theta_s^\top\sigma^\top + y_sV_y(W_s, y_s, 0)\sigma_y^\top)\,dZ_s,\end{aligned} \tag{49}$$

where

$$\begin{aligned}\mathscr{L}V &= \tfrac{1}{2}\theta_t^\top\sigma^\top\sigma\theta_t V_{WW}+\theta_t^\top\sigma^\top\sigma_y y_t V_{Wy}+\tfrac{1}{2}\sigma_y^\top\sigma_y y^2 V_{yy}\\ &\quad+(rW_t+\theta_t^\top(\mu-r\mathbf{1})+\delta(W_t-B_t)-c_t+(1-R_t)y_t)V_W+\mu_y V_y-(\rho+\delta)V.\end{aligned}$$

By the definitions of V, φ, $(c^*, B^*, \theta^*, R^*, W^*)$, and the fact that $\varphi(x,0)$ satisfies (36)–(39), we obtain that the first integral is always nonpositive for any feasible policy (c, B, θ, R) and is equal to zero for the claimed optimal policy $(c^*, B^*, \theta^*, R^*)$. By (48), the third term in (49) is always nonpositive for every feasible retirement policy R_t and equal to zero for the claimed optimal policy R_t^*. In addition, using the expressions for the claimed optimal θ_t^*, V, B_t^*, and W_t^*, we have that, under the claimed optimal policy, the stochastic integral is a martingale because y_t is a geometric Brownian motion; with $\mathscr{C}=0$, x_t is bounded between $\underline{x}$ and $\bar{x}$ before retirement; and x_t is also a geometric Brownian motion after retirement. This shows that M_t is a local supermartingale for all feasible policies and a martingale for the claimed optimal policy.

Next, we show that M_t is actually a supermartingale for all feasible policies. First, we show that $(1-\gamma)V(W,y,0)\geq 0$ for every feasible policy. By (29) and (30), $V_W(W,y,0)=y^{-\gamma}x>0$ and thus $V(W,y,0)$ increases in W. If $\gamma<1$, then $V(W,y,0)\geq 0$ because $V(W,y,0)\geq V(W,y,1)\geq 0$. If $\gamma>1$, then $V(W,y,0)<0$ because $V(W,y,0)\leq V(\bar{W},y,0)=V(\bar{W},y,1)<0$. Therefore, $(1-\gamma)V(W,y,0)\geq 0$ for every feasible policy.

If $\gamma<1$, then $V(W_t,y_t,0)>0$ and the local supermartingale M_t is then always nonnegative and thus a supermartingale.

Suppose $\gamma>1$. By (49), there exists an increasing sequence of stopping times $\tau_n\to\infty$ such that $M_0\geq E[M_{\tau_n\wedge t}]$, i.e.,

$$\begin{aligned}V(W_0,y_0,0) \geq E\int_s^{\tau_n\wedge t} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+V(W_s,y_s,1)\,dR_s\right]\\ +E[(1-R_{\tau_n\wedge t})e^{-(\rho+\delta)(\tau_n\wedge t)}V(W_{\tau_n\wedge t},y_{\tau_n\wedge t},0)].\end{aligned} \tag{50}$$

Because the integrand in the integral of (50) is always negative, this integral is monotonically decreasing in time. In addition,

$$\begin{aligned}0 &\geq (1-R_t)e^{-(\rho+\delta)t}V(W_t,y_t,0)\\ &\geq e^{-(\rho+\delta)t}V(0,y_t,0)\\ &= V(0,1,0)e^{-(\rho+\delta)t}{y_t}^{1-\gamma}\\ &\geq V(0,1,0)N_t,\end{aligned} \tag{51}$$

where

$$N_t\equiv e^{-\frac{1}{2}(1-\gamma)^2\sigma_y^2 t+(1-\gamma)\sigma_y Z_t} \tag{52}$$

is a martingale with $E[N_t]=1$, the second inequality follows from V being negative and increasing in W and $W_t>0$, the equality follows from the form of V as defined by (29) and (30), and the last inequality follows from $V(0,1,0)<0$ and $\beta_2>0$. In addition, $V(0,1,0)>-\infty$.

Therefore, taking $n\to\infty$ in (50), by the monotone convergence theorem for the first term and Lemma 3.2 for the second term, we have

$$\begin{aligned}V(W_0,y_0,0) \geq E\int_0^{t} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+V(W_s,y_s,1)\,dR_s\right]\\ +E[(1-R_t)e^{-(\rho+\delta)t}V(W_t,y_t,0)].\end{aligned}$$

That is, $M_0\geq E[M_t]$ for any $t\geq 0$. Because this argument applies to any time $s\leq t$, we have $M_s\geq E_s[M_t]$ for any $t\geq s$. Thus, M_t is a supermartingale for all feasible policies.

2. Because $(1-\gamma)V(W,y,0)\geq 0$ for every feasible policy, we have

$$\begin{aligned}0 &\leq \lim_{t\to\infty}E[(1-R_t)e^{-(\rho+\delta)t}(1-\gamma)V(W_t,y_t,0)]\\ &=\lim_{t\to\infty}E[(1-R_t)e^{-(\rho+\delta)t}y_t^{1-\gamma}(1-\gamma)(\varphi(x_t,0)-x_t\varphi_x(x_t,0))]\\ &\leq\lim_{t\to\infty}E[L_1 e^{-(\rho+\delta)t}y_t^{1-\gamma}+\hat{\eta}e^{-(\rho+\delta)t/\gamma}\xi_t^b]\\ &=0\end{aligned} \tag{53}$$

for some constant L_1, where the second inequality follows from the fact that when $\mathscr{C}=0,x_t$, $\varphi(x_t,0)$, and $\varphi_x(x_t,0)$ are all bounded and $R_t=1$ for $t>\tau^*$. The last equality in (53) follows from the conditions that $\nu>0$ and $\beta_2>0$.

Therefore, for the claimed optimal policy, we obtain

$$\lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}V(W_t,y_t,0)]=0.$$

For any feasible policy, if $\gamma<1$, then $V(W,y,0,\mathscr{C})>0$ and therefore (47) holds. If $\gamma>1$, because $\beta_2>0$, we have $\lim_{t\to\infty}E[e^{-(\rho+\delta)t}y_t{}^{1-\gamma}]=0$. Therefore, taking the limit as $t\to\infty$ in (51), we have that (47) also holds. This completes the proof. □

LEMMA 3.4. *Suppose $\nu>0$, $\beta_1>0$, $\beta_2>0$, and $\beta_3>0$. Then, there exists a unique solution $\zeta^*\in(0,1)$ to Equation (23) and*

$$\zeta^*<\bar{\zeta}=\min\left(\left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\right)^{\gamma},1\right).$$

PROOF OF LEMMA 3.4. Because $\nu>0$, $\beta_1>0$, $\beta_2>0$, and $\beta_3>0$,

$$\alpha_+>1>b>\alpha_-,\qquad \alpha_-<0.$$

Next, because $\zeta^{b-\alpha_+}$ dominates $\zeta^{1-\alpha_+}$ as $\zeta\to 0$, we have

$$\lim_{\zeta\to 0}q(\zeta)=\lim_{\zeta\to 0}-\frac{1-K^{-b}}{b(1+\delta k^{-b})}(\alpha_--b)(\alpha_+-1)\zeta^{b-\alpha_+}=+\infty.$$

If $\bar{\zeta}=1$, it is easy to verify that

$$q(\bar{\zeta})=-\frac{(\alpha_+-1)(\alpha_--1)(\alpha_+-\alpha_-)(K^{-b}+\delta k^{-b})}{\alpha_+\alpha_-(1+\delta k^{-b})}<0.$$

If $\bar{\zeta}=((1-K^{-b})/(b(1+\delta k^{-b})))^{\gamma}<1$, then we have

$$\frac{1-K^{-b}}{b(1+\delta k^{-b})}\bar{\zeta}^{b-\alpha_+}-\frac{1}{\alpha_+}=\bar{\zeta}^{1-\alpha_+}-\frac{1}{\alpha_+},\qquad \frac{1-K^{-b}}{b(1+\delta k^{-b})}\bar{\zeta}^{b-\alpha_-}-\frac{1}{\alpha_-}=\bar{\zeta}^{1-\alpha_-}-\frac{1}{\alpha_-},\qquad\text{and}$$

$$\bar{\zeta}^{1-\alpha_+}>1>\frac{1}{\alpha_+}.$$

It follows that

$$q(\bar{\zeta})=-\frac{1}{\gamma}\left(\bar{\zeta}^{1-\alpha_+}-\frac{1}{\alpha_+}\right)\left(\bar{\zeta}^{1-\alpha_-}-\frac{1}{\alpha_-}\right)(\alpha_+-\alpha_-)<0.$$

Then, by continuity of q, there exists a solution $\zeta^*\in(0,\bar{\zeta})$ such that $q(\zeta^*)=0$. Suppose there exists another solution $\hat{\zeta}\in[0,1)$ such that $q(\hat{\zeta})=0$. Let $V(W,y,0)$ and $\bar{W}$ be the value function and boundary, respectively, corresponding to ζ^* and let $\hat{V}(W,y,0)$ and $\hat{\bar{W}}$ be the value function and boundary, respectively, corresponding to $\hat{\zeta}$. Without loss of generality, suppose $\bar{W}>\hat{\bar{W}}$. Because $\hat{\bar{W}}$ is the retirement boundary, the value function corresponding to $\hat{\zeta}$ for $\bar{W}>W>\hat{\bar{W}}$ is equal to $V(W,y,1)$. However, Lemma 3.1 implies that $V(W,y,0)>V(W,y,1)$ for any $W<\bar{W}$. This implies that $\hat{\bar{W}}$ cannot be the optimal retirement boundary, which contradicts Theorem 3.2. Therefore, the solution to Equation (23) is unique. □

We are now ready to prove Theorem 3.2.

PROOF OF THEOREM 3.2. If $R_0=1$, then Problem 2 is identical to Problem 1. Therefore, the optimality of the claimed optimal strategy follows from Theorem 3.1. In addition, as noted in (9), we have

$$V(W,y,1)=v(W,y,T),$$

where $v(W,y,T)$ (independent of T) is the value function after retirement for Problem 1. From now on, we assume w.l.o.g. that $R_0=0$. It is tedious but straightforward to use the generalized Itô's lemma and Equations (17)–(23) and (27)–(28) to verify that the claimed optimal strategy W_t^*, c_t^*, θ_t^*, and R_t^* in Theorems 3.1 and 3.2 satisfy the budget constraint (6). In addition, by Lemma 3.1, x_0 exists and is unique and W_t^* satisfies the borrowing constraint in each problem. Furthermore, by Lemma 3.4, there is a unique solution to (23).

By Doob's optional sampling theorem, we can restrict attention w.l.o.g. to the set of feasible policies that implement the optimal policy stated in Theorem 1 after retirement. The utility function for such a strategy can be written as

$$E\int_0^\infty e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+V(W_s,y_s,1)\,dR_s\right].$$

By Lemma 3.3, M_t is a supermartingale for any feasible policy (c,B,θ,R) and a martingale for the claimed optimal policy (c^*,B^*,θ^*,R^*), which implies that $M_0\ge E[M_t]$, i.e.,

$$V(W_0,y_0,0)\ge E\int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+V(W_s,y_s,1)\,dR_s\right] + E[(1-R_t)e^{-(\rho+\delta)t}V(W_t,y_t)], \tag{54}$$

with equality for the claimed optimal policy. In addition, by Lemma 3.3, we also have that

$$\lim_{t\to\infty}E[(1-R_t)e^{-(\rho+\delta)t}V(W_t,y_t)]\ge 0$$

with equality for the claimed optimal policy.

Therefore, taking the limit as $t\uparrow\infty$ in (54), we have

$$V(W_0,y_0,0)\ge E\int_0^\infty e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+V(W_s,y_s,1)\,dR_s\right]$$

with equality for the claimed optimal policy (c^*,B^*,θ^*,R^*). This completes the proof. □

Theorem 2 provides essentially complete solutions because the solution for x_0, given W_0, requires only a one-dimensional monotone search to solve Equation (25) and, for ζ, one only needs to solve Equation (23).

We next provide results on computing the market value of human capital at any point in time, which is useful for understanding much of the economics in the paper.

PROPOSITION 3.1. *Consider the optimal policies stated in Theorems 3.1–3.2. After retirement, the market value of the human capital (i.e., the capitalized labor income) is zero. Before retirement, in Theorem 3.1, the market value of the human capital is*

$$H(y_t,t)=g(t)y_t,$$

where y_t and $g(t)$ are given in (1) and (3). In Theorem 3.2, if $\mathscr{C}=1$, then the market value of the human capital is

$$H(x_t,y_t)=\frac{y_t}{\beta_1}(-\underline{x}^{1-\alpha_-}x_t^{\alpha_--1}+1)\quad \textit{and},$$

if $\mathscr{C}=0$, then the market value of the human capital is

$$H(x_t,y_t)=\frac{y_t}{\beta_1}(Ax_t^{\alpha_--1}+Bx_t^{\alpha_+-1}+1),$$

where

$$A=\frac{(1-\alpha_+)\underline{x}^{1-\alpha_-}\bar{x}^{\alpha_+-\alpha_-}}{(\alpha_+-1)\bar{x}^{\alpha_+-\alpha_-}-(\alpha_--1)\underline{x}^{\alpha_+-\alpha_-}}$$

and

$$B=\frac{(\alpha_--1)\underline{x}^{1-\alpha_-}}{(\alpha_+-1)\bar{x}^{\alpha_+-\alpha_-}-(\alpha_--1)\underline{x}^{\alpha_+-\alpha_-}}.$$

PROOF. For Problem 1, by Itô's lemma, (1), (3), (4), (11), and simple algebra,

$$d(\xi_t g(t)y_t)=-\xi_t(1-R_t)y_t\,dt+\xi_t g(t)y_t(\sigma_y-\kappa)^\top dZ_t.$$

Furthermore,

$$E\int_0^t(\xi_s g(s)y_s)^2(\sigma_y-\kappa)^\top(\sigma_y-\kappa)\,ds<\infty$$

because $\xi_s y_s$ is a standard lognormal diffusion and the other factors are bounded for any t and zero for $t>T$. Therefore, the local martingale

$$\xi_t g(t)y_t+\int_0^t\xi_s(1-R_s)y_s\,ds=g(0)y_0+\int_{s=0}^t\xi_s g(s)y_s(\sigma_y-\kappa)^\top dZ_s \tag{55}$$

is a martingale that is constant for $t > T$. Picking any $\mathcal{T} > \max(t, T)$, the definition of a martingale implies that

$$\xi_t g(t) y_t + \int_0^t \xi_s (1 - R_s) y_s \, ds = E_t\left[\xi_{\mathcal{T}} g(\mathcal{T}) y_{\mathcal{T}} + \int_0^{\mathcal{T}} \xi_s (1 - R_s) y_s \, ds\right].$$

Now, $\mathcal{T} > \max(t, T)$ implies that $g(\mathcal{T}) = 0$, and the integral on the left-hand side is known at t. Therefore, we can subtract the integral from both sides and divide both sides by ξ_t to conclude

$$\begin{aligned} g(t) y_t &= \frac{1}{\xi_t} E_t\left[\int_t^{\mathcal{T}} \xi_s (1 - R_s) y_s \, ds\right] \\ &= \frac{1}{\xi_t} E_t\left[\int_t^{\infty} \xi_s (1 - R_s) y_s \, ds\right], \end{aligned}$$

where the second equality follows from the fact that $R_s \equiv 1$ for $s > \mathcal{T}$.

For the cases with voluntary retirement, because there is no more labor income after retirement, the market value of human capital after retirement is zero. We next prove the claims for after retirement. Using the expressions of H and the dynamics of x_t and y_t, it can be verified that, for $x_t > \underline{x}$ when $\mathscr{C} = 1$ and for $\underline{x} < x_t < \bar{x}$ when $\mathscr{C} = 0$, we have that the change in the market value of human capital plus the flow of labor income will be given by

$$\begin{aligned} d\big(\xi_t H(x_t, y_t)\big) + \xi_t y_t \, dt &= \xi_t\big(\tfrac{1}{2}\beta_3 x_t^2 H_{xx} - (\beta_1 - \beta_2 - \beta_3) x_t H_x - \beta_1 H + y_t\big) dt \\ &\quad + \xi_t\big(x_t H_x \sigma_x^\top + H(\sigma_y^\top - \kappa^\top)\big) dZ_t. \end{aligned} \tag{56}$$

The drift term in (56) is equal to zero after plugging in the expressions for H (if $\mathscr{C} = 0$, the additional local time term at $\bar{x}$ from applying the generalized Itô's lemma is also equal to zero because it can be verified that $H_x(\bar{x}, y_t) = 0$). This implies that

$$\mathcal{M}_t \equiv \xi_t H(x_t, y_t) + \int_0^t \xi_s y_s \, ds$$

is a local martingale. In addition, there exists a constant $0 < L < \infty$ such that

$$|\xi_t (x_t H_x \sigma_x^\top + H(\sigma_y^\top - \kappa^\top))| < L \xi_t y_t$$

because $\alpha_- < 0$ and $x_t > \underline{x}$ if $\mathscr{C} = 1$ and $\underline{x} < x_t \le \bar{x}$ if $\mathscr{C} = 0$. Because both ξ_t and y_t are geometric Brownian motions, we have that $\mathcal{M}_t$ is actually a martingale. Recall the definition (26) of the optimal retirement time τ^*. For all $t \le \tau^*$ we have

$$\xi_t H(x_t, y_t) + \int_0^t \xi_s y_s \, ds = E_t\left[\xi_{\tau^*} H(\underline{x}, y_{\tau^*}) + \int_0^{\tau^*} \xi_s y_s \, ds\right],$$

which implies that

$$H(x_t, y_t) = \xi_t^{-1} E_t\left[\int_t^{\tau^*} \xi_s y_s \, ds\right]$$

because it can be easily verified that $H(\underline{x}, y) = 0$. Therefore, H as specified in Proposition 3.1 is indeed the market value of the future labor income. □

The following result shows that because of retirement flexibility, human capital may have a negative beta even when labor income correlates positively with market risk.

Proposition 3.2. *As an investor's financial wealth W increases, the investor's human capital H decreases in Problem 2. Furthermore, if $\sigma_y < \kappa/\gamma$, then human capital has a negative beta measured relative to any locally mean-variance–efficient risky portfolio.*

The following lemma is useful for the proof of Propositions 3.2 and 3.4.

Lemma 3.5. *Suppose $\mathscr{C} = 1$, $\nu > 0$, $\beta_1 > 0$, $\beta_2 > 0$, and $\beta_3 > 0$. Then,*
(i) *$\hat{\psi}(x)$ is strictly decreasing and strictly convex.*
(ii) *$\psi(x)$ is strictly convex and $\psi_x(x) \le 1/\beta_1$.*
(iii) *$\forall x \ge 0$, we have $\psi(x) \ge \hat{\psi}(x)$ and $\forall x \ge \underline{x}$, we have $\psi_x(x) \ge \hat{\psi}_x(x)$.*
(iv) *Given (24), there exists a unique solution $x_0 > 0$ to (25). In addition, W_t^* defined in (28) satisfies the borrowing constraint (8).*

Proof of Lemma 3.5. (i) This follows from direct differentiation because $\hat{\eta}>0$ and $b-1<0$. (ii) First, because $\nu>0$, $\beta_1>0$, $\beta_2>0$, and $\beta_3>0$, it is straightforward to use the definitions of α_+ and α_- to show that

$$\alpha_+>1>b>\alpha_-, \qquad \alpha_-<0, \qquad \text{and} \qquad A_+>0.$$

Then, the claimed results also follow from direct differentiation. (iii) This follows from a similar argument to that of part (ii) of Lemma 3.1. (iv) By part (i), part (ii), and $\psi_x(\underline{x})=\hat{\psi}_x(\underline{x})$, $\varphi'(x,R,\mathscr{C})$ is continuous and strictly increasing in x. By an inspection of (33) and (34), $\varphi_x(x,R,\mathscr{C})$ takes on all values that are less than or equal to $1/\beta_1$. Because $y_0>0$, there exists a unique solution $x_0>0$ to (25) for each $W_0\ge -y_0/\beta_1$. Also, because $\varphi_x(x,R,\mathscr{C})\le 1/\beta_1$, (28) implies that $W_t^*>-(1-R_t)y_t/\beta_1$, $\forall t\ge 0$. □

Proof of Proposition 3.2. First, as shown in Lemmas 3.1 and 3.5, the dual value function φ defined in Theorem 3.2 is convex and thus the wealth level W_t defined in Theorem 2 decreases with the dual variable x_t. By Proposition 3.1, differentiating the expression for human capital with respect to x_t for the case $\mathscr{C}=1$ yields that human capital is increasing in x_t because $\alpha_-<0$. Therefore, human capital decreases with financial wealth W for $\mathscr{C}=1$ in Problem 2. For $\mathscr{C}=0$ in Problem 2, differentiating human capital H with respect to x, we have that, before retirement,

$$\begin{aligned}\frac{\partial H(x,y)}{\partial x} &= \frac{y}{\beta_1}(A(\alpha_- -1)x^{\alpha_- -1}+B(\alpha_+ -1)x^{\alpha_+ -1})\\ &= \frac{y}{\beta_1}\frac{(\alpha_+-1)(1-\alpha_-)\underline{x}^{1-\alpha_-}x^{\alpha_+-2}}{(\alpha_+-1)\bar{x}^{\alpha_+-\alpha_-}-(\alpha_--1)\underline{x}^{\alpha_+-\alpha_-}}\left(\left(\frac{\bar{x}}{x}\right)^{\alpha_+-\alpha_-}-1\right)>0,\end{aligned} \tag{57}$$

where the second equality follows from the expressions of A and B in Proposition 3.1 and the inequality follows from the fact that $\alpha_+>1>\alpha_-$ and $x<\bar{x}$. Thus, human capital decreases with financial wealth W also for $\mathscr{C}=0$. Furthermore, because $\sigma_x=\gamma\sigma_y-\kappa$, if $\sigma_y<\kappa/\gamma$, then, as the market risk Z_t increases, x_t decreases and therefore human capital decreases by (57), i.e., human capital has a negative beta. □

The following result shows that retirement flexibility tends to increase stock investment.

Proposition 3.3. *Suppose $\sigma_y=0$ and $\mu>r\mathbf{1}$. Then, the fraction of total wealth $W+H$ invested in the risky asset in Problem 2 when $\mathscr{C}=1$ is greater than that in Problem 1.*

Proof. By Theorem 3.1, the fraction of total wealth $W+H$ invested in the risky asset in Problem 1 is constantly equal to $((\sigma^\top\sigma)^{-1}(\mu-r\mathbf{1}))/\gamma$.

With $\mathscr{C}=1$, by Theorem 3.2 and Proposition 3.1, we have

$$\frac{\theta}{W+H}=\frac{(\sigma^\top\sigma)^{-1}(\mu-r\mathbf{1})}{\gamma}\frac{\gamma A_+\alpha_-(\alpha_--1)x^{\alpha_--1}+\eta x^{b-1}}{-A_+\alpha_-x^{\alpha_--1}+\eta x^{b-1}-1/\beta_1\underline{x}^{1-\alpha_-}x^{\alpha_--1}}.$$

Plugging in the expressions for A_+ and $\underline{x}$ and using the fact that $\alpha_-<b$, we have

$$\frac{\gamma A_+\alpha_-(\alpha_--1)x^{\alpha_--1}+\eta x^{b-1}}{-A_+\alpha_-x^{\alpha_--1}+\eta x^{b-1}-1/\beta_1\underline{x}^{1-\alpha_-}x^{\alpha_--1}}>1. \quad \square$$

A retirement decision is critical for an investor's consumption and investment policies. The following proposition shows that the presence of the no-borrowing constraint tends to make an investor retire earlier.

Proposition 3.4. *In Problem 2, the retirement wealth threshold for $\mathscr{C}=1$ is higher than for $\mathscr{C}=0$.*

Proof. Let A_-, A_+, and $\underline{x}$ be defined as in Theorem 3.2. To make the exposition clear, we now make their dependence on $\mathscr{C}$ explicit, i.e., using $A_-(\mathscr{C})$, $A_+(\mathscr{C})$, and $\underline{x}(\mathscr{C})$ instead:

$$h(x,\mathscr{C})=A_+(\mathscr{C})x^{\alpha_-}+A_-(\mathscr{C})x^{\alpha_+}-\eta\frac{x^b}{b}+\frac{1}{\beta_1}x+\hat{\eta}\frac{x^b}{b}.$$

We prove by contradiction. Suppose $\underline{x}(0)\le\underline{x}(1)$. By Lemmas 3.1 and 3.5, we have $h(\underline{x}(0),0)=h_x(\underline{x}(0),0)=0$, $h(\underline{x}(1),1)=h_x(\underline{x}(1),1)=0$. From the proof of Lemma 3.1, we have $h_x(x,0)>0$ for all $x\in(\underline{x}(0),\bar{x}(0)]$. By (22) and (44), we have $\underline{x}(1)<\bar{x}(0)$. Therefore,

$$h(\underline{x}(1),0)\ge 0=h(\underline{x}(1),1) \qquad \text{and} \qquad h_x(\underline{x}(1),0)\ge 0=h_x(\underline{x}(1),1). \tag{58}$$

The first equation of (58) implies that

$$A_+(0)\underline{x}(1)^{\alpha_-}+A_-(0)\underline{x}(1)^{\alpha_+}\geq A_+(1)\underline{x}(1)^{\alpha_-},$$

which, in turn, implies

$$A_+(0)>A_+(1) \tag{59}$$

because $A_-(0)<0$ as shown in Lemma 3.1. On the other hand, the second equation of (58) implies that

$$A_+(0)\alpha_-\underline{x}(1)^{\alpha_--1}+A_-(0)\alpha_+\underline{x}(1)^{\alpha_+-1}\geq A_+(1)\alpha_-\underline{x}(1)^{\alpha_--1},$$

which, in turn, implies

$$A_+(0)<A_+(1) \tag{60}$$

because $A_-(0)<0$ and $\alpha_+>0$. Result (60) contradicts (59). This shows that we must have $\underline{x}(0)>\underline{x}(1)$. Because, at retirement, the financial wealth is equal to $-y\varphi_x(\underline{x}(\mathscr{C}),1,\mathscr{C})$ for Problem 2 and because $-y\varphi_x(\underline{x}(\mathscr{C}),1,\mathscr{C})=\hat{\eta}\underline{x}(\mathscr{C})^{-1/\gamma}$, we must have that the financial wealth level W at retirement for $\mathscr{C}=1$ is higher than for $\mathscr{C}=0$. □

One measure that is useful for examining the life cycle investment policy is the expected time to retirement. The following proposition shows how to compute this measure.

Proposition 3.5. *In Problem 2, suppose that an investor is not retired and that $\mu_x-\frac{1}{2}\sigma_x^2<0$. Then, the expected time to retirement for the optimal policy is*

$$E_t[\tau^*\mid x_t=x]=\frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2-\mu_x},\quad \forall x_t>\underline{x}$$

for $\mathscr{C}=1$ and

$$E_t[\tau^*\mid x_t=x]=\frac{\underline{x}^m-x^m}{(\frac{1}{2}\sigma_x^2-\mu_x)m\bar{x}^m}+\frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2-\mu_x},\quad \forall x_t\in[\underline{x},\bar{x}] \tag{61}$$

for $\mathscr{C}=0$, where

$$m=1-\frac{2\mu_x}{\sigma_x^2}.$$

Proof of Proposition 3.5. First, we prove the result for $\mathscr{C}=1$ in Theorem 3.2. Recall that

$$dx_s=x_s(\mu_x\,ds+\sigma_x^\top dZs),$$

which implies that

$$x_s=x_t\exp\big[(\mu_x-\tfrac{1}{2}\sigma_x^2)(s-t)+\sigma_x(Z_s-Z_t)\big].$$

Because $x_t>\underline{x}$ and $\mu_x-\frac{1}{2}\sigma_x^2<0$, we have $\tau^*<\infty$ almost surely (see, for example, Karatzas and Shreve [4, p. 349]). Let

$$f(x)\equiv\frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^\top\sigma_x-\mu_x}.$$

Then, by Itô's lemma, for any stopping time $\mathscr{T}\geq t$, we have

$$f(x_{\mathscr{T}})+\int_t^{\mathscr{T}}1\,ds=f(x_t)+\int_t^{\mathscr{T}}\left(\frac{1}{2}\sigma_x^\top\sigma_x x_s^2 f_{xx}+\mu_x x_s f_x+1\right)ds+\int_t^{\mathscr{T}}\frac{\sigma_x^\top}{\frac{1}{2}\sigma_x^\top\sigma_x-\mu_x}dZ_s, \tag{62}$$

which implies that $f(x_{\mathscr{T}})+\int_t^{\mathscr{T}}1\,ds$ is a martingale because it can be easily verified that the drift term is zero, given the definition of $f(x)$, and the stochastic integral is a scaled Brownian motion and thus a martingale. Thus, taking $\mathscr{T}=t+\tau^*$ and taking the expectation in (62), we get

$$f(x)=E_t[\tau^*\mid x_t=x]$$

because $x_{t+\tau^*}=\underline{x}$ and $f(\underline{x})=0$. A similar argument applies to the case $\mathscr{C}=0$; note that whenevaluated at $x=\bar{x}$, the first derivative of the right-hand side of (61) with respect to x is zero and x_t is bounded. This completes the proof. □

The following proposition shows how the expected time to retirement is related to human capital and financial wealth, which can help explain the graphical solutions presented in the previous section.

Proposition 3.6. *Suppose that an investor is not retired and that $\frac{1}{2}\sigma_x^2-\mu_x>0$. Then, as the expected time to retirement increases, financial wealth decreases and human capital increases.*

Proof. By Proposition 3.5, it can be easily verified that the expected time to retirement is increasing in x. By (57), we have that human capital is increasing in x and Proposition 3.2 then implies that financial wealth is decreasing in x. Therefore, the claim holds. □

4. Conclusion. We examine the impact of retirement flexibility and borrowing constraints against future labor income on optimal consumption and investment policies. We solve two alternative models almost explicitly (at least parametrically up to at most a constant) and provide verification theorems that are proved using a combination of the dual approach and an analysis of the boundary. In addition, we also obtain and prove some interesting comparative statics.

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