Lifetime consumption and investment: Retirement and constrained borrowing

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Abstract

Retirement flexibility and inability to borrow against future labor income can significantly affect optimal consumption and investment. With voluntary retirement, there exists an optimal wealth-to-wage ratio threshold for retirement and human capital correlates negatively with the stock market even when wages have zero or slightly positive market risk exposure. Consequently, investors optimally invest more in the stock market than without retirement flexibility. Both consumption and portfolio choice jump at the endogenous retirement date. The inability to borrow limits hedging and reduces the value of labor income, the wealth-to-wage ratio threshold for retirement, and the stock investment.
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1. Introduction

How does retirement affect consumption and investment? Our paper offers two new results:

- Consumption jumps at retirement because preferences are different when not working (and preferences are not additively separable over consumption and leisure).
- Investment jumps at retirement because retirement is irreversible and human capital's beta does not go to zero as the agent approaches the retirement boundary.

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The first result resolves an empirical puzzle in the literature, and the second result is an empirical prediction that is consistent with anecdotal evidence.[1] We have other results that refine or extend ideas in the literature:

- When wages are not related to market returns, agents' tendency to work longer in expensive states in which the market is down gives labor income a negative beta that makes portfolio choice even more aggressive when young than predicted by the idea that the total portfolio equals bond-like human capital plus financial capital chosen to manage overall risk exposure.
- The above result can be reversed when wages move with the market.
- When the risk in human capital is due partly to wage uncertainty, hedging of human capital is less effective when it is not possible to borrow against future labor income or when the correlation between the wage rate and the market is low, and is dampened even when the no-borrowing constraint is not currently binding.

These results are derived analytically in a consistent framework that yields rich empirical predictions and still hold even when labor income is unspanned by the financial market. We hope that these analysis and extensions will lend themselves to the study of policy questions in insurance, pensions, and retirement.

We solve three models to isolate the effects on the optimal consumption and investment strategy of retirement flexibility with and without borrowing against future labor income. We derive almost explicit solutions (at least parametrically up to at most two constants) in all three models. Except for retirement flexibility and borrowing constraints these models share common features: irreversible retirement,[2] a constant mortality rate, different marginal utility per unit of consumption before and after retirement, possibly stochastic labor income, bequest, and actuarially fair life insurance.[3]

The first model serves as a benchmark, it has an exogenous mandatory retirement date and allows limited borrowing.[4] The second model considers voluntary retirement and also allows limited borrowing. This model seems intractable in the primal, so we solve it in the dual (i.e., as a function of the marginal utility of wealth) and obtain an explicit parametric solution up to a constant that is easy to determine numerically. We show that there exists a critical wealth-to-wage ratio above which it is optimal to retire. In addition, if labor income does not have a highly positive market exposure, human capital (the present value of future labor income) has a negative beta with any efficient portfolio ("the market"). This is because if the wage is nearly constant, it

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[1] The U.S. News and World Report (October 2004, "Preserving your portfolio," p. 66) quotes a pension consultant of Hewitt Associates who says that investors have a strange tendency to be "overly aggressive until the day they retire. Then they become overly conservative. It's like a light switch."

[2] Irreversibility is a stark assumption that emphasizes the fact that an employee's value is much less working part-time than full-time (see, e.g., Gustman and Steinmeier [7]). Also, the marketability of an employee declines rapidly while not working. Many variations are possible; for example, we have solved a model allowing part-time employment at a lower wage after retirement. The formulation in this paper has the merit of producing clean results that are easy to interpret.

[3] In general, many age-dependent factors may induce retirement, including declining health, reduced productivity, short remaining life expectancy, and institutions such as public and private pension plans. In this paper, we abstract from explicit age-dependent factors and focus on the pure impact of wealth and portfolio considerations.

[4] See Lazear [15] for why mandatory retirement may be optimal. An alternative model of mandatory retirement that allows for early retirement is more complicated because of the extra time dimension, but can be solved using the randomization method employed by Liu and Loewenstein [16] (see also Panageas and Farhi [19]). Our simpler assumption is a better benchmark because we can solve the model exactly and it is easier to compare with the other models.

is optimal to work longer in expensive states, i.e., when the market is down, and work less when the market is up. In the absence of human capital risk, it is optimal to hold a constant fraction of total wealth (financial wealth plus human capital) in the market. With human capital risk, it is optimal to have an additional hedging component in the portfolio. Since human capital has a negative beta when retirement is flexible, the investor invests more aggressively in the market in this case to hedge against the human capital risk. If, however, the wage varies significantly with the market, the result can be reversed because human capital can now have a positive beta, which implies a less aggressive portfolio strategy since some required beta comes from labor income.

The third model also considers voluntary retirement, but in contrast to the second case, prohibits the agent from borrowing against future labor income. This restriction limits hedging labor income risk in the stock market because the optimal hedge would cause financial wealth to become negative in many states. Limited hedging makes working longer less attractive and therefore optimal retirement comes sooner. As a result, the beta of human capital falls in magnitude. When the wage is nearly constant and thus human capital has a negative beta, limited hedging implies that the optimal portfolio is less aggressive than that in the second case, not only at the borrowing boundary but also away from the boundary. If the wage varies significantly with the market so that human capital has a positive beta, inability to hedge fully makes the optimal portfolio invest more in the market.

When labor income is unspanned by the financial market, the effectiveness of hedging with stock investment is also reduced. Thus the unspanning of labor income produces a similar impact on the optimal stock investment to that of the no-borrowing constraint.

It has been widely documented that consumption jumps at retirement (e.g., Banks et al. [1], Bernheim et al. [4]). Our model offers an explanation of this puzzle. The jump in consumption comes from the discrete change in labor supplied at retirement and a lack of additive separability of preferences over labor and consumption. For these reasons, the marginal utility of consumption at a given consumption level changes at retirement but the cost of consumption (implicit in the state-price density) is continuous, implying that optimal consumption jumps at retirement. Our model also predicts that portfolio weights jump at retirement, which is consistent with some anecdotal evidence. The jump in investment choice follows from the irreversibility of retirement and the curious fact that the beta of human capital does not approach zero as the retirement boundary nears (but is of course zero after retirement). The beta of the total wealth jumps at retirement and therefore the investment also jumps at retirement. Most of the existing literature does not produce discrete jumps of consumption or investment at retirement. For example, Lachance [14] considers a similar problem with endogenous retirement. However, in contrast to this paper, she assumes that the utility function is additively separable in labor and consumption, which implies that consumption is continuous across retirement.

Financial advisors often advise investors to invest more in the stock market when young and to shift gradually into the riskless asset as they age. Two main justifications are provided in the literature. Bodie et al. [5] (BMS) show that if investors can frequently change working-hours, then labor income will be negatively perfectly correlated with the stock market and therefore the young should invest more in the stock market, because they can work longer hours if market goes down. However, working-hours are typically inflexible and consistent with this, an extensive empirical literature shows that labor income has a very low correlation with the stock market (e.g., Heaton and Lucas [9]). Therefore this working-hour flexibility is unlikely the main justification for the traditional advice. In contrast, Jagannathan and Kocherlakota [11] (JK) argue that total capital is human capital (which is bond-like) plus financial capital (whose market risk can be

chosen). To keep the overall mix constant, financial capital has to have a high beta on market risk when young (when total wealth consists mostly of human capital) but a more modest beta on market risk when old (when total wealth consists mostly of financial capital).

Our analysis contributes to this literature by providing two more important factors that affect the validity of the traditional advice. Our model implies that portfolio choice is not just a function of age and the relative size of financial wealth and earnings capacity may be more important. If we neglect these factors, we show that retirement flexibility has a positive effect on the validity of the traditional advice and the no-borrowing constraint has a negative one. Our analysis complements BMS by showing that even though wage rate itself might be uncorrelated with the stock market, human capital can be significantly negatively correlated with the market given voluntary retirement. This retirement flexibility, like working-hour flexibility, can make it optimal for the young to invest more in the stock. Compared to JK, our analysis suggests that the traditional advice may be valid for an even larger class of labor income distributions, because retirement flexibility induces an incremental negative correlation between human capital and the stock market and this negative correlation can offset some positive correlation between wage rate and the stock market. However, this observation must be placed in the context that in addition to age the optimal portfolio strategy also depends on other factors such as financial wealth and earnings capacity. In contrast to our model, neither BMS nor JK considers the retirement decision, which is arguably one of the most important life-cycle decisions. In addition, our model accounts for the inflexibility of labor supply choice before retirement, the imperfect correlation between labor income and stock return, and the ability to choose when to retire.

This paper also contains technical innovations that lead to explicit parametric solutions (up to two constants) and analytical comparative statics. In particular, we combine the dual approach of Pliska [20], He and Pagès [8], and Karatzas and Wang [12] with an analysis of the boundary to obtain a problem we can solve in a parametric form even if no known solution exists in the primal problem. Having an explicit solution allows us to derive analytically the impact of parameter changes and, more importantly, allows us to prove a verification theorem (given in the companion paper Dybvig and Liu [6]), showing that the first-order (Bellman equation) solution is a true solution to the choice problem. Proving a verification theorem in our model is more subtle than it might seem, because of (1) the nonconvexity introduced by the retirement decision, (2) the market incompleteness (from the agent's view) caused by the no-borrowing constraint, and (3) the technical problems caused by utility unbounded above or below. In particular, traditional verification theorems based on dynamic programming (Fleming–Richel) or a separation theorem (Slater condition) do not seem to apply to our model, and instead our proof uses a hybrid of the two techniques, patched together using optional sampling.[5]

The literature on life-cycle consumption and investment is extensive. Jun Liu and Neis [17] and Basak [2] consider the optimal consumption and investment problem with endogenous working hours. Similar to Lachance [14], they do not consider any borrowing constraint against future labor income. Sundaresan and Zapatero [21] investigate the effects of pension plans on retirement policies with an emphasis on the valuation of pension obligations. Khitatrakun [13] shows that individuals not affected by institutional constraints respond to a positive wealth shock by retiring or expecting to retire earlier than previously expected. Gustman and Steinmeier [7] also find a positive correlation between wealth and retirement. These findings are consistent with the em-

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[5] For example, Panageas and Farhi [19] sketch a dynamic programming proof of a verification theorem in a similar model, but their approach does not seem to work in our case or theirs.

pirical implications of this paper, in particular the implication that a worker will retire when the wealth-to-income ratio is high enough.

The rest of the paper is organized as follows. Section 2 describes the formal choice problems used in most of the paper. Section 3 presents analytical results and comparative statics. Section 4 discusses the case with unspanned labor income. Section 5 provides graphical illustration and more discussion of the main results. We offer some discussions on possible further extensions in Section 6 and Section 7 closes the paper.

2. Choice problems

Our general goal is to provide tractable workhorse models that can be used to analyze various issues related to life-cycle consumption, investment, retirement, and insurance. This paper focuses on stationary models that can be solved more-or-less explicitly. This section poses the formal decision problems for these stationary models.

The choice problems make many of the assumptions that are common in continuous-time financial models, such as a constant risk-free rate and lognormal risky asset returns. Other assumptions are not standard but seem particularly appropriate for analysis of life-cycle consumption and investment. For example, our model includes mortality and bequest as well as the disutility of working. In addition, an investor earns labor income with a potentially stochastic wage rate before retirement and can also purchase life insurance or term annuity throughout life.

All the models in this paper consider irreversible retirement without return to part-time or full-time work after retirement. Nothing prevents the addition of these other features to the model, but we choose to focus instead on the essential nonconvexity that indicates half-time work is much less valuable than full-time work in some positions.

In our main analysis we consider the following three cases:

Benchmark Fixed retirement date and free borrowing against wages (Problem 1 and Theorem 1).
VR ("Voluntary Retirement") Free choice of retirement date and free borrowing against wages (Problem 2 and Theorem 2).
VRNBC ("Voluntary Retirement with No-Borrowing Constraint") Free choice of retirement date but no borrowing against wages (Problem 3 and Theorem 3).

The benchmark case is a close relative of the Merton [18] model with i.i.d. returns and constant relative risk aversion. Moving to the VR case isolates the impact of making retirement flexible. Subsequently moving to the VRNBC case isolates the impact of the no-borrowing constraint.

Below are the three choice problems corresponding to the above three cases. An explanation of the notation immediately follows the problem statements.

Problem 1 (*Benchmark*). *Given initial wealth W_0, initial income from working y_0, and time-to-retirement T with associated retirement indicator function $R_t = \iota(T \leqslant t)$, choose adapted nonnegative consumption $\{c_t\}$, adapted portfolio $\{\theta_t\}$, and adapted nonnegative bequest $\{B_t\}$, to maximize expected utility of lifetime consumption and bequest*

$$E\left[\int_{t=0}^{\tau_d} e^{-\rho t}\left((1 - R_t)\frac{c_t^{1-\gamma}}{1-\gamma} + R_t\frac{(Kc_t)^{1-\gamma}}{1-\gamma}\right)dt + e^{-\rho\tau_d}\frac{(kB_{\tau_d})^{1-\gamma}}{1-\gamma}\right], \tag{1}$$

subject to the budget constraint

$$W_t = W_0 + \int_{s=0}^{t} \left(rW_s\, ds + \theta_s^\top \left((\mu - r\mathbf{1})\, ds + \sigma^\top dZ_s \right) + \delta(W_s - B_s)\, ds - c_s\, ds \right.$$
$$\left. + (1 - R_s) y_s\, ds \right), \tag{2}$$

the labor income process

$$y_t \equiv y_0 \exp\left[\left(\mu_y - \frac{\sigma_y^\top \sigma_y}{2} \right) t + \sigma_y^\top Z_t \right], \tag{3}$$

and the no-borrowing-without-repayment constraint

$$W_t \geqslant -g(t) y_t, \tag{4}$$

where

$$g(t) \equiv \begin{cases} \left(\frac{1 - e^{-\beta_1(T-t)}}{\beta_1} \right)^+ & \text{if } \beta_1 \neq 0, \\ (T - t)^+ & \text{if } \beta_1 = 0, \end{cases} \tag{5}$$

$$\beta_1 \equiv r + \delta - \mu_y + \sigma_y^\top \kappa \tag{6}$$

is the effective discount rate for labor income and assumed to be positive, and

$$\kappa \equiv \left(\sigma^\top \right)^{-1} (\mu - r\mathbf{1}) \tag{7}$$

is the price of risk.

Problem 2 (VR). *Given initial wealth W_0, initial income from working y_0, and initial retirement status R_{0-}, choose adapted nonnegative consumption $\{c_t\}$, adapted portfolio $\{\theta_t\}$, adapted nonnegative bequest $\{B_t\}$, and adapted nondecreasing retirement indicator $\{R_t\}$,[6] to maximize the expected utility of lifetime consumption and bequest* (1) *subject to the budget constraint* (2), *the labor income process before retirement* (3), *and the no-borrowing-without-repayment constraint*

$$W_t \geqslant -(1 - R_t) \frac{y_t}{\beta_1}. \tag{8}$$

Problem 3 (VRNBC). *The same as Problem* 2, *except that the no-borrowing-without-repayment constraint is replaced by the stronger no-borrowing constraint*

$$W_t \geqslant 0. \tag{9}$$

The uncertainty in the model comes from two sources: the standard vector Wiener process Z_t and the Poisson arrival of mortality at a fixed hazard rate δ. The Poisson arrival time is denoted as τ_d and is independent of the Wiener process Z_t. We allow the investor to trade in one risk-free asset and multiple risky assets ("stocks"). The Wiener process Z_t has dimensionality equal to the number of linearly independent risky returns, and maps into security returns through the constant mean vector μ and the constant nonsingular standard deviation matrix σ. For most of the paper, we will assume that local changes in the labor income y are spanned by local returns on the assets, but in Section 4, we will relax this assumption to allow for unspanned labor income. The

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[6] By "indicator," we mean a right-continuous process taking values 0 and 1. By nondecreasing we mean that for $0 < s < t$, $R_{0-} \leqslant R_0 \leqslant R_s \leqslant R_t$.

investor can also purchase insurance. Insurance is assumed to be fairly priced at the mortality rate δ per unit of coverage. When $W - B < 0$, this is term life coverage purchased for a premium of $\delta(B - W)$ per unit time and a receipt of $B - W$ at death. If $W - B > 0$, then this resembles a term version of a life annuity since it trades wealth in the event of death for a cash inflow while living.

The retirement status at time t is given by the retirement indicator R_t, which is 1 after retirement and 0 before retirement. Retirement is exogenously fixed at time T in Problem 1 and endogenously chosen in Problems 2 and 3. In all cases, retirement is right-continuous (technically convenient) and nondecreasing (retirement is irreversible). The state variable R_{0-} is the retirement status at the beginning of the investment horizon. Its value may be different from R_0 since if not retired at the outset ($R_{0-} = 0$), it may still be optimal to retire immediately ($R_0 = 1$). We assume the investor always works full time before retirement, and we take retirement to be irreversible: for $0 < s < t$, $R_{0-} \leqslant R_0 \leqslant R_s \leqslant R_t$.[7] Irreversible retirement is not the only modeling choice. For example, completely flexible hours have been considered by Jun Liu and Neis [17]. In their model, the agent can move freely in and out of retirement, and can vary hours continuously when working. We focus instead on the nonconvexity of working: there are many types of jobs for which working half-time is worth a lot less to a company than working full-time.

The investor derives utility from intertemporal consumption and bequest.[8] The investor has a constant relative risk aversion (CRRA), time additive utility function (1) with a subjective time discount rate ρ. The constant $K > 1$ indicates preferences for not working, in the sense that the marginal utility of consumption is greater after retirement than before retirement. Preference for not working could result from a disutility of work, or household production, or cost savings. For example, when working, there may not be enough time to shop for bargains or prepare meals or take a cruise. The constant $k > 0$ measures the intensity of preference for leaving a large bequest, while the limit $k^{1-\gamma} \to 0$ implements the special case with no preference for bequest.

The terms in the integrand of the wealth equation (2) are mostly familiar. The first term says that if all wealth is invested in the risk-free assets, the rate of return is r. The dollar investment θ_t in the risky asset entails risk exposure $\theta_t^\top \sigma^\top dZ_t$ with mean excess return $\theta_t^\top (\mu - r\mathbf{1}) \, dt$ (where $\mathbf{1}$ is a vector of 1's with dimension equal to the number of risky assets). The term $\delta(B_t - W_t) \, dt$ represents the insurance premium we have already discussed, $c_t \, dt$ denotes payment for consumption, and $(1 - R_t) y_t \, dt$ determines labor income, which implies no income after retirement.

In general, it is a subtle question what kind of constraint to include in an infinite-horizon portfolio problem to rule out borrowing without repayment and doubling strategies. Fortunately, simple and reasonable constraints suffice in our models. The no-borrowing-without-repayment constraints (4) and (8) specify that the level of indebtedness ($= \max(-W_t, 0)$) can never exceed earnings potential. The two constraints differ because the earnings potential differs from the case with a fixed retirement date to the case with voluntary retirement.[9] Problem 3 replaces the no-borrowing-without-repayment constraint with the stronger no-borrowing constraint (9).

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[7] It is not hard to solve intermediate cases. For example, we have solved an example in which the worker can only work full-time before retirement but is free to work flexible hours at a lower wage after retirement. The results are qualitatively similar to what we obtain in this paper.

[8] Having bequest motive is not important for our main results. We include it in the models for generality.

[9] It can be shown that as long as arbitrage strategies such as the doubling strategy are ruled out, the solutions to the two models remain the same. In particular, one can also impose an identical constraint (e.g., the less stringent one in (4) and (8)) across the two models without affecting the solutions. Therefore, the difference in the solutions across these two models is Not due to the difference in the borrowing constrains.

To summarize the differences across the problems, moving from the benchmark Problem 1 to the VR Problem 2, the fixed retirement date T ($R_t = \iota(t \geqslant T)$) is replaced by free choice of retirement date (R_t a choice variable), along with a technical change in the calculation of the maximum value of future labor income ((4) to (8)). Moving from the VR Problem 2 to the VRNBC Problem 3 replaces the no-borrowing-without-repayment constraint (8) with a no-borrowing constraint (9).

3. The analytical solution and comparative statics

Let

$$\nu \equiv \frac{\gamma}{\rho + \delta - (1 - \gamma)\left(r + \delta + \frac{\kappa^\top \kappa}{2\gamma}\right)}. \tag{10}$$

For our solutions, we will assume $\nu > 0$, which is also the condition for the corresponding Merton problem to have a solution.[10]

Theorem 1 *(Benchmark). Suppose $\nu > 0$ and that the no-borrowing-without-repayment constraint is satisfied with strict inequality at the initial values*:

$$W_0 > -g(0)y_0. \tag{11}$$

Then in the solution to the investor's Problem 1, *the optimal wealth process is*

$$W_t^* = f(t)y_t x_t^{-1/\gamma} - g(t)y_t, \tag{12}$$

the optimal consumption policy is

$$c_t^* = K^{-bR_t} f(t)^{-1}\left(W_t^* + g(t)y_t\right), \tag{13}$$

the optimal trading strategy is

$$\theta_t^* = \frac{(\sigma^\top \sigma)^{-1}(\mu - r\mathbf{1})}{\gamma}\left(W_t^* + g(t)y_t\right) - \sigma^{-1}\sigma_y g(t)y_t, \tag{14}$$

and the optimal bequest policy is

$$B_t^* = k^{-b} f(t)^{-1}\left(W_t^* + g(t)y_t\right), \tag{15}$$

where

$$b \equiv 1 - 1/\gamma, \tag{16}$$

$$f(t) \equiv (\hat{\eta} - \eta)\exp\left(-\frac{1 + \delta k^{-b}}{\eta}(T - t)^+\right) + \eta, \tag{17}$$

$$\eta \equiv \left(1 + \delta k^{-b}\right)\nu, \tag{18}$$

$$\hat{\eta} \equiv \left(K^{-b} + \delta k^{-b}\right)\nu, \tag{19}$$

$$x_t \equiv \left(\frac{W_0 + g(0)y_0}{y_0 f(0)}\right)^{-\gamma} e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t + \sigma_x^\top Z_t}, \tag{20}$$

$$\mu_x \equiv -(r - \rho) - \frac{1}{2}\gamma(1 - \gamma)\sigma_y^\top \sigma_y + \gamma\mu_y - \gamma\sigma_y^\top \kappa, \tag{21}$$

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[10] If $\nu < 0$, then the investor can achieve infinite utility by delaying consumption. If relative risk aversion $\gamma > 1$, ν is always positive, but for $1 > \gamma > 0$, whether ν is positive depends on the other parameters.

and

$$\sigma_x \equiv \gamma \sigma_y - \kappa. \tag{22}$$

Furthermore, the value function for the problem is

$$V(W, y, t) = f(t)^\gamma \frac{(W + g(t)y)^{1-\gamma}}{1 - \gamma}.$$

Proof. The proof uses a separating hyperplane to separate preferred consumptions from the feasible consumptions. The feasibility of the claimed optimum follows from direct substitution. Given (11), W^* is well-defined by (12). It is tedious but straightforward to verify the budget equation (2) using Itô's lemma and the claimed form of the strategy $(c^*, \theta^*, B^*, W^*)$ in (12)–(15), various definitions (5)–(7) and (12)–(22), and the definition of labor income (3). Note that $W_0^* = W_0$ by (12) and (20). The no-borrowing-without-repayment constraint (4) follows from the positivity of $f(t)$ and x, and the definition of W^* in (12).

We start by noting the state-price density and pricing results, both for labor income and for consumption and bequest. Define the state price density process ξ by

$$\xi_t \equiv e^{-(r+\delta+\frac{1}{2}\kappa^\top \kappa)t - \kappa^\top Z_t}. \tag{23}$$

This is the usual state-price density but adjusted to condition on living, given the mortality rate δ and fair pricing of long and short positions in term life insurance.

It can be shown that $g(t)y_t$ is the value at t of subsequent labor income, where g is defined in (5). (Note that $g(t) \equiv 0$ for $t \geqslant T$.) Furthermore, we have that the present value of future consumption and bequest is less than or equal to the initial wealth:

$$E\left[\int_0^\infty \xi_t (c_t + \delta B_t) \, dt\right] \leqslant W_0 + g(0)y_0, \tag{24}$$

for any feasible strategy, with equality for our claimed optimum.

We then have, after integrating out the mortality risk, for any feasible strategy (c, θ, B),

$$E\left[\int_0^\infty e^{-(\rho+\delta)t} \left(\frac{(K^{R_t} c_t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t)^{1-\gamma}}{1-\gamma}\right) dt\right]$$

$$\leqslant E\left[\int_0^\infty e^{-(\rho+\delta)t} \left(\frac{(K^{R_t} c_t^*)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^*)^{1-\gamma}}{1-\gamma}\right) dt\right]$$

$$+ \frac{x_0}{y_0^\gamma} E\left[\int_0^\infty \xi_t \left(c_t + \delta B_t - c_t^* - \delta B_t^*\right) dt\right]$$

$$\leqslant E\left[\int_0^\infty e^{-(\rho+\delta)t} \left(\frac{(K^{R_t} c_t^*)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^*)^{1-\gamma}}{1-\gamma}\right) dt\right], \tag{25}$$

where the first inequality follows direct verification from plugging in the expressions of c_t^* and B_t^* and the second inequality follows from the budget constraint (24) for all strategies and equality for the claimed optimum. This says that the claimed optimum dominates all other feasible

strategies. We showed previously that the claimed optimum is feasible, so it must indeed be optimal.   □

Unlike Problem 1, Problems 2 and 3 do not seem to have explicit solutions in terms of the primal variables. However, we provide explicit solutions (up to at most two constants) in terms of marginal utility in Theorems 2 and 3. Recall the definition of ν in (10) and β_1 in (6), and define

$$\beta_2 \equiv \rho + \delta + \frac{1}{2}\gamma(1-\gamma)\sigma_y^\top \sigma_y - (1-\gamma)\mu_y \tag{26}$$

and

$$\beta_3 \equiv (\gamma\sigma_y - \kappa)^\top(\gamma\sigma_y - \kappa). \tag{27}$$

Then, here is the solution for the VR case.

Theorem 2 (VR). *Suppose $\nu > 0$, $\beta_1 > 0$, $\beta_2 > 0$, and that the no-borrowing-without-repayment constraint holds with strict inequality at the initial condition:*

$$W_0 > -(1 - R_{0-})\frac{y_0}{\beta_1}. \tag{28}$$

The solution to the investor's Problem 2 can be written in terms of the dual variable x_t (a normalized marginal utility of consumption). Specifically, let the dual variable be defined by

$$x_t \equiv x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t + \sigma_x^\top Z_t}, \tag{29}$$

where x_0 solves

$$-y_0\varphi_x(x_0, R_0) = W_0, \tag{30}$$

where

$$\varphi(x, R) = \begin{cases} -\hat{\eta}\dfrac{x^b}{b} & \text{if } R = 1 \text{ or } x \leqslant \underline{x}, \\ A_+ x^{\alpha_-} - \eta\dfrac{x^b}{b} + \dfrac{1}{\beta_1}x & \text{otherwise,} \end{cases} \tag{31}$$

where b, η, and $\hat{\eta}$ are as defined in Theorem 1 (in Eqs. (16), (18), and (19)), and

$$A_+ \equiv \frac{1}{\gamma(b - \alpha_-)\beta_1}\underline{x}^{1-\alpha_-}, \tag{32}$$

the optimal retirement boundary is

$$\underline{x} = \left(\frac{(\eta - \hat{\eta})(b - \alpha_-)\beta_1}{b(1 - \alpha_-)}\right)^\gamma, \tag{33}$$

where

$$\alpha_- \equiv \frac{\beta_1 - \beta_2 + \frac{1}{2}\beta_3 - \sqrt{(\beta_1 - \beta_2 + \frac{1}{2}\beta_3)^2 + 2\beta_2\beta_3}}{\beta_3}. \tag{34}$$

Then the optimal consumption policy is

$$c_t^* = K^{-bR_t^*} y_t x_t^{-1/\gamma}, \tag{35}$$

the optimal trading strategy is

$$\theta_t^* = y_t\left[(\sigma^\top\sigma)^{-1}(\mu - r\mathbf{1})x_t\varphi_{xx}(x_t, R_t^*) - \sigma^{-1}\sigma_y(\gamma x_t\varphi_{xx}(x_t, R_t^*) + \varphi_x(x_t, R_t^*))\right], \tag{36}$$

the optimal bequest policy is

$$B_t^* = k^{-b} y_t x_t^{-1/\gamma}, \tag{37}$$

the optimal retirement policy is

$$R_t^* = \iota\{t \geqslant \tau^*\}, \tag{38}$$

the corresponding retirement wealth threshold is

$$\overline{W}_t = -y_t \varphi_x(\underline{x}, 0), \tag{39}$$

and the optimal wealth is

$$W_t^* = -y_t \varphi_x(x_t, R_t^*), \tag{40}$$

where

$$\tau^* = (1 - R_0) \inf\{t \geqslant 0 : x_t \leqslant \underline{x}\}. \tag{41}$$

Furthermore, the value function is

$$V(W, y, R) = y^{1-\gamma}(\varphi(x, R) - x\varphi_x(x, R)), \tag{42}$$

where x solves

$$-y\varphi_x(x, R) = W. \tag{43}$$

See the proof after Theorem 3.

The following theorem provides an almost explicit solution to the VRNBC case with the no-borrowing constraint.

Theorem 3 *(VRNBC). Suppose $\nu > 0$, $\beta_1 > 0$, $\beta_2 > 0$, and that initial wealth is strictly positive:*

$$W_0 > 0. \tag{44}$$

The solution to the investor's Problem 3 is similar to the solution to Problem 2, and can be written in terms of the new dual variable x_t defined by

$$x_t = \frac{x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t - \sigma_x^\top Z_t}}{\max(1, \sup_{0 \leqslant s \leqslant \min(t, \tau^*)} x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)s - \sigma_x^\top Z_s}/\bar{x})}, \tag{45}$$

where x_0 solves

$$-y_0 \varphi_x(x_0, R_0) = W_0, \tag{46}$$

and μ_x and σ_x are the same as in Theorem 2 (as given by (21) and (22)). The new dual value function is

$$\varphi(x, R) = \begin{cases} -\hat{\eta}\frac{x^b}{b} & \text{if } R = 1 \text{ or } x \leqslant \underline{x}, \\ A_+ x^{\alpha_-} + A_- x^{\alpha_+} - \eta\frac{x^b}{b} + \frac{1}{\beta_1}x & \text{otherwise}, \end{cases} \tag{47}$$

where

$$A_- = \frac{\eta(b - \alpha_-)}{\alpha_+(\alpha_+ - \alpha_-)}\bar{x}^{b-\alpha_+} - \frac{1 - \alpha_-}{\alpha_+(\alpha_+ - \alpha_-)\beta_1}\bar{x}^{1-\alpha_+}, \tag{48}$$

$$A_+ = \frac{\eta(\alpha_+ - b)}{\alpha_-(\alpha_+ - \alpha_-)}\bar{x}^{b-\alpha_-} - \frac{\alpha_+ - 1}{\alpha_-(\alpha_+ - \alpha_-)\beta_1}\bar{x}^{1-\alpha_-}, \tag{49}$$

the x value at which the financial wealth is zero is

$$\bar{x} = \left(\frac{\left(\frac{\eta - \hat{\eta}}{b} \zeta^{b - \alpha_-} - \frac{\eta}{\alpha_-} \right)(\alpha_+ - b)\beta_1}{\left(\zeta^{1 - \alpha_-} - \frac{1}{\alpha_-} \right)(\alpha_+ - 1)} \right)^{\gamma}, \tag{50}$$

the optimal retirement boundary

$$\underline{x} = \zeta \bar{x}, \tag{51}$$

where $\zeta \in (0, 1)$ is the unique solution to $q(\zeta) = 0$, where

$$q(\zeta) \equiv \left(\frac{1 - K^{-b}}{b(1 + \delta k^{-b})} \zeta^{b - \alpha_-} - \frac{1}{\alpha_-} \right) \left(\zeta^{1 - \alpha_+} - \frac{1}{\alpha_+} \right)(\alpha_+ - b)(\alpha_- - 1)$$

$$- \left(\frac{1 - K^{-b}}{b(1 + \delta k^{-b})} \zeta^{b - \alpha_+} - \frac{1}{\alpha_+} \right) \left(\zeta^{1 - \alpha_-} - \frac{1}{\alpha_-} \right)(\alpha_- - b)(\alpha_+ - 1), \tag{52}$$

and

$$\alpha_+ = \frac{\beta_1 - \beta_2 + \frac{1}{2}\beta_3 + \sqrt{(\beta_1 - \beta_2 + \frac{1}{2}\beta_3)^2 + 2\beta_2\beta_3}}{\beta_3}. \tag{53}$$

Then given the new dual variable x_t and the new dual value function, the rest of the form of the solution are given by (35) through (43) in Theorem 2.

Proof of Theorems 2 and 3. We only provide a sketch of the proof that contains the main steps.[11] If $R_0 = 1$, then Problems 2 and 3 are identical to Problem 1. Therefore, the optimality of the claimed optimal strategy follows from Theorem 1. From now on, we assume w.l.o.g. that $R_0 = 0$. It is tedious but straightforward to use the generalized Itô's lemma, Eqs. (31)–(40), and (47)–(53) to verify that the claimed optimal strategy W_t^*, c_t^*, θ_t^*, and R_t^* in these two theorems satisfy the budget constraint (2). In addition, it can be shown that x_0 exists and is unique and W_t^* satisfies the borrowing constraint in each problem. Furthermore, there is a unique solution to (52).

By Doob's optional sampling theorem, we can restrict attention w.l.o.g. to the set of feasible policies that implement the optimal policy stated in Theorem 1 after retirement, and after integrating out the mortality risk, the utility function for such a strategy can be written as

$$E \int_0^\infty e^{-(\rho + \delta)s} \left[(1 - R_s) \left(\frac{c_s^{1 - \gamma}}{1 - \gamma} + \delta \frac{(kB_s)^{1 - \gamma}}{1 - \gamma} \right) ds + V(W_s, y_s, 1) \, dR_s \right]. \tag{54}$$

Accordingly, define

$$M_t = \int_0^t e^{-(\rho + \delta)s} \left[(1 - R_s) \left(\frac{c_s^{1 - \gamma}}{1 - \gamma} + \delta \frac{(kB_s)^{1 - \gamma}}{1 - \gamma} \right) ds + V(W_s, y_s, 1) \, dR_s \right]$$

$$+ (1 - R_t) e^{-(\rho + \delta)t} V(W_t, y_t, 0). \tag{55}$$

One can show that M_t is a supermartingale for any feasible policy (c, B, R, W) and a martingale for the claimed optimal policy (c^*, B^*, R^*, W^*), which implies that $M_0 \geqslant E[M_t]$, i.e.,

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[11] For the detailed proof, see a companion paper Dybvig and Liu [6] that focuses on proofs of all the analytical results.

$$V(W_0, y_0, 0) \geqslant E \int_0^t e^{-(\rho+\delta)s} \left[(1 - R_s) \left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s, y_s, 1) dR_s \right]$$

$$+ E\left[(1 - R_t)e^{-(\rho+\delta)t} V(W_t, y_t, 0) \right], \tag{56}$$

and with equality for the claimed optimal policy. In addition, it can be shown that

$$\lim_{t\to\infty} E\left[(1 - R_t)e^{-(\rho+\delta)t} V(W_t, y_t, 0) \right] \geqslant 0,$$

with equality for the claimed optimal policy.

Therefore, taking the limit as $t \uparrow \infty$ in (56), we have

$$V(W_0, y_0, 0) \geqslant E \int_0^\infty e^{-(\rho+\delta)s} \left[(1 - R_s) \left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s, y_s, 1) dR_s \right],$$

with equality for the claimed optimal policy (c^*, B^*, R^*). This completes the proof. $\quad\square$

Theorems 2 and 3 provide essentially complete solutions, since the solution for x_0 given W_0 requires only a one-dimensional monotone search to solve Eqs. (30) and (46) and for ζ one only needs to solve Eq. (52).

We next provide results on computing the market value of human capital at any point in time, which is useful for understanding much of the economics in the paper. Proofs for all these results are available in Dybvig and Liu [6].

Proposition 1. *Consider the optimal policies stated in Theorems* 1–3. *After retirement, the market value of the human capital is zero. Before retirement, in Theorem* 1 *the market value of the human capital is*

$$H(y_t, t) = g(t) y_t,$$

where y_t and $g(t)$ are given in (3) and (5); in Theorem 2 *the market value of the human capital is*

$$H(x_t, y_t) = \frac{y_t}{\beta_1} \left(-\underline{x}^{1-\alpha_-} x_t^{\alpha_--1} + 1 \right),$$

where y_t, β_1, x_t, $\underline{x}$, and α_- are given in (3), (6), (20), (33), and (34); and in Theorem 3 *the market value of the human capital is*

$$H(x_t, y_t) = \frac{y_t}{\beta_1} \left(A x_t^{\alpha_--1} + B x_t^{\alpha_++1} + 1 \right),$$

where y_t, β_1, α_-, x_t, and α_+, are given in (3), (6), (34), (45), and (53), and where

$$A = \frac{(1 - \alpha_+)\underline{x}^{1-\alpha_-} \bar{x}^{\alpha_+-\alpha_-}}{(\alpha_+ - 1)\bar{x}^{\alpha_+-\alpha_-} - (\alpha_- - 1)\underline{x}^{\alpha_+-\alpha_-}}$$

and

$$B = \frac{(\alpha_- - 1)\underline{x}^{1-\alpha_-}}{(\alpha_+ - 1)\bar{x}^{\alpha_+-\alpha_-} - (\alpha_- - 1)\underline{x}^{\alpha_+-\alpha_-}}.$$

The following result shows that because of the retirement flexibility human capital may have a negative beta, even when the labor income correlates positively with the market risk.

Proposition 2. *As the investor's financial wealth W increases, the investor's human capital H decreases in both Problem 2 and Problem 3. Furthermore, if $\sigma_y < \kappa/\gamma$, then human capital has a negative beta measured relative to any locally mean-variance efficient risky portfolio.*

The following result shows that retirement flexibility tends to increase stock investment.

Proposition 3. *Suppose $\sigma_y = 0$ and $\mu > r$. Then the fraction of total wealth $W + H$ invested in the risky asset in Problem 2 is greater than that in Problem 1.*

Proposition 3 shows that in the absence of labor income risk, an investor with retirement flexibility always invests a greater faction of the total wealth in the stock than an investor without such flexibility if the risk premium is positive. Intuitively, in the absence of labor income risk, the investor with a mandatory retirement date T invests a constant fraction of the total wealth irrespective of the time to retirement T, as confirmed by Theorem 1. Retirement flexibility introduces a negative correlation between human capital and the market as shown in Proposition 2, and therefore the investor invests more to hedge against the human capital risk.

The retirement decision is critical for the investor's consumption and investment policies. The following proposition shows that the presence of no-borrowing constraint tends to make an investor retire earlier.

Proposition 4. *The retirement wealth threshold for Problem 2 is higher than that for Problem 3.*

One measure that is useful for examining the life cycle investment policy is the expected time to retirement. The following proposition shows how to compute this measure.

Proposition 5. *Suppose that the investor is not retired and $\frac{1}{2}\sigma_x^2 - \mu_x > 0$. Then the expected time to retirement for the optimal policy is*

$$E_t[\tau^* | x_t = x] = \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2 - \mu_x}, \quad \forall x_t > \underline{x}$$

in Theorem 2 and is

$$E_t[\tau^* | x_t = x] = \frac{\underline{x}^m - x^m}{(\frac{1}{2}\sigma_x^2 - \mu_x)m\bar{x}^m} + \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2 - \mu_x}, \quad \forall x_t \in [\underline{x}, \bar{x}] \tag{57}$$

in Theorem 3, where

$$m = 1 - \frac{2\mu_x}{\sigma_x^2}.$$

The following proposition shows how the expected-time-to-retirement is related to human capital and financial wealth, which can help explain the graphical illustrations to be shown later (e.g., Figs. 5 and 7).

Proposition 6. *Suppose that the investor is not retired and $\frac{1}{2}\sigma_x^2 - \mu_x > 0$. Then as the expected-time-to-retirement increases, financial wealth decreases and human capital increases.*

4. Imperfectly correlated labor income

Next we consider the case where labor income is not spanned. Specifically, assume

$$(\forall t \geqslant 0) \quad \frac{dy_t}{y_t} = \mu_y \, dt + \sigma_y \, dZ_t + \hat{\sigma}_y \, d\hat{Z}_t, \tag{58}$$

where $\hat{Z}_t$ is a one-dimensional Brownian motion independent of Z_t. The primal problem is difficult to solve due to a singular boundary condition at $W_t = 0$. We therefore examine this case also using the dual approach.

Let a convex and decreasing function $\varphi(x, R)$ be such that the value function $V(W, y, R) = y^{1-\gamma}(\varphi(x, R) - x\varphi_x(x, R))$, where x solves $-y\varphi_x(x, R) = W$. Then after retirement, $\varphi(x, 1)$ is the same as the one in Problems 1–3. After straightforward simplification, the HJB equation for $\varphi(x, 0)$ becomes

$$\frac{1}{2}\beta_3 x^2 \varphi_{xx}(x, 0) - (\beta_1 - \beta_2)x\varphi_x(x, 0) - \beta_2\varphi(x, 0) - \frac{1}{2}\hat{\sigma}_y^2 \frac{\varphi_x^2(x, 0)}{\varphi_{xx}(x, 0)}$$

$$- \left(1 + \delta k^{-b}\right)\frac{x^b}{b} + x = 0, \tag{59}$$

where

$$\beta_1 = r + \delta - \mu_y + \sigma_y \kappa^\top + \gamma\hat{\sigma}_y^2, \tag{60}$$

$$\beta_2 = \rho + \delta + \frac{1}{2}\gamma(1 - \gamma)\left(\sigma_y^\top \sigma_y + \hat{\sigma}_y^2\right) - (1 - \gamma)\mu_y, \tag{61}$$

and β_3 is the same as in (27). Note that if the labor income correlates perfectly with the risky asset market, i.e., $\hat{\sigma}_y = 0$, then this ODE reduces to the corresponding equation for the previous section.

We need to solve ODE (59) subject to some boundary conditions. Different from the case with perfectly correlated labor income, the HJB ODE (59) is fully nonlinear and an explicit form for the value function seems unavailable. However, this nonlinear ODE with free boundaries can be easily solved numerically, as we show later in Figs. 7 and 8.

5. Graphical illustration of the solution

In this section, we present graphical illustration and more detailed discussion of our main results.[12]

Figs. 1 and 2 show the optimal stock trading and consumption strategy. Fig. 1 shows the optimal risky asset position in the three cases, per unit of total wealth, as a function of financial wealth. Total wealth equals financial wealth W plus human capital H. The horizontal line shows the optimal portfolio choice for the benchmark case with (1) a fixed retirement date of 20 years from now and (2) free borrowing against future wages. In the benchmark case, it is as if all future wage income were capitalized before the investor invests a constant fraction of the total wealth in the risky asset regardless of the time to retirement T, as in the Merton model. Moving to the VR case, the negative beta of the human capital resulted from retirement flexibility induces a larger equity position, because equity provides a hedge against human capital risk. The no-borrowing

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[12] Although our model allows for multiple risky assets, for simplicity we consider only one risky asset in all the figures.
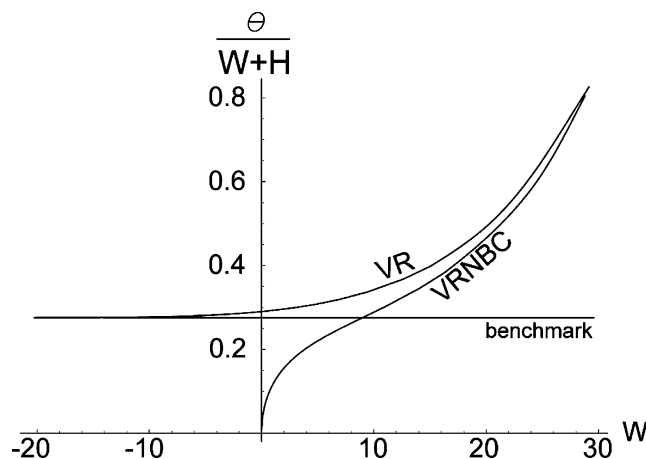
Fig. 1. Equity per total wealth against financial wealth. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.
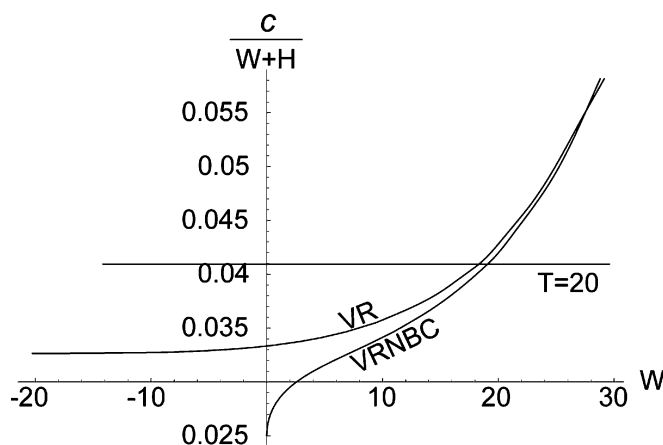

Fig. 2. Consumption per total wealth against financial wealth. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

constraint in the VRNBC case restricts the transfer of wealth across states, and indeed, it is relatively pointless to take a significant position in equities when financial wealth is low, since that would just imply bumping into the no-borrowing constraint more frequently. Therefore, investment in the risky asset is reduced compared to the VR case.

Fig. 2 shows the consumption rate, normalized by the total wealth, as a function of financial wealth. In the benchmark case, it is optimal to consume a constant fraction of the total wealth, and this constant fraction is independent of financial wealth. Moving to the VR case, the addition of flexible retirement leads to a higher consumption rate when wealth is higher, since high wealth implies the expectation of imminent retirement, low human capital, and lessened demand for consumption after retirement because the marginal utility per unit of consumption is higher after retirement. Moving to the VRNBC case, consumption significantly lessens at low wealth levels, due to higher saving rates for protecting against market downturns.[13]

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[13] Note that the two curves for the VR and VRNBC cases cross at high wealth level. This is because with the no-borrowing constraint, the investor retires earlier and so the human capital is lower as wealth level approaches the retirement wealth threshold.

Table 1
Comparative statics.

Parameter	$\bar{W}$	$\frac{c(\tau)}{\bar{W}}$	$\frac{c(\tau^+)}{\bar{W}}$	$\frac{\theta(\tau)}{\bar{W}}$	$\frac{\theta(\tau^+)}{\bar{W}}$	$\bar{W}_{VR}$	$\frac{\theta_{VR}(\tau_{VR}^+)}{\bar{W}_{VR}}$
Base case	28.81	0.058	0.028	0.8	0.28	29.16	0.83
$\gamma = 3.5$	28.98	0.057	0.026	0.69	0.24	29.19	0.71
$\gamma = 2.5$	28.91	0.058	0.030	0.95	0.33	29.52	0.98
$\mu = 0.04$	28.03	0.056	0.027	0.72	0.21	28.14	0.73
$\mu = 0.06$	29.26	0.061	0.029	0.89	0.34	30.06	0.92
$\sigma = 0.15$	29.38	0.065	0.031	1.40	0.59	30.76	1.48
$\sigma = 0.30$	27.97	0.060	0.030	0.53	0.15	28.07	0.53
$\delta = 0.02$	28.81	0.058	0.028	0.80	0.28	29.16	0.83
$\delta = 0.03$	20.96	0.075	0.036	0.81	0.28	21.01	0.82
$\rho = 0.008$	29.08	0.057	0.027	0.78	0.28	29.39	0.80
$\rho = 0.012$	28.54	0.059	0.028	0.83	0.28	28.94	0.85
$K = 2.5$	35.93	0.053	0.029	0.71	0.28	36.22	0.72
$K = 3.5$	24.45	0.063	0.027	0.89	0.28	24.85	0.92
$k = 0.025$	33.59	0.050	0.024	0.74	0.28	33.91	0.75
$k = 0.075$	26.87	0.062	0.030	0.84	0.28	17.86	0.76
$\mu_y = 0.01$	34.82	0.058	0.028	1.10	0.28	38.02	1.23
$\mu_y = -0.01$	25.56	0.058	0.028	0.58	0.28	25.57	0.58

Base case parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

An interesting point shown in Table 1 (but not visible in Fig. 2) is that consumption jumps down at retirement if the relative risk aversion coefficient is greater than 1. In our model, consumption jumps on the retirement date because the marginal utility per unit of consumption changes after retirement. This change could result from household production, reduced work-related expenses, or just a different preference for consumption when more leisure is available. Our finding concurs with the empirical evidence that consumption drops at retirement (the so-called Retirement–Consumption puzzle documented by [1] and [4]). The existing literature proposes several explanations, such as the lack of a forward-looking and unpredictable retirement date. Our model provides an alternative explanation of the Retirement–Consumption puzzle where, in contrast to other explanations, the investor is forward-looking and the retirement date is predictable.

Fig. 3 illustrates one of our main results, the negative beta of human capital, where human capital is the market value of future labor income in the optimal solution (see Proposition 2 in Section 3). More specifically, we first generate 1000 sample paths for the market index (stock price) according to

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ_t,$$

and the corresponding human capital process for the VR case. We then plot human capital against the market index at the fixed time $t = 30$. Fig. 3 shows that human capital decreases with the market index before retirement. After retirement, the human capital is always 0 and the market moves randomly up and down. This finding implies that human capital has a negative beta, provided that the volatility of the wage rate is small. Intuitively, the effective wage is equal to the wage times the state price; when the market is down, the state price is high, and thus it pays to work longer
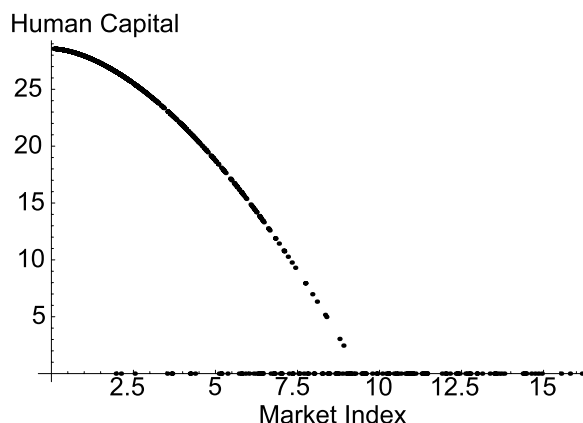
Fig. 3. Human capital against market index. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.
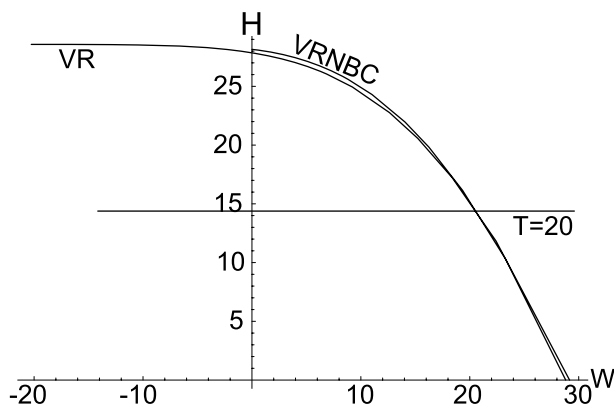

Fig. 4. Human capital against financial wealth. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

than when the market is down.[14] Consistent with this intuition, Fig. 4 shows that when financial wealth is low, the agent works longer and increases thereby the value of human capital.

Stock brokers have traditionally advised customers that younger people should invest more in the stock market than older people who are close to retirement. To explain the traditional rule, the existing literature resorts to negatively correlated labor income (e.g., Bodie et al. [5]) or return predictability (e.g., Merton [18]). Our model provides an alternative justification of the traditional rule even when the wage rate is uncorrelated with the market and the investment opportunity set is constant. Intuitively, with voluntary retirement, as long as the wage rate exposure to the market is small, the human capital has a negative beta, as shown in Fig. 3. This negative beta implies that the young will invest more in the stock market than the old, because the young have a greater human capital risk to be hedged by stock investment than the old. Fig. 5, which plots the fraction of financial wealth invested in stock against the expected-time-to-retirement (see Proposition 6), confirms this intuition. It shows that in general, an investor should invest more when young (i.e., expected time to retirement is long) than when old, even when the wage rate is uncorrelated with the market. The presence of a no-borrowing constraint reduces the stock investment, but

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[14] When the wage rate is positively correlated with the market and highly volatile, the higher state price may be offset by the smaller wage, with ambiguous impact on the effective wages and hence the value of human capital.
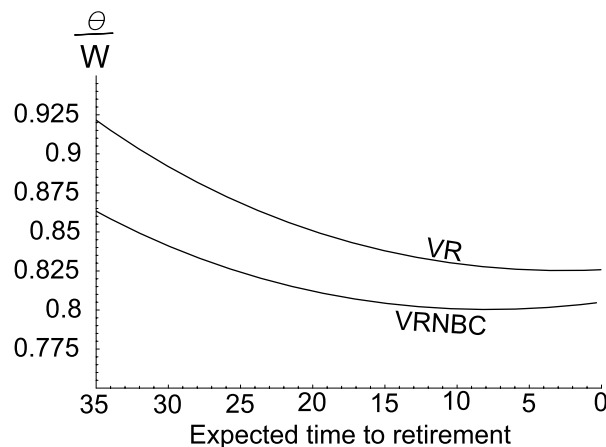
Fig. 5. Equity holdings against expected time to retirement. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.
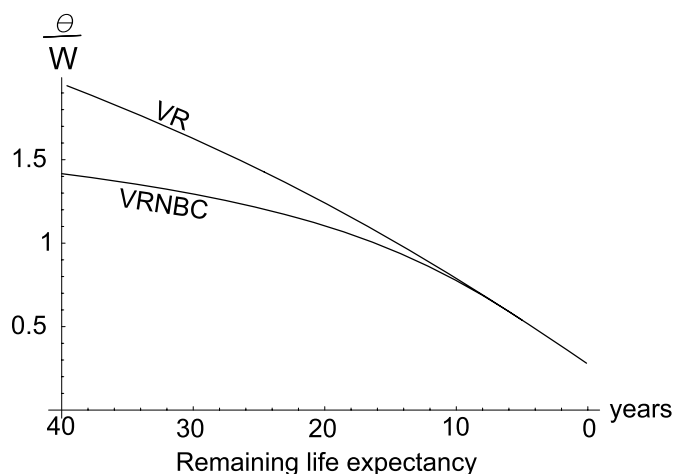

Fig. 6. Equity holdings against remaining life expectancy. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, $W_0 = 10$, and $y_0 = 1$.

does not change this "life-cycle" pattern. Since the expected-time-to-retirement is endogenous, we also plot the optimal trading strategy against the remaining life expectancy in Fig. 6. This figure supports the finding in Fig. 5. In particular, it shows that as the remaining life expectancy decreases, an investor should invest less in the stock market, consistent with the traditional rule.

So far, we have assumed that the wage rate is constant. Fig. 7 shows the effect of the labor income riskiness on the portfolio choice. It shows that if an agent's wage rate is highly volatile and positively correlated with the market risk, then the optimal trading strategy is the opposite of the traditional rule, i.e., one should invest less when young than when old. In extreme cases, it may even be optimal for the young to short the market, even when the market risk premium is positive.[15] Intuitively, when labor income is highly volatile and moves with the market, it reduces risky asset demand for the direct reason that human capital is part of the agent's portfolio; it also reduces risky asset demand for the indirect reason that the agent is less inclined to work longer when the market is low, diminishing the human capital risk and thus the hedging demand.

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[15] The qualitative results remain the same when we plot the trading strategy in this case against the remaining life expectancy as in Fig. 6. We omit the figure to save space.
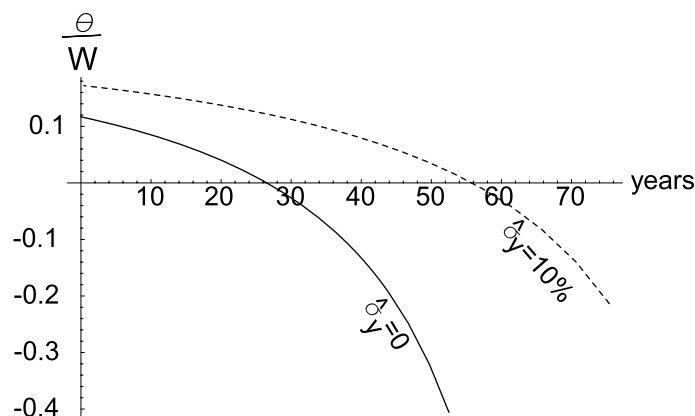
Fig. 7. Equity holdings against expected time to retirement with risky labor income. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $\mu_y = 0$, $\sigma_y = 0.1$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, and $y_0 = 1$.
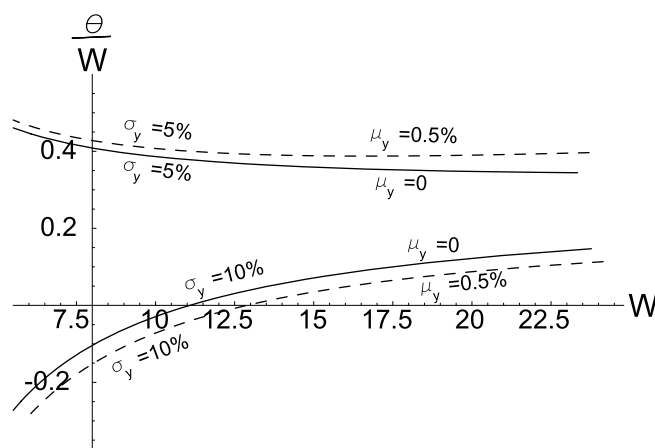

Fig. 8. Equity holdings against financial wealth with risky labor income. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $\hat{\sigma}_y = 0.005$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, and $y_0 = 1$.

The dotted curves in Fig. 7 show a case with idiosyncratic volatility in the income so that the labor income is not locally spanned by market returns. The absence of spanning reduces the effectiveness of the hedging using the stock market and thus tends to moderate hedging demand for the risky asset, but the qualitative results remain unchanged.

Our model suggests that the traditional investment rule applies to investors whose income is not highly sensitive to the market performance. For those who have significant exposure to the market risk (such as entrepreneurs), however, our model predicts that the reverse rule is optimal. This prediction supports empirical findings in the extensive literature on entrepreneurs' stock market invest strategies. For example, Heaton and Lucas [10] find that (1) entrepreneurs who are close to retirement invest more in stock; (2) as the growth rate of the proprietary income increases, the fraction of liquid wealth in stock decreases; and (3) their investment drops after retirement. We can interpret the investor in our model as an entrepreneur and the labor income y_t in our model as the proprietary income from his firm. Fig. 8 plots the optimal investment strategy against financial wealth for different levels of market sensitivity σ_y and income growth rate μ_y. This figure shows that when the market sensitivity σ_y is high, an investor should invest more when close to retirement (i.e., has greater financial wealth). In addition, when the sensitivity is high, a higher growth rate of the proprietary income (i.e., higher μ_y) dictates lower stock investment. Intuitively, as the growth rate increases, the market exposure of the human capital

increases and thus the entrepreneur should invest less in the stock market. Furthermore, Table 1 shows that stock investment drops after retirement because of the loss of the hedging from labor income after retirement.[16] Therefore, our model provides possible explanations for all the above findings of Heaton and Lucas [10].

6. Some further discussions

It would be nice to add more state variables to the model. For example, it has long been known that wages are sticky and it is reasonable that they respond to shocks in the stock market, but with a delay. One model that has this feature is to assume that wages are of the form[17]

$$y_t = A_t e^{z_t},$$

where

$$\frac{dA_t}{A_t} = \mu_A \, dt + \sigma_A \, dZ_t,$$

and z_t follows a mean reverting process

$$dz_t = -\rho(z_t - \bar{z}) \, dt + \sigma_z \, d\hat{Z}_t,$$

where $\rho > 0$, $\bar{z}$, and σ_z are constants. This formulation implies that

$$\frac{dy_t}{y_t} = \left(\mu_A + \frac{1}{2}\sigma_z^2 - \rho(z_t - \bar{z}) \right) dt + \sigma_A \, dZ_t + \sigma_z \, d\hat{Z}_t.$$

To choose the parameters μ_A, σ_A, $\rho > 0$, $\bar{z}$, and σ_z, we might think that the drift in wages pushes them up or down so that labor's share of rents follows a stationary process, and consequently we might assume $\sigma_A = \sigma$ and μ_A is equal to the stock drift less the dividend payout rate plus the rate of new capital formation less the population growth rate.

Unfortunately, models with additional state variables seem almost impossible to be solved analytically given current tools and numerical solution is also very difficult. We can, however, say something about the solution based on economic principles. When wealth is low, we are very likely far from the retirement horizon, and the correlation of labor income looking forward is very important and the solution should look similar to the solution with correlation between the stock market and wages. One important difference is that we care both about the wage now and about where the wage is going, so the threshold wealth for retirement will be higher, leaving fixed the current wage, when the wage is expected to rise ($z_t - \bar{z} < 0$) than when it is expected to fall ($z_t - \bar{z} > 0$). Near the horizon, things are more subtle because we may be close to retirement but some important paths may delay retirement. However, given that expected time to retirement is small the changing mean is probably less important and the solution should be more similar to our case when the correlation is 0. Admittedly these results are speculative and await rigorous verification.

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[16] If the correlation between labor income and the financial market is highly positive, then the drop in the risky asset investment at retirement can be potentially smaller.

[17] This model can specialize to one with wages and market value co-integrated. Benzoni et al. [3] study a similar model with cointegration.

7. Conclusion

We have constructed tractable models to examine how retirement flexibility and a borrowing constraint affect life-cycle consumption and investment. Retirement flexibility causes human capital to correlate negatively with the stock market if labor income does not have highly positive market exposure. Both consumption and portfolio choice jump at the voluntary retirement date because of the difference in preferences and the option to work longer before retirement. The inability to borrow against future income limits the value of retirement flexibility and, concomitantly, the stock investment. Our models suggest that those whose labor income has low market exposure should subscribe to the traditional life-cycle investment rule. However, for those, such as entrepreneurs, whose labor income has highly positive market exposure, the opposite of the traditional rule applies.

We abstract from many age-dependent and institutional factors, such as health and social securities, and focus on the pure effect of wealth and earnings. Empirically testable implications are largely consistent with existing empirical studies. We hope these models will prove useful for analyzing retirement, pension, and insurance.

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References

[1] J. Banks, R. Blundell, S. Tanner, Is there a retirement–savings puzzle? Amer. Econ. Rev. 88 (1998) 769–788.

[2] S. Basak, On the fluctuations in consumption and market returns in the presence of labor and human capital: An equilibrium analysis, J. Econ. Dynam. Control 23 (1999) 1029–1064.

[3] L. Benzoni, P. Collin-Dufresne, R.S. Goldstein, Portfolio choice over the life-cycle when the stock and labor markets are cointegrated, J. Finance 62 (2007) 2123–2167.

[4] B.D. Bernheim, J. Skinner, S. Weinberg, What accounts for the variation in retirement wealth among U.S. households? Amer. Econ. Rev. 91 (2001) 832–857.

[5] Z. Bodie, R.C. Merton, W.F. Samuelson, Labor supply flexibility and portfolio choice in a life cycle model, J. Econ. Dynam. Control 16 (1992) 427–449.

[6] P.H. Dybvig, H. Liu, Verification theorems for models of optimal consumption and investment with retirement and constrained borrowing, working paper, Washington University in St. Louis, 2009.

[7] A.L. Gustman, T.L. Steinmeier, A structural retirement model, Econometrica 54 (1986) 555–584.

[8] H. He, H.F. Pagès, Labor income, borrowing constraints, and equilibrium asset prices: A duality approach, Econ. Theory 3 (1993) 663–696.

[9] J. Heaton, D. Lucas, Market frictions, savings behavior, and portfolio choice, Macroecon. Dynam. 1 (1997) 76–101.

[10] J. Heaton, D. Lucas, Portfolio choice and asset prices: The importance of entrepreneurial risk, J. Finance 55 (2000) 1163–1198.

[11] R. Jagannathan, N.R. Kocherlakota, Why should older people invest less in stocks than younger people? Fed. Reserve Bank Minneapolis Quart. Rev. 20 (1996) 11–23.

[12] I. Karatzas, H. Wang, Utility maximization with discretionary stopping, SIAM J. Control Optim. 39 (2000) 306–329.

[13] S. Khitatrakun, Wealth and the timing of retirement, working paper, University of Wisconsin–Madison, 2002.

[14] M.E. Lachance, Retirement income insurance: A do-it-yourself approach, working paper, San Diego State University, 2004.

[15] E.P. Lazear, Why is there mandatory retirement? J. Polit. Economy 87 (1979) 1261–1284.

[16] H. Liu, M. Loewenstein, Optimal portfolio selection with transaction costs and finite horizons, Rev. Finan. Stud. 15 (2002) 805–835.

[17] J. Liu, E. Neis, Endogenous retirement, endogenous labor supply, and wealth shocks, working paper, UCLA, 2002.

[18] R.C. Merton, Lifetime portfolio selection under uncertainty: The continuous time case, Rev. Econ. Statist. 51 (1969) 247–257.

[19] S. Panageas, E. Farhi, Saving and investing for early retirement: A theoretical analysis, J. Finan. Econ. 83 (2007) 87–121.

[20] S.R. Pliska, A stochastic calculus model of continuous trading: Optimal portfolio, Math. Oper. Res. 11 (1986) 371–382.

[21] S. Sundaresan, F. Zapatero, Valuation, optimal asset allocation and retirement incentives of pension plans, Rev. Finan. Stud. 10 (1997) 631–660.