Multiple Birds, One Stone: Can Portfolio Rebalancing Contribute to Disposition Effect-related Trading Patterns? *

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Journal of Economic Literature Classification Numbers: G11, H24, K34, D91.

Keywords: disposition effect, portfolio rebalancing, learning, transaction costs

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Abstract

Extant theories on the disposition effect are largely silent on most of the related trading patterns, including the V-shape results for probabilities of buying and selling against unrealized profit. On the other hand, portfolio rebalancing and learning have been shown to be important, even for retail investors. We show that if expected returns are unknown and transaction costs are nonzero, then portfolio rebalancing *alone* can predict the disposition effect and many of the related trading patterns, including the V-shape results. Our model also provides new empirically testable predictions related to the disposition effect.

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1 Introduction

The disposition effect, which is the tendency of investors to sell winners while holding onto losers, has been widely documented in the empirical literature. For example, using data containing 10,000 stock investment accounts in a U.S. discount brokerage from 1987 to 1993, Odean conducts a careful set of tests of the disposition effect hypothesis in his seminal work of Odean (1998). He concludes that the disposition effect exists across years and investors.[1] Closely related to the disposition effect, Ben-David and Hirshleifer (2012) show that the plots of the probabilities of selling and of buying more of some existing shares against unrealized profit both exhibit V-shape patterns, i.e., as the magnitudes of unrealized profits/losses increase, these probabilities also increase. Theories based on prospect theory, mental accounting, regret aversion, and gain/loss realization utility to explain the disposition effect have dominated the literature.[2] However, it is difficult for extant theories to explain the V-shape patterns found by Ben-David and Hirshleifer (2012). In addition, as far as we know, there have been no theoretical models that have been proposed to explain other well-documented disposition effect-related patterns, such as: 1) investors may sell winners that subsequently outperform losers that they hold (e.g., Odean (1998)); 2) the disposition effect is stronger for less sophisticated investors (e.g., Dhar and Zhu (2006)); 3) the disposition effect may increase with return volatility (e.g., Kumar (2009)); and 4) investors are reluctant to repurchase stocks previously sold for a loss, as well as stocks that have appreciated in price subsequent to a prior sale (e.g., Strahilevitz, Odean, and Barber (2011)).

Another strand of literature documents that portfolio rebalancing is an important driver behind even retail investors' trading. For example, Calvet, Campbell, and Sodini (2009) find strong household-level evidence of active rebalancing by retail investors in Sweden. Using a sample of Japanese retail investors from 2013 to 2016, Komai, Koyano, and Miyakawa (2018) find that investors tend to conduct contrarian trades, as predicted by standard portfolio rebalancing models. In addition, trading patterns consistent with rational learning by investors have been

[1] See also, Shefrin and Statman (1985), Grinblatt and Keloharju (2001), Kumar (2009), Ivković and Weisbenner (2009), and Engelberg, Henriksson, and Williams (2018).

[2] See, for example, Shefrin and Statman (1985), Odean (1998), Barberis and Xiong (2009), Ingersoll and Jin (2013), Chang, Solomon, and Westerfield (2016), and Frydman, Hartzmark, and Solomon (2018). While these theories do seem to offer a promising framework for understanding the disposition effect, the possible link has almost always been discussed in informal terms, with one notable exception. Using a rigorous model, Barberis and Xiong (2009) demonstrate that assuming prospect theory utility on realized gains/losses can potentially predict a disposition effect.

widely documented. For example, Grinblatt and Keloharju (2001) report that past returns and historical price patterns affect trading decisions in ways that are consistent with rational learning. Kandel, Ofer, and Sarig (1993) and Banerjee (2011) provide evidence that investors learn about information contained in asset prices and revise their trading strategy accordingly. Furthermore, even though transaction costs have declined in recent years, bid-ask spreads and other trading costs (e.g., time costs) remain significant, especially for retail investors. As a result, most investors still do not trade frequently because even very small transaction costs can make them trade infrequently (e.g., Davis and Norman (1990), Liu (2004)).

As extant literature has shown (e.g., Odean (1998)), portfolio rebalancing without any market friction cannot explain the disposition effect. Based on the aforementioned empirical evidence on the importance of portfolio rebalancing, learning, and transaction costs, we develop an optimal portfolio rebalancing model with transaction costs and incomplete information in the form of unknown expected returns to examine whether rational portfolio rebalancing in the presence of these frictions can help explain the disposition-effect and the related findings. We show that, indeed, in the presence of some frictions, such as transaction costs and incomplete information, portfolio rebalancing alone can lead to the disposition effect and many of the related trading patterns, including the V-shape patterns found by Ben-David and Hirshleifer (2012). The driving forces behind these results are the presence of, and the interaction between, the "exposure effect" (i.e., the effect of keeping the stock risk exposure within a certain range) and the "learning effect" (i.e., the effect of learning about the expected return from past returns). While we believe that behavioral types of explanations are essential in understanding the disposition effect and the related trading patterns, our finding that one stone (i.e., portfolio rebalancing) can potentially "kill" multiple birds (i.e., explain most of the disposition-effect related patterns) suggests that portfolio rebalancing may also constitute a significant driving force behind these results and thus complement the extant theories.

More specifically, we consider a portfolio rebalancing model in which a small retail investor (i.e., one who has no price impact) can trade a risk-free asset and multiple risky assets ("stocks") to maximize the expected utility from the final wealth at a finite horizon.[3] The stocks' expected returns are unknown and the investor Bayes updates the conditional distributions of the ex-

[3] As in the existing literature on the disposition effect (e.g., Shefrin and Statman (1985), Barberis and Xiong (2009), and Ingersoll and Jin (2013)), we use a partial equilibrium setting because most of the empirical studies on the disposition effect and the related trading patterns focus on retail investors, whose trading unlikely affects market prices.

pected returns after observing past returns. Trading the stocks is subject to small proportional transaction costs. We characterize the solution and compute various measures related to the disposition effect using numerical and Monte Carlo simulation methods.

We show that the optimal rebalancing strategy implied by our model exhibits the disposition effect with similar magnitudes to those found in empirical studies. For example, for a reasonable set of parameter values, the ratio of the number of realized gains to the number of all gains (realized gains plus paper gains), i.e., PGR, is approximately 0.239, while the ratio of the number of realized losses to the number of all losses (realized losses plus paper losses), i.e., PLR, is approximately 0.057. As a comparison, Odean (1998) reports these ratios as 0.148 and 0.098, respectively (Table I in Odean (1998)). In addition, among all sales, gains account for more than 86%, which also indicates that an investor is much more likely to realize a gain than a loss.

The main driving force for the disposition effect displayed in our model is the "exposure effect." Intuitively, for a risk-averse utility maximizing investor, it is optimal to keep the exposure to a stock within a certain range to trade off risks and returns. A rise in the price of a stock results in a gain and increases the investor's risk exposure to this stock. If the exposure increases above an upper bound, then it is optimal to sell and thus realize a gain. A fall in the stock price results in a loss and decreases the investor's risk exposure. If the exposure decreases below a lower bound, then it is optimal to buy, *not sell*, additional shares. It is this asymmetry (i.e., selling with a large gain, but buying with a large loss) due to the exposure effect that makes investors realize gains more often than losses. On the other hand, if the exposure after a gain or loss is still within the bounds, the investor does not trade, due to the presence of transaction costs. Because selling a stock with a loss requires the upper bound of the risk exposure to be reached after a decline in the stock price and buying additional shares after a loss requires the lower bound to be reached, investors hold onto losers after small losses. Thus, the combination of the exposure effect and the presence of transaction costs makes investors tend to sell winners and hold onto losers, consistent with existing empirical findings. Because the exposure effect exists for any risk-averse utility maximizing investors, the above qualitative results apply to all risk-averse preferences, such as CRRA, CARA, and Epstein-Zin preferences.[4]

[4] For example, for CARA preferences, it is optimal to keep the dollar amount in a stock in a range, and for CRRA preferences and Epstein-Zin preferences, it is optimal to keep the fraction of wealth in a stock in a range as in our model. For all of these preferences, it is optimal to sell when the stock price rises sufficiently, and to buy when the stock price decreases sufficiently.

Ben-David and Hirshleifer (2012) demonstrate that the probability of buying more and of selling are both greater for positions with larger paper gains or larger paper losses.[5] Theories based on the static prospect theory, or regret aversion, predict that the larger the loss, the less likely it is for investors to sell, and the larger the gain, the less likely it is for investors to buy, which is opposite of the V-shape pattern.

We show that the V-shape patterns for both the purchase probability and the sale probability are consistent with the optimal trading strategy in our portfolio rebalancing model. Intuitively, two opposing forces exist in our model: the "exposure effect" and the "learning effect." As previously explained, the exposure effect tends to make investors sell after a large gain but buy after a large loss ("buy low, sell high"), exhibiting a contrarian trading strategy. In contrast, the learning effect tends to make investors buy after a large gain and sell after a large loss ("buy high, sell low"), exhibiting a momentum trading strategy. This is because investors revise upward their estimate of expected returns after gains and do the opposite after losses. The patterns of the probability of selling increasing with the magnitude of gains and the probability of buying increasing with the magnitude of losses are driven by the exposure effect. On the other hand, because there is a greater increase (decrease) in the estimate of the expected return after observing a large gain (loss), the probability of buying more (selling) is greater for a large gain (loss) than for a small gain (loss).[6] Thus, the patterns of the probability of buying more increasing with the magnitude of gains and the probability of selling increasing with the magnitude of losses are driven by the learning effect. The relative strength of the two effects determines the trading direction. It is the coexistence of the exposure effect and the learning effect that drives the V-shape patterns.

Moreover, in contrast to the existing literature, our model can generate many other disposition effect-related patterns documented in empirical studies, such as those four stated at the end of the first paragraph. As in the previous results, the driving forces behind these results are also the presence of, and the interaction between, the exposure effect and the learning effect. For example, less sophisticated investors may learn more slowly than more sophisticated investors.

[5] Note that the evidence that the probability of selling increases with loss magnitudes is not contradictory to the disposition effect, because the unconditional probability of selling losers is still smaller than that of selling winners. An (2016) and An and Argyle (2016) find that stocks with both large unrealized gains and large unrealized losses outperform others in the following month. This finding may be consistent with V-shape trading patterns.

[6] The conditional volatility of the expected return deterministically decreases over time. Its impact is largely dominated by the impact of the change in the estimate of the expected return, especially when there are large return shocks. See Section 3 for more detailed discussions.

As a result, the learning effect may be smaller and the disposition effect may be stronger for less sophisticated investors. We explain the intuitions for these results in detail in Section 3.

It is well known that, with capital gains tax, realizing losses sooner and deferring capital gains can provide significant benefits (e.g., Constantinides (1983)). This force acts against the disposition effect. We demonstrate that, consistent with the empirical findings of Lakonishok and Smidt (1986), the disposition effect can still arise in an optimal portfolio rebalancing model with capital gains tax and transaction costs. Intuitively, when a stock's price appreciates sufficiently, the investor's risk exposure to this stock can become too high, and the benefit of lowering the exposure by a sale can dominate the benefit of deferring the realization of gains. In addition, with transaction costs, realizing losses immediately is no longer optimal, and deferring even large capital losses may be optimal. This is because the extra time value obtained from realizing losses sooner can be outweighed by the necessary transaction cost payment, even when the transaction cost is small.

Our model offers some new empirically testable predictions for future studies. For example, our model predicts that: (1) conditional on return volatility, the magnitude of the disposition effect is greater for stocks for which there is more public information, because for these stocks much is already known, and thus the learning effect is smaller; (2) investors with a more diversified portfolio or a better hedged portfolio have a weaker disposition effect, because for these portfolios the exposure effect is smaller; and (3) the V-shaped trading patterns are more pronounced for stocks with less public information, because the learning effect is stronger for these stocks.

Assuming that an investor can obtain a burst of reference-dependent utility from a sale in a dynamic prospect theory setting, Ingersoll and Jin (2013) demonstrate that the probability of selling can increase with the magnitude of losses because there is a benefit of realizing losses to reset references in this dynamic setting.[7] Peng (2017) attributes the V-shaped selling pattern to irrational extrapolation of past returns. In contrast, as previously explained, the mechanism in our model for the pattern is through the learning effect. In addition, these studies do not explain why the probability of buying more increases with the magnitude of gains, and do not attempt to explain many of the other disposition-effect related findings.

Although we consider a small investor whose trades have no price impact and thus adopt a

[7] For a discussion of realization utility theory, see also Barberis and Xiong (2012).

partial equilibrium approach, the disposition effect can arise in equilibrium (e.g., Basak (2005), Dorn and Strobl (2009)). For example, Dorn and Strobl (2009) demonstrate that, in the presence of information asymmetry, the less informed become contrarians while the more informed become momentum traders in equilibrium. The less informed investors in their model may represent retail investors. They trade in the same way as the investor in our model, and thus displays the disposition effect in equilibrium. More generally speaking, the fact that, in equilibrium for each investor who sells there must be a counterparty who buys, does not imply that there is no disposition effect on average. This is because it is possible that a greater number of retail investors with a stronger disposition effect trade with a small number of institutional investors, for example, and most of the studies of the disposition effect and the related trading patterns focus on retail investors.

The remainder of the paper proceeds as follows. In the next section, we present the main model and theoretical analysis. In Section 3, we numerically solve the model and conduct simulations to illustrate that our model can generate most of the disposition-effect related patterns. We also show that the disposition effect can exist even with capital gains tax. We conclude with Section 4, and all proofs are provided in the Appendix.

2 The Model

2.1 Economic setting

We consider the optimal investment problem of a small retail investor (i.e., a price taker) who maximizes the expected constant relative risk averse (CRRA) utility from the final wealth at some finite time $T > 0$.[8] The investor can invest in one risk-free asset ("bond") and $N \geq 1$ risky assets ("stocks"). For $i = 1, ..., N$, we assume that the price of Stock i evolves as follows:

$$\frac{dS_{it}}{S_{it}} = \mu_i dt + \sigma_i dB_{it}^S, \tag{1}$$

where μ_i and σ_i are constants and the Brownian motion B_{it}^S is independent of B_{jt}^S for $j \neq i$.[9] For $i = 1, ..., N$, while σ_i is known, μ_i may be unobservable. This reflects the fact that the expected

[8] Including intertemporal consumption would not qualitatively change our results because, as will become clear later, the main driving forces of our results remain the same even with intertemporal consumptions.

[9] This assumption allows us to obtain a good approximate solution in the presence of transaction costs. We provide some analyses of the impact of return correlation in Section 3.3.3.

returns of stocks are difficult to estimate from a finite sample. The investor starts with an independent (of all Brownian motions in this model), prior normal distribution of $N(z_{i0}; V_i(0))$ for μ_i, where z_{i0} and $V_i(0)$ are known constants, and updates this distribution to $N(z_{it}; V_i(t))$ by learning from past stock returns, where:

$$z_{it} = E[\mu_i | \mathcal{F}_t] \tag{2}$$

is the conditional mean,

$$V_i(t) = E[(\mu_i - z_{it})^2 | \mathcal{F}_t] \tag{3}$$

is the conditional variance, and $\mathcal{F}_t$ is the augmented filtration generated by $\{S_{ju} : u \leq t, 1 \leq j \leq N\}$. According to the standard filtering theory (e.g., Lipster and Shiryayev (2001)), z_{it} evolves according to the following process:

$$dz_{it} = \sigma_{zi}(t) d\hat{B}^S_{it}, \tag{4}$$

where $\sigma_{zi}(t) = \frac{V_i(t)}{\sigma_i}$, $V_i(t)$ satisfies:

$$\frac{dV_i(t)}{dt} = -\left(\frac{V_i(t)}{\sigma_i}\right)^2, \tag{5}$$

and $\hat{B}^S_{it}$ is an observable innovation process satisfying:

$$d\hat{B}^S_{it} = \frac{1}{\sigma_i}(\mu_i - z_{it})dt + dB^S_{it}. \tag{6}$$

Equation (6) implies that the stock-price process (1) can be rewritten as:

$$\frac{dS_{it}}{S_{it}} = z_{it}dt + \sigma_i d\hat{B}^S_{it}, \tag{7}$$

which is the price process observed in the investor's filter.

2.1.1 Discussion of the model

In our model, there is neither serial correlation nor mean reversion in the stock returns. The solution to Equation (5) is:

$$V_i(t) = \frac{\sigma_i^2 V_i(0)}{\sigma_i^2 + V_i(0)\, t}, \tag{8}$$

hence, Equation (4) becomes:

$$dz_{it} = \frac{\sigma_i V_i(0)}{\sigma_i^2 + V_i(0)\, t} d\hat{B}_{it}^S, \tag{9}$$

which is equivalent to:

$$dz_{it} = \frac{V_i(0)}{\sigma_i^2 + V_i(0)\, t} \left(\frac{dS_{it}}{S_{it}} - z_{it} dt \right). \tag{10}$$

Since $\frac{V_i(0)}{\sigma_i^2 + V_i(0)t} > 0$, Equation (10) suggests that the changes in z_{it} (i.e., the investor's conditional expectation of μ_i) are driven by the instantaneous realized return dS_{it}/S_{it} in excess of the current estimate of the expected return $z_{it}dt$. In particular, the investor will increase her estimate of the stock's expected return if and only if:

$$\frac{dS_{it}}{S_{it}} > z_{it} dt. \tag{11}$$

In other words, a realized return that is better (worse) than the expected return makes the investor increase (decrease) her estimated expected return. We term the effect of this learning from the past stock prices on the trading strategy the "learning effect." We show later that this learning effect can be important for predicting some empirically documented trading patterns.

It is clear from Equations (8) and (10) that for any $i = 1, 2, ..., N$, if we set $V_i(0) = 0$ and $z_{i0} = \mu_i$, then our model is equivalent to assuming that Stock i's expected return is observable, and thus there is no learning effect for the stock. Therefore, our model nests the case of observable expected returns as a special case. Later, we use the comparison between the observable case and the unobservable case to clearly identify the role of the learning effect in driving various predicted trading patterns.

2.2 The investor's problem

For $i = 1, 2, ..., N$, the investor can buy Stock i at the ask price $S_{it}^A = (1 + \theta_i) S_{it}$ and sell the stock at the bid price $S_{it}^B = (1 - \alpha_i) S_{it}$, where $\theta_i \geq 0$ and $0 \leq \alpha_i < 1$ represent the proportional transaction cost rates for trading Stock i.

Let Y_{it} be the dollar amount invested in Stock i for $i = 1, 2, ..., N$, X_t be the dollar amount invested in the bond, and D_{it} and I_{it} with $D_{i0-} = I_{i0-} = 0$ be nondecreasing, right continuous adapted processes that represent the cumulative dollar amount of sale and purchase,

respectively. Then, we have the following budget constraints:

$$dX_t = rX_t dt + \sum_{i=1}^{N}(1-\alpha_i)dD_{it} - \sum_{i=1}^{N}(1+\theta_i)dI_{it}, \tag{12}$$

$$dY_{it} = Y_{it}z_{it}dt + Y_{it}\sigma_i d\hat{B}_{it}^S + dI_{it} - dD_{it}, \quad i = 1, ..., N. \tag{13}$$

In addition, since short-sales are either too costly or too risky for most retail investors, we assume that the investor cannot short-sell,[10] i.e.:

$$Y_{it} \geq 0, \quad i = 1, ..., N. \tag{14}$$

The investor's problem is to choose her optimal policy $\{(D_{it}, I_{it}) : i = 1, ..., N\}$ among all of the admissible policies to maximize her expected CRRA utility from the terminal net wealth at time T, i.e.:

$$E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right], \tag{15}$$

subject to Equations (4), (12), and (13) and the short-sale constraint (14), as well as the solvency condition:

$$W_t \geq 0, \tag{16}$$

where $\gamma > 0$ and $\gamma \neq 1$ is the investor's constant relative risk aversion coefficient, and:

$$W_t = X_t + \sum_{i=1}^{N}(1-\alpha_i)Y_{it} \tag{17}$$

is the investor's net after-liquidation wealth level at time t.

2.3 The HJB equation and an approximate solution

We define the investor's value function as follows:

$$J(x, \mathbf{y}, \mathbf{z}, t) = \sup_{\{(I_i, D_i): i=1,...,N\}} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\mathcal{F}_t\right], \tag{18}$$

subject to Equations (4), (12), and (13) and the short-sale constraint (14), as well as the solvency condition (16), where $\mathbf{y} = (y_1, ..., y_N)$ and $\mathbf{z} = (z_1, ..., z_N)$. Under certain regular-

[10] Consistent with this high cost or high risk, investors rarely short-sell. For example, the results of Anderson (1999) and Boehmer, Jones, and Zhang (2008) imply that only approximately 1.5% of short-sales come from individual investors.

ity conditions, $J(x, \mathbf{y}, \mathbf{z}, t)$ must satisfy the following Hamilton-Jacobi-Bellman (HJB) partial differential equation:

$$\max\left\{\mathcal{L}_0 J, \quad \max_{1\le i\le N} \mathcal{B}_{0i} J, \quad \max_{1\le i\le N} \mathcal{S}_{0i} J\right\} = 0, \tag{19}$$

on the domain $\Omega = \{t \in [0, T], x \in R, y_i \ge 0, z_i \in R, i = 1, 2, ..., N\}$, with terminal condition:

$$J(x, \mathbf{y}, \mathbf{z}, T) = \frac{1}{1-\gamma}\left(x + \sum_{i=1}^{N}(1-\alpha_i)y_i\right)^{1-\gamma}, \tag{20}$$

where:

$$\begin{aligned}\mathcal{L}_0 J \quad = \quad & \frac{\partial J}{\partial t} + rx\frac{\partial J}{\partial x} + \sum_{i=1}^{N} z_i y_i \frac{\partial J}{\partial y_i} + \sum_{i=1}^{N}\frac{1}{2}\sigma_i^2 y_i^2 \frac{\partial^2 J}{\partial y_i^2} \\ & + \sum_{i=1}^{N}\frac{1}{2}\sigma_{zi}(t)^2 \frac{\partial^2 J}{\partial z_i^2} + \sum_{i=1}^{N}\sigma_i\sigma_{zi}(t) y_i \frac{\partial^2 J}{\partial y_i \partial z_i},\end{aligned} \tag{21}$$

$$\mathcal{B}_{0i} J = \frac{\partial J}{\partial y_i} - (1+\theta_i)\frac{\partial J}{\partial x}, \tag{22}$$

and:

$$\mathcal{S}_{0i} J = (1-\alpha_i)\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y_i}. \tag{23}$$

The solution to Equation (19) splits the solution domain Ω into several regions. In particular:

$$BR_i \equiv \{(x, \mathbf{y}, \mathbf{z}, t) : \mathcal{B}_{0i} J = 0\} \tag{24}$$

denotes the Buy Region (BR) of Stock i;

$$SR_i \equiv \{(x, \mathbf{y}, \mathbf{z}, t) : \mathcal{S}_{0i} J = 0\} \tag{25}$$

denotes the Sell Region (SR) of Stock i; and

$$NTR_i \equiv \{(x, \mathbf{y}, \mathbf{z}, t) : \mathcal{B}_{0i} J < 0, \ \ \mathcal{S}_{0i} J < 0\} \tag{26}$$

denotes the No-Trade Region (NTR) of Stock i.

We first provide a verification argument which characterizes the optimal trading strategies.

***Proposition* 1:** *(Verification theorem) Let $V(x, \mathbf{y}, \mathbf{z}, t)$ be a solution to Equation (19) with terminal condition (20) satisfying certain regularity conditions, with the respective trading regions*

defined by (24) - (26). Then, the optimal trading policy (I_{it}^*, D_{it}^*) *for Stock* i, $i = 1, ..., N$, *is given by:*

$$I_{it}^* = \int_0^t 1_{\{(X_s, \mathbf{Y}_s, \mathbf{z}_s, s) \in \partial BR_i \bigcap \partial NTR_i\}} dI_{is}^*, \tag{27}$$

and:

$$D_{it}^* = \int_0^t 1_{\{(X_s, \mathbf{Y}_s, \mathbf{z}_s, s) \in \partial SR_i \bigcap \partial NTR_i\}} dD_{is}^*, \tag{28}$$

and $V(x, \mathbf{y}, \mathbf{z}, t)$ *coincides with the value function* $J(x, \mathbf{y}, \mathbf{z}, t)$.

Unfortunately, due to the curse of dimensionality, the HJB equation (19) is extremely difficult to solve numerically.[11] We thus resort to implementing a trading strategy which is a good approximation to the optimal. To illustrate the approximation idea, we first show that, under the independence assumption, the optimal position in each stock can be independently solved when the transaction cost is absent.

Proposition 2: *(Decomposition of risk exposure without transaction cost) Suppose that there is no transaction cost for any stock, i.e.,* $\alpha_i = \theta_i = 0$ *for* $i = 1, 2, ..., N$. *Then, the optimal fraction of total wealth* W_t *invested in Stock* i *in the model with* N *stocks equals the optimal fraction when the investor can only invest in the risk-free asset and Stock* i.

Even without any transaction costs, directly solving the high dimensional HJB equation numerically to obtain the reliable optimal trading strategy in the presence of a large number of stocks would be infeasible. Proposition 2 suggests that, in the absence of transaction cost, if the stocks' return processes are independent, then we can decompose the optimal investment problem with N stocks into N optimal investment problems with a single stock. This result makes it feasible and reliable to solve for the optimal trading strategy for a large number of stocks. Motivated by this result, we perform the same decomposition in the presence of *small* transaction costs. More specifically, for any stock, we solve a model with this particular stock and one risk-free asset to compute the investor's optimal fraction of wealth invested in this stock in the presence of transaction cost. We use the obtained optimal fraction in this one-stock model to approximate the optimal fraction of total wealth in the N-stock model. Although the optimal fraction obtained in the one-stock model is suboptimal for the N-stock model, it is nevertheless a good approximation when the transaction cost rates are reasonably small.[12]

[11] For example, with N stocks, the HJB equation involves $2 \times N$ spacial variables plus one temporal variable even after a dimensional reduction, making finding a reliable solution almost infeasible when $N \geq 3$.

[12] In Appendix A.3, we compute the optimal trading strategy in a two-stock case, in which one stock has an

It should be noted that, although we solve for the optimal fraction of wealth invested in a stock using the one-stock model for each stock, fluctuations in other stocks' prices do influence the trading decision of a particular stock, because these fluctuations will change the total wealth, and thus change the optimal dollar amount that should be invested in the stock. For example, consider a scenario in which there are only two stocks and each has an optimal weight of 40% in the total wealth. After a drop in the price of Stock 1, the fraction of wealth invested in Stock 2 is now higher than 40%, and thus Stock 2 may need be sold to rebalance. This is different from the model with a CARA preference and uncorrelated stocks as studied in Liu (2002), who shows that the optimal dollar amount invested in a stock is independent of other stocks.

3 Analysis of the Trading Policy and the Disposition Effect-related Patterns

In this section, we provide a comprehensive numerical analysis of the model. Specifically, we examine the investor's trading strategy and its implications for various aspects of disposition effect patterns.

3.1 Baseline parameter values

In the baseline case, we capture stock level heterogeneity in a risk-return profile by assuming that the investor holds two types of stocks. Type I stocks have greater volatilities than Type II stocks. In particular, we assume that the investor invests in three Type I stocks with $\sigma_i = 0.3$ for $1 \leq i \leq 3$, and three Type II stocks with $\sigma_i = 0.2$ for $4 \leq i \leq 6$. Thus, there are six stocks in the investor's portfolio, i.e., $N = 6$.[13] The true expected returns for Type I and Type II stocks are respectively 0.1 and 0.06 (used only for simulations), but as assumed in the model,

observable constant expected return, and demonstrate that the results obtained from the approximately optimal strategy are indeed close to those obtained from the optimal strategy. In an earlier version of the paper, we solved the *optimal* trading strategy for the same model, but with CARA preferences where we obtained the same qualitative and similar quantitative results on the disposition effect-related patterns.

[13] In Odean (1998)'s sample, the median number of stocks held by investors is four. However, it can be easily shown that, in a model with four stocks, at least one of the percentage of gains realized (PGR) or the percentage of losses realized (PLR) will be no less than 1/4. This suggests that the empirical magnitude of the disposition effect (around 0.15 as found by Odean (1998)) must be from investors who hold a larger number of stocks. One can demonstrate that to have PGR and PLR values below 0.2, one needs at least six stocks in the representative investor's portfolio, which is what we assume in the base case. We also stress that assuming different return processes across stocks is not crucial for the overall disposition effect. The purpose of this parameterization is to show that heterogeneity in stock return processes can help generate a broad range of disposition effect-related patterns.

Table 1: Baseline parameter values.

This table summarizes the baseline parameter values that we use to illustrate our results.

Parameter	Symbol	Baseline value
Investment horizon (years)	T	5
Relative risk-aversion coefficient	γ	6
Risk-free rate	r	0.01
True but unobservable expected return	μ_i	0.1, $1 \leq i \leq 3$ 0.06, $4 \leq i \leq 6$
Return volatility	σ_i	0.3, $1 \leq i \leq 3$ 0.2, $4 \leq i \leq 6$
The prior on the expected returns	z_{i0}	0.1, $1 \leq i \leq 3$ 0.06, $4 \leq i \leq 6$
	V_{i0}	0.0025, $1 \leq i \leq 3$ 0.0016, $4 \leq i \leq 6$
Proportional transaction cost rates	$\theta_i = \alpha_i$	0.005, $i = 1, ..., 6$

the investor may not know these values. The risk-free rate is set at $r = 0.01$. The investor is assumed to have a relative risk aversion level of $\gamma = 6$. The proportional transaction cost rate for both purchase and sale is 50 basis points for all stocks, i.e., $\alpha_i = \theta_i = 0.005$ for $i = 1, ..., N$. The investor's initial wealth level is set at \$100,000. Furthermore, we assume that the investor's prior estimates of the expected returns are $z_{i0} = 0.1$ for $i = 1, 2, 3$ and $z_{i0} = 0.06$ for $i = 4, 5, 6$, and the investor has greater uncertainty in the estimate of more volatile stocks. Accordingly, we set $V_i(0) = 0.0025$ for $1 \leq i \leq 3$, and $V_i(0) = 0.0016$ for $4 \leq i \leq 6$.[14]

We summarize these baseline parameter values in Table 1. Our baseline parameterization allows us to examine various disposition effect-related patterns under a unified setting. For example, the differential risk-return profiles of these two types of stocks allow us to examine the magnitude of the disposition effect for stocks with different volatilities, and to examine the difference in the *ex-post* performance of sold stocks. The overall disposition effect, however, is insensitive to this choice of parameter values.

[14] Because it is difficult to precisely determine parameter values for a representative investor's portfolio, to show robustness of our results, we have conducted extensive numerical analyses utilizing a wide range of parameter values. We find similar qualitative results in these analyses. We do not report them in the paper to save space, but they are available from the authors.

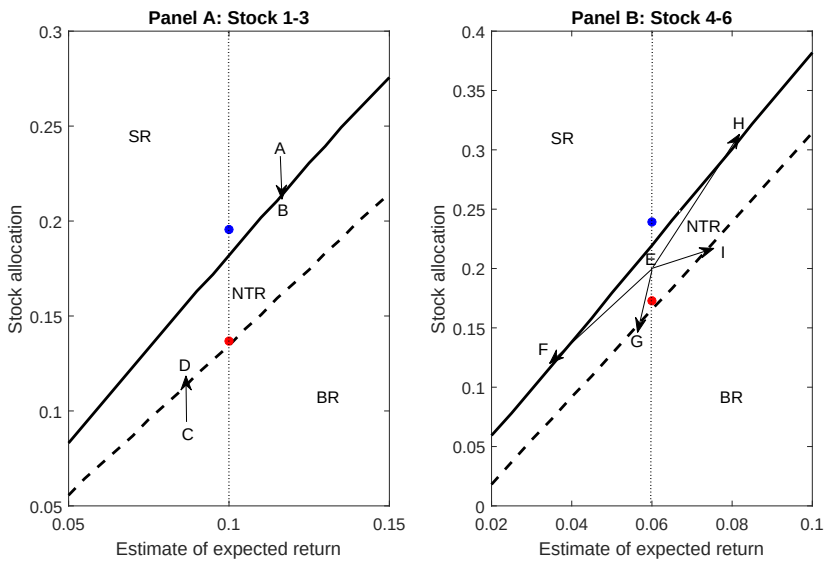

Figure 1: Optimal trading policy.
This figure shows the optimal trading policy of each type of stock at time $t = 2.5$ years. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$. The blue (red resp.) dot in each subfigure is the optimal selling (buying resp.) boundary when the expected returns are observable.

3.2 Rebalancing strategy

In this section, we illustrate the main features of the rebalancing strategy.

It is well-known that, in the presence of transaction costs, it is optimal for the investor to maintain the weight of each stock within a proper range.[15] We plot in Figure 1 the boundaries that represent such ranges, at time $t = 2.5$ years, as a function of z_{it}, i.e., the conditional mean of the expected return μ_i.

When a stock's weight in the portfolio enters the Sell Region due to fluctuations in prices, the investor *sells* the minimal amount of this stock necessary for its weight in the portfolio to be pushed down to the No-Trade Region (e.g., A to B in the left subfigure). When a stock's weight enters the Buy Region, the investor *buys* the minimal amount of additional shares of this stock required for its weight in the portfolio to be pushed up to the No-Trade Region (e.g., C to D in the left subfigure). In contrast, when the stock's weight is inside the No-Trade Region, it is optimal not to trade. In addition, as the conditional mean z_{it} increases, the conditional expected return of the stock also increases. The investor then desires a larger exposure to this stock, and thus the No-Trade Region shifts upward.

[15] See, for example, Davis and Norman (1990), Shreve and Soner (1994), and Liu and Loewenstein (2002).

Furthermore, Figure 1 suggests that selling can occur after either a gain (e.g., E to H in the right subfigure) or a loss (e.g., E to F in the right subfigure). Similarly, buying more shares can take place after either a loss (e.g., E to G in the right subfigure) or a gain (e.g., E to I in the right subfigure). As the sell (buy) boundary is more likely to be reached after a rise (drop) in a stock's price, an asymmetry exists between the trading direction after a gain (i.e., more likely to sell) versus the trading direction after a loss (i.e., more likely to buy).[16] Intuitively, it is optimal for the investor to keep the exposure to a stock within a range. As the price rises, the exposure increases and thus the investor has an incentive to sell. In contrast, as the price drops, the exposure decreases and thus the investor has an incentive to *buy more*. We term this asymmetric effect of optimal exposure range on the trading direction as the "exposure effect." As we will later demonstrate, in our model it is the exposure effect that drives the disposition effect. Because of transaction costs, the investor does not sell immediately after a stock becomes a winner or buy immediately after a stock becomes a loser. Instead, she holds a winner or a loser for a period of time until the gain or loss is sufficiently high. This is consistent with the empirical finding that investors usually do not realize penny gains and hold onto losers without purchasing more shares immediately (Odean (1998)).

Figure 1 also shows the optimal trading boundary when the expected returns are observable. In this case, the optimal trading boundary can be represented by two points at $z_{it} = \mu_i$ for any $t \in [0, T]$ and any $1 \leq i \leq N$. In particular, the blue dot in each subfigure represents the sell boundary, while the red dot denotes the buy boundary. Unlike in the case with unobservable expected returns where sales and purchases of a stock can occur after either a gain or a loss in this stock, a sale of a stock in the observable case cannot occur after a loss in this stock, and a purchase of a stock cannot occur after a gain in this stock without changes in the price of another stock. In the observable case, a sale of a stock can occur after a loss only if the loss is from a different stock (which pushes the fraction of wealth invested in the first stock high enough to reach the sell boundary), and similarly a purchase of a stock can occur after a gain only if the gain is from a different stock. This implies an even stronger asymmetry between the trading directions for winners and losers, and thus a stronger disposition effect in the observable case, as we show later.

[16] The investor sells with a loss only when a decrease in the conditional mean z_{it} significantly reduces the expected return and lowers the sell boundary after a drop in the stock price.

3.3 The disposition effect and related patterns

In this section, we examine in detail whether our model predicts a disposition effect and related patterns.

3.3.1 The disposition effect

To determine whether the widely documented disposition effect is consistent with the optimal trading strategies implied by our model, we conduct simulations of these trading strategies, keeping track of quantities, such as purchase prices, sale prices, and transaction times. Following Odean (1998), each day that a sale takes place, we compare the selling price for each stock sold to its average purchase price to determine whether that stock is sold for a gain or a loss. Each stock that is in that portfolio at the beginning of that day, but is not sold, is counted as a paper (unrealized) gain or loss, or neither. This is determined by comparing the stock's highest and lowest prices for that day to its average purchase price. If its daily low is above its average purchase price, it is counted as a paper gain; if its daily high is below its average purchase price, it is counted as a paper loss; and if its average purchase price lies between the high and the low, neither a gain nor a loss is counted. On days when no sales take place, no gains or losses (realized or paper) are counted.[17]

The disposition effect in the full sample. For each simulated path of the stocks and on each day, using the above definitions, we compute the number of realized gains/losses (# Realized Gains/Losses) and the number of paper gains/losses (# Paper Gains/Losses) for the optimal trading strategy. Then, we sum these numbers across all simulated paths to calculate the following ratios,[18] as used by Odean (1998):

$$\begin{aligned} PGR &= \frac{\#\text{Realized Gains}}{\#\text{Realized Gains} + \#\text{Paper Gains}}, \\ PLR &= \frac{\#\text{Realized Losses}}{\#\text{Realized Losses} + \#\text{Paper Losses}}. \end{aligned}$$

[17] As in Odean (1998), when a sale occurs, we assume that the average purchasing price of the remaining shares does not change. For a robustness check, we also use alternative counting methods, such as first-in-first-out, last-in-first-out, and highest-purchase-price-first-out for the purpose of computing the average purchasing price for the current position. We find that the results are similar.

[18] Similar to Barberis and Xiong (2009), we assume that each sample path is corresponding to the realization in a trading account.

Table 2: Disposition effect measures

This table shows the disposition effect measures for the observable and the unobservable cases: A1 and A2 for the full sample of sales; B1 and B2 for the subsample of sales in which there is no new purchase in the following three weeks; and C1 and C2 for the subsample of sales in which there is at least one stock being completely sold. The results are obtained from 10,000 simulated paths for each stock. $DE \equiv PGR - PLR$ and $DER \equiv PGR/PLR$. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$ and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$. $V_i(0) = 0$ for $i = 1, ..., 6$ for the observable case. The symbol *** indicates a statistical-significance level of 1%.

		Observable case	
	A1: Full sample	B1: No-new purchase subsample	C1: Complete sale subsample
PGR	0.258	0.242	N.A.
PLR	0.023	0.032	N.A.
DE	0.235***	0.210***	N.A.
DER	11.355***	7.661***	N.A.
PGL	0.949	0.933	N.A.
		Unobservable case	
	A2: Full sample	B2: No-new purchase subsample	C2: Complete sale subsample
PGR	0.239	0.238	0.004
PLR	0.057	0.061	0.474
DE	0.182***	0.177***	-0.470***
DER	4.199***	3.926***	0.008***
PGL	0.865	0.852	0.015

We also compute the fraction of sales that are gains, i.e.:

$$PGL = \frac{\#\text{Realized Gains}}{\#\text{Realized Gains} + \#\text{Realized Losses}}.$$

We report these values in Parts A1 and A2 of Table 2 for the observable expected return and the unobservable expected return cases, respectively. These values suggest that the disposition effect documented in the existing literature is consistent with the optimal portfolio rebalancing strategy implied by our model. For example, Table I of Odean (1998) reports a PLR of 0.098 and a PGR of 0.148. In comparison, our model with six stocks implies a PLR of 0.023 and a PGR of 0.258, with small standard errors that have been omitted from the table. The disposition effect measure $DE \equiv PGR - PLR$ is equal to 0.235 and is statistically significant at 1%. We also report an alternative disposition effect measure $DER \equiv PGR/PLR$, which uses the ratio of the two fractions. As shown in Table 2, the results are qualitatively similar. In addition, among all sales, gains realizations account for 94.9%. As the results in Part A2 show, in the unobservable case where there is the learning effect, the disposition effect is reduced but still statistically significant and comparable to the findings of Odean (1998).

The main intuition for our results is as follows. The disposition effect in our model is driven

by the exposure effect, i.e., the effect of the need to keep stock risk exposure within a certain range. If the risk exposure increases beyond the range after an increase in a stock price, the investor sells with a gain. If the risk exposure decreases beyond the range after a decrease in a stock price, however, the investor *buys* additional shares instead of selling. Thus, the exposure effect makes the investor sell after a sufficient increase in stock price, but buy after a sufficient decrease in stock price. This asymmetry in trading directions for winners and losers implies one aspect of the disposition effect, i.e., investors sell winners more frequently than losers. The other aspect of the disposition effect, i.e., investors tend to hold onto losers (rather than buying more), follows from the presence of transaction costs, which makes it costly to buy immediately after a stock becomes a loser.

As an alternative way of explaining the disposition effect result, note that selling a stock with a loss requires that the sell boundary be reached after a drop in the stock price. However, *ceteris paribus*, after a decrease in the stock price, the (higher) sell boundary is less likely to be reached than the (lower) buy boundary, whereas after an increase in the stock price, the (higher) sell boundary is more likely to be reached than the (lower) buy boundary. In addition, because stocks that are bought have positive expected returns, overall gains occur more often than losses do. Consequently, the investor sells more often to realize gains than to realize losses, which is consistent with the disposition effect.

If the expected returns are unobservable, then the learning effect, i.e., the effect of the investor's revision of the conditional distribution of the expected return after a change in the stock price, is also at work. If the stock price goes up (down), the investor revises upward (downward) the estimate of the expected return. The conditional variance of the expected return decreases deterministically and monotonically with time. Thus, the learning effect tends to make the investor buy after a stock price increase and tends to make the investor sell after a stock price decrease if the decrease in the conditional expected return dominates the decrease in the conditional variance. Therefore, the learning effect can counteract against the exposure effect. Indeed, as shown in Parts A1 and A2, the disposition effect is stronger in the observable case because the counteracting learning effect is absent in this case. Because learning about means is very slow, the learning effect is on average much smaller than the exposure effect, the exposure effect dominates unconditionally, and thus the disposition effect is still strongly

significant, even in the unobservable case.[19]

The above finding that the learning effect tends to decrease the disposition effect may shed some light on the empirical evidence that the disposition effect is stronger among less sophisticated investors (e.g., Dhar and Zhu (2006)). This is because less sophisticated investors may learn more slowly about the true expected returns through past returns than more sophisticated investors, and thus the learning effect is weaker and the disposition effect is stronger for less sophisticated investors. For naive investors who do not learn at all, the disposition effect is the greatest, whether they happen to have the correct estimate of the expected return or not.

The disposition effect in sales not followed by purchase. Odean (1998) demonstrates that, among the sales after which there were no purchases of another stock in three weeks, the disposition effect still appears. Because in most of the existing portfolio rebalancing models (e.g., Merton (1971)) selling a stock without immediately purchasing others is unlikely to be optimal, Odean (1998) concludes that portfolio rebalancing is unlikely to explain the disposition effect in this subsample. While it is true that an investor always immediately buys another stock after a sale of a stock in the absence of transaction costs, in the presence of transaction costs, however, it can be optimal for an investor to sell a stock without purchasing another for an extended period of time. This is because, as long as other stock positions are inside their no-transaction regions, it is not optimal for the investor to buy any additional amount of these stocks, even after a sale of another stock.

To determine if our model could generate the disposition effect in the subsample with no immediate purchases of other stocks after selling one, we computed the PLR and PGR ratios when restricted to this subsample. Parts B1 and B2 of Table 2 display the results, which are similar to those obtained for the full sample. For example, Panel B2 of Table 2 demonstrates that, across all sample paths without a new purchase in three weeks after a sale, PGR is equal to 0.238, PLR is equal to 0.061, and DE is equal to 0.177 with high statistical significance. As in the full sample case, the results in the observable case are stronger. These results suggest that the disposition effect found in the no-new-purchase subsamples that Odean (1998) considers can be consistent with the portfolio rebalancing strategies implied by a model such as ours.

The disposition effect in complete sales. Odean (1998) demonstrates that, in the subsample in which the investor sells the entire position of at least one stock, the disposition

[19] As we show later, the learning effect can dominate in some states.

effect still appears.[20] Because in most of the existing portfolio rebalancing models (e.g., Merton (1971)) selling the entire position of a stock is not optimal, Odean (1998) concludes that portfolio rebalancing is unlikely to explain the disposition effect in this subsample. A similar analysis is conducted, and the same conclusion is reached by Engelberg, Henriksson, and Williams (2018).

Indeed, our baseline case does not generate the disposition effect in this subsample, as shown by Panel C2 of Table 2. The reason is that complete sales occur only when the price of a stock declines so much that its conditional expected return turns negative. As a result, complete sales more likely follow a loss if all stocks have constant expected returns.[21] However, if there is a stock (e.g., Stock 7) with a mean-reverting expected return to the investor's portfolio, then we are able to generate the disposition effect in this subsample.[22] For this stock, with a large positive shock on price, its instantaneous expected return can be driven below the risk-free rate. This is consistent with the evidence for stock-level mean-reversion, conditional on large price changes (e.g., Zawadowski, Andor, and Kertesz (2006), Dunis, Laws, and Rudy (2010)). As a result, completely liquidating this stock can also be driven by a large stock price increase (in addition to learning about the expected return after a large price drop). With a reasonable set of parameter values for this stock and the same parameter values for the other six stocks, we obtain $PGR = 0.167$, $PLR = 0.093$, and thus $DE = 0.074$ among the subsample with complete sales of at least one stock. As before, in this subsample, the disposition effect is still driven by the exposure effect.[23]

3.3.2 Additional measures related to the disposition effect

The reverse disposition effect. Odean (1998) also documents a reverse disposition effect,

[20] Such complete sales represent about 4% of the sample of all sales.

[21] In the case with observable expected returns, it is never optimal to liquidate the entire position on a stock due to its known positive risk premium. As a result, we put N.A. in Part C1.

[22] For this stock, the price dynamics is given as follows:

$$\frac{dS_{it}}{S_{it}} = (\mu_i + \xi_{it})dt + \sigma_i dB^S_{it}, \tag{29}$$

where ξ_{it} follows an Ornstein-Uhlenbeck process with zero mean (without loss of generality), i.e.:

$$d\xi_{it} = -g_i \xi_{it} dt + \nu_i dB^{\xi}_{it}. \tag{30}$$

We assume that the Brownian motions (B^S_{it}, B^{ξ}_{it}) are correlated with coefficient ρ_i, and they are independent of all other Brownian motions in the model.

[23] We note that a mean-reverting expected return is not necessary for the disposition effect within a sample with complete sales. For example, in a previous version of the paper, we show that the disposition effect is consistent with investors' trading pattern in the presence of committed consumption (as in Liu (2014)). To conserve space, we do not include this alternative model or its results in this paper, but they are available from the authors.

Table 3: Alternative disposition effect measures

This table shows the alternative disposition effect measures. The results are obtained from 10,000 simulated paths for each stock. Parts A1 and A2 report the reverse disposition effect measures; and Parts B1 and B2 report the average holding time of winners and losers. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$. $V_i(0) = 0$ for $i = 1, ..., 6$ for the observable case. The symbol *** indicates a statistical-significance level of 1%.

Observable case			
A1: Reverse disposition effect		B1: Average holding time (years)	
$PLPA$	0.282	Winners	0.242
$PGPA$	0.099	Losers	1.135
Difference	0.183***	Difference	0.893***
Unobservable case			
A2: Reverse disposition effect		B2: Average holding time (years)	
$PLPA$	0.254	Winners	0.571
$PGPA$	0.124	Losers	1.404
Difference	0.130***	Difference	0.833***

i.e., relative to winning stocks, investors have a higher tendency to purchase additional shares of losing stocks. This is clearly consistent with portfolio rebalancing, which predicts that, after a drop in price, an investor is more likely to buy the stock to increase risk exposure. To confirm this intuition, we calculate the two measures $PLPA$ and $PGPA$ used by Odean (1998):

$$\begin{aligned} PGPA &= \frac{\#\text{Gains Purchased Again}}{\#\text{Gains Purchased Again} + \#\text{Gains Potentially Purchased Again}}, \\ PLPA &= \frac{\#\text{Losses Purchased Again}}{\#\text{Losses Purchased Again} + \#\text{Losses Potentially Purchased Again}}. \end{aligned}$$

These measures are similar to PGR and PLR, except that they are computed at the time when a purchase, instead of a sale, is made. For example, #Gains Purchased Again is the number of times when a purchase is made on a stock that has a gain as of the purchasing time, and #Gains Potentially Purchased Again is the number of other stocks that have a paper gain, but are not purchased again at the aforementioned purchasing time. Odean (1998) reports $PLPA = 0.135$ and $PGPA = 0.094$. Using the baseline parameter values in Table 1, we obtain $PLPA = 0.254$, and $PGPA = 0.124$ for the case with unobservable expected returns (reported in Part A2 of Table 3). The reverse disposition effect is stronger in the observable case as shown in Part A1 of the same table, because of the absence of the learning effect. This suggests that the reverse disposition effect is also consistent with optimal portfolio rebalancing.

Holding time of winners and losers. Another reflection of the disposition effect is that

the average holding time of losers is greater than that of winners. We compute by simulation the average holding time between the last time when a stock investment has a gain or loss and the first sale time of the stock afterwards, and report the results in Parts B1 and B2 of Table 3. In our baseline case, on average, it takes 0.571 years to realize a gain and 1.404 years to realize a loss. This is because selling a loser after a drop in the price of this stock requires that its estimated expected return drops enough to offset the price decrease so that the sell boundary is reached, which occurs much less frequently than selling a winner after an increase in the stock price since the price increase helps reach the sell boundary. In the observable case, even though holding times of both winners and losers are shorter because of the reduced uncertainty, the average holding time of losers is still longer than that of winners. These results confirm that, for portfolio rebalancing purposes, holding losers substantially longer than winners can indeed be optimal.

3.3.3 The effect of return correlation

In order to facilitate the construction of a rebalancing strategy, so far we have assumed that the stocks' returns are uncorrelated. In this subsection, we examine how the disposition effect measures change with return correlation. For tractability, we consider the case in which there are only two stocks with observable constant expected returns.[24] In this case, we use the optimal rebalancing strategy (instead of the approximately optimal strategy) to conduct our analysis.

Intuitively, when the returns of stocks are correlated, the prices of these stocks tend to move along the same or the opposite directions, depending on the sign of the correlation. When the two stocks' returns are positively correlated, a realized gain in one stock is likely associated with a paper gain in the other stock, and similarly a realized loss in one stock is likely accompanied by a paper loss in the other stock. As a result, both the PGR and PLR tend to decrease. Compared to PLR, however, PGR reduces faster since stocks carry a positive risk premium and paper gains increase more than paper losses. Consequently, the disposition effect measure decreases when return correlation increases, and so does the reverse disposition effect. By a similar intuition, when the correlation is negative, both the disposition effect and the reverse disposition effect become stronger.

We plot in Figure 2 the disposition effect measure and the reverse disposition effect measure

[24] When the expected returns are observable, incorporating correlation is straightforward. Thus, we omit the details of the model in this case. They are available from the authors upon request.

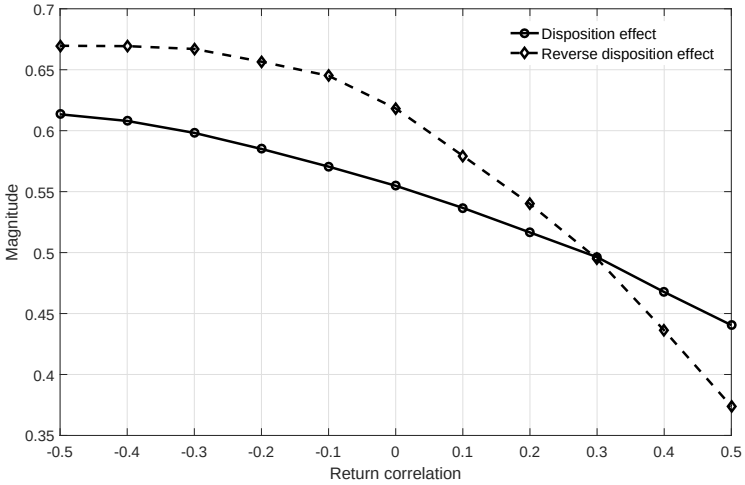

Figure 2: Disposition effect and return correlation.
This figure shows how the disposition effect measures DE and RDE change with the return correlation. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 2$, $\mu_1 = 0.1$, $\sigma_1 = 0.3$, $\mu_2 = 0.06$, $\sigma_2 = 0.2$, $\alpha_i = \theta_i = 0.005$, for $i = 1, 2$. The expected returns of both stocks are assumed to be observable.

against the return correlation from -0.5 to 0.5. Consistent with the above intuition, compared with the uncorrelated case, the disposition effect measure and the reverse disposition effect measure are lower with positive correlations, but higher with negative correlations. For example, when the return correlation increases from 0 to 0.5, the disposition effect measure decreases from 0.561 to 0.441, and the reverse disposition effect measure decreases from 0.623 to 0.374. When the return correlation decreases from 0 to -0.5, the disposition effect measure increases from 0.561 to 0.613, and the reverse disposition effect measure increases from 0.623 to 0.670.

3.3.4 The *ex-post* return pattern

Studies such as Odean (1998) have found that investors tend to sell winners that subsequently outperform losers that they continue to hold, which could indicate that investors sell winners too soon and hold onto losers too long. The existing literature has interpreted this evidence as supporting the argument that displaying the disposition effect is costly to investors. We next demonstrate that, for portfolio rebalancing purposes, it can be optimal for investors to sell winners that subsequently outperform losers that they have retained.

We report in Table 4 the average *ex-post* returns of stocks sold as winners and of those held as losers in simulations of our model. The table shows that selling winners whose future expected returns are greater than those of the losers held can be optimal. For example, over

Table 4: *Ex-post* returns

This table shows the average *ex-post* returns of the stocks sold as winners and of the stocks held as losers. The results are obtained from 10,000 simulated paths. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$. The symbol *** indicates a statistical-significance level of 1%.

	Over the next 84 trading days	Over the next 252 trading days
Stocks sold as winners	3.16%	9.89%
Stocks held as losers	2.70%	8.10%
Difference	0.46%***	1.79%***

the next 84 days after a sale, the average return of the winners sold is 0.46% higher than the losers held. Over the next 252 days, the return gap grows to 1.79%. This result is due to a straightforward mechanism at work: stocks with higher expected returns (i.e., Stocks 1-3 in our baseline case) are more likely to be sold as winners because it is more often that the exposure in these stocks exceeds the sell boundary as a result of the faster expected growth in their prices; in contrast, stocks with lower expected returns (i.e., Stocks 4-6 in our baseline case) are more likely to become losers to be held onto than those with higher expected returns. This mechanism implies that the average *ex-post* returns of the sold winners can exceed those of the held losers.

3.3.5 The impact of higher volatility on the disposition effect

Kumar (2009) investigates stock-level determinants of the disposition effect and finds that the disposition effect is stronger for stocks with higher volatility.[25] Kumar argues that this is consistent with behavioral biases being stronger for more volatile stocks. We next demonstrate that this disposition effect pattern can also be a result of portfolio rebalancing. We separately calculate the disposition effect measure of the high volatility group (i.e., Stocks 1-3) and of the low volatility group (i.e., Stocks 4-6). We report in Table 5 the disposition effect measures of these two groups, which indicate a stronger disposition effect among the stocks with high volatility.

The main driving force behind the above result is the greater exposure effect for a more volatile stock. As volatility increases, the sell boundary is reached more frequently, as indicated by the shorter average duration between sales. Consequently, gains are realized more often.

[25] To the extent that mutual funds have less volatile returns than individual stocks do, this is consistent with the finding that the trading in mutual funds exhibits a weaker disposition effect.

Table 5: The disposition effect and volatility

This table shows the disposition effect measures among the high volatility group (Stocks 1-3) and the low volatility group (Stocks 4-6). The results are obtained from 10,000 simulated price paths for each stock. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$. The symbol *** indicates a statistical-significance level of 1%.

	High volatility (Stocks 1-3)	Low volatility (Stocks 4-6)
Average duration between sales	0.114	0.180
PGR	0.381	0.082
PLR	0.084	0.033
DE	0.296***	0.049***
$\Delta(DE)$	0.247***	

Because losses are more likely followed by purchases, this implies a greater exposure effect and thus a stronger disposition effect for more volatile stocks.

When investors need to learn about the expected returns, the learning effect is also at work. As indicated by Equation (9), the strength of the learning effect increases with $\frac{\sigma_i V_i(0)}{\sigma_i^2 + V_i(0)\, t}$, i.e., the sensitivity of the revision of the conditional expected return dz_{it} to the realized shock $d\hat{B}_{it}^S$. Thus the strength of the learning effect is hump-shaped in the return volatility σ_i. More specifically, when the volatility is low, the learning effect increases with the volatility, but when the volatility is high, the learning effect decreases as the volatility increases. The parameter values used in Table 5 are such that the learning effect decreases with the volatility, and thus the change in the learning effect after an increase in the volatility also leads to a stronger disposition effect. On the other hand, we find that even when the learning effect increases with volatility, and thus tends to decrease the disposition effect, the disposition effect still increases with volatilities for a wide range of parameter values. This is because learning about the expected returns is slow, and thus the impact of the increase in the exposure effect on the disposition effect dominates that of the increase in the learning effect.

The empirical studies conducted by Kumar (2009) are on the stock level. Based on the above discussions, our model offers a new prediction: investors with a more diversified portfolio or a better hedged portfolio have a weaker disposition effect. This is because a more diversified portfolio or a better hedged portfolio tends to have a lower volatility, and thus a weaker exposure effect.

3.4 The V-shaped trading patterns and distribution of realized returns

Ben-David and Hirshleifer (2012) demonstrate that the probability of selling and of buying more are both greater for positions with larger unrealized gains and larger unrealized losses, i.e., the plots of these probabilities against paper profit exhibit V-shaped patterns. In contrast, extant theories based on the static prospect theory and regret aversion predict that the larger the loss, the less likely investors are to sell, and the larger the gain, the less likely they are to buy, which are both opposite to the V-shape patterns. We next demonstrate that the V-shape patterns are consistent with the optimal trading strategy in our model because of the interactions between the exposure effect and the learning effect. Assuming that an investor can obtain a burst of reference-dependent utility from a sale in a dynamic-prospect-theory setting, Ingersoll and Jin (2013) demonstrate that the probability of selling may increase with the magnitude of losses. Different from the mechanism in our model, the driving force in their model is that realizing large losses resets the reference points to lower levels, which can potentially increase the reference-dependent utility. In contrast, Frydman, Hartzmark, and Solomon (2018) find convincing empirical evidence that investors do not reset their reference points after sales if they buy new assets shortly after the sales. In addition, Ingersoll and Jin (2013) cannot explain why the probability of buying more increases with the magnitude of gains.

We plot the probability of selling and the probability of buying more as a function of past annualized returns obtained from holding shares in Figure 3 for both the observable and the unobservable cases.[26] This figure demonstrates that, if the expected returns are unobservable, then both the selling probability and the buying probability against past returns implied by our model can display V-shape patterns, consistent with empirical evidence in Ben-David and Hirshleifer (2012).[27] In Figure 4, we show that the V-shape patterns remain present under various alternative parameter values for parameters, such as the mean and variance of the investor's prior on the stocks' expected returns and the return volatility. These results suggest that the V-shape patterns may be robust to alternative parameterization of our model with unknown expected returns.

As we discussed previously, there are two possibly opposing effects at work in our model

[26] See Section A.4 for details on how we calculate the probabilities of selling and buying given the magnitude of paper gains or losses.

[27] Note that Figure 3 is also consistent with the disposition effect, because the unconditional probability of selling is lower for a loss compared to that for a gain of the same magnitude.

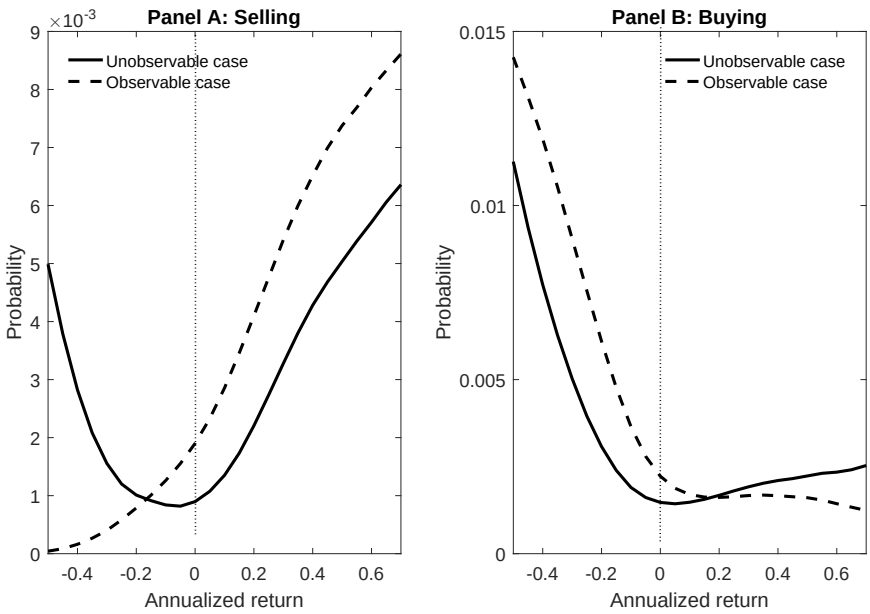

Figure 3: V-shape in the probability of selling or of buying shares.
This figure shows the probability of selling or of buying shares against the up-to-date annualized return. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$. $V_i(0) = 0$ for $i = 1, ..., 6$ for the observable case.

with unobservable expected returns. The first one is the exposure effect, which tends to make the investor sell (buy) after an increase (a decrease) in exposure following a positive (negative) return. The second one is the learning effect, which tends to counteract against the exposure effect. The intuition behind the V-shape pattern results is as follows:

1. When there is a gain. As the gain increases, the learning effect increases the probability of buying and decreases the probability of selling, while the exposure effect does the opposite. The V-shape for selling is conditional on selling, i.e., only among the paths in which selling is optimal. Therefore, for the right half of the V-shape for selling (i.e., conditional selling after a gain), the exposure effect which promotes selling a gain dominates the learning effect which promotes the opposite, and the net effect increases with the magnitude of the gain. The V-shape for buying is conditional on buying, i.e., only among the paths in which buying is optimal. Therefore, for the right half of the V-shape for buying (i.e., conditional on buying more after a gain), the learning effect which promotes buying more after a gain dominates the exposure effect which promotes the opposite, and the net effect increases with the magnitude of the gain.

2. When there is a loss. As the loss increases, the learning effect increases the probability of selling and decreases the probability of buying, while the exposure effect does the opposite. The V-shape for selling is conditional on selling, i.e., only among the paths in which selling is optimal. Therefore, for the left half of the V-shape for selling (i.e., conditional selling after a loss), the learning effect which promotes selling a loss dominates the exposure effect which promotes the opposite, and the net effect increases with the magnitude of the loss. The V-shape for buying is conditional on buying, i.e., only among the paths in which buying is optimal. Therefore, for the left half of the V-shape for buying (i.e., conditional on buying after a loss), the exposure effect which promotes buying more after a loss dominates the learning effect which promotes the opposite, and the net effect increases with the magnitude of the loss.

3. The slope asymmetry between the right and left parts of the V-Shape results follows from the slowness in learning about the means,[28] i.e., as the return magnitude changes, the learning effect changes relatively slowly compared to the exposure effect.

Consistent with the above intuition, the dashed lines in Figure 3 show that if the expected returns are observable, then the V-shape patterns disappear. This is because, in this case, there is no learning effect, and therefore the exposure effect makes the probability of selling increase monotonically and the probability of buying more decrease monotonically with the past returns. This highlights the importance of learning in understanding the V-shape patterns. Our model thus offers another new prediction: the V-shaped trading patterns are less pronounced for stocks with more public information, such as S&P 500 stocks, because for these stocks much is already known and as a result the learning effect is weaker.

Our model can also shed light on another finding of Ben-David and Hirshleifer (2012) that the V-shape patterns are more prominent for stocks with shorter holding periods. In Figure 5, we plot the probabilities of selling or buying more as functions of past annualized returns for a holding period of less than three months at the top ("short period") and a holding period of more than three months at the bottom ("long period"). This indicates that the V-shape pattern is indeed more prominent for stocks with short holding periods. The intuition is that the learning effect is stronger at early stages of learning. As the holding period increases, the investor gradually learns about the expected returns with greater accuracy, and thus the

[28] One prediction is that the faster one learns, the less asymmetric the V-shape curves.

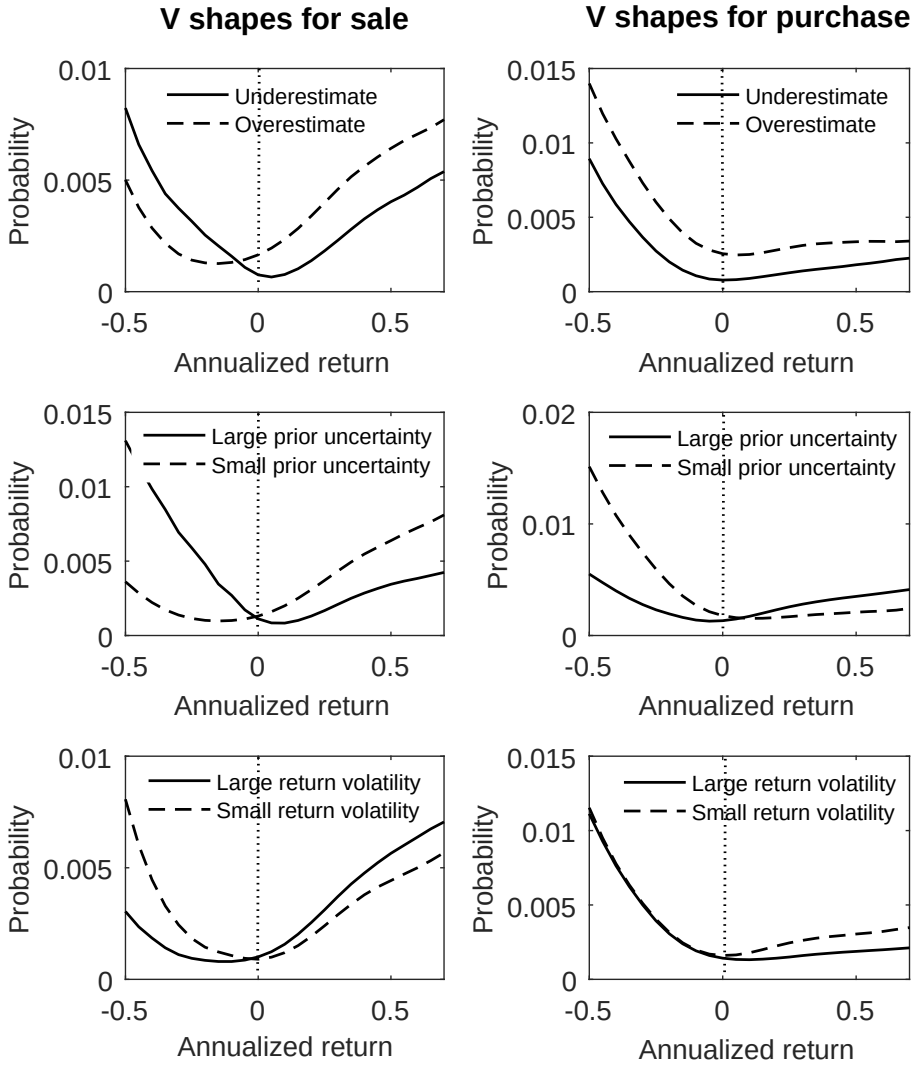

Figure 4: V-shapes for alternative parameter values.
This figure shows the probability of selling or of buying shares against the up-to-date annualized return, for various alternative parameter values in the model. Baseline parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$. For the two subfigures on the top, "Overestimate" is the case with $z_{i0} = \mu_i + 0.02$, $i = 1, ..., N$, and "Underestimate" is the case with $z_{i0} = \mu_i - 0.02$, $i = 1, ..., N$. For the two subfigures in the middle, "Large prior uncertainty" is the case in which $V_i(0) = 0.005$ for $i = 1, 2, 3$, and $V_i(0) = 0.0032$ for $i = 4, 5, 6$. "Small prior uncertainty" is the case in which $V_i(0) = 0.00125$ for $i = 1, 2, 3$, and $V_i(0) = 0.0008$ for $i = 4, 5, 6$. For the two subfigures at the bottom, "Large return volatility" is the case in which $\sigma_i = 0.33$ for $i = 1, 2, 3$, and $\sigma_i = 0.22$ for $i = 4, 5, 6$. "Small return volatility" is the case in which $\sigma_i = 0.27$ for $i = 1, 2, 3$, and $\sigma_i = 0.18$ for $i = 4, 5, 6$.

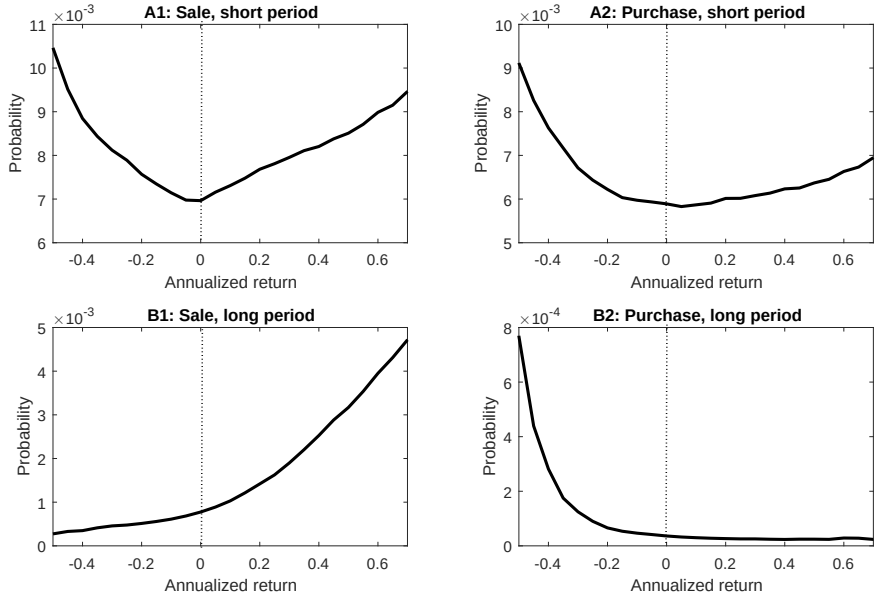

Figure 5: V-shapes for short and long holding period.
This figure shows the probability of selling or of buying shares against the up-to-date annualized return, for sales and purchases with short or long holding period, respectively. Baseline parameter values: $T = 1$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$.

learning effect weakens over time and the V-shape patterns become less prominent.

Ben-David and Hirshleifer (2012) also find that the empirical distribution of realized returns is hump-shaped with a maximal value in the domain of gains (see Figure 4 in the Appendix of Ben-David and Hirshleifer (2012)). We plot in Figure 6 the distribution of realized returns generated by our model via 10,000 simulated sample paths. Figure 6 shows that the distribution of realized returns implied by our model is also hump-shaped, consistent with the empirical finding of Ben-David and Hirshleifer (2012). The reason for the hump-shape in our model is that the investor optimally keeps her risk exposure in a certain range, and thus sales are most likely to occur when the magnitude of a gain is just large enough to push her risk exposure out of the optimal range. Therefore, more realized returns concentrate around this critical magnitude and the rest have lower probability density, which implies that the distribution of realized returns exhibits a humped shape.

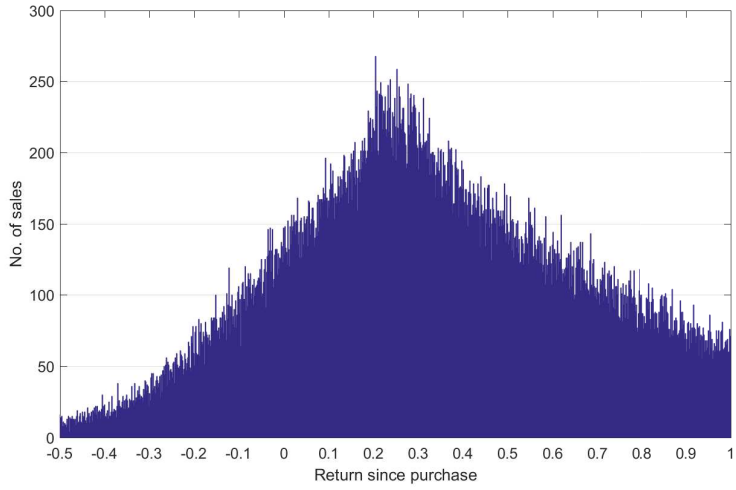

Figure 6: Distribution of realized returns.
This figure presents the distribution of realized returns predicted by our model. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$.

3.5 The repurchase pattern

Strahilevitz, Odean, and Barber (2011) find that investors are reluctant to repurchase stocks previously sold for a loss and stocks that have appreciated in price subsequent to a prior sale. Strahilevitz, Odean, and Barber (2011) attribute this repurchase pattern to the emotional impact of past trading activities.

Following the approach outlined in Strahilevitz, Odean, and Barber (2011), we simulate our model to compute the proportion of prior losers repurchased ($PLRP$), the proportion of prior winners repurchased ($PWRP$), the proportion of stocks that have gone up in prices since the last sale at the time of the repurchase (PUR), and the proportion of stocks that have gone down in price since the last sale at the time of the repurchase (PDR).[29] Strahilevitz, Odean, and Barber (2011) find that $PLPR < PWPR$ and $PUR < PDR$. In Table 6, we report these repurchase measures from the simulation using the baseline parameter values. Overall, we find that these measures from our model agree with the empirical findings of Strahilevitz, Odean, and Barber (2011).

The intuition is as follows. Selling a loser is typically triggered by a substantial decrease in the investor's estimate of the stock's expected return, while selling a winner is more often

[29] We use the notation $PLRP$ to denote the proportion of prior losers repurchased to distinguish from the disposition effect-related measure PLR.

Table 6: Repurchase effect measures

This table shows the repurchase effect measures. The results are obtained from 10,000 simulated paths for each stock. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$. The symbol *** indicates a statistical-significance level of 1%.

A: Previous winners or losers		B: Winners or losers since last sale	
$PLRP$	0.227	PDR	0.435
$PWRP$	0.304	PUR	0.057
Difference	-0.077***	Difference	0.378***

driven by a price increase. Because changes in the estimate of expected return are slow, it takes a longer time to repurchase a loser sold. This implies that $PLPR < PWPR$.

On the other hand, the exposure effect implies that, after a sale, the investor will only repurchase a stock when the (lower) buy boundary is reached, which is more likely to occur after a drop in stock price. Thus, the investor is more likely to repurchase stocks that have depreciated in value since the last sale. This is why our model predicts that $PUR < PDR$.

3.6 The effect of capital gains tax

It is well known that, with capital gains tax and full capital loss tax rebate, loss realization is beneficial while gains realization becomes more costly (Constantinides (1983)). As a result, the disposition effect will be reduced if one takes capital gains tax into account.

In this subsection, we examine the effect of capital gains tax on the disposition effect.[30] We use the same stock-by-stock approximation approach as previously.[31]

Trading strategy. We plot the trading boundaries for Stock 1 against the basis-price ratio $\frac{k_1}{(1-\alpha_1)y_1}$ in Figure 7, where k_1 is the total cost basis of Stock 1, fixing the estimate of the expected return at its true value $z_1 = \mu_1$ and assuming a capital gains tax rate of 15%.[32] Note that there is a gain after liquidation if and only if $\frac{k_1}{(1-\alpha_1)y_1} < 1$. As in the case without capital gains tax, the investor should maintain the stock exposure within a certain range, as suggested by the Sell Region (SR) above the sell boundary and the Buy Region (BR) below the

[30] To keep the conciseness of the exposition, we relegate to Appendix A.4 the details of the model with capital gains tax.

[31] Portfolio choice problems with multiple stocks are difficult to solve because of the significant increase in the number state variables even after simplifying approximations such average basis and full rebate for capital losses (see, e.g., Gallmeyer, Kaniel, and Tompaidis (2006)).

[32] Since the main features of the trading strategies of other stocks are similar, we do not show them to save space.

buy boundary.

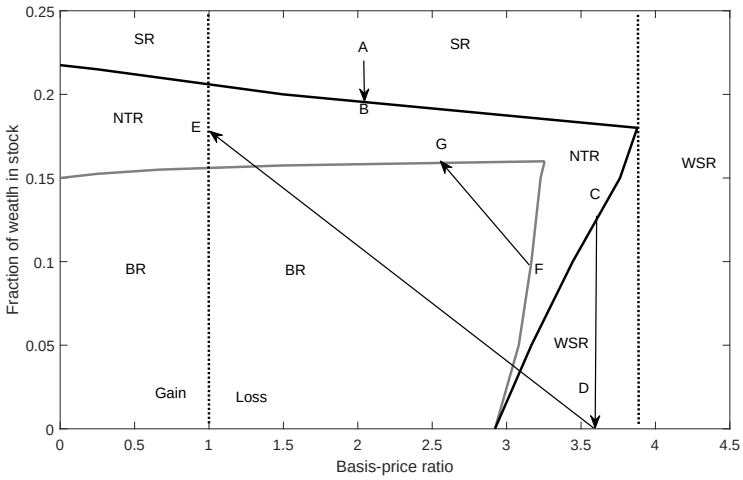

Figure 7: Trading strategy with capital gains tax.
This figure shows a snapshot of the trading boundaries at time $t = 2.5$ years in the presence of capital gains tax with fully rebatable capital losses. The region labeled "WSR" is the wash-sale region. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $\mu_1 = 0.1$, $\sigma_1 = 0.3$, $V_1(0) = 0.0025$, and $\alpha_1 = \theta_1 = 0.005$. The conditional estimate of the return predictor is set at $z_1 = 0.1$, and the capital gains tax rate is $\tau = 0.15$.

In contrast to the case with capital gains tax but without transaction costs (as considered by the existing literature, e.g., Constantinides (1983)), Figure 7 shows that it can be optimal to defer the realization of even large capital losses, as indicated by the no-transaction region (NTR) to the right of the vertical line at $\frac{k_1}{(1-\alpha_1)y_1} = 1$. In addition, even when it is optimal to realize capital losses, the optimal realization can be only a fraction of the losses, e.g., point A to point B. This is because the time value of the tax rebate can be smaller than the transaction costs required, and realizing only part of the losses can avoid the transaction costs needed for buying back shares. Only when the capital losses are large enough does the investor immediately realize all losses and buy back some shares to achieve the optimal risk exposure (e.g., point C to point D, and then to point E).

Due to the presence of transaction costs, when the investor has capital losses, it can be optimal to purchase more without first realizing losses (e.g., from F to G). This is because capital loss on a stock reduces the exposure to this stock, and selling first to realize losses and then buying back some to achieve the desired exposure would incur too much transaction cost. In such a case, the investor purchases shares to maintain a desirable exposure to the stock without engaging in any loss selling.

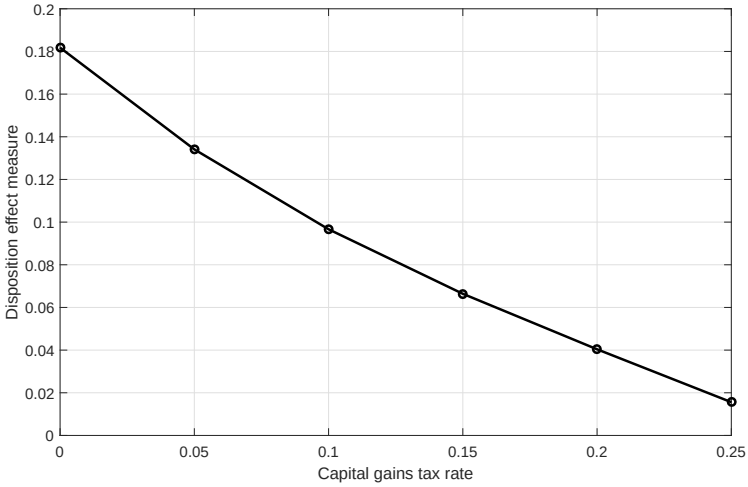

Figure 8: Capital gains tax and the disposition effect.
This figure shows the disposition effect measure $DE = PGR - PLR$ for various capital gains tax rates. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 6$; $\mu_i = 0.1$, $\sigma_i = 0.3$, $z_{i0} = 0.1$, and $V_i(0) = 0.0025$ for $i = 1, 2, 3$; $\mu_i = 0.06$, $\sigma_i = 0.2$, $z_{i0} = 0.06$, and $V_i(0) = 0.0016$ for $i = 4, 5, 6$; $\alpha_i = \theta_i = 0.005$, for $i = 1, ..., 6$.

The disposition effect. In Figure 8, we plot the disposition effect measure DE against capital gains tax rate. As anticipated, the presence of capital gains tax increases the investor's propensity to realize losses and decreases her propensity to realize gains, thus reducing the disposition effect. However, a significant disposition effect still exists, even with capital gains tax. For example, as shown in Figure 8, with a capital gains tax rate of 25%, the disposition effect measure $DE = 0.016$, which is still highly statistically significant. On the other hand, as tax rates increase, the DE monotonically decreases. This can be consistent with Dhar and Zhu (2006)'s finding that investors with higher income are less prone to the disposition effect, because these investors are subject to a higher capital gains tax rate.

It should be noted that, in this subsection, we assume that the capital losses are fully rebatable to simplify our analysis. In practice, however, the U.S. tax code stipulates a limited tax rebate of up to $3,000 in losses per year. This feature would reduce the benefit of realizing losses, and thus would increase the magnitude of the disposition effect significantly.

To summarize, although capital gains tax tends to reduce the disposition effect, the disposition effect is still significant even with capital gains tax. The main intuition is that the exposure effect is still present and in the presence of transaction costs, it is optimal to defer even some large capital losses even when capital losses are fully rebatable. Therefore, the disposition effect is still present, although the magnitude of the disposition effect is reduced due to the additional

benefit of realizing losses and deferring gains.

4 Conclusions

The disposition effect, i.e., the tendency of investors to sell winners while holding onto losers, has been widely documented. Most of the existing theories that attempt to explain the disposition effect (mostly based on prospect theory, mental accounting, and regret aversion) cannot explain why the plots of the probabilities of buying more and of selling against paper profit both exhibit V-shape patterns, as shown by Ben-David and Hirshleifer (2012). In addition, they are largely silent on other well-documented disposition-effect related patterns, such as investors selling winners that subsequently outperform losers that they hold, the disposition effect being greater for stocks with greater volatilities, and the disposition effect being stronger for less sophisticated investors.

Based on the empirical evidence on the relevance of portfolio rebalancing, learning, and transaction costs for retail investors, we propose an optimal portfolio rebalancing model with learning and transaction costs to show that the disposition effect and many of the related patterns, including the V-shaped trading patterns, are consistent with the optimal trading strategies implied by our model. Our finding that portfolio rebalancing alone predicts the disposition effect and many of the related patterns suggests portfolio rebalancing might complement existing theories in understanding the disposition effect and the related patterns.

In addition to matching most of the disposition-effect related findings in the literature, our model also offers some new testable predictions. For example, our model predicts that: (1) conditional on return volatility, the magnitude of the disposition effect is greater for stocks for which there is more public information; (2) investors with a more diversified portfolio or a better hedged portfolio have a weaker disposition effect; and (3) the V-shaped trading patterns are more pronounced for stocks with less public information.

Our central message is that, although various types of behavioral biases are likely to exist among some investors, there can well be a rational component in the disposition-effect and the related trading patterns. How to separate the rational portfolio rebalancing and behavioral components of the disposition effect and its related findings constitutes an interesting empirical question for future studies.

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Appendix

The contents of this appendix are arranged as follows. In Section A.1 and A.2, we collect the proofs for our analytical results. In Section A.3, we show that, in some special cases of the model in which we can compute the optimal trading strategy, the disposition effect measures from the optimal trading strategies are very close to what we obtain in the main text using the approximately optimal trading strategy. In Section A.4, we provide details for the calculation of the probabilities of selling and buying. In Section A.5, we present the model with capital gains tax.

A.1 Proof of Proposition 1

Proof. Since this proof is very similar to those in the standard literature, such as Davis and Norman (1990) and Shreve and Soner (1994), we only sketch the main steps of the proof.

For any admissible trading policy $(\mathbf{I}_t, \mathbf{D}_t) = (I_{it}, D_{it} : i = 1, ..., N)$, let $(\mathbf{I}_t^c, \mathbf{D}_t^c) = (I_{it}^c, D_{it}^c : i = 1, ..., N)$ be its continuous component, $(\Delta\mathbf{I}_t, \Delta\mathbf{D}_t) = (\Delta I_{it}, \Delta D_{it} : i = 1, ..., N)$ be its discontinuous component (jump component), $(1+\theta) = (1+\theta_1, ..., 1+\theta_N)$, $(1-\alpha) = (1-\alpha_1, ..., 1-\alpha_N)$, and $(X_t, \mathbf{Y}_t)$ be the associated sub-wealth processes. Under regularity conditions, applying the generalized version of Ito's lemma to $V(X_t, \mathbf{Y_t}, \mathbf{Z_t}, t)$ yields:

$$
\begin{aligned}
& V(X_T, \mathbf{Y}_T, \mathbf{Z}_T, T) - V(X_t, \mathbf{Y}_t, \mathbf{Z}_t, t) \\
= & \int_t^T \mathcal{L}_0^s V ds + \sum_{i=1}^N \int_t^T \mathcal{B}_{0i} V dI_{is}^c + \sum_{i=1}^N \int_t^T \mathcal{S}_{0i} V dD_{is}^c \\
& + \sum_{i=1}^N \int_t^T \frac{\partial V}{\partial y_i} \sigma_i Y_{is} d\hat{B}_{is}^S + \sum_{i=1}^N \int_t^T \frac{\partial V}{\partial z_i} \sigma_{zi}(s) d\hat{B}_{is}^S \\
& + \sum_{t \le s \le T} [V(X_s - (1+\theta)\Delta\mathbf{I}_s, \mathbf{Y}_s + \Delta\mathbf{I}_s, \mathbf{Z}_s, s) - V(X_s, \mathbf{Y}_s, \mathbf{Z}_s, s)] \\
& + \sum_{t \le s \le T} [V(X_s + (1-\alpha)\Delta\mathbf{D}_s, \mathbf{Y}_s - \Delta\mathbf{D}_s, \mathbf{Z}_s, s) - V(X_s, \mathbf{Y}_s, \mathbf{Z}_s, s)], \qquad \text{(A-1)}
\end{aligned}
$$

where all of the partial derivatives on the right-hand side are evaluated at point $(X_s, \mathbf{Y}_s, \mathbf{Z}_s, s)$. Taking expectation in the above formula, the first three terms on the right-hand side are non-positive due to the HJB equation, the fourth and fifth terms on the right-hand side equal zero

under regularity conditions, and the last two summations are non-positive by applying the mean-value theorem and using the HJB equation. Therefore, we have

$$V(X_t, \mathbf{Y}_t, \mathbf{Z}_t, t) \geq E[V(X_T, \mathbf{Y}_T, \mathbf{Z}_T, T)] = E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]. \tag{A-2}$$

Due to the arbitrariness of $(I_{it}, D_{it} : i = 1, ..., N)$, $V(x, \mathbf{y}, \mathbf{z}, t)$ must be no less than the value function $J(x, \mathbf{y}, \mathbf{z}, t)$.

We now proceed to the second step of the proof. Using the trading policy specified in this proposition, the expectations of all of the terms on the right-hand side of (A-1) become zero. Therefore, we have:

$$V(X_t, \mathbf{Y}_t, \mathbf{Z}_t, t) = E\left[\frac{(W_T^*)^{1-\gamma}}{1-\gamma}\right], \tag{A-3}$$

where W_T^* is the terminal net wealth level generated by the specified trading policy. Therefore, the definition of the value function indicates that $V(x, \mathbf{y}, \mathbf{z}, t)$ must be no greater than the value function $J(x, \mathbf{y}, \mathbf{z}, t)$.

In conclusion, $V(x, \mathbf{y}, \mathbf{z}, t)$ must coincide with the value function $J(x, \mathbf{y}, \mathbf{z}, t)$, and $(\mathbf{I}_t^*, \mathbf{D}_t^*)$ is the optimal trading policy. □

A.2 Proof of Proposition 2

Proof. In the absence of transaction cost, we can choose the investor's wealth W_t, instead of $(X_t, Y_{1t}, ..., Y_{Nt})$, as a state variable. The budget constraint on W_t is given by:

$$dW_t = W_t\left(rdt + \sum_{i=1}^{N} \pi_{it}(z_{it} - r)dt + \pi_{it}\sigma_i d\hat{B}_{it}^S\right), \tag{A-4}$$

where π_{it} is the fraction of total wealth invested in Stock i at time t, satisfying $\pi_{it} \geq 0$ due to the short-sale constraint. Let $\Phi(W, z_1, ..., z_N, t)$ be the value function, then the associated HJB

equation is:

$$\sup_{\pi_i \geq 0, 1 \leq i \leq N} \left\{ \frac{\partial \Phi}{\partial t} \quad + \quad (r + \sum_{i=1}^{N} (z_i - r)\pi_i) W \frac{\partial \Phi}{\partial W} + \frac{1}{2} \sum_{i=1}^{N} \pi_i^2 \sigma_i^2 W^2 \frac{\partial^2 \Phi}{\partial W^2} \right.$$
$$\left. + \quad \frac{1}{2} \sum_{i=1}^{N} \sigma_{zi}(t)^2 \frac{\partial^2 \Phi}{\partial z_i^2} + W \sum_{i=1}^{N} \pi_i \sigma_i \sigma_{z_i}(t) \frac{\partial^2 \Phi}{\partial W \partial z_i} \right\} = 0, \tag{A-5}$$

with the following terminal condition:

$$\Phi(W, z_1, ..., z_N, T) = \frac{1}{1-\gamma} W^{1-\gamma}. \tag{A-6}$$

Due to the homogeneity property of the CRRA preference and the linearity of Equation (A-4), there exists a function $h(z_1, ..., z_N, t)$, such that:

$$\Phi(W, z_1, ..., z_N, t) = \frac{W^{1-\gamma}}{1-\gamma} e^{(1-\gamma)(r(T-t) + h(z_1, ..., z_N, t))}. \tag{A-7}$$

By substitution, it is straightforward to show that $h(z_1, ..., z_N, t)$ satisfies the following equation:

$$\sup_{\pi_i \geq 0, 1 \leq i \leq N} \left\{ \frac{\partial h}{\partial t} \quad + \quad \sum_{i=1}^{N} \left((z_i - r)\pi_i - \frac{\gamma}{2} \pi_i^2 \sigma_i^2 \right) + \sum_{i=1}^{N} (\pi_i \sigma_i \sigma_{z_i}(t)(1-\gamma)) \frac{\partial h}{\partial z_i} \right.$$
$$\left. + \quad \frac{1}{2} \sum_{i=1}^{N} \sigma_{zi}(t)^2 \left(\frac{\partial^2 h}{\partial z_i^2} + (1-\gamma) \left(\frac{\partial h}{\partial z_i} \right)^2 \right) \right\} = 0, \tag{A-8}$$

with a terminal condition:

$$h(z_1, ..., z_N, T) = 0. \tag{A-9}$$

Now, suppose there are N functions $h^i(z_i)$, $i = 1, ..., N$, satisfying the following equation:

$$\sup_{\pi_i \geq 0} \left\{ \frac{\partial h^i}{\partial t} \quad + \quad (z_i - r)\pi_i - \frac{\gamma}{2} \pi_i^2 \sigma_i^2 + (\pi_i \sigma_i \sigma_{z_i}(t)(1-\gamma)) \frac{\partial h^i}{\partial z_i} \right.$$
$$\left. + \quad \frac{1}{2} \sigma_{zi}(t)^2 \left(\frac{\partial^2 h^i}{\partial z_i^2} + (1-\gamma) \left(\frac{\partial h^i}{\partial z_i} \right)^2 \right) \right\} = 0, \tag{A-10}$$

with a terminal condition:

$$h^i(z_i, T) = 0. \tag{A-11}$$

Define $H(z_1, ..., z_N, t) = \sum_{i=1}^{N} h^i(z_i, t)$, then it is easy to verify that $H(z_1, ..., z_N, t)$ satisfies Equation (A-8), and hence we have:

$$h(z_1, ..., z_N, t) = \sum_{i=1}^{N} h^i(z_i, t). \tag{A-12}$$

Therefore, the optimal allocation to Stock i is given by:

$$\pi_i(t, z_{it}) = \left(\frac{z_i - r + (1-\gamma)\sigma_i \sigma_{zi}(t) \frac{\partial h^i}{\partial z_i}(z_{it}, t)}{\gamma \sigma_i^2} \right)^+, \tag{A-13}$$

which only depends on the instantaneous estimate of Stock i's expected return, i.e., z_{it}, and the calendar time t. □

A.3 Are the disposition-effect related measures similar using the optimal trading strategy?

In this appendix, we demonstrate that, in some special cases where we can solve for the optimal rebalancing strategy, the disposition-effect related measures using the optimal trading strategies are similar to what we obtain in the main text. In particular, we consider the case in which there are only two stocks with constant expected returns. The expected return of the first (second) stock is observable (unobservable) to the investor. In this case, the associated value function involves three spacial variables and one temporal variable after dimensional reduction, and it is feasible to solve for the optimal strategy numerically.

We show in Table 7 the disposition-effect related measures generated by the approximately optimal rebalancing strategy or by the optimal rebalancing strategy. We focus on the measures in the full sample of sales. The results suggest that these disposition effect measures are indeed very close. For example, the optimal trading strategy generates a disposition effect measure of $DE = 0.360$, and the approximately optimal trading strategy generates a disposition effect measure of $DE = 0.347$. The percentage error is smaller than 4%. This suggests that our results obtained from the approximately optimal rebalancing strategy are reliable.

Table 7: Disposition effect-related measures

This table presents the disposition effect-related measures. The results are obtained from 10,000 simulated paths for each stock from Monte Carlo simulations of the model. Panel A indicates the average disposition effect measures generated by the approximately optimal trading strategy used in the main text; and Panel B shows the average disposition effect measures generated by the optimal trading strategy. The investor holds two stocks, both with constant expected return. The first stock's expected return is observable, while the second stock's is not. Parameter values: $T = 5$, $\gamma = 6$, $r = 0.01$, $N = 2$, $\mu_1 = 0.1$, $\sigma_1 = 0.3$, $\mu_2 = 0.06$, $z_{20} = 0$, $V_2(0) = 0.0016$, $\sigma_2 = 0.2$, $\alpha_i = \theta_i = 0.005$, for $i = 1, 2$. The symbol *** indicates a statistical-significance level of 1%.

	A: Approximately optimal strategy	B: Optimal strategy
Disposition effect		
PGR	0.603	0.608
PLR	0.256	0.248
DE	0.347***	0.360***
Reverse disposition effect		
PLPA	0.658	0.657
PGPA	0.352	0.346
RDE	0.306***	0.311***
Average holding time		
Gains	0.457	0.479
Losses	1.358	1.461
Ratio	3.049***	2.972***
***Ex-post* returns**		
84 days		
Sold winners	0.034	0.033
Held losers	0.023	0.023
Difference	0.011***	0.010***
252 days		
Sold winners	0.103	0.102
Held losers	0.067	0.068
Difference	0.036***	0.034***

A.4 Calculation of probability of selling and buying

To calculate the probability of selling or buying shares of any stock within certain ranges of realized return, we first choose a realized return bracket $[R_{min}, R_{max}]$ and divide it into equally spaced subintervals, so that $R_{min} = R_0 < R_1 < ... < R_n = R_{max}$. We simulate daily price data for each stock, and perform the following calculations: along each sample path ω, for each day t when a particular stock i is in the investor's portfolio, we compute the annualized continuously compounded return, $Ret_{it}(\omega)$, obtained by holding this stock by:

$$Ret_{it}(\omega) = \frac{1}{H_{it}(\omega)} \log\left(\frac{S_{it}(\omega)}{A_{it}(\omega)}\right),$$

where $S_{it}(\omega)$ is the spot price of this stock, $A_{it}(\omega)$ is the average purchase price of this stock, and $H_{it}(\omega)$ is the average holding time of this stock, all up to day t.[33]

We define three indicator functions for the corresponding events as follows:

$$s^j_{it}(\omega) = 1_{\{\text{Stock } i \text{ is sold on day } t \text{ and } R_j \leq Ret_{it}(\omega) < R_{j+1}\}},$$

$$b^j_{it}(\omega) = 1_{\{\text{More of Stock } i \text{ is bought on day } t \text{ and } R_j \leq Ret_{it}(\omega) < R_{j+1}\}}$$

and:

$$R^j_{it}(\omega) = 1_{\{R_j \leq Ret_{it}(\omega) < R_{j+1}\}}.$$

We then calculate the frequency:

$$f^S_j = \frac{\sum_{i,t,\omega} s^j_{it}(\omega)}{\sum_{i,t,\omega} R^j_{it}(\omega)} \tag{A-14}$$

as the probability that a sale takes place with the realized return in the jth bracket, and

[33] Because multiple purchases and sales of the same stock can occur along a sample path, we use the average holding time to annualize the returns to make them more comparable. To understand the mechanism of the average holding time system, consider the following simple example: assume on day 1, the investor purchases 10 shares; on day 11, the investor purchases five more shares. Then, the average holding time of each share is $(10 \times (11-1) + 5 \times 0)/(10+5) = 6.67$ days on day 11. Assume that the investor does not make any transaction between day 12 and 15, then on day 15, the average holding time is $(10 \times (15-1) + 5 \times (15-11))/(10+5) = 10.67$ days.

similarly:

$$f_j^B = \frac{\sum_{i,t,\omega} b_{it}^j(\omega)}{\sum_{i,t,\omega} R_{it}^j(\omega)}, \tag{A-15}$$

as the probability that a purchase takes place with the realized return in the jth bracket.

A.5 A model with capital gains tax

In the presence of (fully rebatable) capital gains tax, we need to keep tracking the total costs of purchasing each stock. Let K_{it} be the total costs of purchasing Stock i up to time t, the investor's budget constraints then read:

$$dX_t = rX_t dt + \sum_{i=1}^{N} f(Y_{it-}, K_{it-}) \frac{dD_{it}}{Y_{it-}} - \sum_{i=1}^{N} (1+\theta_i) dI_{it}, \tag{A-16}$$

$$dY_{it} = Y_{it} z_{it} dt + Y_{it} \sigma_i d\hat{B}_{it}^S + dI_{it} - dD_{it}, \tag{A-17}$$

$$dK_{it} = (1+\theta_i) dI_{it} - K_{it-} \frac{dD_{it}}{Y_{it-}}. \tag{A-18}$$

where:

$$f(Y_{it}, K_{it}) = (1-\alpha_i) Y_{it} - \tau \left[((1-\alpha_i) Y_{it} - K_{it}) \right] \tag{A-19}$$

is the total proceeds that the investor would obtain if she sold the entire position on Stock i at time t, and τ is the capital gains tax rate. Therefore, the investor's time t net wealth is:

$$W_t = X_t + \sum_{i=1}^{N} f(Y_{it}, K_{it}). \tag{A-20}$$

The investor's problem is again to choose her optimal policy $\{(D_{it}, I_{it}) : i = 1, ..., N\}$ among all of the admissible policies to maximize her expected CRRA utility from the terminal net wealth at some finite time T, i.e.:

$$E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right], \tag{A-21}$$

subject to Equations (4), (A-16), and (A-17), and (A-18), and the short-sale constraint as well as the solvency condition.