Stock Portfolio and Housing Choice when the Stock and Housing Markets are Cointegrated  

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ABSTRACT

The well documented stock-market nonparticipation and the highly negative correlation between stock and housing investment for the U.S. households are puzzling, because stock returns are high and the stock and housing markets have a low contemporaneous correlation. In this paper, we show that even though the stock and housing markets have a low contemporaneous correlation, the long-term stock and housing returns are significantly correlated because they are cointegrated. This long-term return correlation implies that households significantly increase housing expenditure, reduce stock investment, and may choose not to participate in the stock market at all if they face shortsale constraints. In addition, for a given level of wealth, the critical level of the participation cost above which households never participate in the stock market is much lower. Our model can thus potentially help explain both the puzzle of the stock-market nonparticipation and the puzzle of the highly negative correlation between stock and housing investment. We also show some empirical evidence that is supportive of the predictions of our model.

JEL classification: C02, G11

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1 Introduction

Only a small fraction of households participate—directly or indirectly through mutual funds—in the stock market. For example, in the United States, only 43% of the households own stocks either directly or indirectly (e.g., through retirement plans), while in India, the number is a mere 8%. This non-participation is puzzling, because standard models of lifetime consumption and portfolio choice predict that all households, no matter how risk averse they are or how much wealth they have, should invest in stocks (Samuelson 1969; Merton 1969, 1971; Arrow 1971). Another piece of empirical evidence is the highly negative correlation between housing investment and stock investment across countries and across time. For example, in 2015, the cross-country correlation between housing and stock investments is about -0.59 among 17 countries including developed countries like the U.S.A. and the UK, and developing countries like China and India, and the cross-time correlation is about -0.71 in the U.S.A. (See Appendix for details). This is also puzzling, because the contemporaneous correlation between stock and housing prices is low (about 0.07) and the standard theories predict low correlations between housing investment and stock investment. In this study, we show that stock and housing markets are cointegrated and this cointegration can help explain both of these puzzles. We also show some empirical evidence that is supportive of the predictions of our model.

We consider the optimal consumption and portfolio choice problem of a household in a continuous-time setting with a risk-free asset, a stock, and two consumption goods: housing and perishable goods, subject to shortsale constraints on the stock and housing. Unlike the existing literature, we study the impact of the cointegration between the stock and the housing markets on the optimal investment policy in the stock and the housing markets and the optimal perishable consumption policy. Calibrated to the U.S. data, our model shows that the presence of cointegration between stock and housing markets significantly affects households’ investment and consumption decisions. In particular, households may choose not to participate in the stock market even when there is no participation cost and the expected stock excess return is highly positive and significantly greater than the expected housing excess return. In addition, the participation cost needed for house-
holds to never participate in the stock market is significantly smaller than without cointegration. Moreover, even when households do participate in the stock market, the investment amount is significantly reduced because of the cointegration. Furthermore, we find that the stock investment and the housing investment are highly negatively correlated, even when the stock price and the house price are independent (and thus standard theories predict zero correlation between stock investment and housing investment). These results are robust to consideration of the high illiquidity in the housing market and the option of renting a house. Our model can thus potentially help explain the significant nonparticipation in stock markets and the highly negative correlation between stock investment and housing investment.

The main intuition is as follows. Even though the contemporaneous correlation between the stock and the housing returns is close to zero, the presence of cointegration is effectively increasing the long run correlation between the stock and the housing markets. For example, the correlation between 5-year stock and house returns equals 0.2841 and the correlation between 10-year stock and house returns equals 0.4589. Therefore, there is a stronger substitution effect between the two in the long run. In particular, when the conditional expected return of housing is high relative to that of the stock, households optimally borrow in the riskfree market to increase the house size. In these states, households would like to short sell the stock to finance the purchase of a house of an even greater size, but due to the shortsale constraints, the best households can do is to stay away from the stock market. This is why households may choose nonparticipation in the stock market even if the unconditional expected return in the stock market is much greater than that in the housing market and there is no participation cost. Even when households do participate in the stock market, they invest less than in the case without cointegration, because of the substitution effect of owning a house and possibly the extra consumption benefit of investing in a house. When there is a participation cost, because of the substitution effect of investing in a house, the critical participation cost above which households choose never to participate in the stock market is much smaller than

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1 This is consistent with Fischer and Stamos (2013) and Corradin et al. (2014). Fischer and Stamos (2013) show that the households choose higher housing-to-net-worth ratio in good states of housing market cycles (Table 3 and Figure 1). Corradin et al. (2014) show that the housing portfolio share immediately after moving to a more valuable house is higher during periods of high expected growth in house prices (Figure 4, Table 3, and Table 7).
that when there is no cointegration. The highly negative correlation between stock investment and house investment implied by our model also follows from the substitution effect of housing investment for stock investment in the long run. In addition, if housing is also a consumption good, then housing has a dual role: consumption and investment. This dual role magnifies the demand for housing and reduces stock investment further. Allowing a housing rental market may make our results even stronger, because with access to the rental market, households may optimally choose to buy even bigger houses (and further reduce stock investment) and rent out part of the houses. This way households can enjoy more the benefit of greater housing investment when the conditional expected return of housing is high relative to that of the stock. The incorporation of housing market illiquidity does not change our results significantly and can even enhance them, because with illiquidity in the housing market, households stay in the houses longer and thus what matters is the correlation between the stock and housing in a longer run, which is greater and thus the substitution effect is greater.

To the best of our knowledge, although various types of cointegration between the stock and housing markets have been found in the existing literature (see e.g. Anoruo and Braha 2008, Tsai et al. 2012), this paper is the first to study how this cointegration affects household investment behavior and can help explain the puzzle of non/limited participation in stock markets and the puzzle of highly negative correlation between stock and housing investment. In addition, different from the existing literature on the cointegration tests for the two markets, we are the first to use the Johansen trace test to establish the cointegration in the form of the stationarity of the log of the ratio of the housing price to the stock price raised to an empirically estimated power.

One prediction of our model is that as the degree of cointegration between housing and stock markets increases, stock investment decreases and stock-market nonparticipation increases. To see if this prediction has empirical support, we utilize the U.S. cross-state variations of the degree of cointegration, stock investment and the stock-market nonparticipation to examine the relationship among the three. Using PSID data at family level in 2015 and 2017 waves, we calculate the average value of equity in stocks, the average ratio of financial wealth invested in stocks, and the proportion
of interviewed families that do not invest in stocks for each state. We find that consistent with our model prediction, as the degree of cointegration increases, stock investment decreases and non-participation in the stock market increases.

In the existing literature, there are several existing explanations for the nonparticipation puzzle. Vissing-Jørgensen (2002) shows that a moderate participation cost can explain half of the nonparticipation observed in data. Cocco (2005) finds that housing crowds out stock holdings which, together with a sizable stock market entry cost, can explain stock market nonparticipation early in life. Yao and Zhang (2005) examine the substitution and diversification effect of equity investment through an optimal dynamic portfolio decision model for households who acquire housing services from either renting or owning a house. They predict that housing investment has a negative effect on stock market participation. Kraft et al. (2017) solve a rich life-cycle model of household decisions. After considering housing habit, they obtain that stock investments are low or zero for many young agents and then gradually increase as they age. Linnainmaa (2005) argues that short-sale constraints combined with learning can generate nonparticipation even when the constraints are not binding at present. Ambiguity aversion, disappointment aversion, and behavioral, cognitive and psychological constraints are also offered as possible explanations of the nonparticipation puzzle (e.g., Epstein and Schneider 2008; Cao, Wang, and Zhang 2005; Ang, Bekaert, and Liu 2005; Andersen and Nielsen 2011). Our model complements these extant theories by incorporating the cointegration between the stock and the housing markets, and may strengthen their explanatory power. For example, our model suggests that the participation cost in Vissing-Jørgensen (2002) and ambiguity aversion in Cao, Wang, and Zhang (2005) required to explain nonparticipation would be significantly smaller if cointegration were incorporated.

Our paper also relates to recent papers on housing decisions. Hemert (2010) investigates household interest rate risk management by solving a life-cycle asset allocation model that includes mortgage and bond portfolio choice and finds some hedge between housing and interest rate. Fischer and Stamos (2013) set up a regime switching model with slow-moving time variation in expected housing returns and find that homeownership rates and the share of net worth in home increase in
good states of housing market cycles. Corradin et al. (2014) show that higher expected growth rates in house prices cause house (stock) investment to increase (decrease), but stock investment is still significant even with a high risk aversion.

As for the puzzle on the highly negative correlation between stock and housing ownership/investment, although some studies (e.g., Cocco 2005; Yao and Zhang 2005) imply a negative relationship, no extant studies have shown whether the magnitudes of the correlations in their models can be as large as those observed in data.

The paper is organized as follows. Section 2 describes the benchmark cointegration model. Section 3 provides estimation of cointegration parameter values. In Section 4 we quantitatively illustrate that the cointegration between stock and house markets leads to non/limited participation in stocks and highly negative correlation between stock and housing investment. Section 5 demonstrates the robustness of our results to the option of renting and to the presence of house illiquidity. In Section 6 we provide empirical evidence that is supportive of the predictions of our model. Section 7 concludes the paper. Some empirical facts on nonparticipation and correlations, and HJB equations plus all the proofs are provided in the Appendix.

2 The Model

We consider a continuous time model where an investor maximizes her expected utility from consuming a perishable consumption good and also possibly from consuming the service flow provided by a house. In addition to trading houses and the perishable consumption good in the goods markets, the investor can also trade a risky stock and a risk-free bond in the financial market without any transaction costs.
2.1 Financial Markets

The bond grows at a constant risk-free rate $r$. The stock’s price $S_t$ evolves according to the following dynamics:

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dB_{St},$$

(2.1)

where $\mu_S > r$ is a constant representing the instantaneous expected return, $\sigma_S$ is a constant representing the instantaneous volatility of stock return, and $B_{St}$ is a one-dimensional standard Brownian motion.

2.2 The housing market

To simplify the analysis, we start by assuming that by selling, buying, remodelling, and expansion, the investor can continuously adjust the house size without transaction costs. The housing size depreciates at a rate of $\delta \geq 0$, i.e., if the investor does not adjust the house size $A_t$, the dynamics of $A_t$ follows

$$dA_t = -\delta A_t dt.$$  (2.2)

Let $H_t$ denote the price of house per square footage. Different from the existing literature, we allow the stock and housing markets to be cointegrated. More specifically, in a similar spirit to that in Benzoni et al. (2007), we assume that the following log ratio denoted by

$$I_t = \log H_t - \lambda \log S_t$$  (2.3)

for some positive constant $\lambda$, follows a mean-reverting process

$$dI_t = k(\bar{I} - I_t) dt + \sigma_H dB_{Ht} - \nu_S dB_{St},$$  (2.4)

2. When houses are indivisible and buying/selling a house incurs transaction cost like in Grossman and Laroque (1990), the investor’s problem becomes much more complicated and is considered in Section 5.2 to show the robustness of our results.
where the constant $k \geq 0$ measures the degree of cointegration, $\bar{I}$ denotes the long-term mean, $\sigma_H$ and $\nu_S$ are conditional volatilities, $B_{Ht}$ is another standard Brownian motion reflecting the uncertainty in housing price and is independent of $B_{St}$. Our specification (2.4) for the log ratio $I_t$ reflects a long-run cointegration between the stock and housing markets, because it implies that the log ratio of the house price to the stock price raised to the power of $\lambda$ tends to the long run mean $\bar{I}$ as time passes. When $I_t > \bar{I}$, the house price tends to decrease over time relative to the stock price, whereas when $I_t < \bar{I}$, the opposite is true.

By Ito’s formula and (2.1), we have

$$\frac{dS_{t}^{\lambda}}{S_{t}^{\lambda}} = \left(\lambda \mu_S + \frac{1}{2} \lambda (\lambda - 1) \sigma_S^2\right) dt + \lambda \sigma_S dB_{St}.$$  \hspace{1cm} (2.5)

From (2.3), we find that $H_t = e^{h S_{t}^{\lambda}}$, which yields that

$$\frac{dH_{t}}{H_{t}} = \mu_H(I_t)dt + \sigma_H dB_{Ht} + (\lambda \sigma_S - \nu_S) dB_{St},$$  \hspace{1cm} (2.6)

where

$$\mu_H(I) = \mu_{H0} + k(\bar{I} - I), \quad \mu_{H0} = \lambda \mu_S + \frac{1}{2} \sigma_H^2 + \frac{1}{2} \nu_S^2 + \frac{1}{2} \lambda (\lambda - 1) \sigma_S^2 - \lambda \sigma_S \nu_S.$$  \hspace{1cm} (2.7)

In this paper, we focus on the analysis of the effect of the cointegration between the housing and the stock markets. Accordingly, we will assume that the contemporaneous correlation between housing and stock returns is zero, i.e., $\lambda \sigma_S = \nu_S$. In the next section, we show that $I_t$ for some empirically estimated value of $\lambda$ is indeed mean-reverting. As in the existing literature, we find that the contemporaneous correlation between housing and stock returns is almost zero. The focus of cointegration between stock and housing markets distinguishes our model from others that ignore such cointegration (see e.g., Fischer and Stamos 2013; Corradin et al. 2014).

We assume the independence of housing and stock risks since the correlation $\log \left(\frac{H_{t}}{H_{t-1}}\right)$ and $\log \left(\frac{S_{t}}{S_{t-1}}\right)$ is only 7.14%, as shown in Section 3. Moreover, setting the correlation to even 10% would not change the main quantitative results in this subsection.
Note that with $\lambda \sigma_S = \nu_S$, although the contemporaneous correlation between housing and stock returns is zero, the housing and the stock markets are linked. For example, after a positive shock in $B_{St}$ to the stock return, the log ratio $I_t$ decreases (equation (2.4)), which in turn increases the conditional expected return of housing $\mu_H(I_t)$ (equation (2.7)). In this sense, the stock and the housing markets tend to move together when they are positively cointegrated.

2.3 Preferences

The investor derives utility not only from the perishable consumption good that serves as the numeraire but also possibly from the housing service flow that is proportional to the size of the house. Thus different from the financial assets, in addition to the role of an investment vehicle, increasing the size of the house also directly increases utility. Following the existing literature (see e.g. Damgaard et al. 2003; Cocco et al. 2005; Yao and Zhang 2005; Kraft and Munk 2011; Fischer and Stamos 2013; Corradin et al. 2014), we assume that the investor’s preferences over the housing and non-housing goods are represented by the nonseparable Cobb-Douglas utility function, which takes the following form

$$U(C, A) = \frac{1}{1 - \gamma} \left( C^{1-\theta} A^{\theta} \right)^{1-\gamma},$$

where $A$ now represents the service flow from the house, $^4 C$ represents the perishable consumption, $\theta \geq 0$ measures the preference for housing relative to non-housing consumption goods, and $\gamma > 0$ is the constant relative risk aversion coefficient.

2.4 The investor’s optimization problem

Let $W_t$ be the total wealth in bonds, stocks, and housing measured in units of the perishable consumption good at time $t$, and $\zeta_t$, $h_t$, and $c_t$ denote the fraction of the total wealth $W_t$ in stock, housing, and perishable goods, respectively. According to equations (2.1), (2.2), and (2.6), $W_t$

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$^4$We assume the service flow is proportional to the house size and equal to $\alpha A_t$, where $\alpha$ is set to 1 without loss of generality.
satisfies the following stochastic differential equation\(^5\)

\[
d\frac{W_t}{W_t} = [r - c_t + \zeta_t(\mu_S - r) + h_t(\mu_H(I_t) - \delta - r)] dt + \sigma_S \zeta_t dB_{St} + \sigma_H h_t dB_{Ht}.
\] (2.8)

Following the existing literature (e.g., Gomes and Michaelides 2005; Cocco et al. 2005; Polkovnichenko 2007; Munk and Sørensen 2010; Wachter and Yogo 2010; Lynch and Tan 2011; Flavin and Yamashita 2011), we assume that the investor cannot short sell stock or houses, that is, \(\zeta_t \geq 0\) and \(h_t \geq 0\).\(^6\) However, the investor can borrow against her house, up to a fraction \((1 - l)\) of the current value of housing, i.e., \(\zeta_t + lh_t \leq 1\), where \(l \in (0, 1)\) is a constant, representing the maximum leverage allowed.

The investor chooses the perishable consumption fraction \(c_t\), the stock portfolio weight \(\zeta_t\), and the housing portfolio weight \(h_t\) to maximize her expected utility from consumptions of the perishable good and the housing service from time 0 to the first jump time \(T\) of an independent Poisson process with intensity \(\delta_M\), which represents the mortality rate of the investor.\(^7\) Let \(\mathcal{A}\) denote the set of all admissible strategies \((c_t, \zeta_t, h_t)\), i.e., the strategies that satisfy the budget constraint (2.8), the shortsale constraint \(\zeta_t \geq 0\) and \(h \geq 0\), and the limited borrowing constraint \(\zeta_t + lh_t \leq 1\), for given processes (2.4) and (2.6). Noting that \(h_t = \frac{A_t}{W_t}\), the value function is then defined as

\[
\Psi(W, H, I) : = \max_{(c_t, \zeta_t, h_t) \in \mathcal{A}} \mathbb{E} \left[ \int_0^T e^{-\beta t} \left( c_t^{\frac{1-\theta}{1-\gamma}} h_t^{\theta} W_t \right)^{1-\gamma} dt \right] \\
= \max_{(c_t, \zeta_t, h_t) \in \mathcal{A}} \mathbb{E} \left[ \int_0^\infty e^{-\left(\beta + \delta_M\right) t} \left( c_t^{\frac{1-\theta}{1-\gamma}} h_t^{\theta} W_t \right)^{1-\gamma} dt \right],
\] (2.9)

\(^5\)In an earlier version of the paper, we also solved a model with stochastic labor income and obtained the same qualitative results. This analysis is not reported in this version to save space, but available from the authors.

\(^6\)Note that the optimal \(c_t\) must be strictly positive because of the CRRA utility function form.

\(^7\)The assumption of a random horizon reduces the time dependence of the optimal strategies. Using a deterministic horizon would not change our qualitative results. In addition, as Liu and Lowenstein (2002) suggest, the optimization problem with a random horizon can be a good approximation for a deterministic horizon when the expected horizon is long.
which satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

\[
\max_{c, \zeta, h \geq 0, \zeta + h \leq 1} \left\{ \left( \frac{1}{2} \sigma_S^2 \zeta^2 + \frac{1}{2} \sigma_H^2 h^2 \right) W^2 \Psi_{WW} + \frac{1}{2} \sigma_S^2 H^2 \Psi_{HH} + \frac{1}{2} (\lambda^2 \sigma_S^2 + \sigma_H^2) \Psi_{II} 
+ (\sigma_H^2 h - \lambda \sigma_S^2 \zeta) W \Psi_W + \sigma_H^2 H h W \Psi_{WH} + \sigma_H^2 H \Psi_{HI} 
+ [r - c + (\mu_S - r) \zeta + (\mu_H(I) - \delta - r) h] W \Psi_W 
+ \mu_H(I) H \Psi_H + k(\bar{I} - I) \Psi_I - (\beta + \delta_M) \Psi + \frac{(c^{1-\theta}(h/H)^\theta W)^{1-\gamma}}{1-\gamma} \right\} = 0
\]

for \( W > 0, H > 0, \) and \( I \in \mathbb{R} \).

Using the homogeneity property of the value function, we can reduce the dimensionality of the problem to one by the following transformation:

\[
\Psi(W, H, I) = \frac{1}{1-\gamma} W^{1-\gamma} H^{-\theta(1-\gamma)} e^{(1-\gamma) u(I)},
\]

for some function \( u(\cdot) \). It can be shown that the function \( u(I) \) satisfies:

\[
\max_{c, \zeta, h \geq 0, \zeta + h \leq 1} \left\{ \frac{1}{2} (\lambda^2 \sigma_S^2 + \sigma_H^2) [u'' + (1-\gamma) u']^2 + \left[ (\sigma_H^2 h - \lambda \sigma_S^2 \zeta - \theta \sigma_H^2) (1-\gamma) + k \bar{I} - k I \right] u' 
- \frac{1}{2} \gamma (\sigma_S^2 \zeta^2 + \sigma_H^2 h^2) + \frac{1}{2} \sigma_H^2 \theta (\theta (1-\gamma) + 1) - \sigma_H^2 \theta (1-\gamma) h + r - c + (\mu_S - r) \zeta 
+ (\mu_H(I) - \delta - r) h - \theta \mu_H(I) - \frac{\beta + \delta_M}{1-\gamma} + \frac{1}{1-\gamma} e^{(1-\theta)(1-\gamma) I^\theta(1-\gamma) e^{-(1-\gamma) u}} \right\} = 0
\]

for \( I \in \mathbb{R} \). With this formulation, we simplify the problem to solving (2.10) for \( u(I) \). Because of the presence of cointegration (i.e., \( k \neq 0 \)), the choice of the stock portfolio weight \( \zeta \) and the choice of the housing investment weight are jointly determined because both depend on the function \( u(I) \) and its derivatives. In the special case where \( k = 0 \), i.e., there is no cointegration between stock and house price, \( u(I) \) is a constant and thus the choice of the stock portfolio weight \( \zeta \) and the choice of the housing investment weight \( h \) are independent. We have the following lemma in this special case.\(^8\)

\(^8\)Similar to the Merton’s problem, to guarantee the existence of a solution, we impose (2.11), because if this fails, then the household can achieve unbounded utility by delaying consumption.
Lemma 1. Suppose $k = 0$ and

$$\beta + \delta_M - (1-\gamma) \left\{ r + \frac{1}{2} \sigma_H^2 \theta (\theta (1-\gamma) + 1) - \mu_{H_0} \theta + \frac{(\mu_S - r)^2}{2 \gamma \sigma_S^2} + \frac{\mu_{H_0} - \delta - r - \sigma_H^2 \theta (1-\gamma))^2}{2 \gamma \sigma_H^2} \right\} > 0.$$ 

We have $\Psi(W, H, I) \equiv K_1 - \gamma W \left( 1-\gamma \right) H^{-\theta (1-\gamma)}$, where

$$K = (h^*)^{\theta (1-\gamma)} \eta^{1-p} \left[ -\frac{1}{2} \gamma \sigma_S^2 (\zeta^*)^2 + (\mu_S - r) \zeta^* - \frac{1}{2} \gamma \sigma_H^2 (h^*)^2 + (\mu_{H_0} - \delta - r - \sigma_H^2 \theta (1-\gamma)) h^* \right. \\
\left. + \frac{1}{2} \sigma_H^2 \theta (\theta (1-\gamma) + 1) + r - \mu_{H_0} \theta - \frac{\beta + \delta_M}{1-\gamma} \right]^{-(1-p)},$$

with $p = (1-\theta)(1-\gamma)$, $\eta = (1-\theta) \frac{1}{1-\gamma}$, and $(\zeta^*, h^*)$ being the optimal stock and house investment satisfying

$$\left\{ -\frac{1}{2} \gamma \sigma_S^2 s^2 + (\mu_S - r) s - \frac{1}{2} \gamma \sigma_H^2 h^2 + (\mu_{H_0} - \delta - r - \sigma_H^2 \theta (1-\gamma)) h \\
- K \frac{1}{1-\gamma} \eta h^{\theta (1-\gamma)} \right\}.$$ 

3 Cointegration Test

In this section, we aim to test whether there is cointegration between stock and housing markets and estimate the cointegration degree if there is. The sources of the stock market data and the house price series are the Standard & Poor’s and Case-Shiller home price index respectively, both of which are inflation adjusted to November 2019 dollars. We use the annual data on December 1 from 1890 to 2017.

Before conducting the test, we first estimate the contemporaneous correlation between the housing and stock returns in our data set. We find that the correlation between $\log \left( \frac{S_t}{S_{t-1}} \right)$ and $\log \left( \frac{H_t}{H_{t-1}} \right)$ is 7.14%, which is consistent with the findings in the existing literature and leads to our simplifying assumption that housing price and stock price have zero contemporaneous correlation, i.e., $\lambda \sigma_S = \nu_S$, made to highlight the effect of cointegration.
We then apply the two-step approach in Engle and Granger (1986) to test the existence of cointegration effect between housing and stock returns. In the first step the normalized cointegration vector \((1, -\lambda)\) is calculated by OLS (i.e., a linear regression model) on \(\log H_t\) and \(\log S_t\). In the second step a stationary test (e.g. Augmented Dickey-Fuller test) is run on the residuals of linear regression in first step. If the residuals turn out to be stationary, then the null hypothesis that there is no cointegration should be rejected. The testing result on our data set shows that there exists cointegration between housing and stock returns and the OLS estimator \(\hat{\lambda}_{OLS}\) equals 0.2471. However, one of the disadvantages of Engle-Granger OLS estimator is that \(\hat{\lambda}_{OLS}\) can be substantially biased in small samples and not efficient in the second step. As an improvement, Johansen (1988,1991) proposed the trace test and the maximal eigenvalue test.

We use the trace test proposed by Johansen (1988,1991) to examine whether the stock and the housing markets are cointegrated, because this test is an improvement over the two-step test proposed by Engle and Granger (1986). The normalized cointegration vector \((1, -\lambda)\) is estimated by MLE, which is asymptotically normal and super consistent. Using this method on our data set, we find that the Johansen MLE estimator \(\hat{\lambda}_{MLE}\) equals 0.2695. We set \(\lambda = \hat{\lambda}_{MLE} = 0.2695\) in benchmark calibration throughout the paper.\(^9\) The trace test shows that the residual process \(I_t = \log H_t - \lambda \log S_t\) follows an AR(1) model:

\[
I_{t+\Delta t} = m + \phi I_t + \epsilon_{t+\Delta t},
\]

where \(\Delta t\) is the time between adjacent observations with \(m = 0.5998\) and \(\phi = 0.8180\). Then, we can compare equations (2.4) and (3.1) to imply the speed of the mean-reversion coefficient \(k\), the mean \(\bar{I}\), and variance \(\sigma_H^2\) in equation (2.4). Observe that equation (2.4) can be written as

\[
dI_t = k(\bar{I} - I_t)dt + \sqrt{\sigma_H^2 + \nu_S^2}dB_{Ht},
\]

\(^9\)We also conducted the two-step Engle and Granger test. The results are similar. For example, the estimate for \(\lambda\) in this alternative test is 0.2471 and setting \(\lambda = 0.2471\) in the benchmark calibration does not significantly change our main result.
where $B_t$ is a new Brownian motion. Equation (3.2) is equivalent to

$$d(I_t e^{kt}) = \bar{I}d(e^{kt}) + \sqrt{\sigma^2_H + \nu_S^2}e^{kt}dB_t,$$

which in discrete time can be approximated by

$$I_{t+\Delta t}e^{k\Delta t} - I_t = \bar{I}(e^{k\Delta t} - 1) + \sqrt{\sigma^2_H + \nu_S^2} \sqrt{\frac{1}{2k}(e^{2k\Delta t} - 1)} \varepsilon_t,$$

where $\varepsilon_t \sim N(0, 1)$. Comparing with (3.1), we have

$$k = -\log(\phi)/\Delta t,$$

$$\bar{I} = m/(1 - e^{-k\Delta t}),$$

$$\sigma^2_H = \frac{2k \text{Var}(\varepsilon_t)}{1 - e^{-2k\Delta t}} - \nu_S^2.$$

Moreover, based on equation (2.7), we can get the value of $\mu_{H0}$.

Figure 1 shows the probability distribution function of $\mu_H(I)$ based on the historic data from 1890 to 2017, which we use to produce the default parameter values. From this figure, we find that from 1890 to 2017, the values of $\mu_H(I)$ were concentrated in the interval $(-0.0785, 0.0709)$.

![Figure 1: Probability distribution function of $\mu_H(I_t)$.](image)

As noted before, even if the stock and housing markets are contemporaneously uncorrelated,
the two markets will be correlated for a longer horizon if they are cointegrated. To see if the set of parameter values estimated above are reasonable, we next compute the model implied long term correlations between stock and housing returns for horizons of 1 year, 5 years, and 10 years. We then compare these correlations with the corresponding empirical correlations in the data. Table 1 shows that the model implied correlations match well with the empirical correlations, which suggests the estimated model reflects the data reasonably well.

4 Numerical Analysis

4.1 Parameter Values

In Table 2 we report the default parameter values for our numerical analysis. We set the interest rate after inflation adjusted at $r = 0.59\%$ based on the 5 year real interest rates, the stock risk premium at 2.81\% (i.e., the stock return $\mu = 3.39\%$), and the standard deviation of stock return at $\sigma_S = 14.11\%$, according to the estimate result of the Standard & Poors 500 index portfolio inflation adjusted to November 2019 dollars. The coefficient of relative risk aversion is set at $\gamma = 10$ to approximately match the stock holdings relative to financial wealth observed in the Panel Study of Income Dynamics (PSID) and Survey of Income and Program Participation (SIPP) survey in literature. The parameter $\theta$ that measures the degree to which the household values housing consumption relative to non-housing consumption, is set at 0.3 to be consistent with the average share of household housing expenditure in the United States (see e.g., Corradin et al. 2014). Households can short bonds to finance homeownership and the minimum housing down payment for homeowners is 20\%, which implies that $l = 0.2$. The value of $\lambda, k, \bar{I},$ and $\sigma_H$ are
estimated in Section 3.

Table 2: **Parameter values used for benchmark calibration**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Riskless rate</td>
<td>$r$</td>
<td>0.0059</td>
</tr>
<tr>
<td>Mean return on stocks</td>
<td>$\mu_S$</td>
<td>0.0339</td>
</tr>
<tr>
<td>Std. stock return</td>
<td>$\sigma_S$</td>
<td>0.1411</td>
</tr>
<tr>
<td>Weight of cointegration</td>
<td>$\lambda$</td>
<td>0.2095</td>
</tr>
<tr>
<td>Degree of mean reversion</td>
<td>$k$</td>
<td>0.1976</td>
</tr>
<tr>
<td>Long-term mean</td>
<td>$\bar{I}$</td>
<td>3.2162</td>
</tr>
<tr>
<td>Long-term Std. house return</td>
<td>$\sigma_H$</td>
<td>0.0791</td>
</tr>
<tr>
<td>Long-term mean house return</td>
<td>$\mu_{H0}$</td>
<td>0.0096</td>
</tr>
<tr>
<td>Time preference rate</td>
<td>$\beta$</td>
<td>0.0059</td>
</tr>
<tr>
<td>Mortality rate</td>
<td>$\delta_M$</td>
<td>0.05</td>
</tr>
<tr>
<td>Risk aversion</td>
<td>$\gamma$</td>
<td>10</td>
</tr>
<tr>
<td>The preference of housing</td>
<td>$\theta$</td>
<td>0.3</td>
</tr>
<tr>
<td>Depreciation rate on house size</td>
<td>$\delta$</td>
<td>0</td>
</tr>
<tr>
<td>Collateral parameter</td>
<td>$l$</td>
<td>0.2</td>
</tr>
</tbody>
</table>

### 4.2 Optimal Investment and Consumption Policies

This section demonstrates how cointegration between stock and housing markets affects optimal investment and consumption policies. In particular, we show that comparing with a model that ignores cointegration, investors invest significantly less in the stock market and significantly more in housing. With cointegration, they may even choose not to participate in the stock market at all, even when there is no participation cost and the unconditional expected return of the stock is greater than that of housing. In addition, because of the cointegration, the stock investment and the housing investment display a strong negative correlation over time. Our model can thus help explain the observed non/limited participation in stock markets and the strong negative correlation between stock investment and housing investment.

#### 4.2.1 Correlation between stock and housing investment

The optimal investment in the stock, in housing, and in the bond are all functions of the conditional expected return of housing $\mu_H(I)$. Accordingly, in Figure 2 we plot the optimal investment in housing and stock (Panels (i) and (ii)) and in bond (Panel (iii)) against $\mu_H(I)$ with and without cointegration. Panel (i) of Figure 2 suggests that when stock investment increases, housing
Figure 2: **Optimal investment policy.** In panels (i) and (ii), $h^*$ (the blue-solid line) is the optimal weight of house investment and the blue-dotted line is the optimal weight of house investment when $k = 0$. They are plotted on the right vertical axis. $\zeta^*$ (the red-solid line) is the optimal weight of stock investment and the corresponding red-dotted line is the optimal weight of stock investment when $k = 0$. They are plotted on the left vertical axis. Panel (iii) plots the optimal weight of bond investment. Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{I} = 3.2162$, $\sigma_H = 0.0791$, $\mu_{H_0} = 0.0096$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$. 

(i) Optimal stock and house investment

(ii) Optimal stock and house investment, $\theta = 0$

(iii) Optimal bond investment
investment decreases, and vice versa. This pattern suggests that consistent with empirical evidence illustrated in the Appendix, housing investment and stock investment are negatively correlated for each household over time. Intuitively, in the presence of the cointegration effect (i.e., $k > 0$), the expected return of housing is time varying and stochastic. Although the long term expected return of housing $\mu_{H0}$ is low, the conditional expected return of housing can be high relative to that of the stock in some states (e.g., when the log ratio $I_t$ is low). Knowing that housing and stock markets are cointegrated and thus correlated in long term, the investor levers up more to increase the house size by borrowing more (as indicated by the corresponding negative bond holdings in Panel (iii)) and decreasing stock investment in these states, consistent with the finding of Fischer and Stamos (2013). When the conditional expected return of housing $\mu_H(I)$ is low, the reverse is true. These changes in the relative conditional expected returns and the cointegration between stock and housing cause the negative correlation between stock investment and housing investment.

One may suspect that the increase in the housing investment and the decrease in the stock investment are mainly due to the assumption that housing is not only an investment vehicle but also a consumption good. Panel (ii) of Figure 2 suggests that there is still a negative correlation between stock investment and housing investment even when the investor does not derive utility directly from housing (i.e., when $\theta = 0$). In contrast, if there were no cointegration, then the investor would always invest constant and positive fractions of wealth in stock and in housing, as indicated by the dashed lines in Panels (i) and (ii). Thus, without cointegration, the correlation between the fractions of wealth invested in stock and housing would be zero. Therefore, it is not the additional role of housing as a consumption good that drives the negative correlation result, rather, the key driver is the cointegration between the stock and the housing markets. The existing literature ignores the cointegration and as a result, given the low contemporaneous correlation between the stock and housing returns, it cannot explain the highly negative correlation between stock and housing investment. This contrast with the existing literature suggests the importance of cointegration in helping solve the negative correlation puzzle.

We next estimate the magnitude of the correlation implied by our model through simulations.
For this purpose, we set the initial log ratio $I_0$ to be its long-term average $\bar{I}$, then simulate 10,000 paths of the process $I_t$ by (2.4), compute stock investment, housing investment, and their correlation for each path, and then average them across all paths. We present the obtained results in Table 3 for various parameter values. Consistent with Figure 2 and empirical evidence, we find a highly negative correlation between housing and stock investment, for example, -0.8496 in the base case. When there is no cointegration, however, the correlation is zero. In addition, the presence of cointegration decreases average stock investment and increases average housing investment. Thus the presence of cointegration may help explain the puzzle of a highly negative correlation between stock and housing investment. When the investor has a greater preference for housing service consumption (i.e., $\theta$ is larger), stock investment decreases, housing investment increases, and the correlation between the two becomes more negative. When the investor cannot borrow against house equity (e.g., $l = 1$), the housing investment decreases and the stock investment increases.

With a lower risk aversion (e.g., $\gamma = 5$), both the housing investment and the stock investment increase. In both cases, the stock market nonparticipation region shrinks and thus the magnitudes of the correlation increase.

Table 3: **Simulation.** Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{I} = 3.2162$, $\sigma_H = 0.0791$, $\mu_{H0} = 0.0096$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Stock Investment ($\zeta^*$)</th>
<th>House Investment ($h^*$)</th>
<th>Investment Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Median</td>
<td>Std</td>
</tr>
<tr>
<td><strong>Base case</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$k = 0$</td>
<td>0.1412</td>
<td>0.1412</td>
<td>0</td>
</tr>
<tr>
<td>$k = 0.1976$</td>
<td>0.0109</td>
<td>0.0090</td>
<td>0.0079</td>
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<tr>
<td>$\theta = 0$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$k = 0$</td>
<td>0.1412</td>
<td>0.1412</td>
<td>0</td>
</tr>
<tr>
<td>$k = 0.1976$</td>
<td>0.0264</td>
<td>0.0249</td>
<td>0.0113</td>
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<tr>
<td>$\theta = 0.6$</td>
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<td></td>
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<tr>
<td>$k = 0$</td>
<td>0.1412</td>
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<tr>
<td>$k = 0.1976$</td>
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<tr>
<td>$l = 1$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$k = 0$</td>
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<tr>
<td>$k = 0.1976$</td>
<td>0.0494</td>
<td>0.0508</td>
<td>0.0143</td>
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<tr>
<td>$\gamma = 5$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$k = 0$</td>
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<td>0.2823</td>
<td>0</td>
</tr>
<tr>
<td>$k = 0.1976$</td>
<td>0.0623</td>
<td>0.0624</td>
<td>0.0275</td>
</tr>
</tbody>
</table>
4.2.2 Stock market nonparticipation

Panel (i) of Figure 2 also shows that even with an adjusted stock risk premium at 2.81% and no participation cost for stock investment, an investor may choose not to participate in the stock market at all, and “underinvest” even when she does choose to participate, which is consistent with the empirically documented non/limited stock market participation. In particular, when $\mu_H(I) > \mu^*_H$, the threshold value of the conditional expected return of housing for nonparticipation, the investor optimally chooses not to participate (i.e., $\zeta^*_t = 0$). The probability of non-participation in the stock market can be written as

$$P(\mu_H(I) \geq \mu^*_H), \quad (4.1)$$

where the distribution of $I$ is shown in Fig 1. In Panel (i) of Figure 2, the non-participation threshold $\mu^*_H$ equals 0.00, which implies that the probability of non-participation equals 0.6. Thus, there is a significant probability of non-participation even when the beta of the housing market is zero and the unconditional expected market risk premium of housing $\mu_H - \delta$ is near zero. The intuition is straightforward. Given the cointegration between stock and housing, the investor levers up to increase the house size by borrowing more and decreasing stock investment in the states where the conditional expected return of housing is high. In addition, even though the expected stock return is high compared to housing return, the investor still would like to short sell the stock if possible to provide funds for further increasing the housing size. Because of the shortsale constraint, however, the best the investor could do is to stop participating in the stock market. When the conditional expected return of housing decreases (equivalently when $I_t$ rises), the investor reduces investment in housing and begins to invest in the stock for the relatively higher expected return in the stock.

Figure 2 suggests that there is still stock market nonparticipation even when the investor does not derive utility directly from housing (i.e., when $\theta = 0$). In contrast, if there were no cointegration, then the investor would always invest in stock, as indicated by the dashed lines in Panels.
(i) and (ii). Thus, without cointegration, the investor would always participate in the stock market. Therefore, same as the negative correlation result, it is not the additional role of housing as a consumption good that drives the nonparticipation result. It is the cointegration between the stock and the housing markets that causes the nonparticipation. In addition, Panel (ii) of Figure 2 also suggests that consistent with empirical evidence, there is a significant negative correlation between house ownership and stock ownership, i.e., when an investor owns a house, it is more likely that she does not own a stock, and vice versa.\(^ {10} \)

The non-participation threshold \( \mu^{*}_H \) and the probability of non-participation depend on the model parameters, as shown in Figures 3 and 4. Figure 3 shows that when \( \theta \) increases, housing investment becomes more important and valuable, because in addition to financial returns housing investment also provides higher marginal utility if \( \theta \) is larger. As a consequence, the non-participation threshold \( \mu^{*}_H \) moves down, which implies a higher probability of non-participation in the stock market. Recall that the parameter \( k \) measures the strength of cointegration effect: if \( k \) is larger, the stock price and the housing price tend to move closer together. Stronger cointegration enhances the substitution effect of housing market risk for stock market risk, and thus the portfolio share of wealth in stocks decreases, leading to a lower non-participation threshold \( \mu^{*}_H \) and a higher probability of non-participation in the stock market. If the investor does not get direct utility from housing (i.e., \( \theta = 0 \)), housing is less attractive, thus the non-participation threshold \( \mu^{*}_H \) is greater and the probability of non-participation in the stock market is lower, but still significant (e.g., 0.4 when \( k = 0.1976 \)).

In Figure 4, we plot the non-participation threshold \( \mu^{*}_H \) and the probability of non-participation against risk aversion \( \gamma \) and mortality rate \( \delta_M \). Figure 4 shows that when the investor becomes more risk-averse, the non-participation threshold \( \mu^{*}_H \) decreases and the probability of nonparticipation increases. This is because the investor is less willing to invest in the stock. If housing does not contribute directly to utility (i.e., \( \theta = 0 \)), then the non-participation threshold \( \mu^{*}_H \) is higher and

\(^{10}\text{Given the Cobb-Douglas utility function we use, as long as housing is a direct consumption good (i.e., } \theta > 0, \text{ the investor always owns a house. The main intuition behind our results suggests that there would be a negative correlation between house ownership and stock ownership even when housing is a direct consumption good if another form of utility function was used such that it was not necessary that the investor always owns a house.} \)
the stock market nonparticipation probability is lower. The non-participation threshold $\mu^*_H$ and the probability of non-participation also vary with other model parameters but we find that the effect of changing other parameter values is relatively small. For example, with a higher mortality rate $\delta_M$, the investor consumes more perishable goods, which only slightly increases the non-participation as shown in Panel (iii)–(iv) of Figure 4.

4.2.3 Optimal perishable good consumption

After examining the impact of cointegration on risk-taking in the housing and the stock market, we next turn to its impact on an investor’s consumption of the perishable good. To illustrate this impact, we plot the optimal perishable good consumption against the conditional expected return of housing $\mu_H(I)$ with and without cointegration in Figure 5. Figure 5 shows that the fraction of wealth spent in the perishable good consumption first decreases and then increases with the conditional expected return of housing $\mu_H(I)$. The intuition is as follows. Changes in the conditional expected return of housing have two opposing effects: the substitution effect and the wealth effect. When the conditional expected return of housing is high, the investor invests more in housing and the substitution effect tends to decrease the perishable good consumption. On the other hand, because of the greater return from the housing investment, the wealth tends to grow faster, which tends to increase the perishable good consumption. When the conditional expected return of housing is low, the investor invests less in housing and thus the substitution effect tends to increase the perishable good consumption. On the other hand, because of the lower expected return from the housing investment, the wealth growth tends to be slower, which tends to decrease the perishable good consumption. Therefore, whether the perishable good consumption increases or not depends on which effect dominates. As shown in Figure 5, when the conditional expected return of housing is low, households consume at a higher rate, indicating the substitution effect dominates because the investor significantly decreases housing investment and thus the marginal utility from the perishable consumption is much higher. In contrast, if the conditional expected return of housing is high, the perishable consumption rate is also high. This is because the much
Figure 3: **Non-participation against $\theta$ and $k$.** When the conditional expected return of housing $\mu_H(I)$ is above $\mu^*_H$, there is non-participation in the stock market. The probability of non-participation is defined in (4.1). Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{I} = 3.2162$, $\sigma_H = 0.0791$, $\mu_{H0} = 0.0096$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$. 
Figure 4: Non-participation against $\gamma$ and $\delta_M$. When the conditional expected return of housing $\mu_H(I)$ is above $\mu_H^*$, there is non-participation in stocks. The probability of non-participation is defined in (4.1). Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{I} = 3.2162$, $\sigma_H = 0.0791$, $\mu_{H0} = 0.0096$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$. 
greater expected return from the housing investment significantly increases the wealth effect which becomes the dominant effect. This explains the nonmonotonicity of the perishable consumption in the conditional expected return of housing $\mu_H(I)$.

If housing does not contribute directly to the investor’s utility (i.e., $\theta = 0$), then the perishable good consumption increases with the conditional expected return of housing $\mu_H(I)$. This result is driven by the absence of the substitution effect between housing and the perishable good when the investor does not get direct utility from housing. As the conditional expected return of housing increases, there is only wealth effect, and as a result, the perishable consumption increases. In addition, for a large range of the conditional expected return of housing $\mu_H(I)$, the investor consumes more perishable good because the marginal utility of perishable good consumption is higher ($1 - \theta$ is greater when $\theta = 0$). However, when the conditional expected return of housing $\mu_H(I)$ is low, the investor reduces significantly housing and the substitution effect leads to an increase in perishable good consumption when $\theta = 0.3$, while the perishable good consumption when $\theta = 0$ goes down due to the wealth effect. This is why the investor consumes less perishable good when the conditional expected return of housing $\mu_H(I)$ is low.
4.3 Cost of Ignoring Cointegration

In this subsection, we analyze the equivalent wealth loss from ignoring the cointegration between stock and housing markets. If an investor ignores the cointegration effect between stock and housing markets, i.e., assumes $k = 0$, the house price becomes a geometric Brownian motion:

$$dH_t = \mu^0_H H_t dt + \sigma^0_H dB_t.$$  \hspace{1cm} (4.2)

Using the home price index on Dec. 1 from 1890 to 2017 inflation adjusted to November 2019 dollars and assuming no cointegration, we have the following new estimates: $\mu^0_H = 0.0068$ and $\sigma^0_H = 0.0711$. We denote by $\Psi^0(W, H)$ the value function of the model that ignores cointegration.

The equivalent wealth loss $\Delta W$ of cointegration can be defined as

$$\Psi(W - \Delta W, H, \bar{I}) = \Psi^0(W, H),$$  \hspace{1cm} (4.3)

where $\Psi$ is the value function of model that correctly incorporates the cointegration in (2.9). We plot the equivalent wealth loss as a fraction of the initial wealth from ignoring the cointegration against the preference for housing parameter $\theta$ in Figure 6. Figure 6 shows that ignoring the cointegration can be costly to an investor. For example, at $\theta = 0.3$, the equivalent wealth loss is about 80% of the initial wealth. In addition, Figure 6 indicates that the equivalent wealth loss is nonmonotonic in the housing preference parameter $\theta$. Intuitively, when $\theta$ is small, the investor invests less in housing and thus the cointegration is less important. When $\theta$ is large, the investor invests a lot in housing but a small amount in stock, and thus also cares less about the long term correlation between the two.

4.4 Stock market participation cost

So far we have shown that an investor may choose not to participate in the stock market in some states of the world (i.e., when the conditional expected return of housing $\mu_H(I)$ is high) and such
decision is independent of the wealth level. In practice, however, some investors never participate in the stock market and the nonparticipation rate decreases with the wealth level. To be consistent with these empirical evidence, we now extend our model to include a one-time, fixed participation cost (e.g., cost of attention, stress, information processing) for participation in the stock market. More specifically, to participate in the stock market from time 0 to time $T$, an investor must pay a one-time cost of $\eta$ at time 0. For a given level of participation cost $\eta$, we can solve for the critical wealth level $\bar{W}$ below which an investor will choose never to participate. Note that even after an investor pays the participation cost $\eta$ at time 0, the investor may still choose not to participate when the conditional expected return of housing is low, as we have shown above.

Let $\bar{W}$ be the critical wealth level below which the investor never participates in the stock market. For given $\eta$, $H_0$ and $I_0$, the critical wealth level $\bar{W}$ at time 0 then solves

$$\Psi(\bar{W} - \eta, H_0, I_0) = \Psi_0(\bar{W}, H_0, I_0).$$
where \( \Psi_0(W, H, I) \) is the value function when an investor is prohibited from ever investing in the stock market. Because of homogeneity, the solution \( \tilde{W} \) is independent of \( H_0 \) and is only a function of \( \eta \) and \( I_0 \). We can then compute the minimum value of \( \tilde{W}(\eta, I_0) \) across all \( I_0 \), i.e.,

\[
\tilde{W}^*(\eta) = \inf_{I_0} \tilde{W}(\eta, I_0).
\]

Alternatively, we can solve for the critical value of the participation cost \( \eta \) above which an investor chooses never to participate in the stock market for given \( W_0, H_0, \) and \( I_0 \).

\[
V((1 - \eta)W_0, H_0, I_0) = V_0(W_0, H_0, I_0).
\]

Because of homogeneity, the solution \( \eta \) is independent of \( H_0 \) and \( W_0 \) and is only a function of \( I_0 \), we thus denote it as \( \eta(I_0) \).

In Figure 7, we plot the minimum critical wealth level \( \tilde{W}^* \) against the participation cost \( \eta \) with and without cointegration. This figure shows that with cointegration, the minimum critical wealth level below which an investor will choose never to participate in the stock market is much greater than that without cointegration. For example, if the participation cost is $1,000, then an investor with an initial wealth above $7,400 will choose to participate in the stock market when there is no cointegration, but when there is cointegration, even those investors who have as much as $44,000 will choose never to participate. If the investor does not get utility directly from housing (i.e., \( \theta = 0 \)), then the critical wealth level \( \tilde{W}^* \) is lower, because housing is less attractive and the investor prefers to invest more in the stock for a given wealth level.

In Figure 8, we plot the critical participation cost \( \eta \) as a fraction of the initial wealth against the initial conditional expected return of housing \( \mu_H(I_0) \) with and without cointegration. This figure shows that with cointegration, the critical participation cost above which an investor will choose never to participate in the stock market is much smaller than that without cointegration. For example, when there is no cointegration, it needs the participation cost to be as large as 13.53% of the initial wealth to deter an investor from stock market participation. In contrast, with cointegration,
Figure 7: Wealth threshold $W$ under which non-participation is optimal against $\eta$, both in units of $1,000$. Default parameters are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{I} = 3.2162$, $\sigma_H = 0.0791$, $\mu_{H0} = 0.0096$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$.

Figure 8: $\eta$ against $\mu_H(I_0)$. Default parameters are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{I} = 3.2162$, $\sigma_H = 0.0791$, $\mu_{H0} = 0.0096$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$. 
if $\mu_H(I_0) = 0.01$, the participation cost only needs to be about 0.38% of the initial wealth to deter an investor from ever participating in the stock market. In addition, with cointegration, this cost is always below 1.8% for any value of $\mu_H(I_0)$. These findings suggest that the presence of cointegration can significantly increase the nonparticipation rate in the stock market. If the investor does not get utility directly from housing (i.e., $\theta = 0$), then the critical participation cost as a fraction of the initial wealth $\eta$ is higher, because housing becomes less attractive, the investor is more willing to invest in the stock, and thus requires a higher participation cost for nonparticipation.

There are many studies aimed at explaining the nonparticipation puzzle. For example, Vissing-Jørgensen (2002) shows that a moderate participation cost can explain half of the nonparticipation observed in data. Cocco (2005) finds that housing crowds out stock holdings which, together with a sizable stock market entry cost, can explain stock market nonparticipation early in life. Our model complements these extant theories by incorporating the cointegration between the stock and the housing markets, and may strengthen their explanatory power. In particular, our model suggests that the magnitudes of the participation costs in Vissing-Jørgensen (2002) and Cocco (2005) and the degree of ambiguity aversion in Cao, Wang, and Zhang (2005) required to explain nonparticipation would be significantly smaller if cointegration were incorporated.

5 Robustness to the option of renting and the housing market illiquidity

In this section, we examine whether our results are robust to the option of renting and the housing market illiquidity.

5.1 Option of Renting

In our calibrated model with $\theta = 0.3$, housing serves the dual role of an investment vehicle and a consumption good. We have shown that even when the investor does not get utility directly from housing (i.e., $\theta = 0$), our main results on non/limited participation and the highly negative
correlation between stock and housing investment remain valid. On the other hand, one may argue that allowing households to rent might significantly weaken the need for buying a house and thus increase the stock market participation and investment. In this subsection, we show the robustness of our results in the presence of renting option.

Assume that the household can rent housing units at a rent proportional to the price of the rented property. We denote the rental rate by $\kappa_R$. Let $h_O$ and $h_R$ be the fraction of wealth $W$ in houses owned and rented, respectively. Therefore, the dynamics of wealth $W$ becomes

$$
\frac{dW_t}{W_t} = \left[r - c_t + \zeta_t(\mu_S - r) + h_Ot(\mu_H(I_t) - \delta - r) - \kappa_R h_Rt\right] dt \\
+ \sigma_S \zeta_t dB_{St} + \sigma_H h_Ot dB_{Ht}.
$$

(5.1)

The net units of housing at time $t$ are $(h_O + h_R)W_t/H$ and thus, the household’s intertemporal utility follows

$$
\int_0^T e^{-\beta t} \frac{(c_t^{1-\theta}(h_O+ h_R)/H_t)^{\theta} W_t^{1-\gamma}}{1-\gamma} dt.
$$

The household chooses the perishable consumption rate $c_t$, the portfolio weights $\zeta_t$, $h_O$, and $h_R$ of the stock, the housing owned, and the housing rented respectively to maximize her intertemporal utility. Let $A_2$ denote the set of all admissible strategies, that is, strategies $(c_t, \zeta_t, h_O, h_R)$ satisfying standard integrability conditions, the wealth constraint $W_t \geq 0$, the consumption constraints $c_t \geq 0$, the short selling constraints $\zeta_t, h_O, h_O + h_R \geq 0$, and the limited borrowing constraint $\zeta_t + lh_O \leq 1$. Note that $h_R$ could be positive or negative. Investors rent houses if $h_R > 0$ and rent out part of their owned house if $h_R < 0$. The value function is defined as

$$
\Psi(W, H, I) : = \max_{(c_t, \zeta_t, h_O, h_R) \in A_2} E \left[ \int_0^T e^{-\beta t} \frac{(c_t^{1-\theta}(h_O+ h_R)/H_t)^{\theta} W_t^{1-\gamma}}{1-\gamma} dt \right]
$$

(5.2)

subject to processes (2.4), (2.6), (5.1), and wealth constraint $W_t \geq 0$. The corresponding HJB equation is given in Appendix A.1.
We also consider a case where investors cannot afford to buy a house (e.g., because a house has a minimum size and investors’ wealth is low) and thus have to rent. In this case, we assume that these investors can invest in securities like REITS which have the same price process as the housing price $H_t$.

We use a rental rate of 6.7% as estimated by Fischer and Stamos (2013) as our default parameter value. Other choice of rental rate will also be considered. The house ownership is affected by the conditional expected return of housing $\mu_H(I)$, as shown in Figure 9. Figure 9 shows that the weight of house owned increases with the conditional expected return of housing $\mu_H(I)$, while the weight of house rented decreases and crosses zero with respect to $\mu_H(I)$. Interestingly, the non/limited stock participation result is strengthened when renting is allowed and investors can buy houses. Recall that when the conditional expected return of housing $\mu_H(I)$ is high, investors would like to short stock to finance a purchase of a larger house, but due to the shortsale constraint, this is impossible. With the rental market opened, investors can now buy a larger house and finance part of the purchase with rents from renting out part of the house bought. As a result, compared to the model that does not allow renting, investors prefer to buy a larger house and do not participate in the stock market with a greater probability (almost equal to one, as defined in subsection 4.2.2).

When investors cannot buy houses but can buy securities like REITS, Figure 9 shows that our main results that investors may choose not to participate in the stock market and the investment in the stock market and the real estate market (i.e., REITS) is significantly negatively correlated still hold. On the other hand, the investment in stock is greater and the probability of nonparticipation is lower, compared to the case where investors can afford to buy houses, because buying REITS does not directly contribute to utility from consumption and thus investors invest less the real estate market.
Figure 9: **House ownership and stock investment.** $h^*_O$, $h^*_R$, and $\zeta^*$ are the optimal weight of houses owned, houses rented, and stocks when renting is allowed, respectively. $h^*$ and $\zeta^*$ are the optimal housing and stock investment when renting is not allowed, as shown in Subsection 4.2. We also consider the case when investors can only rent but can invest in REITS. The optimal REITS investment is shown as $h^*_\text{REITS}$ in the plot. Note that with renting option but without cointegration, the optimal $h^*_O$, $h^*_R$, and $\zeta^*$ are 1.4113, −1.1239, and 0.1412, respectively. Default parameter values: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{I} = 3.2162$, $\sigma_H = 0.0791$, $\mu_H = 0.0096$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$, and $\kappa_R = 6.7\%$

To analyze the effect of renting option and the rental rate $\kappa_R$ on house and stock investment and on the correlation between the two, we do the simulation similarly as before. The result is reported in Table 4 which shows that as the rental rate increases, stock investment decreases and housing/REITS investment increases. In addition, the correlation of the two also decreases. However, as before, the correlations are still significantly negative.
Table 4: **Simulation.** Default parameter values: \( r = 0.0059, \mu_S = 0.0339, \sigma_S = 0.1411, \lambda = 0.2695, \bar{I} = 3.2162, \sigma_H = 0.0791, \mu_H0 = 0.0096, \beta = 0.0059, \delta_M = 0.05, \gamma = 10, \theta = 0.3, \delta = 0, l = 0.2, \) and \( \kappa_R = 6.7\% \).

### Panel A: with housing and renting

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Stock Investment ((\zeta^*))</th>
<th>House Owned ((h^*_O))</th>
<th>House Rented ((h^*_R))</th>
<th>Investment Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 0 )</td>
<td>Mean</td>
<td>Median</td>
<td>Std</td>
<td>Mean</td>
</tr>
<tr>
<td>( \kappa_R = 6.7% )</td>
<td>0.1412</td>
<td>0.1412</td>
<td>0</td>
<td>1.4113</td>
</tr>
<tr>
<td>( k = 0.1976 )</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>2.7415</td>
</tr>
<tr>
<td>( \kappa_R = 3.35% )</td>
<td>0.1412</td>
<td>0.1412</td>
<td>0</td>
<td>0.8704</td>
</tr>
<tr>
<td>( k = 0.1976 )</td>
<td>0.0011</td>
<td>0.0001</td>
<td>0.0020</td>
<td>2.0941</td>
</tr>
<tr>
<td>( \kappa_R = 1.675% )</td>
<td>0.1412</td>
<td>0.1412</td>
<td>0</td>
<td>0.6525</td>
</tr>
<tr>
<td>( k = 0.1976 )</td>
<td>0.0038</td>
<td>0.0022</td>
<td>0.0046</td>
<td>1.8161</td>
</tr>
</tbody>
</table>

### Panel B: with REITS and renting

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Stock Investment ((\zeta^*))</th>
<th>REITS Invested ((h^*_{REITS}))</th>
<th>House Rented ((h^*_R))</th>
<th>Investment Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 0 )</td>
<td>Mean</td>
<td>Median</td>
<td>Std</td>
<td>Mean</td>
</tr>
<tr>
<td>( \kappa_R = 6.7% )</td>
<td>0.1412</td>
<td>0.1412</td>
<td>0</td>
<td>0.3296</td>
</tr>
<tr>
<td>( k = 0.1976 )</td>
<td>0.0620</td>
<td>0.0616</td>
<td>0.0212</td>
<td>0.7933</td>
</tr>
<tr>
<td>( \kappa_R = 3.35% )</td>
<td>0.1412</td>
<td>0.1412</td>
<td>0</td>
<td>0.3296</td>
</tr>
<tr>
<td>( k = 0.1976 )</td>
<td>0.0620</td>
<td>0.0611</td>
<td>0.0211</td>
<td>0.7910</td>
</tr>
<tr>
<td>( \kappa_R = 1.675% )</td>
<td>0.1412</td>
<td>0.1412</td>
<td>0</td>
<td>0.3296</td>
</tr>
</tbody>
</table>
5.2 Illiquidity in the Housing Market

In our main model, for simplicity of analysis and exposition, we assume there is no transaction cost for buying or selling houses. However, in practice, trading in the housing market can incur significant transaction costs and this may change the effect of the cointegration because of the lower frequency of trading in the housing market. To address this potential concern, we consider the effect of the housing market illiquidity in this section. Similar to Grossman and Laroque (1990), we assume that to change the house size, the investor has to first sell her old house and then purchase a new house with the preferred size, and the investor must pay a transaction cost that is proportional to the value of the house sold. More specifically, if the investor wants to buy a new house at time $\tau_i$, she needs to sell her original house and pay a transaction cost of $\alpha A_{\tau_i} - H_{\tau_i}$, where $\alpha \in [0, 1)$ represents the proportional transaction cost rate, $A_{\tau_i}$ is the size of the original house sold at time $t_i$, and $H_{t_i}$ is the market house price at that time. After selling, the household buys a new house with size $A_i$. Define $\tilde{W}_i$ as the financial wealth invested in bonds and stocks, $\pi_t$ as the dollar amount invested in stocks, and $\tilde{C}_t$ as the perishable good consumption. We have

$$d\tilde{W}_t = [r\tilde{W}_t - \tilde{C}_t + \pi_t(\mu_S - r)]dt + \pi_t \sigma_S dB_{St}, \ t \neq \tau_i, \quad (5.3)$$

$$dA_t = -\delta A_t dt, \ t \neq \tau_i, \quad (5.4)$$

$$\tilde{W}_{\tau_i} = \tilde{W}_{\tau_i-} + (1 - \alpha)A_{\tau_i-}H_{\tau_i} - A_{\tau_i}H_{\tau_i}, \ i = 1, 2, ..., \quad (5.5)$$

$$A_{\tau_i} = A_i, \ i = 1, 2, .... \quad (5.6)$$

The investor’s objective function in the presence of the illiquid house market is to choose per-period consumption $\tilde{C}_t$, stock investment $\pi_t$, and housing quantity $A_t$ to maximize her expected utility, i.e.,

$$\Phi(\tilde{W}, A, H, I) = \max_{\tilde{C}_t, \pi_t \geq 0, (\tau_i, A_i)} \mathbb{E} \left[ \int_0^\infty e^{-(\beta + \delta) t} \left( \frac{\tilde{C}_t^{1-\theta} A_t^\theta}{1 - \gamma} \right)^{1-\gamma} dt \right] \quad (5.7)$$

subject to processes (5.3)–(5.6), (2.6), (2.4), the solvency constraint $\tilde{W}_t + (1 - \alpha)A_tH_t > 0$ and
the leverage constraint \( \pi_t + l(1 - \alpha)A_t H_t \leq \tilde{W}_t + (1 - \alpha)A_t H_t \).

By the homogeneity property, we can make the following transformation

\[
\Phi(\tilde{W}, A, H, I) = \frac{1}{1 - \gamma} (\tilde{W} + (1 - \alpha)AH)^{1-\gamma}H^{-\theta(1-\gamma)}e^{(1-\gamma)\phi(h, I)}; \quad h = \frac{(1 - \alpha)AH}{\tilde{W} + (1 - \alpha)AH},
\]

where \( h \) is the ratio of house value to the net wealth. The corresponding HJB equation and the iterative algorithm for solving it are given in Appendix A.2.

Following Corradin et al. (2014), we set the housing transaction cost to be \( \alpha = 10\% \) of the unit’s value as a baseline parameter value, including commissions, legal fees, the time cost of searching, and the direct cost of moving possessions. The numerical result is shown in Figure 10.

In the presence of transaction costs, there exists an optimal buying ratio \( h_B(I) \), an optimal selling ratio \( h_S(I) \), and an optimal target ratio of house value to net wealth \( h^*(I) \). When the ratio of the house value to the net wealth is below the optimal buying ratio \( h_B(I) \), the investor optimally sells her current house and purchases a bigger one such that the new ratio of the house value to the net wealth jumps upward to the optimal target level \( h^*(I) \). When the ratio of the house value to the net wealth is above the optimal selling ratio \( h_S(I) \), the investor optimally sells her current house and purchases a smaller one such that the new ratio of the house value to the net wealth jumps downward to the optimal target level \( h^*(I) \). The area between \( h_B(I) \) and \( h_S(I) \) is the no-trading region. When the ratio of the house value to the net wealth falls inside this area, the investor does not sell the current house to buy a new one, and the function \( \phi(h, I) \) satisfies the corresponding HJB equation. Value-matching and smooth-pasting conditions hold at the two bounds \( h_B(I) \) and \( h_S(I) \), and an optimality condition holds at the target point \( h^*(I) \). All of these free boundaries depend on the log ratio \( I_t \).

We plot the optimal ratios of the stock value to the net wealth in red-dotted, red-solid, and red-dashed lines, when the ratios of the house value to the net wealth are respectively \( h_B(I) \), \( h^*(I) \), and \( h_S(I) \). Figure 10 suggests that the presence of a significant illiquidity in the housing market does not change our main result that with cointegration, non/limited participation in the stock market
Figure 10: **Optimal stock and house investment.** The blue-solid line is the optimal target ratio of house value to net wealth \( h^*(I) \). The blue-dashed and blue-dotted lines are the optimal house selling ratio \( h_S(I) \) and buying ratio \( h_B(I) \), respectively. These three lines are on the left Y-axis. The red-solid, red-dashed, and red-dotted lines are the optimal ratio of stock to the net wealth when the ratios of house value to the net wealth equals \( h^*(I) \), \( h_S(I) \), and \( h_B(I) \). When there is no cointegration, i.e., \( k = 0 \), the optimal housing selling ratio \( h_S = 0.8020 \), the optimal house buy ratio \( h_B = 0.2667 \), the optimal house target ratio \( h^* = 0.4333 \), and the optimal stock ratio equals 0.1737, 0.1648, and 0.1508 when house ratio equals \( h_B \), \( h_S \), and \( h^* \), respectively. Default parameter values are from Table 2: \( r = 0.0059 \), \( \mu_S = 0.0339 \), \( \sigma_S = 0.1411 \), \( \lambda = 0.2695 \), \( \bar{I} = 3.2162 \), \( \sigma_H = 0.0791 \), \( \mu_{H0} = 0.0096 \), \( \beta = 0.0059 \), \( \delta_M = 0.05 \), \( \gamma = 10 \), \( \theta = 0.3 \), \( \delta = 0 \), \( l = 0.2 \), \( \alpha = 0.10 \).

can be optimal, and the house investment and stock investment are negatively correlated.\(^{11}\)

To examine the average impact of cointegration, similar to Kraft and Munk (2011), we set the initial financial wealth is \( \tilde{W}_0 = 20 \), representing $20,000 in line with the median net worth and before-tax income statistics for young individuals according to the 2013 SCF. The initial unit house price \( H_0 = 0.25 \), which corresponds to $250 per square foot. The initial value of the log ratio \( I_t \) is set to be \( I_0 = \bar{I} \). We then simulate 10,000 paths of processes \( I_t, H_t, \) and \( \tilde{W}_t \) by (2.4), (2.6), and (5.3), respectively. The policy \( \{ \pi_t, \bar{C}_t, (\tau_i, A_i) \} \) is chosen from the optimal ones derived by maximizing the objective function. The results presented in Table 5 are averages over

\(^{11}\)Similar to Grossman and Laroque (1990), when the housing level is at the optimal target, the corresponding stock investment is the lowest. This is because the risk aversion of the investor is the highest at the target level, since it takes a significant amount of time before the house size can be changed.
the 10,000 paths generated. Table 5 indicates that the presence of significant illiquidity does not change the result that the cointegration effect on average lowers stock investment and increases housing investment or the result that stock investment and house investment are highly negatively correlated (a mean correlation coefficient of -0.80).

Table 5: Simulation. This table reports the simulated average of stock and house investments as percentages of net wealth. Default parameter values are from Table 2: \( r = 0.0059, \mu_S = 0.0339, \sigma_S = 0.1411, \lambda = 0.2695, \bar{I} = 3.2162, \sigma_H = 0.0791, \mu_{H0} = 0.0096, \beta = 0.0059, \delta_M = 0.05, \gamma = 10, \theta = 0.3, \delta = 0, l = 0.2, \alpha = 0.10. \)

| Parameters | Stock Investment | | House Investment | | Investment correlation |
|---|---|---|---|---|---|---|---|
| Mean | Median | Std | Mean | Median | Std | Mean | Median | Std |
| \( \theta = 0 \) | | | | | | | |
| \( k = 0 \) | 0.1221 | 0.1217 | 0.0019 | 0.1808 | 0.1699 | 0.0319 | 0.3333 | 0.5783 | 0.5213 |
| \( k = 0.1976 \) | 0.0314 | 0.0165 | 0.0401 | 0.8490 | 1.2649 | 0.5905 | -0.8013 | -0.8095 | 0.0747 |
| \( \theta = 0.3 \) | | | | | | | |
| \( k = 0 \) | 0.1144 | 0.1146 | 0.0030 | 0.3863 | 0.3807 | 0.0322 | -0.6347 | -0.9270 | 0.5392 |
| \( k = 0.1976 \) | 0.0129 | 0.0105 | 0.0292 | 1.4969 | 1.5976 | 0.3341 | -0.8481 | -0.8644 | 0.0772 |

6 Model Prediction and Empirical Evidence

Our model predicts that as the cointegration between housing and stock prices increases, stock investment decreases and stock-market nonparticipation increases. To see if this prediction has empirical support, we next utilize the U.S. cross-state variations of the degree of cointegration, stock investment and the stock-market nonparticipation to examine the relationship among the three. We use the PSID data of family level in 2015 and 2017 waves, totally more than 9,000 observations. The value of stock holding is extracted from variable ER65368 in 2015 wave and ER71445 in 2017 wave. The financial wealth is calculated as the sum of equity in stocks and the value in safe account, where the value in safe account is the money amount in checking or savings accounts, money market funds, certificates of deposit, government bonds, or treasury bills. In 2015 wave, the value in safe account is extracted from variable ER61772. In 2017 wave, the value in safe account is extracted from variable ER67826. We then calculate the average value of equity in stocks, the average ratio of financial wealth invested in stocks,\(^\text{12}\) and the proportion of interviewed

\(^{12}\text{We can also calculate the equity investment ratio as risky assets divided by the sum of risky assets and safe assets. The risky assets comprise stock holdings, IRAs, and annuity holdings. The safe assets include other assets (net of debt}
families that do not invest in stocks for each state. To analyze the cointegration between house and stock markets in each state, we refer to the monthly Housing Price Index data from 1975 to 2019 from the Federal Housing Finance Agency (FHFA). The strength of cointegration is measured by the parameter $k$. Note that the larger the value of $k$, the stronger the cointegration between housing and stock markets.

Figure 11 shows that consistent with our model prediction, as the degree of cointegration increases, stock investment decreases and non-participation in the stock market increases. For example, Connecticut has low cointegration between stock and housing prices, high stock investment and low stock-market nonparticipation.


Figure 11: Cointegration effect on stock investment. Abbreviation is used for each state. For example, “CA” refers to California. The red line is the linear regression result. It is observed that the higher the strength of cointegration between stock and house markets, the lower the stock share and the higher the non-participation ratio. The value p-value reports the significance test result of the coefficient of $k$ in the linear regression.

7 Conclusion

In this paper, we consider the optimal joint choice of stock portfolio and housing of a household when the stock and housing markets are cointegrated. We show that in the presence of cointegration—such as bonds, insurance, etc), checking balances, and savings balances, less the principal on the primary residence. The result is similar and not reported in this version to save space, but available from the authors.
igration, households significantly reduce stock investment and increase housing investment. As a result they may choose not to participate in the stock market at all even when there is no participation cost and the expected return from housing is lower than that of the stock. In the presence of participation cost, the critical wealth level below which households never participate in the stock market is much higher than that in the absence of cointegration and the critical participation cost level above which households never participate in the stock market is much smaller than that in the absence of cointegration. These results are robust to extensions that incorporate rental alternatives and housing market illiquidity. Our model complements the existing studies and can potentially help explain both the puzzle of stock market non/limited-participation and the puzzle of the highly negative correlation between stock and housing investment. We also show empirical evidence that is supportive of the predictions of our model.
References


Appendix

In this appendix, we plot some figures to show empirical evidence of the correlations between stock market participation/investment and houseownership/investment across countries and across time. We then provide proofs of the analytical results in the main text.

Empirical evidence

Figure 12 plots stockownership (including direct and indirect ownership) against houseownership across 17 countries in 2015, with the red line showing the OLS regression line. This figure suggests that significant stock market nonparticipation is an international phenomenon, although standard portfolio choice theories predict close to 100% participation. The highest participation rate is about 61% (UK), with the lowest being about 8% (India). Despite the exceptional stock returns in the U.S., the participation rate in the U.S. is only about 43%. In addition, Figure 12 suggests that as houseownership increases, stockownership tends to decrease, with a correlation between the two of about -0.59.

Figure 13 plots the stock investment and housing investment across 20 countries in 2015. Figure 13 shows a strong pattern of negative correlation between the stock investment and the housing investment. Indeed the correlation between the two is -0.62 on average across these 20 countries.

Figure 14 plots the correlation between stock investment and housing investment across time over the period of 2000 to 2017 for 16 countries. Figure 14 suggests that stock investment and housing investment are highly negatively correlated across time for most countries. The correlation coefficient is about -0.71 on average. This highly negative correlation is puzzling because it is well known that the contemporaneous correlation between stock and housing prices is low (around 0.07) and standard optimal investment theories imply a low negative correlation of around -0.18.

The only exceptions are Australia, Canada, and Singapore where the correlations are positive. The positive correlations found in these countries may be related to the special immigration and real estate policies in these countries. For example, for Australia and Canada, immigration is encouraged and immigrants tend to be wealthy who invests both in housing and stocks.
Figure 12: **Houseownership and Stockownership across Countries (2015).**

Figure 13: **Nonfinancial Wealth and Equity across Countries (2015)**
Figure 14: Housing and Stock Investment Correlations Across Time (2000-2017)

### A.1 HJB equation for the model with option of renting

The associated HJB equation for the household’s optimization problem (5.2) is

\[
\max_{c, \zeta, h \geq 0; \zeta + h \leq 1; h_R} \left\{ \left( \frac{1}{2} \sigma_S^2 \zeta^2 + \frac{1}{2} \sigma_H^2 h_O^2 \right) W^2 \Psi_{WW} + \frac{1}{2} \sigma_H^2 H^2 \Psi_{HH} + \frac{1}{2} (\lambda^2 \sigma_S^2 + \sigma_H^2) \Psi_{II} \right. \\
+ \left( \sigma_H^2 h_O - \lambda \sigma_S^2 \zeta \right) W \Psi_{WI} + \sigma_H^2 h_O W \Psi_{WH} + \sigma_H^2 H \Psi_{HI} \\
+ \left[ r - c + (\mu_S - r) \zeta + (\mu_H(I) - \delta - r + \kappa_R) h_O - \kappa_R(h_O + h_R) \right] W \Psi_W \\
+ \mu_H(I) H \Psi_H + k(\bar{I} - I) \Psi_I - (\beta + \delta_M) \Psi + \left( \frac{c^{1-\theta}((h_O + h_R)/H)^\theta W}{1 - \gamma} \right) \right\} = 0
\]  

(A.1-1)

for \( W > 0, H > 0, I \in \mathbb{R} \).

Using the homogeneity property of the value function, we can reduce the dimensionality of the problem by the following transformation:

\[
\Psi(W, H, I) = \frac{1}{1 - \gamma} W^{1-\gamma} H^{-\theta(1-\gamma)} e^{(1-\gamma)u(I)}
\]
for some function \( u \). Then equation (A.1-1) can be reduced to
\[
\max_{c, \zeta, h_0 \geq 0; \zeta + h_0 \leq 1; h_R} \left\{ \frac{1}{2} (\lambda^2 \sigma_S^2 + \sigma_H^2)[u_{II} + (1 - \gamma)u_I^2] + \left[ (\sigma_H^2 h_O - \lambda \sigma_S^2 \zeta - \theta \sigma_H^2)(1 - \gamma) + k I - k I \right]u_I \\
- \frac{1}{2} \gamma (\sigma_S^2 \zeta^2 + \sigma_H^2 h_O^2) + \frac{1}{2} \sigma^2 \theta (\theta (1 - \gamma) + 1) - \sigma_H^2 \theta (1 - \gamma) h_O + r - c + (\mu_S - r)\zeta \\
+ (\mu_H(I) - \delta - r + \kappa_R)h_O - \kappa_R(h_O + h_R) - \theta \mu_H(I) - \frac{\beta + \delta_M}{1 - \gamma} \\
+ \frac{1}{1 - \gamma} e^{(1 - \theta)(1 - \gamma)}(h_O + h_R)^{(1 - \gamma)e^{-(1 - \gamma)u}} \right\} = 0, \quad I \in \mathbb{R}.
\]

A.2 HJB equation for the model with an illiquid housing market

The value function as given in (5.7) satisfies the following HJB equation
\[
\max_{\tilde{W}, 0 \leq \pi \leq \tilde{W} + (1 - \alpha)AH} \left\{ \left( \frac{1}{2} \lambda^2 \sigma_S^2 \Psi_{W} + \frac{1}{2} \sigma_H^2 H^2 \Psi_{HH} + \frac{1}{2} (\lambda^2 \sigma_S^2 + \sigma_H^2) \Psi_{II} - \lambda \sigma_S^2 \pi \Psi_{W} \right) \\
+ \sigma_H^2 H \Psi_{HI} + [r \tilde{W} - \tilde{C} + (\mu_S - r)\pi] \Psi_{W} + \mu_H(I) H \Psi_H \\
+ k(\bar{I} - I) \Psi_I - \delta A \Psi_A - (\beta + \delta_M) \Psi + \left( \tilde{C}^{1 - \theta} A^\theta \right)^{1 - \gamma} \right\},
\]
\[
\max_{0 \leq Q \leq \tilde{W} + (1 - \alpha)AH} \Psi(\tilde{W} + (1 - \alpha)AH - QH, Q, H, I) - \Psi(\tilde{W}, A, H, I) = 0
\]
in \( \Omega = \{ (\tilde{W}, A, H, I) : \tilde{W} + (1 - \alpha)AH > 0, A > 0, H > 0, I \in \mathbb{R} \} \).

By the homogeneity property, we can do the following transformation
\[
\Phi(\tilde{W}, A, H, I) = \frac{1}{1 - \gamma} (\tilde{W} + (1 - \alpha)AH)^{1 - \gamma} H^{-\theta(1 - \gamma)} e^{(1 - \gamma)\phi(h, I)}, \quad h = \frac{(1 - \alpha)AH}{\tilde{W} + (1 - \alpha)AH},
\]
where \( h \) is the ratio of house value to the net wealth. The new function \( \phi(h, I) \) satisfies the following HJB equation
\[
\max_{h'} \left\{ \mathcal{L} \phi, \log(1 - ah) + \max_{h'} \phi(h', I) - \phi \right\} = 0, \quad (A.2-2)
\]

47
where
\[
L\phi = \max_{\zeta \in [0, 1-\bar{h}]} \left\{ \frac{1}{2} [\sigma^2_S \zeta^2 + \sigma^2_H (1 - h)^2] h^2 [\phi_h + (1 - \gamma) \phi_h^2] + \frac{1}{2} (\lambda \sigma^2_S + \sigma^2_H) [\phi_I + (1 - \gamma) \phi_I^2] + \lambda \sigma^2_S \zeta + \sigma^2_H (1 - h) \right\}
\]

with \( \zeta = \pi / (\tilde{W} + (1 - \alpha) A H) \) and the optimal consumption satisfies
\[
\tilde{c}^* = \frac{\tilde{C}^*}{\tilde{W} + (1 - \alpha) A H} = (1 - \theta) \frac{1}{\tau - \rho} (1 - h \phi_h) - \frac{1}{\tau - \rho} h^{\frac{\theta(1 - \gamma)}{1 - \rho} - \frac{1 - \gamma}{1 - \rho}} \phi.
\]

We can solve (A.2-2) numerically. Define \( M(I) := \sup_h \phi(h, I) \). The algorithm is given as follows:

1. Set an initial guess of \( M_0(I) \);
2. Given \( M_i(I) \), use the penalty method with finite difference scheme to solve
   \[
   \max \left\{ L\phi, \log(1 - \alpha h) + M_i(I) - \phi \right\} = 0, h \in (0, \bar{h}),
   \]
   where \( \bar{h} \) is the upper bond set numerically;
3. Let \( M_{i+1}(I) := \sup_{h'} \phi(h', I) \);
4. If \( ||M_{i+1} - M_i|| < \) tolerance then stop; otherwise go to Step 2.

### A.2.1 Verification Theorem

In this subsection, we provide the verification theorem below that captures households’ optimal investment policy. We focus on the general illiquid model in Subsection 5.2, while the liquid benchmark model in Section 2 is a degenerated case.
Theorem 2. (Verification Theorem) Let $\Psi(\tilde{W}, A, H, I)$ be a smooth solution to the HJB equation (A.2-1) and satisfy the transversality condition, i.e.,

$$\lim_{t \to \infty} E \left[ e^{-(\beta+\delta)t} \Psi(\tilde{W}_t, A_t, H_t, I_t) \right] = 0, \quad \forall \tilde{W}_t + (1 - \alpha) A_t H_t > 0, A_t > 0, H_t > 0, I_t \in \mathbb{R}. \quad (\text{Verification Theorem})$$

In addition, let

$$C_t^*(\tilde{W}_t, A_t, H_t, I_t) = \left( \frac{\Psi_W A^{-\theta(1-\gamma)}(1-\theta)(1-\gamma)}{1-\theta} \right)^{1/(1-\theta)(1-\gamma)},$$

$$\pi_t^*(\tilde{W}_t, A_t, H_t, I_t) = \arg \max_{0 \leq \pi_t \leq \tilde{W}_t+(1-\alpha)A_t H_t} \left\{ \frac{1}{2} \sigma^2 \pi^2 \Psi_{\tilde{W}\tilde{W}} + \lambda \sigma^2 \pi \Psi_{\tilde{W}I} + (\mu_S - r) \pi \Psi_{\tilde{W}} \right\},$$

where $\eta = (1 - \theta) \frac{1}{(1-\theta)(1-\gamma)} - \frac{(1-\theta)(1-\gamma)}{1-\gamma}$, all the partial derivatives are evaluated at $(\tilde{W}_t, A_t, H_t, I_t)$, and

$$Q_t^*(\tilde{W}_t-, A_t-, H_t, I_t) = \arg \max_{0 < Q \leq \tilde{W}_t-+(1-\alpha)A_t H_t} \Psi(\tilde{W}_t- + (1 - \alpha)A_t H_t - Q_t H_t, Q_t, H_t, I_t),$$

$$\tau_t^* = \inf \{ t > \tau_{t-1} : \Psi(\tilde{W}_t-, A_t-, H_t, I_t) = \Psi(\tilde{W}_t- + (1 - \alpha)A_t H_t - Q_t^* H_t, Q_t^*, H_t, I_t) \},$$

$$A_t^* = Q_{\tau_t^*}^*(\tilde{W}_{\tau_t^*}, A_{\tau_t^*}, H_{\tau_t^*}, I_{\tau_t^*}).$$

Then $\Psi(\tilde{W}, A, H, I)$ coincides with the value function as given in (5.7), and $\Theta^* = \{ C_t^*, \pi_t^*, (\tau_t^*, A_t^*) \}$ is the corresponding optimal investment policy.

Proof. Let $\Psi(\tilde{W}, A, H, I)$ be a smooth solution to the HJB equation (A.2-1) and satisfy the transversality condition given in Theorem 2. Given any admissible investment policy $\Theta = \{ C_t, \pi_t, (\tau_t, A_t) \}$, we denote by $(\tilde{W}_t, A_t, H_t, I_t)$ the stochastic processes generated by policy $\Theta$ for notional convenience. Define $\mathcal{O}_n := \{ (\tilde{W}, A, H, I) : \frac{1}{n} \leq \tilde{W} + (1 - \alpha) AH \leq n, \frac{1}{n} \leq A \leq n, \frac{1}{n} \leq H \leq n, |I| \leq n \}$ and a sequence of stopping times $T_n := n \wedge \inf \{ t \geq 0 : (\tilde{W}_t, A_t, H_t, I_t) \notin \mathcal{O}_n \}$. 

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By the generalized Ito’s formula,

\[ e^{-(\beta + \delta)T_n} \Psi(\widetilde{W}_{T_n}, A_{T_n}, H_{T_n}, I_{T_n}) \]

\[ = \Psi(\widetilde{W}, A, H, I) - \int_0^{T_n} e^{-(\beta + \delta)u} \left( \frac{\tilde{C}^{1-\theta} A^\theta_u}{1 - \gamma} \right) du \]

\[ + \int_0^{T_n} e^{-(\beta + \delta)u} \mathcal{L} \Psi(\widetilde{W}_u, A_u, H_u, I_u) du \]

\[ + \int_0^{T_n} e^{-(\beta + \delta)u} (\sigma_u S \Psi \widetilde{W}_u - \lambda S \Psi I_u) (\widetilde{W}_u, A_u, H_u, I_u) dB_{su} \]

\[ + \int_0^{T_n} e^{-(\beta + \delta)u} (\sigma_u H \Psi H_u + \sigma_u \Psi I_u) (\widetilde{W}_u, A_u, H_u, I_u) dB_{Ht} \]

\[ + \sum_{\tau_i < T_n} e^{-(\beta + \delta)\tau_i} \left( \Psi(\widetilde{W}_{\tau_i}, A_{\tau_i}, H_{\tau_i}, I_{\tau_i}) - \Psi(\widetilde{W}_{\tau_i-}, A_{\tau_i-}, H_{\tau_i}, I_{\tau_i}) \right), \tag{A.2-3} \]

where

\[ \mathcal{L} \Psi = \frac{1}{2} \sigma^2 \bar{W}^2 \Psi \widetilde{W} + \frac{1}{2} \sigma_H^2 H^2 \Psi H + \frac{1}{2} (\lambda^2 \sigma_S^2 + \sigma_H^2) \Psi I - \lambda \sigma_S^2 \Psi \widetilde{W} I 

+ \sigma_H^2 H \Psi H + [r \widetilde{W} - \tilde{C} + (\mu_S - r) \pi] \Psi \widetilde{W} + \mu_H(I) H \Psi H 

+ k(\bar{I} - I) \Psi I - \delta A \Psi A - (\beta + \delta M) \Psi + \left( \frac{\tilde{C}^{1-\theta} A^\theta}{1 - \gamma} \right). \]

Since \( \Psi \) is the solution to HJB equation (A.2-1), the second integral term and the last term in the RHS of (A.2-3) are nonpositive. Bellman’s principle of optimality suggests that these two terms equal zero under the optimal policy \( \Theta^\ast = \{ \tilde{C}^\ast_t, \pi^\ast_t, (\tau^\ast_t, A^\ast_t) \} \). The two Ito integrals under expectation equal zero because \( \Psi \tilde{W}, \Psi H, \Psi I \) are bounded when \( (\widetilde{W}_u, A_u, H_u, I_u) \) is in the bounded domain \( \mathcal{O}_n \) during \( [0, T_n] \). Taking expectation in (A.2-3), we have

\[ \Psi(\widetilde{W}, A, H, I) \geq E \left[ \int_0^{T_n} e^{-(\beta + \delta)u} \left( \frac{\tilde{C}^{1-\theta} A^\theta_u}{1 - \gamma} \right) du \right] + E \left[ e^{-(\beta + \delta)T_n} \Psi(\widetilde{W}_{T_n}, A_{T_n}, H_{T_n}, I_{T_n}) \right]. \]

As analyzed above, the equality above holds only for the claimed optimal strategy \( \Theta^\ast = \{ \tilde{C}^\ast_t, \pi^\ast_t, (\tau^\ast_t, A^\ast_t) \} \). As \( n \to \infty \), \( T_n \) tends to infinity with probability 1. By the transversality condition of \( \Psi \) and the
dominant convergence theorem, the first expectation above converges to the original utility function $E \left[ \int_0^\infty e^{-(\beta+\delta t)} u \left( \frac{e^{1-\theta \hat{A} t}}{1-\gamma} \right)^{1-\gamma} dt \right]$ and the second expectation goes to zero. Equality holds for the claimed optimal policy $\Theta^*$ and $\Psi$ coincides with original objective function. This completes the proof. 

\[ \square \]

### A.3 Conditional expected return of housing and stock

The sources of the stock market data and the house price series are the Standard & Poor’s and Case Shiller home price index on Dec. 1 from 1890 to 2017 respectively, both of which are inflation adjusted Nov 2019 dollars. As shown in Section 3, the residual process $I_t = \log H_t - \lambda \log S_t$ with $\lambda = 0.2695$. To compare the conditional expected return of housing and stock, we analyze the growth rates of house price and stock price when $I_t$ falls below its long-term limit $\bar{I}$. We divide the whole sample periods into two groups by the sign of $I_t$. The result is shown in Figure 15. The upper panel shows the residual process $I_t = \log H_t - 0.2695 \log S_t$. The middle panel depicts the house growth rate $\log(H_{t+1}/H_t)$ at the state of $I_t < \bar{I}$. The lower panel depicts the stock growth rate $\log(S_{t+1}/S_t)$ at the state of $I_t < \bar{I}$. The average house growth rate when $I_t < \bar{I}$ from observation year 1890 (1953) to 2017 equals 0.0119 (0.0125), while the average stock growth rate when $I_t < \bar{I}$ from observation year 1890 (1953) to 2017 equals −0.0063 (0.0068). It shows that the house price grows faster than the stock price in the state of $I_t < \bar{I}$, supporting that when the conditional expected return of housing is high relative to that of the stock, households would like to stay away from the stock market, as analyzed in the main body of our paper.
Figure 15: **Growth rate of house price and stock price.** The upper panel shows the residual process $I_t$. The middle panel depicts the house growth rate $\log(H_{t+1}/H_t)$ at the state of $I_t < \bar{I}$. The lower panel depicts the stock growth rate $\log(S_{t+1}/S_t)$ at the state of $I_t < \bar{I}$. The average house growth rate when $I_t < \bar{I}$ from 1890 (1953) to 2017 equals $0.0119 (0.0125)$, while the average stock growth rate when $I_t < \bar{I}$ from 1890 (1953) to 2017 equals $-0.0063 (0.0068)$. This implies that the house price grows faster than the stock price when the ratio of house price to stock price is low.