Optimal Consumption and Investment with Cointegrated Stock and Housing Markets [*]

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ABSTRACT

The well-documented nonparticipation in the stock market by many households and the highly negative correlation between stock and housing investment are puzzling. We show that stock and housing markets are cointegrated, and thus households significantly increase housing expenditure, reduce stock investment, and may choose nonparticipation in the stock market at all if they face short-sale constraints. Our model can thus potentially help explain both the puzzle of stock market nonparticipation and the puzzle of the highly negative correlation between stock and housing investment. We also show some empirical evidence that is supportive of the model's main implications.

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1 Introduction

Only a small fraction of households participate, directly or indirectly, in the stock market. For example, in the United States, only 43% of households own stocks either directly or indirectly (e.g., through retirement plans), while in India, the number is a mere 8%. This limited participation is puzzling because standard models of lifetime consumption and portfolio choice predict that all households, no matter how risk-averse they are or how little wealth they have, should invest in stocks (Samuelson, 1969; Merton, 1969, 1971; Arrow, 1971). Another piece of empirical evidence is the highly negative correlation between housing investment and stock investment across countries and across time. For example, in 2015, the cross-country correlation between housing and stock investments is about -0.59 among 17 countries, including developed countries like the United States and the United Kingdom, and developing countries like China and India, and the cross-time correlation is about -0.71 in the United States (see Appendix for details). This is also puzzling, because the contemporaneous correlation between stock and housing prices is low (about 0.07), and the standard theories predict low correlations between housing investment and stock investment. In this study, we show that even though stock and housing markets have a low contemporaneous correlation, they are significantly cointegrated and this cointegration can help explain both of these puzzles. We also show some empirical evidence that is supportive of the main prediction of our model.

More specifically, we consider the optimal consumption and portfolio choice problem of a household in a continuous-time setting with a risk-free asset, a stock, and two consumption goods: a perishable good and housing service, subject to short-sale constraints on stock and housing investment. Unlike the existing literature, we first show that the stock and housing markets are significantly cointegrated and then study the impact of this cointegration on the optimal investment and consumption policy. Calibrated to the U.S. data, our model shows that the presence of cointegration between stock and housing markets significantly affects households' investment and consumption decisions. In particular, households may choose not to participate in the stock market even when there is no participation cost and the expected excess return on the stock is highly positive and significantly greater than that on the housing. In addition, the participation cost needed for households to never participate in the stock market is significantly smaller than without cointegration. Moreover, even when households do participate in the stock market, the investment amount is significantly reduced because of the cointegration. Furthermore, we find that the stock and housing investments are highly negatively correlated, even when the stock price and house price are independent (and thus standard theories predict zero correlation between stock investment and housing investment). These results are robust to the consideration of the option of renting a house (instead of owning one) and the high illiquidity in the housing market. Our model can thus potentially help explain the significant nonparticipation in stock markets and the highly negative correlation between stock investment and housing investment.

The main intuition is as follows. Even though the contemporaneous correlation between the stock and housing returns is close to zero, the presence of cointegration results in a significant and positive long-run correlation between the stock and housing markets. For example, the correlation between the 5-year stock and housing returns equals 0.2841 and the correlation between the 10-year stock and housing returns is as high as 0.4589.[1] Therefore, there is a strong substitution effect between the housing and stock investments if a household's investment horizon is long (e.g., 10 years). It is this substitution effect that drives our main results. For example, when the conditional expected return of housing is high relative to that of the stock,[2] households optimally borrow in the risk-free market to increase the size of their house.[3] In these states, in addition to borrowing in the bond market, households would also like to short sell the stock to finance the purchase of an even bigger house. However, due to the short-sale constraints, the best households can do is to stay away from the stock market. This is why households may choose nonparticipation in the stock market even if the stock market's unconditional expected return is much greater than that in the housing market and there is no participation cost. In addition, even when households do participate in the stock market, they invest less than in the case without cointegration, because owning a house already exposes a household to some stock market risk in the long run. When there is a participation cost, because of the indirect stock market exposure from investing in a house (due to the long-run correlation), the critical participation cost above which households choose never to participate in the stock market is much smaller than when there is no cointegration. The highly negative correlation between stock investment and house investment implied by our model also follows from this long-run substitution effect of housing investment for stock investment. In addition, if housing is also a consumption good, then it has a dual role: consumption and investment. This dual role magnifies the demand for housing and reduces stock investment further. Allowing a housing rental market may make our results even stronger, because with access to the rental market, households may optimally choose to buy even bigger houses (further reducing

[1] Modeling the driving force behind the empirical evidence that the correlation between the stock and housing markets increases with the horizon is out of this paper's scope. We suspect this horizon dependent correlation could be consistent with the existence of common factors that affect both the stock and housing markets (e.g., underlying production technology), whose effect is confounded by short-term noises, and thus appears to be statistically small in the short-run. However, the effect becomes statistically and economically significant in the long run after the noises are averaged out.

[2] See Section A.2 for empirical evidence for the time-varying conditional expected return of the housing investment relative to the stock investment.

[3] This is consistent with Fischer and Stamos (2013) and Corradin, Fillat, and Vergara-Alert (2014). Fischer and Stamos (2013) show that the households choose a higher housing-to-net-worth ratio in good states of housing market cycles (Table 3 and Figure 1). Corradin, Fillat, and Vergara-Alert (2014) show that the housing portfolio share immediately after moving to a more valuable house is higher during periods of high expected growth in house prices (Figure 4, Table 3, and Table 7).

stock investment) and rent out part of the houses to finance the purchae. This way, households can benefit more when the conditional expected return of housing is high relative to that of stocks.[4] The incorporation of housing market illiquidity does not change our main results either and can even enhance them. This is because, with illiquidity in the housing market, households stay in the same houses for a longer period of time, and for cointegrated processes, the correlation increases with duration. As a result, the substitution effect of housing investment for stock investment increases.

To the best of our knowledge, although various types of cointegration between the stock and housing markets have been found in the existing literature (see, e.g., Anoruo and Braha, 2008; Tsai, Lee, and Chiang, 2012), this paper is the first to study how this cointegration affects household investment behavior and can help explain the puzzle of non-/limited participation in stock markets as well as the puzzle of the highly negative correlation between stock and housing investment. In addition, unlike the existing literature on the cointegration tests for the two markets, we are the first to use the Johansen trace test to establish cointegration in the form of the stationarity of the log of the ratio of the housing price to the stock price raised to an empirically estimated power.

The main prediction of our model is that as the degree of cointegration between housing and stock markets increases, stock investment decreases and stock market nonparticipation increases. To see if this prediction has empirical support, we utilize the U.S. cross-state variations of the degree of conintegration, stock investment, and stock market nonparticipation to examine the relations among the three. Using data from the Panel Study of Income Dynamics (PSID) at the family level in the 2015 and 2017 waves, we calculate the average value of equity in stocks, the average ratio of financial wealth invested in stocks, and the proportion of interviewed families that do not invest in stocks for each state. We find that, consistent with the model prediction, as the degree of cointegration increases, stock investment decreases and nonparticipation in the stock market increases. (Cointegration can help to explain the distinguished investment behavior across different states. We also need to claim that the risk aversion and other heterogeneous factors induce different investment behavior for investors living in the same state.)

In the existing literature, there are several explanations for the nonparticipation and limited participation puzzle. Benzoni, Collin-Dufresne, and Goldstein (2007) consider the impact of the cointegration between labor income and stock market return. They find that because of the cointegration, investors with labor income invest less in the stock market. Despite the different economic contexts, the substitution effect between stock investment and housing in our model is qualitatively similar to that between labor income and stock investment in their model, except the cointegration

[4]Even when households cannot afford to buy a house and thus have to rent, our main results still hold as long as they can invest in the housing market through securities such as the Case-Shiller House Index futures, because the driving force behind our main results is the substitution effect between the stock market and housing market investment, which exists regardless of homeownership.

vector.[5] Vissing-Jørgensen (2002) shows that a moderate participation cost can explain half of the nonparticipation observed in the data. Cocco (2005) finds that housing crowds out stockholdings, which, together with a sizable stock market entry cost, can explain stock market nonparticipation early in life. Yao and Zhang (2005) examine the substitution and diversification effect of equity investment through an optimal dynamic portfolio decision model for households that acquire housing services from either renting or owning a house. They predict that housing investment has a negative effect on stock market participation. Kraft, Munk, and Wagner (2017) propose a rich life-cycle model of household decisions. After considering housing habit, they obtain that stock investments are low or zero for many young agents and then gradually increase as they age. Linnainmaa (2005) argues that short-sale constraints combined with learning can generate nonparticipation even when the constraints are not binding at present. Ambiguity aversion, disappointment aversion, and behavioral, cognitive and psychological constraints are also offered as possible explanations for the nonparticipation puzzle (e.g., Epstein and Schneider, 2008; Cao, Wang, and Zhang, 2005; Ang, Bekaert, and Liu, 2005; Andersen and Nielsen, 2011). Unlike our paper, all these studies ignore the cointegration between the stock and the housing markets. Our model complements these extant theories and may strengthen their explanatory power. For example, our model suggests that the participation cost in Vissing-Jørgensen (2002) and ambiguity aversion in Cao, Wang, and Zhang (2005) required to explain nonparticipation would be significantly smaller if cointegration were incorporated.

Our paper also relates to recent papers on housing decisions. Hemert (2010) investigates household interest rate risk management with a life-cycle asset allocation model that includes mortgage and bond portfolio choice and finds some hedge between housing and interest rate. Fischer and Stamos (2013) set up a regime-switching model with slow-moving time variation in expected housing returns and find that homeownership rates and the share of net worth in a home increase in good states of housing market cycles. Corradin, Fillat, and Vergara-Alert (2014) show that higher expected growth rates in house prices cause house (stock) investment to increase (decrease), but stock investment is still significant even with a high risk aversion.

As for the puzzle concerning the highly negative correlation between stock and housing ownership/investment, although some studies (e.g., Cocco, 2005; Yao and Zhang, 2005) imply a negative relationship, no extant studies have shown whether the magnitudes of the correlations in their models can be as large as those observed in data.

The rest of the paper is organized as follows. Section 2 describes the benchmark cointegration model. Section 3.1 shows that the stock and housing markets are cointegrated and provides an

[5] Benzoni, Collin-Dufresne, and Goldstein (2007) assume that the difference between income return and stock return is a stationary process such that the cointegration vector simply equals $(1, 1)$. However, regarding to housing return and stock return based on the real empirical data, the cointegration vector should to be written as $(1, \lambda)$, where λ is the cointegration factor that needs to be estimated. See Section 3.1 for more details.

estimation of cointegration parameter values. In Section 4, we quantitatively illustrate that the cointegration between the stock and housing markets leads to non-/limited participation in the stock market and a highly negative correlation between stock and housing investment. Section **??** demonstrates the robustness of our results to the option of renting and to the presence of house illiquidity. In Section 3.2, we provide some empirical evidence that is supportive of the predictions of our model. Section 5 concludes the paper. Empirical facts on nonparticipation and correlations, some Hamilton-Jacobi-Bellman (HJB) equations, and all the proofs are provided in the Appendix.

2 The Model

We consider a continuous-time model where a small household (i.e., with no price impact) maximizes its expected utility from consuming a perishable consumption good and possibly consuming a house's service flow. In addition to trading houses and the perishable consumption good in the goods markets, the household can also trade a risky stock and a risk-free bond in the financial market without any transaction costs.

2.1 Financial markets

The bond grows at a constant risk-free rate r. The stock's price S_t evolves according to the following dynamics:[6]

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dB_{St}, \tag{2.1}$$

where $\mu_S > r$ is a constant representing the instantaneous expected return, σ_S is a constant representing the instantaneous volatility of a stock return, and B_{St} is a one-dimensional standard Brownian motion.

2.2 The housing market

To simplify the analysis, we start by assuming that by selling, buying, remodelling, and expanding, the household can continuously adjust the house size without transaction costs.[7] Let I_t denote the national housing price index per square footage at time t. Unlike the existing literature, we allow the stock and housing markets to be cointegrated. More specifically, in a similar spirit to Benzoni,

[6] Like Benzoni, Collin-Dufresne, and Goldstein (2007), we can start with a dividend process and a cointegrated housing service process and specify a pricing kernel to generate the stock price process and the cointegrated housing process described later. The derivation is omitted here to save space, but is available from the authors.

[7] When houses are indivisible and buying/selling a house incurs a transaction cost, as in Grossman and Laroque (1990), the household's problem becomes much more complicated; this scenario is considered in Section 4.2 to show the robustness of our results.

Collin-Dufresne, and Goldstein (2007), we assume that the following log-ratio denoted by

$$R_t = \log I_t - \lambda \log S_t \tag{2.2}$$

for some positive constant λ follows a mean-reverting process

$$dR_t = k(\bar{R} - R_t)dt + \sigma_I dB_{It} - \nu_S dB_{St}, \tag{2.3}$$

where the constant $k \geq 0$ measures the degree of cointegration, $\bar{R}$ denotes the long-term mean, σ_I and ν_S are conditional volatilities, and B_{It} is another standard Brownian motion reflecting the uncertainty in the aggregate housing price index and is independent of B_{St}. In the next section, we show that R_t for some empirically estimated value of λ is indeed mean-reverting. Our specification (2.3) for the log-ratio R_t reflects a long-run cointegration between the stock and housing markets, because it implies that the log-ratio of the house price index to the stock price raised to the power of λ tends to the long-run mean $\bar{R}$ as time passes. When $R_t > \bar{R}$, the house price index tends to decrease over time relative to the stock price, whereas when $R_t < \bar{R}$, the opposite is true.

By (2.2), we have $I_t = e^{R_t} S_t^{\lambda}$, and hence

$$\frac{dI_t}{I_t} = \mu_I(R_t)dt + \sigma_I dB_{It} + (\lambda\sigma_S - \nu_S)dB_{St}, \tag{2.4}$$

where

$$\mu_I(R) = \mu_{I0} + k(\bar{R} - R), \;\; \mu_{I0} = \lambda\mu_S + \frac{1}{2}\sigma_I^2 + \frac{1}{2}\nu_S^2 + \frac{1}{2}\lambda(\lambda - 1)\sigma_S^2 - \lambda\sigma_S\nu_S. \tag{2.5}$$

In this paper, we focus on the analysis of the effect of the cointegration between the housing and stock markets. In addition, as shown in the existing literature and our later analysis in Section 3.1, the contemporaneous correlation between housing and stock returns is almost zero. Accordingly, we will simply assume that the contemporaneous correlation between housing and stock returns is zero, i.e., $\lambda\sigma_S = \nu_S$.[8] The focus of cointegration between stock and housing markets distinguishes our model from others that ignore such cointegration (see e.g., Fischer and Stamos, 2013; Corradin, Fillat, and Vergara-Alert, 2014).

Note that with $\lambda\sigma_S = \nu_S$, although the contemporaneous correlation between housing and stock returns is zero, the housing and stock markets are linked. For example, after a positive shock in B_{St} to the stock return, the log-ratio R_t decreases (equation (2.3)), which in turn increases the conditional expected return of housing $\mu_I(R_t)$ (equation (2.5)). It is this cointegration that drives our main results.

[8] Increasing the correlation would make our results even stronger.

We assume that there is a continuum of households, and each household can buy and sell houses in its local housing market. For household i, the local housing price $H_{it} = I_t \epsilon_{it}$, where ϵ_{it} represents idiosyncratic risk faced by household i and follows

$$\frac{d\epsilon_{it}}{\epsilon_{it}} = \mu_{\epsilon i} dt + \sigma_{\epsilon_i} dB_{it}, \tag{2.6}$$

where $\mu_{\epsilon i}$ and $\sigma_{\epsilon i}$ are constants and B_{it} is a Brownian motion that is independent across i and of all other risks. This implies that

$$\frac{dH_{it}}{H_{it}} = \mu_H(R_t)dt + \sigma_I dB_{It} + \sigma_{\epsilon i} dB_{it}, \tag{2.7}$$

where $\mu_H(R) = \mu_I(R) + \mu_{\epsilon i} = \mu_{H0} + k(\bar{R} - R)$ with $\mu_{H0} = \mu_{I0} + \mu_{\epsilon i}$.

2.3 Preferences

A household derives utility not only from the perishable consumption good that serves as the numeraire but also possibly from the housing service flow that is proportional to the house size. Thus, unlike financial assets, in addition to the role of an investment vehicle, a house may also directly contribute to utility. We assume that the service flow from a house is proportional to the house size A_t and equal to ιA_t, where $\iota > 0$ is a constant and set to 1 without loss of generality. Following the existing literature (see e.g. Damgaard, Fuglsbjerg, and Munk, 2003; Cocco, Gomes, and Maenhout, 2005; Yao and Zhang, 2005; Kraft and Munk, 2011; Fischer and Stamos, 2013; Corradin, Fillat, and Vergara-Alert, 2014), we assume that the household's preferences over the housing service flow and nonhousing goods take the following nonseparable Cobb-Douglas utility form:

$$U(C, A) = \frac{1}{1-\gamma} \left(C^{1-\theta} A^{\theta} \right)^{1-\gamma},$$

where C represents the perishable good consumption, $\theta \geq 0$ measures the preference for housing, and $\gamma > 0$ is the constant relative risk aversion coefficient.

2.4 The household's optimization problem

Compared to stock investment, trading in the housing market can incur significant transaction costs. Following Grossman and Laroque (1990), we assume that to change the house size, a household has to first sell the old house and then purchase a new one of the preferred size, and the household must pay a transaction cost that is proportional to the value of the house sold. More specifically, if the household wants to buy a new house at time τ_i, it is necessary to sell the original house first and pay a transaction cost of $\alpha A_{\tau_i -} H_{\tau_i}$, where $\alpha \in [0, 1)$ represents the proportional transaction cost rate, A_{t_i-} is the size of the original house sold at time t_i-, and H_{t_i} is the market price of the house at that time. After selling, the household buys a new house with size $A_i \geq 0$.[9] Define $\widetilde{W}_t$ as the financial wealth invested in bonds and stocks, π_t as the dollar amount invested in stocks, and C_t as the perishable good consumption. We have

$$\begin{aligned}
d\widetilde{W}_t &= [r\widetilde{W}_t - C_t + \pi_t(\mu_S - r)]dt + \pi_t\sigma_S dB_{St},\ t \neq \tau_i, && (2.8)\\
dA_t &= -\delta A_t dt,\ t \neq \tau_i, && (2.9)\\
\widetilde{W}_{\tau_i} &= \widetilde{W}_{\tau_i-} + (1-\alpha)A_{\tau_i-}H_{\tau_i} - A_{\tau_i}H_{\tau_i},\ i = 1, 2, ..., && (2.10)\\
A_{\tau_i} &= A_i,\ i = 1, 2, ... && (2.11)
\end{aligned}$$

where δ is the depreciation rate of housing.

Following the existing literature (e.g., Gomes and Michaelides, 2005; Cocco, Gomes, and Maenhout, 2005; Polkovnichenko, 2007; Munk and Sørensen, 2010; Wachter and Yogo, 2010; Lynch and Tan, 2011; Flavin and Yamashita, 2011), we assume that the household cannot short sell stock or houses, that is, $\pi_t \geq 0$ and $A_t \geq 0$.[10] However, the household can borrow against the house, up to a fraction $(1-l)$ of the liquidated value of housing, i.e.,

$$\pi_t + l(1-\alpha)A_t H_t \leq \widetilde{W}_t + (1-\alpha)A_t H_t, \tag{2.12}$$

where $l \in (0, 1]$ is a constant, representing the maximum leverage allowed for house purchases.

The household chooses the perishable consumption C_t, the stock investment π_t, and the housing size A_t to maximize the expected utility from consumption of the perishable good and the housing service from time 0 to the first jump time $\mathcal{T}$ of an independent Poisson process with intensity δ_M, which represents the mortality rate of the household.[11] Let $\mathcal{A}$ denote the set of all admissible strategies (C_t, π_t, A_t), i.e., the strategies that satisfy the budget constraint (2.8), the short-sale constraint $\pi_t \geq 0$ and $A_t \geq 0$, the limited borrowing constraint (2.12), and the solvency constraint $\widetilde{W}_t + (1-\alpha)A_t H_t > 0$, for given processes (2.3), (2.7)–(2.11). We define the value function as

$$\begin{aligned}
\Psi(\widetilde{W}, A, H, R) :&= \max_{C_t,\ \pi_t \geq 0,\ (\tau_i, A_i)} \mathbb{E}\left[\int_0^{\mathcal{T}} e^{-\beta t} \frac{\left(C_t^{1-\theta} A_t^{\theta}\right)^{1-\gamma}}{1-\gamma} dt\right]\\
&= \max_{C_t,\ \pi_t \geq 0,\ (\tau_i, A_i)} \mathbb{E}\left[\int_0^{\infty} e^{-(\beta+\delta_M)t} \frac{\left(C_t^{1-\theta} A_t^{\theta}\right)^{1-\gamma}}{1-\gamma} dt\right] && (2.13)
\end{aligned}$$

[9]As in practice, we assume only sellers pay the real estate agent fee.

[10]Note that the optimal C_t must be strictly positive because of the utility function form.

[11]The assumption of a random horizon eliminates the time dependence of the optimal strategies. Using a deterministic horizon would not change our qualitative results. In addition, as Liu and Lowenstein (2002) suggest, the optimization problem with a random horizon can be a good approximation for a deterministic horizon when the expected horizon is long.

The value function as given in (2.13) satisfies the following HJB equation:

$$\begin{aligned}
\max\Bigg\{ &\max_{C,\, 0\le\pi\le\widetilde{W}+(1-l)(1-\alpha)AH} \Big(\frac{1}{2}\sigma_S^2\pi^2\Psi_{\widetilde{W}\widetilde{W}} + \frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)H^2\Psi_{HH} + \frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)\Psi_{RR} \\
&- \lambda\sigma_S^2\pi\Psi_{\widetilde{W}R} + \sigma_I^2 H\Psi_{HR} + [r\widetilde{W} - C + (\mu_S - r)\pi]\Psi_{\widetilde{W}} + \mu_H(R)H\Psi_H \\
&+ k(\bar{R}-R)\Psi_R - \delta A\Psi_A - (\beta+\delta_M)\Psi + \frac{\left(C^{1-\theta}A^{\theta}\right)^{1-\gamma}}{1-\gamma}\Big), \\
&\max_{0<Q\le\frac{\widetilde{W}+(1-\alpha)AH}{l(1-\alpha)H+\alpha H}} \Psi(\widetilde{W}+(1-\alpha)AH - QH, Q, H, R) - \Psi(\widetilde{W}, A, H, R)\Bigg\} = 0
\end{aligned} \tag{2.14}$$

in Solvency region $\Omega = \{(\widetilde{W}, A, H, R) : \widetilde{W} + (1-\alpha)AH > 0, A > 0, H > 0, R \in \mathbb{R}\}$.

By the homogeneity property, we can make the following transformation:

$$\Psi(\widetilde{W}, A, H, R) = \frac{1}{1-\gamma}(\widetilde{W} + (1-\alpha)AH)^{1-\gamma}H^{-\theta(1-\gamma)}e^{(1-\gamma)\phi(h,R)}, \quad h = \frac{(1-\alpha)AH}{\widetilde{W}+(1-\alpha)AH},$$

where h is the ratio of house value to net wealth, and $\phi(h, R)$ is a function to be determined. The corresponding HJB equation and the iterative algorithm for solving it are given in Appendix A.1.

3 Empirical Analysis

In this section, we do empiricial analysis based on various data source to testify the following three hypotheses:

- There exists cointegration between US stock and housing market.
- The cointegration may contribute to low stock participation in nationwide.
- The correlation between stock and housing investment is highly negative, in both national and individual levels.

3.1 Cointegration Test

In this section, we aim to test whether there is cointegration between stock price and housing index and estimate the cointegration degree if there is. The sources of the stock market data and the house price index series are Standard & Poor's and the Case-Shiller Home Price Indices (CSI), respectively, both of which are inflation-adjusted to year 2019.[12] The annual data from 1890 to

[12]The data can be downloaded from `http://www.econ.yale.edu/~shiller/data` and `http://www.econ.yale.edu/~shiller/data/Fig3-1.xls`. The data after year 2016 are supplemented from online sources.

2019 are shown in Fig 1.

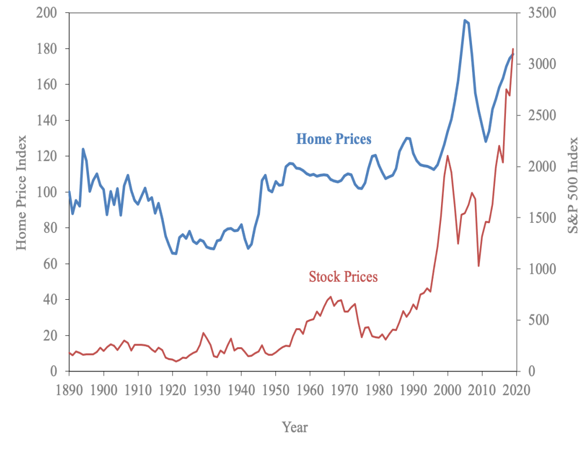


Figure 1: **Historical stock and housing index, inflation adjusted to year 2019.**

Before conducting the test, we first estimate the contemporaneous correlation between the housing index and stock returns in our data set. We find that the correlation between $\log\left(\frac{S_t}{S_{t-1}}\right)$ and $\log\left(\frac{I_t}{I_{t-1}}\right)$ is 7.14%, which is consistent with the findings in the existing literature and leads to our simplifying assumption that the housing price index and the stock price have zero contemporaneous correlation (i.e., $\lambda\sigma_S = \nu_S$). In Fig 1, we may observe the information transition through the stock market to the housing market, implying a delayed reaction of housing price to the shocks in stock market. Tsai, Lee, and Chiang (2012) state that such transition mechanism is due to the wealth effect but the reverse transition is statistically unlikely. To reflect the asymmetrical long-term relationship of the two markets, we formulate the housing return as a function of R_t in (2.5), while keeping stock return as constant μ_S.

We use the trace test proposed by Johansen (1988,1991) to examine whether the stock and housing markets are cointegrated, because this test is an improvement over the two-step test proposed by Engle and Granger (1987).[13] Note that the normalized cointegration vector $(1, -\lambda)$ is chosen such that the residual process $R_t = \log I_t - \lambda \log S_t$ becomes stationary. The value of λ is estimated using Maximum Likelihood Estimation (MLE) methods. The estimates are asymptotically normal and super consistent under Johansen test. Using this method on our data set, we find that the Johansen MLE estimator $\hat{\lambda}^{MLE}$ equals 0.2640. As a result, we set $\lambda = \hat{\lambda}^{MLE} = 0.2640$ in benchmark calibration throughout the paper.[14] The trace test shows that the residual process

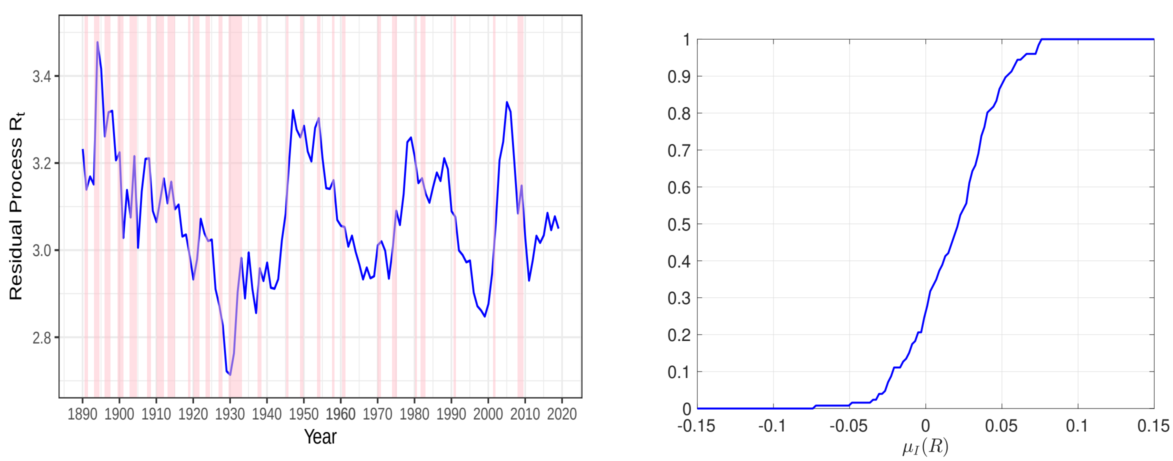
[13] The two-step test of Engle and Granger (1987) is also called augmented Engle-Granger (AEG) test. The AEG test first runs a regression of house prices on the fundamentals and then conducts a unit-root test–the augmented Dickey–Fuller (ADF) test on the residuals.

[14] We also conducted the AEG test. The results are similar. For example, the estimate for λ in this alternative test is 0.2378 and setting $\lambda = 0.2378$ in the benchmark calibration does not significantly change our main results.

$R_t = \log I_t - \lambda \log S_t$ follows a stationary AR(1) process:

$$R_{t+\Delta t} = m + \phi R_t + \epsilon_{t+\Delta t}, \tag{3.1}$$

where Δt is the time between adjacent observations with $m = 0.4511$ and $\phi = 0.8535$. Then, we can compare equations (2.3) and (3.1) to imply the speed of the mean-reversion coefficient k, the mean $\bar{R}$, and the variance σ_I in equation (2.3). The residual process R_t is plotted in the left panel of Fig 2. We observe that R_t increases in recession periods due to the plunge in stock market and an increasing attempt of home purchase under lower mortgage rates. The right panel of Fig 2 shows the probability distribution function of $\mu_I(R)$ based on the historic data from 1890 to 2019, which we use to calibrate some default parameter values. From this figure, we find that the values of $\mu_I(R)$ were concentrated in the interval $(-0.0785, 0.0709)$. The optimal strategy in our model shows that households tend to increase housing investment as well as decrease stock investment when the residual R increases (or equivalently, when $\mu_I(R)$ decreases). Particularly, households have low (or even zero) participation in stocks when the value of R is higher above its long-term average $\bar{R}$ (or equivalently, when $\mu_I(R)$ is above μ_{I0}).



(i) Residual process R_t

(ii) Probability distribution function of $\mu_I(R)$.

Figure 2: **Residual process R_t and probability density function of housing index return.** The left panel plots the residual process $R_t = \log I_t - 0.2640 \log S_t$ and the historical recession periods (pink rectangles). The right panel shows the probability distribution function of housing index return $\mu_I(R)$ defined in (2.5) using the historical data of house price index.

As noted before, even if the stock and housing markets are contemporaneously uncorrelated, the two markets will be correlated for a longer horizon if they are cointegrated. To see if the set of parameter values estimated above is reasonable, we next compute the model-implied long-term correlations between stock and housing index returns for horizons of 1 year, 5 years, and 10 years by simulation. We then compare these correlations with the corresponding empirical correlations in the data. Table 1 shows that the model-implied correlations match well with the empirical correlations, which suggests the estimated model reflects the data reasonably well.

Table 1: **Long-term correlation**

	Correlation between house and stock return		
	1-year	5-year	10-year
Historical data observation	0.0714	0.2841	0.4589
Cointegration model implied	0.0874	0.2807	0.4483

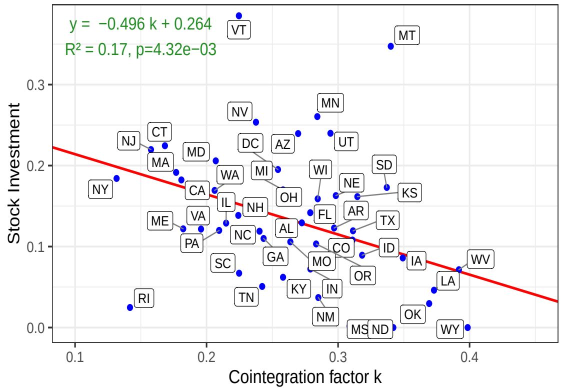
3.2 Cross-State Cointegration Effect on Stock Investment

In United States, only 43% of households participate directly or indirectly in the stock markets. We claim that a stronger degree of cointegration between housing and stock markets contributes to lower stock market participation. To see if this hypothesis has any empirical support, we next utilize the U.S. cross-state variations of the degree of conintegration, stock investment, and stock market nonparticipation to examine the relations among the three. We use the PSID data of family level in 2015 and 2017 waves, totalling more than 9,000 observations. The value of stockholding is extracted from variable ER65368 in the 2015 wave and ER71445 in the 2017 wave. Financial wealth is calculated as the sum of equity in stocks and the value in safe account, where the value in safe account is the money amount in checking and savings accounts, money market funds, certificates of deposit, government bonds, or treasury bills. In the 2015 wave, the value in safe account is extracted from variable ER61772. In the 2017 wave, the value in safe account is extracted from variable ER67826. We then calculate the average value of equity in stocks, the average ratio of financial wealth invested in stocks,[15] and the proportion of interviewed families that do not invest in stocks for each state. To analyze the cointegration between housing and stock markets in each state, we refer to the monthly state-level Housing Price Index data from 1975 to 2019 from the Federal Housing Finance Agency (FHFA) and estimate the strength of cointegration parameter k for each state using similar cointegration test and estimation as subsection 3.1. Note that the larger the value of k, the stronger the cointegration between the housing and stock markets.

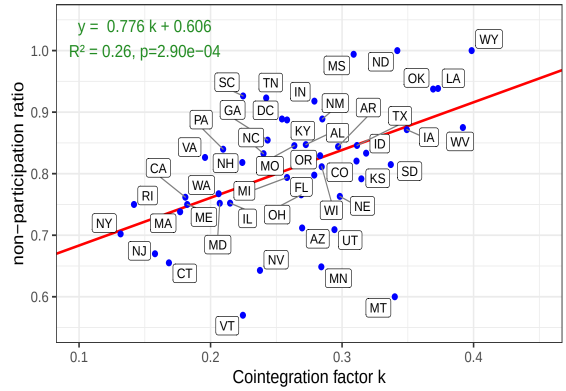
Figure 3 shows that, at the state level, as the degree of cointegration increases, stock investment decreases and nonparticipation in the stock market increases. For example, Connecticut has a low cointegration between stock and housing prices; as a result, it has high stock investment

[15] We can also calculate the equity investment ratio as risky assets divided by the sum of risky assets and safe assets. The risky assets comprise stockholdings, IRAs, and annuity holdings. The safe assets include other assets (net of debt, such as bond and insurance), checking balances, and savings balances, less the principal on the primary residence. The result is similar and not reported here to save space, but available from the authors.

and low stock market nonparticipation on average. The result shows plausible evidences that the cointegration effect may help explain the low (or zero) particiaption in stock market.



(i) Cointegration effect and stock investment across states.



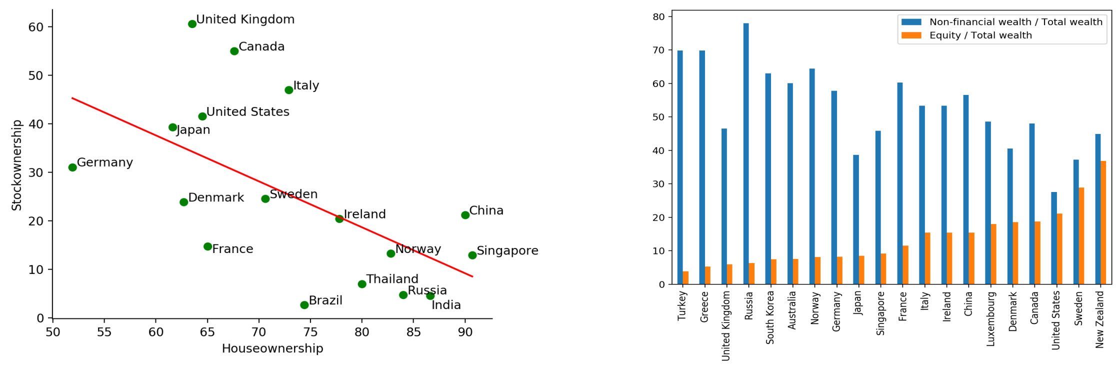
(ii) Cointegration effect and nonparticipation across states.

Figure 3: **Cointegration effect on stock investment.** Data come from state-level FHFA's Housing Price Index from 1975 to 2019 and PSID of family level in 2015 and 2017 waves. The names of states are abbreviated; for example, "CA" refers to California. The red line is the linear regression result. It is observed that the higher the strength of cointegration between the stock and housing markets, the lower the stock share and the higher the nonparticipation ratio. The p-value reports the significance test result of the coefficient of k in the linear regression.

3.3 Negative Correlation Between Stock and Housing Investment

In this subsection, we explore the correlation between stock and housing investment in both nation-level and individual-level. We first compare the homeownership and stock ownership across different countries, and then calculate the stock and housing investment in different countries. Moreover, we calculate the corresponding correlation of stock and housing investment across time in both national level and individual level.

Using the data in Grout, Megginson, and Zalewska (2009), the left panel of Figure 4 plots stock ownership (including direct and indirect ownership) against homeownership across 17 countries in 2015, with the red line showing the OLS regression. This figure suggests that significant stock market nonparticipation is an international phenomenon, although standard portfolio choice theories predict close to 100% participation. The highest participation rate is about 61% (United Kingdom) and the lowest about 8% (India). Despite exceptional stock returns, the participation rate in the United States is only about 43%. In addition, the left panel suggests that as homeownership increases, stock ownership tends to decrease, with a correlation between the two of about -0.59. The right panel of Figure 4 plots stock investment and housing investment across 20 countries in



(i) Homeownership and stock ownership across countries (2015)

(ii) Nonfinancial wealth and equity across countries (2015)

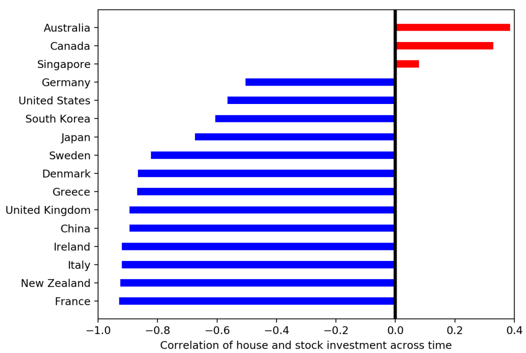
Figure 4: **Ownership and investment in stock and housing across countries.**

2015. It shows a strong pattern of negative correlation between stock investment and housing investment. Indeed, the correlation between the two is -0.62 on average across these 20 countries. Data are collected from the global wealth databook 2018 revealed by Credit Suisse.

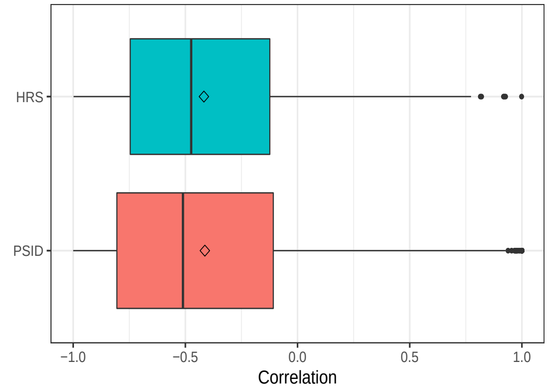
Further, using Credit Suisse's research report, we calculate the correlation between stock investment and housing investment across time over the period from 2000 to 2017 for 16 countries, as shown in the left panel of Figure 5. It suggests that stock investment and housing investment are highly negatively correlated across time for most countries.[16] The correlation coefficient is about -0.71 on average. This highly negative correlation is puzzling because it is well known that the contemporaneous correlation between stock and housing prices is low (around 0.07) and standard optimal investment theories imply a low negative correlation of around zero. We want to further testify if the highly negative correlation of stock and housing investment in national level stands in individual level as well. We use 1999-2017 waves of PSID to construct the time series of stock and housing investments for each household, in such a way the individual's correlation of stock and housing investment can be calculated. However, the PSID tracks (young) individuals over a long period of time but the number of individuals is relatively small. To strengthen the result, we also refer to the Health and Retirement Study (HRS) in 1992-2016 waves,[17] the advantage of which over PSID is a larger number of older individuals. The individual-level correlation is shown in the right panel of Figure 5. It is shown that both of the PSID and HRS data yields high negative

[16]The only exceptions are Australia, Canada, and Singapore, where the correlations are positive. The positive correlations found in these countries may be related to special immigration and real estate policies. For example, for Australia and Canada, immigration is encouraged and immigrants, who invest in both housing and stocks, tend to be wealthy.

[17]The data of HRS we used is RAND HRS 2016v1, the codebook of which is available at `https://www.rand.org/content/dam/rand/www/external/labor/aging/dataprod/randhrsimp1992_2016v1.pdf`.



(i) Housing and stock investment correlations across time (2000-2017)



(ii) Boxplot of individual's stock and housing investment (PSID: 1999-2017; HRS: 1992-2016)

Figure 5: **Correlation of stock and housing investment in national and individual levels.**

correlation (-0.5) on average in individual level.

4 Numerical Analysis

4.1 Parameter values

In Table 2, we report the default parameter values for our numerical analysis. According to Shiller's inflation adjusted data of real interest, S&P index price, and home price index, we set the inflation-adjusted interest rate at $r = 1.70\%$, the stock risk premium at 6.33% (i.e., the stock return $\mu = 4.63\%$), and the standard deviation of stock return at $\sigma_S = 17.46\%$. The coefficient of relative risk aversion is set at $\gamma = 10$ to approximately match the stockholdings relative to financial wealth observed in the PSID and Survey of Income and Program Participation (SIPP) literature. The parameter θ that measures the degree to which the household values housing consumption is set at 0.3 to be consistent with the average share of household housing expenditure in the United States (see, e.g., Corradin, Fillat, and Vergara-Alert, 2014). Households can short bonds to finance homeownership and the minimum housing down payment for homeowners is 20%, which implies that $l = 0.2$. The values of λ, k, $\bar{R}$, and σ_I are estimated in Section 3.1. We estimate the idiosyncratic risk of local house price by comparing the 52 states' house price indices to the national house price index. For each state, we assume that the ratio of the annual state house price index divided by the national house price index follows a geometric Brownian motion. Then, we average across states to get the estimated parameters: $\mu_{\varepsilon i} = 0.0056$ and $\sigma_{\varepsilon i} = 0.0316$. One limitation of the state-level idiosyncratic risk is that we can only get the optimal investment and consumption policy of a representative household in a state and cannot have heterogeneity within a state. If we were able to obtain zip code-level housing prices, we would be able to obtain such heterogeneity.

Table 2: **Parameter values used for benchmark calibration**

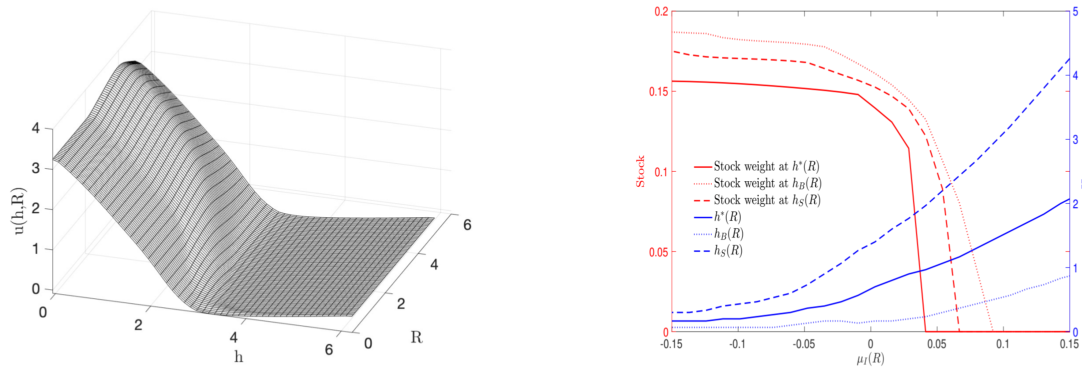
Variable	Symbol	Value
Riskless rate	r	0.0170
Stock expected return	μ_S	0.0633
Stock volatility	σ_S	0.1746
Weight of cointegration	λ	0.2640
Degree of cointegration	k	0.1584
Long-term log-ratio mean	$\bar{R}$	3.0800
Housing index volatility	σ_I	0.0791
Housing index return long-term mean	μ_{I0}	0.0188
Idiosyncratic housing price risk mean	$\mu_{\varepsilon i}$	0.0056
Idiosyncratic housing price risk volatility	$\sigma_{\varepsilon i}$	0.0316
Time discount rate	β	0.0170
Mortality rate	δ_M	0.02
Risk aversion coefficient	γ	10
The preference for housing	θ	0.3
Depreciation rate for housing	δ	0
Housing collateral rate	l	0.2

4.2 Illiquidity in the housing market

Following Corradin, Fillat, and Vergara-Alert (2014), we set the housing transaction cost to be $\alpha = 10\%$ of the unit's value as a baseline parameter value, including commissions, legal fees, the time cost of searching, and the direct cost of moving possessions. The numerical result is shown in Figure 6. In the presence of transaction costs, there exist an optimal buying ratio $h_B(R)$, an optimal selling ratio $h_S(R)$, and an optimal target ratio of house value to net wealth $h^*(R)$. When the ratio of house value to net wealth is below the optimal buying ratio $h_B(R)$, the household optimally sells the current house and purchases a bigger one such that the new ratio of house value to net wealth jumps upward to the optimal target level $h^*(R)$. When the ratio of house value to net wealth is above the optimal selling ratio $h_S(R)$, the household optimally also sells the current house but purchases a smaller one such that the new ratio of house value to net wealth jumps downward to the optimal target level $h^*(R)$. The area between $h_B(R)$ and $h_S(R)$ is the no-trading region. When the ratio of house value to net wealth falls inside this area, the household does not trade in the housing market.

The function $\phi(h, R)$ satisfies the corresponding HJB equation specified in the Appendix. Value-matching and smooth-pasting conditions hold at the two bounds $h_B(R)$ and $h_S(R)$, and an optimality condition holds at the target point $h^*(R)$. All of these free boundaries depend on the log-ratio R_t.

We plot the optimal ratios of stock value to net wealth $h_B(R)$, $h^*(R)$, and $h_S(R)$ in red-dotted, red-solid, and red-dashed lines, respectively. Figure 6 suggests that the presence of significant



(i) $\phi(h, R)$ (ii) Optimal strategy

Figure 6: **Optimal stock and housing investment with illiquid housing.** The left panel shows the optimal value of $\phi(h, R)$, solving (A.1-1) using penalty method. In the right panel, the blue-solid line is the optimal target ratio of house value to net wealth $h^*(R)$. The blue-dashed and blue-dotted lines are the optimal house selling ratio $h_S(R)$ and buying ratio $h_B(R)$, respectively. These three lines are on the right Y-axis. The red-solid, red-dashed, and red-dotted lines are the optimal ratio of stock to net wealth when the ratios of house value to net wealth equal $h^*(R)$, $h_S(R)$, and $h_B(R)$. When there is no cointegration, i.e., $k = 0$, the optimal housing selling ratio $h_S = 0.8000$, the optimal house buy ratio $h_B = 0.2333$, the optimal house target ratio $h^* = 0.4667$, and the optimal stock ratio equals 0.17078, 0.15767, and 0.14256 when the house ratio equals h_B, h_S, and h^*, respectively. Default parameter values are from Table 2: $r = 0.0170$, $\mu_S = 0.0633$, $\sigma_S = 0.1746$, $\lambda = 0.2640$, $\bar{R} = 3.0800$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0188$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = r$, $\delta_M = 0.02$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$, and $\alpha = 0.10$.

illiquidity in the housing market does not change our main result that with cointegration, non-/limited participation in the stock market can be optimal, and the house investment and stock investment are negatively correlated.[18]

To examine the average impact of cointegration, similar to the simulation in Subsection **??**, we set the initial value of the log-ratio R_t to be $R_0 = \bar{R}$. We then simulate 10,000 paths of processes R_t, H_{it}, and $\widetilde{W}_t$ by (2.3), (2.7), and (2.8), respectively. The policy $\{\pi_t, \tilde{C}_t, (\tau_i, A_i)\}$ is chosen from the optimal ones derived by maximizing the objective function. The results presented in Table 3 are averages across 52 states. Table 3 indicates that the presence of significant illiquidity does not change the results: (1) the cointegration effect on average lowers stock investment and increases housing investment and (2) stock investment and house investment are highly negatively correlated (with a mean correlation coefficient of -0.80).

[18]Similar to Grossman and Laroque (1990), when the housing level is at the optimal target, the corresponding stock investment is the lowest. This is because the risk aversion of the household is the highest at the target level, since it takes a significant amount of time before the house size can be changed.

Table 3: **Simulation with illiquid housing.** This table reports the simulated average of stock and house investments as percentages of net wealth. Default parameter values are from Table 2: $r = 0.0170$, $\mu_S = 0.0633$, $\sigma_S = 0.1746$, $\lambda = 0.2640$, $\bar{R} = 3.0800$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0188$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0170$, $\delta_M = 0.02$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$, and $\alpha = 0.10$.

Parameters		Stock Investment			House Investment			Investment correlation		
		Mean	Median	Std	Mean	Median	Std	Mean	Median	Std
$\theta = 0$	$k = 0$	0.1633	0.1605	0.0120	0.2694	0.2789	0.0416	0.0801	0.3352	0.3273
	$k = 0.1976$	0.0598	0.0560	0.0506	0.6598	0.6818	0.4531	-0.6324	-0.7270	0.1383
$\theta = 0.3$	$k = 0.1562$	0.1547	0.1478	0.0097	0.5168	0.5715	0.0580	-0.0227	0.2487	0.3600
	$k = 0.1976$	0.0411	0.0473	0.0168	0.8057	0.8548	0.1305	-0.7984	-0.8115	0.0910

5 Conclusion

In this paper, we consider the optimal joint choice of stock portfolio and housing of a household when the stock and housing markets are cointegrated. We show that in the presence of cointegration, households significantly reduce stock investment and increase housing investment. As a result, they may choose not to participate in the stock market at all even when there is no participation cost and the unconditional expected return of housing is lower than that of the stock. In the presence of participation cost, the critical wealth level below which households never participate in the stock market is much higher than that in the absence of cointegration, and the critical participation cost level above which households never participate in the stock market is much smaller than that in the absence of cointegration. These results are robust to extensions that incorporate rental alternatives and housing market illiquidity. Our model complements existing studies and can potentially help explain both the puzzle of stock market non-/limited participation and the puzzle of the highly negative correlation between stock and housing investment. We also show empirical evidence that is supportive of some predictions of our model. In particular, across the 50 states of the United States, as the degree of cointegration increases, stock investment decreases and nonparticipation in the stock market increases.

References

Andersen, S. and Nielsen, K.M. (2011). Participation Constraints in the Stock Market: Evidence from Unexpected Inheritance Due to Sudden Death. *Review of Financial Studies*. **24**, 1667–1697.

Ang, A., Bekaert, G., and Liu, J. (2005). Why Stocks May Disappoint? *Journal of Financial Economics*. **76**, 471–508.

Anoruo, E. and Braha, H. (2008). Housing and Stock Market Returns: An Application of GARCH Enhanced VECM. *The IUP Journal of Financial Economics.* **2**, 30–40.

Arrow, K.J. (1971). Essays in the Theory of Risk-Bearing. Markham Economics Series. Markham Pub. Co.

Benzoni L., Collin-Dufresne P., and Goldstein, R. (2007). Portfolio Choice over the Life-Cycle When the Stock and Labor Markets Are Cointegrated. *Journal of Finance.* **62**, 2123–2167.

Cao, H.H., Wang, T., and Zhang, H.H. (2005). Model Uncertainty, Limited Market Participation, and Asset Prices. *Review of Financial Studies.* **18**, 1219–1251.

Cocco, J. F. (2005). Portfolio Choice in the Presence of Housing. *Review of Financial Studies.* **18**, 535–567.

Cocco, J. F., Gomes, F. J., and Maenhout, P. J. (2005). Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies.* **18**, 491–533.

Corradin, S., Fillat, J. L., and Vergara-Alert, C. (2014). Optimal Portfolio Choice with Predictability in House Prices and Transaction Costs. *Review of Financial Studies.* **27**, 823–880.

Dai, M. and Zhong, Y. (2010). Penalty Methods for Continuous-Time Portfolio Selection with Proportional Transaction Costs. *Journal of Computational Finance.* **13,** 1–31.

Damgaard, A., Fuglsbjerg, B., and Munk, C. (2003). Optimal Consumption and Investment Strategies with a Perishable and an Indivisible Durable Consumption Good. *Journal of Economic Dynamics and Control.* **28**, 209–253.

Engle, R.F., and Granger, W.J. (1987). Co-integration and Error Correction: Representation, Estimation, and Testing. *Econometrica.* **55**, 251–276.

Epstein, L. and Schneider, M. (2008). Ambiguity, Information Quality, and Asset Pricing. *Journal of Finance.* **63**, 197–228.

Fischer, M. and Stamos, M. (2013). Optimal Life Cycle Portfolio Choice with Housing Market Cycles. *Review of Financial Studies.* **26**, 2311–2352.

Flavin, M. and Yamashita, T. (2011). Owner-Occupied Housing: Life-Cycle Implications for the Household Portfolio. *American Economic Review.* **101**, 609–614.

Gomes, F. and Michaelides, A. (2005). Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence. *Journal of Finance.* **60**, 869–904.

Grossman, S.J. and Laroque, G. (1990). Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods. *Econometrica.* **58**, 25–51.

Grout, P., Megginson, W., and Zalewska. A.(2009). One Half-Billion Shareholders and Counting-Determinants of Individual Share Ownership Around the World. Available at `https://www.semanticscholar.org/paper/One-Half-Billion-Shareholders-and-of-Individual-the-Grout-Megginson/892d9f8dd9fa001602612144ee57067d1aabc08f`

Hemert, O.V. (2010). Household Interest Rate Risk Management. *Real Estate Economics.* **38**, 467–505.

Johansen, S. (1988). Statistical Analysis of Cointegration Vectors. *Journal of Economic Dynamics and Control.* **12**, 231–254.

Johansen, S. (1991). Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models. *Econometrica* **59**, 1551–1580.

Kraft, H. and Munk, C. (2011). Optimal Housing, Consumption, and Investment Decisions over the Life-Cycle. *Management Science.* **57**, 1025–1041.

Kraft, H., Munk, C., and Wagner, S. (2017). Housing Habits and Their Implications for Life-Cycle Consumption and Investment. *Review of Finance.* **22**, 1737–1762.

Linnainmaa, J. (2005). The Individual Day Trader. Working Paper, UCLA.

Liu, H. and Loewenstein, M. (2002). Optimal Portfolio Selection with Transaction Costs and Finite Horizons. *Review of Financial Studies.* **15**, 805–835.

Lynch, A. W. and Tan, S. (2011). Labor Income Dynamics at Business-Cycle Frequencies: Implications for Portfolio Choice. *Journal of Financial Economics.* **101**, 333–359.

Merton, R. (1969). Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *The Review of Economics and Statistics.* **51**, 247–257.

Merton, R. (1971). Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory.* **3**, 373–413.

Munk, C. and Sørensen, C. (2010). Dynamic Asset Allocation with Stochastic Income and Interest Rates. *Journal of Financial Economics.* **96**, 433–462.

Polkovnichenko, V. (2007). Life-Cycle Portfolio Choice with Additive Habit Formation Preferences and Uninsurable Labor Income Risk. *Review of Financial Studies.* **20**, 83–124.

Samuelson, P. (1969). Lifetime Portfolio Selection by Dynamic Stochastic Programming. *The Review of Economics and Statistics.* **51**, 239–246.

Tsai, I.C., Lee, C.F., and Chiang, M.C. (2012). The Asymmetric Wealth Effect in the U.S. Housing and Stock Markets: Evidence from the Threshold Cointegration Model. *Journal of Estate Finance and Economics.* **45**, 1005–1020.

Vissing-Jørgensen, A. (2002). Limited Asset Market Participation and the Elasticity of Intertemporal Substitution. *Journal of Political Economy.* **110**, 825–853.

Wachter, J. A. and Yogo, M. (2010). Why Do Household Portfolio Shares Rise in Wealth? *Review of Financial Studies.* **23**, 3929–3965.

Yao, R. and Zhang, H. H. (2005). Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints. *Review of Financial Studies.* **18**, 197–239.

Appendix

In this appendix, we plot some figures to show empirical evidence of the correlations between stock market participation/investment and homeownership/investment across countries and across time. We then provide proofs of the analytical results in the main text.

Empirical Evidence

A.1 HJB Equation for the Model with an Illiquid Housing Market

The value function as given in (2.13) satisfies the following HJB equation:

$$
\begin{aligned}
\max\Big\{ & \max_{\tilde{C},\, 0\le\pi\le\widetilde{W}+(1-l)(1-\alpha)AH} \Big(\frac{1}{2}\sigma_S^2\pi^2\Psi_{\widetilde{W}\widetilde{W}} + \frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)H^2\Psi_{HH} + \frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)\Psi_{RR} \\
& - \lambda\sigma_S^2\pi\Psi_{\widetilde{W}R} + \sigma_I^2 H\Psi_{HR} + [r\widetilde{W} - \tilde{C} + (\mu_S - r)\pi]\Psi_{\widetilde{W}} + \mu_H(R)H\Psi_H \\
& + k(\bar{R}-R)\Psi_R - \delta A\Psi_A - (\beta+\delta_M)\Psi + \frac{\left(\tilde{C}^{1-\theta}A^{\theta}\right)^{1-\gamma}}{1-\gamma}\Big), \\
& \max_{0<Q\le\frac{\widetilde{W}+(1-\alpha)AH}{l(1-\alpha)H+\alpha H}} \Psi(\widetilde{W}+(1-\alpha)AH - QH, Q, H, R) - \Psi(\widetilde{W}, A, H, R)\Big\} = 0
\end{aligned}
\tag{A.1-1}
$$

in $\Omega = \{(\widetilde{W}, A, H, R) : \widetilde{W} + (1-\alpha)AH > 0, A > 0, H > 0, R \in \mathbb{R}\}$.

By the homogeneity property, we can do the following transformation:

$$
\Psi(\widetilde{W}, A, H, R) = \frac{1}{1-\gamma}(\widetilde{W}+(1-\alpha)AH)^{1-\gamma}H^{-\theta(1-\gamma)}e^{(1-\gamma)\phi(h,R)}, \quad h = \frac{(1-\alpha)AH}{\widetilde{W}+(1-\alpha)AH},
$$

where h is the ratio of house value to net wealth. The new function $\phi(h, R)$ satisfies the following HJB equation:

$$
\max\left\{\mathcal{L}\phi, \quad \max_{0\le h'\le 1/l}\left(\phi(h', R) + \log\frac{1-\alpha}{1-\alpha+\alpha h'}\right) - \phi(h, R)\right\} = 0, \tag{A.1-2}
$$

where

$$\begin{aligned}\mathcal{L}\phi = \max_{\zeta\in[0,1-lh]} \Big\{ &\frac{1}{2}[\sigma_S^2\zeta^2 + (\sigma_I^2+\sigma_{\varepsilon i}^2)(1-h)^2]h^2[\phi_{hh} + (1-\gamma)\phi_h^2] + \frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)[\phi_{RR}+(1-\gamma)\phi_R^2] \\
&+ [\lambda\sigma_S^2\zeta + \sigma_I^2(1-h)]h[\phi_{hR}+(1-\gamma)\phi_h\phi_R] + \big[\gamma\sigma_S^2\zeta^2 - (\sigma_I^2+\sigma_{\varepsilon i}^2)(\theta(1-\gamma)+\gamma h)(1-h) \\
&- (\mu_S - r)\zeta + (\mu_H(R) - r - \delta)(1-h)\big]h\phi_h + \big[(1-\gamma)\sigma_I^2(h-\theta) - \lambda(1-\gamma)\sigma_S^2\zeta \\
&+ k(\bar{R}-R)\big]\phi_R - \frac{1}{2}\gamma\sigma_S^2\zeta^2 + \frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(\theta(1-\gamma)+1) - \frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)(2\theta(1-\gamma)+\gamma h)h \\
&+ r(1-h) + (\mu_S - r)\zeta + \mu_H(R)(h-\theta) - \delta h - \frac{\beta+\delta_M}{1-\gamma} - \eta(1-h\phi_h)^{-\frac{p}{1-p}}h^{\frac{\theta(1-\gamma)}{1-p}}e^{-\frac{1-\gamma}{1-p}\phi}\Big\}\end{aligned}$$

with $p = (1-\theta)(1-\gamma)$, $\eta = \frac{1}{1-\gamma}\left(\frac{(1-\alpha)^{\theta(1-\gamma)}}{1-\theta}\right)^{\frac{p}{p-1}} - \left(\frac{(1-\alpha)^{\theta(1-\gamma)}}{1-\theta}\right)^{\frac{1}{p-1}}$, $\zeta = \pi/(\widetilde{W} + (1-\alpha)AH)$ and the optimal consumption satisfies

$$\tilde{c}^* = \frac{\tilde{C}^*}{\widetilde{W}+(1-\alpha)AH} = (1-\theta)^{\frac{1}{1-p}}(1-\alpha)^{\frac{\theta(1-\gamma)}{1-p}}(1-h\phi_h)^{-\frac{1}{1-p}}h^{\frac{\theta(1-\gamma)}{1-p}}e^{-\frac{1-\gamma}{1-p}\phi}.$$

We can solve (A.1-2) numerically. Define $M(R) := \max_{0\le h\le 1/l}\left\{\phi(h,R) + \log\left(\frac{1-\alpha}{1-\alpha+\alpha h}\right)\right\}$. The algorithm is given as follows:

(1) Set an initial guess of $M_0(R)$;

(2) Given $M_i(R)$, use the penalty method with finite difference scheme[19] to solve

$$\max\Big\{\mathcal{L}\phi, \quad M_i(R)-\phi\Big\} = 0, h\in(0,1/l);$$

(3) Let $M_{i+1}(R) := \max_{0\le h'\le 1/l}\left\{\phi(h',R) + \log\left(\frac{1-\alpha}{1-\alpha+\alpha h'}\right)\right\}$;

(4) If $||M_{i+1} - M_i|| <$ tolerance then stop; otherwise go to Step 2.

A.1.1 Verification theorem

In this subsection, we provide the verification theorem that captures households' optimal investment policy. We focus on the general illiquid model in Subsection 4.2, while the liquid benchmark model in Section 2 is a degenerated case.

[19]For the penalty method, see, e.g., Dai and Zhong (2010).

Theorem 1. ***(Verification Theorem)*** *Let $\Psi(\widetilde{W}, A, H, R)$ be a smooth solution to the HJB equation (A.1-1) and satisfy the transverality condition, i.e.,*

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-(\beta+\delta_M)t}\Psi(\widetilde{W}_t, A_t, H_t, R_t)\right] = 0,\ \forall \widetilde{W}_t + (1-\alpha)A_t H_t > 0, A_t > 0, H_t > 0, R_t \in \mathbb{R}.$$

In addition, let

$$\begin{aligned}
\tilde{C}_t^*(\widetilde{W}_t, A_t, H_t, R_t) &= \left(\frac{\Psi_W A^{-\theta(1-\gamma)}}{1-\theta}\right)^{\frac{1}{(1-\theta)(1-\gamma)-1}}, \\
\pi_t^*(\widetilde{W}_t, A_t, H_t, R_t) &= \underset{0\le\pi_t\le\widetilde{W}_t+(1-l)(1-\alpha)A_t H_t}{\arg\max} \Big\{\frac{1}{2}\sigma_S^2\pi^2\Psi_{\widetilde{W}\widetilde{W}} - \lambda\sigma_S^2\pi\Psi_{\widetilde{W}R} + (\mu_S - r)\pi\Psi_{\widetilde{W}} \\
&\qquad - \delta A\Psi_A - \eta A^{\frac{-\theta(1-\gamma)}{(1-\theta)(1-\gamma)}}\Psi_W^{\frac{(1-\theta)(1-\gamma)}{(1-\theta)(1-\gamma)-1}}\Big\},
\end{aligned}$$

where $\eta = (1-\theta)^{\frac{1}{1-(1-\theta)(1-\gamma)}} - \frac{(1-\theta)^{\frac{(1-\theta)(1-\gamma)}{1-(1-\theta)(1-\gamma)}}}{1-\gamma}$, all the partial derivatives are evaluated at $(\widetilde{W}_t, A_t, H_t, R_t)$, and

$$\begin{aligned}
Q_t^*(\widetilde{W}_{t-}, A_{t-}, H_t, R_t) &= \underset{0<Q\le\frac{\widetilde{W}_{t-}+(1-\alpha)A_{t-}H_t}{\alpha H_t + l(1-\alpha)H_t}}{\arg\max} \Psi(\widetilde{W}_{t-} + (1-\alpha)A_{t-}H_t - Q_t H_t, Q_t, H_t, R_t), \\
\tau_i^* &= \inf\{t > \tau_{i-1}^* :\ \Psi(\widetilde{W}_{t-}, A_{t-}, H_t, R_t) = \Psi(\widetilde{W}_{t-} + (1-\alpha)A_{t-}H_t - Q_t^* H_t, Q_t^*, H_t, R_t)\}, \\
A_i^* &= Q_{\tau_i^*}^*(\widetilde{W}_{\tau_i^*-}, A_{\tau_i^*-}, H_{\tau_i^*}, R_{\tau_i^*}).
\end{aligned}$$

Then, $\Psi(\widetilde{W}, A, H, R)$ coincides with the value function as given in (2.13), and $\Theta^* = \{\tilde{C}_t^*, \pi_t^*, (\tau_i^*, A_i^*)\}$ is the corresponding optimal investment policy.

Proof. Given any admissible investment policy $\Theta = \{\tilde{C}_t, \pi_t, (\tau_i, A_i)\}$, we denote by $(\widetilde{W}_t, A_t, H_t, R_t)$ the stochastic processes generated by policy Θ for notional convenience. Define $\mathcal{O}_n := \{(\widetilde{W}, A, H, R) : \frac{1}{n} \le \widetilde{W} + (1-\alpha)AH \le n, \frac{1}{n} \le A \le n, \frac{1}{n} \le H \le n, |R| \le n\}$ and a sequence of stoppting times $T_n := n \wedge \inf\{t \ge 0 : (\widetilde{W}_t, A_t, H_t, R_t) \notin \mathcal{O}_n\}$.

By the generalized Ito's formula,

$$
\begin{aligned}
& e^{-(\beta+\delta_M)T_n}\Psi(\widetilde{W}_{T_n}, A_{T_n}, H_{T_n}, R_{T_n}) \\
= & \Psi(\widetilde{W}, A, H, R) - \int_0^{T_n} e^{-(\beta+\delta_M)u}\frac{\left(\tilde{C}_u^{1-\theta}A_u^{\theta}\right)^{1-\gamma}}{1-\gamma}du \\
& + \int_0^{T_n} e^{-(\beta+\delta_M)u}\mathcal{L}\Psi(\widetilde{W}_u, A_u, H_u, R_u)du \\
& + \int_0^{T_n} e^{-(\beta+\delta_M)u}\big(\pi_u\sigma_S\Psi_{\widetilde{W}} - \lambda\sigma_S\Psi_R\big)(\widetilde{W}_u, A_u, H_u, R_u)dB_{Su} \\
& + \int_0^{T_n} e^{-(\beta+\delta_M)u}\big(\sigma_I H_t\Psi_H + \sigma_I\Psi_R\big)(\widetilde{W}_u, A_u, H_u, R_u)dB_{Iu} \\
& + \int_0^{T_n} e^{-(\beta+\delta_M)u}\sigma_{\varepsilon i}H_t\Psi_H(\widetilde{W}_u, A_u, H_u, R_u)dB_{iu} \\
& + \sum_{\tau_i<T_n} e^{-(\beta+\delta_M)\tau_i}\big(\Psi(\widetilde{W}_{\tau_i}, A_{\tau_i}, H_{\tau_i}, R_{\tau_i}) - \Psi(\widetilde{W}_{\tau_i-}, A_{\tau_i-}, H_{\tau_i}, R_{\tau_i})\big),
\end{aligned}
\tag{A.1-3}
$$

where

$$
\begin{aligned}
\mathcal{L}\Psi = & \frac{1}{2}\sigma_S^2\pi^2\Psi_{\widetilde{W}\widetilde{W}} + \frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)H^2\Psi_{HH} + \frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)\Psi_{RR} - \lambda\sigma_S^2\pi\Psi_{\widetilde{W}R} \\
& + \sigma_I^2 H\Psi_{HR} + [r\widetilde{W} - \tilde{C} + (\mu_S - r)\pi]\Psi_{\widetilde{W}} + \mu_H(R)H\Psi_H \\
& + k(\bar{R}-R)\Psi_R - \delta A\Psi_A - (\beta+\delta_M)\Psi + \frac{\left(\tilde{C}^{1-\theta}A^{\theta}\right)^{1-\gamma}}{1-\gamma}.
\end{aligned}
$$

Since Ψ is the solution to HJB equation (A.1-1), the second integral term and the last term in the right-hand side of (A.1-3) are nonpositive. Bellman's principle of optimality suggests that these two terms equal zero under the optimal policy $\Theta^* = \{\tilde{C}_t^*, \pi_t^*, (\tau_i^*, A_i^*)\}$. The three Ito integrals under expectation equal zero because $\Psi_{\widetilde{W}}, \Psi_H, \Psi_R$ are bounded when $(\widetilde{W}_u, A_u, H_u, R_u)$ is in the bounded domain $\mathcal{O}_n$ during $[0, T_n]$. Taking expectation in (A.1-3), we have

$$
\Psi(\widetilde{W}, A, H, R) \geq E\left[\int_0^{T_n} e^{-(\beta+\delta_M)u}\frac{\left(\tilde{C}_u^{1-\theta}A_u^{\theta}\right)^{1-\gamma}}{1-\gamma}du\right] + E\left[e^{-(\beta+\delta_M)T_n}\Psi(\widetilde{W}_{T_n}, A_{T_n}, H_{T_n}, R_{T_n})\right].
$$

As analyzed above, the equality above holds only for the claimed optimal strategy $\Theta^* = \{\tilde{C}_t^*, \pi_t^*, (\tau_i^*, A_i^*)\}$. As $n \to \infty$, T_n tends to infinity with probability 1. By the transversality condition of Ψ and the dominant convergence theorem, the first expectation above converges to the original utility function $E\left[\int_0^{\infty} e^{-(\beta+\delta_M)u}\frac{\left(\tilde{c}_t^{1-\theta}A_t^{\theta}\right)^{1-\gamma}}{1-\gamma}dt\right]$ and the second expectation goes to zero. Equality holds for the claimed optimal policy Θ^* and Ψ coincides with the original objective function. This completes the proof. $\square$

A.2 Conditional Expected Return of Housing and Stock

The sources of the stock market data and the house price series are Standard & Poor's and the Case Shiller Home Price Indicies, respectively, on December 1 from 1890 to 2017; both are in inflation-adjusted Novemeber 2019 dollars. As shown in Section 3.1, the residual process $R_t = \log I_t - \lambda \log S_t$ with $\lambda = 0.2695$. To compare the conditional expected return of housing and stock, we analyze the growth rates of house price and stock price when R_t falls below its long-term limit $\bar{R}$. We divide the whole sample periods into two groups by the sign of R_t. The result is shown in Figure 7. The upper panel shows the residual process $R_t = \log I_t - 0.2695 \log S_t$. The middle panel depicts the house growth rate $\log(I_{t+1}/I_t)$ at the state of $R_t < \bar{R}$. The lower panel depicts the stock growth rate $\log(S_{t+1}/S_t)$ at the state of $R_t < \bar{R}$. The average house growth rate when $R_t < \bar{R}$ from observation year 1890 (1953) to 2017 equals 0.0126 (0.0139), while the average stock growth rate when $R_t < \bar{R}$ from observation year 1890 (1953) to 2017 equals -0.0028 (0.0132). It shows that the house price grows faster than the stock price in the state of $R_t < \bar{R}$, supporting the notion that when the conditional expected return of housing is high relative to that of stock, households prefer to stay away from the stock market, as analyzed in the main body of our paper.

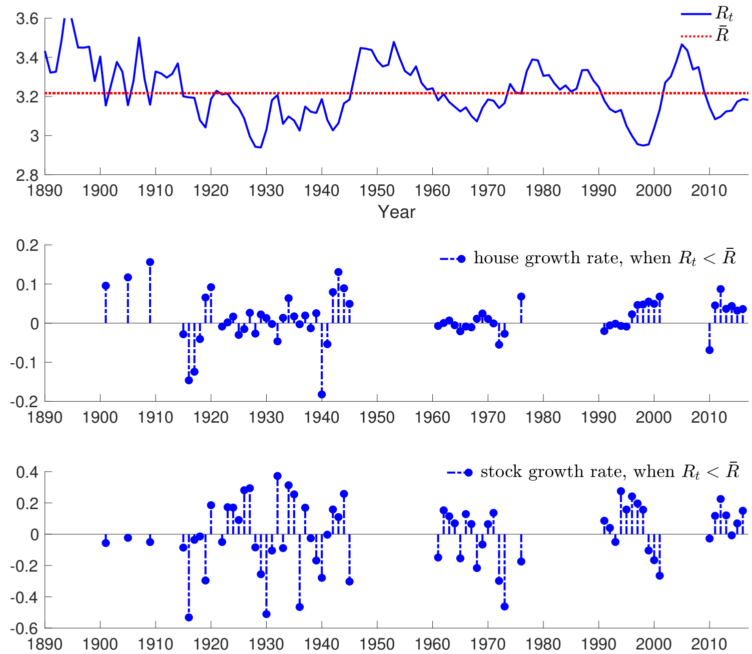


Figure 7: **Growth rate of house price and stock price.** The upper panel shows the residual process R_t. The middle panel depicts the house index growth rate $\log(I_{t+1}/I_t)$ at the state of $R_t < \bar{R}$. The lower panel depicts the stock growth rate $\log(S_{t+1}/S_t)$ at the state of $R_t < \bar{R}$. The average house growth rate when $R_t < \bar{R}$ from 1890 (1953) to 2017 equals 0.0126 (0.0139), while the average stock growth rate when $R_t < \bar{R}$ from 1890 (1953) to 2017 equals -0.0028 (0.0132). This implies that the house price grows faster than the stock price when the ratio of house price to stock price is low.