Circuit Breakers and Contagion*

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ABSTRACT

Circuit breakers based on indices are commonly imposed in financial markets to reduce market crashes and volatility in bad times. We develop a dynamic equilibrium model with multiple stocks to study how circuit breakers affect joint stock price dynamics and cross-stock contagion. We show that circuit breakers can cause crash contagion, volatility contagion, and high correlations among otherwise independent stocks, especially in bad times. Our analysis suggests that circuit breakers rules might have exacerbated the market plunges and extreme volatility triggered by the COVID-19 pandemic. We propose an alternative circuit breaker approach that does not cause cross-stock contagion.

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1. Introduction

Circuit breakers in financial markets based on indices are widely implemented in many countries (e.g., the United States, France, Canada, and China) as one of the measures aimed at stabilizing market prices in bad times. In most cases, when the percentage decline in a market index reaches a regulatory threshold, the circuit breaker is triggered and trading is halted for a period of time for the entire market. Recent COVID-19 fears triggered circuit breakers multiple times across many countries including the United States, Japan, and South Korea. For example, circuit breakers on the S&P 500 were triggered twice during the week of March 9, 2020 and plunged almost 10% on March 12, 2020. In a dramatic move, Chinese regulators removed a four-day-old circuit breakers rule after it was triggered twice in the week of January 7, 2016. One open question in the existing literature on circuit breakers (e.g., Chen, Petukhov, and Wang (2017), Greenwald and Stein (1991), Subrahmanyam (1994)) is how circuit breakers affect the systemic risk caused by stock return correlations and market-wide contagion in bad times. In this paper, we develop a continuous-time asset pricing equilibrium model to shed some light on this important issue.

Contrary to regulatory goals, we show that in bad times, circuit breakers can cause crash contagion, volatility contagion, and high correlations among otherwise independent stocks, and can significantly accelerate market decline and increase market volatility. Our analysis suggests that the circuit breakers rules might have significantly exacerbated the international market plunges and extreme volatility triggered by the COVID-19 pandemic, because of the contagion effect. Our analysis can also help explain the concurrence of the implementation of the circuit breakers rule and the significant market tumble in the week of January 7, 2016 in Chinese stock markets. Our model suggests that market-wide circuit breakers may be a source of financial contagion and a channel through which idiosyncratic risks become systemic risks. We propose an alternative circuit breaker approach based on individual stocks, rather than an index, that does not cause either correlation or any contagion.[1]

In our model, investors can invest in one risk-free asset and two risky assets ("stocks") with independent jump diffusion dividend processes to maximize their expected utility from their final wealth at time T. Investors have heterogeneous beliefs about the drift

[1]Needless to say, circuit breakers may play a positive role in stabilizing markets. For example, they may reduce the effect of overreaction, panic, and herding on stock prices. Our model does not consider some potential benefits of market closure that can potentially justify the imposition of circuit breakers and is designed to shed light on some potential costs of circuit breakers.

of a dividend process and the disagreement between two types of investors is stochastic. To highlight the role of circuit breakers, we assume that the investors have exponential preferences so that, in the absence of circuit breakers, the equilibrium stock returns are independent. The stock market is subject to a market-wide circuit breaker rule in the sense that if the sum of the two stock prices (the index) reaches a threshold, the entire stock market is closed until T.

The intuition for our main results that circuit breakers increase return correlations, cause contagion of crash and volatility, accelerate market decline, and raise market volatility is as follows. After the circuit breaker is triggered, the market is closed, and thus risk sharing is reduced, which in turn causes stock prices to be likely lower than those without market closure. Therefore, when an idiosyncratic negative shock to the price of one stock occurs, the sum of stock prices (in general, the index of the market) gets smaller, the probability of reaching the circuit breaker threshold increases, and thus the price of the other stock may also decrease in anticipation of the more likely market closure. This link through the circuit breaker induces the positive return correlation, even though stocks are independent in the absence of the circuit breaker. When the idiosyncratic shock is large, and thus the index becomes close to the circuit breaker, this increase in the correlation is even greater because the likelihood of market closure is much higher. In the extreme case where one stock crashes and the circuit breaker is triggered, the price of the other stock must jump down to the after-market-closure level. This results in crash contagion. Because, after some stocks fall in prices, the index gets closer to the circuit breaker threshold, other stock prices also fall due to the fear of market closure, which in turn drives the index even closer to the threshold, and so on. It is this vicious cycle that may increase market volatility. In addition, as one stock becomes more volatile (e.g., due to an increase in the volatility of its dividend), the likelihood of triggering the circuit breaker becomes greater, and thus the prices of other stocks also become more volatile. This explains why a crash of one stock may cause another stock to crash and volatility can transmit across stocks even though stocks are independent in the absence of circuit breakers. These contagion effects may transform idiosyncratic risks into systemic risks.

Our results suggest that to reduce the contagion effects and the systemic risks, it is better to impose circuit breakers on individual stocks. In this alternative approach, the threshold is based on individual stock returns: when a stock's circuit breaker is triggered, only trading in this single stock is halted. This alternative approach does not increase correlations or cause any form of contagion. We show that with this alternative approach, stock prices are generally higher, a market-wide large decline is less likely, and systemic risk is lower, compared to those with circuit breakers imposed on an index.

In the model, we assume there are only two stocks in the index on which the circuit breakers are based. One possible concern is that in practice indices typically consist of hundreds of stocks (if not more) and therefore it is unlikely that one stock's fall would trigger the fall of many other stocks. On the other hand, in bad times, markets typically focus on a small number of key factors such as Federal Reserve decisions and major economic news. Each of the two stocks in our model represent a large group of stocks that are significantly exposed to a common risk factor in bad times. When there is a bad shock in the risk factor, the prices of the large group of stocks go down, which can drag down another large group of stocks through the circuit breakers connection even though the latter group of stocks is not exposed to the risk factor.

Our paper is motivated by the seminal paper Chen, Petukhov, and Wang (2017). Using a dynamic asset pricing model with a single stock, Chen, Petukhov, and Wang (2017) are the first to show in a dynamic equilibrium setting that, contrary to some of the main goals of regulators, a downside circuit breaker may lower stock price, increase market volatility, and accelerate market decline (which they call the "magnet effect"). Our analysis is an extension of their model to a dynamic equilibrium model with multiple stocks to examine the cross-stock contagion effect of circuit breaker rules. We show that the magnet effect they find is robust to a setting with multiple stocks and becomes even stronger because of the contagion effect. An important difference from their model is that we incorporate a jump risk, which can be significant in bad times. Because of the jump risk, circuit breakers have a possible price limit effect (i.e., stock prices cannot fall below the circuit breaker threshold level) and the market can crash. We show that, in the presence of jump risk, a crash in one stock can cause a crash in an otherwise independent stock, and circuit breakers can reduce market volatility in some states due to the price limit effect. In addition, in Chen, Petukhov, and Wang (2017) the main mechanism through which circuit breakers affect price dynamics is the difference in leverage before and after market closure. Before market closure, investors face no constraints on leverage, but after market closure they cannot lever at all to ensure solvency during the closure. As a result, investors need to completely unlever when the circuit breaker is triggered, which magnifies the effect of market closure. In this paper, there is no constraint on leverage either before or after market closure. Our results suggest that, even in the absence of leverage constraints, the magnet effect is still present and circuit breakers can still have a large impact on price dynamics.

Among other theoretical work related to circuit breakers, Greenwald and Stein (1991) show that in a market with limited participation, circuit breakers can help coordinate trading for market participants. Subrahmanyam (1994) demonstrates that circuit breakers can increase price volatility because investors may shift their trades to earlier periods with a lower liquidity supply if there is information asymmetry. Hong and Wang (2000) examine the impact of periodic exogenous market closure on asset prices and show that their model produces rich patterns of trading and returns consistent with empirical findings.

Many empirical studies find evidence against advocates of circuit breakers (including market-wide circuit breakers, price limits, and trading pauses). For example, exploiting Nasdaq order book data, Hautsch and Horvath (2019) show that trading pauses cause extra volatility and reduce price stability and liquidity after the pause, but enhance price discovery during the break. Kim and Rhee (1997) find evidence from Tokyo Stock Exchange data suggesting that the price limit system may be ineffective in the sense that price limits may cause higher volatility levels, prevent prices from efficiently reaching their equilibrium level, and interfere with trading. Lauterbach and Ben-Zion (1993) examine the behavior of the Israeli stock market to study the performance of circuit breakers during the October 1987 crash. They find that circuit breakers reduced the next-day opening order imbalance and the initial price loss; however, they had no effect on the long-run response. Lee, Ready, and Seguin (1994) examine the effect of firm-specific New York Stock Exchange (NYSE) trading halts and find that trading halts do not reduce either volume or price volatility during the post-halt period. Goldstein and Kavajecz (2004) focus on the NYSE during the October 1997 market break and demonstrate the magnet effect, that is, an acceleration of activity approaching the market-wide circuit breaker.[2]

Unlike the existing literature, this paper studies impacts of market-wide circuit breakers on the dynamic interactions among multiple stocks. Even though circuit breakers are designed exclusively to stabilize markets in bad states, we find that market-wide circuit breakers can have significant crash and volatility contagion effects, *especially in bad states*. To the best of our knowledge, this prediction is new to both the theoretical and

[2]A few other studies on market halts focus on other related issues. For example, Ackert, Church, and Jayaraman (2001) conduct an experimental study to analyze the effects of mandated market closures and temporary halts on market behavior. Corwin and Lipson (2000) study order submission strategies of traders around market halts, providing a detailed description of the mechanics of trading halts and identifying traders who provide liquidity. Christie, Corwin, and Harris (2002) study the impact on post-halt market prices of Nasdaq's alternative halt and reopening procedures. Their results are consistent with the hypothesis that increased information transmission during the halt reduces post-halt uncertainty.

the empirical literature on circuit breakers.

The rest of this paper is organized as follows. In Section 2, we formulate the basic market model. In Section 3, we solve the market equilibrium in the absence of circuit breakers. In Section 4, we study the market equilibrium when there are circuit breakers. In Section 5, we quantitatively examine the impact of circuit breakers on equilibrium prices and their correlation. Section 6 concludes. All proofs are provided in the Appendix.

2. The Model

We consider a continuous-time exchange economy over the finite time interval $[0, T]$. Investors can trade two stocks, Stock 1 and Stock 2, and one risk-free asset. Each of the two stocks in our model represents a group of stocks that share the same significant risk exposure in bad times. The risk-free asset has a net supply of zero and the interest rate can be normalized to zero because there is no intertemporal consumption in our model. The total supply of each stock is one and each stock pays only a terminal dividend at time T. The dividend processes are exogenous and publicly observed. Uncertainty about dividends is represented by a standard Brownian motion Z_t and an independent standard Poisson process N_t defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. An augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by Z_t and N_t.

There is a continuum of investors of Types A and B in the economy, with a mass of 1 for each type. For $i = A, B$ and $j = 1, 2$, Type i investors are initially endowed with θ^i_{j0} shares of Stock j but no risk-free asset, with $0 \leq \theta^i_{j0} \leq 1$ and $\theta^A_{j0} + \theta^B_{j0} = 1$. For Type A investors, the probability measure is $\mathbf{P}^A$, which is the same as the physical probability measure $\mathbf{P}$. Under Type A's probability measure, Stock 1's dividend process evolves as:

$$dD_{1,t} = \mu^A_1 dt + \sigma dZ_t, \tag{1}$$

and Stock 2's dividend process follows a jump process with drift:

$$dD_{2,t} = \mu_2 dt + \mu_J dN_t, \tag{2}$$

with $D_{j,0} = 1$ for $j = 1, 2$. Stock 1's dividend growth rate μ^A_1, Stock 1's dividend volatility σ, and Stock 2's dividend growth rate μ_2 are all constants. The Poisson process N_t has a constant jump intensity of κ and a constant jump size of μ_J.

Relative to Type A investors, Type B investors have different beliefs about the dividend process $D_{1,t}$ and employ a different probability measure $\mathbf{P}^B$, under which the dividend process $D_{1,t}$ evolves as

$$dD_{1,t} = \mu_{1,t}^B dt + \sigma dZ_t^B, \tag{3}$$

where Z_t^B is a Brownian motion under measure $\mathbf{P}^B$, and $\mu_{1,t}^B \equiv \mu_1^A + \delta_t$ for a stochastic process δ_t that measures the disagreement between Type A and Type B investors about the growth rate of the dividend process $D_{1,t}$. The Radon-Nikodym derivative between the two probability measures is therefore defined as follows.

$$\eta_T = \frac{d\mathbf{P}^B}{d\mathbf{P}^A}|_{\mathcal{F}_T} = e^{\int_0^T \frac{\delta_t}{\sigma} dZ_t - \int_0^T \frac{\delta_t^2}{2\sigma^2} dt}. \tag{4}$$

For the disagreement process δ_t, we assume that under the probability measure $\mathbf{P}^A$:

$$d\delta_t = -k(\delta_t - \bar{\delta})dt + \nu dZ_t, \tag{5}$$

where $\bar{\delta}$ is the constant long-time average of the disagreement (which could be zero), k measures the speed of mean reversion, and ν is the volatility of the disagreement.[3]

For simplicity and without loss of the main economic insights, we assume that there is no disagreement about the dividend process $D_{2,t}$ between Type A and Type B investors.[4] Circuit breakers have two direct effects: The first is the *market closure effect*, i.e., investors cannot trade for a period of time after circuit breakers are triggered; the second is the *price limit effect*, i.e., stock prices cannot fall below the circuit-breaker threshold levels. As we show later, without a jump in a dividend process, the price limit effect would be absent because prices move continuously to the levels implied by the fundamentals. Without stochastic disagreement, the market closure effect would be absent in our model because investors would not trade after time zero even without circuit breakers. Thus we propose the above two dividend processes to capture these two direct effects in the simplest way.[5]

It follows from (1) and (3) that $Z_t^B = Z_t - \frac{\delta_t}{\sigma} dt$ and is thus independent of N_t.

[3]In the Appendix, we show that this δ_t process is consistent with Kalman filtering when Type B investors do not know the expected growth rate of Stock 1's dividend.

[4]In a previous version of the paper, we also allowed disagreement on the stochastic process followed by Dividend 2. However, this more complicated model did not provide new insight into the effects of circuit breakers. Another alternative, assuming disagreement about the dividend process $D_{2,t}$ but no disagreement about the dividend process $D_{1,t}$, also yields the same qualitative conclusions, but is more complex. We therefore simply assume there is no disagreement about the dividend process $D_{2,t}$ in order to focus on the main ideas.

[5]Using a jump diffusion dividend process for both stocks would not change our conclusions, but complicates analysis.

Hereafter, we use the convention $\mathbb{E}^i[\cdot]$ to denote the expectation under the probability measure $\mathbf{P}^i$ for $i \in \{A, B\}$.

To isolate the impact of circuit breakers on stock return correlations, we assume that, for $i \in \{A, B\}$, Type i investors have constant absolute risk averse (CARA) preferences over the terminal wealth W_T^i at time T:

$$u(W_T^i) = -\exp(-\gamma W_T^i),$$

where $\gamma > 0$ is the absolute risk aversion coefficient. With CARA preferences, there is no wealth effect and therefore in the absence of circuit breakers, it can be shown that returns of the two stocks would be independent.

Trading in the stocks is subject to a market-wide circuit breaker rule as explained next. Let $S_{j,t}$ denote the price of Stock $j = 1, 2$ at time $t \leq T$ and the index $S_t = S_{1,t} + S_{2,t}$ denote the sum of the two prices.[6] Define the circuit breaker trigger time

$$\tau = \inf\{t : S_t \leq h, t \in [0, T)\},$$

where h is the circuit breaker threshold (hurdle). At the circuit breaker trigger time τ, the market is closed until T,[7] which results in the market closure effect, and the index $S_{1,t} + S_{2,t}$ of stock prices cannot go below h, which leads to the price limit effect. In practice, the circuit breaker threshold h is typically equal to a percentage of the previous day's closing level. In this paper, we set $h = (1 - \alpha)S_0$ for a constant α (e.g., $\alpha = 0.07$ for the level 2 market closure in the Chinese stock markets and for the level 1 market closure in the U.S. market).

3. Equilibrium without Circuit Breakers

As a benchmark case, we first solve for the equilibrium stock prices when there is no circuit breaker in place in the market. To do so, it is convenient to solve the planner's

[6]Using a different form of the combination of the stock prices as the index would not change our main results, as long as the index is increasing in both stock prices.

[7]Assuming that markets can reopen after being halted for a period of time would not change the qualitative results on contagion. Quantitatively, the results are close in very bad times, because the fear of market closure is similar whether the closure is long or relatively short in very bad times.

problem:

$$\max_{W_T^A, W_T^B} \mathbb{E}_0^A[u(W_T^A) + \xi\eta_T u(W_T^B)], \tag{6}$$

subject to the budget constraint $W_T^A + W_T^B = D_{1,T} + D_{2,T}$, where ξ is a constant depending on the initial wealth weights of the two types of investors.

From the first order conditions, we obtain:

$$W_T^A = \frac{1}{2\gamma}\log(\frac{1}{\xi\eta_T}) + \frac{1}{2}(D_{1,T} + D_{2,T}), \tag{7}$$

$$W_T^B = -\frac{1}{2\gamma}\log(\frac{1}{\xi\eta_T}) + \frac{1}{2}(D_{1,T} + D_{2,T}). \tag{8}$$

Given the utility function $u(x) = -e^{-\gamma x}$, the state price density under Type A investors' beliefs is

$$\pi_t^A = \mathbb{E}_t^A[\zeta u'(W_T^A)] = \mathbb{E}_t^A[\gamma\zeta e^{-\gamma W_T^A}] = \gamma\zeta\xi^{\frac{1}{2}}\mathbb{E}_t^A[\eta_T^{\frac{1}{2}} \cdot e^{-\frac{\gamma}{2}(D_{1,T}+D_{2,T})}], \tag{9}$$

for some constant ζ. Therefore, the stock price in equilibrium is given by

$$\hat{S}_{j,t} = \frac{\mathbb{E}_t^A\left[\pi_T^A D_{j,T}\right]}{\mathbb{E}_t^A[\pi_T^A]} = D_{j,t} + \frac{\mathbb{E}_t^A\left[\pi_T^A(D_{j,T} - D_{j,t})\right]}{\mathbb{E}_t^A[\pi_T^A]}, \quad j = 1, 2. \tag{10}$$

Since the two dividend processes are independent, Equation (10) can be simplified into

$$\hat{S}_{1,t} = \frac{\mathbb{E}_t^A[\pi_{1,T}^A D_{1,T}]}{\mathbb{E}_t^A[\pi_{1,T}^A]}, \qquad \hat{S}_{2,t} = \frac{\mathbb{E}_t^A[\pi_{2,T}^A D_{2,T}]}{\mathbb{E}_t^A[\pi_{2,T}^A]}, \tag{11}$$

where $\pi_{1,t}^A = \mathbb{E}_t^A[\eta_T^{1/2} \cdot e^{-\frac{\gamma}{2}D_{1,T}}]$, $\pi_{2,t}^A = \mathbb{E}_t^A[\eta_T^{1/2} e^{-\frac{\gamma}{2}D_{2,T}}]$. Thus, the two prices can be computed separately when there are no circuit breakers, which implies that stock returns are independent.

Next, we derive the equilibrium prices in closed form for the two stocks. Then, we examine the impact of the jump and the stochastic disagreement on the market equilibrium.

For Stock 1, the disagreement process is governed by the mean-reverting process (5). The formula of equilibrium price $\hat{S}_{1,t}$ can be derived analytically and is presented in the following proposition.

PROPOSITION 1. *When there are no circuit breakers, the equilibrium price of Stock*

1 is:

$$\hat{S}_{1,t} = D_{1,t} + \mu_1^A (T-t) - 2\left(\frac{dA(t;\gamma)}{d\gamma} + \frac{dC(t;\gamma)}{d\gamma}\delta_t\right), \tag{12}$$

where $A(t;\gamma)$ and $C(t;\gamma)$ are given in Appendix A.

Proposition 1 shows that, in addition to the dividend payment, disagreement also affects the price of Stock 1. As a result, the instantaneous volatility of the stock price $\hat{S}_{1,t}$ is not the same as that of the dividend process.

To show the importance of stochastic disagreement, we next show what would happen if the disagreement were constant, that is, $\delta_t = \delta_0$ for all $t \in [0,T]$. In this case, the equilibrium price would simplify to

$$\hat{S}_{1,t} = D_{1,t} + \frac{\mu_1^A + \mu_1^B}{2}(T-t) - \frac{\gamma}{2}\sigma^2 (T-t).$$

Thus, the equilibrium price of Stock 1 would be determined by the average beliefs of Type A and B investors on the growth rate of the dividend and the volatility of the stock price would be the same as the volatility of its dividend. Moreover, by applying Ito's lemma to the wealth process $W_t^A = \frac{\mathbb{E}_t^A[\pi_T^A W_T^A]}{\mathbb{E}_t^A[\pi_T^A]}$, we can find that the equilibrium number of shares of Stock 1 held by Type A investors would be equal to

$$\hat{\theta}_{1,t}^A = \frac{1}{2} - \frac{1}{2\gamma}\frac{\delta_0}{\sigma^2}, \tag{13}$$

which implies that the equilibrium number of shares of Stock 1 held by Type B investors would be equal to

$$\hat{\theta}_{1,t}^B = \frac{1}{2} + \frac{1}{2\gamma}\frac{\delta_0}{\sigma^2}. \tag{14}$$

Because the number of shares held by investors in the equilibrium would be constant over time if the disagreement were constant, market closure would not have any impact on the equilibrium price in the case of constant disagreement. This result implies that stochastic disagreement is necessary for circuit breakers to have any impact through the market closure channel.

For Stock 2, the equilibrium price is obtained by evaluating the second formula of (11) directly. The result is presented in the following proposition.

PROPOSITION 2. *When there are no circuit breakers, the equilibrium price of Stock 2 is:*

$$\hat{S}_{2,t} = D_{2,t} + \mu_2(T-t) + \kappa\mu_J(T-t)e^{-\frac{\gamma}{2}\mu_J}. \tag{15}$$

Proposition 2 shows that the instantaneous volatility (square root of instantaneous variance) of the equilibrium price of Stock 2 is the same as that of the dividend process because the rest of the terms in (15) are deterministic.[8]

Let $\hat{\theta}^A_{i,t}$ be the optimal shares of Stock i held by Type A investors. Then $dW^A_t = \hat{\theta}^A_{1,t} d\hat{S}_{1,t} + \hat{\theta}^A_{2,t} d\hat{S}_{2,t}$. We can obtain the optimal share holding of Stock 2 for Type A investors as

$$\hat{\theta}^A_{2,t} = \frac{1}{2},$$

which is a constant. This indicates that in the absence of circuit breakers, the equilibrium trading strategy in Stock 2 for all investors is to buy and hold (similar to Stock 1 when the disagreement is constant). Therefore, in the presence of circuit breakers, there is no market closure effect for Stock 2.

4. Equilibrium with Circuit Breakers

In this section, we study equilibrium prices when the circuit breaker rule is imposed in the market. We first solve for the indirect utility functions at the circuit breaker trigger time τ by maximizing investors' expected utility at $\tau \leq T$:

$$\max_{\theta^i_{1,\tau}, \theta^i_{2,\tau}} \mathbb{E}^i[u(W^i_\tau + \theta^i_{1,\tau}(S_{1,T} - S_{1,\tau}) + \theta^i_{2,\tau}(S_{2,T} - S_{2,\tau}))], \ i \in \{A, B\}, \tag{16}$$

with the market clearing condition $\theta^A_{j,\tau} + \theta^B_{j,\tau} = 1$ and the terminal condition $S_{j,T} = D_{j,T}$, where $\theta^i_{j,\tau}$ is the optimal number of shares of Stock j held by Type i investors at time τ, for $i \in \{A, B\}$ and $j = 1, 2$.

If the circuit breaker is triggered by a continuous decline in Stock 1's price, then the after-closure prices of both stocks will reflect their respective fundamental values. If the circuit breaker is triggered by a jump in Stock 2 price, we assume that investors can trade the stocks one more time, and in addition, Stock 1 can trade to reach its fundamental value, but Stock 2's price may be limited by the circuit breaker threshold.

[8] If there were stochastic disagreement on Stock 2's dividend, then the volatility of the equilibrium price of Stock 2 would be different from that of the dividend process.

The rationale behind this assumption is that the jump can be viewed as an approximation of a deterministic steep decline (less than but very close to a 90-degree drop) and during the fast decline, Stock 1 can trade freely instantly and thus reach its fundamental value before Stock 2's price is limited by the circuit breaker threshold.[9]

Exploiting the dynamics of $D_{j,t}$ and evaluating the expectation in the above optimization problems, we reach a system of equations that determine $\theta^i_{j,\tau}$ for $i \in \{A, B\}, j = 1, 2$. Then the equilibrium prices are obtained through market clearing conditions. We summarize the result in the following proposition.

PROPOSITION 3. *Suppose that the market is halted at a stopping time $\tau < T$.*
(1) For Stock 1, the market clearing price at τ is given by

$$S^c_{1,\tau} = D_{1,\tau} + \mu^A_1(T-\tau) - \gamma\theta^A_{1,\tau}\sigma^2(T-\tau),$$

where the optimal share holding of Type A investors is

$$\theta^A_{1,\tau} = \frac{-\frac{1}{\tilde{k}}(1-e^{\tilde{k}(\tau-T)})\delta_\tau - \frac{k\bar{\delta}}{\tilde{k}}(T-\tau-\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}}) + I_\tau}{I_\tau + \gamma\sigma^2(T-\tau)}, \tag{17}$$

with $\tilde{k} = k - \frac{\nu}{\sigma}$ and

$$I_\tau = -\gamma\sigma^2(\tau-T) + \frac{2\nu\sigma\gamma}{\tilde{k}}(T-\tau-\frac{1-e^{\tilde{k}(\tau-T)}}{\tilde{k}}) + \frac{\nu^2\gamma}{\tilde{k}^2}(T-\tau-2\frac{1-e^{\tilde{k}(\tau-T)}}{\tilde{k}} + \frac{1-e^{2\tilde{k}(\tau-T)}}{2\tilde{k}}).$$

If $\tilde{k} = 0$, the optimal share holding is simplified into:[10]

$$\theta^A_{1,\tau} = \frac{1}{\gamma}\left(\frac{\gamma\sigma^2 - \gamma\nu\sigma(\tau-T) + \frac{1}{2}k\bar{\delta}(\tau-T) + \frac{\nu^2\gamma}{3}(\tau-T)^2 - \delta_\tau}{-\nu\sigma(\tau-T) + \frac{\nu^3}{3}(\tau-T)^2 + 2\sigma^2}\right).$$

(2) For Stock 2, the market clearing price is given by

$$S^c_{2,\tau} = D^*_{2,\tau} + \mu_2(T-\tau) + \kappa\mu_J e^{-\frac{\gamma}{2}\mu_J}(T-\tau),$$

where $D^*_{2,\tau} \in [D_{2,\tau}, D_{2,\tau-})$ is such that $S^c_{1,\tau} + S^c_{2,\tau} = h$. The optimal share holding of Type A investors at τ is specified in Section 4.1.

[9]Using another way of dividing the price limit effect does not produce qualitatively different results on contagion.

[10]It can be verified that as $\tau \to T^-$, $\theta^A_{1,\tau} \to \frac{1}{2} - \frac{\delta_T}{2\gamma\sigma^2}$, which coincides with the optimal share holding of Stock 1 by Type A in the case of constant disagreement.

The proofs are collected in Appendix B.

By the definition of $D^*_{2,\tau}$, we see that $S^c_{2,\tau} \geq \hat{S}_{2,\tau}$, that is, the market clearing price of Stock 2 is not less than the equilibrium price in the absence of circuit breakers. This is because the circuit breaker prevents the price from falling beyond the threshold. As a consequence, we show at the end of Appendix B that if there is only Stock 2 in the market, its equilibrium price $S_{2,t}$ in the presence of circuit breakers should not be less than $\hat{S}_{2,t}$, the equilibrium price in the absence of circuit breakers. The difference $S_{2,t} - \hat{S}_{2,t}$ reflects the price limit effect.

If the circuit breaker is triggered by a decline in the price of a stock with a continuous dividend process, investors can continuously adjust the valuation to reflect the fundamentals represented by the dividend process, and thus the price limit effect is zero. In contrast, when the circuit breaker is triggered by a jump in the price of a stock caused by a jump in its dividend, the price is stopped at the threshold level, and thus there is a strictly positive price limit effect almost surely.

In contrast, depending on parameter values, Stock 1's price at τ may be higher or lower than that without circuit breakers. Intuitively, (1) circuit breakers do not have a price limit effect on Stock 1 because Stock 1's dividend does not jump; and (2) the market closure effect of circuit breakers can increase or decrease the stock price compared to the case without circuit breakers, because market closure may reduce expected net sales or expected net purchases depending on the distribution of the disagreement and the share holdings at the trigger time.

4.1 Optimal Share Holding of Stock 2 at τ

At τ, the equilibrium share holding for Stock 1 is exactly given by Prop. 3, because circuit breakers do not cause any price distortion in Stock 1's price due to the continuity of Stock 1's dividend, even when circuit breakers are triggered by a jump in the price of Stock 2. For Stock 2, if it is the diffusion of Stock 1 that triggers a circuit breaker, the share holding for Stock 2 at τ is the same as $\hat{\theta}_{2,\tau}$ ($= \frac{1}{2}$, the share holding in the case of no circuit breakers), as derived in Appendix B.2. However, if the market closure is caused by a jump in Stock 2's dividend, because of the price limit effect of the circuit breakers, Stock 2's price may not fall by the full amount that reflects the fundamentals. In other words, the circuit breaker rule distorts Stock 2's price and the equilibrium share holding of Stock 2 is a corner solution.

What makes it complicated is that, for Stock 2, there is almost surely no equilibrium where both Type A and Type B investors maximize their individual utilities and the market clears, because of the potential price limit effect of the circuit breakers. We introduce a mechanism to determine the equilibrium share holding of Stock 2 when the price limit effect is strictly positive as a result of a jump in Stock 2's dividend.

Let $\tilde{\theta}^i_{2,\tau}$ denote the share holding that would maximize the individual utility of Type i at τ (the expression of $\tilde{\theta}^i_{2,\tau}$ is given by (B.10) in Appendix B) and $\theta^i_{2,t}$ denote the equilibrium share holding at $t \leq \tau$, $i = A, B$. Then, if Type i investors would like to sell/buy more than other investors would like to buy/sell respectively, the equilibrium trading amount at τ is equal to the smaller amount that other investors would like to buy/sell. If, on the other hand, all investors would like to trade in the same direction, then no one can trade. More precisely, the equilibrium share holding of Stock 2 at τ is determined by the following rule:[11]

- If $(\theta^A_{2,\tau-} - \tilde{\theta}^A_{2,\tau}) \cdot (\theta^B_{2,\tau-} - \tilde{\theta}^B_{2,\tau}) > 0$, then $\theta^i_{2,\tau} = \theta^i_{2,\tau-}$, $i \in \{A, B\}$;
- otherwise
  - If $|\theta^A_{2,\tau-} - \tilde{\theta}^A_{2,\tau}| \leq |\theta^B_{2,\tau-} - \tilde{\theta}^B_{2,\tau}|$, then $\theta^A_{2,\tau} = \tilde{\theta}^A_{2,\tau}$ and $\theta^B_{2,\tau} = 1 - \tilde{\theta}^A_{2,\tau}$.
  - If $|\theta^A_{2,\tau-} - \tilde{\theta}^A_{2,\tau}| > |\theta^B_{2,\tau-} - \tilde{\theta}^B_{2,\tau}|$, then $\theta^A_{2,\tau} = 1 - \tilde{\theta}^B_{2,\tau}$ and $\theta^B_{2,\tau} = \tilde{\theta}^B_{2,\tau}$.

Having obtained the market clearing prices and the optimal share holdings at the circuit breaker trigger time τ, we now characterize the circuit breaker trigger time τ.

4.2 Circuit Breaker Trigger Time τ

The circuit breaker trigger time τ can be characterized using the dividend values. Because the market is closed when the sum of prices reaches the threshold h, we have

$$\begin{aligned} h &= S^c_{1,\tau} + S^c_{2,\tau} \\ &= D_{1,\tau} + D^*_{2,\tau} + \left(\mu^A_1 + \mu_2 - \gamma\sigma^2\theta^A_{1,\tau} + \kappa\mu_J e^{-\gamma\mu_J/2}\right)(T - \tau) \\ &\geq D_{1,\tau} + D_{2,\tau} + \left(\mu^A_1 + \mu_2 - \gamma\sigma^2\theta^A_{1,\tau} + \kappa\mu_J e^{-\gamma\mu_J/2}\right)(T - \tau). \end{aligned}$$

It follows that we may define the stopping time τ using the dividend processes as follows.

[11] As an alternative mechanism, we may assume that the price jump occurs so quickly that nobody can adjust their holdings before the market is halted and thus $\theta^i_{2,\tau} = \theta^i_{2,\tau-}$, $i \in \{A, B\}$. In our numerical analysis, we find that the results are qualitatively the same and quantitatively very close under these two mechanisms. We discuss this issue extensively in Appendix F.

PROPOSITION 4. *Let h be the threshold. Define a stopping time*

$$\tau = \inf\{t \geq 0 : D_{1,t} + D_{2,t} \leq \underline{D}(t)\},$$

where

$$\underline{D}(t) = h - \left(\mu_1^A + \mu_2 - \gamma\sigma^2\theta_{1,t}^A + \kappa\mu_J e^{-\gamma\mu_J/2}\right)(T-t).$$

Then the circuit breaker is triggered at time τ when $\tau < T$.

Note that $D_{1,t} + D_{2,t}$ is a jump diffusion process; thus, the trigger time τ is the first time the jump-diffusion process hits $\underline{D}(t)$.

4.3 Equilibrium Prices before τ

After obtaining the market clearing prices and the optimal portfolios at τ, we now study the equilibrium stock prices at $t < \tau \wedge T$. For $i \in \{A, B\}$, let

$$G_\tau^i(\theta_{1,\tau}^i, \theta_{2,\tau}^i) = G_{1,\tau}^i + G_{2,\tau}^i,$$

where $G_{1,\tau}^i$ and $G_{2,\tau}^i$ are given by (B.5),(B.6), and (B.11) in Appendix B. It can be shown that the indirect utility function of Type $i \in \{A, B\}$ at τ can be written as follows:

$$V^i(W_\tau^i, \tau) = \max_{\theta_{1,\tau}^i, \theta_{2,\tau}^i} \mathbb{E}^i[u(W_\tau^i + \theta_{1,\tau}^i(S_{1,T} - S_{1,\tau}) + \theta_{2,\tau}^i(S_{2,T} - S_{2,\tau}))] = -e^{-\gamma(W_\tau^i + G_\tau^i(\theta_{1,\tau}^i, \theta_{2,\tau}^i))}.$$

Then we are ready to solve the planner's problem:

$$\max_{W_{T\wedge\tau}^A, W_{T\wedge\tau}^B} \mathbb{E}_0^A[V^A(W_{T\wedge\tau}^A, T\wedge\tau) + \xi\eta_{T\wedge\tau}V^B(W_{T\wedge\tau}^B, T\wedge\tau)], \tag{18}$$

subject to the wealth constraint $W_{T\wedge\tau}^A + W_{T\wedge\tau}^B = S_{1,T\wedge\tau} + S_{2,T\wedge\tau}$.

Similar to the case without circuit breakers, it follows from the first order conditions and the wealth constraint that

$$W_{T\wedge\tau}^A = \frac{1}{2\gamma}\log\left(\frac{1}{\xi\eta_{T\wedge\tau}}\right) + \frac{1}{2}(S_{1,T\wedge\tau} + S_{2,T\wedge\tau}) + \frac{G_{T\wedge\tau}^B - G_{T\wedge\tau}^A}{2}, \tag{19}$$

$$W_{T\wedge\tau}^B = -\frac{1}{2\gamma}\log\left(\frac{1}{\xi\eta_{T\wedge\tau}}\right) + \frac{1}{2}(S_{1,T\wedge\tau} + S_{2,T\wedge\tau}) + \frac{G_{T\wedge\tau}^A - G_{T\wedge\tau}^B}{2}. \tag{20}$$

In addition, the state price density under Type A investors' beliefs is

$$\begin{aligned}\pi_t^A &= \mathbb{E}_t^A[\zeta(V^A(W_{T\wedge\tau}^A, T\wedge\tau))'] = \mathbb{E}_t^A[\gamma\zeta e^{-\gamma(W_{T\wedge\tau}^A + G_{T\wedge\tau}^A)}] \\ &= \gamma\zeta\mathbb{E}_t^A[\eta_{T\wedge\tau}^{1/2}\cdot e^{-\frac{\gamma}{2}(S_{1,T\wedge\tau}+S_{2,T\wedge\tau}+G_{T\wedge\tau}^B+G_{T\wedge\tau}^A)}],\end{aligned} \tag{21}$$

for some constant ζ, where $(V^A(W_{T\wedge\tau}^A, T\wedge\tau))'$ denotes the marginal utility of wealth. Thus, the stock price at $t < T\wedge\tau$ in equilibrium is given by

$$S_{j,t} = \frac{\mathbb{E}_t^A[\pi_{T\wedge\tau}^A S_{j,T\wedge\tau}]}{\mathbb{E}_t^A[\pi_{T\wedge\tau}^A]}, \quad j = 1,2, \tag{22}$$

with

$$S_{j,T\wedge\tau} = \begin{cases} D_{j,T}, & \text{if } \tau \geq T, \\ S_{j,\tau}^c, & \text{if } \tau < T. \end{cases} \tag{23}$$

In Equation (22), because the stopping time τ depends on the circuit breaker threshold h, the equilibrium prices $S_{1,t}$ and $S_{2,t}$ also depend on h. On the other hand, in practice, h depends on the initial stock prices $S_{1,0}$ and $S_{2,0}$, because $h = (1-\alpha)(S_{1,0}+S_{2,0})$ (e.g., $\alpha = 0.07$ for China). Therefore, to obtain the equilibrium prices $S_{1,t}$ and $S_{2,t}$, we need to solve the following fixed point problem $S_{1,0}$ and $S_{2,0}$:

$$S_{j,0} = \frac{\mathbb{E}_0^A[\pi_{T\wedge\tau}^A S_{j,T\wedge\tau}]}{\mathbb{E}_0^A[\pi_{T\wedge\tau}^A]}, j = 1,2, \tag{24}$$

where the right hand side is a function of the initial stock prices $S_{1,0}$ and $S_{2,0}$.

In addition, the wealth process of Type A investors is

$$W_t^A = \frac{\mathbb{E}_t^A[\pi_{T\wedge\tau}^A W_{T\wedge\tau}^A]}{\mathbb{E}_t^A[\pi_{T\wedge\tau}^A]}, t < T\wedge\tau. \tag{25}$$

Suppose that

$$dW_t^A = \bar{\theta}_{1,t}^A dS_{1,t} + \bar{\theta}_{2,t}^A dS_{2,t},$$

where $\bar{\theta}_{1,t}^A$ and $\bar{\theta}_{2,t}^A$ are share holdings of Type A for Stock 1 and Stock 2, respectively. We can recover the share holdings of Stock j at t by calculating quantities of $\mathbb{E}_t^A[dW_t^A \cdot dS_{j,t}]$, $\mathbb{E}_t^A[dS_{1,t}\cdot dS_{2,t}]$, and $\mathbb{E}_t^A[(dS_{j,t})^2]$, for $j = 1,2$.

The following proposition guarantees the existence and uniqueness of a solution to the above fixed point problem.

PROPOSITION 5. *If the initial equilibrium index value* $\hat{S}_{1,0} + \hat{S}_{2,0}$ *is positive in the* *absence of circuit breakers, there exists a unique solution to the fixed point problem (24) in the presence of circuit breakers.*

We leave the proof to Appendix D.

In the next section, we numerically compute the equilibrium prices and analyze the impact of circuit breakers.

5. Impact of Circuit Breakers

In this section, we examine the impact of circuit breakers on the dynamics of the market. The default parameter values for numerical analysis are set as follows, where daily growth rates and volatilities are used.

$$\begin{aligned}
&\mu_1^A = 0.10/250, && \sigma = 0.03, && \nu = 0.12, \\
&k = 4, && \delta_0 = -0.12, && \bar{\delta} = -0.12, \\
&\mu_2 = 0.10/250, && \mu_J = -0.1, && \kappa = 0.25, \\
&\gamma = 1, && \alpha = 0.07, && T = 1.
\end{aligned}$$

Given these parameter values, the disagreement δ_t evolves as a random walk with constant drift under Type B investors' probability measure. Because $\delta_0 < 0$, Type B investors initially under-estimate the growth rate of Dividend 1. Since our main goal is to examine the impact of circuit breakers in bad times when the market is volatile and the crash probability of some stocks is high (e.g., the U.S. market in the week of March 9, 2020 and the Chinese stock market in early January of 2016), we set the jump frequency high and the jump size large, along with a high volatility of Stock 1's dividend. Because of the CARA preferences, the initial share endowment of the investors does not affect the equilibrium. The circuit breaker is triggered when the sum of two prices (i.e., the index) first reaches the threshold $(1-\alpha)(S_{1,0}+S_{2,0})$, i.e., drops 7% from the initial value.

One alternative to the market-wide circuit breakers is to impose a circuit breaker separately on each stock (instead of on an index). With this separate circuit breaker on each stock, if a circuit breaker for a stock is triggered, only the trading in the corresponding stock is halted. For example, when the circuit breaker of Stock 1 is triggered, only the trading of Stock 1 is halted, but trading in Stock 2 is unaffected. Obviously, with separate circuit breakers, equilibrium prices remain independent, in sharp contrast to the case of market-wide circuit breakers. Let $S_{j,t}^{sep}, j = 1, 2$ denote the equilibrium prices of Stock j in this benchmark. We compare the impact of circuit breakers on the stock prices when they are on an index and when they are on individual stocks.

5.1 Equilibrium Prices

By Prop. 1 and Prop. 2, we obtain the initial equilibrium prices $\hat{S}_{1,0} = 0.9717, \hat{S}_{2,0} = 0.9741$ in the absence of circuit breakers. When there are separate circuit breakers on individual stocks, the equilibrium prices are $S^{sep}_{1,0} = 0.9711$, $S^{sep}_{2,0} = 0.9800$. The price of Stock 1 at time 0 with circuit breakers is lower than that without circuit breakers, because market closure prevents efficient risk sharing. On the other hand, Stock 2's price with a separate circuit breaker (i.e., $S^{sep}_{2,0}$) is always higher than the one without a circuit breaker (i.e., $\hat{S}_{2,0}$) because the price limit effect of a circuit breaker prevents the price from falling to the full extent after a jump of the dividend.

By solving the fixed point problem numerically, in the presence of market-wide circuit breakers, we obtain the equilibrium prices $S_{1,0} = 0.9715, S_{2,0} = 0.9745$. Compared with the prices without circuit breakers, Stock 1 price is lower and Stock price 2 is higher, and the opposite is true compared with the prices with separate circuit breakers. This is because the market closure effect for Stock 1 and the price limit effect for Stock 2 spill over to the other stock in equilibrium.

5.2 Crash Contagion

Because the circuit breaker based on a stock index is triggered when the index reaches a threshold, a crash in a group of stocks (e.g., from a downward jump in their dividends) may trigger the circuit breaker and cause the entire market to be closed down. As a result, the prices of otherwise independent stocks may also jump down because of the sudden market closure. This pattern of cross-stock serial crashes is called *crash contagion*.

Figure 1 presents the sum of stock prices generated by the same sample path of dividends under different circuit breaker implementations. In this sample path, the market-wide circuit breaker is triggered by a jump in Stock 2's price S_2 as a result of a jump in the dividend process D_2. The sum of prices without circuit breakers ($\hat{S}_t$, green dash line) jumps down to a value below the circuit breaker threshold (h, dot line). Because of the price limit effect, the sum of prices with market-wide circuit breakers (S_t, red sold line) stops at the threshold. This shows that circuit breakers do have the function of price support in bad times. As a result, the index level with circuit breakers is higher than that without any circuit breakers. However, compared to the separate circuit breakers rule, the net price limit effect is smaller. This is because with separate circuit breakers,

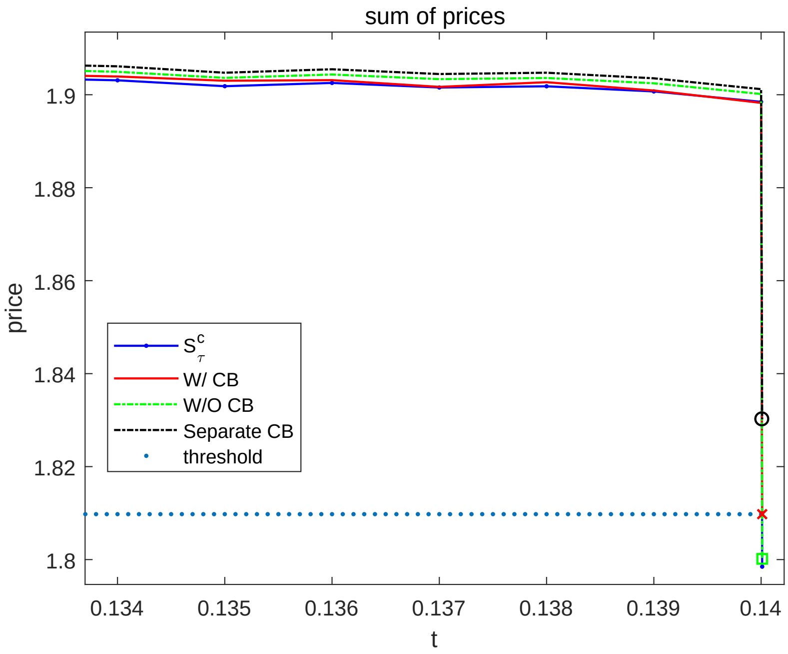

Figure 1. A sample path of the sum of prices. The market is halted early at the time when the solid red line (the sum) touches the threshold at the red cross. In this sample path, the breaker is triggered by a sudden jump occurring in the dividend of Stock 2.

the price limit effect of circuit breakers is kept, while the market-wide closure effect is avoided.

Figure 2 separates out the two individual stock prices using the same sample path as in Figure 1. For Stock 2, its price jumps down toward to the market clearing price $S^c_{2,\tau}$ (the red cross point). For Stock 1, even though there is no jump in its dividend process, its price also jumps down because the price functions in the dividend before the jump and after the jump are different. This figure illustrates that market-wide circuit breakers can cause crash contagion across otherwise independent stocks.

Figures 1 and 2 use a particular sample path to illustrate the possibility of crash contagion. In Figure 3, we plot the distribution of Stock 1's price change conditional on a jump in Stock 2's price that triggers the circuit breaker and holding Stock 1's dividend constant at the crash time (red line) and the distribution of Stock 1's price change with no circuit breaker in place (green line). Figure 3 shows that without a circuit breaker, the price change of Stock 1, which is independent of Stock 2, is normally distributed with mean zero. In contrast, in the presence of circuit breakers, after a crash of Stock 2 that triggers the circuit breaker, Stock 1's price always goes down and the magnitude of the

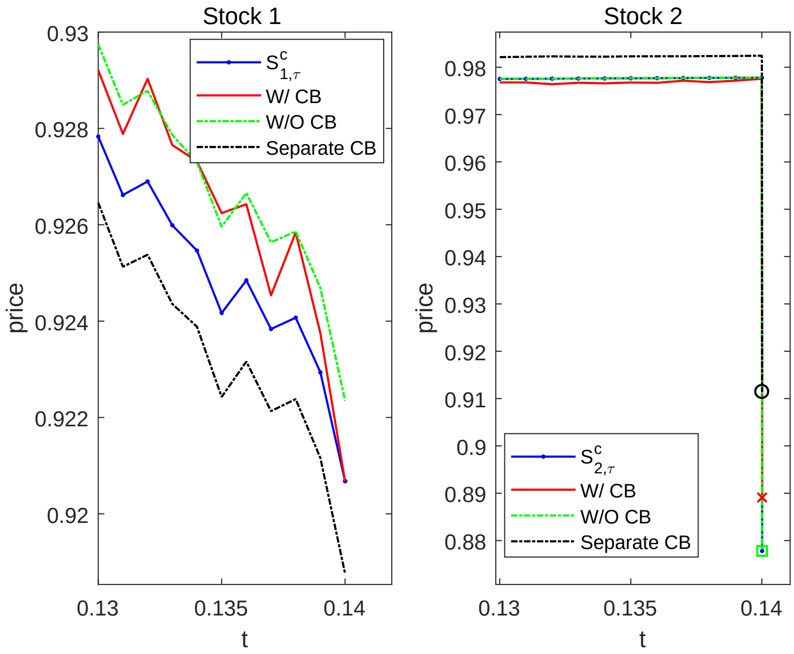

Figure 2. The two individual prices. Using the sample path as in Figure 1, the circuit breaker is triggered by a jump occurring in the price $S_{2,t}$ (the right panel).

drop can be significant. Note that this change is clearly caused by the crash of Stock 2 because Stock 1's dividend is held constant when the crash occurs. Therefore, this distribution represents the distribution of the crash magnitudes in Stock 1 caused by the crash in Stock 2.[12]

Our findings are consistent with what happened in March 2020 in the U.S. stock market. The circuit breaker of the U.S. market was triggered four times in March 2020. The first one occurred at 9:34:13 am on March 9th, less than five minutes after the market opened. The second occurred at 9:35:43 on March 12th. Four days later, the market lasted only one second on March 16th before the circuit breaker was triggered. The fourth time occurred at 12:56:17 pm on March 18th. To illustrate that our results are consistent with what happened around the circuit breaker trigger time, we use high-frequency prices of the components of the S&P 500 index during the 10 minutes before the circuit breaker was triggered on March 18th, 2020 and sort components by their total dollar trading volumes. Simple regression of the return of the top 25%–50% of stocks inside the S&P 500 index on the lagged return of the top 25% of stocks suggests that, in

[12]Because a crash in Stock 2 occurs randomly, the crash in Stock 1 caused by Stock 2's crash is also random, even though Stock 1's dividend is kept constant when the crash occurs.

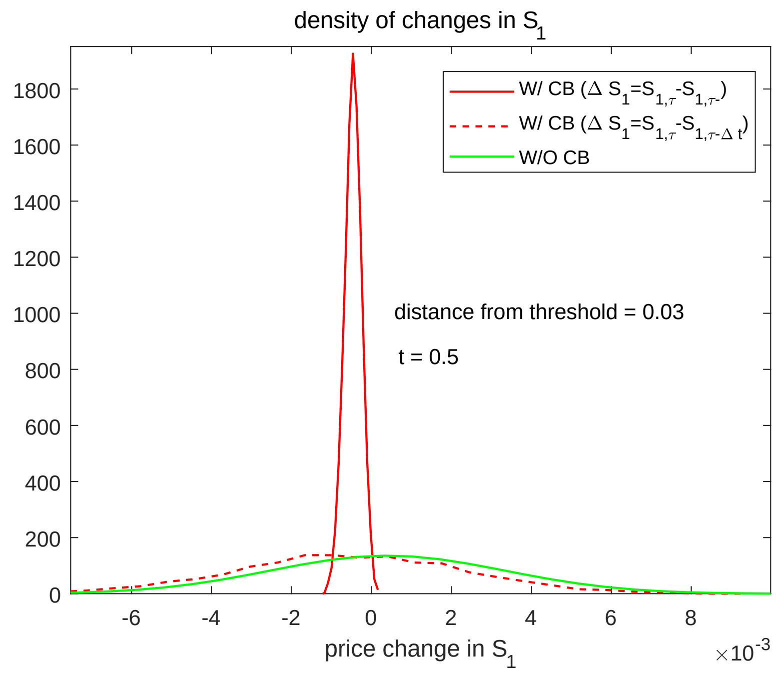

Figure 3. Distribution of changes in Stock 1's price when the circuit breaker is triggered by a jump in Stock 2's price. In the presence of a circuit breaker, the distribution is skewed negatively. Results for two methods of measuring the changes are presented. Meanwhile, in the absence of circuit breakers the price changes follows a normal distribution.

the market crash of March 18th, 2020, the crash of the top 25%–50% of stocks followed that of the top 25% of stocks. Figure 4 depicts how returns of S&P 500 component stocks moved during the 10 minute period right before the market was halted.

Let Rt_t and Rb_t be the time t returns of the top 25% of stocks and the top 25%–50% of stocks, respectively. We obtain the regression result as follows.

$$\begin{aligned} Rb_t &= -0.01 + 0.5Rt_{t-\Delta t} + 0.1Rt_t + \epsilon_t, \\ t\text{-}stat &: (-9) \quad (10.8) \qquad\quad (2.2) \end{aligned}$$

where Δt equals one second. The regression result indicates that those stocks with relatively low trading volumes followed the moves of those with high trading volumes. While this does not prove the causal relationship, it suggests a pattern that is consistent with cross-stock contagion.

A similar illustration is shown in Figure 5 for stocks in the China Securities Index (CSI) 300 index on January 4th, 2016 when the Chinese stock market crashed. Simple regression of the return of the top 25%-50% of stocks inside the CSI 300 index on the

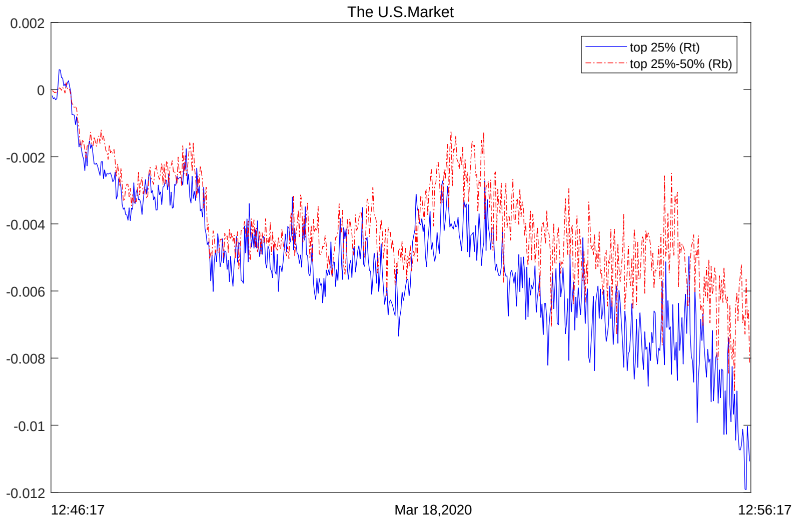

Figure 4. Evidence of contagion in real markets: the United States.

lagged return of the top 25% of stocks yields

$$\begin{aligned} Rb_t =& 10^{-4} + 0.78 Rt_{t-\Delta t} + 0.25 Rt_t + \epsilon_t, \\ t\text{-}stat:& \ (0.67) \quad (2.85) \qquad\quad (0.92) \end{aligned}$$

where Δt equals 3 seconds. This result suggests that, in the market crash of January 4th, 2016, the crash of the top 25%–50% of stocks followed that of the top 25% of stocks.

5.3 Increased Correlations

With circuit breakers based on indices, a discrete jump (crash) in a stock is not necessary to adversely affect otherwise independent stocks. Intuitively, even after a small decline in the price of a stock, the index gets closer to the circuit breaker threshold and thus the market is more likely to be closed early, which may lower the prices of otherwise independent stocks, which in turn makes the index even closer to the circuit breaker threshold, entering into a vicious circle. This contagion magnitude is typically smaller than that caused by a crash in a stock in normal times, but can become much more significant and create strong correlations when the circuit breaker is close to being triggered because of the magnified vicious circle effect. We next show that a gradual change in the price of a stock can indeed affect the price of another stock and can also cause high correlations

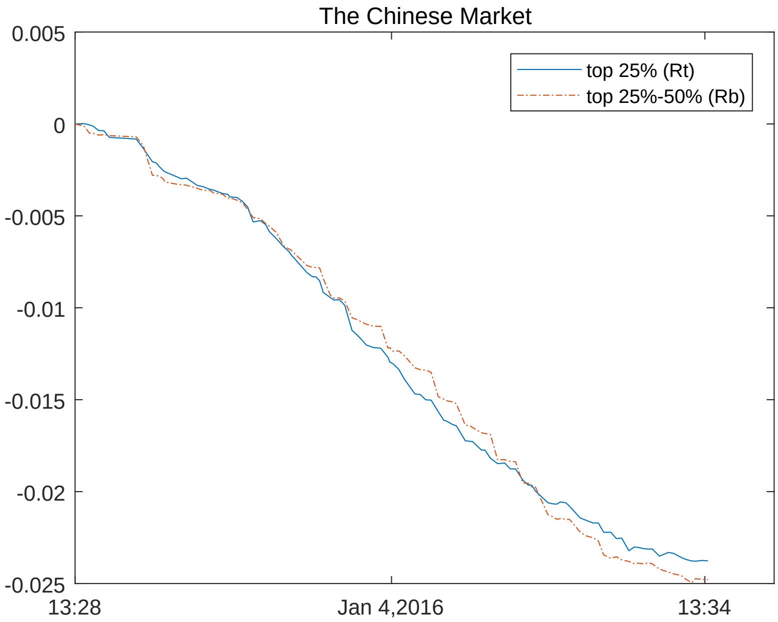

Figure 5. Evidence of contagion in real markets: China.

among otherwise independent stocks when close to the circuit breaker threshold.

Figures 6 and 7 show how the same variables as those in Figures 1 and 2 change along a different sample path, where the circuit breaker is triggered by a small change in Stock 1's price due to a decline in its dividend. Unlike the sample paths illustrated in Figures 1 and 2, prices do not jump in Figures 6 and 7 because there is no jump in dividends. On the other hand, as the right sub-figure of Figure 7 shows, Stock 2's price is adversely affected by the decline in Stock 1's price. Figures 1–7 suggest that they become positively correlated when the circuit breakers are close to being triggered, even though the two stocks are independent in the absence of circuit breakers. As in the case of a jump–triggered market closure, the prices under the separate circuit breakers rule (black dot-dash lines) are higher than those with market-wide circuit breakers.

Consistent with our intuition, Figure 8 shows that the correlation between the two prices with circuit breakers increases significantly as the index gets very close to the threshold.[13] When the index is far from the threshold, the correlation becomes close

[13] In the figure, "distance from threshold" is defined as the value of the index in exceed of the threshold. Because the equilibrium index level is determined jointly by the dividend levels of the two stocks, the way to vary the distance is not unique. In all the figures in this paper that plot against the distance to threshold we fix $D_{1,t}$ and vary $D_{2,t}$. We also used alternative ways such as fixing $D_{1,t}$ and varying $D_{2,t}$ and find similar results.

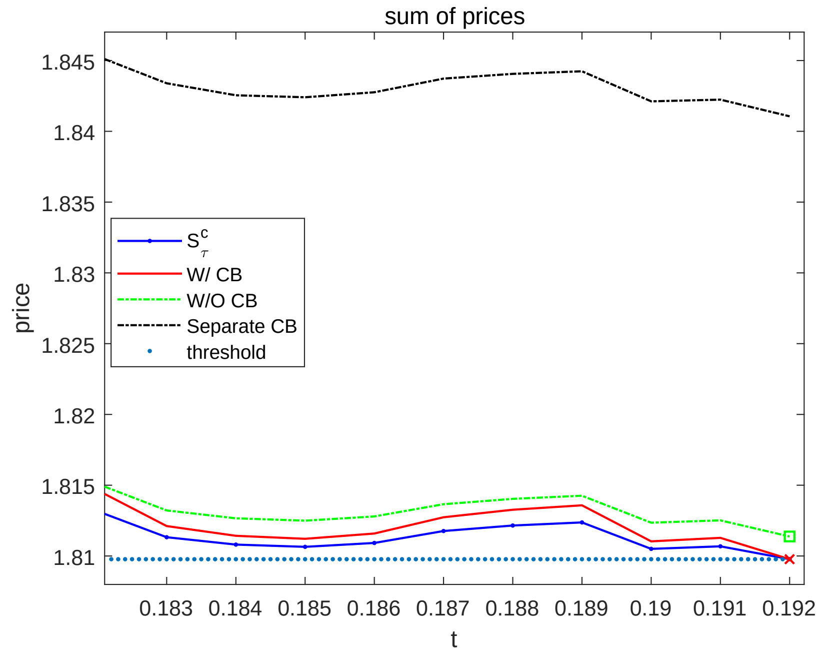

Figure 6. A sample path of the sum of prices along which the circuit breaker is triggered by Stock 1.

to zero, because the correlation without circuit breakers is zero. In addition, when the potential market closure duration is large ($T-t$ is large), the impact of the circuit breakers on the correlation is even greater, because the fear of a market closure is stronger when the potential market closure duration is longer. For example, conditional on the same distance of 0.02 from the threshold, if it is later in the day at $t = 0.75$, then the correlation is 0.2, while it is around 0.55 if it is early in the day at $t = 0$.

Surprisingly, Figure 8 shows that the correlation can become negative when the distance from the threshold is greater, before it approaches zero eventually. This negative correlation is due to the price limit effect of the circuit breaker. To help explain this channel, we plot stock price and the index in Figure 9 against changes in Stock 1's dividend. As Stock 1's dividend increases, the price of Stock 1 increases (blue starred line), as expected. However, the price of Stock 2 changes non-monotonically (red starred line). When Stock 1's dividend is very low such that a small change in either stock price would trigger the circuit breaker, the likelihood of the circuit breaker being triggered by a jump in Stock 2's dividend is relatively small because the probability of a jump is low. Recall that the price limit effect is strictly positive only when the circuit breaker is triggered by a jump in Stock 2's dividend. This implies that the present value of the price limit

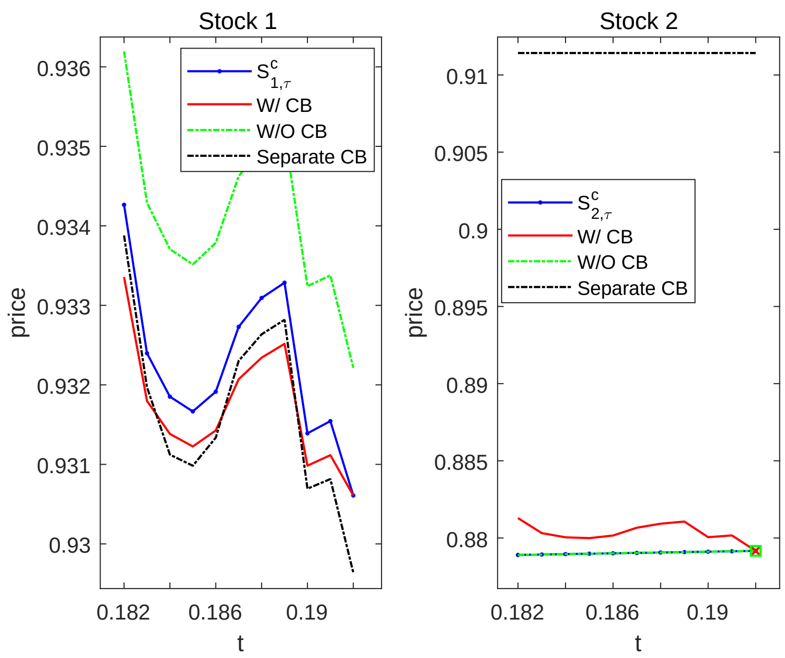

Figure 7. The individual stock prices along the same sample path as in the previous figure.

effect of the circuit breaker is small. As Stock 1's dividend increases, the price of Stock 1 increases, and thus the distance from the circuit breaker threshold is increased. It becomes more likely that only a jump in Stock 2's dividend can trigger the circuit breaker. Therefore, the price limit effect increases, which in turn increases the price of Stock 2. However, when Stock 1's dividend is too large, the index moves far away from the threshold and thus even a jump in Stock 2's dividend would not trigger the circuit breaker. Therefore, the price limit effect eventually approaches zero when D_1 is high enough. This explains the nonmonotonicity of the price of Stock 2 in D_1, which in turn implies that the correlation is positive when the index is close to the threshold, turns negative when the index is further away, and converges to zero when the index is far enough, as shown in Figure 8.

5.4 Acceleration of Market Decline: The Magnet Effect

Circuit breakers are implemented to protect the market from a fast decline. Contrary to this intention, Chen, Petukhov, and Wang (2017) show in a single-stock setting that circuit breakers can accelerate a stock price decline compared to the case without circuit breakers. This acceleration is what is called the "magnet effect" by

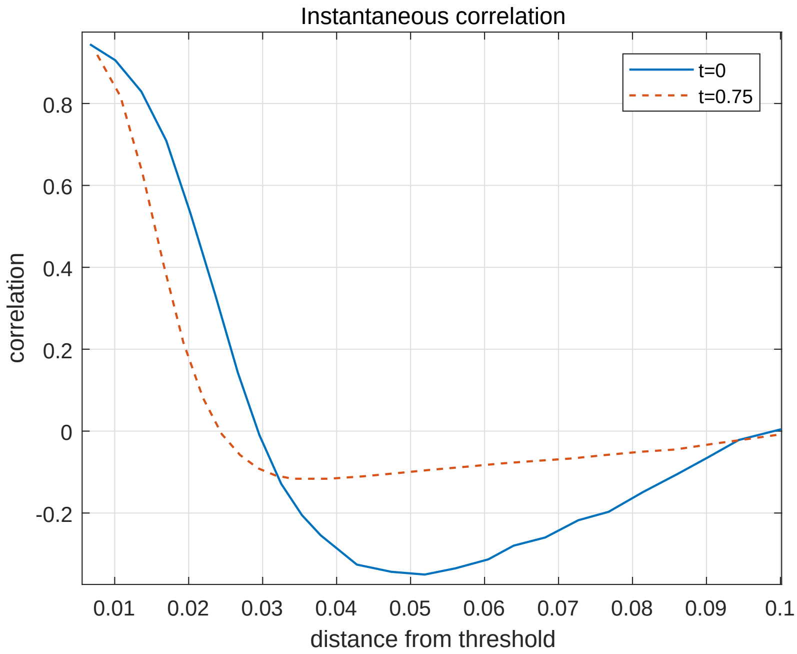

Figure 8. Instantaneous correlation.

Chen, Petukhov, and Wang (2017). However, it is not clear how the presence of multiple stocks affects this magnet effect. Our following results suggest that, in the presence of circuit breakers on stock indices, the probability of falling to the index threshold compared to the case without circuit breakers is also increased, so the magnet effect found by Chen, Petukhov, and Wang (2017) is robust to a multiple-stock setting and the absence of leverage constraints after market closure. On the other hand, with separate circuit breakers, such a probability may be reduced in some cases.

Figure 10 shows the probabilities of reaching the circuit breaker index threshold in a given time interval with circuit breakers on the index, with separate circuit breakers on individual stocks, and without circuit breakers. It suggests that the probability of falling to the index threshold when there is a circuit breaker on the index (red dash-dot lines) is higher than that without any circuit breakers (blue solid lines), which is in turn higher than that when circuit breakers are on individual stocks (black dash lines). This is because with circuit breakers on indices, when one stock goes down, the distance to the circuit breaker threshold is shorter and the likelihood of an early market closure is greater. As a result, other stock prices tend to go down, which in turn drags the index further downward, resulting in a downward accelerating vicious circle, contrary to regulators'

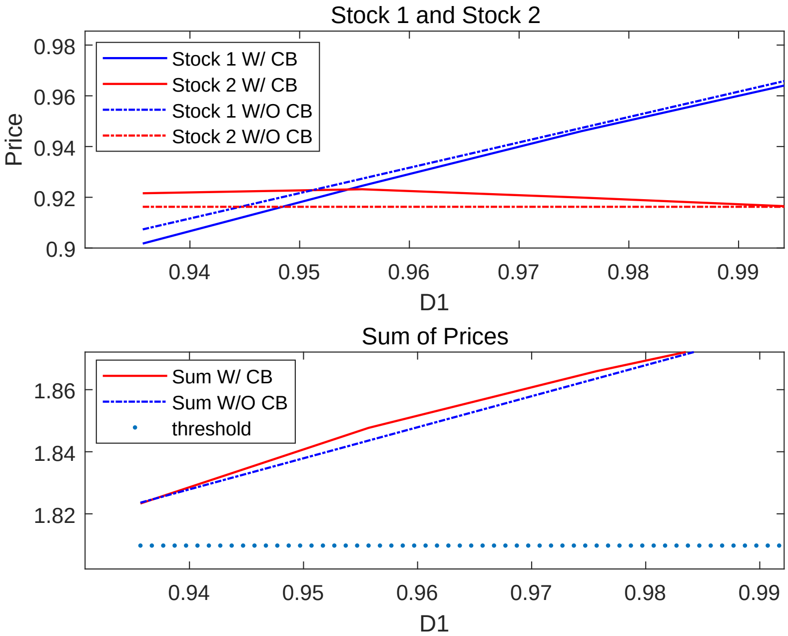

Figure 9. This figure illustrates why the correlation is positive when the threshold is close and why it becomes positive when the distance is larger. Eventually, $S_{2,t}$ approaches a constant and the correlation becomes almost zero.

intention. Because of the contagion effect across stocks, the magnet effect in a model with multiple stocks like ours can be stronger than that found in the single-stock setting of Chen, Petukhov, and Wang (2017), ceteris paribus. In addition, when the potential market closure duration is longer (e.g., at $t = 0$), this magnet effect is even stronger. The main driving force for the magnet effect in Chen, Petukhov, and Wang (2017) is the fear that one has to liquidate a levered position at the market closure time because after market closure, leverage is prohibited by the solvency requirement. In contrast, in this paper there is no change in the leverage level allowed before and after market closure. Our results show that circuit breakers on indices can accelerate a market decline even without the de-leverage channel.

In contrast, if circuit breakers are imposed on individual stocks, the probability of falling to the index threshold can be lower than that without circuit breakers. This is because individual circuit breakers prevent corresponding stock prices from falling below their individual stock price thresholds and thus can decrease the probability of falling to the index threshold compared to the case without circuit breakers. These results suggest that separate circuit breakers can potentially slow down a market-wide decline, while circuit breakers on indices tend to do the opposite.

Another measure of the magnet effect is how fast stock prices go down as the index

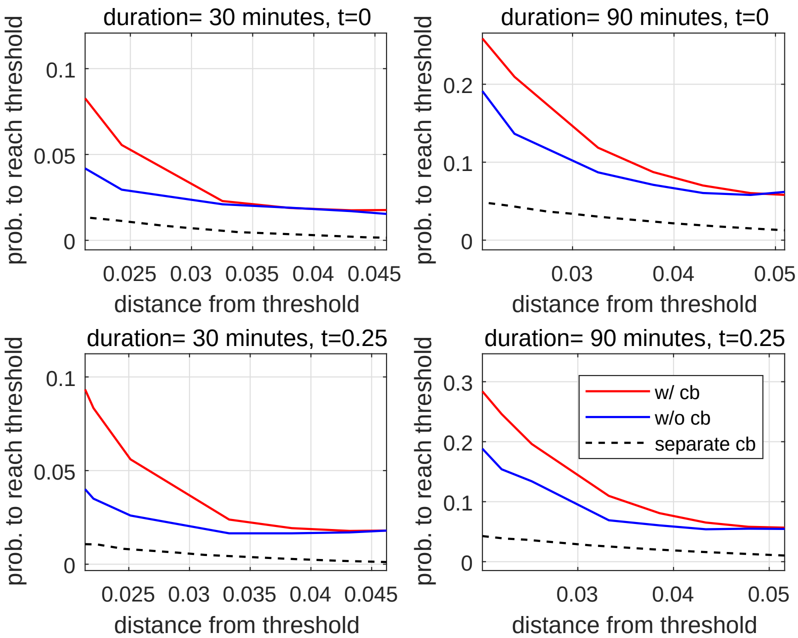

Figure 10. This figure shows the probability that prices will reach the threshold with or without a circuit breaker.

gets close to the circuit breaker threshold. In Figure 11, we plot the average prices against the time to market closure using simulated sample paths. More specifically, we simulate a large number of sample paths of dividends, compute the corresponding stock prices, and identify the circuit breaker trigger times for each sample path. Then we calculate the average stock prices across all the sample paths at a given time prior to market closure. The downward-concave shapes displayed in Figure 11 imply that as the index gets closer to the threshold, stock prices fall faster. Figure 12 plots the average time it takes for the index to fall by 1% against the distance to the threshold. It implies that the falling speed increases as the index gets closer to the threshold. These patterns are consistent with those observed in real markets, such as the U.S. market in the week of March 12th, 2020 and the January 2016 Chinese market when circuit breakers were first implemented and then abandoned after four days.

5.5 Volatility Contagion

One of the regulatory goals of the circuit breaker is to reduce market volatility. We next examine the circuit breaker's impact on stock volatility and volatility contagion.

In Figure 13, we plot the ratios of the volatility with circuit breakers to that without circuit breakers against the index's distance from the circuit breaker's threshold at two

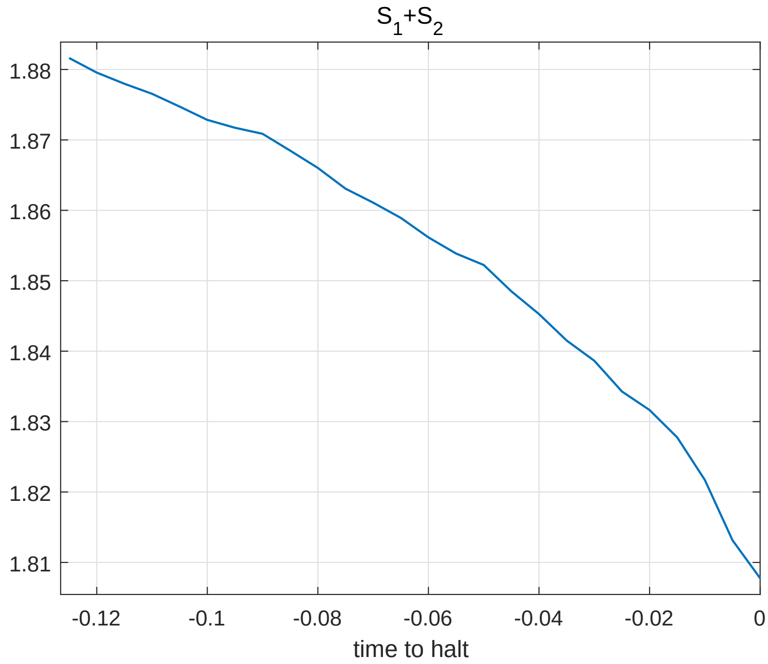

Figure 11. This figure shows average stock prices during a short period immediately prior to the early closure of the market caused by Stock 1.

different time points $t = 0$ and $t = 0.5$. The figure suggests that, contrary to the regulatory goal, circuit breakers can increase stock volatility. In addition, we find that if the time to the end of the day is longer (i.e., the potential market closure duration is greater), volatilities are even larger. When the distance from the threshold is sufficiently large, instantaneous volatilities approach the corresponding levels in the absence of circuit breakers.

Unlike Chen, Petukhov, and Wang (2017), however, as shown in Figure 13, circuit breakers can reduce Stock 1 volatility when the index is sufficiently close to the threshold. For Stock 2, its volatility is reduced for a wider range.[14] This reduction is due to the price limit effect, which is absent in Chen, Petukhov, and Wang (2017). Intuitively, circuit breakers can limit the price drop and thus reduce price fluctuation. This volatility reduction effect in some states is consistent with the conventional wisdom for imposing circuit breakers: trading halts can reduce market volatility.

Next, we show that circuit breakers can also cause volatility contagion, i.e., an increase in the volatility of one stock can cause an increase in that of another. Figure 14 plots the volatility of Stock 1 against that of Stock 2 as we change the volatility of Stock 1's dividend for three levels of distance to the threshold. Figure 14 indicates that, indeed,

[14] Recall that the volatility when the distance is large is close to that in the absence of circuit breakers.

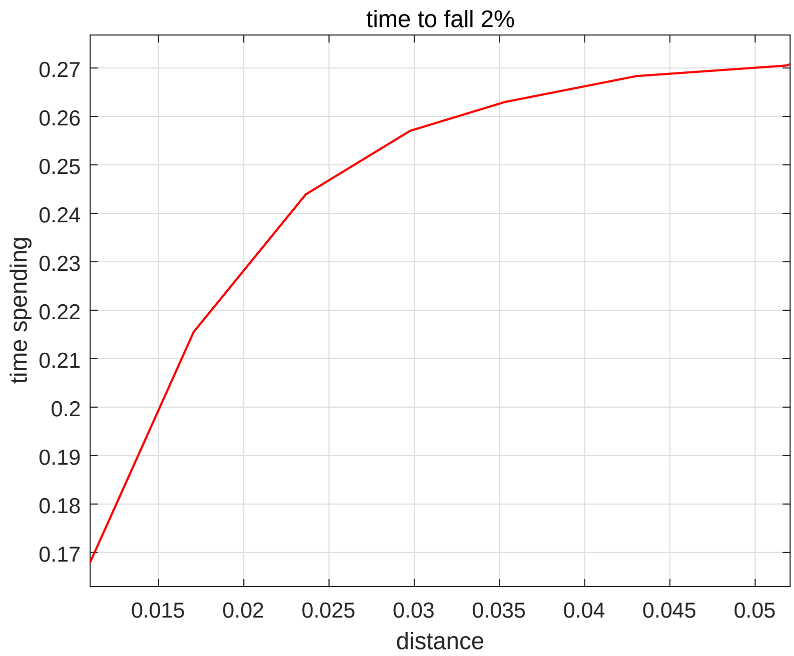

Figure 12. This figure shows the average time spent for the sum of prices dropping 2% at various distances from the threshold.

a higher volatility of Stock 1's dividend causes a higher volatility of Stock 2, and in addition, this increase in the volatility of Stock 2 gets magnified when the distance to the threshold is shorter. This volatility contagion can amplify market-wide volatility, which is contrary to the purpose of circuit breakers.

6. Correlated Dividends

In the preceding sections, the dividend processes are assumed to be uncorrelated and we show that a strong correlation of the stock prices can emerge due to circuit breakers. One may suspect that, if the dividend processes are already correlated, then the additional correlation caused by the circuit breakers may be small and thus the effect of circuit breakers in increasing correlation is small in practice. To address this concern, we now briefly discuss an extended model where the dividend processes are correlated (the detailed derivation can be found in Appendix E) by assuming a diffusion term in the dynamics of Stock 2's dividend:

$$dD_{2,t} = \mu_2 dt + \sigma_2 dZ_t + \mu_J dN_t.$$

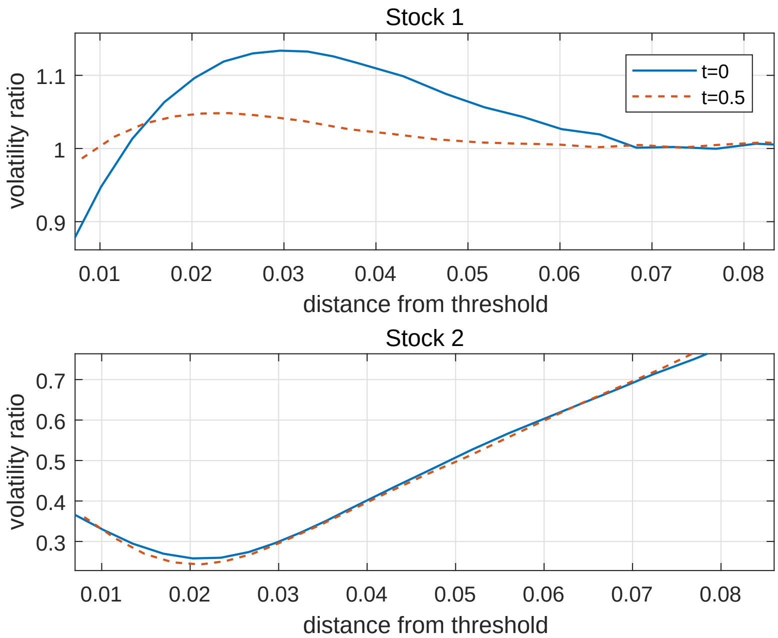

Figure 13. This figure plot the ratios of volatility with circuit breakers to that without circuit breakers against the distance from the circuit breaker threshold.

Then the two dividend processes are correlated with correlation (suppose $\sigma_1 > 0$)

$$\rho_D = \frac{\sigma_2}{\sqrt{\sigma_2^2 + \kappa\mu_J^2}}. \tag{26}$$

Figure 15 compares correlations of stock prices for different correlation coefficients ρ_D of the dividend processes.

The figure suggests that even when the dividends are correlated, the presence of circuit breakers can significantly increase the correlation of stock prices. For example, when the dividends' correlation coefficient is 0.2, the increase in the correlation is still as high as 0.6. Furthermore, the presence of circuit breakers can even make negatively correlated stocks in the absence of circuit breakers ($\sigma_1 < 0$, $\sigma_2 < 0$) become positively correlated. This reversal is because as the index gets close to the threshold, the common fear for market closure offsets the effect of the negatively correlated dividends and as a result the correlation turns positive.

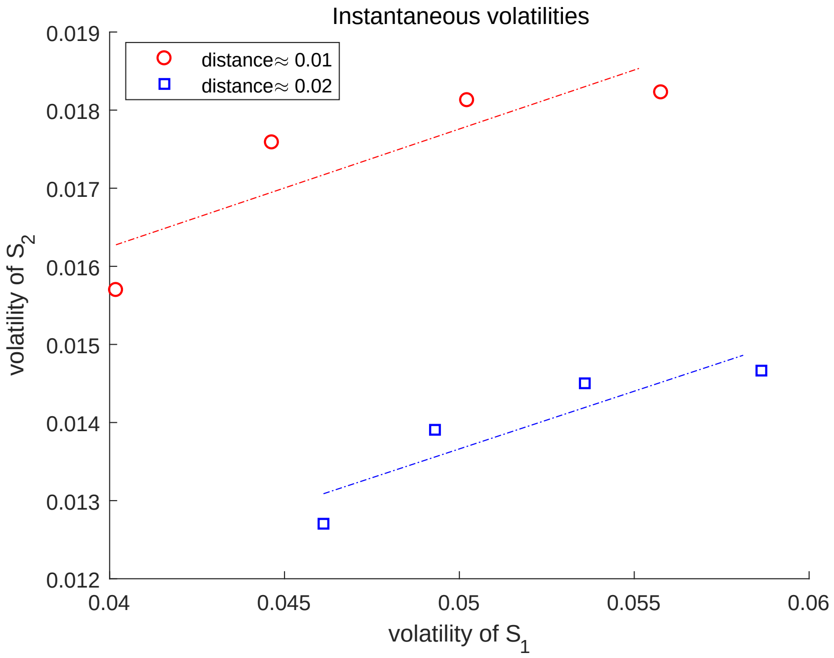

Figure 14. Volatilities of the two prices are independent in the absence of circuit breakers. This figure shows volatilities are linearly correlated in the presence of circuit breakers. A higher volatility of Stock 1 corresponds to a higher volatility of Stock 2. Moreover, if the threshold is closer, the volatility of Stock 2 is even larger.

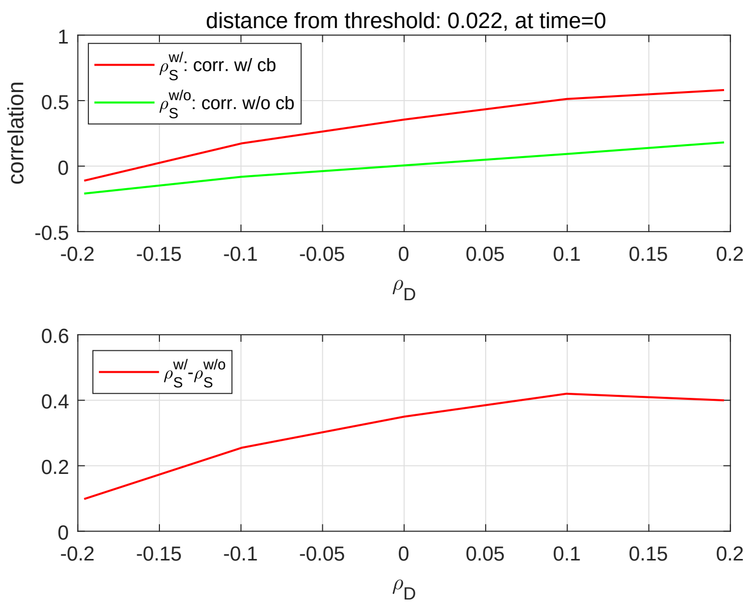

Figure 15. This figure shows the impact of a correlation in dividend processes on the correlation in stock prices. All correlations are calculated when the distance from threshold is around 0.05.

7. Conclusion

Circuit breakers based on indices are commonly imposed in financial markets to prevent market crashes and reduce volatility in bad times. We develop a continuous-time equilibrium model with multiple stocks to study how circuit breakers affect joint stock price dynamics, cross-stock contagion, and market volatility. Contrary to regulatory goals, we show that in bad times, circuit breakers can cause crash contagion, volatility contagion, and high correlations among otherwise independent stocks, and can significantly increase volatility and accelerate market decline. Our analysis suggests that international market plunges triggered by the COVID-19 pandemic may have been exacerbated by circuit breakers rules because of the contagion effect of these circuit breakers. Our model shows that market-wide circuit breakers may be a source of financial contagion and a channel through which idiosyncratic risks become systemic risks, especially in bad times. An alternative circuit breaker approach based on individual stock returns instead of indices would alleviate such problems.

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Appendix

A Price of Stock 1: Without Circuit Breakers

We assume that the disagreement process δ_t is stochastic and follows Equation (5). When there are no circuit breakers, the equilibrium price of Stock 1 is independent of Stock 2 because of independent dividend processes. The price of Stock 1 can be obtained in closed-form as follows.

We first evaluate $\mathbb{E}_t^A[\pi_{1,T}^A]$. Ignoring constants, we need to calculate

$$\mathbb{E}_t^A[\eta_{1,T}^{1/2} e^{-\frac{\gamma}{2}D_{1,T}}] = \mathbb{E}_t^A[e^{Y_{1,T}}] \cdot f(t),$$

where $f(t)$ is a deterministic function and,

$$Y_{1,T} = \int_0^T (\frac{\delta_s}{2\sigma} - \frac{\gamma\sigma}{2})dZ_s + \int_0^T (-\frac{\delta_s^2}{4\sigma^2})ds.$$

Conjecture $F(t, y, \delta, \delta^2) = e^{A(t)+B(t)y+C(t)\delta+\frac{H(t)}{2}\delta^2} = \mathbb{E}^A[e^{Y_T}|Y_t = y, \delta_t = \delta]$, with $A(T) = C(T) = H(T) = 0$ and $B(T) = 1$. Substituting the conjecture into the moment generating function of the process (Y_t, δ_t) and collecting the coefficients of y, δ, δ^2 and constants, we obtain four ordinary different equations:

$$\begin{aligned}
A'(t) + \frac{1}{8}\gamma^2\sigma^2 B(t)^2 + k\bar{\delta}C(t) + \frac{\nu^2}{2}(C(t)^2 + H(t)) - \frac{\gamma\sigma\nu}{2}B(t)C(t) &= 0, \\
B'(t) &= 0, \\
C'(t) - \frac{\gamma}{4}B(t)^2 + k\bar{\delta}H(t) - kC(t) + C(t)H(t)\nu^2 + \frac{\nu}{2\sigma}B(t)C(t) - \frac{\gamma\sigma\nu}{2}B(t)H(t) &= 0, \\
\frac{H'(t)}{2} - \frac{1}{4\sigma^2}B(t) + \frac{B(t)^2}{8\sigma^2} - kH(t) + \frac{\nu^2}{2}H(t)^2 + \frac{\nu B(t)H(t)}{2\sigma} &= 0.
\end{aligned}$$

The solution of the ODE system is obtained as follows.

$$
\begin{aligned}
B(t) &= 1,\\
H(t) &= \frac{e^{(D^+-D^-)v^2(t-T)}-1}{e^{(D^+-D^-)v^2(t-T)}D^- - D^+}D^+D^-,\\
C(t) &= \int_t^T e^{\int_t^s f(x)ds} g(s)ds = \frac{1}{\Delta(D^- - D^+e^{2\Delta(T-t)})}\\
&\quad \cdot \left(-\frac{\gamma}{4}((D^+ + D^-)e^{\Delta(T-t)} - D^+e^{2\Delta(T-t)} - D^-) - (k\bar{\delta} - \frac{\sigma\nu\gamma}{2})D^+D^-(e^{\Delta(T-t)}-1)^2\right),\\
A(t) &= \int_T^t (-\frac{1}{8}\gamma^2\sigma^2 - k\bar{\delta}C(s) - \frac{v^2}{2}(C(s)^2 + H(s)) + \frac{\gamma}{2}v\sigma C(s))ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta =& \sqrt{k^2 + \frac{v^2}{2\sigma^2} - \frac{vk}{\sigma}},\\
D^{\pm} =& \frac{k - \frac{v}{2\sigma} \pm \sqrt{k^2 + \frac{v^2}{2\sigma^2} - \frac{vk}{\sigma}}}{v^2},\\
f(t) =& -k + v^2H(t) + \frac{v}{2\sigma},\\
g(t) =& -\frac{\gamma}{4} + k\bar{\delta}H(t) - \frac{\gamma\sigma v}{2}H(t).
\end{aligned}
$$

Then

$$
\mathbb{E}_t^A[e^{Y_T}] = F(t, y, \delta, \delta^2; \gamma) = e^{A(t)+B(t)y+C(t)\delta+\frac{H(t)}{2}\delta^2}.
$$

Next, we consider the first derivative of F with respect to γ to obtain $\mathbb{E}_t^A[e^{Y_T}Z_T]$. We define

$$
A(t;\gamma) = A(t), C(t;\gamma) = C(t).
$$

Note that

$$
\begin{aligned}
\frac{dB(t)}{d\gamma} =& \frac{dH(t)}{d\gamma} = 0,\\
\frac{dC(t;\gamma)}{d\gamma} =& \int_t^T e^{\int_t^s f(x)dx}[-\frac{1}{4} - \frac{\sigma v}{2}H(s)]ds,\\
\frac{dA(t;\gamma)}{d\gamma} =& \int_T^t (\frac{-\sigma^2\gamma}{4} - k\bar{\delta}\frac{dC(s;\gamma)}{d\gamma} - v^2C(s;\gamma)\frac{dC(s;\gamma)}{d\gamma} + \frac{v\sigma}{2}C(s;\gamma) + \frac{\gamma v\sigma}{2}\frac{dC(s;\gamma)}{d\gamma})ds.
\end{aligned}
$$

Hence

$$
\mathbb{E}_t^A[e^{Y_T}Z_T] = -\frac{2}{\sigma}\frac{d}{d\gamma}\mathbb{E}_t^A[e^{Y_T}] = -\frac{2}{\sigma}\frac{d}{d\gamma}F(t, y, \delta, \delta^2; \gamma).
$$

Finally, the stock price in the equilibrium is given by

$$\begin{aligned}\hat{S}_{1,t} &= \frac{\mathbb{E}_t^A[\pi_{1,T}^A D_{1,T}]}{\mathbb{E}_t^A[\pi_{1,T}^A]} = \frac{\mathbb{E}_t^A[\pi_{1,T}^A D_{1,T}]}{F} = D_{1,0} + \mu_1^A T - 2\frac{\frac{dF}{d\gamma}}{F} \\ &= D_0 + \mu^A T - 2(\frac{dA(t;\gamma)}{d\gamma} + \frac{dy}{d\gamma} + \frac{dC(t;\gamma)}{d\gamma}\delta_t) \\ &= D_{1,0} + \mu_1^A T - 2(\frac{dA(t;\gamma)}{d\gamma} - \frac{\sigma}{2} Z_t + \frac{dC(t;\gamma)}{d\gamma}\delta_t).\end{aligned}$$

The last equality above holds because $Y_t = \int_0^t (\frac{\delta_s}{2\sigma} - \frac{\gamma\sigma}{2}) dZ_s + \int_0^t (-\frac{\delta_s^2}{4\sigma^2}) ds$ and $Y_t = y$ yield $dy/d\gamma = -1/2\sigma Z_t$. By $D_{1,t} = D_{1,0} + \mu_1^A t + \sigma Z_t$ (μ_1^A is constant), we obtain

$$\hat{S}_{1,t} = D_{1,t} + \mu_1^A (T-t) - 2(\frac{dA(t;\gamma)}{d\gamma} + \frac{dC(t;\gamma)}{d\gamma}\delta_t). \tag{A.1}$$

In case δ_t is constant, i.e., $v = k = 0$ and $\delta_t \equiv \delta_0$, we find that $dA(t)/d\gamma = -\sigma^2\gamma/4(t-T)$ and $dC(t;\gamma)/d\gamma = -1/4(T-t)$. Thus, $\hat{S}_{1,t} = D_{1,t} + \mu_1^A(T-t) + (\delta_0/2 - \sigma^2\gamma/2)(T-t)$. This is the equilibrium price of Stock 1 in the case of constant disagreement.

Since $H(t) \to 0$ as $t \to T$, we see that $dC(t;\gamma)/d\gamma$ is negative when $T-t$ is small. Thus, it follows (A.1) that the instantaneous volatility of the stock price $\sigma_{\hat{S}} = \sigma - 2\frac{dC(t)}{d\gamma}\nu$ is greater than the dividend volatility σ when $T-t$ is small, given ν is positive.

B Market Clearing Prices

B.1 Stock 1: Stochastic Disagreement

In the presence of circuit breakers, we cannot obtain the equilibrium price of Stock 1 directly. In this section, we derive the market clearing price of Stock 1 when a circuit breaker is triggered and the market is closed early. Because the two dividend processes are independent and we assume no leverage constraints when the market is halted, the market clearing prices for the two stocks are independent of each other.

The disagreement δ_t is stochastic following (5), therefore $\mu_{1,t}^B = \delta_t + \mu_1^A$ is stochastic as well. In the presence of a circuit breaker, we solve for the market clearing price when the market is halted. To do so, we solve the utility maximization problem

$$\max_{\theta_{1,\tau}^A} \mathbb{E}_\tau^A[-e^{-\gamma W_T^A}],$$

subject to $W_T^A = \theta_\tau^A(D_{1,T} - S_{1,\tau}) + W_\tau^A$, where W_t^A is the wealth of Type A investors at time t.

Using the dynamics $D_{1,T} = D_{1,\tau} + \mu_1^A(T-\tau) + \sigma(Z_T - Z_\tau)$, we obtain the optimal portfolio of agent A as follows.

$$\theta_{1,\tau}^A = \frac{D_{1,\tau} - S_{2,\tau} + \mu_1^A(T-\tau)}{\gamma\sigma^2(T-\tau)}. \tag{B.1}$$

Next, we study the utility maximization problem of agent B:

$$\max_{\theta_{1,\tau}^B} \mathbb{E}_\tau^B[-e^{-\gamma(W_\tau^B + \theta_{1,\tau}^B(D_{1,T} - S_{1,\tau}))}].$$

We first prove the following lemma.

Lemma B1. *Suppose θ is a constant, then*

$$\mathbb{E}_t^B[e^{-\gamma\theta D_{1,T}}] = e^{A(t,\theta) + B(t,\theta)D_{1,t} + C(t,\theta)\delta_t},$$

where

$$\begin{aligned}
A(t,\theta) &= \gamma\theta\mu_1^A(t-T) - \frac{\sigma^2}{2}\gamma^2\theta^2(t-T) + \frac{1}{\tilde{k}}(-\gamma\theta k\bar{\delta} + \nu\sigma\gamma^2\theta^2)(T - t - \frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}}) \\
&\quad + \frac{\nu^2\gamma^2\theta^2}{2\tilde{k}^2}(T - t - 2\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}} + \frac{1-e^{2\tilde{k}(t-T)}}{2\tilde{k}}), \\
B(t,\theta) &= -\gamma\theta, \\
C(t,\theta) &= \frac{-\gamma\theta}{\tilde{k}}(1 - e^{\tilde{k}(t-T)}),
\end{aligned}$$

with $\tilde{k} = k - \nu/\sigma$. In particular, if $\tilde{k} = 0$, then

$$\begin{aligned}
A(t,\theta) &= \gamma\theta\mu_1^A(t-T) - \frac{\sigma^2}{2}\gamma^2\theta^2(t-T) + \frac{1}{2}(-\gamma\theta k\bar{\delta} + \gamma^2\theta^2\nu\sigma)(t-T)^2 - \frac{\nu^2\gamma^2\theta^2}{6}(t-T)^3, \\
B(t,\theta) &= -\gamma\theta, \\
C(t,\theta) &= \gamma\theta(t-T).
\end{aligned}$$

Lemma B1 can be proved by using the moment generating function of process $D_{1,t}$ and δ_t and solving an ODE system. Detailed deviations are omitted here.

By the lemma,

$$\mathbb{E}^B_\tau[-e^{-\gamma(W^B_\tau+\theta^B_{1,\tau}(D_{1,T}-S_{1,\tau}))}] = -e^{-\gamma W^B_\tau} e^{A(t,\theta^B_{1,\tau})+C(t,\theta^B_{1,\tau})\delta_\tau} e^{-\gamma\theta^B_{1,\tau}(D_{1,\tau}-S_{1,\tau})}.$$

Then the FOC with respect to $\theta^B_{1,\tau}$ yields that

$$\gamma S_{1,\tau} - \gamma D_{1,\tau} + \frac{\partial A(\tau,\theta^B_{1,\tau})}{\partial\theta^B_{1,\tau}} + \frac{\partial C(\tau,\theta^B_{1,\tau})}{\partial\theta^B_{1,\tau}}\delta_\tau = 0$$

or

$$S_{1,\tau} - D_{1,\tau} + \mu^A_1(\tau-T) - \frac{1}{\tilde{k}}(1-e^{\tilde{k}(\tau-T)})\delta_\tau - \bar{\delta}\frac{k}{\tilde{k}}(T-\tau-\frac{1-e^{\tilde{k}(\tau-T)}}{\tilde{k}}) + \theta^B_{1,\tau}I(\tau) = 0, \tag{B.2}$$

where

$$I(t) = -\gamma\sigma^2(t-T) + \frac{2\nu\sigma\gamma}{\tilde{k}}(T-t-\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}}) + \frac{\nu^2\gamma}{\tilde{k}^2}(T-t-2\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}} + \frac{1-e^{2\tilde{k}(t-T)}}{2\tilde{k}}).$$

It follows (B.1) that

$$S_{1,\tau} = D_{1,\tau} + \mu^A_1(T-\tau) - \theta^A_{1,\tau}\gamma\sigma^2(T-\tau). \tag{B.3}$$

Together with (B.2) and the market clearing condition $\theta^A_{1,\tau} + \theta^B_{1,\tau} = 1$, we obtain the optimal share holding of Type A for Stock 1 at the time of market closure.

$$\theta^A_{1,\tau} = \frac{-\frac{1}{\tilde{k}}(1-e^{\tilde{k}(\tau-T)})\delta_\tau - \bar{\delta}\frac{k}{\tilde{k}}(T-t-\frac{1-e^{\tilde{k}(\tau-T)}}{\tilde{k}}) + I(\tau)}{I(\tau)+\gamma\sigma^2(T-t)}. \tag{B.4}$$

Therefore, we find the market clearing price $S_{1,\tau}$ by (B.3) where $\theta^A_{1,\tau}$ is given by (B.4).

In particular, in the case $\tilde{k}=0$ (or $k=\nu/\sigma$),

$$\theta^A_{1,\tau} = \frac{1}{\gamma}\left(\frac{\gamma\sigma^2 - \gamma\nu\sigma(\tau-T) + \frac{1}{2}k\bar{\delta}(\tau-T) + \frac{\nu^2\gamma}{3}(\tau-T)^2 - \delta_\tau}{-\nu\sigma(\tau-T) + \frac{\nu^2}{3}(\tau-T)^2 + 2\sigma^2}\right),$$

and substituting it into (B.1), it follows that

$$S_{1,\tau} = D_{1,\tau} + \mu_1^A(T-\tau) + \frac{\gamma\sigma^2 - \gamma\nu\sigma(\tau-T) + \frac{1}{2}k\bar{\delta}(\tau-T) + \frac{\nu^2\gamma}{3}(\tau-T)^2 - \delta_\tau}{-\nu\sigma(\tau-T) + \frac{\nu^2}{3}(\tau-T)^2 + 2\sigma^2}\sigma^2(\tau-T).$$

Finally, it is worthy mentioning that $S_{1,\tau}$ may not be larger than $\hat{S}_{1,\tau}$ (the equilibrium price in the absence of circuit breakers at time τ). In fact, for a relative large positive δ_0 and small ν (say, less than half of the volatility σ), the coefficient of δ_t in (B.3) can always be less than the coefficient of δ_t in the formula of $\hat{S}_{1,\tau}$. Thus, along with a small γ, we can always have $S_{1,\tau} < \hat{S}_{1,\tau}$. Under these conditions, the market clearing price with circuit breakers can always be smaller than the price without circuit breakers at time τ.

Denote the market clearing price of Stock 1 by $S_{1,\tau}^c$. Then by (B.3),

$$S_{1,\tau}^c = D_{1,\tau} + \mu_1^A(T-\tau) - \gamma\theta_{1,\tau}^A\sigma^2(T-\tau).$$

In addition, we obtain the value function of Type B investors:

$$V_1^B(\tau, W_\tau^B) = \max_{\theta_{1,\tau}^B} \mathbb{E}_\tau^B[e^{-\gamma(W_\tau^B + \theta_{1,\tau}^B(D_{1,T} - S_{1,\tau}))}] = e^{-\gamma W_\tau^B} e^{-\gamma G_{1,\tau}^B},$$

where $-\gamma G_{1,\tau}^B = -\gamma\theta_{1,\tau}^B(D_{1,\tau} - S_{2,\tau}) + A(\tau, \theta_{1,\tau}^B) + C(\tau, \theta_{1,\tau}^B)\delta_\tau$, or

$$G_{1,\tau}^B = \theta_{1,\tau}^B(D_{1,\tau} - S_{2,\tau}) - \frac{1}{\gamma}A(\tau, \theta_{1,\tau}^B) - \frac{1}{\gamma}C(\tau, \theta_{1,\tau}^B)\delta_\tau, \tag{B.5}$$

and the value function of Type A investors:

$$V_1^A(\tau, W_\tau^A) = \max_{\theta_{1,\tau}^A} \mathbb{E}_\tau^A[e^{-\gamma(W_\tau^A + \theta_{1\tau}^A(D_{1,T} - S_{1,\tau}))}] = e^{-\gamma W_\tau^A} e^{-\gamma G_{1,\tau}^A},$$

where $-\gamma G_{1,\tau}^A = -\gamma\theta_{1,\tau}^A(D_{1,\tau} - S_{1,\tau}) - \gamma\theta_{1,\tau}^A\mu_1^A(T-\tau) + \frac{\gamma^2(\theta_{1,\tau}^A)^2}{2}\sigma^2(T-\tau) = -\frac{\gamma^2(\theta_{1,\tau}^A)^2}{2}\sigma^2(T-\tau)$, or

$$G_{1,\tau}^A = \theta_{1,\tau}^A(D_{1,\tau} - S_{1,\tau}) + \theta_{1,\tau}^A\mu_1^A(T-\tau) - \frac{\gamma(\theta_{1,\tau}^A)^2}{2}\sigma^2(T-\tau) = \frac{\gamma(\theta_{1,\tau}^A)^2}{2}\sigma^2(T-\tau). \tag{B.6}$$

B.2 Stock 2: Jump

Note that for Stock 2 there is no disagreement on the dynamics of dividend process $D_{2,t}$, which follows

$$D_{2,t} = D_{2,0} + \mu_2 t + \mu_J dN_{2,t}. \tag{B.7}$$

In the text, the equilibrium price $\hat{S}_{2,t}$ has been derived when there are no circuit breakers. In this Appendix, we derive the market clearing price when a circuit breaker is triggered.

Suppose that the circuit breaker is triggered at $\tau < T$. The individual optimization problem of Type $i \in \{A, B\}$ investors at τ is:

$$V_2^i(W_\tau^i, \tau) = \max_{\theta_{2,\tau}^i} \mathbb{E}_\tau^i[-\exp(-\gamma(W_\tau^i + \theta_{2,\tau}^i(D_{2,T} - S_{2,\tau})))], \tag{B.8}$$

subject to the market clearing condition $\theta_{2,\tau}^A + \theta_{2,\tau}^B = 1$, where W_τ^i is the wealth owned by Type i investors at time τ. It follows that

$$\begin{aligned}\mathbb{E}_\tau^i[u(W_\tau^i + \theta_{j,\tau}^i(D_{2,T} - S_{2,\tau}))] &= \mathbb{E}_\tau^i[u(W_\tau^i + \theta_{2,\tau}^i(D_{2,T} - D_{2,\tau}) + \theta_{2,\tau}^i(D_{2,\tau} - S_{2,\tau}))] \\ &= -e^{-\gamma W_\tau^i} e^{-\gamma\theta_{2,\tau}^i \mu_2 (T-\tau) - \gamma\theta_{2,\tau}^i (D_{2,\tau} - S_{2,\tau}) + \kappa(T-\tau)(e^{-\gamma\theta_{2,\tau}^i \mu_J} - 1)}.\end{aligned}$$

The first order conditions with respect to $\theta_{2,\tau}^i$ for $j \in \{1, 2\}, i \in \{A, B\}$ and the market clearing conditions yield that

$$0 = D_{2,\tau} - S_{2,\tau} + \mu_2(T - \tau) + \kappa\mu_J(T - \tau)e^{-\gamma\theta_{2,\tau}^A \mu_J}. \tag{B.9}$$

Along with the market clearing condition: $\theta_{2,\tau}^A + \theta_{2,\tau}^B = 1$, we find $\theta_{2,\tau}^A = \theta_{2,\tau}^B = \frac{1}{2}$. This is the same as the optimal share holding of Stock 2 of agent A in the absence of circuit breakers.

Then it follows (B.9) that

$$S_{2,\tau} = D_{2,\tau} + \mu_2(T - \tau) + \kappa\mu_J(T - \tau)e^{-\frac{\gamma}{2}\mu_J}.$$

There may be a jump of $D_{2,t}$ occurring at $t = \tau$. For such a case, the price of Stock 2 is limited by the threshold h and $S_{2,\tau} = h - S_{1,\tau}$. We define $D_{2,\tau}^* \in [D_{2,\tau}, D_{2,\tau-})$, such that

$$D_{2,\tau}^* = h - S_{1,\tau} - \mu_2(T - \tau) - \kappa\mu_J(T - \tau)e^{-\frac{\gamma}{2}\mu_J}.$$

Thus, in general the market clearing price of Stock 2 is

$$S_{2,\tau}^{c} := h - S_{1,\tau} = D_{2,\tau}^{*} + \mu_2(T-\tau) + \kappa\mu_J(T-\tau)e^{-\frac{\gamma}{2}\mu_J}.$$

When the stock price is $S_{2,\tau}^{c}$, the first order condition (B.9) cannot be satisfied in general. In other words, the individual utility cannot be maximized at $\theta_{2,\tau}^{i} = \frac{1}{2}$. Define

$$\tilde{\theta}_{2,\tau}^{i} = \frac{-1}{\gamma\mu_J}\log\left(\frac{S_{2,\tau}^{c} - D_{2,\tau} - \mu_2(T-\tau)}{\kappa\mu_J(T-\tau)}\right), \tag{B.10}$$

which maximizes the individual utility function of Type i investors, $i = A, B$. It follows the facts of $D_{2,\tau}^{*} \geq D_{2,\tau}$ and no disagreement on $D_{2,t}$ that $\tilde{\theta}_{2,\tau}^{A} = \tilde{\theta}_{2,\tau}^{B} \leq \frac{1}{2}$. In other words, due to the threshold, the price of Stock 2 cannot drop to a correct level corresponding to the true value of its dividend right after a jump. Thus, the investors would like to hold fewer shares of the stock.

In addition, define

$$\begin{aligned} G_{2,\tau}^{i} =& \max_{\theta_{2,\tau}^{i}} \mathbb{E}_{\tau}^{i}[-\exp(-\gamma + \theta_{2,\tau}^{i}(D_{2,T} - S_{2,\tau}))] \\ =& \theta_{2,\tau}^{i}\mu_2(T-\tau) + \theta_{2,\tau}^{i}(D_{2,\tau} - S_{2,\tau}^{c}) - \frac{\kappa}{\gamma}(T-\tau)(e^{-\gamma\theta_{2,\tau}^{i}\mu_J} - 1). \end{aligned} \tag{B.11}$$

Then, the value function of Type i investors at τ can be expressed in terms of $W_{2,\tau}^{i}$ and $G_{2,\tau}^{i}$ as follows.

$$V_2^{i}(W_{\tau}^{i}, \tau) = -e^{-\gamma W_{\tau}^{i}} e^{-\gamma G_{2,\tau}^{i}}, i \in \{A, B\}.$$

By the expressions of $G_{2,\tau}^{A}$ and $G_{2,\tau}^{B}$, it is easy to see that

$$S_{2,\tau}^{c} + G_{2,\tau}^{A} + G_{2,\tau}^{B} = \mu_2(T-\tau) + D_{2,\tau} - \frac{\kappa}{\gamma}(T-\tau)(e^{-\gamma\mu_J\theta_{2,\tau}^{A}} + e^{-\gamma\mu_J\theta_{2,\tau}^{B}} - 2),$$

which does not depend on $S_{2,\tau}^{c}$ directly. The quantity depends on the optimal share holdings at τ: $\theta_{2,\tau}^{A}$ and $\theta_{2,\tau}^{B}$, which are determined by the mechanism introduced in Section 4.1.

We consider a case where there is only Stock 2 in the market. Since there is no disagreement on the dividend process $D_{2,\tau}$, the optimal share holding of Type A or B must be 1/2 at any time $t < \tau$. According to the mechanism, at τ both of the optimal share holdings are 1/2 as well (nobody can sell or buy when the circuit breaker is triggered).

On the one hand, the state price density for $t < \tau$ is

$$\pi_{2,t}^A = \mathbb{E}_t^A[\zeta(V_2^A(W_{T\wedge\tau}^A, T\wedge\tau))'] = \mathbb{E}_t^A[\gamma\zeta e^{-\gamma(W_{T\wedge\tau}^A + G_{T\wedge\tau}^A)}] = \gamma\zeta\mathbb{E}_t^A[e^{-\frac{\gamma}{2}(S_{2,T\wedge\tau}+G_{T\wedge\tau}^B+G_{T\wedge\tau}^A)}],$$

where ζ is a constant, is the same as the state price density $\hat{\pi}_{2,t}$ in the absence of circuit breakers, given $\theta_{2,\tau}^A = \theta_{2,\tau}^B = 1/2$. On the other hand, due to the the threshold, the market clearing price $S_{2,\tau}^c$ is not less than $\hat{S}_{2,\tau}$. Therefore, at $t < \tau$ the equilibrium price $S_{2,t}$ in the presence of circuit breakers is greater than or equal to the equilibrium price $\hat{S}_{2,\tau}$ in the absence of circuit breakers. This indicates the price limit effect.

C Learning and Heterogeneous Beliefs

Suppose

$$dD_t = \mu_t dt + \sigma d\bar{Z}_t.$$

The dividend D_t is observable but the growth rate μ_t is not. Agents A and B infer the value of μ_t through the information from the dividend. Assume that

$$d\mu_t = -k(\mu_t - \bar{\mu})dt + \sigma_\mu d\bar{Z}_t,$$

and $\mu_0 \sim N(a_0, b_0)$, a normal distribution with mean a_0 and standard deviation b_0. Agent $i \in \{A, B\}$ believes $k = k^i, \bar{\mu} = \bar{\mu}^i, \sigma_\mu = \sigma_\mu^i, a_0 = a_0^i, b_0 = b_0^i$. Both of them learn μ_t through $\{D_s\}_{s=0}^t$. Let $\mu_t^A = \mathbb{E}^A[\mu_t|\{D_s\}_{s=0}^t]$ and $\mu_t^B = \mathbb{E}^B[\mu_t|\{D_s\}_{s=0}^t]$. Then following the standard filtering results, we have (under the assumption: $\mu_t|\{D_s\}_{s=0}^t \sim N(\hat{\mu}, \nu)$)

$$d\mu_t^A = -k^A(\mu_t^A - \bar{\mu}^A)dt + \nu^A dZ_t^A,$$
$$d\mu_t^B = -k^B(\mu_t^B - \bar{\mu}^B)dt + \nu^B dZ_t^B,$$

where $dZ_t^i = \frac{1}{\sigma}(dD_t - \mu_t^i dt), i = A, B$. Then

$$dD_t = \mu_t^A dt + \sigma dZ_t^A, \qquad dD_t = \mu_t^B dt + \sigma dZ_t^B.$$

Therefore, $Z_t^B + \frac{\delta_t}{\sigma_t}t$ is equal to Z_t^A almost surely, where $\delta_t = \mu_t^B - \mu_t^A$. In other words, $Z_t^B + \frac{\delta_t}{\sigma_t}t$ is a standard Brownian motion under agent A's probability measure $\mathbf{P}^A$.

Thus,

$$d\mu_t^B = -k^B(\mu_t^B - \bar{\mu}^B)dt - \frac{\nu^B}{\sigma}\delta_t dt + \nu^B dZ_t^A.$$

So we can obtain the general dynamics of the stochastic disagreement δ_t under learning. To validate the setting adopted in this paper, we let $\nu^A = 0$, $k^A = 0$, and $\mu_t^A = \mu^A$. That is, we assume that Type A investors take the long-time mean of the growth rate as the estimation and impose no learning. Then it follows that

$$\begin{aligned} d\delta_t = d(\mu_t^B - \mu^A) &= -(k^B + \frac{\nu^B}{\sigma})\delta_t dt - k^B(\mu^A - \bar{\mu}^B)dt + \nu^B dZ_t^A \\ &= -k^B \delta_t dt + k^B(\bar{\mu}^B - \mu^A)dt + \nu^B dZ_t^B. \end{aligned}$$

Further, let $k^B + \nu^B/\sigma = k$, $\nu^B = \nu$ and $(\bar{\mu}^B - \mu^A)/k = \bar{\delta}/\kappa^B$; we have reached the mean-reverting disagreement process assumed in the paper.

D The Fixed Point Problem

We prove the existence and uniqueness of a solution to the fixed point problem. First of all, based on the explicit expressions of the prices, we restrict the model parameters and the initial conditions (e.g., $D_{1,0}, D_{2,0}$) and assume that both $\hat{S}_{j,0}$ (the price without circuit breakers) and $S_{j,0}^c$ (the market clearing price) are positive for each $j = 1, 2$.

Recall that $S_{1,0}, S_{2,0}$ impact valuation of the expectations through the sum $S_{1,0} + S_{2,0}$ only. When the initial stock prices are $S_{1,0}$ and $S_{2,0}$, the threshold h is $(S_{1,0} + S_{2,0})(1-\alpha)$. So, we define

$$f_j(S_{1,0} + S_{2,0}) = \frac{\mathbb{E}_0^A[\pi_{T\wedge\tau}^A S_{j,T\wedge\tau}]}{\pi_0^A}, j = 1, 2.$$

and define a function $f : \mathcal{R} \to \mathcal{R}^2$ such that $f(S_{1,0} + S_{2,0}) = (f_1(S_{1,0} + S_{2,0}), f_2(S_{1,0} + S_{2,0}))^\top$, where $\top$ denotes the transpose of a vector. Then the fixed point problem is expressed as follows.

$$(S_{1,0}, S_{2,0})^\top = f(S_{1,0} + S_{2,0}).$$

Define $g(x) = f_1(x) + f_2(x) - x$, where $x \in \mathcal{R}$. When the threshold is zero, the circuit breaker is hardly triggered. Thus the equilibrium prices are close to the prices $\hat{S}_{0,1}$ and $\hat{S}_{2,0}$ respectively in the absence of circuit breakers. Given positive $\hat{S}_{0,1}$ and $\hat{S}_{2,0}$, we can obtain (specifically, for a sufficiently small volatility of $D_{1,t}$ and jump intensity of $D_{2,t}$): $g(0) = f_1(0) + f_2(0) > 0$. On the other hand, if the threshold is the sum of the market clearing prices $S_{1,0}^c + S_{2,0}^c$, the market is stopped immediately and the equilibrium prices must be the market clearing prices exactly. Thus, $g(\frac{S_{1,0}^c + S_{2,0}^c}{1-\alpha}) = f_1(\frac{S_{1,0}^c + S_{2,0}^c}{1-\alpha}) + f_2(\frac{S_{1,0}^c + S_{2,0}^c}{1-\alpha}) - \frac{S_{1,0}^c + S_{2,0}^c}{1-\alpha} = S_{1,0}^c + S_{2,0}^c - \frac{S_{1,0}^c + S_{2,0}^c}{1-\alpha} < 0$. It can be shown that $g(x)$ is a continuous function. Hence, there exists $x^* \in (0, \frac{S^c_{1,0}+S^c_{2,0}}{1-\alpha})$, such that $g(x^*) = 0$. Thus $f_1(x^*) + f_2(x^*) = x^*$.

Now define $(S^*_{1,0}, S^*_{2,0})^\top = f(x^*)$. Then $x^* = f_1(x^*) + f_2(x^*) = S^*_{1,0} + S^*_{2,0}$ and

$$(S^*_{1,0}, S^*_{2,0})^\top = f(x^*) = f(S^*_{1,0} + S^*_{2,0}).$$

Thus $(S^*_{1,0}, S^*_{2,0})^\top \in \mathcal{R}^2$ is a solution to the fixed problem. The existence is proved.

Next, we show that the solution is unique. To do so, it is sufficient to show that $g(x)$ is monotonic. For the sake of notional simplicity, we ignore super-script "A" of expectations and π^A_t below.

Let $D_0 = D_{1,0} + D_{2,0}$. Given an exogenous threshold h and initial dividend sum value D_0, let $S^{h,D_0}_t = S^c_{1,t} + S^c_{2,t}$, where $S^c_{1,t}$ and $S^c_{2,t}$ are the market clearing prices; let $\tau(h, D_0)$ denote the stopping time; and let π^{h,D_0}_t be the state price density, i.e.

$$\pi^{h,D_0}_t = (\eta_t)^{1/2} e^{-\frac{\gamma}{2} S^{h,D_0}_t} \cdot e^{\frac{G^A_t + G^B_t}{2}}.$$

We redefine

$$g(x) = g(x; D_0) = \frac{\mathbb{E}[\pi_{\tau(h,D_0)\wedge T} S^{h,D_0}_{\tau(h,D_0)\wedge T}]}{\mathbb{E}[\pi^A_{\tau(h,D_0)\wedge T}]} - x,$$

where $h = x(1-\alpha)$. Observe that $\tau(h, D_0) = \tau(0, D_0 - h)$ because the stopping time is determined by D_t and δ_t only. Then the market clearing (sum) price $S^{h,D_0}_{\tau(h,D_0)} = S^{0,D_0-h}_{\tau(0,D_0-h)} + h$ by the expressions of $S^c_{j,\tau}$, $j = 1, 2$. In addition, by the definition of G^i_τ, we see that $G^i_{\tau(h,D_0)} = G^i_{\tau(0,D_0-h)}$, $i = A, B$. Therefore

$$\pi^{h,D_0}_{\tau(h,D_0)} = e^{-\frac{\gamma}{2} h} \cdot \pi^{0,D_0-h}_{\tau(0,D_0-h)}. \tag{D.1}$$

Thus,

$$\begin{aligned} g(x; D_0) &= \frac{\mathbb{E}[\pi^{h,D_0}_{\tau(h,D_0)\wedge T} \cdot (S^{h,D_0}_{\tau(h,D_0)\wedge T} - x)]}{\mathbb{E}[\pi^{h,D_0}_{\tau(h,D_0)\wedge T}]} \\ &= \frac{\mathbb{E}[\pi^{0,D_0-h}_{\tau(0,D_0-h)\wedge T} \cdot (S^{0,D_0-h}_{\tau(0,D_0-h)\wedge T} - x + h)]}{\mathbb{E}[\pi^{0,D_0-h}_{\tau(0,D_0-h)\wedge T}]} \\ &= g(0; D_0 - h) - x + h = g(0; D_0 - h) - \alpha x. \end{aligned}$$

Given $h_1 < h_2$, we have $\tau(0, D_0 - h_1) \geq \tau(0, D_0 - h_2)$. Then,

$$\begin{aligned}
&\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)}] = \mathbb{E}[\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)}|\tau(0, D_0 - h_2)]] = \mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_2)}] \\
&= \mathbb{E}[(\eta_{\tau(0,D_0-h_2)\wedge T})^{1/2} e^{-\frac{\gamma}{2} S^{0,D_0-h_1}_{\tau(0,D_0-h_2)\wedge T}} \cdot e^{\frac{G^A_{\tau(0,D_0-h_2)\wedge T} + G^B_{\tau(0,D_0-h_2)\wedge T}}{2}}] \\
&= \mathbb{E}[(\eta_{\tau(0,D_0-h_2)\wedge T})^{1/2} e^{-\frac{\gamma}{2} S^{0,D_0-h_2}_{\tau(0,D_0-h_2)\wedge T}} \cdot e^{\frac{G^A_{\tau(0,D_0-h_2)\wedge T} + G^B_{\tau(0,D_0-h_2)\wedge T}}{2}} \cdot e^{-\gamma/2(h_2-h_1)}] \\
&= \mathbb{E}[\pi^{0,D_0-h_2}_{0,D_0-h_2}] e^{-\gamma/2(h_2-h_1)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)} \cdot S^{0,D_0-h_1}_{\tau(0,D_0-h_1)}] &= \mathbb{E}[\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)} \cdot S^{0,D_0-h_1}_{\tau(0,D_0-h_1)}|\tau(0, D_0 - h_2)]] \\
&= \mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_2)} \cdot S^{0,D_0-h_1}_{\tau(0,D_0-h_2)}] e^{-\gamma/2(h_2-h_1)} \\
&\geq \mathbb{E}[\pi^{0,D_0-h_2}_{\tau(0,D_0-h_2)} \cdot S^{0,D_0-h_2}_{\tau(0,D_0-h_2)}] e^{-\gamma/2(h_2-h_1)}.
\end{aligned}$$

Finally, let $x_1 < x_2$ and $h_1 = x_1(1-\alpha), h_2 = x_2(1-\alpha)$. It follows that

$$\begin{aligned}
g(x_1; D_0) = g(0; D_0 - x_1) - \alpha x_1 &= \frac{\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)\wedge T} \cdot S^{0,D_0-h_1}_{\tau(0,D_0-h_1)\wedge T}]}{\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)\wedge T}]} - \alpha x_1 \\
&\geq \frac{\mathbb{E}[\pi^{0,D_0-h_2}_{\tau(0,D_0-h_2)\wedge T} \cdot S^{0,D_0-h_2}_{\tau(0,D_0-h_2)\wedge T}]}{\mathbb{E}[\pi^{0,D_0-h_2}_{\tau(0,D_0-h_2)\wedge T}]} - \alpha x_1 \\
&= g(0; D_0 - h_2) = g(x_2; D_0) + \alpha x_2 - \alpha x_1 > g(x_2; D_0).
\end{aligned}$$

Thus, $g(\cdot, D_0)$ is monotonic. This completes the proof of uniqueness.

E The Case of Correlated Dividend Processes

To impose a correlation between dividend processes, we assume that: under $\mathbf{P}^A$,

$$dD_{1,t} = \mu_1^A dt + \sigma_1 dZ_t, \tag{E.1}$$

$$dD_{2,t} = \mu_2 dt + \sigma_2 dZ_t + \mu_J dN_t, \tag{E.2}$$

and under $\mathbf{P}^B$:

$$dD_{1,t} = \mu_1^B dt + \sigma_1 dZ_t^B, \tag{E.3}$$

$$dD_{2,t} = \mu_2 dt + \frac{\sigma_2}{\sigma_1}\delta_t dt + \sigma_2 dZ_t^B + \mu_J dN_t, \tag{E.4}$$

where $\mu_1^B = \mu_1^A + \delta_t$ and

$$d\delta_t = -k(\delta_t - \bar{\delta})dt + \nu dZ_t,$$

or

$$d\delta_t = -k(\delta_t - \bar{\delta})dt + \frac{\nu}{\sigma_1}\delta_t dt + \nu dZ_t^B.$$

Then the two dividend processes are correlated with instantaneous correlation

$$\rho = \frac{\sigma_2}{\sqrt{\sigma_2^2 + \kappa\mu_J^2}}.$$

E.1 The Equilibrium Prices without Circuit Breakers

The pricing formula has the same expression as that in the uncorrelated case.

$$\hat{S}_{j,t} = \mathbb{E}_t^A\left[\frac{\pi_T^A D_{j,T}}{\mathbb{E}_t^A[\pi_T^A]}\right], j = 1, 2,$$

where $\pi_T^A = \gamma\zeta\mathbb{E}_t^A[\eta_T^{1/2} \cdot e^{-\frac{\gamma}{2}(D_{1,T}+D_{2,T})}]$. However, the two prices cannot be evaluated separately anymore because the two dividend processes are correlated ($\sigma_2 \neq 0$). Similar to the case of $\rho = 0$, the equilibrium prices in closed form can be derived.

E.2 The Equilibrium Prices with Circuit Breakers

We derive the market clearing prices when the market is closed early due to the circuit breaker.

Type A investors need to maximize the individual utility function

$$\max_{\theta_{1,\tau}^A, \theta_{2,\tau}^A} \mathbb{E}_t^A[-e^{-\gamma(\theta_{1,\tau}^A(D_{1,T}-S_{1,\tau})+\theta_{2,\tau}^A(D_{2,T}-S_{2,\tau}))}].$$

It results in first order conditions:

$$-\gamma(D_{1,\tau}-S_{1,\tau})-\gamma\mu_1^A(T-\tau)+\gamma^2(\theta_{1,\tau}^A\sigma_1+\theta_{2,\tau}^A\sigma_2)\sigma_1(T-\tau)=0, \tag{E.5}$$

$$-\gamma(D_{2,\tau}-S_{2,\tau})-\gamma\mu_2(T-\tau)+\gamma^2(\theta_{1,\tau}^A\sigma_1+\theta_{2,\tau}^A\sigma_2)\sigma_2(T-\tau)-\gamma\mu_J\kappa^A e^{-\gamma\theta_2^A\mu_J}=0. \tag{E.6}$$

For Type B investors, the optimization problem is

$$\max_{\theta_{1,\tau}^B,\theta_{2,\tau}^B} \mathbb{E}_t^B[-e^{-\gamma(\theta_{1,\tau}^B(D_{1,T}-S_{1,\tau})+\theta_{2,\tau}^B(D_{2,T}-S_{2,\tau}))}].$$

We first obtain an expression for the following expectation for any real numbers x and y:

$$\mathbb{E}_t^B[e^{x\int_t^T\delta_s ds+y(Z_T^B-Z_t^B)}]=e^{A(t;x,y)+C(t;x)\delta_t},$$

where

$$\begin{aligned}A(t;x,y)&=\frac{y^2}{2}(T-\tau)+k\bar{\delta}\int_t^T C(s;x)ds+\frac{\nu^2}{2}\int_t^T C(s;x)^2ds+y\nu\int_t^T C(s;x)ds,\\ C(t;x)&=\frac{x}{k-\frac{\nu}{\sigma_1}}(1-e^{(k-\frac{\nu}{\sigma_1})(\tau-T)}).\end{aligned}$$

Then let $y=-\gamma(\theta_{1,\tau}^B\sigma_1+\theta_{2,\tau}^B\sigma_2)$ and $x=-\gamma(\theta_{1,\tau}^B+\theta_{2,\tau}^B\frac{\sigma_2}{\sigma_1})$; we obtain the first order conditions for the maximization problem of Type B:

$$-\gamma(D_{1,\tau}-S_{1,\tau})-\gamma\mu_1^A(T-\tau)+\frac{dA(t;x,y)}{d\theta_{1,\tau}^B}+\frac{dC(t;x)}{d\theta_{1,\tau}^B}\delta_t=0, \tag{E.7}$$

$$-\gamma(D_{2,\tau}-S_{2,\tau})-\gamma\mu_2(T-\tau)-\gamma\kappa\mu_J(T-\tau)e^{-\gamma\theta_{2,\tau}^B\mu_J}+\frac{dA(t;x,y)}{d\theta_{2,\tau}^B}+\frac{dC(t;x)}{d\theta_{2,\tau}^B}\delta_t=0. \tag{E.8}$$

Along with the market clearing condition $\theta_{j,\tau}^A+\theta_{j,\tau}^B=1, j=1,2$, the four first order conditions determine the solution $S_{1,\tau}^*, S_{2,\tau}^*, (\theta_{1,\tau}^A)^*, (\theta_{2,\tau}^A)^*$, that is the market clearing prices and the share holdings at the market early closure time τ, respectively.

Next, as in the case of uncorrected dividend processes, we obtain the indirect utility functions for Type A and Type B investors and the state price density. The equilibrium stock prices at $t<\tau$ can be evaluated numerically by solving a fixed point problem similar to (24).

F The Issue of Share Holding at τ

In the text, a mechanism is introduced to determine the share holding of Stock 2 at time τ because the two types of agents cannot optimize both of their utilities while making the market clear due to the price limit effect of circuit breakers. By the mechanism, the share holding of Stock 2 at time τ depends on the information at $\tau-$. As a result, the indirect utility or the state price density in the pricing formula is dependent on the share holding at $\tau-$. Since stock prices at any time t before τ are given by the formula (22), it turns out that we have to solve a functional fixed point problem as follows.

Let $\theta^A_{2,t} = f(D_{1,t}, D_{2,t}, t)$ denote the share holding of Stock 2 at any time $t < \tau$. By the scheme, we can find the share holding at τ, which can be written as

$$\theta^A_{2,\tau} = g(\theta^A_{2,\tau-}, D_{1,\tau}, D_{2,\tau}) = g(f(D_{1,\tau-}, D_{2,\tau-}, \tau-), D_{1,\tau}, D_{2,2\tau}),$$

where the function $g(\cdot,\cdot,\cdot)$ is known according to the scheme. Then we need to find the expression of $f(\cdot,\cdot,\cdot)$ such that, the stock prices obtained by (22) and the wealth processes obtained by (25) ensure that the share holding is exactly given by $\theta^A_{2,t} = f(D_{1,t}, D_{2,t}, t)$ for $t \le \tau-$.

Undoubtedly, it is difficult to solve this functional fixed point problem in general. In practice, we consider a linear approximation

$$f(D_1, D_2, t) = a(t) + b(t)D_1 + c(t)D_2,$$

where $a(t), b(t)$ and $c(t)$ are deterministic coefficients. In addition, we adopt an algorithm for numerical approaches as follows.

- *Step 1. At t=$T - \Delta t$, we set $\theta^A_{1,t} = \hat{\theta}^A_{1,t}, \theta^A_{2,t} = \hat{\theta}_{2,t} = \frac{1}{2}$, because it is shown that, as $t \to T-$, the two share holdings approach the same values as those in the absence of circuit breakers.*
- *Step 2. Calculate $\theta^A_{1,t}$ and $\theta^A_{2,t}$ for $t = T - 2\Delta t, T - 3\Delta t$,...,0 sequentially. For each t, we use Monte Carlo simulations to calculate $\mathbb{E}[\Delta W^A_t \Delta S_{i,t}]$ and $\mathbb{E}[\Delta S_{i,t} \Delta S_{j,t}]$, where $\Delta S_{i,t} = S_{i,t+\Delta t} - S_{i,t}$ and $\Delta W^A_t = W^A_{t+\Delta t} - W^A_t$, for $i, j = 1, 2$ Then, we*

calculate the share holdings at t by the formula

$$\begin{pmatrix} \theta^A_{1,t} \\ \theta^A_{2,t} \end{pmatrix} = \begin{pmatrix} \mathbb{E}[\Delta S_{1,t}\Delta S_{1,t}] & \mathbb{E}[\Delta S_{2,t}\Delta S_{1,t}] \\ \mathbb{E}[\Delta S_{1,t}\Delta S_{2,t}] & \mathbb{E}[\Delta S_{2,t}\Delta S_{2,t}] \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{E}[\Delta W^A_t \Delta S_{1,t}] \\ \mathbb{E}[\Delta W^A_t \Delta S_{2,t}] \end{pmatrix}.$$

We repeat this procedure for a set of $D_{1,t}, D_{2,t}$ and obtain a set of share holdings. Then a linear regression gives us coefficients of the linear dependence $a(t), b(t), c(t)$.

The above numerical algorithm is applied for a given threshold h. To find the initial stock price S_0, the whole procedure is iterated to solve the fixed point problem (24).

Applying the mechanism and algorithm introduced above, we find that the optimal share holding of Stock 2 at $t < \tau$ is not far away from 1/2, and thus so is the optimal share holding at τ. Besides, a jump occurs in one day at a relatively low frequency. As a consequence, we do not find the equilibrium prices determined by the above mechanism to be significantly different from that obtained by the simple rule ($\theta^A_{2,\tau} = \frac{1}{2}$). The difference is actually in the order of 10^{-4}. In contrast, the price limit effect that prevents the price of Stock 2 from dropping too much impacts the equilibrium price significantly and usually makes a difference at the first decimal digit (10^{-1}) of the equilibrium stock price, compared to that in the absence of circuit breakers.

For fast simulations, in the numerical analysis of the text, we assume the simple rule that the shares held by Type A agents at τ are 1/2. This is equivalent to assuming that when a price jump triggers the circuit breaker, both types of agents agree to hold the optimal share holdings (1/2) as if the price had jumped to the equilibrium level in the absence of circuit breakers, although the price is kept at the threshold and the half-half share holdings are not optimal for both types of agents.