Circuit Breakers and Contagion*

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December, 2018

ABSTRACT

Circuit breakers based on indices are commonly imposed in financial markets to prevent market crashes and reduce volatility in bad times. We develop a continuous-time equilibrium model with multiple stocks to study how circuit breakers affect joint stock price dynamics, cross-stock contagion, and market volatility. Contrary to the regulatory goals and consistent with what happened in recent Chinese markets, we show that in bad times, circuit breakers can cause crash contagion, volatility contagion, and high correlations among otherwise independent stocks. They can also significantly increase market volatility and accelerate market decline. We propose an alternative circuit breaker approach that does not cause either correlation or any contagion.

JEL classification:

Keywords: Circuit breaker, contagion, volatility, market crash

*For helpful comments, we thank Asaf Manela and seminar participants at Fudan University, University of Southern California, and Washington University in St. Louis. Authors can be reached at liuh@wustl.edu and zeng.xudong@mail.shufe.edu.cn.

1. Introduction

Circuit breakers based on indices are widely implemented in financial markets as one of measures aimed at stabilizing market prices in bad times (e.g., circuit breakers implemented in U.S., France, Canada, and China). In most cases, when the percentage decline in a market index reaches a regulatory threshold, the breaker is triggered and trading is halted for a period of time for the entire market. In a dramatic move, Chinese regulators removed a four-day old circuit breakers rule after it was triggered twice in the week of January 7, 2016 and the Shanghai stock markets tumbled seven percent within half an hour of opening on January 7, 2016. This event has reignited research interest in the impact of circuit breakers on financial markets (e.g., Chen, Petukhov, and Wang (2017)). One open question is how circuit breakers affect the systemic risk caused by stock return correlations and market-wide contagion in bad times. In this paper, we develop a continuous-time asset pricing equilibrium model to provide some insight on this important issue.

Contrary to the regulatory goals, we show that in bad times, circuit breakers can cause crash and volatility contagion and high correlations among otherwise independent stocks, can significantly increase market volatility, and can accelerate market decline. Our analysis helps explain the concurrence of the implementation of the circuit breakers rule and the significant market tumble in the week of January 7, 2016 in Chinese stock markets. Our model suggests that market-wide circuit breakers may be a source of financial contagion and a channel through which idiosyncratic risks become systemic risks. We propose an alternative circuit breaker approach based on individual stocks rather than an index that does not cause either correlation or any contagion.

In our model, investors can invest in one risk free asset and two risky assets ("stocks") with independent jump diffusion dividend processes to maximize their expected utility from their final wealth at time T. Investors have heterogeneous beliefs on the drift and the jump parameters. To highlight the role of circuit breakers, we assume that the investors have exponential preferences so that in the absence of circuit breakers, the equilibrium stock returns are independent. The stock market is subject to a market-wide circuit breaker rule in the sense that if the sum of the two stock prices (the index) reaches a threshold, the entire stock market is closed until T.

The intuition for our main result that circuit breakers increase return correlations and cause volatility and crash contagion is as follows. After the circuit breaker is triggered, market is closed, risk sharing is reduced and thus stock prices may be lower than those without market closure. Therefore, when an idiosyncratic negative shock to the price of one stock occurs, the sum of stock prices (or the index of the market) gets smaller, the probability of reaching the circuit breaker threshold increases, and thus the price of the other stock may also decrease for the fear of the more likely market closure. This link through the circuit breaker induces the positive return correlation, even though stocks are independent in the absence of the circuit breaker. When the idiosyncratic shock is large and thus the index becomes close to the circuit breaker, this increase in the correlation is even greater because the likelihood of market closure is much higher. In the extreme case where one stock crashes and the circuit breaker is triggered, the price of the other stock must jump to the after-market-closure level. Because after some stocks fall in prices, the index gets closer to the circuit breaker threshold, other stock prices fall, which drives the index even closer to the threshold, so on and so forth. It is this vicious cycle effect that accelerates market-wide decline. In addition, as one stock becomes more volatile (e.g., due to an increase in the volatility of its dividend), the likelihood of triggering the circuit breaker gets greater, and thus the prices of other stocks also become more volatile. This explains why a crash of one stock may cause another stock to crash and volatility can transmit across stocks even though stocks are independent in the absence of circuit breakers. These contagion effects increase the systemic risk.

Our results suggest that to reduce the contagion effects and the systemic risk, it is better to impose circuit breakers on individual stocks. In this alternative approach, the threshold is based on individual stock returns and when a stock's circuit breaker is triggered, only trading in this single stock is halted. This alternative approach does not increase correlations or cause any form of contagion. We show that with this alternative approach, stock prices are generally higher, a market-wide large decline is less likely, and systemic risk is lower.

In the model, we assume there are only two stocks in the index on which the circuit breakers are based. One possible concern is that in practice indices typically consist of hundreds of stocks (if not more) and therefore it is unlikely one stock's fall would trigger the fall of many other stocks. In bad times, markets typically focus on a small number of key factors such as Federal reserve decisions and major economic news. The risk for each of the two stocks in our model can represent a large group of stocks that are significantly exposed to a common risk factor in bad times. When there is a bad shock in the risk factor, the prices of the large group of stocks go down, which can drag down another large group of stocks through the circuit breakers connection even though the latter group of stocks are not exposed to the risk factor.

The closest work to ours is the seminal paper Chen, Petukhov, and Wang (2017). Using a dynamic asset pricing model with one stock, Chen, Petukhov, and Wang (2017) find that a downside circuit breaker may lower stock price and increase market volatility, contrary to one of the main goals of regulators. In addition, as a consequence of higher volatility, a market with circuit breakers can more likely decline sufficiently to reach the trigger thresholds than without. Different from our research focus, Chen, Petukhov, and Wang (2017) restricts their analysis to the single stock case without jump risk and thus does not examine the effect of circuit breakers on stock return correlations or market-wide crash or volatility contagion. In addition, in Chen, Petukhov, and Wang (2017) the main mechanism through which circuit breakers affect asset dynamics is the difference in leverage before and after market closure. Before market closure, investors face no leverage constraint, but after market closure investors cannot lever at all to guarantee a finite expected CRRA utility. As a result, investors need to completely unlever when the circuit breaker is triggered, which magnifies the effect of market closure. In this paper leverage is allowed before and after market closure. We show that even in the absence of leverage constraints, circuit breakers can still have large impact on price dynamics.

Among other theoretical work related to circuit breakers, Greenwald and Stein (1991)show that in a market with limited participation, circuit breakers can help coordinate trading for market participants. Subrahmanyam (1994) demonstrates that circuit breakers can increase price volatility because investors may shift their trades to earlier periods with lower liquidity supply if there is information asymmetry. Hong and Wang (2000) examine the impact of periodic exogenous market closure on asset prices.

This paper also contributes to the literature in several additional aspects. First, we solve an equilibrium model with jumps. The result may be related to the vast literature of jump models, see e.g. Das and Uppal (2004), Eraker (2004), Eraker, Johannes, and Polson (2003), Liu, Longstaff, and Pan (2003), Roll (1998). Second, we show how correlation arises given a circuit breaker. This result can be inspirational for research of correlation, see e.g. Asai and McAleer (2009), Buraschi, Porchia, and Trojani (2010). Third, the present work complement the existing studies of circuit breakers, see e.g. Greenwald and Stein (1991), Subrahmanyam (1994), Hong and Wang (2000), Chen, Petukhov, and Wang (2017). Fourth, we provide a novel approach for studying contagion. Among a few of relevant literature, Aït-Sahalia, Cacho-Diaz, and Laeven (2015) and Aït-Sahalia, Cacho-Diaz, and Laeven (2015) propose a type of financial contagion transmitting through self/mutually-exciting jumps. We show that contagion may be a consequence of circuit breakers.

The rest of this paper is organized as follows. In Section 2, we formulate the basic market model. In Section 3, we provide the solution of market equilibrium for the market in the absence of circuit breakers. In Section 4, we study the market equilibrium when there are circuit breakers. In Section 5, we quantitatively examine the impact of circuit breakers on equilibrium prices and their correlation. Section 6 concludes. All proofs are provided in the appendix.

2. Model

We consider a continuous-time exchange economy over the finite time interval $[0, T]$. There are two stocks, Stocks 1 and 2, and one risk-free asset in the economy that investors can trade. Each of the two stocks in our model represents a group of stocks that have the same significant risk exposure in bad times. The risk-free asset has a net supply of zero. The total supply of each stock is one and each stock pays a terminal dividend at time T. The dividend processes are exogenous and publicly observed. Uncertainty about dividends is represented by a standard Brownian motion Z_t and an independent standard Poisson process N_t defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by Z_t and N_t.

There are a continuum of investors of Types A and B in the economiy, with a mass of 1 for each type. For $i = A, B$ and $j = 1, 2$. Type i investors are initially endowed with θ^i_{j0} shares of Stock j but no riskless asset, with $0 \leq \theta^i_{j0} \leq 1$ and $\theta^A_{j0} + \theta^B_{j0} = 1$. Type A and Type B investors have heterogeneous beliefs on the dividend processes. Type A investors have a probability measure $\mathbf{P}^A$ which is the same as the natural probability measure $\mathbf{P}$. Under Type A's probability measure, Stock 1's dividend process evolves as:

$$dD_{1,t} = \mu^A_1 dt + \sigma dZ_t, \tag{1}$$

and Stock 2's dividend process follows a jump process with drift:

$$dD_{2,t} = \mu_2 dt + \mu_J dN_t \tag{2}$$

with $D_{j,0} = 1$ for $j = 1, 2$, where Stock 1 dividend growth rate μ^A_1, Stock 1 dividend volatility σ, and Stock 2 drift μ_2 are all constants. The Poisson process N_t has a constant jump intensity of κ^A under Type A's probability measure and a constant jump size of μ_J. Circuit breakers have two direct effects: The first is the *market closure effect*, i.e., investors cannot trade after circuit breakers are triggered; The second is the *price limit effect*, i.e., stock prices cannot fall below the circuit-breaker threshold levels. As we show later, without a jump in a dividend process, the price limit effect would be missing. Without the diffusion risk, the market closure effect would be missing. Thus we propose the above two dividend processes to capture these two direct effects in the simplest way.[1]

Relative to Type A investors, Type B have different beliefs on the dividend processes and employ a different probability measure $\mathbf{P}^B$. The Randon-Nikodym derivative $\eta_T = d\mathbf{P}^B/d\mathbf{P}^A|_{\mathcal{F}_T} = \eta_{1,T}\eta_{2,T}$, where

$$\eta_{1,T} = e^{\int_0^T \frac{\delta_t}{\sigma} dZ_t - \int_0^T \frac{\delta_t^2}{2\sigma^2} dt}, \qquad \eta_{2,T} = e^{-(\kappa^B - \kappa^A)T} (\frac{\kappa^B}{\kappa^A})^{N_T}, \tag{3}$$

κ^B is the constant jump intensity in the view of Type B investors and the process $\delta_t \equiv \mu_{1,t}^B - \mu_1^A$ measures the disagreement between Type A and Type B investors about the growth rate of the dividend process $D_{1,t}$. For the disagreement process δ_t, we assume that

$$d\delta_t = -k(\delta_t - \bar{\delta})dt + \sigma_\delta dZ_t, \tag{4}$$

where $\bar{\delta}$ is the constant long-time average of the disagreement, k measures the speed of mean reversion, and σ_δ is the volatility of the disagreement.[2]

It follows from the Randon-Nikodym derivative that in the view of Type B investors, the two dividend processes follow

$$dD_{1,t} = \mu_{1,t}^B dt + \sigma dZ_t^B, \tag{5}$$

$$dD_{2,t} = \mu_2 dt + \mu_J dN_t^B, \tag{6}$$

[1] Using a jump diffusion dividend process for both stocks would not change our conclusions, but complicates analysis.

[2] In the appendix, we show that this δ_t process is consistent with Kalman filtering when Type B investors do not know the expected growth rate of Stock 1 dividend.

where

$$\mu_{1,t}^B = \mu_1^A + \delta_t, \tag{7}$$

$$Z_t^B = Z_t - \frac{\delta_t}{\sigma} dt, \tag{8}$$

and under the probability measure $\mathbf{P}^B$, Z_t^B is a standard Brownian motion, N_t^B is a standard Poisson process with intensity κ^B, and the two processes are independent. Hereafter, we use the convention $\mathbb{E}^i[\cdot]$ to denote the expectation under the probability measure $\mathbf{P}^i$ for $i \in \{A, B\}$.

To isolate the impact of circuit breakers on stock return correlations, we assume that for $i \in \{A, B\}$, Type i investors have constant absolute risk averse (CARA) preferences over the terminal wealth W_T^i at time T:

$$u(W_T^i) = -\exp(-\gamma W_T^i),$$

where $\gamma > 0$ is the absolute risk aversion coefficient. With CARA preferences, there is no wealth effect and therefore in the absence of circuit breakers, returns of the two stocks are independent.

Trading in the stocks is subject to a market-wide circuit-breaker rule as explained next. Let $S_{j,t}$ denote the price of Stock $j = 1, 2$ at time $t \le T$ and the index $S_t = S_{1,t} + S_{2,t}$ denote the sum of the two prices. Define the circuit-breaker trigger time

$$\tau = \inf\{t : S_t \le h, t \in [0, T]\},$$

where h is the circuit breaker threshold. At the circuit-breaker trigger time τ, the market is closed until T (market closure effect) and the sum $S_{1,t} + S_{2,t}$ of stock prices cannot go below h (price limit effect). In practice, the circuit breaker threshold h is typically equal to a fraction of the previous day's closing level. In this paper, we set $h = (1 - \alpha)S_0$ for a constant α (e.g., $\alpha = 0.07$ as in the Chinese stock markets).

3. Equilibrium without Circuit Breakers

As a benchmark case, we first solve for the equilibrium stock prices when there is no circuit-breaker in place in the market. To do so, it is convenient to solve the planner's problem:

$$\max_{W_T^A, W_T^B} \mathbb{E}_0^A[u(W_T^A) + \xi\eta_T u(W_T^B)], \tag{9}$$

subject to the wealth constraint $W_T^A + W_T^B = D_{1,T} + D_{2,T}$, where ξ is a constant depending on the initial wealth weights of the agents and can be determined in equilibrium.

From the first order conditions, we obtain:

$$W_T^A = \frac{1}{2\gamma}\log(\frac{1}{\xi\eta_T}) + \frac{1}{2}(D_{1,T} + D_{2,T}), \tag{10}$$

$$W_T^B = -\frac{1}{2\gamma}\log(\frac{1}{\xi\eta_T}) + \frac{1}{2}(D_{1,T} + D_{2,T}). \tag{11}$$

Given the utility function $u(x) = -e^{-\gamma x}$, the state price density under Type A investors' beliefs is

$$\pi_t^A = \mathbb{E}_t^A[\zeta u'(W_T^A)] = \mathbb{E}_t^A[\gamma\zeta e^{-\gamma W_T^A}] = \gamma\zeta\xi^{\frac{1}{2}}\mathbb{E}_t^A[\eta_T^{\frac{1}{2}} \cdot e^{-\frac{\gamma}{2}(D_{1,T}+D_{2,T})}], \tag{12}$$

for some constant ζ. Therefore, the stock price in equilibrium is given by

$$\hat{S}_{j,t} = \frac{\mathbb{E}_t^A\left[\pi_T^A D_{j,T}\right]}{\mathbb{E}_t^A[\pi_T^A]} = D_{j,t} + \frac{\mathbb{E}_t^A\left[\pi_T^A(D_{j,T} - D_{j,t})\right]}{\mathbb{E}_t^A[\pi_T^A]}, \quad j = 1, 2. \tag{13}$$

Since the two dividend processes are independent, Equation (13) can be simplified into

$$\hat{S}_{1,t} = \frac{\mathbb{E}_t^A[\pi_{1,T}^A D_{1,T}]}{\mathbb{E}_t^A[\pi_{1,T}^A]}, \quad \hat{S}_{2,t} = \frac{\mathbb{E}_t^A[\pi_{2,T}^A D_{2,T}]}{\mathbb{E}_t^A[\pi_{2,T}^A]}, \tag{14}$$

where $\pi_{j,t}^A = \mathbb{E}_t^A[\eta_{j,T}^{1/2} \cdot e^{-\frac{\gamma}{2}D_{j,T}}]$ for $j = 1, 2$, i.e., the two prices can be computed separately.

Next, we derive the equilibrium prices in closed-form for the two stocks. Then, we examine the impact of the jump and the stochastic disagreement on the market equilibrium.

For Stock 1, the disagreement process is governed by the mean-reverting process (4). The formula of equilibrium price $\hat{S}_{1,t}$ can be derived analytically and is presented in the following proposition.

PROPOSITION 1. *When there are no circuit breakers, the equilibrium price of Stock 1 is:*

$$\hat{S}_{1,t} = D_{1,t} + \mu_1^A(T-t) - 2\left(\frac{dA(t;\gamma)}{d\gamma} + \frac{dC(t;\gamma)}{d\gamma}\delta_t\right), \tag{15}$$

where $A(t;\gamma)$ and $C(t;\gamma)$ are given in the appendix.

In the special case of constant disagreement ($\delta_t = \delta_0$ for all $t \in [0, T]$), the equilibrium price is simplified to

$$\hat{S}_{1,t} = D_{1,t} + \frac{\mu_1^A + \mu_1^B}{2}(T-t) - \frac{\gamma}{2}\sigma^2(T-t).$$

Thus, the equilibrium price of Stock 1 is determined by the average beliefs of Type A and B investors on the growth rate of dividend and the volatility of the stock price is the same as the volatility of its dividend. Moreover, by applying the Ito's lemma to the wealth process $W_t^A = \frac{\mathbb{E}_t^A[\pi_T^A W_T^A]}{\mathbb{E}_t^A[\pi_T^A]}$, we can find that the equilibrium number of shares of Stock 1 held by Type A investors is equal to

$$\hat{\theta}_{1,t}^A = \frac{1}{2} - \frac{1}{2\gamma}\frac{\delta_0}{\sigma^2}, \tag{16}$$

which implies that the equilibrium number of shares of Stock 1 held by Type B investors is equal to

$$\hat{\theta}^B_{1,t} = \frac{1}{2} + \frac{1}{2\gamma}\frac{\delta_0}{\sigma^2}. \tag{17}$$

Because the number of shares held by investors in the equilibrium is constant over time, market closure would not have any impact on the equilibrium price in the case of constant disagreement. This result implies that stochastic disagreement is necessary for circuit breakers to have any impact through the market closure channel.

For Stock 2, it can be shown that

$$\mathbb{E}^A_t[\pi^A_{2,T} D_{2,T}] = \mathbb{E}^A_t[\eta^{1/2}_{2,T} e^{-\gamma D_{2,T}/2}] \cdot \left(D_{2,t} + (\mu^A_2 - \kappa^A \mu_J)(T-t) + \sqrt{\kappa^A \kappa^B}\mu_J (T-t) e^{-\frac{\gamma}{2}\mu_J} \right).$$

Then by Equation (14), we have the equilibrium price of Stock 2 as in the following proposition.

PROPOSITION 2. *When there are no circuit breakers, the equilibrium price of Stock 2 is:*

$$\hat{S}_{2,t} = D_{2,t} + \mu_2(T-t) + \sqrt{\kappa^A \kappa^B}\mu_J (T-t) e^{-\frac{\gamma}{2}\mu_J}. \tag{18}$$

Proposition 2 shows that the equilibrium price is affected by the heterogenous beliefs through the geometric average of beliefs of Type A and Type B investors on the jump intensity. In addition, the instantaneous volatility (square root of instantaneous variance) of the equilibrium price under $\mathbf{P}^A$ is the same as that of the dividend process because the rest of the terms in (18) are deterministic.

Let $\hat{\theta}^A_{j,t}$ be the optimal shares of Stock i held by Type A investors. Then $dW^A_t = \hat{\theta}_{1,t} d\hat{S}_{1,t} + \hat{\theta}_{2,t} d\hat{S}_{2,t}$. Applying Ito's formula to $W^A_t = \mathbb{E}^A_t[\pi^A_T W^A_T]/\pi^A_t$ and collecting the coefficients of stochastic terms, we obtain the optimal shares holding of Stock 2 for Type A investors as follows.

$$\hat{\theta}^A_{2,t} = \frac{1}{2} - \frac{1}{2\gamma\mu_J}\log(\frac{\kappa^B}{\kappa^A}). \tag{19}$$

This shows that in the absence of circuit breakers, the equilibrium trading strategy in Stock 1 for all investors is to buy and hold. Therefore, in the presence of circuit breakers, there is no market closure effect on Stock 2.

4. Equilibrium with Circuit Breakers

In this section, we study the equilibrium prices when the circuit breaker rule is imposed in the market. We first solve for the indirect utility function at the circuit breaker trigger time τ by maximizing investors' expected utility at $\tau < T$:

$$\max_{\theta^i_{1,\tau},\theta^i_{2,\tau}} \mathbb{E}^i[u(W^i_\tau + \theta^i_{1,\tau}(S_{1,T} - S_{1,\tau}) + \theta^i_{2,\tau}(S_{2,T} - S_{2,\tau}))],\ i \in \{A, B\}, \tag{20}$$

with the market clearing condition $\theta^A_{j,\tau} + \theta^B_{j,\tau} = 1$ and terminal condition $S_{j,T} = D_{j,T}$, where $\theta^i_{j,\tau}$ is the optimal number of shares of Stock j held by Type i investors at time τ, for $j = 1, 2$.

Exploiting the dynamics of $D_{j,T}$ and evaluating the expectation in the above optimization problem, we obtain a system of equations that determine $\theta^i_{j,\tau}$ and $S_{j,\tau}$ for $i \in \{A, B\}$, $j = 1, 2$. We summarize the result in the following proposition.

PROPOSITION 3. *Suppose that the market is halted at a stopping time $\tau < T$.*

(1) For Stock 1, the market clearing price at τ is given by

$$S^c_{1,\tau} = D_{1,\tau} + \mu^A_1(T - \tau) - \theta^A_{1,\tau}\gamma\sigma^2(T - \tau),$$

where the optimal shares holding of Type A is

$$\theta^A_{1,\tau} = \frac{-\frac{1}{\tilde{k}}(1-e^{\tilde{k}(\tau-T)})\delta_\tau - \bar{\delta}(T-\tau-\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}}) + I_\tau}{I_\tau + \gamma\sigma^2(T-\tau)}, \tag{21}$$

with $\tilde{k} = k - \frac{\nu}{\sigma}$ *and*

$$I_\tau = -\gamma\sigma^2(\tau-T) + \frac{2\nu\sigma\gamma}{\tilde{k}}(T-\tau-\frac{1-e^{\tilde{k}(\tau-T)}}{\tilde{k}}) + \frac{\nu^2\gamma}{\tilde{k}^2}(T-\tau-2\frac{1-e^{\tilde{k}(\tau-T)}}{\tilde{k}} + \frac{1-e^{2\tilde{k}(\tau-T)}}{2\tilde{k}}).$$

If $\tilde{k} = 0$, *the optimal shares holding is simplified to:*[3]

$$\theta^A_{1,\tau} = \frac{1}{\gamma}\frac{(\gamma\sigma^2 - \gamma\nu\sigma(\tau-T) + \frac{\nu^2\gamma}{3}(\tau-T)^2 - \delta_\tau)}{-\nu\sigma(\tau-T) + \frac{\nu^3}{3}(\tau-T)^2 + 2\sigma^2}.$$

(2) For Stock 2, the market clearing price is given by

$$S^c_{2,\tau} = D^*_{2,\tau} + \mu_2(T-\tau) + \sqrt{\kappa^A\kappa^B}\mu_J e^{-\frac{\gamma}{2}\mu_J}(T-\tau),$$

where $D^*_{2,\tau} \in [D_{2,\tau}, D_{2,\tau-})$ *is such that* $S^c_{1,\tau} + S^c_{2,\tau} = h$.

The optimal shares holding of Stock 2 at τ *is the same as that in the absence of circuit breakers, i.e.,*

$$\theta^A_{2,\tau} = \hat{\theta}^A_{2,\tau}. \tag{22}$$

By the definition of $D^*_{2,\tau}$, we see that $S^c_{2,\tau} \geq \hat{S}_{2,\tau}$, that is, the market clearing price of Stock 2 is not less than the equilibrium price in the absence of circuit breakers. This is because the circuit breaker prevents the price from falling beyond the threshold. As a consequence, $S_{2,t}$, the equilibrium price of Stock 2 in the presence of circuit breakers should not be less than $\hat{S}_{2,t}$, the equilibrium price in the absence of circuit breakers. We will refer to this impact of

[3]It is easy to verify that as $\tau \to T^-$, $\theta^A_{1,\tau} \to \frac{1}{2} - \frac{\delta_T}{2\gamma\sigma^2}$, which is the optimal shares holding of Stock 1 by Type A in the case of constant disagreement.

circuit breakers on equilibrium prices as *price limit effect*. If a stock's dividend is continuous, investors can continuously adjust the valuation to reflect the fundamentals represented by the dividend process, and thus the price limit effect is zero. In contrast, when a jump occurs and the price is stopped at the threshold level, the price with a circuit breaker is strictly higher than the price without the circuit breaker, and thus there is a strictly positive price limit effect.

In contrast, depending on parameter values, Stock 1's price at τ may be higher or lower than that without circuit breakers. Intuitively, circuit breakers do not have price limit effect on Stock 1 because Stock 1's dividend does not jump. The market closure effect of circuit breakers can increase or decrease the stock price compared to the case without circuit breakers because market closure may reduce sale initiated trades more than purchase initiated trades and vice versa.

Having obtained the market clearing prices at the circuit breaker trigger time τ, we turn to characterize τ.

4.1 The Circuit Breakers Trigger Time and Shares Holing at the Trigger Time

The circuit breaker trigger time τ can be characterized using the dividend values. Because the market is closed when the sum of prices reaches the threshold h, we have

$$
\begin{aligned}
h &= S^c_{1,\tau} + S^c_{2,\tau} \\
&= D_{1,\tau} + D^*_{2,\tau} + \left(\mu^A_1 + \mu_2 - \gamma\sigma^2\theta^A_{1,\tau} + \kappa^A\phi'(-\gamma\theta^A_{2,\tau})\right)(T-\tau) \\
&\geq D_{1,\tau} + D_{2,\tau} + \left(\mu^A_1 + \mu_2 - \gamma\sigma^2\theta^A_{1,\tau} + \kappa^A\phi'(-\gamma\theta^A_{2,\tau})\right)(T-\tau),
\end{aligned}
$$

where $\phi(x) = e^{x\mu_J}$. It follows that we may define the stopping time τ using the dividend processes as follows.

PROPOSITION 4. *Let h be a threshold. Define a stopping time*

$$\tau = \inf\{t \geq 0 : D_{1,t} + D_{2,t} \leq \underline{D}(t)\},$$

where

$$\underline{D}(t) = h - \left(\mu_1^A + \mu_2 - \gamma\sigma^2\theta_{1,\tau}^A + \kappa^A \mu_J e^{-\gamma\theta_{2,\tau}^A \mu_J}\right)(T - t).$$

Then the circuit breaker is triggered at time τ when $\tau < T$.

Note that

$$D_{1,t} + D_{2,t} - \underline{D}(t) = D_{1,0} + D_{2,0} + (\mu_1^A + \mu_2)T - \gamma\sigma^2\theta_{1,\tau}^A T + \kappa^A \mu_J e^{-\gamma\theta_{2,\tau}^A \mu_J} T - h$$
$$+ \sigma Z_t + \mu_J N_t + (\gamma\sigma^2\theta_{1,\tau}^A - \kappa^A \mu_J e^{-\gamma\theta_{2,\tau}^A \mu_J})t.$$

Thus, the trigger time τ can be characterized as the first hitting time of zero of a jump-diffusion process with a stochastic drift.

4.1.1 Equilibrium Shares Holding

At τ, the equilibrium shares holding for Stock 1 is exactly given by Prop. 3, because circuit breakers do not cause any price distortion in Stock 1 price due to the continuity of Stock 1's dividend. For Stock 2, if it is the diffusion of Stock 1 that triggers the circuit breaker, the shares holding for Stock 2 at τ is the same as $\hat{\theta}_{2,\tau}$, the shares holding in the case of no circuit breakers. However, if the market closure is caused by a jump of Stock 2's dividend, because of the price limit effect of the circuit breakers, Stock 2 price may not fall to the full amount as in the case of no circuit breakers. In other words, the circuit breaker rule distorts Stock 2's price and the equilibrium shares holding of Stock 2 is a corner solution.

We next introduce a mechanism to determine the equilibrium shares holding of Stock 2 when the price limit effect is strictly positive as a result of a jump in Stock 2's dividend. Let $\tilde{\theta}_{2,\tau}^i$ denote the shares holding that would maximize the individual utility of Type i at

τ and $\theta^i_{2,t}$ denote the equilibrium shares holding at t, $i = A, B$. Then if Type i investors would like to sell/buy more than other investors would like to buy/sell, then the equilibrium trading amount at τ is equal to the smaller amount that would like to be bought/sold by other investors. If, on the other hand, all investors would like to trade in the same direction, then no one can trade. More precisely, the equilibrium shares holding of Stock 2 at τ is determined by the following rule:

- If $(\theta^A_{2,\tau-} - \tilde{\theta}^A_{2,\tau}) \cdot (\theta^B_{2,\tau-} - \tilde{\theta}^B_{2,\tau}) > 0$, then $\theta^i_{2,\tau} = \theta^i_{2,\tau-}$, $i \in \{A, B\}$;
- otherwise
  - If $|\theta^A_{2,\tau-} - \tilde{\theta}^A_{2,\tau}| \leq |\theta^B_{2,\tau-} - \tilde{\theta}^B_{2,\tau}|$, then $\theta^A_{2,\tau} = \tilde{\theta}^A_{2,\tau}$ and $\theta^B_{2,\tau} = 1 - \tilde{\theta}^A_{2,\tau}$.
  - If $|\theta^A_{2,\tau-} - \tilde{\theta}^A_{2,\tau}| > |\theta^B_{2,\tau-} - \tilde{\theta}^B_{2,\tau}|$, then $\theta^A_{2,\tau} = 1 - \tilde{\theta}^B_{2,\tau}$ and $\theta^B_{2,\tau} = \tilde{\theta}^B_{2,\tau}$.

4.2 The Equilibrium Price before τ

After obtaining the market clearing prices and the optimal portfolios at τ in closed-form, we now turn to study the equilibrium stock prices at $t < \tau \wedge T$. For $i \in \{A, B\}$, let

$$G^i_\tau(\theta^i_{1,\tau}, \theta^i_{2,\tau}) = -\frac{1}{\gamma} \log(\mathbb{E}^i_\tau[e^{-\gamma(\theta^i_{1,\tau}(S_{1,T}-S_{1,\tau})+\theta^i_{2,\tau}(S_{2,T}-S_{2,\tau}))}]).$$

Then the indirect utility function of Type $i \in \{A, B\}$ at τ is

$$V^i(W^i_\tau, \tau) = -e^{-\gamma(W^i_\tau + G^i_\tau(\theta^i_{1,\tau}, \theta^i_{2,\tau}))}.$$

Given the optimal shares holding $\theta^i_{j,\tau}$ at τ, the exact expression of G^i_τ can be found explicitly. Then, we solve the planner's problem:

$$\max_{W^A_{T\wedge\tau}, W^B_{T\wedge\tau}} \mathbb{E}^A_0[V^A(W^A_{T\wedge\tau}, T\wedge\tau) + \xi\eta_\nu V^B(W^B_{T\wedge\tau}, T\wedge\tau)] \tag{23}$$

subject to the wealth constraint $W^A_{T\wedge\tau} + W^B_{T\wedge\tau} = S_{1,T\wedge\tau} + S_{2,T\wedge\tau}$.

Similar to the case without circuit breakers, it follows from the first order conditions and the budget constraint that

$$W^A_{T\wedge\tau} = \frac{1}{2\gamma}\log(\frac{1}{\xi\eta_{T\wedge\tau}}) + \frac{1}{2}(S_{1,T\wedge\tau} + S_{2,T\wedge\tau}) + \frac{G^B_{T\wedge\tau} - G^A_{T\wedge\tau}}{2}, \tag{24}$$

$$W^B_{T\wedge\tau} = -\frac{1}{2\gamma}\log(\frac{1}{\xi\eta_{T\wedge\tau}}) + \frac{1}{2}(S_{1,T\wedge\tau} + S_{2,T\wedge\tau}) + \frac{G^A_{T\wedge\tau} - G^B_{T\wedge\tau}}{2}. \tag{25}$$

In addition, the state price density under Type A investors' beliefs is

$$\begin{aligned}\pi^A_t &= \mathbb{E}^A_t[\zeta(V^A(W^A_{T\wedge\tau}))'] = \mathbb{E}^A_t[\gamma\zeta e^{-\gamma(W^A_{T\wedge\tau}+G^A_{T\wedge\tau})}] \\ &= \gamma\zeta\mathbb{E}^A_t[\eta^{1/2}_{T\wedge\tau}\cdot e^{-\frac{\gamma}{2}(S_{1,T\wedge\tau}+S_{2,T\wedge\tau}+G^B_{T\wedge\tau}+G^A_{T\wedge\tau})}],\end{aligned} \tag{26}$$

for some constant ζ. Thus, the stock price in equilibrium is given by

$$S_{j,t} = \frac{\mathbb{E}^A_t[\pi^A_{T\wedge\tau}S_{j,T\wedge\tau}]}{\mathbb{E}^A_t[\pi^A_{T\wedge\tau}]}, \quad j = 1, 2, \tag{27}$$

with

$$S_{j,T\wedge\tau} = \begin{cases} D_{j,T}, & if\ \tau \geq T, \\ S^c_{j,\tau}, & if\ \tau < T. \end{cases} \tag{28}$$

The wealth process of Type A investors is

$$W^A_t = \frac{\mathbb{E}^A_t[\pi^A_{T\wedge\tau}W^A_{T\wedge\tau}]}{\mathbb{E}^A_t[\pi^A_{T\wedge\tau}]}, t < T\wedge\tau.$$

Suppose that

$$dW^A_t = M_t dt + \theta^A_{1,t}dS_{1,t} + \theta^A_{2,t}dS_{2,t},$$

for some process M_t, where $\theta_{1,t}$ and $\theta_{2,t}$ are shares holdings of Type A for Stock 1 and Stock 2 respectively. We can recover the shares holdings at t by quantities of $\mathbb{E}^A_t[dW^A_t \cdot dS_{j,t}]$, $\mathbb{E}^A_t[dS_{1,t} \cdot dS_{2,t}]$, and $\mathbb{E}^A_t[(dS_{j,t})^2]$, $j = 1, 2$.

Finally, to obtain the equilibrium price we let the threshold depend on the initial stock prices, that is, $h = (1-\alpha)(S_{1,0}+S_{2,0})$. Then the initial stock prices can be found by solving a fixed point problem

$$S_{j,0} = \frac{\mathbb{E}_0^A[\pi_{T\wedge\tau}^A S_{j,T\wedge\tau}]}{\mathbb{E}_0^A[\pi_{T\wedge\tau}^A]}. \tag{29}$$

The following proposition shows the existence and uniqueness of a solution to the above fixed point problem.

PROPOSITION 5. *If the initial equilibrium index value $\hat{S}_{10}+\hat{S}_{20}$ is positive in the absence of circuit breakers, there exits a unique solution to the fixed point problem (29) in the presence of circuit breakers.*

We numerically evaluate the equilibrium prices and illustrate impacts of circuit breakers on the market equilibrium in the next section.

5. Impact of Circuit Breakers

In this section, we examine impact of circuit breakers on the dynamics of market. The model default parameter values for numerical analysis are as follows, where growth rates and volatilities are daily ones.

$$\begin{array}{llll}
\mu_1^A = 0.10/250, & \sigma = 0.06, & & \\
\delta_0 = 0.06, & \nu = 0.06, & k = \frac{\nu}{\sigma} = 1, & \bar{\delta} = 0.06, \\
\mu_2 = 0.10/250, & \mu_J = -0.25, & \kappa^A = 1, & \kappa^B = 0.1, \\
\gamma = 0.1, & \alpha = 0.07. & &
\end{array}$$

Given these parameter values, the disagreement δ_t evolves as a random walk under Type B investors' probability measure. Type B over-estimates the growth rate of dividend 1 (in the beginning) and under-estimates the downside jump frequency of dividend 2. Since our main goal is to examine the impact of circuit breakers in bad times when market is volatile and the crash probability of some stocks is high (e.g., the Chinese stock market early January of 2016), we set the jump frequency high and the jump size large, along with a high volatility of Stock 1's dividend. Because of the CARA preferences, the initial share endowment of the investors does not affect the equilibrium. The circuit breaker is triggered when the sum of two prices (i.e., the index) first reaches the threshold $(1-\alpha)(S_{1,0}+S_{2,0})$, i.e., drops 7% from the initial value.

One alternative to the market-wide circuit breakers is to impose a circuit breaker separately on each stock (instead of on an index). With this separate circuit breaker on each stock, if a circuit breaker for a stock is triggered, only trading in the corresponding stock is halted. For example, when the circuit breaker of Stock 1 is triggered, only trading of Stock 1 is halted, but trading in Stock 2 is unaffected. Obviously, with separate circuit breakers, equilibrium prices remain independent, in sharp contrast to the case of market-wide circuit breakers. Let $\tilde{S}_{j,t}, j = 1, 2$ denote the equilibrium prices of Stock j in this benchmark. We compare the impact of circuit breakers on the stock prices when they are on an index and when they are on individual stocks.

5.1 Equilibrium Prices

By Prop. 1 and Prop. 2, we obtain the initial equilibrium price $\hat{S}_{1,0} = 1.0246, \hat{S}_{2,0} = 0.9108$ in the absence of circuit breakers. When there are separate circuit breakers on individual stocks, the equilibrium prices are $\tilde{S}_{1,0} = 1.0270$, $\tilde{S}_{2,0} = 0.9708$, both of which are greater than those in the absence of circuit breakers. As noted before, the market closure effect of a circuit breaker can also decrease the equilibrium price because it may restrict the purchasing pressure due to the disagreement. Thus $\tilde{S}_{1,0}$ may be lower or higher than $\hat{S}_{1,0}$, depending on values of the model parameters (e.g., those related to the disagreement process). On the other hand, Stock 2 price with a separate circuit breaker (i.e., $\tilde{S}_{2,0}$) is always higher than the one without a circuit breaker (i.e., $\hat{S}_{2,0}$) because the price limit effect of a circuit breaker prevents the price from falling down to the full extent after a jump of the dividend.

By solving the fixed point problem numerically, in the presence of market-wide circuit breakers, we obtain the equilibrium prices $S_{1,0} = 1.0241, S_{2,0} = 0.9486$. This shows that equilibrium stock prices with a market-wide circuit breaker are lower than those with separate circuit breakers. The reason is that with separate circuit breakers, a stock would not be adversely impacted by a fall in another otherwise independent stock, which can happen with a market-wide circuit breaker.

5.2 Crash Contagion

Because the circuit breaker based on a stock index is triggered when the index reaches a threshold, a crash in a group of stocks (e.g., from a downward jump in their dividends) may trigger the circuit breaker and cause the entire market to be closed down. As a result, the prices of otherwise independent stocks may also jump down because of the sudden market closure. This pattern of cross-stock serial crashes is called *crash contagion*.

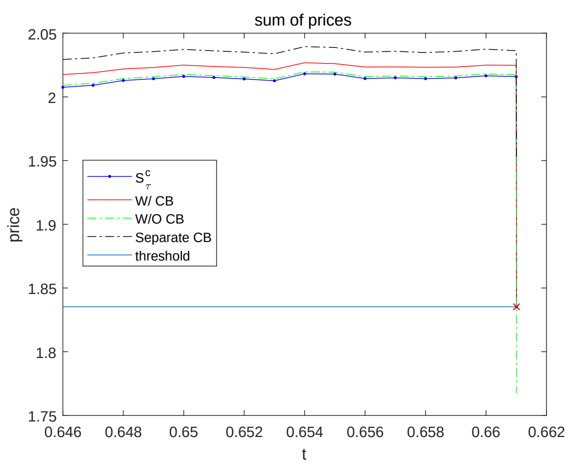

Figure 1. A sample of the sum of prices. The market is early halted at the time when the red line (the sum) touches the threshold at the red cross. In this sample, the breaker is triggered by a sudden jump occurring in dividend of stock 2.

Figure 1 presents the sum of stock prices generated by the same sample of dividends under different circuit breaker implementations. In this sample, the market-wide circuit breaker is triggered by a jump in Stock 2 price S_2 as a result of a jump in the dividend process D_2.

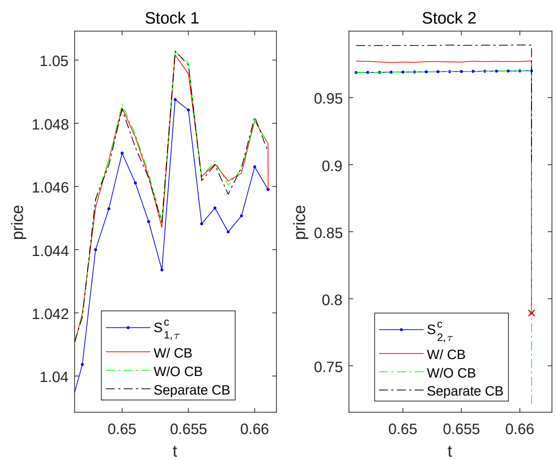

Figure 2. The two individual prices. In the sample as Figure 1, the circuit breaker is triggered by a jump occurring in the price $S_{2,t}$ (the right panel).

The sum of prices without circuit breakers ($\hat{S}_t$, green line) jumps down to a value below the circuit breaker threshold (light blue line). Because of the price limit effect, the sum of prices with market-wide circuit breakers (S_t, red line) stops at the threshold. This shows that circuit breakers do have the function of price support in bad times. As a result, the index level with circuit breakers is higher than that without any circuit breakers. However, compared to the separate circuit breakers rule, the net price limit effect is smaller. this is because with separate circuit breakers, the price limit effect of circuit breakers is kept, while the market-wide closure effect is avoided.

Figure 2 separates out the two individual stock prices using the same sample as in Figure 1. For Stock 2, its price jumps down toward to the market clearing price $S^c_{2,\tau}$ (the red cross point). For Stock 1, even though there is no jump in its dividend process, its price also jumps down because the circuit breaker is triggered and the liquidity in the market vanishes due to the market closure. This figure illustrates that market-wide circuit breakers can cause crash contagion across otherwise independent stocks.

Figures 1 and 2 use a particular sample path to illustrate the possibility of crash contagion. In Figure 3, we plot the distribution of Stock 1 price change conditional on a jump in Stock 2 price that triggers the circuit breaker and holding Stock 1's dividend constant at the crash time (red line) and the distribution of Stock 1 price change with no circuit breaker in place (green line). Figure 3 shows that without a circuit breaker, the price change of Stock 1, which is independent of Stock 2, is normally distributed with mean zero. In contrast, in the presence of circuit breakers, after a crash of Stock 2 that triggers the circuit breaker, Stock 2 price always goes down and the magnitude of the drop can be significant. Note that this change is only from the crash of Stock 2 because Stock 1's dividend is held constant when the crash occurs. Therefore, this distribution represents the distribution of the crash magnitudes in Stock 1 caused by the crash in Stock 2.[4]

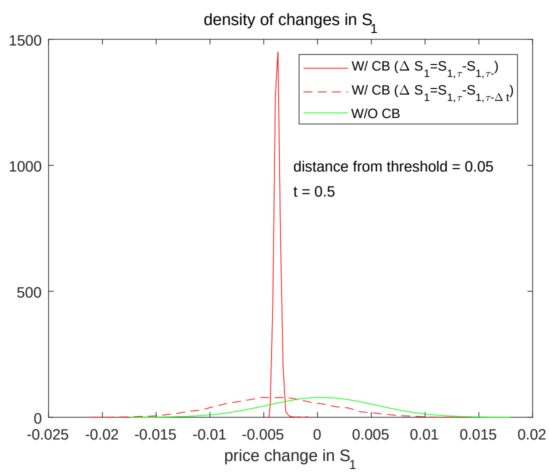

Figure 3. Distribution of price changes in stock 1 when the circuit breaker is triggered by a jump in the stock price 2. In the presence of circuit breaker, the distribution is skewed negatively. Meanwhile, in the absence of circuit breakers the price changes seem converging to a normal distribution.

Our above findings are consistent with what happened on January 4th, 2016 in Chinese stock markets. January 4th 2016 was the first day of the implementation of a circuit breaker rule in Chinese stock markets which stipulates that the entire market is closed if the CSI 300 index falls by 7% from the previous day close. Using high-frequency prices of the components of CSI 300 during six minutes before the circuit breaker was triggered on January 4th, we sort components by their total dollar trading volumes. Simple regression of the return of the top 25%-50% stocks inside the CSI 300 index on the lagged return of the top 25% stocks

[4]Because Stock 1 price is random when the crash in Stock 2 occurs, the crash in Stock 1 caused by Stock 2 crash is also random, even though Stock 1's dividend is kept constant when the crash occurs.

suggests that in the market crash of January 4th, 2016, the crash of the top 25%-50% stocks followed that of the top 25% stocks (t-statistics in parenthesis). This result is reported in Figure 4.

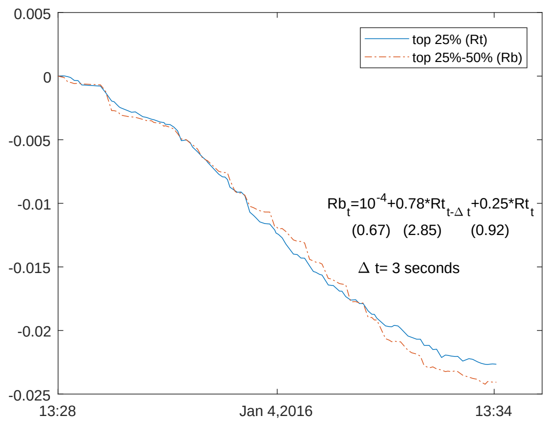

Figure 4. Evidence of contagion in real markets.

5.3 Increased Correlations

With circuit breakers based-on indices, a discrete jump (crash) in a stock is not necessary for adversely affecting otherwise independent stocks. Intuitively, even after a small decline in the price of a stock, the index gets closer to the circuit breaker threshold and thus the market is more likely to be closed early, which may lower the prices of otherwise independent stocks, which in turn makes the index even closer to the circuit breaker threshold, and enters into a vicious circle. This contagion magnitude is typically smaller than that caused by a crash in a stock in normal times, but can get significant and create strong correlations when the circuit breaker is close to be triggered because of the magnified vicious circle effect. We next show that a gradual change in the price of a stock can indeed affect the price of another stock and can cause high correlations among otherwise independent stocks when close to the circuit breaker threshold.

Figures 5 and 6 show how the same variables as in Figures 1 and 2 change along a different sample path where the circuit breaker is triggered by small changes of Stock 1 price due to a decline in its dividend. Different from the sample paths illustrated in Figures 1 and 2, prices do not jump in Figures 5 and 6 because there is no jump in dividends in Figures 5 and 6. On the other hand, as the right sub-figure of Figure 6 shows, Stock 2 price is adversely affected by the decline in the Stock 1's price. Figures 1–6 suggests that although the two stocks are independent in the absence of circuit breakers, they become positively correlated when the circuit breakers are close to be triggered, regardless of the occurrence of a crash of a stock. As in the case of jump triggered market closure, the prices under the separate circuit breakers rule (Black dot-dash lines) are higher than those with market-wide circuit breakers.

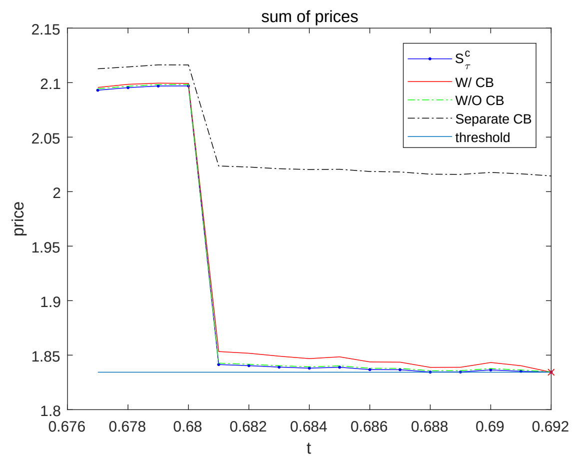

Figure 5. A sample path of the sum of prices along which the circuit breaker is triggered by Stock 1.

Consistent with our intuition, Figure 7 shows that correlation of the two prices with circuit breakers increases significantly as the index gets very close to the threshold. When the index is far from the threshold, the correlation becomes close to zero, because the correlation without circuit breakers is zero.[5] In addition, when the potential market closure duration is large ($T - t$ is large), the impact of the circuit breakers on the correlation is even bigger, because the fear for market closure kicks is stronger when the potential market closure

[5]We also find that the impact on the correlation tends to be greater the further away from the end of day.

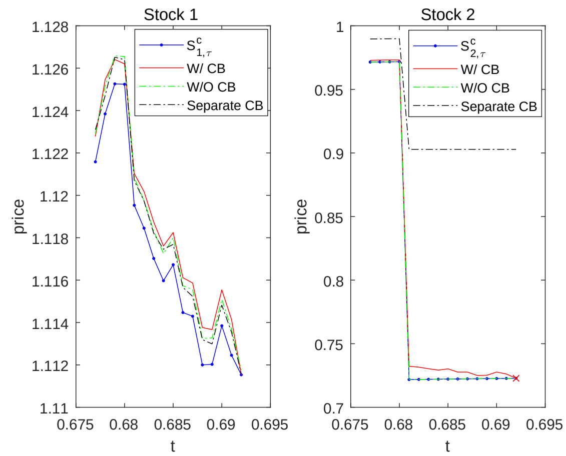

Figure 6. The individual stock prices along the same sample path as in the previous figure.

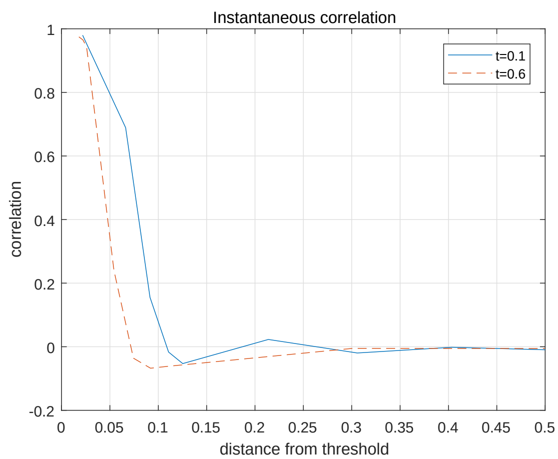

Figure 7. Instantaneous correlation.

duration is longer. For example, conditional on the same distance of 0.05 from the threshold, if it is later in the day at $t = 0.6$, then the correlation is 0.3, while it is 0.8 if it is early in the day at $t = 0.1$.

Surprisingly, Figure 7 shows that correlation can turn negative when the distance from the threshold is greater, before it approaches zero eventually. This negative correlation is due to the price limit effect of the circuit breaker. To help explain this channel, we plot stock price and the index in Figure 8 against changes in Stock 1's dividend. As Stock 1's dividend increases, the price of Stock 1 increases (blue starred line), as expected. However, the price of Stock 2 changes non-monotonically (red starred line). When Stock 1's dividend is very low such that a small change in either stock price would trigger the circuit breaker, the likelihood of the circuit breaker being triggered by a jump in Stock 2's dividend is relatively small because the probability of a jump is low. Recall that the price limit effect is strictly positive only when the circuit breaker is triggered by a jump in Stock 2's dividend. This implies the present value of the price limit effect of the circuit breaker is small. As Stock 1's dividend increases, the price of Stock 1 increases, and thus the distance from the circuit breaker threshold is increased. It becomes more likely that only a jump in Stock 2's dividend can trigger the circuit breaker. Therefore the price limit effect increases which in turns increases the price of Stock 2. However, when Stock 1's dividend is too large, the index becomes far away from the threshold, which makes even a jump in Stock 2's dividend would not trigger the circuit breaker. Therefore the price limit effect eventually approaches zero when D_1 is high enough. This explains the nonmonotonicity of the price of Stock 2 in D_1, which implies that the correlation is positive when the index is close to the threshold, turns negative when the index is further away, and converges to zero when the index is far enough, as shown in Figure 7.

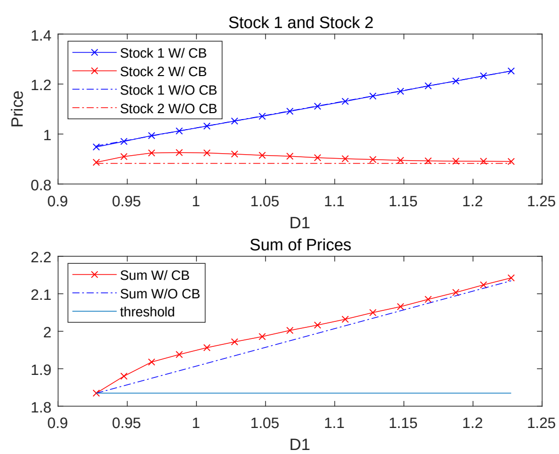

Figure 8. This figure illustrates why the correlation is positive when the threshold is close and why it turns to be positive when the distance is larger. Eventually, $S_{2,t}$ approaches to a constant and the correlation becomes almost zero.

5.4 Acceleration of Market Decline: the Magnet Effect

Circuit breakers are implemented to prevent market from a fast decline. Contrary to this intention, Chen et. al. (2018) shows in a single stock setting that circuit breakers can accelerate stock price decline compared to the case without circuit breakers. This acceleration is what is called the "magnet effect" by Chen et. al. (2018). However, it is not clear whether this conclusion in a single stock setting applies to the case with multiple stocks. As we show next, in a multiple stock setting whether circuit breakers increase or decrease the probability of falling to the index threshold compared to the case without circuit breakers depends on whether they are imposed on indices or on individual stocks.

Figure 9 shows probabilities to reach the circuit breaker index threshold in a given time interval with circuit breakers on the index, with circuit breakers on individual stocks, and without circuit breakers. Figure 9 suggests that the probability of falling to the index threshold when there is a circuit breaker on the index (red lines) is higher than that without any circuit breakers (blue lines), which is in turn higher than that when circuit breakers are on individual stocks (black lines). This is because with circuit breakers on indices, when one stock goes down, the distance to the circuit breaker threshold is shorter, the likelihood of an early market closure is greater. As a result, other stock prices tend to go down, which in turn drag the index further downward, and result in a downward accelerating vicious circle, contrary to regulators' intention. In addition, when the potential market closure duration is longer (e.g., at $t = 0$), this magnet effect is even stronger. The main driving force for the magnet effect in Chen et. al. (2018) is the fear that one has to liquidate a levered position at the market closure time because after market closure, leverage is prohibited by the solvency requirement. In contrast, in this paper, there is no change in the leverage level allowed before and after market closure. Our results show that circuit breakers on indices can accelerate a market decline even without the de-leverage effect.

In contrast, if circuit breakers are imposed on individual stocks, the probability of falling to the index threshold is lower than that without circuit breakers. This is because individual circuit breakers prevent corresponding stock prices from falling below their individual stock price thresholds and decrease the probability of the index threshold compared to the case without circuit breakers. These results suggest that separate circuit breakers can slow down market wide decline, while circuit breakers on indices do the opposite.

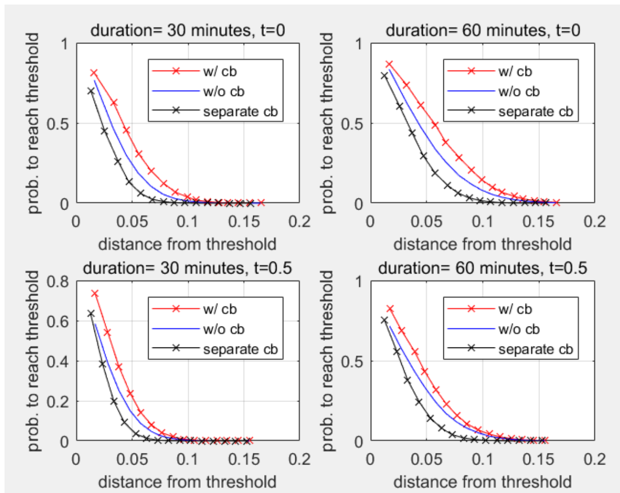

Figure 9. This figure shows probabilities of prices to reach the threshold with or without a circuit breaker.

Another measure of the magnet effect is how fast stock prices go down as the index gets close to the circuit breaker threshold. In Figure 10, we plot the average prices against the time to market closure using simulated sample paths. More specifically, we simulate a large number of sample paths of dividends, compute the corresponding stock prices and identify the circuit breaker trigger times for each sample path. Then we calculate the average stock prices across all the sample paths at a given time prior to market closure. The downward-concave shapes displayed in Figure 10 implies that as the index gets closer to the threshold, stock prices fall faster. Figure 11 plots the average time it takes for the index to fall by 1% against the distance to threshold. It implies that the falling speed increases as the index gets closer to the threshold. These patterns are consistent with what were observed in real markets, such as the January 2016 Chinese market when circuit breakers were first implemented and then abandoned after 4 days.

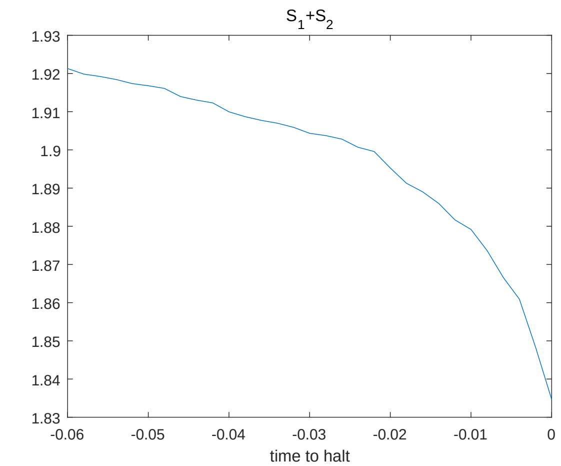

Figure 10. This figure shows the average stock prices during a short time period right prior to the early closure of market caused by Stock 1.

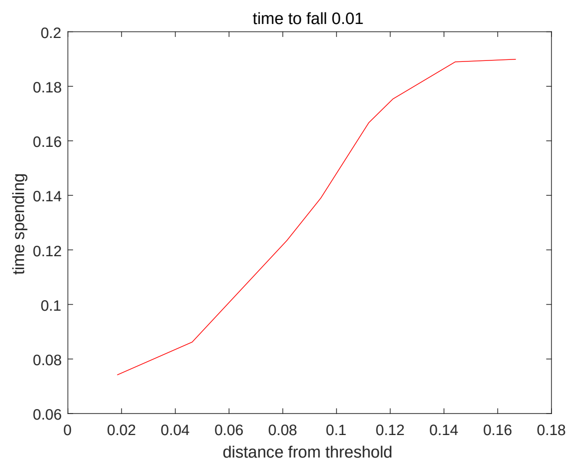

Figure 11. This figure shows the average falling speeds of prices during the short time periods right prior to the early closure of market.

5.5 Increased Volatility and Volatility Contagion

One of the regulatory goal of the circuit breaker is to reduce volatility. We next examine what is the impact of the circuit breaker on stock volatility. In Figure 12, we plot the volatilities against the index's distance from the circuit breaker's threshold. Figure 12 suggests that opposite to the regulatory goal, circuit breakers can increase stock volatility in bad times when the circuit breaker is close to be triggered. This increased volatility is caused by the magnet effect explained above. In addition, we find that if time to the end of day is longer (i.e., the potential market closure duration is greater), volatilities are even larger.

When distance from the threshold is sufficiently large, instantaneous volatilities approach the corresponding levels in the absence of circuit breakers.

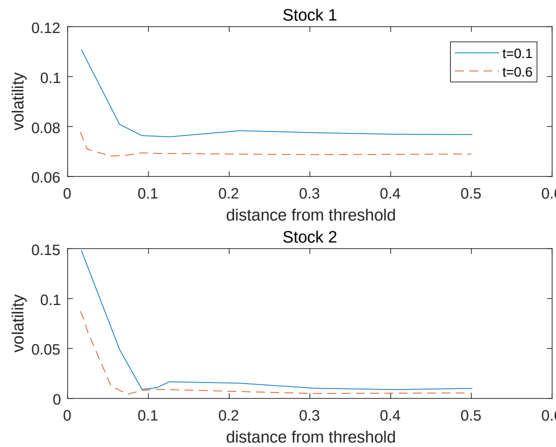

Figure 12. Instantaneous volatilities tend to be higher when the circuit breaker is more likely to be triggered.

Next we show that circuit breakers can also cause volatility contagion, i.e., an increase in the volatility of one stock can cause an increase in the volatility of another stock. Figure 13 plots the volatility of Stock 1 against the volatility of Stock 2 as we change the volatility of Stock 1's dividend for three levels of distance to threshold. Figure 13 indicates that indeed a higher volatility of Stock 1's dividend causes a higher volatility of Stock 2, and in addition, this increase in the volatility of Stock 2 gets magnified when the distance to the threshold is shorter. This volatility contagion can amplify market-wide volatility, which is also against what circuit breakers are designed for.

6. Correlated Dividends

In the preceding sections, the dividend processes are assumed to be uncorrelated and we show that a strong correlation of the stock prices can emerge due to circuit breakers. One concern might be that, if the dividend processes are already correlated, then the additional correlation caused by the circuit breakers may be small. To address this concern, we now briefly discuss an extended model where the dividend processes are correlated (The detailed

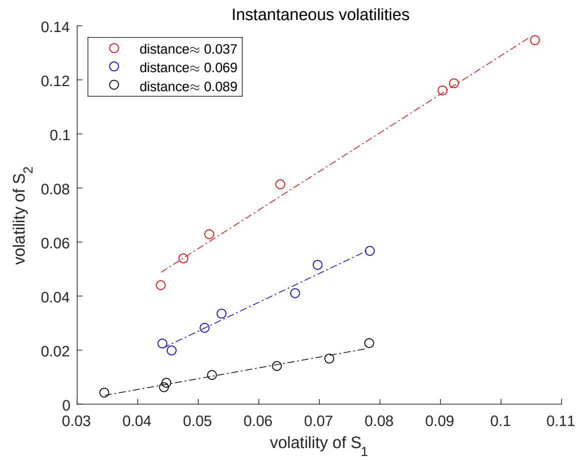

Figure 13. Volatilities of the two prices are independent in the absence of circuit breakers. This figure shows volatilities are linearly correlated in the presence of circuit breakers. A higher volatility of stock 1 corresponds to a higher volatility of stock 2. Mover over, if the threshold is closer, volatility of stock 2 is even larger.

derivation can be found in the appendix) by assuming a diffusion term in the dynamics of Stock 2's dividend:

$$dD_{2,t} = \mu_2 dt + \sigma_2 dZ_t + \mu_J dN_t.$$

Then the two dividend processes are correlated with correlation

$$\rho_D = \frac{\sigma_2}{\sqrt{\sigma_2^2 + \kappa^A \mu_J^2}}. \tag{30}$$

Figure 14 compares correlations of stock prices for different correlation coefficients ρ_D of dividend processes. The figure suggests that even when the dividends are correlated, the presence of circuit breakers can significantly increase the correlation of stock prices. For example, when the dividends correlation coefficient is 0.2, the increase in the correlation is still as high as 0.75. On the other hand, when dividends are highly negatively correlated ($\sigma_2 << 0$), the presence of circuit breakers can make the stock prices even more negatively correlated. This is because the price limit effect of circuit breakers cuts the effective jump size in the stock price, and thus changes of stock prices come more from the diffusion parts which have a correlation coefficient of -1. In the extreme, if the stock price jump size was reduced to zero, then the price correlation would be equal to -1, similar to what is implied

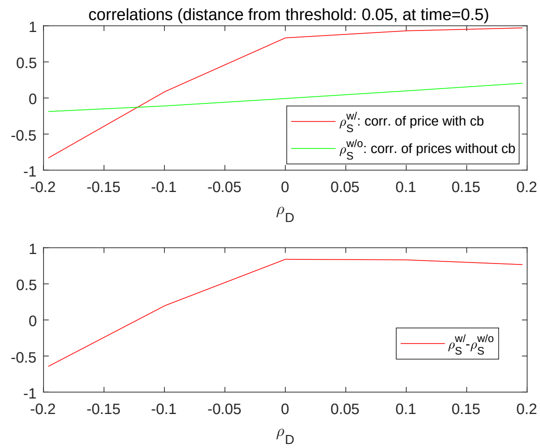

Figure 14. This figure compares how correlation of stock prices are impacted by correlation of dividend processes. All correlations are calculated when distance from threshold is around 0.05.

by Equation (30).

7. Conclusion

Circuit breakers based on indices are commonly imposed in financial markets to prevent market crashes and reduce volatility in bad times. We develop a continuous-time equilibrium model with multiple stocks to study how circuit breakers affect joint stock price dynamics, cross-stock contagion, and market volatility. Contrary to the regulatory goals, we show that in bad times, circuit breakers can cause crash and volatility contagion and high correlations among otherwise independent stocks, can significantly increase market volatility, and can accelerate market decline. Our analysis helps explain the concurrence of the implementation of the circuit breakers rule and the significant market tumble in the week of January 4, 2016 in Chinese stock markets, and also the quick suspension of the circuit breaker rule 3 days later. Our model suggests that market-wide circuit breakers may be a source of financial contagion and a channel through which idiosyncratic risks become systemic risks, especially in bad times. An alternative circuit breaker approach based on individual stock returns instead of indices would alleviate such problems.

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Appendix

A Stock 1: Stochastic Disagreement

We assume that the disagreement process δ_t is stochastic and follows the equation (4). When there are no circuit breakers, the equilibrium price is obtained in closed-form. In the presence of circuit breakers, we first find out the market clearing price at the time of market closure and then evaluate the equilibrium price numerically.

A.1 Without Circuit Breakers

When there are no circuit breakers, the equilibrium price be obtained in closed-form as follows.

We first evaluate $\mathbb{E}_t^A[\pi_{1,T}^A]$. Ignoring constants, we need to calculate

$$\mathbb{E}_t^A[\eta_{1,T}^{1/2} e^{-\frac{\gamma}{2} D_{1,T}}] = \mathbb{E}_t^A[e^{Y_{1,T}}] \cdot f(t),$$

where $f(t)$ is a deterministic function and,

$$Y_{1,T} = \int_0^T (\frac{\delta_s}{2\sigma} - \frac{\gamma\sigma}{2}) dZ_s + \int_0^T (-\frac{\delta_s^2}{4\sigma^2}) ds.$$

For the sake of conventional simplicity, we surpass the subscription dependence of i and consider $\mathbb{E}_t^A[e^{Y_T}]$ hereafter in this section.

Conjecture $F(t, y, \delta, \delta^2) = e^{A(t)+B(t)y+C(t)\delta+\frac{H(t)}{2}\delta^2} = \mathbb{E}^A[e^{Y_T} | Y_t = y, \delta_t = \delta]$, with $A(T) = C(T) = H(T) = 0$ and $B(T) = 1$. Substituting the conjecture into the moment generating function of the process (Y_t, δ_t) and collecting coefficients of y, δ, δ^2 and constant, we obtain four ordinary different equations:

$$
\begin{aligned}
A'(t) + \frac{1}{8}\gamma^2\sigma^2 B(t)^2 + k\bar{\delta}C(t) + \frac{\nu^2}{2}(C(t)^2 + H(t)) - \frac{\gamma\sigma\nu}{2}B(t)C(t) &= 0, \\
B'(t) &= 0, \\
C'(t) - \frac{\gamma}{4}B(t)^2 + k\bar{\delta}H(t) - kC(t) + C(t)H(t)\nu^2 + \frac{\nu}{2\sigma}B(t)C(t) - \frac{\gamma\sigma\nu}{2}B(t)H(t) &= 0, \\
\frac{H'(t)}{2} - \frac{1}{4\sigma^2}B(t) + \frac{B(t)^2}{8\sigma^2} - kH(t) + \frac{\nu^2}{2}H(t)^2 + \frac{\nu B(t)H(t)}{2\sigma} &= 0.
\end{aligned}
$$

The solution of the ODE system is obtained as follows.

$$
\begin{aligned}
B(t) &= 1, \\
H(t) &= \frac{e^{(D^+ - D^-)v^2(t-T)} - 1}{e^{(D^+ - D^-)v^2(t-T)}D^- - D^+}D^+D^-, \\
C(t) &= \int_t^T e^{\int_t^s f(x)ds} g(s)ds = \frac{1}{\Delta(D^- - D^+e^{2\Delta(T-t)})} \\
&\quad \cdot \left(-\frac{\gamma}{4}((D^+ + D^-)e^{\Delta(T-t)} - D^+e^{2\Delta(T-t)} - D^-) - (k\bar{\delta} - \frac{\sigma\nu\gamma}{2})D^+D^-(e^{\Delta(T-t)} - 1)^2\right), \\
A(t) &= \int_T^t (-\frac{1}{8}\gamma^2\sigma^2 - k\bar{\delta}C(s) - \frac{v^2}{2}(C(s)^2 + H(s)) + \frac{\gamma}{2}v\sigma C(s))ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta =& \sqrt{k^2\frac{v^2}{2\sigma^2} - \frac{vk}{\sigma}}, \\
D^{\pm} =& \frac{k - \frac{v}{2\sigma} \pm \sqrt{k^2 + \frac{v^2}{2\sigma^2} - \frac{vk}{\sigma}}}{v^2}, \\
f(t) =& -k + v^2H(t) + \frac{v}{2\sigma}, \\
g(t) =& -\frac{\gamma}{4} + k\bar{\delta}H(t) - \frac{\gamma\sigma v}{2}H(t).
\end{aligned}
$$

Then

$$
\mathbb{E}_t^A[e^{Y_T}] = F(t, y, \delta, \delta^2; \gamma) = e^{A(t) + B(t)y + C(t)\delta + \frac{H(t)}{2}\delta^2}.
$$

Next, we consider the first derivative of F with respect to γ to obtain $\mathbb{E}_t^A[e^{Y_T}Z_T]$. Note that

$$\begin{aligned}
\frac{dB(t)}{d\gamma} =& \frac{dH(t)}{d\gamma} = 0, \\
\frac{dC(t)}{d\gamma} =& \int_t^T e^{\int_t^s f(x)dx}[-\frac{1}{4} - \frac{\sigma v}{2}H(s)]ds, \\
\frac{dA(t)}{d\gamma} =& \int_T^t (-\frac{1}{4}\sigma^2\gamma - k\bar{\delta}\frac{dC(t)}{d\gamma} - v^2 C(s)\frac{dC(s)}{d\gamma} + \frac{1}{2}v\sigma C(s) + \frac{\gamma}{2}v\sigma\frac{dC(s)}{d\gamma})ds.
\end{aligned}$$

Hence

$$\mathbb{E}_t^A[e^{Y_T}Z_T] = -\frac{2}{\sigma}\frac{d}{d\gamma}\mathbb{E}_t^A[e^{Y_T}] = -\frac{2}{\sigma}\frac{d}{d\gamma}F(t, y, \delta, \delta^2; \gamma).$$

Finally, the stock price in the equilibrium is given by

$$\begin{aligned}
\hat{S}_t &= \frac{\mathbb{E}_t^A[\pi_T^A D_T]}{\mathbb{E}_t^A[\pi_T^A]} = \frac{\mathbb{E}_t^A[\pi_T^A D_T]}{F} = D_0 + \mu_i^A T - 2\frac{\frac{dF}{d\gamma}}{F} \\
&= D_0 + \mu^A T - 2(\frac{dA(t)}{d\gamma} + \frac{dy}{d\gamma} + \frac{dC(t)}{d\gamma}\delta_t) \\
&= D_0 + \mu^A T - 2(\frac{dA(t)}{d\gamma} - \frac{1}{2}Z_t + \frac{dC(t)}{d\gamma}\delta_t).
\end{aligned}$$

Since $Y_t = \int_0^t (\frac{\delta_s}{2\sigma} - \frac{\gamma\sigma}{2})dZ_s + \int_0^t(-\frac{\delta_s^2}{4\sigma^2})ds$ and $Y_t = y$, we have $dy/d\gamma = -1/2\sigma Z_t$. By $D_t = D_0 + \mu^A t + \sigma Z_t$ (μ^A is constant), we obtain

$$\hat{S}_t = D_t + \mu^A(T-t) - 2(\frac{dA(t)}{d\gamma} + \frac{dC(t)}{d\gamma}\delta_t). \tag{A.1}$$

In case δ_t is constant, i.e., $v = k = 0$ and $\delta_t \equiv \delta_0$, we find that $dA(t)/d\gamma = -\sigma^2\gamma/4(t-T)$ and $dC(t)/d\gamma = -1/4(T-t)$. Thus, $\hat{S}_t = D_t + \mu^A(T-t) + (\delta_0/2 - \sigma^2\gamma/2)(T-t)$. This is the equilibrium price of stock 1 in the case of constant disagreement.

Since $H(t) \to 0$ as $t \to T$, we see that $dC(t)/d\gamma$ is negative when $T-t$ is small. Thus, it follows (A.1) that the instantaneous volatility of the stock price $\sigma_{\hat{S}} = \sigma - 2\frac{dC(t)}{d\gamma}\nu$, is greater than the dividend volatility σ when $T-t$ is small.

A.2 With Circuit Breakers

Because the two dividend processes are independent and we assume no leverage constraints when the market is halted, the market clearing prices for the two stocks are independent of each other. Hence, we only need to consider one stock and surpass the subscription dependence of i.

The disagreement δ_t is stochastic following (4), therefore $\mu_t^B = \delta_t + \mu^A$ is stochastic as well. In the presence of a circuit breaker, we solve for the market clearing price when the market is halted. To do so, we solve the utility maximization problem

$$\max_{\theta_\tau^A} \mathbb{E}_\tau^A[-e^{-\gamma(W_T^A)}],$$

subject to $W_T^A = \theta_\tau(D_T - S_\tau) + W_\tau^A$, where W_t^A is the wealth of agent A at time t.

Using the dynamics $D_T = D_\tau + \mu^A(T-\tau) + \sigma(Z_T - Z_\tau)$, we obtain the optimal portfolio of agent A as follows.

$$\theta_\tau^A = \frac{D_\tau - S_\tau + \mu^A(T-\tau)}{\gamma\sigma^2(T-\tau)}. \tag{A.2}$$

Next, we study the utility maximization problem of agent B:

$$\max_{\theta_\tau^B} \mathbb{E}_\tau^B[-e^{-\gamma(W_\tau^B + \theta_\tau^B(D_T - S_\tau))}].$$

We first prove the following lemma.

Lemma Suppose θ is a constant, then

$$\mathbb{E}_t^B[e^{-\gamma\theta D_T}] = e^{A(t,\theta)+B(t,\theta)D_t+C(t,\theta)\delta_t},$$

where

$$
\begin{aligned}
A(t,\theta) &= \gamma\theta\mu^A(t-T) - \frac{\sigma^2}{2}\gamma^2\theta^2(t-T) + (-\gamma\theta\bar{\delta} + \frac{\nu\sigma\gamma^2\theta^2}{\tilde{k}})(T-t-\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}}), \\
&\quad + \frac{\nu^2\gamma^2\theta^2}{2\tilde{k}^2}(T-t-2\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}} + \frac{1-e^{2\tilde{k}(t-T)}}{2\tilde{k}}), \\
B(t,\theta) &= -\gamma\theta, \\
C(t,\theta) &= \frac{-\gamma\theta}{\tilde{k}}(1-e^{\tilde{k}(t-T)}).
\end{aligned}
$$

with $\tilde{k} = k - \nu/\sigma$. In particular, if $\tilde{k} = 0$, then

$$
\begin{aligned}
A(t,\theta) &= \gamma\theta\mu^A(t-T) - \frac{\sigma^2}{2}\gamma^2\theta^2(t-T) + \gamma^2\theta^2\nu\sigma(t-T)^2 - \frac{\nu^2\gamma^2\theta^2}{6}(t-T)^3, \\
B(t,\theta) &= -\gamma\theta, \\
C(t,\theta) &= \gamma\theta(t-T).
\end{aligned}
$$

The above lemma can be proved by using the moment generating function of process D_t and δ_t and solving an O.D.E. system. Detailed deviations are omitted here.

By the lemma,

$$
\mathbb{E}^B_\tau[-e^{-\gamma(W^B_\tau+\theta^B_\tau(D_T-S_\tau))}] = -e^{-\gamma W^B_\tau}e^{A(t,\theta)+C(t,\theta)\delta_t}e^{-\gamma\theta^B_\tau(D_t-S_t)}.
$$

Then the F.O.C. with respect to θ^B_τ yields that

$$
\gamma S_t - \gamma D_t + \frac{\partial A(t,\theta)}{\partial\theta} + \frac{\partial C(t,\theta)}{\partial\theta}\delta_t = 0
$$

or

$$
S_\tau - D_\tau + \mu^A(\tau-T) - \frac{1}{k}(1-e^{k(\tau-T)})\delta_\tau - \bar{\delta}(T-\tau-\frac{1-e^{\tilde{k}(\tau-T)}}{\tilde{k}}) + \theta^B_\tau I(\tau) = 0, \quad \text{(A.3)}
$$

where

$$I(t) = -\gamma\sigma^2(t-T) + \frac{2\nu\sigma\gamma}{\tilde{k}}(T-t-\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}}) + \frac{\nu^2\gamma}{\tilde{k}^2}(T-t-2\frac{1-e^{\tilde{k}(t-T)}}{\tilde{k}} + \frac{1-e^{2\tilde{k}(t-T)}}{2\tilde{k}}).$$

It follows (A.2) that

$$S_\tau = D_\tau + \mu^A(T-\tau) - \theta_\tau^A\gamma\sigma^2(T-\tau). \tag{A.4}$$

Together with (A.3) and the market clearing condition $\theta_\tau^A + \theta_\tau^B = 1$, we obtain the optimal shares holding of Type A for stock 1 at the time of market closure.

$$\theta_\tau^A = \frac{-\frac{1}{\tilde{k}}(1-e^{k(\tau-T)})\delta_\tau - \bar{\delta}(T-t-\frac{1-e^{\tilde{k}(\tau-T)}}{\tilde{k}}) + I(\tau)}{I(\tau) + \gamma\sigma^2(T-t)}. \tag{A.5}$$

Therefore, we find the market clearing price S_τ by (A.4) where θ_τ^A is given by (A.5).

In particular, in the case $\tilde{k} = 0$ (or $k = \nu/\sigma$),

$$\theta_\tau^A = \frac{1}{\gamma}\frac{(\gamma\sigma^2 - \gamma\nu\sigma(\tau-T) + \frac{\nu^2\gamma}{3}(\tau-T)^2 - \delta_\tau)}{-\nu\sigma(\tau-T) + \frac{\nu^2}{3}(\tau-T)^2 + 2\sigma^2},$$

and substituting it into (A.2), it follows that

$$S_\tau = D_\tau + \mu^A(T-\tau) + \frac{\gamma\sigma^2 - \gamma\nu\sigma(\tau-T) + \frac{\nu^2\gamma}{3}(\tau-T)^2 - \delta_\tau}{-\nu\sigma(\tau-T) + \frac{\nu^2}{3}(\tau-T)^2 + 2\sigma^2}\sigma^2(\tau-T).$$

It is worthy mentioning that given positive disagreement this equilibrium price is strictly ($\tau < T$) lower than the equilibrium price in the absence of circuit breakers (constant disagreement): $\hat{S}_\tau = D_\tau + \mu^A(T-\tau) - \gamma\hat{\theta}_\tau^A\sigma^2(T-\tau)$.

In fact, for a relative large positive δ_0 and small ν (say less than half of the volatility σ), the coefficient of δ_t in (A.4) can be always less than the coefficient of δ_t in $\hat{S}_\tau$. Thus along with a small γ, we can always have $S_1^\tau < \hat{S}_{1,t}$. Under these conditions, the equilibrium price with circuit breakers can be always smaller than the price without circuit breakers.

Next, we compute the indirect utility functions that will be used to find the equilibrium prices for any time $t < \tau$. Denote

$$V^B(\tau, W^B_\tau) = \max_{\theta^B_\tau} \mathbb{E}^B_\tau[e^{-\gamma(W^B_\tau + \theta^B_\tau(D_T - S_\tau))}] = e^{-\gamma W^B_\tau} e^{-\gamma G^B_\tau},$$

where

$$-\gamma G^B = -\gamma \theta^B_\tau (D_\tau - S_\tau) + A(\tau, \theta^B_\tau) + C(\tau, \theta^B_\tau)\delta_\tau.$$

Denote

$$V^A(\tau, W^A_\tau) = \max_{\theta^A_\tau} \mathbb{E}^A_\tau[e^{-\gamma(W^A_\tau + \theta^A_\tau(D_T - S_\tau))}] = e^{-\gamma W^A_\tau} e^{G^A_\tau},$$

where

$$-\gamma G^A_\tau = -\gamma \theta^A_\tau (D_\tau - S_\tau) - \gamma \theta^A_\tau \mu^A (T-\tau) + \frac{(\gamma^2 \theta^A_\tau)^2}{2}\sigma^2(T-\tau) = -\frac{\gamma^2(\theta^A_\tau)^2}{2}\sigma^2(T-\tau).$$

Then the equilibrium price of stock 1 is obtained by

$$S_{1,t} = \frac{\mathbb{E}^A_t[\pi^A_{1,T\wedge\tau} S_{1,T\wedge\tau}]}{\mathbb{E}^A_t[\pi^A_{1,T\wedge\tau}]}, \quad t < T\wedge\tau,$$

where $\pi^A_{1,T\wedge\tau} = e^{-\gamma W^A_{1,T\wedge\tau}} = \eta^{1/2}_{1,T\wedge\tau} e^{-\gamma/2 S_{1,T\wedge\tau}} e^{-\gamma/2(G^A_{T\wedge\tau}/2 + G^B_{T\wedge\tau})}$.

Denote the market clearing price of stock 1 by $S^c_{1,\tau}$. Then

$$S^c_{1,\tau} = D_{1,\tau} + \mu^A_1(T-\tau) - \gamma\theta^A_{1,\tau}\sigma^2(T-\tau).$$

B Stock 2: Jump

Note that for the stock 2 the disagreement is constant. The dividend process follows

$$D_{2,t} = D_{2,0} + \mu_2 t + \sigma Z_{2,t} + \mu_J dN_{2,t}. \tag{B.1}$$

In the text, the equilibrium price $\hat{S}_t$ has been derived when there are no circuit breakers. In this appendix, we derive the market clearing price in the presence of a circuit breaker.

The following formulas are useful for calculations related to a Poisson jump process:

$$\begin{aligned} \mathbb{E}[e^{\alpha N_t} = e^{\lambda_J t(e^{\alpha}-1)}] \\ \mathbb{E}[e^{\alpha N_t} N_t] = e^{\lambda_J t(e^{\alpha}-1)} \lambda_J t e^{\alpha} \\ \mathbb{E}[e^{\alpha J}] = e^{\lambda_J t(\phi(\alpha)-1)} \\ \mathbb{E}[e^{\alpha J} J] = \lambda_J t \phi'(\alpha) e^{\lambda_J t(\phi(\alpha)-1)} \end{aligned}$$

where $J_t = \sum_i^{N_t} Y_i$ and $\phi(\alpha) = \mathbb{E}[e^{\alpha Y}]$.

Suppose that the circuit breaker is triggered at $\tau < T$. We maximize the individual utility of agent $i \in \{A, B\}$ at τ:

$$\max_{\theta^i_{2,\tau}} \mathbb{E}^i_\tau[-\exp(-\gamma(W^i_\tau + \theta^i_{2,\tau}(D_{2,T} - S_{2,\tau})))] \tag{B.2}$$

with the market clearing condition $\theta^A_{2,\tau} + \theta^B_{2,\tau} = 1$.

Let

$$-e^{-\gamma W^i_\tau} e^{-\gamma G^i_\tau} := \mathbb{E}^i_\tau[u(W^i_\tau + \theta^i_{j,\tau}(D_{2,T} - S_{2,\tau}))] = \mathbb{E}^i_\tau[u(W^i_\tau + \theta^i_{2,\tau}(D_{2,T} - D_{2,\tau}) + \theta^i_{2,\tau}(D_{2,\tau} - S_{2,\tau}))].$$

It follows that

$$-\gamma G_\tau^i = -\gamma\theta_{2,\tau}^i\mu_2(T-\tau) - \gamma\theta_{2,\tau}^i(D_{2,\tau} - S_{2,\tau}) + \kappa^A(T-\tau)(e^{-\gamma\theta_{2,\tau}^i\mu_J} - 1). \tag{B.3}$$

The first order conditions with respect to $\theta_{2,\tau}^i$ for $j \in \{1,2\}, i \in \{A,B\}$ and the market clearing conditions yield that

$$0 = D_{2,\tau} - S_{2,\tau} + \mu_2(T-\tau) + \kappa^A\mu_J(T-\tau)e^{-\gamma\theta_{2,\tau}^A\mu_J}, \tag{B.4}$$

$$0 = D_{2,\tau} - S_{2,\tau} + \mu_2(T-\tau) + \kappa^B\mu_J(T-\tau)e^{-\gamma\theta_{2,\tau}^B\mu_J}. \tag{B.5}$$

Thus, $\theta_{2,\tau}^A$ and $\theta_{2,\tau}^B$ solve

$$\mu_2 + \kappa_2^A\mu_J e^{-\gamma\theta_{2,\tau}^A\mu_J} = \mu_2 + \kappa_2^B\mu_J e^{-\gamma\theta_{2,\tau}^B\mu_J} \tag{B.6}$$

and the market clearing condition: $\theta_{2,\tau}^A + \theta_{2,\tau}^B = 1$. It follows that

$$\theta_{2,\tau}^A = \frac{1}{2} - \frac{1}{2\gamma\mu_J}\log(\frac{\kappa^B}{\kappa^A}),$$

identical to (19), the optimal shares holding of stock 2 of agent A, in the absence of circuit breakers.

Then it follows (B.4) that

$$S_{2,\tau} = D_{2,\tau} + \mu_2(T-\tau) + \kappa^A\mu_J(T-\tau)e^{-\gamma\theta_{2,\tau}^A)\mu_J}.$$

There may be a jump of $D_{2,t}$ occurring at $t=\tau$. For such a case, the price of stock 2 is limited by the threshold h and $S_{2,\tau} = h - S_{1,\tau}$. We define $D_{2,\tau}^* \in [D_{2,\tau-}, D_{2,\tau}]$, such than

$$D_{2,\tau}^* = h - S_{1,\tau} - \mu_2(T-\tau) - \kappa^A\mu_J(T-\tau)e^{-\gamma\theta_{2,\tau}^A\mu_J}.$$

Thus, the market clearing price of stock 2 is

$$S_{2,\tau}^c := S_{2,\tau} = h - S_{1,\tau} = D_{2,\tau}^* + \mu_2(T-\tau) + \kappa^A \mu_J (T-\tau) e^{-\gamma \theta_{2,\tau}^A \mu_J}.$$

In addition, substituting the above expression into (B.3), we obtain G_τ^i for the indirect utility functions of Type A and B investors. Moreover,

$$-\frac{\gamma}{2}(G_\tau^A + G_\tau^B) = -\frac{\gamma}{2}\mu_2(T-\tau) - \frac{\gamma}{2}(D_{2,\tau} - S_{2,\tau}^c) + \sqrt{\kappa^A \kappa^B} e^{-\frac{\gamma}{2}\mu_J}(T-\tau) - \frac{\kappa^A + \kappa^B}{2}(T-\tau).$$

C Learning and Heterogeneous Beliefs

Suppose

$$dD_t = \mu_t dt + \sigma d\bar{Z}_t.$$

The dividend D_t is observable but the growth rate μ_t is not. Agent A and agent B infer the value of μ_t through the information of dividend. Assume that

$$d\mu_t = -k(\mu_t - \bar{\mu})dt + \sigma_\mu d\bar{Z}_t,$$

and $\mu_0 \sim N(a_0, b_0)$, a normal distribution with mean a_0, standard deviation b_0. Agent $i \in \{A, B\}$ believes $k = k^i, \bar{\mu} = \bar{\mu}^i, \sigma_\mu = \sigma_\mu^i, a_0 = a_0^i, b_0 = b_0^i$. Both of them learn μ_t through $\{D_s\}_{s=0}^t$. Let $\mu_t^A = \mathbb{E}^A[\mu_t|\{D_s\}_{s=0}^t]$ and $\mu_t^B = \mathbb{E}^B[\mu_t|\{D_s\}_{s=0}^t]$. Then following the standard filtering results, we have (under the assumption: $\mu_t|\{D_s\}_{s=0}^t \sim N(\hat{\mu}, \nu)$)

$$d\mu_t^A = -k^A(\mu_t^A - \bar{\mu}^A)dt + \nu^A dZ_t^A,$$
$$d\mu_t^B = -k^B(\mu_t^B - \bar{\mu}^B)dt + \nu^B dZ_t^B,$$

where $dZ_t^i = \frac{1}{\sigma}(dD_t - \mu_t^i dt), i = A, B$. Then

$$dD_t = \mu_t^A dt + \sigma dZ_t^A,$$

$$dD_t = \mu_t^B dt + \sigma dZ_t^B.$$

Therefore, $Z_t^B + \frac{\delta_t}{\sigma_t}t$ is equal to Z_t^A almost surely, where $\delta_t = \mu_t^B - \mu_t^A$. In other words, $Z_t^B + \frac{\delta_t}{\sigma_t}t$ is a standard Brownian motion under agent A's probability measure $\mathbf{P}^A$.

Thus,

$$d\mu_t^B = -k^B(\mu_t^B - \bar{\mu}^B)dt - \frac{\nu^B}{\sigma}\delta_t dt + \nu^B dZ_t^A.$$

Further assume $k^B = k^A = k$, then

$$d\delta_t = -(k + \frac{\nu^B}{\sigma})\delta_t + k\bar{\delta}dt + (\nu^B - \nu^A)dZ_t^A = -(k + \frac{\nu^A}{\sigma})\delta_t + k\bar{\delta} + (\nu^B - \nu^A)dZ_t^B,$$

where $\bar{\delta} = \bar{\mu}^A - \bar{\mu}^B$ is a constant.

In the above we derives the general dynamics of stochastic disagreement under learning. To validate the setting adopted in this paper, we let $\nu^A = 0$,$k^A = 0$, and $\mu_t^A = \mu^A$. That is, we assume that Type A investors take the long-time mean of the growth rate as the estimation and impose no learning. Then it follows that

$$\begin{aligned} d\delta_t = d(\mu_t^B - \mu^A) &= -(k^B + \frac{\nu^B}{\sigma})\delta_t dt - k^B(\mu^A - \bar{\mu}^B)dt + \nu^B dZ_t^A \\ &= -k^B\delta_t dt + k^B(\bar{\mu}^B - \mu^A)dt + \nu^B dZ_t^B. \end{aligned}$$

Further, let $k^B + \nu^B/\sigma = k$, $\nu^B = \nu$, and $(\bar{\mu}^B - \mu^A)/k = \bar{\delta}$, we reach the mean-reverting disagreement process assumed in the paper.

D The Case of Correlated Dividend Processes

To impose correlation between dividend processes, we assume that: under $\mathbf{P}^A$,

$$dD_{1,t} = \mu_1^A dt + \sigma_1 dZ_t, \tag{D.1}$$

$$dD_{2,t} = \mu_2 dt + \sigma_2 dZ_t + \mu_J dN_t, \tag{D.2}$$

and under $\mathbf{P}^B$:

$$dD_{1,t} = \mu_1^B dt + \sigma_1 dZ_t^B, \tag{D.3}$$

$$dD_{2,t} = \mu_2 dt + \frac{\sigma_2}{\sigma_1}\delta_t dt + \sigma_2 dZ_t^B + \mu_J dN_t^B, \tag{D.4}$$

where $\mu_1^B = \mu_1^A + \delta_t$ and

$$d\delta_t = -k(\delta_t - \bar{\delta})dt + \nu dZ_t,$$

or

$$d\delta_t = -k(\delta_t - \bar{\delta})dt + \frac{\nu}{\sigma_1}\delta_t dt + \nu dZ_t^B.$$

Then the two dividend processes are correlated with instantaneous correlation (under $\mathbf{P}^A$)

$$\rho = \frac{\sigma_2}{\sqrt{\sigma_2^2 + \mu_J^2 \kappa^A}}$$

or (under $\mathbf{P}^B$)

$$\rho = \frac{\sigma_2}{\sqrt{\sigma_2^2 + \mu_J^2 \kappa^B}}$$

D.1 The equilibrium prices without circuit breakers

The pricing formula has the same expression as that of the uncorrelated case.

$$\hat{S}_{j,t} = \mathbb{E}_t^A \left[\frac{\pi_T^A D_{j,T}}{\mathbb{E}_t^A [\pi_T^A]} \right], j = 1, 2,$$

where $\pi_T^A = \gamma\zeta\mathbb{E}_t^A[\eta_T^{1/2} \cdot e^{-\frac{\gamma}{2}(D_{1,T}+D_{2,T})}]$. However, the two prices cannot be evaluated separately any more because the two dividend processes are correlated ($\sigma_2 \neq 0$). It is not difficult to derive a closed-form solution, though.

D.2 The equilibrium prices with circuit breakers

What we need are the market clearing prices when the market is early closed due to the circuit breaker.

Agent A needs to maximize the individual utility function

$$\max_{\theta_{1,\tau}^A,\theta_{2,\tau}^A} \mathbb{E}_t^A[-e^{-\gamma\theta_{1,\tau}^A(D_{1,T}-S_{1,\tau})+\theta_{2,\tau}^A(D_{2,T}-S_{2,\tau})}].$$

It results in a first order condition:

$$-\gamma(D_1 - S_1) - \gamma\mu_1^A(T-\tau) + \gamma^2(\theta_1^A\sigma_1 + \theta_2^A\sigma_2)\sigma_1(T-\tau) = 0, \tag{D.5}$$

$$-\gamma(D_2 - S_2) - \gamma\mu_2(T-\tau) + \gamma^2(\theta_1^A\sigma_1 + \theta_2^A\sigma_2)\sigma_2(T-\tau) - \gamma\mu_J\kappa^A e^{-\gamma\theta_2^A\mu_J} = 0. \tag{D.6}$$

For agent B, the maximization problem is

$$\max_{\theta_{1,\tau}^B,\theta_{2,\tau}^B} \mathbb{E}_t^B[-e^{-\gamma\theta_{1,\tau}^B(D_{1,T}-S_{1,\tau})+\theta_{2,\tau}^B(D_{2,T}-S_{2,\tau})}].$$

We first evaluate the expectation for any real numbers x and y:

$$\mathbb{E}_t^B[e^{x\int_t^T \delta_s ds + y(Z_T^B - Z_t^B)}] = e^{-x\int_0^t \delta_s ds - yZ_t^B} e^{A(t)+C\delta_t},$$

where

$$A(t) = \frac{\beta^2}{2}(T-\tau) + k\bar{\delta}\int_t^T C(s)ds + \frac{\nu^2}{2}\int_t^T C(s)^2 ds + \beta\nu\int_t^T C(s)ds,$$
$$C(s) = \frac{\alpha}{k-\frac{\nu}{\sigma}}(1-e^{(k-\frac{\nu}{\sigma_1})(\tau-T)}).$$

With $y = -\gamma(\theta_1^B\sigma_1 + \theta_2^B\sigma_2)$ and $x = -\gamma(\theta_1^B + \theta_2^B\frac{\sigma_2}{\sigma_2})$, the first order conditions for the maximization problem are

$$-\gamma(D_{1,\tau} - S_{1,\tau}) - \gamma\mu_1^A(T-\tau) + \frac{dA}{d\theta_1^B} + \frac{dC}{d\theta_{1,\tau}^B}\delta_t = 0, \quad \text{(D.7)}$$
$$-\gamma(D_2 - S_2) - \gamma\mu_2(T-\tau) - \gamma\kappa^B\mu_J(T-\tau)e^{-\gamma\theta_{2,\tau}^B\mu_J} + \frac{dA}{d\theta_{2,\tau}^B} + \frac{dC}{d\theta_{2,\tau}^B}\delta_t = 0 \quad \text{(D.8)}$$

Along with the market clearing condition $\theta_{j,\tau}^A + \theta_{j,\tau}^B = 1, j = 1,2$, the four first order conditions determine the solution $S_{1,\tau}^*, S_{2,\tau}^*, (\theta_{1,\tau}^A)^*, (\theta_{2,\tau}^A)^*$, that are the market clearing prices and the shares holding at the market early closure time, respectively.

The equilibrium stock prices at $t < \tau$ are given by the same formulas as in the case of uncorrected dividends and they can be evaluated numerically.

E Solution of the Fixed Point Problem

We prove the existence and uniqueness of a solution to the fixed point problem. First of all, based on the explicit expressions of the prices, we restrict the model parameters and the initial conditions (e.g. $D_{1,0}, D_{2,0}$) and assume that both $\hat{S}_{j,0}$(the price without circuit breakers) and $S^c_{j,0}$ (the market clearing price) are positive for each $j = 1, 2$.

Recall that $S_{1,0}, S_{2,0}$ impact on valuation of the expectations through the sum $S_{1,0} + S_{2,0}$ only. When the initial stock prices are $S_{1,0}$ and $S_{2,0}$, the threshold h is $(S_{1,0} + S_{2,0})(1 - \alpha)$. So we define

$$f_j(S_{1,0} + S_{2,0}) = \frac{\mathbb{E}^A_0[\pi^A_{T\wedge\tau} D_{j,T\wedge\tau}]}{\pi^A_0}, j = 1, 2.$$

and define a function $f : \mathcal{R} \to \mathcal{R}^2$ such that $f(S_{1,0} + S_{2,0}) = (f_1(S_{1,0} + S_{2,0}), f_2(S_{1,0} + S_{2,0}))'$, where $'$ denotes transpose of a vector. Then the fixed point problem is expressed as follows.

$$(S_{1,0}, S_{2,0})' = f(S_{1,0} + S_{2,0}).$$

Define $g(x) = f_1(x) + f_2(x) - x$ where $x \in \mathcal{R}$. When the threshold is zero, the equilibrium prices are the prices with circuit breakers. Therefore, $g(0) = f_1(0) + f_2(0) = \hat{S}_{1,0} + \hat{S}_{2,0} > 0$. In addition, if the threshold is the sum of the market clearing prices $S^c_{1,0} + S^c_{2,0}$, the market is stopped immediately and the equilibrium prices must be the market clearing prices exactly. Thus, $g(\frac{S^c_{1,0}+S^c_{2,0}}{1-\alpha}) = f_1(\frac{S^c_{1,0}+S^c_{2,0}}{1-\alpha}) + f_2(\frac{S^c_{1,0}+S^c_{2,0}}{1-\alpha}) - \frac{S^c_{1,0}+S^c_{2,0}}{1-\alpha} = S^c_{1,0} + S^c_{2,0} - \frac{S^c_{1,0}+S^c_{2,0}}{1-\alpha} < 0$. It can be shown that $g(x)$ is a continuous function. Hence, there exists $x^* \in (0, \frac{S^c_{1,0}+S^c_{2,0}}{1-\alpha})$, such that $g(x^*) = 0$. Thus $f_1(x^*) + f_2(x^*) = x^*$.

Now define $(S^*_{1,0}, S^*_{2,0})' = f(x^*)$. Then $x^* = f_1(x^*) + f_2(x^*) = S^*_{1,0} + S^*_{2,0}$ and

$$(S^*_{1,0}, S^*_{2,0})' = f(x^*) = f(S^*_{1,0} + S^*_{2,0}).$$

Thus $(S^*_{1,0}, S^*_{2,0})' \in \mathcal{R}^2$ is a solution to the fixed problem. The existence is proved.

Next, we show that the solution is unique. To do so, it is sufficient to show that $g(x)$ is monotonic. For the sake of notional simplicity, we ignore super-script "A" of expectations and π_t^A below.

Let $D_0 = D_{1,0} + D_{2,0}$. Given an exogenous threshold h and initial dividend sum value D_0, let $S_t^{h,D_0} = S_{1,t}^c + S_{2,t}^c$, where $S_{1,t}$ and $S_{2,t}$ are market clearing prices; Let $\tau(h, D_0)$ denote the stopping time; let π_t^{h,D_0} be the state price density, i.e.

$$\pi_t^{h,D_0} = (\eta_t)^{1/2} e^{-\frac{\gamma}{2} S_t^{h,D_0}} \cdot e^{\frac{G_t^A + G_t^B}{2}}.$$

We redefine

$$g(x) = g(x; D_0) = \frac{\mathbb{E}[\pi_{\tau(h,D_0)\wedge T} S^{h,D_0}_{\tau(h,D_0)\wedge T}]}{\mathbb{E}[\pi^A_{\tau(h,D_0)\wedge T}]} - x,$$

where $h = x(1 - \alpha)$. Observe that $\tau(h, D_0) = \tau(0, D_0 - h)$ because the stopping time is determined by D_t and δ_t only. Then the market clearing (sum) price $S^{h,D_0}_{\tau(h,D_0)} = S^{0,D_0-h}_{\tau(0,D_0-h)} + h$ by the expressions of $S^c_{j,\tau}$, $j = 1, 2$. In addition, by the definition of G^i_τ, we see that $G^i_{\tau(h,D_0)} = G^i_{\tau(0,D_0-h)}$, $i = A, B$. Therefore

$$\pi^{h,D_0}_{\tau(h,D_0)} = e^{-\frac{\gamma}{2} h} \cdot \pi^{0,D_0-h}_{\tau(0,D_0-h)}. \tag{E.1}$$

Thus,

$$\begin{aligned}
g(x; D_0) &= \frac{\mathbb{E}[\pi^{h,D_0}_{\tau(h,D_0)\wedge T} \cdot (S^{h,D_0}_{\tau(h,D_0)\wedge T} - x)]}{E_0[\pi^{h,D_0}_{\tau(h,D_0)\wedge T}]} \\
&= \frac{\mathbb{E}[\pi^{0,D_0-h}_{\tau(0,D_0-h)\wedge T} \cdot (S^{0,D_0-h}_{\tau(0,D_0-h)\wedge T} - x + h)]}{\mathbb{E}[\pi^{0,D_0-h}_{\tau(0,D_0-h)\wedge T}]} \\
&= g(0; D_0 - h) - x + h = g(0; D_0 - h) - \alpha x.
\end{aligned}$$

Given $h_1 < h_2$, we have $\tau(0, D_0 - h_1) \geq \tau(0, D_0 - h_2)$. Then,

$$
\begin{aligned}
&\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)}] = \mathbb{E}[\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)}|\tau(0, D_0 - h_2)]] = \mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_2)}] \\
&= \mathbb{E}[(\eta_{\tau(0,D_0-h_2)\wedge T})^{1/2} e^{-\frac{\gamma}{2} S^{0,D_0-h_1}_{\tau(0,D_0-h_2)\wedge T}} \cdot e^{\frac{G^A_{\tau(0,D_0-h_2)\wedge T} + G^B_{\tau(0,D_0-h_2)\wedge T}}{2}}] \\
&= \mathbb{E}[(\eta_{\tau(0,D_0-h_2)\wedge T})^{1/2} e^{-\frac{\gamma}{2} S^{0,D_0-h_2}_{\tau(0,D_0-h_2)\wedge T}} \cdot e^{\frac{G^A_{\tau(0,D_0-h_2)\wedge T} + G^B_{\tau(0,D_0-h_2)\wedge T}}{2}} \cdot e^{-\gamma/2(h_2-h_1)}] \\
&= \mathbb{E}[\pi^{0,D_0-h_2}_{0,D_0-h_2}] e^{-\gamma/2(h_2-h_1)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)} \cdot S^{0,D_0-h_1}_{\tau(0,D_0-h_1)}] &= \mathbb{E}[\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)} \cdot S^{0,D_0-h_1}_{\tau(0,D_0-h_1)}|\tau(0, D_0 - h_2)]] \\
&= \mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_2)} \cdot S^{0,D_0-h_1}_{\tau(0,D_0-h_2)}] e^{-\gamma/2(h_2-h_1)} \\
&\geq \mathbb{E}[\pi^{0,D_0-h_2}_{\tau(0,D_0-h_2)} \cdot S^{0,D_0-h_2}_{\tau(0,D_0-h_2)}] e^{-\gamma/2(h_2-h_1)}.
\end{aligned}
$$

Finally, let $x_1 < x_2$ and $h_1 = x_1(1-\alpha), h_2 = x_2(1-\alpha)$. It follows that

$$
\begin{aligned}
g(x_1; D_0) = g(0; D_0 - x_1) - \alpha x_1 &= \frac{\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)\wedge T} \cdot S^{0,D_0-h_1}_{\tau(0,D_0-h_1)\wedge T}]}{\mathbb{E}[\pi^{0,D_0-h_1}_{\tau(0,D_0-h_1)\wedge T}]} - \alpha x_1 \\
&\geq \frac{\mathbb{E}[\pi^{0,D_0-h_2}_{\tau(0,D_0-h_2)\wedge T} \cdot S^{0,D_0-h_2}_{\tau(0,D_0-h_2)\wedge T}]}{\mathbb{E}[\pi^{0,D_0-h_2}_{\tau(0,D_0-h_2)\wedge T}]} - \alpha x_1 \\
&= g(0; D_0 - h_2) = g(x_2; D_0) + \alpha x_2 - \alpha x_1 > g(x_2; D_0).
\end{aligned}
$$

This completes the proof.