Additions to Proofs in “A Rational Expectations Theory of Kinks in Financial Reporting” by I. Guttman, O. Kadan, and E. Kandel (Accounting Review, 2006)

Some parts of the proofs in the published version were omitted to conserve space. This document includes the omitted parts.

Complete Proof of Existence and Uniqueness of a Pooling Interval (including the uniqueness part)

It is sufficient to show that there exists a unique $a \in \mathbb{R}$, such that $E(\tilde{x}|\tilde{x} \in [a, a + \frac{\alpha c}{\beta}]) = a + \frac{3\alpha c}{4\beta}$. As $b = a + \frac{\alpha c}{\beta}$ we shall denote the conditional expectation by $d(a) = E(\tilde{x}|\tilde{x} \in [a, a + \frac{\alpha c}{\beta}])$ instead of $d(a, b)$. Thus, we will show that there exists a unique $a \in \mathbb{R}$, such that $d(a) = a + \frac{3\alpha c}{4\beta}$. The conditional expectation $d(a)$ is the expectation of a truncated normal random variable over the interval $[a, a + \frac{\alpha c}{\beta}]$. It is well known (see Johnson, Kotz and Balakrishnan (1994)) that $d(a)$ may be expressed using the following formula:

$$d(a) = x_0 - \sigma^2 \frac{f(a + \frac{\alpha c}{\beta}) - f(a)}{F(a + \frac{\alpha c}{\beta}) - F(a)} \quad a \in \mathbb{R}. \tag{1}$$

Also, notice that the first derivative of the normal density satisfies:

$$f'(x) = -\frac{x - x_0}{\sigma^2} f(x). \tag{2}$$

The following two lemmas are needed in order to establish the existence of the required a.

Lemma 1 *The following holds for any $s > 0$:*

$$\begin{aligned} \lim_{x \to \infty} \frac{f(x+s)}{f(x)} &= 0 \\ \lim_{x \to -\infty} \frac{f(x+s)}{f(x)} &= \infty. \end{aligned}$$

Proof. For any $s > 0$, and $x \in \mathbb{R}$ we have

$$\frac{f(x+s)}{f(x)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{(x+s-x_0)^2}{2\sigma^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{(x-x_0)^2}{2\sigma^2}} = \exp - \frac{s(2x - 2x_0 + s)}{2\sigma^2}.$$

The result follows by taking the appropriate limits. ■

Lemma 2 *The following holds:*

$$\begin{aligned} \lim_{a \to \infty} [d(a) - a] &= 0 \\ \lim_{a \to -\infty} [d(a) - a] &= \frac{\alpha c}{\beta}. \end{aligned}$$

Proof. By (1), and using L'Hopital's law we have

$$\begin{aligned}
\lim_{a\to\infty} d(a) &= x_0 - \sigma^2 \lim_{a\to\infty} \frac{f(a+\frac{\alpha c}{\beta}) - f(a)}{F(a+\frac{\alpha c}{\beta}) - F(a)} = x_0 - \sigma^2 \lim_{a\to\infty} \frac{f'(a+\frac{\alpha c}{\beta}) - f'(a)}{f(a+\frac{\alpha c}{\beta}) - f(a)} \\
&= x_0 + \sigma^2 \lim_{a\to\infty} \frac{\frac{a+\frac{\alpha c}{\beta}-x_0}{\sigma^2} f(a+\frac{\alpha c}{\beta}) - \frac{a-x_0}{\sigma^2} f(a)}{f(a+\frac{\alpha c}{\beta}) - f(a)} \\
&= x_0 + \lim_{a\to\infty} \frac{(a+\frac{\alpha c}{\beta}-x_0) f(a+\frac{\alpha c}{\beta}) - (a-x_0) f(a)}{f(a+\frac{\alpha c}{\beta}) - f(a)} \\
&= \lim_{a\to\infty} \left[a + \frac{\alpha c}{\beta} \frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta}) - f(a)} \right]
\end{aligned}$$

Now, by plugging $s = \frac{\alpha c}{\beta}$ in Lemma 1 it follows that

$$\lim_{a\to\infty} d(a) - a = \frac{\alpha c}{\beta} \lim_{a\to\infty} \frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta}) - f(a)} = \frac{\alpha c}{\beta} \lim_{a\to\infty} \frac{1}{1 - \frac{f(a)}{f(a+\frac{\alpha c}{\beta})}} = 0,$$

as required.
As for the second part, repeating the previous analysis we obtain

$$\lim_{a\to-\infty} d(a) = \lim_{a\to-\infty} \left[a + \frac{\alpha c}{\beta} \frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta}) - f(a)} \right].$$

Using Lemma 1 it follows that

$$\lim_{a\to-\infty} [d(a) - a] = \frac{\alpha c}{\beta} \lim_{a\to-\infty} \frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta}) - f(a)} = \frac{\alpha c}{\beta} \lim_{a\to-\infty} \frac{1}{1 - \frac{f(a)}{f(a+\frac{\alpha c}{\beta})}} = \frac{\alpha c}{\beta},$$

as required. ■

It is now easy to prove the existence of a required a. Indeed, define $H(a) \equiv d(a) - a - \frac{3\alpha c}{4\beta}$. From Lemma 2 it follows that $\lim\limits_{a\to-\infty} H(a) = \frac{\alpha c}{4\beta} > 0$, and $\lim\limits_{a\to\infty} H(a) = -\frac{3\alpha c}{4\beta} < 0$. Thus, from the continuity of $H(a)$ we conclude that there exists an $a \in \mathbb{R}$ such that $H(a) = 0$.

Our next step is to prove the uniqueness of the chosen a. We shall accomplish this by showing that $H(a)$ is strictly decreasing, namely, that $d'(a) < 1$. For brevity we shall assume that $x_0 = 0$. This shortens the presentation and has no effect on the results.

From Lemma 2: $\lim_{a\to\infty} d'(a) = \lim_{a\to-\infty} d'(a) = 1$. Also, denote $k(a) \equiv d(a) - a$. Notice that, for all $a \in \mathbb{R}$: $0 \le d(a) \le \frac{\alpha c}{\beta}$, and $k'(a) = d'(a) - 1$. Also from Lemma 2: $\lim_{a\to\infty} d(a) = 0$, $\lim_{a\to-\infty} d(a) = \frac{\alpha c}{\beta}$.

Differentiating $d(a)$ and using the fact that $f'(x) = -\frac{x}{\sigma^2} f(x)$ we obtain

$$\begin{aligned} d'(a) &= \frac{(a+\frac{\alpha c}{\beta})f(a+\frac{\alpha c}{\beta}) - af(a)}{F(a+\frac{\alpha c}{\beta}) - F(a)} + \sigma^2 \left(\frac{f(a+\frac{\alpha c}{\beta}) - f(a)}{F(a+\frac{\alpha c}{\beta}) - F(a)} \right)^2 \qquad (3) \\ &= -\frac{a}{\sigma^2} d(a) + \frac{d(a)^2}{\sigma^2} + \frac{\frac{\alpha c}{\beta} f(a+\frac{\alpha c}{\beta})}{F(a+\frac{\alpha c}{\beta}) - F(a)} = \frac{1}{\sigma^2} d(a)k(a) + \frac{\frac{\alpha c}{\beta} f(a+\frac{\alpha c}{\beta})}{F(a+\frac{\alpha c}{\beta}) - F(a)}. \end{aligned}$$

Using this equation we can evaluate $d'(\cdot)$ at $a = -\frac{\alpha c}{2\beta}$. We accomplish this in the next lemma.

Lemma 3 $d'(-\frac{\alpha c}{2\beta}) = \frac{\frac{\alpha c}{\beta} f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})} < 1.$

Proof. From (3) we have:

$$d'(-\frac{\alpha c}{2\beta}) = \frac{1}{\sigma^2} d(-\frac{\alpha c}{2\beta}) k(-\frac{\alpha c}{2\beta}) + \frac{\frac{\alpha c}{\beta} f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})}.$$

Since $f(\frac{\alpha c}{2\beta}) = f(-\frac{\alpha c}{2\beta})$ we have: $d(-\frac{\alpha c}{2\beta}) = 0$; therefore $d'(-\frac{\alpha c}{2\beta}) = \frac{\frac{\alpha c}{\beta} f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})}$. Now, suppose on the contrary that $\frac{\frac{\alpha c}{\beta} f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})} \geq 1$. Since $\frac{\alpha c}{\beta} > 0$, this implies $\frac{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})}{\frac{\alpha c}{\beta}} \leq f(\frac{\alpha c}{2\beta})$. However, by the mean value theorem we have $\frac{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})}{\frac{\alpha c}{\beta}} = \frac{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})}{\frac{\alpha c}{2\beta} - (-\frac{\alpha c}{2\beta})} = f(\xi)$ for some $\xi \in (-\frac{\alpha c}{2\beta}, \frac{\alpha c}{2\beta})$. But since f is normal with mean zero, it follows that $f(\xi) > f(\frac{\alpha c}{2\beta})$ for all $\xi \in (-\frac{\alpha c}{2\beta}, \frac{\alpha c}{2\beta})$. This constitutes a contradiction. ■

In order to proceed we need the following lemma, which uses the symmetry of the normal distribution.

Lemma 4 $k(a) = \frac{\alpha c}{2\beta}$ *if and only if* $a = -\frac{\alpha c}{2\beta}$.

Proof. If $a = -\frac{\alpha c}{2\beta}$ then by the symmetry of f around 0: $f(a) = f(a + \frac{\alpha c}{\beta})$, and thus $d(a) = 0$, and $k(a) = \frac{\alpha c}{2\beta}$.

To prove the "only if" part of the lemma recall that $d(a) = \frac{\int_a^{a+\frac{\alpha c}{\beta}} f(x)dx}{F(a+\frac{\alpha c}{\beta}) - F(a)}$. Denote $\Delta(a) \equiv d(a) - (a + \frac{\alpha c}{2\beta})$. We may write

$$\begin{aligned} \Delta(a) &= \frac{1}{F(a+\frac{\alpha c}{\beta}) - F(a)} \int_a^{a+\frac{\alpha c}{\beta}} (x - (a + \frac{\alpha c}{2\beta})) f(x) dx \\ &= \frac{\int_a^{a+\frac{\alpha c}{2\beta}} (x - (a+\frac{\alpha c}{2\beta})) f(x) dx + \int_{a+\frac{\alpha c}{2\beta}}^{a+\frac{\alpha c}{\beta}} (x - (a+\frac{\alpha c}{2\beta})) f(x) dx}{F(a+\frac{\alpha c}{\beta}) - F(a)}. \end{aligned}$$

Changing variables in the right hand integral to $\eta = 2a + \frac{\alpha c}{\beta} - x$ we obtain:

$$\Delta(a) = \frac{\int_a^{a+\frac{\alpha c}{2\beta}} (x - (a + \frac{\alpha c}{2\beta}))f(x)dx - \int_a^{a+\frac{\alpha c}{2\beta}} ((a + \frac{\alpha c}{2\beta}) - \eta) f(2a + \frac{\alpha c}{\beta} - \eta)d\eta}{F(a + \frac{\alpha c}{\beta}) - F(a)} \quad (4)$$

$$= \frac{\int_a^{a+\frac{\alpha c}{2\beta}} (x - (a + \frac{\alpha c}{2\beta}))[f(x) - f(2a + \frac{\alpha c}{\beta} - x)]dx}{F(a + \frac{\alpha c}{\beta}) - F(a)}.$$

Now, suppose that $k(a) = \frac{\alpha c}{2\beta}$, namely, $\Delta(a) = 0$, and suppose on the contrary that $a \neq -\frac{\alpha c}{2\beta}$. Consider first the case of $a < -\frac{\alpha c}{2\beta}$. In this case, the symmetry of f around 0 implies that for all $x \in (a, a + \frac{\alpha c}{2\beta}) : f(x) < f(2a + \frac{\alpha c}{\beta} - x)$. But from (4) it follows that $\Delta(a) > 0$ - a contradiction. A similar argument shows that it cannot be the case that $a > -\frac{\alpha c}{2\beta}$. We conclude that $a = \frac{\alpha c}{2\beta}$. ■

The condition $H(a) = 0$ implies that $k(a) = \frac{3\alpha c}{4\beta}$. Thus, Lemma 4 implies that we can assume $a \neq \frac{\alpha c}{2\beta}$. Since $d(a)$ is increasing in a, and $d(\frac{\alpha c}{2\beta}) = 0$, it follows that $d(a) \neq 0$, and $f(a + \frac{\alpha c}{\beta}) \neq f(a)$. Using this observation we can solve (3) for $k(a)$ and obtain

$$k(a) = \frac{\sigma^2 d'(a)}{d(a)} - \frac{\alpha c}{\beta}\sigma^2 \frac{\frac{f(a+\frac{\alpha c}{\beta})}{F(a+\frac{\alpha c}{\beta})-F(a)}}{d(a)} = \frac{\sigma^2 d'(a)}{d(a)} + \frac{\frac{\alpha c}{\beta} f(a + \frac{\alpha c}{\beta})}{f(a + \frac{\alpha c}{\beta}) - f(a)} \quad (5)$$

Differentiating (5) yields

$$k'(a) = \frac{\sigma^2 d''(a)}{d(a)} - \sigma^2 \left(\frac{d'(a)}{d(a)}\right)^2 + \frac{\alpha c}{\beta} \frac{-\frac{a+\frac{\alpha c}{\beta}}{\sigma^2} f(a + \frac{\alpha c}{\beta})[f(a + \frac{\alpha c}{\beta}) - f(a)]}{[f(a + \frac{\alpha c}{\beta}) - f(a)]^2} \quad (6)$$

$$+ \frac{\alpha c}{\beta} \frac{[\frac{a+\frac{\alpha c}{\beta}}{\sigma^2} f(a + \frac{\alpha c}{\beta}) - \frac{a}{\sigma^2} f(a)]f(a + \frac{\alpha c}{\beta})}{[f(a + \frac{\alpha c}{\beta}) - f(a)]^2}$$

$$= \frac{\sigma^2 d''(a)}{d(a)} - \sigma^2 \left(\frac{d'(a)}{d(a)}\right)^2 + \frac{(\frac{\alpha c}{\beta})^2}{\sigma^2} \frac{f(a)f(a + \frac{\alpha c}{\beta})}{[f(a + \frac{\alpha c}{\beta}) - f(a)]^2}$$

The following notation is useful. For all $a \in \mathbb{R}$ denote $Q(a) \equiv \frac{(\frac{\alpha c}{\beta})^2 f(a) f(a+\frac{\alpha c}{\beta})}{[F(a+\frac{\alpha c}{\beta})-F(a)]^2}$. Notice that $Q(a) > 0$ for all a. The next lemma shows that $Q(\cdot)$ is bounded from above by 1.

Lemma 5 *For all $a \in \mathbb{R}$, $Q(a) < 1$.*

Proof. Differentiating Q we obtain for all $a \in \mathbb{R}$:

$$\begin{aligned} Q'(a) &= (\frac{\alpha c}{\beta})^2 \frac{-\frac{a}{\sigma^2} f(a) f(a+\frac{\alpha c}{\beta}) - \frac{a+\frac{\alpha c}{\beta}}{\sigma^2} f(a) f(a+\frac{\alpha c}{\beta})}{[F(a+\frac{\alpha c}{\beta}) - F(a)]^2} \\ &\quad -(\frac{\alpha c}{\beta})^2 \frac{2(f(a+\frac{\alpha c}{\beta}) - f(a)) f(a) f(a+\frac{\alpha c}{\beta})}{[F(a+\frac{\alpha c}{\beta}) - F(a)]^3} \\ &= -\frac{a}{\sigma^2} Q(a) - \frac{a+\frac{\alpha c}{\beta}}{\sigma^2} Q(a) + \frac{2}{\sigma^2} d(a) Q(a) \\ &= \frac{Q(a)}{\sigma^2} \left[2(d(a)-a) - \frac{\alpha c}{\beta} \right]. \end{aligned}$$

Since $Q(a) > 0$ for all a, it follows that $Q'(a) = 0$ if and only if $d(a) - a = \frac{\alpha c}{2\beta}$. And from Lemma 4 we conclude that $Q'(a) = 0$ if and only if $a = -\frac{\alpha c}{2\beta}$. We shall now show that $a = -\frac{\alpha c}{2\beta}$ is a global maximum for Q. Indeed, differentiating Q once again we obtain

$$Q''(a) = \frac{Q'(a)}{\sigma^2} \left[2(d(a)-a) - \frac{\alpha c}{\beta} \right] + \frac{Q(a)}{\sigma^2} [2(d'(a)-1)].$$

It follows that

$$Q''(-\frac{\alpha c}{2\beta}) = \frac{Q(\frac{\alpha c}{2\beta})}{\sigma^2} [2(d'(-\frac{\alpha c}{2\beta}) - 1)].$$

Thus, from Lemma 3 we conclude that $Q''(-\frac{\alpha c}{2\beta}) < 0$, and $a = -\frac{\alpha c}{2\beta}$ is a global maximum. Given this, in order to show that $Q(a) < 1$ for all a, it is sufficient to show that $Q(-\frac{\alpha c}{2\beta}) < 1$. Indeed:

$$Q(-\frac{\alpha c}{2\beta}) = \frac{(\frac{\alpha c}{\beta})^2 f(\frac{\alpha c}{2\beta})^2}{[F(a+\frac{\alpha c}{\beta}) - F(a)]^2}.$$

However, from Lemma 3 we know that $d'(-\frac{\alpha c}{2\beta}) = \frac{\frac{\alpha c}{\beta} f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})} < 1$; therefore $Q(-\frac{\alpha c}{2\beta}) = (d'(-\frac{\alpha c}{2\beta}))^2 < 1$. ■

We are now ready to show that $H(a) = d(a) - a - \frac{3\alpha c}{4\beta}$ is strictly decreasing in a, namely, that $d'(a) < 1$ for all $a \in \mathbb{R}$. We will show that this is true for all $a \in (-\infty, -\frac{\alpha c}{2\beta}]$. A parallel argument shows that this assertion is true also for all $a \in (-\frac{\alpha c}{2\beta}, \infty)$.

Suppose on the contrary that $d'(a) \geq 1$ for some a values in $(-\infty, -\frac{\alpha c}{2\beta}]$. Note that $\lim_{a \to -\infty} d'(a) = 1$, and from Lemma 3, $d'(-\frac{\alpha c}{2\beta}) < 1$. It follows that there exists an $\hat{a} \in (-\infty, -\frac{\alpha c}{2\beta})$ such that $d'(\hat{a}) \geq 1$ and $d''(\hat{a}) = 0$. Substituting $\hat{a}$ in (6) we obtain

$$\begin{aligned} k'(\hat{a}) &\leq \frac{-\sigma^2}{d(\hat{a})^2} + \frac{(\frac{\alpha c}{\beta})^2}{\sigma^2} \frac{f(\hat{a})f(\hat{a}+\frac{\alpha c}{\beta})}{[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} \\ &= -\frac{1}{\sigma^2} \frac{[F(\hat{a}+\frac{\alpha c}{\beta}) - F(\hat{a})]^2}{[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} + \frac{(\frac{\alpha c}{\beta})^2}{\sigma^2} \frac{f(\hat{a})f(\hat{a}+\frac{\alpha c}{\beta})}{[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} \\ &= \frac{(F(\hat{a}+\frac{\alpha c}{\beta}) - F(\hat{a}))^2}{\sigma^2[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} \left[(\frac{\alpha c}{\beta})^2 \frac{f(\hat{a})f(\hat{a}+\frac{\alpha c}{\beta})}{(F(\hat{a}+\frac{\alpha c}{\beta}) - F(\hat{a}))^2} - 1 \right] \\ &= \frac{(F(\hat{a}+\frac{\alpha c}{\beta}) - F(\hat{a}))^2}{\sigma^2[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} (Q(\hat{a}) - 1). \end{aligned}$$

But from Lemma 5 it follows that $Q(\hat{a}) - 1 < 0$, and therefore: $k'(\hat{a}) < 0$, or equivalently $d'(\hat{a}) < 1$ - a contradiction. This shows that there is a unique a that satisfies $H(a) = 0$, as required. ■

Comparative Statics of the Pooling Interval by σ

For brevity we assume $x_0 = 0$.[1] Since we are interested in the impact of σ, we view a and d as functions of σ. Define

$$H(a, \sigma) \equiv d(a, \sigma) - a - \frac{3\alpha c}{4\beta}.$$

The relation between a and σ is given by the implicit equation $H(a, \sigma) = 0$. We have shown that $\frac{\partial H(a,\sigma)}{\partial a} < 0$ for all $a, \sigma \in \mathbb{R}$. By the implicit function theorem we have

$$\frac{\partial a(\sigma)}{\partial \sigma} = -\frac{\frac{\partial H(a,\sigma)}{\partial \sigma}}{\frac{\partial H(a,\sigma)}{\partial a}}.$$

Thus, to show that $\frac{\partial a(\sigma)}{\partial \sigma} < 0$ it is sufficient to show that $\frac{\partial H(a,\sigma)}{\partial \sigma} < 0$. We have

[1] A different choice of x_0 would shift $a(\sigma)$ by a constant, and thus it has no effect on $\frac{\partial a}{\partial \sigma}$.

$$\begin{aligned}
\frac{\partial H(a,\sigma)}{\partial \sigma} &= \frac{\partial d(a,\sigma)}{\partial \sigma} = \frac{\partial}{\partial \sigma}\frac{\int_a^{a+\frac{\alpha c}{\beta}} x f(x)dx}{F(a+\frac{\alpha c}{\beta})-F(a)} = \frac{\partial}{\partial \sigma}\frac{\frac{1}{\sqrt{2\pi\sigma^2}}\int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\frac{1}{\sqrt{2\pi\sigma^2}}\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx} \\
&= \frac{\int_a^{a+\frac{\alpha c}{\beta}} \frac{x^3}{\sigma^3}e^{-\frac{x^2}{2\sigma^2}}dx \cdot \int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx - \int_a^{a+\frac{\alpha c}{\beta}} \frac{x^2}{\sigma^3}e^{-\frac{x^2}{2\sigma^2}}dx \cdot \int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\left[\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx\right]^2} \\
&= \frac{1}{\sigma^3}\left[\frac{\int_a^{a+\frac{\alpha c}{\beta}} x^3e^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx} - \frac{\int_a^{a+\frac{\alpha c}{\beta}} x^2e^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx}\cdot\frac{\int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{\pi^2}{2\sigma^2}}dx}\right] \\
&= \frac{1}{\sigma^3}\left[\frac{\int_a^{a+\frac{\alpha c}{\beta}} x^3f(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)d\pi} - \frac{\int_a^{a+\frac{\alpha c}{\beta}} x^2f(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)dx}\cdot\frac{\int_a^{a+\frac{\alpha c}{\beta}} xf(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)dx}\right] \\
&= \frac{1}{\sigma^3}\left[E(\tilde{x}^3|a\le\tilde{x}\le a+\frac{\alpha c}{\beta}) - E(\tilde{x}^2|a\le\tilde{x}\le a+\frac{\alpha c}{\beta})\cdot E(\tilde{x}|a\le\tilde{x}\le a+\frac{\alpha c}{\beta})\right] \\
&= \frac{1}{\sigma^3}Cov(\tilde{x}^2,\tilde{x}|a\le\tilde{x}\le a+\frac{\alpha c}{\beta}).
\end{aligned}$$

Thus, the sign of $\frac{\partial H(a,\sigma)}{\partial\sigma}$ is equal to the sign of the $Cov(\tilde{y},\tilde{y}^2)$, where $\tilde{y}$ is a random variable obtained from a truncation of $\tilde{x}$ between a and $a+\frac{\alpha c}{\beta}$. It can be shown that this covariance is strictly negative, as required. The proof is contained below in this document.
Given that $a(\sigma)$ is decreasing in σ, we know that $a_0 \equiv \lim_{\sigma\to 0} a(\sigma)$ exists. It is easy to see that for any fixed $a < x_0$ and $b > a$ we have

$$\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a,b]) = \begin{cases} b & x_0 \notin [a,b] \\ x_0 & x_0\in[a,b] \end{cases}. \tag{7}$$

We claim first that there exists an $a_0 > 0$ such that $a_0+\frac{\alpha c}{\beta} > x_0$. Indeed, suppose on the contrary that $a_0+\frac{\alpha c}{\beta}\le x_0$. This implies by (7) that $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a(\sigma),a(\sigma)+\frac{\alpha c}{\beta}]) = a_0+\frac{\alpha c}{\beta}$, contradicting the fact that for all σ, $E(\tilde{x}|\tilde{x}\in[a(\sigma),a(\sigma)+\frac{\alpha c}{\beta}]) = a(\sigma)+\frac{3\alpha c}{4\beta}$. Now, for all $\varepsilon > 0$ sufficiently small we have: $a_0-\varepsilon+\frac{\alpha c}{\beta} > x_0$. Thus, by (7) we have: $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a_0-\varepsilon,a_0-\varepsilon+\frac{\alpha c}{\beta}]) = x_0$. From the continuity of the conditional expectation and since ε is arbitrary we conclude that $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a(\sigma),a(\sigma)+\frac{\alpha c}{\beta}]) = x_0$. And, hence $\lim_{\sigma\to 0} a(\sigma) = x_0-\frac{3\alpha c}{4\beta}$, as required.

As for the case of $\sigma\to\infty$. For all fixed a and b we have: $E(\tilde{x}|\tilde{x}\in[a,b]\to\frac{a+b}{2}$.

Indeed, by applying L'Hopital's law we obtain

$$\begin{aligned}
\lim_{\sigma\to\infty}\left[x_0-\sigma^2\frac{f(b)-f(a)}{F(b)-F(a)}\right] &= x_0-\lim_{\sigma\to\infty}\sigma^2\frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}}-e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\int_a^b e^{\frac{-(x-x_0)^2}{2\sigma^2}}dx} \\
&= x_0-\frac{1}{b-a}\lim_{\sigma\to\infty}\frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}}-e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\frac{1}{\sigma^2}} \\
&= x_0-\frac{1}{b-a}\lim_{\sigma\to\infty}\frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}}-e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\frac{1}{\sigma^2}} \\
&= x_0-\frac{1}{b-a}\lim_{\sigma\to\infty}\frac{\frac{(b-x_0)^2}{\sigma^3}e^{\frac{-(b-x_0)^2}{2\sigma^2}}-\frac{(a-x_0)^2}{\sigma^3}e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{-\frac{2}{\sigma^3}} \\
&= x_0+\frac{(b-x_0)^2-(a-x_0)^2}{2(b-a)}=\frac{a+b}{2}
\end{aligned}$$

This calculation implies that if $a_\infty \equiv \lim_{\sigma\to\infty} a(\sigma)$ were finite, we would have that $d(a(\sigma)) \to a_\infty + \frac{\alpha c}{2\beta}$ - a contradiction to the fact $d(a(\sigma)) = a(\sigma) + \frac{3\alpha c}{4\beta}$ for all σ. ■

The Sign of a Covariance between Y and Y^2 Obtained from Truncation of a Symmetric R.V.

Notations and assumptions:

Assume that X is a symmetric r.v with mean 0. The density of X is $f(x)$ and the CDF is $F(x)$. Assume that Y is the truncation of X on an interval $[a,b]$. The density of Y is $g(y)$. We have

$$g(y) = \frac{f(y)}{F(b)-F(a)}.$$

Theorem 6 *For all $a \neq -b$, $Cov(Y,Y^2) \neq 0$. Moreover, if $b > a > 0$ or $b > -a > 0$ then $Cov(Y,Y^2) > 0$. If $a < b < 0$ or $-a > b > 0$ then $Cov(Y,Y^2) < 0$.*

Proof:

$$\begin{aligned}
Cov(Y,Y^2) &= Cov(X,X^2|X\in[a,b]) = E(x^3|x\in[a,b]) - E(x^2|x\in[a,b])E(x|x\in[a,b]) \\
&= \int_a^b x^3\frac{f(x)}{F(b)-F(a)}dx - \int_a^b x^2\frac{f(x)}{F(b)-F(a)}dx\int_a^b x\frac{f(x)}{F(b)-F(a)}dx
\end{aligned}$$

Since $F(b)-F(a)=\int_a^b \varphi(x)dx > 0$, the sign of $Cov(Y,Y^2)$ is identical to the sign of

$$J(a,b) \equiv \int_a^b f(x)dx \int_a^b x^3 f(x)dx - \int_a^b x^2 f(x)dx \int_a^b x f(x)dx \tag{8}$$

Consider first the case: $b > -a > 0$

The symmetry of f implies that $\int_a^{|a|} x^3 f(x)dx = \int_a^{|a|} x f(x)dx = 0$.Therefore, we can rewrite (8) as:

$$J(a,b) = \int_{|a|}^b x^3 f(x)dx \int_a^b f(x)dx - \int_{|a|}^b x f(x)dx \int_a^b x^2 f(x)dx \tag{9}$$

If it were the case that $b = -a$ then $\frac{\partial J(a,b)}{\partial b} = 0$ and hence $Cov(Y, Y^2) = 0$. Next we show that for all $b > -a > 0$, $\frac{\partial J(a,b)}{\partial b} > 0$. This will indicate that for all $b > -a > 0$, $Cov(Y, Y^2) > 0$.

Differentiation yields

$$\begin{aligned}
\frac{\partial J(a,b)}{\partial b} &= \frac{\partial}{\partial b}\left[\int_{|a|}^b x^3 f(x)dx \int_a^b f(x)dx - \int_{|a|}^b x f(x)dx \int_a^b x^2 f(x)dx\right] \\
&= b^3 f(b) \int_a^b f(x)dx + f(b) \int_{|a|}^b x^3 f(x)dx - b f(b) \int_a^b x^2 f(x)dx - b^2 f(b) \int_{|a|}^b x f(x)dx
\end{aligned}$$

Dividing by $f(b)$ will not change the sign. We obtain

$$\begin{aligned}
& b^3 \int_a^b f(x)dx + \int_{|a|}^b x^3 f(x)dx - b \int_a^b x^2 f(x)dx - b^2 \int_{|a|}^b x f(x)dx \\
= \;& b \int_a^{|a|} \left(b^2 - x^2\right) f(x)dx + \int_{|a|}^b \left(b^3 + x^3 - bx^2 - b^2 x\right) f(x)dx \\
= \;& b \int_a^{|a|} \left(b^2 - x^2\right) f(x)dx + \int_{|a|}^b (b^2 - x^2)(b - x) f(x)dx
\end{aligned}$$

Now, since $|a| < b$ we have: $b^2 > x^2$ for all $a \in [a, |a|]$, therefore, the first integral is positive. Also, it is clear that the second integral is positive. This implies that the whole expression is positive as required.

A symmetric idea is used in the case: $-a > b > 0$ to show that $Cov(Y, Y^2) < 0$.

Consider now the case $b > a > 0$.

The following is a standard theorem in probability theory:

Proposition 7 *Let Y be a random variable, and let $h_1(\cdot)$ and $h_2(\cdot)$ be two increasing functions of Y. Then $Cov(h_1(Y), h_2(Y)) > 0$.*

We can set $h_1(y) = y$, and $h_2(y) = y^2$. It is trivial that h_1 is increasing. The fact that h_2 is increasing follows since $b > a > 0$. This yields the required result.

A parallel argument is used for the case $a < b < 0$.

Out of Equilibrium Beliefs

Contrary to the separating equilibrium, the partially pooling, discontinuous equilibrium relies strongly on out of equilibrium pricing. Some reports will never appear in equilibrium. Since Bayes rule does not apply, the modeler has some leeway in prescribing the beliefs associated with these reports. In the paper we assumed that if investors observe an out-of-equilibrium report $x^R \in (a+\frac{\alpha c}{2\beta}, b) \cup (b, b+\frac{\alpha c}{2\beta})$ then they believe that the manager is "mistakenly" playing the benchmark linear equilibrium. These out of equilibrium beliefs, while sufficient to support the equilibrium, are not necessary, namely, they are too strong. Below, we provide a necessary and sufficient condition for out of equilibrium beliefs to support the partially pooling equilibrium. The fundamental idea here is to find the pricing function that will make types 'a' and 'b' just indifferent between their equilibrium action of reporting b, and providing an out of equilibrium report. It turns out that there exists a unique pricing function of this type.

Our first step is the next lemma showing that a sufficient condition for an out-of-equilibrium pricing function to support our partially pooling equilibrium, is that the types 'a' and 'b' are indifferent between following the equilibrium strategy and deviating from it to an out of equilibrium report.

Lemma 8 *Consider any out-of-equilibrium report $x^R \in (a+\frac{\alpha c}{2\beta}, b) \cup (b, b+\frac{\alpha c}{2\beta})$ combined with an out-of-equilibrium pricing function $P(x^R)$. The following holds:*

1. *If $x^R \in (a+\frac{\alpha c}{2\beta}, b)$, and if type '$a$' is indifferent between the equilibrium report of b, and the out-of-equilibrium report of x^R, then all other types $x' \neq a$ strictly prefer the equilibrium report b over the out-of-equilibrium report x^R.*

2. *If $x^R \in (b, b+\frac{\alpha c}{2\beta})$, and if type '$b$' is indifferent between the equilibrium report of b, and the out-of-equilibrium report of x^R, then all other types $x' \neq b$ strictly prefer the equilibrium report b over the out-of-equilibrium report x^R.*

Proof of Lemma 8

We shall prove Part 1 of the lemma. The proof of Part 2 is symmetric.

Suppose $x^R \in (a+\frac{\alpha c}{2\beta}, b)$ is an out-of-equilibrium report, and let $P(x^R)$ be the price in case a report of x^R is observed. We claim that in this case, if the 'a' type is indifferent between submitting a report of b (equilibrium report) or x^R (deviating), then all other types $x' \neq a$ strictly prefer to stick to their equilibrium report. We shall consider three cases.

Case 1: $x' \in (a, b]$. Since the 'a' type is indifferent between submitting b, and deviating to x^R, we obtain

$$\alpha cd - \beta(b-a)^2 = \alpha P(x^R) - \beta(x^R - a)^2. \tag{10}$$

The payoff to type $x' \in (a, b]$ from reporting x^R is: $\alpha P - \beta(x^R - x')^2$. It follows that the largest benefit from deviating to a report of x^R is incurred when the type is equal to the report, namely: $x' = x^R$. In this case, the payoff in case of deviation is αP, while the payoff on the equilibrium path is: $\alpha cd - \beta(x^R - b)^2$. By (10), the difference between the payoff on the equilibrium path, and the payoff in case of deviation is

$$\begin{aligned}\alpha cd - \beta(x^R - b)^2 - \alpha P &= -\beta(x^R - b)^2 + \beta(b-a)^2 - \beta(x^R - a)^2 \\ &= 2\beta(x^R - a)(b - x^R) > 0.\end{aligned}$$

Thus, type x' strictly prefers to stick to his equilibrium report.

Case 2: $x' < a$. Since the 'a' type is indifferent between submitting $a + \frac{\alpha c}{2\beta}$, and deviating to x^R we obtain

$$\alpha ca - \beta(\frac{\alpha c}{2\beta})^2 = \alpha P(x^R) - \beta(x^R - a)^2. \tag{11}$$

Now, if type x' follows the equilibrium he obtains: $\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2$. If on the other hand he deviates to x^R he obtains: $\alpha P(x^R) - \beta(x^R - x')^2$. Using (11) we obtain that the difference is

$$\begin{aligned}\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2 - \alpha P(x^R) + \beta(x^R - x')^2 &= \alpha cx' - \alpha ca - \beta(x^R - a)^2 + \beta(x^R - x')^2 \\ &= \beta(a - x')(2x^R - x' - a - \frac{\alpha c}{\beta}) \\ &> \beta(a - x')(2(a + \frac{\alpha c}{2\beta}) - x' - a - \frac{\alpha c}{\beta}) \\ &= \beta(a - x')^2 > 0,\end{aligned}$$

where the penultimate inequality follows since $x^R > a + \frac{\alpha c}{2\beta}$. Thus, type x' is better off sticking to the equilibrium strategy.

Case 3: $x' > b$. In Case 1, we have shown that if type 'a' is indifferent between the two alternatives, then type 'b' strictly prefers to stick to the equilibrium. Thus

$$\alpha cb - \beta(\frac{\alpha c}{2\beta})^2 > \alpha P(x^R) - \beta(x^R - b)^2.$$

Therefore

$$\alpha P(x^R) + \beta(\frac{\alpha c}{2\beta})^2 < \alpha cb + \beta(x^R - b)^2.$$

We conclude that

$$\begin{aligned}\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2 - \alpha P(x^R) + \beta(x^R - x')^2 &> \alpha cx' - \alpha cb - \beta(x^R - b)^2 + \beta(x^R - x')^2 \\ &= \beta(x' - b)(x' + b + \frac{\alpha c}{\beta} - 2x^R) \\ &> \beta(x' - b)(x' - b + \frac{\alpha c}{\beta}) > 0,\end{aligned}$$

where the penultimate inequality follows since $x^R < b$. Thus, the deviation is not profitable. This concludes the proof. ■

Based on Lemma 8, the partially pooling equilibrium strategy $\rho_p^*(\cdot)$ is said to be supported by a *tight* pricing function $P(x^R)$, if for all $x^R \in (a + \frac{\alpha c}{2\beta}, b)$, type '$a$' is indifferent between the equilibrium strategy and deviating to x^R, and for all $x^R \in (b, b + \frac{\alpha c}{2\beta})$, type '$b$' is indifferent between following the equilibrium strategy and deviating to x^R. The next proposition shows that there exists a unique tight pricing function. Moreover, a necessary and sufficient condition for any pricing function to support the partially pooling equilibrium is that the out of equilibrium pricing function will lie weakly below the tight pricing function. To see this intuitively, consider Figure 1. The out of equilibrium pricing function used in the paper is represented in this figure by the dotted straight line connecting points A and B, except for a report of b where the price is $cd(a, b)$. The tight pricing function is the dotted curve ADB. The original pricing function lies below the tight one, meaning that under the original out of equilibrium pricing, the investors "punish" the manager more severely then necessary in order to maintain this equilibrium. In general, any out of equilibrium pricing that lies below the curve ADB will support the partially pooling equilibrium. Thus, the tight pricing function is the least restrictive one that still supports this equilibrium. Formally,

Proposition 9 *There exists a unique tight pricing function that supports the partially pooling strategy $\rho_p^*(\cdot)$. This pricing function is monotone increasing and is given by*

$$P_t^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a + \frac{\alpha c}{2\beta} \text{ or } x^R > b + \frac{\alpha c}{2\beta} \\ c \cdot d(a,b) & x^R = b \\ c\left(a - \frac{\alpha c}{4\beta} + \frac{\beta}{\alpha c}(x^R - a)^2\right) & x^R \in [a + \frac{\alpha c}{2\beta}, b) \\ c\left(b - \frac{\alpha c}{4\beta} + \frac{\beta}{\alpha c}(x^R - b)^2\right) & x^R \in (b, b + \frac{\alpha c}{2\beta}] \end{cases}.$$

Moreover, a necessary and sufficient condition for any pricing function $P^*(x^R)$ to support the partially pooling equilibrium is that: $P^*(x^R) = c\left(x^R - \frac{\alpha c}{2\beta}\right)$ if $x^R < a + \frac{\alpha c}{2\beta}$ or $x^R > b + \frac{\alpha c}{2\beta}$, $P^*(x^R) = c \cdot d(a,b)$ if $x^R = b$, and $P^*(x^R) \leq P_t^*(x^R)$ for all $x^R \in (a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta})$.

Proof of Proposition 9

The cases $x^R < a + \frac{\alpha c}{2\beta}$, $x^R > b + \frac{\alpha c}{2\beta}$, and $x^R = b$ are identical to these cases in the paper, and are determined uniquely using Bayes rule. As for the pooling region: for all $x^R \in [a + \frac{\alpha c}{2\beta}, b)$, we look for a pricing function $P_t^*(x^R)$ that makes type 'a' indifferent between deviating to x^R and sticking to the equilibrium. This indifference implies that this pricing function must satisfy

$$\alpha ca - \beta(\frac{\alpha c}{2\beta})^2 = \alpha P_t^*(x^R) - \beta(x^R - a)^2.$$

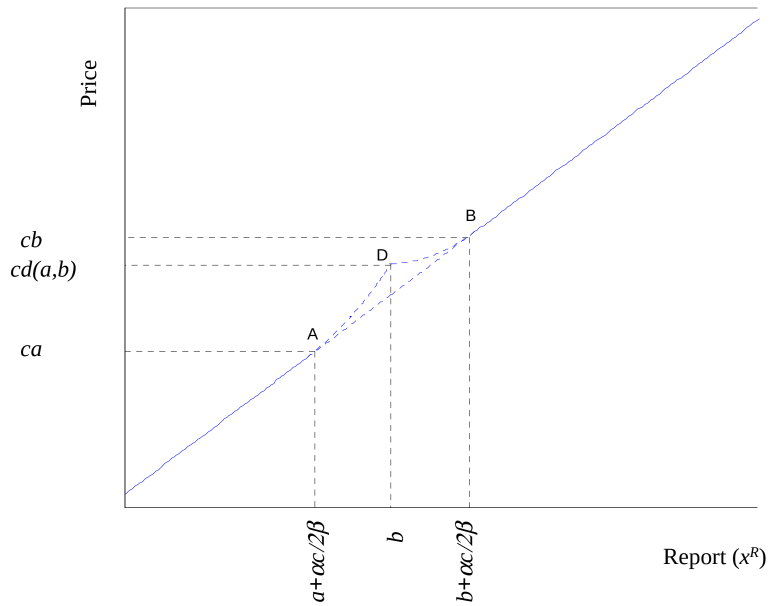

Figure 1: The Tight Pricing Function

Solving for $P_t^*(x^R)$ yields the required result. A similar calculation applies for the case $x^R \in (b, b+\frac{\alpha c}{2\beta})$. Lemma 8 implies that this out-of-equilibrium pricing guarantees that no type will be willing to deviate from the partially pooling strategy $\rho_p^*(\cdot)$.

Now, let $P^*(x^R)$ be any other pricing function. Obviously, it must coincide with $P_t^*(x^R)$ if $x^R < a + \frac{\alpha c}{2\beta}$ or $x^R > b + \frac{\alpha c}{2\beta}$, or $x^R = b$. Now, if $P^*(x^R) \leq P_t^*(x^R)$ for all $x^R \in (a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta})$ then types 'a' and 'b' (weakly) prefer to provide the equilibrium report b compared to any out of equilibrium report. By an argument similar to Lemma 8 this implies that all other types strictly prefer not to deviate from the equilibrium. This establishes sufficiency. To show necessity, suppose on the contrary that $P^*(x^R) > P_t^*(x^R)$ for some $x^R \in (a+\frac{\alpha c}{2\beta}, b)$. Then type '$a$' would prefer deviating and reporting x^R instead of b. Similarly, if $P^*(x^R) > P_t^*(x^R)$ for some $x^R \in (b, b + \frac{\alpha c}{2\beta})$ then type 'b' would deviate.

Now, it is straightforward to verify that $P_t^*(\cdot)$ is monotone increasing. This concludes the proof. ■