Appendix B - Robustness Results

In this Appendix we check the robustness of our results, relaxing assumptions **A.2**, and **A.3** separately.

B.1 Queueing

Proof of Proposition 8

We maintain **A.1** and **A.3** but we allow traders to queue at the best quotes. Assume that traders follow the same order placement strategies as in the equilibrium in which they are not allowed to queue (i.e. the equilibrium is as described in Propositions 3,4,5). We identify below a condition under which these strategies still form an equilibrium when traders are allowed to queue at the inside quotes.

Consider a patient trader who faces a spread equal to n_h. If he improves upon the inside quotes, he optimally chooses a limit order which creates a spread equal to n_{h-1} (this follows from the fact that the other traders behave as described in Propositions 3,4,5). Hence, we only need to find a condition under which this trader is better off improving the price, rather than queuing at the best quotes.

Let $T^*(n_h, 2)$ be the expected waiting time of the trader if he decides to queue by placing an order at the inside quote. The trader is better off undercutting iff

$$n_{h-1}\Delta - T^*(n_{h-1})\delta_P \geq n_h\Delta - T^*(n_h, 2)\delta_P, \quad \forall\, h \geq 1,$$

or

$$(n_h - n_{h-1})\Delta \leq \left[T^*(n_h, 2) - T^*(n_{h-1})\right]\delta_P \quad \forall\, h \geq 1. \tag{30}$$

We now identify a *sufficient* condition under which this no queuing condition holds. We first derive a lower bound for $T^*(n_h, 2)$. Suppose that the trader who decides to queue is a buyer (call him B2). As time priority is enforced, this buyer cannot be executed before the buyer who is posting the best bid price when the spread is n_h (call him B1). The expected waiting time of B1 is equal to $T^*(n_h)$. When B1's order executes, B2 acquires price priority and as buyers and sellers alternate, the next trader is a buyer. Thus from the moment B1's order is executed, it takes *at least* two periods for B2's limit order to execute. It takes *exactly* two periods if and only if the next two traders are impatient. Otherwise, it takes more than 4 periods for B2's order to be executed. We conclude that

$$T^*(n_h, 2) \geq T^*(n_h) + (1-\theta_P)^2(\frac{2}{\lambda}) + (1-(1-\theta_P)^2)\frac{4}{\lambda},$$

which rewrites

$$T^*(n_h, 2) \geq T^*(n_h) + \frac{2}{\lambda} + (1-(1-\theta_P)^2)\frac{2}{\lambda}. \tag{31}$$

Substituting this lower bound for $T^*(n_h, 2)$ into the no-queuing condition (30) we obtain:

$$(n_h - n_{h-1})\Delta \leq (T^*(n_h) - T^*(n_{h-1}))\delta_P + \frac{2}{\lambda}(1 + \theta_P(2-\theta_P))\delta_P \quad \forall\, h \geq 1,$$

or

$$(n_h - n_{h-1})\Delta - (T^*(n_h) - T^*(n_{h-1}))\delta_P \leq \frac{2}{\lambda}(1 + \theta_P(2-\theta_P))\delta_P \quad \forall\, h \geq 1. \tag{32}$$

Recall that in equilibrium:

$$(n_h - n_{h-1})\Delta = CF\left((T^*(n_h) - T^*(n_{h-1}))\frac{\delta_P}{\Delta}\right)\Delta.$$

Hence the Left Hand Side of Inequality (32) is smaller than Δ. We conclude that

$$\Delta \leq \frac{2}{\lambda}(1 + \theta_P(2-\theta_P))\delta_P$$

is a sufficient condition for queuing to be suboptimal. ■

B.2 The Alternating Arrival Assumption

B.2.1 Framework

We maintain assumptions **A.1** and **A.2** but relax assumption **A.3**: we assume that each arrival is either a buyer or a seller with equal probabilities. Suppose that $K = 4$. In this case, there are 3 possible prices in the range $[B, A]$ which can be chosen by limit order submitters. Hence the state of the limit order book can be described by a triplet (x_1, x_2, x_3) where x_i indicates (1) whether a limit order is posted at price $B + i\Delta$ or not and (2) the nature of the limit order (buy or sell) posted at price $B + i\Delta$. Hence x_i belongs to the set $\{b, s, n\}$ where "b" ("s") stands for "buy" ("sell") limit order and "n" stands for "no order" (an empty cell). For instance, (b, n, s) is a limit order book in which (i) one buy limit order is posted at price $B + \Delta$, (ii) no order is posted at price $B + 2\Delta$ and (iii) one sell limit order is posted at price $B + 3\Delta$. The size of the bid ask spread in this book is 2 ticks. Let $T^i_{x_1x_2x_3}$ denote the expected waiting time of the limit order posted at a price equal to $B+i\Delta$ when the state of the book is (x_1, x_2, x_3), just after the last arrival. For example: T^1_{bnn} is the expected waiting time of a buy limit order posted at $B+\Delta$ right after the arrival of an order. Another example: T^3_{bss} is the expected waiting time of the sell limit order posted at $B+3\Delta$ when the state of the limit order book is (b, s, s).

To ascertain that the waiting time function is not recursive consider the following example. Suppose the current state of the book is (n, s, n). A buyer arrives in the market and submits a limit order at price $B+\Delta$. Then the state of the book becomes (b, s, n), and the buyer's expected waiting time is T^1_{bsn}. The bid ask spread in the book (b, s, n) is one tick, hence the next trader must submit a market order. This next trader is a buyer or a seller with equal probabilities. If the next trader is a seller, then our buyer's limit order is cleared. If the next trader is a buyer, the state of the book becomes (b, n, n) and our original buyer has an additional expected waiting time of T^1_{bnn}. It follows that:

$$T^1_{bsn} = \frac{0.5}{\lambda} + 0.5(\frac{1}{\lambda} + T^1_{bnn}).$$

Thus, the expected waiting time of a limit order creating a spread of 1 tick depends on the expected waiting time of this limit order when the book has a larger spread (equal to 3 ticks). This means that the waiting time function does not have a recursive structure and it precludes the solution method that we employed in our original model. Furthermore, the waiting time is a function of the entire structure of the limit order book, not simply the spread. Indeed, in general $T^1_{bsn} \neq T^1_{bss}$ although both books have a bid-ask spread of 1 tick.[32]

As we cannot solve the game by induction, it becomes impossible to solve the model in general. In the next section, we present 3 examples which show that the main results of our model are still obtained when buyers and sellers arrive randomly. There are two properties which simplify the computations, that we present now. As traders must submit price improving orders (**A.2**), a trader's waiting time does not depend on the orders that are behind him in the queue. This implies that:

$$T^3_{bbb} = T^3_{nbb} = T^3_{bnb} = T^3_{nnb}, \tag{33}$$

$$T^2_{bbn} = T^2_{nbn}, \tag{34}$$

$$T^2_{bbs} = T^2_{nbs}. \tag{35}$$

Furthermore, as traders can be buyers or sellers with equal probabilities, waiting times for buyers and sellers are symmetric. For instance: $T^1_{bsn} = T^3_{nbs}$.

B.2.2 Solved Examples

Example 4 - The Homogeneous Case (a strongly resilient book)

One might suspect that the oscillating equilibrium described in Proposition 1 is an artifact of the alternating arrival assumption. The following example shows that it is not. Set $K = 4$, $\Delta = \frac{1}{16}$, $\lambda = 1$, $\delta \overset{def}{=} \delta_P = \delta_I = 0.025$ (we denote the common waiting cost by δ). Now we show that the following order placement strategy forms an equilibrium: (i) when the spread is larger than 1 tick, buyers and sellers submit a 1-limit order and (ii) when the spread is equal to 1 tick, both submit a market order.

We proceed as follows. In the first step we compute the expected waiting times associated with the previous order placement strategy for each limit order in each possible state of the book. In a second step we check that the order placement strategy is optimal given the expected waiting times computed in the first step.

Step 1.

First, we compute T^3_{nnb}. When the state of the book is (n, n, b), the spread is equal to 1 tick. Hence the next trader must submit a market order. If the next trader is a seller, the buy limit order at $B + 3\Delta$ will be cleared. If the next trader is a buyer, the state of the book is unchanged. It follows that:

$$T^3_{nnb} = 0.5 + 0.5 \left[1 + T^3_{nnb}\right],$$

or $T^3_{nnb} = 2$. Using Equation (33), we deduce that: $T^3_{bbb} = T^3_{nbb} = T^3_{bnb} = 2$, and by symmetry: $T^1_{sss} = T^1_{ssn} = T^1_{sns} = T^1_{ssn} = 2$.

Next, we compute T^2_{bbn}. When the state of the book is (b, b, n), the spread is equal to 2 ticks. Therefore, according to the conjectured equilibrium strategy, the next trader will submit a 1-limit order. With probability 0.5 the next trader is a buyer and the state of the book becomes (b, b, b). With probability 0.5 the next trader is a seller and the state of the book becomes (b, b, s). This implies:

$$T^2_{bbn} = 0.5 \left[1 + T^2_{bbb}\right] + 0.5 \left[1 + T^2_{bbs}\right]. \tag{36}$$

The same type of reasoning yields:

$$T^2_{bbs} = 0.5 + 0.5\left[1 + T^2_{bbn}\right], \tag{37}$$

$$T^2_{bbb} = 0.5\left[1 + T^2_{bbb}\right] + 0.5\left[1 + T^2_{bbn}\right]. \tag{38}$$

Solving the system of equations (36), (37), (38) yields: $T^2_{bbn} = 10$. Using equation (34), we deduce that $T^2_{nbn} = 10$. Also by symmetry: $T^2_{nss} = T^2_{nsn} = 10$. Finally, we calculate T^1_{bnn}. Using the conjectured equilibrium strategy we get the following system of equations:

$$\begin{aligned}
T^1_{bnn} &= 0.5\left[1 + T^1_{bnb}\right] + 0.5\left[1 + T^1_{bsn}\right], \\
T^1_{bnb} &= 0.5\left[1 + T^1_{bnb}\right] + 0.5\left[1 + T^1_{bnn}\right], \\
T^1_{bsn} &= 0.5 + 0.5\left[1 + T^1_{bnn}\right].
\end{aligned}$$

Solving these equations yields: $T^1_{bnn} = 10$ and by symmetry: $T^3_{nns} = 10$.

Step 2. Now we check that traders' order placement strategy is optimal given the expected waiting times computed in step 1. For instance consider a trader (say a buyer) who arrives when the state of the book is (n, n, n). He has three options. If he submits a 3-limit order his payoff is: $3\Delta - \delta T^1_{bnn} = 0.1875 - 0.025 \cdot 10 = -0.0625$. If he submits a 2-limit order his payoff is $2\Delta - \delta T^2_{nbn} = 0.125 - 0.025 \cdot 10 = -0.125$. If the trader submits a 1-limit order his payoff is $\Delta - \delta T^3_{nnb} = 0.0625 - 0.025 \cdot 2 = 0.0125$. It follows that the optimal strategy of the trader when the spread is equal to 4 ticks is to submit a 1-limit order as conjectured. We can proceed in the same way to show that the conjectured order placement strategy when the trader faces other states of the book is optimal. Thus, similar to our baseline model we obtain an oscillating equilibrium. The spread is either 4 ticks or 1 tick. The resiliency of the book is equal to 1, as all transactions are performed when the tick size is 1, and the spread reverts immediately to this competitive spread after each deviation.

Example 5 - A Resilient Book (heterogenous traders, high θ_P)

Set: $\Delta = \frac{1}{16}$, $K = 4$, $\theta_P = 0.7$, $\lambda = 1$, $\delta_P = 0.01$, and $\delta_I = 0.07$. We show that the following order placement strategy forms an equilibrium. First, an impatient trader always submits a market order. Second, a patient trader submits (i) a 2-limit order when the spread is equal to 3 or 4 ticks, (ii) a 1-limit order when the spread is equal to 2 ticks and (iii) a market order when the spread is equal to 1 tick. The corresponding order placement strategy for each state of the book and for each type of trader is given in Table 6. We proceed in 2 steps as in Example 4.

Step 1. As in Example 4, using the conjectured equilibrium strategies we can determine the expected waiting times of each limit order in each possible state of the book. This requires solving a number of systems of linear equations. We do not report the computations here to save space.[33] Solving these equations yields the following expected waiting times:

$$\begin{aligned}
&T^3_{bbb} = T^1_{sss} = 2;\ T^2_{bbn} = T^2_{nss} = 6.31;\ T^2_{bbb} = T^2_{sss} = 8.3 \\
&T^2_{bbs} = T^2_{bss} = 4.15;\ T^1_{bnn} = T^3_{nns} = 12.74;\ T^1_{bbn} = T^3_{nss} = 17.76 \\
&T^1_{bbb} = T^3_{sss} = 19.76;\ T^1_{bns} = T^3_{bns} = 10.34;\ T^1_{bbs} = T^3_{bss} = 16.07 \\
&T^1_{bss} = T^3_{bbs} = 6.17;\ T^1_{bsn} = T^3_{nbs} = 7.37.
\end{aligned} \tag{39}$$

Step 2. Using these expressions for the expected waiting times, we can check that the conjectured order placement strategy is optimal for each type of trader. For instance consider a patient trader (say a buyer) who arrives when the state of the book is (n, n, n). He has three options. If he submits a 3-limit order his payoff is: $3\Delta - \delta_P T^1_{bnn} = 0.1875 - 0.01 \cdot 12.74 = 0.0601$. If he submits a 2-limit order his payoff is $2\Delta - \delta_P T^2_{nbn} = 2\Delta - \delta_P T^2_{bbn} = 0.125 - 0.01 \cdot 6.31 = 0.0619$. If the trader submits a 1-limit order his payoff is $\Delta - \delta_P T^3_{nnb} = \Delta - \delta_P T^3_{bbb} = 0.0625 - 0.01 \cdot 2 = 0.0425$. It follows that the optimal strategy of a patient trader when the spread is equal to 4 ticks is to submit a 2-limit order as conjectured. Proceeding in this way, we show that the order placement strategies described in Table 6 are optimal for each type of trader and for each state of the book. Thus, given the high proportion of patient traders they find it optimal to act aggressively and improve the current spread by more than one tick similar to what we have in our base model when buyers and sellers alternate. Notice that when the spread is equal to 4 ticks, it takes a string of 2 patient traders to bring the spread to the competitive level (1 tick here). Thus the resiliency of the book is $R = 0.7^2 = 0.49$.

Insert Table 6 about here

Example 6 - A Weakly Resilient Book (heterogenous traders, low θ_P)

Set: $\Delta = \frac{1}{16}$, $K = 4$, $\theta_P = 0.3$, $\lambda = 1$, $\delta_P = 0.01$, and $\delta_I = 0.07$. We show that the following order placement strategies constitute an equilibrium. An impatient trader always submits a market order. A patient trader submits (i) a limit order reducing the current spread by one tick, provided that the current spread is larger than one tick, and (ii) a market order when the spread is equal to one tick. The corresponding order placement strategy for each state of the book and for each type of trader is given in Table 6.

Step 1. As in Example 4, using the conjectured equilibrium strategies we can determine the expected waiting times of each limit order in each possible state of the book. We obtain:

$$\begin{aligned}
&T^3_{bbb} = T^1_{sss} = 2;\ T^2_{bbn} = T^2_{nss} = 3.41;\ T^2_{bbb} = T^2_{sss} = 5.41 \\
&T^2_{bbs} = T^2_{bss} = 2.706;\ T^1_{bnn} = T^3_{nns} = 4.185;\ T^1_{bbn} = T^3_{nss} = 7.55 \\
&T^1_{bbb} = T^3_{sss} = 9.55;\ T^1_{bns} = T^3_{bns} = 3.919;\ T^1_{bbs} = T^3_{bss} = 6.78 \\
&T^1_{bss} = T^3_{bbs} = 2.96
\end{aligned} \tag{40}$$

Step 2. Using these expressions for the expected waiting times, we can check that the conjectured order placement strategy is optimal for each type of trader. For instance consider a patient trader (say a buyer) who arrives when the state of the book is (n, n, n). He has three options. If he submits a 3-limit order his payoff is: $3\Delta - \delta_P T^1_{bnn} = 0.1875 - 0.01 \cdot 4.185 = 0.1457$. If he submits a 2-limit order his payoff is $2\Delta - \delta_P T^2_{nbn} = 2\Delta - \delta_P T^2_{bbn} = 0.125 - 0.01 \cdot 3.41 = 0.09$. If the trader submits a 1-limit order his payoff is $\Delta - \delta_P T^3_{nnb} = \Delta - \delta_P T^3_{bbb} = 0.0625 - 0.01 \cdot 2 = 0.0425$. It follows that the optimal strategy of the trader when the spread is equal to 4 ticks is to submit a 3-limit order as conjectured. Thus, as in our base model, given the low level of θ_P, patient traders do not act aggressively, and improve the spread by no more than one tick size. Notice that when the spread is equal to 4 ticks, it takes a string of 3 patient traders to bring the spread to the competitive level (1 tick here). Thus the resiliency of the book is $R = 0.3^3 = 0.027$.

B.2.3 Distribution of Spreads

As in our baseline model, the possible states of the limit order book on the equilibrium path form a Markov chain. We can compute the stationary probability distribution of this Markov chain and deduce the stationary distribution of the spread (by grouping all the states of the book with the same spread). The stationary distributions of spreads for the equilibrium described in Examples 5 and 6 are given in Table 7.[34] Observe that, as in the baseline model, the distribution of spreads in Example 5 (large proportion of patient traders) is skewed towards small spreads, while in Example 6 (small proportion of patient traders) it is skewed towards large spreads. It follows that the expected spread is smaller in Example 5 than in Example 6 (1.76 ticks vs. 3 ticks). Again this is as in the model in which buyers and sellers alternate.

Insert Table 7 about here

Table 6 - Equilibrium Strategies in Examples 5 and 6

B_P=Patient Buyer, S_P=Patient Seller, B_I=Impatient Buyer, S_I=Impatient Seller

		Strategies - Ex. 5				Strategies - Ex. 6			
book status	spread (ticks)	B_P	S_P	B_I	S_I	B_P	S_P	B_I	S_I
(b,b,b)	1	0	0	0	0	0	0	0	0
(s,s,s)	1	0	0	0	0	0	0	0	0
(b,s,s)	1	0	0	0	0	0	0	0	0
(b,b,s)	1	0	0	0	0	0	0	0	0
(n,b,s)	1	0	0	0	0	0	0	0	0
(b,s,n)	1	0	0	0	0	0	0	0	0
(n,b,b)	1	0	0	0	0	0	0	0	0
(s,s,n)	1	0	0	0	0	0	0	0	0
(b,b,n)	2	1	1	0	0	1	1	0	0
(n,s,s)	2	1	1	0	0	1	1	0	0
(n,b,n)	2	1	1	0	0	1	1	0	0
(n,s,n)	2	1	1	0	0	1	1	0	0
(b,n,s)	2	1	1	0	0	1	1	0	0
(b,n,n)	3	2	2	0	0	2	2	0	0
(n,n,s)	3	2	2	0	0	2	2	0	0
(n,n,n)	4	2	2	0	0	3	3	0	0

Table 7 - Spread Distribution

Spread size	Probability	
	Example 5	Example 6
1	0.42	0.08
2	0.44	0.21
3	0.10	0.33
4	0.04	0.38