Internet Appendix for "Liquidity Cycles and Make/Take Fees in Electronic Markets" by T. Foucault, O. Kadan, and E. Kandel

In this appendix we provide additional results and proofs mentioned in the paper but unreported there for brevity.

A Proof of Corollary 1

Recall that $\mathcal{V}^*$ is such that:

$$\mathcal{V}^* = h(\mathcal{V}^*, M, N, z), \tag{A.1}$$

where $h(\cdot)$ is defined in Equation (35). It is immediate that $h(\cdot)$ increases in M, decreases in N, and increases in z. As $h(\cdot)$ decreases in $\mathcal{V}^*$, we have

$$\frac{\partial \mathcal{V}^*}{\partial M} > 0, \tag{A.2}$$

$$\frac{\partial \mathcal{V}^*}{\partial N} < 0, \tag{A.3}$$

$$\frac{\partial \mathcal{V}^*}{\partial z} > 0. \tag{A.4}$$

Now, using Equations (12) and (A.3), we conclude that

$$\frac{\partial \mu_i^*}{\partial N} = \frac{-\frac{\partial \mathcal{V}^*}{\partial N} \cdot ((M+1) + (M-1)\mathcal{V}^*)}{(1+\mathcal{V}^*)^3} \left(\frac{\pi_m}{M\beta} \right) > 0.$$

Hence, $\frac{\partial \bar{\mu}^*}{\partial N} > 0$. Furthermore, since $\bar{\tau}^* = \frac{\bar{\mu}^*}{\mathcal{V}^*}$, we have (using (A.3)) that $\frac{\partial \bar{\tau}^*}{\partial N} > 0$. A similar argument shows that $\frac{\partial \bar{\tau}^*}{\partial M} > 0$ and $\frac{\partial \bar{\mu}^*}{\partial M} > 0$.

Now, consider the effect of a change in β on market-takers' monitoring intensities. We have (see Proposition 2),

$$\tau_j^* = \zeta(\mathcal{V}^*) \left(\frac{\pi_t}{N\gamma} \right),$$

where

$$\zeta(\mathcal{V}^*) = \left(\frac{\mathcal{V}^* \left((1+\mathcal{V}^*) N - 1 \right)}{(1+\mathcal{V}^*)^2} \right).$$

Thus

$$\frac{\partial \tau_j^*}{\partial \beta} = \left(\frac{\partial \zeta(\mathcal{V}^*)}{\partial \mathcal{V}^*} \frac{\partial \mathcal{V}^*}{\partial z} \frac{\partial z}{\partial \beta} \right) \left(\frac{\pi_t}{N\gamma} \right)$$

We have $\frac{\partial \zeta(\mathcal{V}^*)}{\partial \mathcal{V}^*} > 0$. Moreover $\frac{\partial \mathcal{V}^*}{\partial z} > 0$ and $\frac{\partial z}{\partial \beta} < 0$. Thus

$$\frac{\partial \tau_j^*}{\partial \beta} < 0,$$

which implies that $\frac{\partial \bar{\tau}^*}{\partial \beta} < 0$. Now, since $\bar{\mu}^* = \mathcal{V}^* \bar{\tau}^*$, we have

$$\frac{\partial \bar{\mu}^*}{\partial \beta} = \mathcal{V}^* \frac{\partial \bar{\tau}^*}{\partial \beta} + \frac{\partial \mathcal{V}^*}{\partial z} \frac{\partial z}{\partial \beta} \bar{\tau}^* < 0,$$

which implies $\frac{\partial \mu_i^*}{\partial \beta} < 0$ for all i. The impact of make/take fees on traders' aggregate monitoring levels is obtained in the same way. The second part of the corollary follows directly from the first part and the definition of the trading rate (Equation (4)). ■

B The Case $M = N = 1$

When there is one market-maker and one market-taker, we can easily obtain a closed-form solutions for traders' monitoring levels and the optimal make/take fee breakdown.

Claim 1 *When $M = N = 1$, the market-maker's monitoring level is $\mu_1^* = \left(1+z^{\frac{1}{3}}\right)^{-2} \cdot \left(\frac{\pi_m}{\beta}\right)$ and the market-taker's monitoring level is $\tau_1^* = \left(1+z^{-\frac{1}{3}}\right)^{-2} \cdot \left(\frac{\pi_t}{\gamma}\right)$.*

Proof of Claim 1. When $M = N = 1$, it is immediate that the solution to Equation (14) in Proposition 2 is $\mathcal{V}^* = z^{\frac{1}{3}}$. The expressions for traders' monitoring levels are then obtained by replacing $\mathcal{V}^*$ by its value in Equations (12) and (13) in Proposition 2. ■

We now derive the optimal make/take fee breakdown for the platform when $M = N = 1$.

Claim 2 *When $M = N = 1$, the trading platform optimally allocates its fee $\bar{c}$ between the market-making side and the market-taking side as follows:*

$$c_m^* = \Delta - \frac{\Gamma - \bar{c}}{1 + r^{\frac{1}{4}}} \quad \text{and } c_t^* = \bar{c} - c_m^*.$$

Proof of Claim 2. There is a one-to-one mapping between the fees charged by the trading platform and the per trade trading profits obtained by the market-making side and the market-taking side, π_m and π_t. Thus, instead of using c_m and c_t as the decision variables of the platform, we can use π_m and π_t. Thus, for a fixed $\bar{c}$, we rewrite the platform's problem as

$$\begin{aligned} &Max_{\pi_m,\pi_t} \frac{\mu_1^* \tau_1^*}{\mu_1^* + \tau_1^*}\bar{c}, \\ &s.t \quad \pi_t + \pi_m = \overline{\pi}. \end{aligned}$$

From Claim 1, we obtain,

$$\frac{\mu_1^*}{\tau_1^*} = z^{\frac{1}{3}} = \left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{\frac{1}{3}}$$

and

$$\mu_1^* = \frac{\pi_m}{\beta}\frac{1}{\left(1+z^{\frac{1}{3}}\right)^2}.$$

Thus, we can rewrite the previous optimization problem as

$$Max_{\pi_m,z} \frac{\mu_1^*}{1+z^{\frac{1}{3}}}\bar{c} \tag{B.1}$$

$$s.t \quad \pi_m\left(1+\frac{\gamma}{\beta z}\right) = \Gamma - \bar{c}. \tag{B.2}$$

$$\text{and } \mu_1^* = \frac{\Gamma - \bar{c}}{\beta\left(1+z^{\frac{1}{3}}\right)^2\left(1+\frac{\gamma}{\beta z}\right)}. \tag{B.3}$$

This problem is equivalent to finding z that minimizes

$$\left(1+z^{\frac{1}{3}}\right)^3\left(\beta+\frac{\gamma}{z}\right).$$

The FOC to this problem imposes

$$-\frac{1}{z^2}\left(\gamma-z^{\frac{4}{3}}\beta\right)\left(z^{\frac{1}{3}}+1\right)^2=0.$$

Hence, the solution is

$$z=\left(\frac{\gamma}{\beta}\right)^{\frac{3}{4}}=r^{\frac{3}{4}}. \tag{B.4}$$

Using the constraint (B.2), we have,

$$\pi_m^*=\frac{\Gamma-\bar{c}}{1+r^{\frac{1}{4}}}. \tag{B.5}$$

The result is then immediate using the fact that $c_m^*=\pi_m^*-\Delta$. ■

C The Optimal Level of Total Fees

The platform's optimization problem can be decomposed into two steps: (i) choose the optimal make/take fees for a given $\bar{c}$ (we solved this problem in the paper); and (ii) choose the optimal $\bar{c}$. Observe that the optimal make/take fees, (c_m^*, c_t^*), increase in $\bar{c}$, and recall that the trading rate decreases in both the make fee and the take fee (Corollary 1). Thus, in the second step, the trading platform faces the standard price-quantity trade-off: by raising $\bar{c}$, the trading platform gets a larger revenue per trade but it decreases the rate at which trades occur. The next result gives the optimal value of $\bar{c}$ for the trading platform in the case of a thick market.

Claim 3 *In the thick market case, the trading platform maximizes its expected profit by setting its total trading fee at* $\bar{c}=\Gamma/2$ *and by splitting this fee between the two sides as described in Proposition 3.*

In contrast to the make/take fees, the optimal total fee for the platform is independent of traders' relative monitoring costs and the relative size of the market-making side. Thus, our results regarding the effect of q, and r hold even if $\bar{c}$ is optimally set by the trading platform.

Proof of Claim 3: We fix $q>0$, and let $N=\frac{M}{q}$. From (43) and (44), we obtain that when fees are set optimally in the thick market case, we have

$$z=\frac{\pi_m}{\pi_t}r=q^{-\frac{1}{3}}r^{\frac{2}{3}},$$

and,

$$\mathcal{V}^{\infty}=(zq)^{\frac{1}{2}}=(rq)^{\frac{1}{3}}. \tag{C.1}$$

Using Corollary 4, Equations (43) and (44) and (C.1) we obtain

$$\mu_i^\infty = \frac{\Gamma - \bar{c}}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^2} \quad \text{and} \quad \tau_j^\infty = \frac{\Gamma - \bar{c}}{\gamma\left(1+(qr)^{-\frac{1}{3}}\right)^2} \quad \text{for } i,j = 1,2,... \tag{C.2}$$

Now, for any given M, maximizing $\mathcal{R}(\mu^*, \tau^*)\bar{c}$ is equivalent to maximizing $\frac{\mathcal{R}(\mu^*,\tau^*)}{M}\bar{c}$, which in turn is equivalent to maximizing $\frac{\mu_1^*}{1+\mathcal{V}^*}\bar{c}$ (using Equation (36) and the fact that $\bar{\mu}^* = M\mu_1^*$) . Denote $\mathcal{H}(\bar{c}) \equiv \frac{\mu_1^*}{1+\mathcal{V}^*}$. Then, to find the optimal total fee $\bar{c}$ in the thick market we need to find the limit as M tends to infinity of

$$\arg\max_{\bar{c}\geq 0} \mathcal{H}(\bar{c})\,\bar{c}.$$

The FOC for a given M is

$$\mathcal{H}(\bar{c}) + \mathcal{H}'(\bar{c})\,\bar{c} = 0. \tag{C.3}$$

Note that $\mathcal{H}$ depends on $\bar{c}$ only through its dependence on μ_1^* and $\mathcal{V}^*$. It follows that

$$\mathcal{H}'(\bar{c}) = \frac{\partial\mathcal{H}}{\partial\mu_1^*}\frac{\partial\mu_1^*}{\partial\bar{c}} + \frac{\partial\mathcal{H}}{\partial\mathcal{V}^*}\frac{\partial\mathcal{V}^*}{\partial\bar{c}} = \frac{1}{1+\mathcal{V}^*}\frac{\partial\mu_1^*}{\partial\bar{c}} - \frac{\mu_1^*}{(1+\mathcal{V}^*)^2}\frac{\partial\mathcal{V}^*}{\partial\bar{c}}. \tag{C.4}$$

Since Equation (C.3) holds for any M, we can take the limit as $M \to \infty$. We have,

$$\lim_{M\to\infty} \mathcal{H}(\bar{c}) = \frac{\mu_1^\infty}{1+\mathcal{V}^\infty} = \frac{\Gamma - \bar{c}}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^3} \quad \text{(using (C.1) and (C.2))}.$$

From (C.1) and (C.2) it also follows that

$$\begin{aligned}
\lim_{M\to\infty} \frac{\partial\mu_1^*}{\partial\bar{c}} &= \frac{\partial\mu_1^\infty}{\partial\bar{c}} = -\frac{1}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^2}, \text{ and} \\
\lim_{M\to\infty} \frac{\partial\mathcal{V}^*}{\partial\bar{c}} &= \frac{\partial\mathcal{V}^\infty}{\partial\bar{c}} = 0.
\end{aligned}$$

Thus, from (C.4),

$$\lim_{M\to\infty} \mathcal{H}'(\bar{c}) = -\frac{1}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^3}.$$

Hence, in the limit (C.3) becomes

$$\frac{\Gamma - \bar{c}}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^3} - \frac{1}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^3}\bar{c} = 0,$$

which gives $\bar{c} = \frac{\Gamma}{2}$. ■

D Optimal Make/Take Fees and the Tick Size

In this section we develop in the detail the analysis that leads to the proof of Proposition 6. We also provide numerical examples that illustrate the intuition of the effects of the tick size on the ask price and the make/take fees.

The first step is to analyze how the ask price, $a_\ell^*(c_m, \theta)$, depends on the make fee when there is a positive tick size. The next lemma shows that this price is simply a step function of the make fee.

Lemma 1 *The optimal ask price is an increasing step function of the make fee c_m. Specifically, there exists a partition of the interval $[\hat{c}_{m1}, \hat{c}_{m1} + (2\ell - 1)\Delta(\ell)]$ into $2\ell - 1$ segments given by $[\hat{c}_{mk}, \hat{c}_{mk+1}]$, where $\hat{c}_{m1}$ is the unique solution to the equation $\mathcal{O}(v_0, \hat{c}_{m1}) = \mathcal{O}(v_0 + \Delta(\ell), \hat{c}_{m1})$, and $\hat{c}_{mk} = \hat{c}_{m1} + (k-1)\Delta(\ell)$ for $k \in \{1, .., 2\ell\}$ such that:*

1. *When $c_m \in (\hat{c}_{mk}, \hat{c}_{mk+1})$, the optimal ask price is unique and given by $a_\ell^*(c_m, \theta) = v_0 + k\Delta(\ell)$ for $k \in \{1, .., 2\ell - 1\}$.*
2. *When $c_m = \hat{c}_{mk+1}$ $(k \in \{1, .., 2(\ell - 1)\})$, both $v_0 + k\Delta(\ell)$ and $v_0 + (k+1)\Delta(\ell)$ are optimal ask prices, and we can set $a_\ell^*(c_m, \theta)$ to any of them.*

Proof of Lemma 1: Since θ is fixed throughout this proof, we omit the argument for θ form $\mathcal{O}(a, c_m, \theta)$ to save space. It is straightforward that the objective function $\mathcal{O}(a, c_m)$ is concave in a and that $\frac{\partial^2 \mathcal{O}(a, c_m)}{\partial a \partial c_m} > 0$. Thus $\mathcal{O}(a, c_m)$ satisfies the Milgrom-Shannon (1994) single-crossing property (SCP): If $\mathcal{O}(a', c_m) \geq \mathcal{O}(a, c_m)$ then $\mathcal{O}(a', c'_m) > \mathcal{O}(a, c'_m)$ for all $a' > a$ and $c'_m > c_m$.

Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ be the floor and ceiling functions, respectively.[32] In the case of a zero tick size ($\ell = \infty$), the solution to the optimization problem (24) is

$$a_\infty^*(c_m, \theta) = v_0 + c_m + (1-\theta)(\Gamma - \bar{c}).$$

As the objective function, $\mathcal{O}(\cdot, \cdot)$, is concave in a, the solution to (24) for a finite ℓ, is

$$a_\ell^*(c_m, \theta) = v_0 + k^*(c_m, \theta)\Delta(\ell),$$

where

$$k^*(c_m, \theta) \in \left\{ \left\lfloor \frac{c_m + (1-\theta)(\Gamma - \bar{c})}{\Delta(\ell)} \right\rfloor, \left\lceil \frac{c_m + (1-\theta)(\Gamma - \bar{c})}{\Delta(\ell)} \right\rceil \right\}.$$

We first show that $k^*(c_m, \theta)$, and thereby $a_\ell^*(c_m, \theta)$ weakly increase in c_m. We proceed by contradiction. Suppose that $c_m < c'_m$ and yet $k^*(c_m, \theta) > k^*(c'_m, \theta)$. As $a_\infty^*(c_m, \theta) < a_\infty^*(c'_m, \theta)$, this is possible only if

$$k^*(c_m, \theta) = \left\lceil \frac{c_m + (1-\theta)(\Gamma - \bar{c})}{\Delta(\ell)} \right\rceil = \left\lceil \frac{c'_m + (1-\theta)(\Gamma - \bar{c})}{\Delta(\ell)} \right\rceil$$

and

$$k^*(c'_m, \theta) = \left\lfloor \frac{c_m + (1-\theta)(\Gamma - \bar{c})}{\Delta(\ell)} \right\rfloor = \left\lfloor \frac{c'_m + (1-\theta)(\Gamma - \bar{c})}{\Delta(\ell)} \right\rfloor. \quad \text{(D.1)}$$

[32] The floor function maps a real number into the greatest integer below (or equal to) this number, and the ceiling function maps a real number into the the smallest integer above (or equal to) this number.

Now, if

$$k^*(c_m,\theta) = \left\lceil \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rceil,$$

then

$$\mathcal{O}\left(v_0 + \left\lceil \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rceil \Delta(\ell), c_m\right) \geq \mathcal{O}\left(v_0 + \left\lfloor \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rfloor \Delta(\ell), c_m\right).$$

Thus, using the SCP, we have

$$\mathcal{O}\left(v_0 + \left\lceil \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rceil \Delta(\ell), c_m'\right) > \mathcal{O}\left(v_0 + \left\lfloor \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rfloor \Delta(\ell), c_m'\right),$$

which implies

$$\mathcal{O}\left(v_0 + \left\lceil \frac{c_m' + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rceil \Delta(\ell), c_m'\right) > \mathcal{O}\left(v_0 + \left\lfloor \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rfloor \Delta(\ell), c_m'\right).$$

But this contradicts (D.1). Thus, $k^*(c_m,\theta)$ and $a_\ell^*(c_m,\theta)$ weakly increase in c_m.

Now let $\underline{c}_m = -(1-\theta)(\Gamma - \bar{c})$ and observe that

$$v_0 + \left\lfloor \frac{\underline{c}_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rfloor \Delta(\ell) = v_0 + \left\lceil \frac{\underline{c}_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rceil \Delta(\ell) = v_0.$$

Similarly, let $\bar{c}_m = \Gamma - (1-\theta)\left(\Gamma - \bar{c}\right)$. Then,

$$\begin{aligned} v_0 + \left\lfloor \frac{\bar{c}_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rfloor \Delta(\ell) &= v_0 + \left\lceil \frac{\bar{c}_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta(\ell)} \right\rceil \Delta(\ell) \\ &= v_0 + 2\ell\Delta(\ell) = v_0 + \Gamma. \end{aligned}$$

As $a_\ell^*(c_m,\theta)$ weakly increases in c_m, $k^*\left(c_m,\theta\right)$ takes all the values between 0 and 2ℓ as c_m goes from $\underline{c}_m$ to $\bar{c}_m$. For $k \in \{1, 2, ...2\ell-1\}$ let $\hat{c}_{mk} \in (\underline{c}_m, \bar{c}_m)$ be the smallest value of c_m such that $a_\ell^*(c_m,\theta) = v_0 + k\Delta(\ell)$. Then, for $c_m \in (\hat{c}_{mk}, \hat{c}_{mk+1})$ we have $a_\ell^*(c_m,\theta) = v_0 + k\Delta(\ell)$. Moreover, by the continuity of $\mathcal{O}\left(\cdot,\cdot\right)$ we have that

$$\mathcal{O}\left(v_0 + k\Delta(\ell), \hat{c}_{mk+1}\right) = \mathcal{O}\left(v_0 + (k+1)\,\Delta(\ell), \hat{c}_{mk+1}\right)$$

for all $k \in \{1, 2, ...2\ell-1\}$. Thus, at the partition points $\{\hat{c}_{mk}\}_{k=2}^{2\ell-1}$, the optimal ask price can be chosen as either $v_0 + k\Delta(\ell)$ or $v_0 + (k+1)\,\Delta(\ell)$. Furthermore, the first partition point $\hat{c}_{m1}$ is determined uniquely by the indifference condition

$$\mathcal{O}\left(v_0, \hat{c}_{m1}\right) = \mathcal{O}\left(v_0 + \Delta(\ell), \hat{c}_{m1}\right).$$

That is, $\hat{c}_{m1}$ is the unique solution to

$$\left(\Gamma - \bar{c} + \hat{c}_{m1}\right)^{\theta}\left(-\hat{c}_{m1}\right)^{1-\theta} = \left(\Gamma - \Delta(\ell) - \bar{c} + \hat{c}_{m1}\right)^{\theta}\left(\Delta(\ell) - \hat{c}_{m1}\right)^{1-\theta},$$

which implies that $\hat{c}_{m1} < 0$.

Now, note that for any number u,

$$\mathcal{O}\left(a, c_m\right) = \mathcal{O}\left(a + u, c_m + u\right).$$

In particular, setting $u = \Delta\left(\ell\right)$,

$$\mathcal{O}\left(v_0 + k\Delta\left(\ell\right), c_m\right) = \mathcal{O}\left(v_0 + \left(k+1\right)\Delta\left(\ell\right), c_m + \Delta\left(\ell\right)\right).$$

Thus, setting $a^*_\ell(c_m, \theta) = v_0 + k\Delta\left(\ell\right)$ is optimal given $c_m \in [\hat{c}_{mk}, \hat{c}_{mk+1}]$ if and only if setting $a^*_\ell(c_m, \theta) = v_0 + \left(k+1\right)\Delta\left(\ell\right)$ is optimal given $c_m \in \left[\hat{c}_{mk} + \Delta\left(\ell\right), \hat{c}_{mk+1} + \Delta\left(\ell\right)\right]$. Hence, $\hat{c}_{mk+1} = \hat{c}_{mk} + \Delta\left(\ell\right)$. In particular, $\hat{c}_{mk} = \hat{c}_{m1} + (k-1)\Delta(\ell)$ for all $k = 1, ..., 2\ell$. ■

Lemma 1 says that the interval of possible make-fees is partitioned into $2\ell - 1$ segments, inducing a weakly increasing step function of optimal ask prices. The optimal ask price $a^*_\ell(c_m, \theta)$ is determined uniquely on the interior of the segments, whereas at the partition points we are free to choose between the left or the right ask prices, as both yield exactly the same gains from trade to makers and takers. Note that Lemma 1 only covers the case in which the make fee is in the interval $[\hat{c}_{m1}, \hat{c}_{m1} + (2\ell - 1)\Delta(\ell)]$. In this case, $a^*_\ell(c_m, \theta)$ is in the range of traders' valuations $[v_0, v_0 + \Gamma]$. For make fees outside this range (e.g., very large rebates for one side), the price $a^*_\ell(c_m, \theta)$ will be outside the range of traders' valuations $[v_0, v_0 + \Gamma]$.[33] Our results hold in this case as well, but it is natural to focus the attention on parameters such that the ask price is always in the range $[v_0, v_0 + \Gamma]$.[34]

To better understand Lemma 1 consider the following numerical example. Set $v_0 = 800$, $\Gamma = 50$, $\bar{c} = 1/10$ (all monetary amounts are in cents). Moreover $\theta = 0.5$. Hence, market-makers and market-takers split the gains from trade equally when the tick size is zero. Finally $\ell = 2$, i.e., the tick size is $\Delta(2) = \frac{\Gamma}{4} = \$\frac{1}{8}$ (12.5 cents). Figure 1 illustrates Lemma 1 for these parameter values.

The partition points in this case are $\hat{c}_{m1} = -18.7$, $\hat{c}_{m2} = -6.2$, and $\hat{c}_{m3} = 6.3$. The solid step-function in the top graph of Figure 1 depicts the optimal ask price $a^*_\ell(c_m, \theta)$ as a function of c_m when the tick size is $\$\frac{1}{8}$ ($\ell = 2$). Between the partition points the optimal price is unique, while at the partition points one can choose either the left or the right price on the grid.[35] The 45° thin line in the graph depicts the optimal ask price as as a function of c_m when the tick size is zero ($\ell = \infty$).

Suppose that $c_m = 0.03$ cents. With a zero tick-size ($\ell = \infty$), the ask price is $a^*_\infty(0.03, 0.5) = 824.98$ and market-makers obtain 50% of the gains from trade. In contrast, with a \$1/8 tick size

[33] For instance, if $c_m < -(1-\theta)(\Gamma - \bar{c})$ then $a^*_\infty(c_m, \theta) < v_0$ (see Equation (25)).

[34] Indeed, suppose that market-makers receive a very large rebate so that they agree to sell at an ask price is less than v_0. By symmetry they would be willing to buy at a bid price higher than v_0, which would lead to an immediate arbitrage opportunity.

[35] For example, if the make-fee is between -6.2 cents and 6.3 cents then the optimal ask price is unique and equal to 825 cents. However, if the make fee is exactly 6.3 cents, then the same division of gains from trade is obtained when the ask price is 825 or 837.5 cents.

($\ell = 2$), the ask price is $a_2^*(0.03, 0.5) = 825$, as reflected in the top graph of Figure 1. Indeed, this is the price on the grid that yields the division of gains from trade which is the closest to that obtained in the absence of a minimum price variation (with this price market-makers obtain 49.95% of the total gains from trade). Now suppose that the make fee decreases by 0.01 cents. With no minimum price variation, the ask price would decrease by the same amount, leaving unchanged the 50/50 split of the gains from trade between makers and takers. However, when the tick size is greater than 0.01 cents, the change in the make fee cannot be fully neutralized and an ask price of 825 is still the price that yields the division of gains from trade which is the closest to that obtained in absence of the tick size. More generally, traders keep trading at 825 as long as the make fee does not exceed 6.3 cents. At this point, traders are indifferent between an ask price of 825 and an ask price of 837.5, and so both are optimal. Once the make fee exceeds 6.3 cents, the optimal ask price becomes 837.5 cents. To sum up, the required minimum price variation renders the ask price less elastic to a change in the make fee, as shown in the top graph of Figure 1.

As a consequence, when the tick size is strictly positive, market-makers' profits depend on the make fee charged by the platform, as shown in the bottom graph of Figure 1. Consider again the previous numerical example and suppose that $c_m = 0.03$. In this case, as explained previously, the ask price is $a_4^*(0.03, 0.5) = 825$ cents. Market-makers earn a surplus of $\pi_m = 24.77$ cents and market-takers earn a surplus of $\pi_t = 25.13$ cents. If instead the platform offers a rebate of 1 cent to the market-makers ($c_m = -1$), the price remains unchanged but the market-makers now have a higher surplus (25.8 cents) and market-takers a lower surplus (24.1 cents). Thus, market-makers (market-takers) will monitor the market more (less) intensively in the second case and the trading rate will be affected, as in the baseline model. One difference compared to the baseline model is that the make/take fee breakdown that maximizes the trading rate is no longer unique, as shown in the next lemma.

Lemma 2 *The trading rate when the platform chooses a make fee $c_m^* \in [\hat{c}_{mk}, \hat{c}_{mk+1}]$ is equal to the trading rate when the platform sets the make fee at $c_m^{**} = c_m^* + n\Delta(\ell)$ for all integers n.*

Proof of Lemma 2: Suppose that (c_m^*, c_t^*) is a make/take fee breakdown that maximizes the trading rate with $c_m^* \in [\hat{c}_{mk}, \hat{c}_{mk+1})$ for some $k \in (1, 2\ell - 1)$. With this fee, traders' profits per trade are

$$\begin{aligned} \pi_m(a_\ell^*(c_m^*, \theta), c_m^*) &= k\Delta(\ell) - c_m^*, \\ \pi_t(a_\ell^*(c_m^*, \theta), c_t^*) &= \Gamma - \bar{c} - k\Delta(\ell) + c_m^* \end{aligned}$$

Now, consider the following make fee: $c_m^{**} = c_m^* + n\Delta(\ell)$ for some integer n. We have $c_m^{**} \in [\hat{c}_{mk+n}, \hat{c}_{mk+n+1})$ since $\hat{c}_{mk} = \hat{c}_{m1} + (k-1)\Delta(\ell)$. Thus, $a_\ell^*(c_m, \theta) = v_0 + (k+n)\Delta(\ell)$. We conclude that

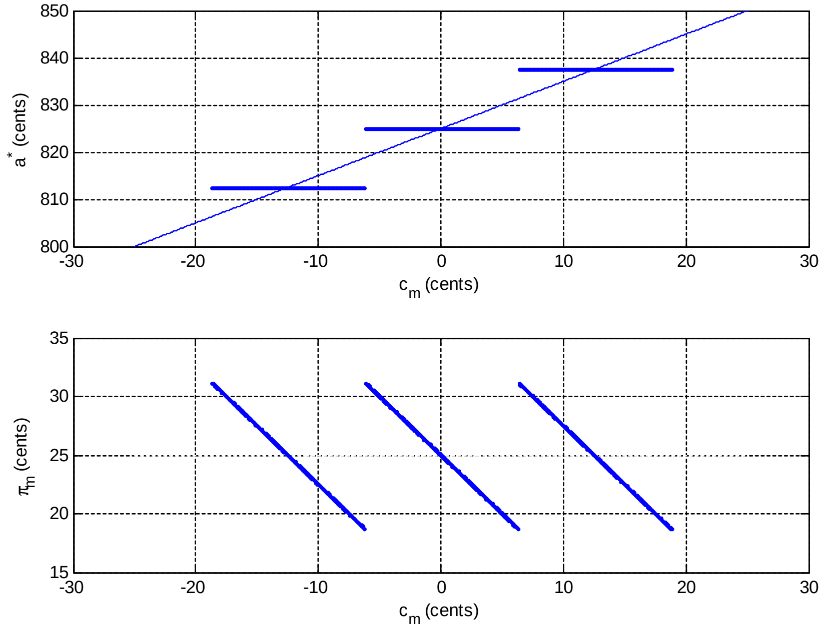

Figure 1: Effect of the make fee on the ask price and market-makers' per trade profit

traders' per trade profits with a fee equal to c_m^{**} are

$$\begin{aligned}\pi_m(a_\ell^*(c_m^{**},\theta),c_m^{**}) &= k\Delta(\ell)-c_m^*,\\ \pi_t(a_\ell^*(c_m^{**},\theta),c_t^{**}) &= \Gamma-\bar{c}-k\Delta(\ell)+c_m^*.\end{aligned}$$

Thus, the fees c_m^* and c_m^{**} result in exactly the same division of gains from trade between market-makers and market-takers. As a consequence, monitoring intensities are identical in both cases and both fees result in the same trading rate. Thus, if (c_m^*, c_t^*) maximizes the trading rate then (c_m^{**}, c_t^{**}) also maximizes the trading rate. ■

The intuition for this result is as follows. Suppose that initially the platform picks its fee in $[\hat{c}_{mk},\ \hat{c}_{mk+1}]$. If the platform adds an integer number of ticks to its make-fee then traders neutralize the effect of this shift in the make fee by adjusting the transaction price by exactly the same number of ticks (Lemma 1). As a consequence, the division of gains from trade between makers and takers is unchanged and traders' monitoring intensities are unchanged as well. Thus, the trading rate is identical in both situations.

We can therefore arbitrarily choose the interval $[\hat{c}_{mk},\ \hat{c}_{mk+1}]$ in which the platform picks its make fee. We find it convenient to set $k=\ell$. Indeed, in this case, the ask price at which trades take place is independent of ℓ since $a_\ell^*(c_m,\theta)=v_0+\ell\Delta(\ell)=v_0+\Delta$ (as $\Delta(\ell)=\Delta/\ell$). Proposition 6 then follows immediately. Indeed, the optimization problem of the platform is identical to its problem in the baseline case, with the additional constraint that $c_m \in [\hat{c}_{m\ell}, \hat{c}_{m\ell+1}]$, where $\hat{c}_{m\ell+1} = \hat{c}_{m\ell} + \Delta(\ell)$. Hence, if the optimal solution in the baseline case satisfies the constraint, then it also solves the constrained problem. This implies that:

$$c_m^*(\ell, r, q) = c_m^* \quad if \quad \hat{c}_{m\ell} < c_m^* < \hat{c}_{m\ell+1}.$$

If instead, the optimal solution in the baseline case does not satisfy the constraint then the constraint binds and the optimal solution for the platform is a corner solution as claimed in Proposition 6.

E Algorithmic Trading and Welfare

In Section 6.2 in the paper, we claim that, for fixed trading fees, a reduction in traders' monitoring costs is a Pareto improvement (i.e., results in a higher expected profit for each type of participant). We now provide a proof of this claim.

Claim 4 *For fixed trading fees, the total expected profit of each participant (market-makers, market-takers, and the trading platform), and therefore, aggregate welfare increases when β or γ decreases.*

Proof of Claim 4: Consider first the aggregate expected profit for market-takers. We have:

$$\Pi_t(\tau_1^*, .., \tau_j^*, ..., \tau_N^*, \bar{\mu}^*; \gamma, \beta, c_m, c_t) = \textstyle\sum_j \Pi_{jt}(\tau_j^*, \bar{\mu}^*; \gamma, \beta, M, N).$$

Thus,

$$\begin{aligned} \frac{d\Pi_t}{d\gamma} &= \textstyle\sum_j \left(\frac{\partial \Pi_{jt}}{\partial \tau_j^*} \frac{\partial \tau_j^*}{\partial \gamma} + \frac{\partial \Pi_{jt}}{\partial \bar{\mu}^*} \frac{\partial \bar{\mu}^*}{\partial \gamma} + \frac{\partial \Pi_{jt}}{\partial \gamma} \right), \\ \frac{d\Pi_t}{d\beta} &= \textstyle\sum_j \left(\frac{\partial \Pi_{jt}}{\partial \tau_j^*} \frac{\partial \tau_j^*}{\partial \beta} + \frac{\partial \Pi_{jt}}{\partial \bar{\mu}^*} \frac{\partial \bar{\mu}^*}{\partial \beta} + \frac{\partial \Pi_{jt}}{\partial \beta} \right). \end{aligned}$$

Now, the envelope theorem implies that $\frac{\partial \Pi_{jt}}{\partial \tau_j^*} = 0$ for all j. Moreover, the cross-side complementarity implies $\frac{\partial \Pi_{jt}}{\partial \bar{\mu}^*} > 0$ for all j, and Corollary 1 yields $\frac{\partial \bar{\mu}^*}{\partial \gamma} < 0$ and $\frac{\partial \bar{\mu}^*}{\partial \beta} < 0$. Last, for all j, $\frac{\partial \Pi_{jt}}{\partial \gamma} = -\frac{1}{2}\left(\tau_j^*\right)^2 < 0$ and $\frac{\partial \Pi_{jt}}{\partial \beta} = 0$. Thus, $\frac{d\Pi_t}{d\gamma} < 0$ and $\frac{d\Pi_t}{d\beta} < 0$. This establishes the first part of Claim 4 for the market-taking side. The proof for the market-making side is parallel. Last, we have proved in Corollary 1 in the paper that the trading rate decreases when β or γ increases. It follows that the expected profit of the platform decreases with β or γ. ■

F Stochastic Needs for Trading

We now provide a detailed analysis of our model when market-takers receive orders from clients at stochastic points in time, as explained in Section 6.1 of the paper. Recall that in this extension, the total gains from trade for the market-maker and the market-taker when a transaction takes place on the platform is $\overline{\pi} = \Gamma - \overline{c} - (1-\alpha)(\Gamma - \Delta)$. We assume that $\overline{\pi} \geq 0$, i.e., $\overline{c} \leq \Gamma - (1-\alpha)\Delta$ as otherwise at least one side would lose money in each transaction (and would therefore not participate to the market). For simplicity, we focus on the case $M = N = 1$.

Monitoring Decisions with Fixed Fees

As in the baseline model, the first step is to analyze traders' monitoring decisions for fixed trading fees. As $M = N = 1$, we denote the market-maker's monitoring intensity by μ and the market-taker's monitoring intensity by τ.

Using the expression for the trading rate given in Equation (23) in the paper, we have that the objective function of the market-maker is

$$\Pi_m = \pi_m \cdot \mathcal{R}\left(\mu, \tau, \kappa\right) - \frac{1}{2}\beta\mu^2 = \frac{\pi_m}{\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\kappa}} - \frac{1}{2}\beta\mu^2$$

and the objective function of the market-taker is

$$\Pi_t = \frac{\pi_t}{\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\kappa}} - \frac{1}{2}\gamma\tau^2.$$

Hence, the first order conditions for the market-maker and market-taker are

$$\beta\mu^3 = \frac{\pi_m}{\left(\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\kappa}\right)^2} \tag{F.1}$$

$$\gamma\tau^3 = \frac{\pi_t}{\left(\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\kappa}\right)^2}. \tag{F.2}$$

We conclude that

$$\frac{\mu}{\tau} = \left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{\frac{1}{3}} = z^{\frac{1}{3}}. \tag{F.3}$$

Using this observation and Equation (F.1), we have that the equilibrium monitoring intensity for the market-maker, μ^*, solves

$$\beta\mu^3\left(\frac{1}{\mu^*}\left(1 + z^{\frac{1}{3}}\right) + \frac{1}{\kappa}\right)^2 = \pi_m. \tag{F.4}$$

That is

$$\mu\left(1 + z^{\frac{1}{3}}\right)^2 + \frac{2\mu^2}{\kappa}\left(1 + z^{\frac{1}{3}}\right) + \frac{\mu^3}{\kappa^2} = \frac{\pi_m}{\beta}. \tag{F.5}$$

Similarly, the equilibrium intensity for the market-taker, τ^*, solves

$$\tau\left(1 + z^{-\frac{1}{3}}\right)^2 + \frac{2\tau^2}{\kappa}\left(1 + z^{-\frac{1}{3}}\right) + \frac{\tau^3}{\kappa^2} = \frac{\pi_t}{\gamma}. \tag{F.6}$$

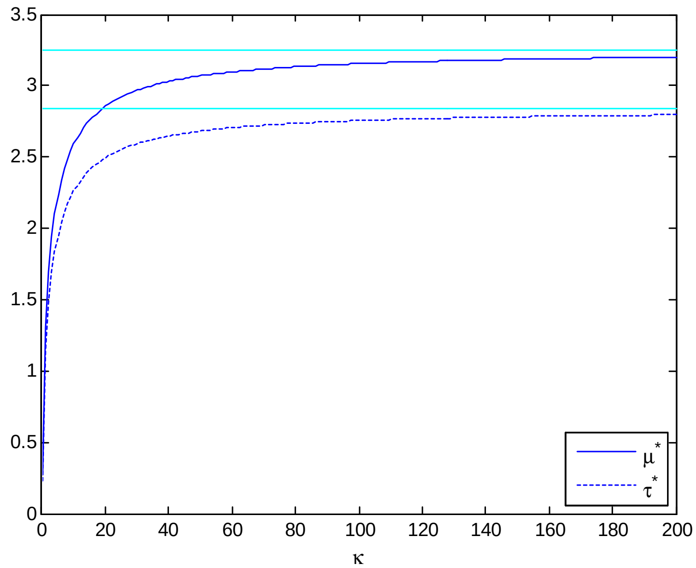

Figure 2: Equilibrium Monitoring Levels and Clients' arrival rate, κ.

It is easy to see that the cubic equations (F.5) and (F.6) have a unique positive solution, (μ^*, τ^*). Furthermore, as κ gets larger, traders' monitoring levels converge to their values in the baseline model. Specifically, (F.5) and (F.6) yield,

$$\lim_{\kappa\to\infty} \mu^* = \left(1+z^{\frac{1}{3}}\right)^{-2} \cdot \left(\frac{\pi_m}{\beta}\right), \text{ and}$$
$$\lim_{\kappa\to\infty} \tau^* = \left(1+z^{-\frac{1}{3}}\right)^{-2} \cdot \left(\frac{\pi_t}{\gamma}\right).$$

This convergence is quite fast as illustrated in Figure 2 where we solve Equations (F.5) and (F.6) numerically for specific parameters of the model, $\Gamma = 25$ cents, $\Delta = 12.5$ cents, $\bar{c} = 0.1$ cent, $\gamma = \beta = 1$. The upward sloping lines depict μ^* and τ^* as a function of κ, whereas the two horizontal lines depict traders' monitoring levels in the baseline case ($\kappa = \infty$).

Applying the implicit function theorem, we can use Equations (F.5) and (F.6) to show that all of the main conclusions in the baseline model still hold in this more general case for any $\kappa > 0$. To start, we will illustrate that the complementarity results in Corollary 1 hold in this case.

Claim 5 *The results of Corollary 1 hold for any* $\kappa > 0$.

Proof of Claim 5: We provide the proof that $\frac{\partial \mu^*}{\partial \gamma} < 0$ and $\frac{\partial \tau^*}{\partial \gamma} < 0$. The proof of the other results in Corollary 1 is similar.

Denote

$$\psi \equiv 1 + z^{\frac{1}{3}}.$$

Then,

$$\frac{\partial \psi}{\partial \pi_m} = \frac{\partial}{\partial \pi_m}\left(1+\left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{\frac{1}{3}}\right) = \frac{1}{3}\left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{-\frac{2}{3}}\frac{\gamma}{\beta}\frac{1}{\pi_t} > 0, \text{ and}$$
$$\frac{\partial \psi}{\partial \pi_t} = \frac{\partial}{\partial \pi_t}\left(1+\left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{\frac{1}{3}}\right) = -\frac{1}{3}\left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{-\frac{2}{3}}\frac{\gamma}{\beta}\frac{\pi_m}{\pi_t^2} < 0.$$

Furthermore,

$$\frac{\partial \psi}{\partial \gamma} = \frac{1}{3}z^{-\frac{1}{3}}\frac{\pi_m}{\pi_t}\frac{1}{\beta} > 0. \tag{F.7}$$

We can rewrite (F.5) as

$$\mu\psi^2 + \frac{2\mu^2}{\kappa}\psi + \frac{\mu^3}{\kappa^2} - \frac{\pi_m}{\beta} = 0. \tag{F.8}$$

Implicitly differentiating (F.8) by γ gives

$$\frac{\partial \mu}{\partial \gamma}\psi^2 + 2\psi\mu\frac{\partial \psi}{\partial \gamma} + \frac{4\mu}{\kappa}\psi\frac{\partial \mu}{\partial \gamma} + \frac{2\mu^2}{\kappa}\frac{\partial \psi}{\partial \gamma} + \frac{3\mu^2}{\kappa^2}\frac{\partial \mu}{\partial \gamma} = 0.$$

Hence,

$$\frac{\partial \mu^*}{\partial \gamma} = -\frac{2\psi\mu^*\frac{\partial \psi}{\partial \gamma} + \frac{2\mu^{*2}}{\kappa}\frac{\partial \psi}{\partial \gamma}}{\psi^2 + \frac{4\mu^*}{\kappa}\psi + \frac{2\mu^{*2}}{\kappa}} < 0,$$

where the inequality follows from (F.7). Also, using (F.3),

$$\frac{\partial \tau^*}{\partial \gamma} = \frac{\partial \mu^*}{\partial \gamma}z^{-\frac{1}{3}} - \frac{1}{3}\mu^* z^{-\frac{2}{3}}\frac{\partial z}{\partial \gamma} < 0.$$

as required. ∎

Now we derive the optimal pricing policy of the platform for all values of κ, that is we provide a proof of Proposition 4 in the paper.

Claim 6 *Suppose $\bar{c} < \Gamma - (1-\alpha)\Delta$. For all parameter values, the optimal make/take fee breakdown does not depend on κ. Moreover, in the thick market case, the optimal make and take fees are*

$$c_m^* = \Delta - \frac{\overline{\pi}}{1+r^{\frac{1}{4}}} \quad \text{and } c_t^* = \bar{c} - c_m^*.$$

where $\overline{\pi} \equiv \pi_m + \pi_t = (\Gamma - \bar{c}) - (1-\alpha)(\Gamma - \Delta)$.

Proof of Claim 6: Remember that $\pi_m = \Delta - c_m$ and $\pi_t = \alpha(\Gamma - \Delta) - c_t$. Thus, there is a one-to-one mapping between traders' profits when a trade takes place and the make/take fees. For this reason, as in the baseline model, for a fixed $\bar{c}$, we can write the platform's problem as

$$Max_{\pi_m,\pi_t}\mathcal{R}(\mu^*,\tau^*)\bar{c} \tag{F.9}$$
$$s.t \quad \pi_t + \pi_m = \overline{\pi}.$$

Let w be the fraction of the total gains from trade that accrues to the market-taker, i.e.,

$$w = \frac{\pi_t}{\overline{\pi}}. \tag{F.10}$$

In equilibrium, $\mathcal{R}(\mu^*, \tau^*) = \frac{1}{\frac{1}{\mu^*}+\frac{1}{\tau^*}+\frac{1}{\kappa}} = \left(\frac{1}{\mu^*}\left(1+z^{\frac{1}{3}}\right)+\frac{1}{\kappa}\right)^{-1}$, where the second equality follows from Equation (F.3). Moreover, using Equation (F.4), we deduce that

$$\left(\frac{1}{\mu^*}\left(1+z^{\frac{1}{3}}\right)+\frac{1}{\kappa}\right)^2 = \frac{\pi_m}{\beta\mu^{*3}} = \frac{(1-w)\,\overline{\pi}}{\beta\cdot\mu^{*3}} \tag{F.11}$$

We conclude that the optimization problem of the platform, F.9, is therefore equivalent to

$$Min_w \ \frac{(1-w)}{\mu^{*3}}. \tag{F.12}$$

Recall that μ^* is the unique solution of Equation (F.4). Clearly, this solution is a function of z. Alternatively it can be written as a function of w since

$$z = \frac{1-w}{w}\frac{\gamma}{\beta}.$$

Using this observation and writing the first order condition for the optimization problem (F.12), we deduce that the optimal w for the platform solves

$$\mu^* + 3\,(1-w^*)\,\frac{\partial\mu^*}{\partial w}\,|_{w=w^*} = 0. \tag{F.13}$$

The L.H.S of this Equation depends on κ since μ^* is also a function of κ (see (F.5)). Yet, we now show that the value of w that solves Equation (F.13) does not depend on κ. To see this, we first rewrite Equation (F.4) as

$$\left(\frac{1+z^{\frac{1}{3}}}{\mu^*}+\frac{1}{\kappa}\right)^2 - \frac{(1-w)\,\overline{\pi}}{\beta\mu^{*3}} = 0.$$

Then, using implicit differentiation with respect to w, we deduce that

$$2\left(\frac{1+z^{\frac{1}{3}}}{\mu^*}+\frac{1}{\kappa}\right)\frac{\frac{1}{3}z^{-\frac{2}{3}}\frac{\partial z}{\partial w}\mu^* - \left(1+z^{\frac{1}{3}}\right)\frac{\partial\mu^*}{\partial w}}{\mu^{*2}} = -\frac{\overline{\pi}}{\beta}\cdot\frac{\mu^*+3\,(1-w)\,\frac{\partial\mu^*}{\partial w}}{\mu^{*4}}. \tag{F.14}$$

Using Equation (F.13), we deduce that the RHS of this Equation is zero when $w = w^*$. Hence, at w^*, Equation (F.14) simplifies to

$$\frac{\partial\mu^*}{\partial w}\,|_{w=w^*} = \frac{\frac{1}{3}z^{-\frac{2}{3}}\frac{\partial z}{\partial w}\mu^*}{1+z^{\frac{1}{3}}}. \tag{F.15}$$

Replacing $\frac{\partial\mu^*}{\partial w}\,|_{w=w^*}$ by this expression in Equation (F.13) we deduce that

$$\mu^* + 3\,(1-w^*)\,\frac{\frac{1}{3}z^{-\frac{2}{3}}\frac{\partial z}{\partial w}\mu^*}{1+z^{\frac{1}{3}}} = 0.$$

That is,

$$1+(1-w^*)\,\frac{z^{-\frac{2}{3}}\frac{\partial z}{\partial w}}{1+z^{\frac{1}{3}}}=0.$$

As $z=\frac{1-w^*}{w^*}\frac{\gamma}{\beta}$, this Equation implicitly characterizes w^*. It does not depend on κ. We deduce that the optimal make/take fee breakdown for the platform (which is fixed by w^*) does not depend on κ, as claimed.

Recall that, in Claim 2 in this Internet Appendix, we have derived the optimal make and take fees for the platform when $M=N=1$ and κ is infinite. We deduce from this characterization that

$$w^*=\frac{r^{\frac{1}{4}}}{1+r^{\frac{1}{4}}}. \tag{F.16}$$

Moreover, we deduce from Equation (F.10) that the optimal make/take fee breakdown is such that

$$\pi_t=w^*(\Gamma-\overline{c}-(1-\alpha)(\Gamma-\Delta)),$$

that is

$$c_t^*=(\Gamma-\Delta)(1-(1-\alpha)(1-w^*))-w^*(\Gamma-\overline{c}).$$

Using Equation (F.16) and the fact $c_t^*+c_m^*=\overline{c}$, we deduce that

$$c_m^*=\Delta-\frac{\overline{\pi}}{1+r^{\frac{1}{4}}}.\ \blacksquare$$

References

[1] Paul Milgrom, P., and C. Shannon, 1994, Monotone Comparative Statics, Econometrica 62, 157-180.