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# Multitask principal–agent problems: Optimal contracts, fragility, and effort misallocation

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## Abstract

We analyze a tractable class of multitask principal–agent problems, such as the one faced by a firm with a manager overseeing several projects. We allow for tasks to be complements or substitutes. We avoid the problems associated with the first-order approach by directly characterizing the shape of the agent's indirect utility function, which exhibits a convex then concave shape in effort. We identify a new source of allocational inefficiency across tasks: excessive concentration, and its consequence, insufficient risk taking. Optimal incentive schemes in our environment are generally “fragile”: small changes in fundamentals can cause the agent's effort to collapse.

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## 1. Introduction

In this paper we address a principal–agent problem in which the principal has many different tasks for the agent to execute, the agent has limited liability, and the marginal product of the agent's effort is constant and independent across tasks. Successes on different tasks may be either substitutes or complements as far as the principal is concerned. We characterize the constrained optimal effort of the agent, along with the contract that implements it. For the case in which

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tasks are complements we identify a new source of allocational inefficiency, namely excessive concentration of effort, and its consequence, insufficient risk taking. We demonstrate that under many circumstances the agent is indifferent between the constrained optimal effort allocation and shirking on all tasks. That is, at the constrained optimum the incentive constraint relating to shirking on all tasks binds.

The class of problems we analyze corresponds to the one faced by a firm where a manager oversees several R&D projects. The principal has several projects or tasks managed by an agent, and the outcome of each task is either success or failure, with the likelihood of success depending on the agent's continuous effort choice in each task. The principal does not observe the agent's effort choices and can only contract on the final output realization, i.e., on the number of successful tasks. We allow for both increasing and decreasing returns in the number of successful tasks, so that tasks may be either complements or substitutes.

There are a number of prominent examples of multitask settings with complementarity among tasks. In the R&D setting, this is the case whenever there are synergies among tasks or when failure in any task disproportionately reduces the output value—as in the failure of the O-rings that led to the space shuttle Challenger disaster, detailed by Kremer [10]. When an auditor checks the balance sheets of multiple divisions of the same firm, failure on just one component of the audit may allow the firm to hide cash from its outside investors. Again, the production function has the O-ring property—failure on one task causes a large drop in output value (of the audit). The relation between the owner (principal) of a bank or financial intermediary and its manager (agent) who must monitor the loans or investments made on behalf of the owner is another example—the principal holds a convex claim on the value of the financial institution. To give a final example, in the case of a regulated and monopolistic “network” industry (electricity distribution, fixed-line telecommunications, etc.) the relevant social welfare function is often convex in the percentage of the network that functions properly. Of course, there are also many situations where the production function is linear or concave—neoclassical production functions, for example—and tasks are substitutes.

In this environment, we identify a new source of allocational inefficiency across tasks. We show that the allocation of effort in which the agent works on a few tasks and completely shirks on the complementary set of tasks—an “all-or-nothing” strategy—yields a concentrated distribution with the lowest output volatility. At the other extreme, an allocation of effort in which the agent distributes effort equally across all tasks yields the “most spread out” distribution, with the highest output volatility. However, in our environment the agent's response to any compensation contract is to adopt a conservative all-or-nothing strategy. Consequently, when the production function features increasing returns to scale, and so output volatility is desired, the agent's allocation of effort is suboptimal. We provide examples in the paper illustrating situations in which this type of inefficiency occurs.

Our approach in this paper is to restrict attention to a special, and yet still quite general, class of multitask problems. In particular, our basic model assumes that the agent is risk-neutral with limited liability; that effort costs are linear; and that the  $n$  different tasks are symmetric.<sup>1</sup> Although admittedly special, many applications are plausibly approximated by this framework.

In return for these assumptions, we are able to comprehensively characterize the solution for the class of environments described. For example, the standard solution method for principal–agent models is the replacement of the agent's incentive compatibility (IC) condition with just the

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<sup>1</sup> We discuss the robustness of our analysis to all three of these assumptions.

local or first-order incentive constraints. However, it has long been appreciated that this so-called *first-order approach* can be problematic. We show that in our multitask setting this approach fails. More interestingly, though, we show that if the local incentive constraint is combined with an “extremal deviation” constraint—the agent does not like the specific deviation of shirking on every task—then the two constraints together are sufficient to guarantee incentive compatibility. The reason why the local incentive constraint by itself fails, but is sufficient when combined with the extremal deviation constraint, is that in our environment the agent’s utility function has a convex then concave shape in total effort.

Our analysis relies on the informational structure of our environment. The information observed by the principal is the number of successes of independent trials with varying probabilities of success. The associated probabilities form a *Pólya frequency sequence*. Pitman [19] provides an excellent review of the properties of such sequences. Our results on effort misallocation use Hoeffding’s [6] analysis of the mean and variance of the number of successes.

We study the contracting problem described under two distinct and commonly used contracting constraints. First, we consider the case in which the principal faces an upper bound on the amount he can pay the agent. This constraint can arise for various reasons. Perhaps most obviously, the principal may simply be budget constrained. Other motivations include the possibility that the principal would default on payments in excess of some amount;<sup>2</sup> a desire to limit the pay of an employee to less than his/her supervisor; external constraints on pay, such as those a board of trustees may impose on a university; and related, public pressure to avoid very large payments to employees such as CEOs.<sup>3</sup> The second contracting constraint we consider is that the principal’s payoff after compensating the agent must be monotonically increasing in output. Innes [11] offers two possible rationales for this restriction: if it is not satisfied, then the principal has the incentive to burn output; and the agent has the incentive to secretly borrow funds to “increase” output.

Many of our results hold for both constraints. The main implication that depends on which constraint is imposed relates to how changes in the incentive scheme affect the agent’s choice of effort. Specifically, if the agent’s compensation is subject to an upper bound then the agent’s total effort is increasing in the strictness of the contract (i.e., the performance level required for a certain payoff), while the opposite holds under monotonicity.

A notable economic property of the class of multitask problems we analyze is that the optimal incentive scheme is often fragile. Small unanticipated changes in fundamentals, such as increases in the agent’s cost of effort, can lead to very large effects on the agent’s effort choice such as complete shirking when the extremal deviation constraint is binding. A possible application is to financial intermediaries. Specifically, in the case of the bank or financial intermediary’s manager monitoring several loans or investments in behalf of its owner, the manager’s actions can affect the probability that each of the investments produces a good return. The agent is compensated with a flat fee if the total return is high enough (the budget constraint case), or by holding a levered equity position (the monotonicity constraint). Our model predicts that the optimal compensation contract is fundamentally fragile, in the sense described above. In this, we concur with the sizeable literature that sees occasional financial crises as *ex post* regrettable yet *ex ante* optimal (see, e.g., Diamond and Rajan [5], or Allen and Gale [1]).

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<sup>2</sup> That is, the principal has limited commitment.

<sup>3</sup> The last three of these are suggested by Kadan and Swinkels [9].

We end this section with a discussion of the most closely related research on multitask principal–agent problems. We refer the readers to the recent survey by Dewatripont, Jewitt, and Tirole [4] for other important papers in the literature that are less related to ours.

Hölmstrom and Milgrom [8] were the first to formally address the multitask principal–agent model. A key insight of their work is that low powered incentive schemes arise as the optimum compensation scheme when there are differences in measurement accuracy: the desirability of providing incentives in one activity decreases with the difficulty of measuring performance in other activities when both activities make competing demands on the agent’s time. Hölmstrom and Milgrom’s model is general in some dimensions, but restrictive in others. In particular, it only allows for linear contracts. We look at a more specific class of problems but consider a more general form of incentive contracts. We show that the optimal contract is highly non-linear, and that some previously unnoted distortions in agent’s effort allocation arise in our model.

MacDonald and Marx [15] analyze a two-task principal–agent model where activities making competing demands on the agent’s time, and the agent prefers (or has lower cost) working on one of the tasks. The tasks are complementary for the principal. Their main result is that the difference in the cost of exerting effort in each task creates a distortion in the allocation of effort. The optimal compensation alleviates this problem by leading the agent to view the activities as complements, and this complementarity is typically achieved using a contract that is non-monotone. The most important difference between our paper and theirs is that the tasks in our model are completely symmetric, and the agent has no preference between them. As such, we isolate a distinct source of inefficient effort concentration.

An important benchmark for our analysis is provided by Laux [13], who analyzes the special case of our model in which (i) neither an upper bound on payments nor a monotonicity constraint is imposed,<sup>4</sup> (ii) the agent’s effort choice on each task is binary, and (iii) the production function is linear. We discuss our relation to his results in detail below (see pages 184 and 191).

The contracting constraints we impose (i.e., an upper bound on payments to the agent and the monotonicity constraint) play a significant role in our analysis. Innes [11] and Matthews [16] have analyzed single-task principal–agent problems under the monotonicity constraint. The main focus of both papers is to show that the optimal contract is debt-like. Kadan and Swinkels [9]<sup>5</sup> study a principal–agent model in which payments to the agent are subject to an upper and lower bound. They characterize the optimal contract, and derive comparative statics for the solution to the cost minimization problem<sup>6</sup> with respect to these bounds. Relative to these papers we characterize the optimal contract beyond its basic shape. In particular, our multitask environment delivers enough structure for us to analyze the agent’s effort choice without imposing the convexity of the distribution function assumption (CDFC, see page 185 below).<sup>7</sup> Indeed, we show that the incentive constraint relating to the agent exerting minimal effort often binds at the optimal

<sup>4</sup> Laux’s Proposition A1 shows that if the principal is subject to a limited liability constraint it is still optimal to assign many tasks to a single agent. However, he does not derive the optimal contract for this case.

<sup>5</sup> See also Hölmstrom [7] and Page [18].

<sup>6</sup> The cost minimization problem consists of minimizing the principal’s expected cost of inducing a given amount of effort.

<sup>7</sup> Kadan and Swinkels impose CDFC. Innes and Matthews do not require this assumption to show the monotonicity constraint implies that the optimal contract is debtlike, and the “face value” of debt in their papers is determined by the principal’s participation constraint. Innes also studies his environment with an upper bound on compensation in place of the monotonicity constraint; for this part of his analysis he too imposes CDFC.

contract. The multitask aspect of our analysis also generates new results on the misallocation of effort, discussed above.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 contains a numerical example highlighting our key results. Section 4 demonstrates that the contracting problem is equivalent to one in which the only incentive constraints are the local and extremal ones. Section 5 derives a comparative static for the agent's effort with respect to the incentive scheme, and uses this result to show that the extremal deviation constraint often binds. Section 6 derives our results on the misallocation of effort. Section 7 considers a limiting version of our economy with an infinite number of tasks, while Section 8 allows for convexity in the agent's cost of effort. Section 9 describes how our analysis would change if the principal's payoff must be monotonically increasing in output. Section 10 considers the case in which either the principal or agent can distinguish between tasks. Finally, Section 11 concludes.

## 2. The model

A principal is endowed with  $n$  symmetric projects and employs an agent to work on them. Each project can result in either success or failure, and the total value of the output is  $f_k \in \mathfrak{N}$ , a function of the number of successful projects  $k = 0, \dots, n$ . The production function  $f_k$  may be concave, linear or convex in  $k$ , representing situations in which there are, respectively, decreasing, constant and increasing returns to success.

The principal pays an agent to undertake (simultaneously) the  $n$  tasks. The agent's actions determine the probability of success in each task. Specifically, for each task  $i$  the agent's effort on that task determines a success probability  $p_i \in [\varepsilon, \pi]$  for that task, where  $0 \leq \varepsilon < \pi \leq 1$ . The expected number of successes is thus  $\sum_i p_i$ . In our basic model we assume that the agent's effort costs are linear in the expected number of successes, and so after normalization the agent's cost of achieving success probability  $p_i$  on task  $i$  is simply  $p_i$ . (In Section 8 we allow for a convex cost of effort.) So the agent effectively chooses a multidimensional effort vector  $\mathbf{p} = (p_1, \dots, p_n) \in [\varepsilon, \pi]^n$ . We write  $e(\mathbf{p})$  for total effort  $\sum_i p_i$ .

The agent's effort allocation is not observable but the number of successful tasks is, and the principal rewards the agent according to the number of successes. Specifically, the principal offers the agent an incentive scheme  $\mathbf{w} = (w_k)_{k=0}^n$  where  $w_k$  is the agent's reward when there are  $k$  successful tasks.<sup>8</sup>

Both the agent and the principal are assumed to be risk neutral.<sup>9</sup> Let  $A_{n,k}(\mathbf{p})$  be the probability of  $k$  successes in the  $n$  tasks under effort  $\mathbf{p}$ . For use throughout the paper, note that  $A_{n,k}(\mathbf{p}) = 0$  if  $k < 0$  or  $k > n$ . The agent's expected compensation given a contract  $\mathbf{w}$  and effort  $\mathbf{p}$  is  $W(\mathbf{p}; \mathbf{w}) \equiv \sum_{k=0}^n w_k A_{n,k}(\mathbf{p})$ . Expected output given effort  $\mathbf{p}$  is  $F(\mathbf{p}) \equiv \sum_{k=0}^n f_k A_{n,k}(\mathbf{p})$ . Thus the principal's utility from contract  $\mathbf{w}$  and agent effort  $\mathbf{p}$  is  $F(\mathbf{p}) - W(\mathbf{p}; \mathbf{w})$ , while the agent's indirect utility function is  $u(\mathbf{p}; \mathbf{w}) \equiv W(\mathbf{p}; \mathbf{w}) - e(\mathbf{p})$ .

<sup>8</sup> In Section 10 below we discuss how our results would change if instead the contract were able to depend on which tasks succeed.

<sup>9</sup> The only result for which agent risk-neutrality is used is Proposition 1, which establishes that the principal always employs a "cutoff" incentive scheme (see (7)). Conditional on a compensation scheme of this type being used, all our other results would hold for general preferences.

The principal's problem is to choose an incentive scheme  $\mathbf{w}$  so as to maximize his utility subject to the agent's incentive constraint

$$\mathbf{p} \in \arg \max_{\tilde{\mathbf{p}} \in [\varepsilon, \pi]^n} u(\tilde{\mathbf{p}}; \mathbf{w}) = \arg \max_{\tilde{\mathbf{p}} \in [\varepsilon, \pi]^n} W(\tilde{\mathbf{p}}; \mathbf{w}) - e(\tilde{\mathbf{p}}). \quad (\text{IC})$$

Moreover, the agent has an outside option  $\underline{u}$ , and so the individual rationality or participation constraint is

$$u(\mathbf{p}; \mathbf{w}) \geq \underline{u}. \quad (\text{IR})$$

We impose several constraints on the incentive schemes or contracts that are allowed. A maintained assumption is that the agent cannot receive a compensation below a certain threshold. The *minimum payment* constraint arises naturally, for example, if the agent has limited liability, if regulations impose minimum wages, or if social norms dictate minimum standards of living. It is well known that without a floor on the agent's wage payment and risk-neutrality, the first-best effort level could be easily achieved. For simplicity we assume that the minimum payment the agent can receive in any state of the world is zero,<sup>10</sup>

$$w_k \geq 0 \quad \text{for all } k = 0, \dots, n. \quad (\text{LL})$$

Under this limited liability assumption, the (IR) constraint may not be binding, as is typically the case in principal agent problems without the (LL) constraint.

As discussed, we analyze the problem under two alternative restrictions on the contracts. The first is a *budget constraint* (BC), which restricts the principal from paying more than  $\bar{w} > 0$  to the agent,

$$w_k \leq \bar{w} \quad \text{for all } k = 0, \dots, n. \quad (\text{BC})$$

The introduction includes a discussion of possible motivations for (BC). In Section 9 below we replace (BC) with the constraint that the principal's payoff must be increasing in the number of successes.<sup>11</sup>

In summary, the principal's optimization problem is to choose an incentive scheme  $\mathbf{w}$  so as to maximize his utility subject to the constraints,

$$\max_{\mathbf{p}, \mathbf{w}} F(\mathbf{p}) - W(\mathbf{p}; \mathbf{w}) \quad \text{such that IC, IR, LL, BC.}$$

We assume that higher effort by the agent is socially efficient, so that the first best level of effort is to work hard on all tasks.<sup>12</sup>

Whenever the agent is indifferent between several effort choices, we assume he picks the one most favorable to the principal. Given the linearity of effort costs, and the fact that the probability  $A_{n,k}(\mathbf{p})$  is linear in each component of  $\mathbf{p}$ , we obtain the following immediate but useful result:

**Lemma 1.** *For any incentive scheme  $\mathbf{w}$  the agent chooses either the minimum or maximum effort on each task.*

<sup>10</sup> The qualitative implications of our model would be unchanged if instead we assumed that the agent's minimum payment were  $\underline{w} \neq 0$ .

<sup>11</sup> Note that (for tractability) we never simultaneously impose BC and the monotonicity constraint. However, the monotonicity constraint implies that all payments to the agent lie below  $f_n$ , and so implicitly contains a mild budget constraint of its own.

<sup>12</sup> Specifically, we assume that the expected output net of effort cost,  $F(\mathbf{p}) - e(\mathbf{p})$ , is increasing in each component of the effort vector  $\mathbf{p}$ .

In light of Lemma 1, we say that the agent *works* (respectively, *shirks*) on a task if he exerts the maximum effort  $\pi$  (respectively, minimum effort  $\varepsilon$ ). Intermediate effort levels enter our analysis only in Section 6, where we compare the agent's allocation of a given amount of effort to the efficient allocation. Notationally, we write  $\mathbf{p}_s^n$  for the effort allocation in which the agent works (exerts effort  $\pi$ ) on  $s$  of  $n$  tasks and shirks (exerts effort  $\varepsilon$ ) on the remaining  $n - s$  tasks. (We omit the superscript  $n$  when we can do so without confusion.) Hence the agent chooses effort allocation  $\mathbf{p}$  if and only if  $\mathbf{p} = \mathbf{p}_s^n$  for some integer  $s$ , and the agent prefers  $\mathbf{p}_s^n$  to all alternative  $\mathbf{p}_{\tilde{s}}^n$ , with strict preference for  $\tilde{s} > s$ .

### 3. Illustrative cases

#### 3.1. A numerical example highlighting our main results

Consider a situation with  $n = 15$  tasks; the production function  $f_k = e^{k/2}$ ; the success probabilities for shirking and working  $\varepsilon = .2$  and  $\pi = .8$ ; the budget-constrained principal's maximum payment  $\bar{w} = 10$ ; and the agent's outside option  $\underline{u} = 0$ . Following Innes [11] (see Proposition 1 below), the principal pays the agent the maximum amount possible,  $\bar{w}$ , if he succeeds on more than a prespecified "cutoff" number of tasks  $m$ ; pays  $\bar{w}$  with probability  $1 - \mu$  if he succeeds on exactly  $m$  tasks; and pays nothing otherwise. The optimal contract has  $m = 6$  and  $\mu = 0.843$ . The agent works on 8 of the 15 tasks while shirking on the remaining 7, so his total effort is  $E = 8 \times .8 + 7 \times .2 = 7.8$ .

Two key results help to determine the optimal contract. First, the agent's expected utility is first convex and then concave in effort for any given cutoff incentive contract (Proposition 2 and Fig. 1). Second, if the agent works on an intermediate number of tasks and the compensation contract is then made tougher (i.e.,  $m + \mu$  increases, so the agent is paid less often), the agent's marginal return to effort increases (Proposition 4 and Fig. 1).

The second of these results implies that the effort level associated with the local maximum of the agent's utility increases when the compensation contract is made tougher. This is illustrated in Fig. 1: the dashed, solid, and dash-dot lines correspond to successively tougher compensation contracts. It is not possible to find the optimal contract using the first-order approach (FOA) here. Focusing on the first-order condition would lead one to conclude that the principal always benefits from making the contract tougher, since the agent works harder and is paid less. However, this leads to the obviously incorrect conclusion that the agent is *never* paid under the optimal contract.

The problem with using the FOA is that when the contract is made tougher the agent may switch to the corner of shirking on all tasks. However, our result that the agent's utility is convex then concave generates a tractable extension of the FOA, in which one needs to add only a single "extremal" incentive constraint. At the optimal contract the local and extremal incentive constraints both bind: the agent is indifferent between the local maximum and the corner solution of shirking on all tasks (see the solid line in Fig. 1).

Finally, at the optimal contract the agent's effort differs from the social optimum in two ways. First, the agent exerts *too little* effort, since he shirks on some tasks. Second, and less obviously, the agent *misallocates* his total effort ( $E = 7.8$ ) across the tasks. Specifically, variance in the number of successes is desirable in this example because the production function is convex. We show in Proposition 6 that variance is maximized by allocating a given amount of effort evenly across all tasks—in this case, exerting effort  $7.8/15 = 0.52$  on each of the 15 tasks. But the agent

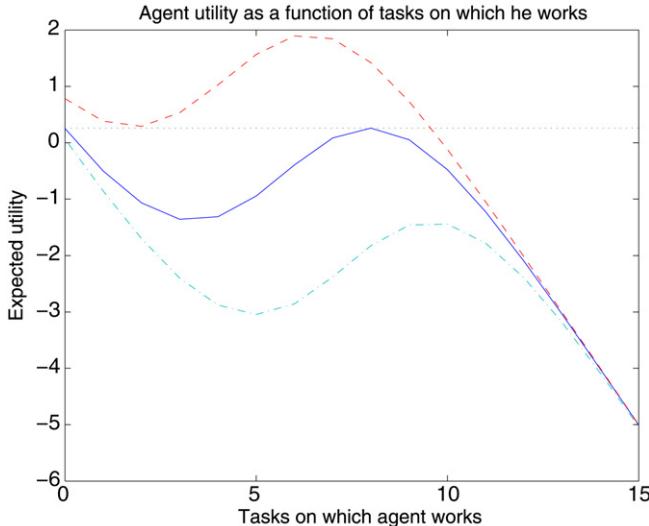


Fig. 1. The dashed, solid, and dash-dot lines represent the agent's expected utility as a function of the number of tasks on which he works, under successively stricter compensation contracts. The solid line corresponds to the optimal contract:  $m = 6$  and  $\mu = .843$ . Under the optimal contract the agent is indifferent between working on 8 tasks (which corresponds to the local maximum of his utility) and shirking on all tasks.

misallocates his effort by working too hard on some tasks and not hard enough on others, which instead minimizes the variance of the number of successes.

### 3.2. The limiting case of infinite tasks ( $n = \infty$ )

Our analysis relies heavily on the two results discussed above: convexity-then-concavity of expected utility in effort, and the effect of contract changes on the agent's marginal return to effort. These results are relatively easy to derive in the limiting case in which the number of tasks is infinite,  $n = \infty$ . We discuss this case informally here, and provide full details in Section 7. In the limit economy, the agent picks total effort  $E$  from some interval, and the probability of  $k$  successes is  $A_k(E) \equiv \frac{E^k e^{-E}}{k!}$  (the Poisson distribution). Under the additional simplifying assumption that contracts make no use of randomization (i.e.,  $\mu = 0$ ),<sup>13</sup> the agent's expected utility is  $\sum_{k=m}^{\infty} \bar{w} A_k(E) - E$ . Because  $A'_k(E) = A_{k-1}(E) - A_k(E)$ , the first and second derivatives of expected utility with respect to effort are  $\bar{w} A_{m-1}(E) - 1$  and  $\bar{w} (A_{m-2}(E) - A_{m-1}(E)) = \bar{w} A_{m-2}(E) (1 - \frac{E}{m-1})$  respectively. So expected utility is convex in effort for low effort levels, and then concave for high effort levels. To understand the second result discussed above, suppose that the agent chooses an interior effort level  $E$  for some contract. From the second-order condition,  $A_{m-2}(E) < A_{m-1}(E)$ . The Poisson distribution is unimodal, and so this condition says  $m - 1$  is below the mode.<sup>14</sup> Consequently, if the principal changes the contract to pay the agent less, i.e., increases  $m$ , the agent's marginal return to effort increases.

<sup>13</sup> See Section 7 for the full argument when  $\mu \neq 0$ .

<sup>14</sup> Our formal analysis handles the case in which the mode is exactly at  $m - 1$ .

## 4. The contracting problem

### 4.1. Preliminaries

For any effort allocation  $\mathbf{p}$ , the distribution of the number of successes among the  $n$  tasks is a Poissonian binomial distribution. (The special case in which all components of  $\mathbf{p}$  coincide is, of course, the standard binomial distribution.) The corresponding sequence of probabilities,  $A_{n,k}(\mathbf{p})$ , is a Pólya frequency sequence (in  $k$ ). Much is known about the properties of such sequences (see Pitman [19] for an excellent overview), of which the most important for our purposes is that they satisfy log-concavity:

$$\frac{A_{n,k}(\mathbf{p})}{A_{n,k-1}(\mathbf{p})} > \frac{A_{n,k+1}(\mathbf{p})}{A_{n,k}(\mathbf{p})}. \quad (1)$$

Throughout the paper, we repeatedly make use of the following decomposition of  $A_{n,k}(\mathbf{p})$ :

$$A_{n,k}(\mathbf{p}) = p_i A_{n-1,k-1}(\mathbf{p}^{-i}) + (1 - p_i) A_{n-1,k}(\mathbf{p}^{-i}), \quad (2)$$

where  $\mathbf{p}^{-i}$  is the  $n - 1$ -vector of probabilities formed from  $\mathbf{p}$  by removing its  $i$ th component. In words, (2) says that the probability of  $k$  successes on  $n$  tasks is equal to the probability of success in the  $i$ th task times the probability of  $k - 1$  successes in the remaining  $n - 1$  tasks, plus the probability of failure in the  $i$ th task times the probability of  $k$  successes in the remaining  $n - 1$  tasks.

An immediate implication of log-concavity is that the sequence  $A_{n,k}(\mathbf{p})$  is unimodal:  $A_{n,k}(\mathbf{p})$  is increasing and then decreasing in  $k$ , with consecutive terms equal at most once. Using (2), it is easily established that the mode is itself increasing in effort, in the following sense (unless otherwise stated, all proofs are in Appendix A).

**Lemma 2.** *If  $A_{n,m}(\mathbf{p}) \geq A_{n,m-1}(\mathbf{p})$  for some  $\mathbf{p} \in [0, 1]^n$  then  $A_{n,m}(\tilde{\mathbf{p}}) \geq A_{n,m-1}(\tilde{\mathbf{p}})$  for any  $\tilde{\mathbf{p}} \in [0, 1]^n$  that is componentwise greater than  $\mathbf{p}$ .*

Log-concavity (1) also combines with the decomposition (2) to imply the monotone likelihood ratio property (MLRP):

**Lemma 3.** *If  $\tilde{\mathbf{p}}$  and  $\mathbf{p}$  are effort allocations such that  $\tilde{\mathbf{p}} \neq \mathbf{p}$  is componentwise greater than  $\mathbf{p}$ , then whenever  $l > k$ ,*

$$\frac{A_{n,l}(\tilde{\mathbf{p}})}{A_{n,l}(\mathbf{p})} > \frac{A_{n,k}(\tilde{\mathbf{p}})}{A_{n,k}(\mathbf{p})}. \quad (\text{MLRP})$$

Lemma 3 is easily proved. Suppose that  $\tilde{\mathbf{p}}$  and  $\mathbf{p}$  differ only in component  $i$ , and  $\tilde{p}_i > p_i$ . Lemma 3 will clearly follow by iteration if we can show that

$$\frac{A_{n,k+1}(\tilde{\mathbf{p}})}{A_{n,k+1}(\mathbf{p})} > \frac{A_{n,k}(\tilde{\mathbf{p}})}{A_{n,k}(\mathbf{p})}. \quad (3)$$

To establish that inequality (3) holds, simply substitute in decomposition (2) and apply (1). (Details are given in Appendices A–C.)

Finally, an important property of (2) is that it enables the effect of a change in effort to be expressed in terms of probability differences for a *given* amount of effort:

$$\frac{\partial A_{n,k}(\mathbf{p})}{\partial p_i} = A_{n-1,k-1}(\mathbf{p}^{-i}) - A_{n-1,k}(\mathbf{p}^{-i}), \quad (4)$$

a particularly useful form of which is<sup>15</sup>

$$A_{n,k}(\mathbf{p}_s^n) - A_{n,k}(\mathbf{p}_{s-1}^n) = (\pi - \varepsilon)(A_{n-1,k-1}(\mathbf{p}_{s-1}^{n-1}) - A_{n-1,k}(\mathbf{p}_{s-1}^{n-1})). \quad (5)$$

The identity (4) is in turn useful in evaluating changes in expected compensation and output, i.e.,

$$\begin{aligned} \frac{\partial}{\partial p_i} W(\mathbf{p}; \mathbf{w}) &= \sum_{k=0}^n (A_{n-1,k-1}(\mathbf{p}^{-i}) - A_{n-1,k}(\mathbf{p}^{-i})) w_k \\ &= \sum_{k=0}^{n-1} (w_{k+1} - w_k) A_{n-1,k}(\mathbf{p}^{-i}), \end{aligned} \quad (6)$$

along with an analogous expression for  $\frac{\partial}{\partial p_i} F(\mathbf{p})$ .

#### 4.2. Benchmark: non-binding (BC)

As a benchmark, consider the optimal contract if (BC) does not bind (that is,  $\bar{w}$  is sufficiently high). Given Lemma 1, this is precisely the problem studied by Laux [13]. He shows that MLRP implies the optimal incentive scheme rewards the agent only when he succeeds on all tasks (i.e.,  $w_k = 0$  if  $k < n$ ). In this case,  $u(\mathbf{p}; \mathbf{w}) = A_{n,n}(\mathbf{p})w_n - e(\mathbf{p})$ , and so from (5) the agent's utility gain from working on  $s$  tasks in place of  $s-1$ , i.e.,  $u(\mathbf{p}_s; \mathbf{w}) - u(\mathbf{p}_{s-1}; \mathbf{w})$ , equals

$$(A_{n,n}(\mathbf{p}_s) - A_{n,n}(\mathbf{p}_{s-1}))w_n - (\pi - \varepsilon) = (\pi - \varepsilon)(A_{n-1,n-1}(\mathbf{p}_{s-1}^{n-1}) - 1).$$

Because  $A_{n-1,n-1}(\mathbf{p}_{s-1}^{n-1})$  is increasing in  $s$  (from (5) again), it follows that  $u(\mathbf{p}_s; \mathbf{w})$  is convex in  $s$ . Thus the agent either works on all tasks or shirks on all tasks under the optimal incentive scheme. Moreover, if the agent works on all tasks, the optimal payment  $w_n$  for success on all tasks is determined either by the (IR) constraint,<sup>16</sup> or from the incentive constraint that the agent prefers working on all tasks to shirking on all tasks. This particular incentive constraint plays an important role in our analysis, and we refer to it as the *extremal* incentive constraint.

When  $\bar{w}$  is finite and the agent's outside option is sufficiently low that (IR) does not bind,<sup>17</sup> (BC) binds whenever the number of tasks  $n$  is sufficiently large. To see this, observe that without constraint (BC) the optimal incentive scheme either pays  $w_{\text{Laux}} \equiv (\pi - \varepsilon)n/(A_{n,n}(\mathbf{p}_n^n) - A_{n,n}(\mathbf{p}_0^n)) = (\pi - \varepsilon)n/(\pi^n - \varepsilon^n)$  when the agent succeeds on all tasks, and induces the agent to work on all tasks; or pays nothing at all, and the agent shirks on all tasks. The principal prefers the former whenever  $n$  is large enough since the agent's surplus  $A_{n,n}(\mathbf{p}_0^n)w_{\text{Laux}}$  approaches zero. So (BC) binds for  $n$  large enough since  $w_{\text{Laux}} \rightarrow \infty$ .

#### 4.3. Cutoff incentive schemes

The remainder of the paper deals with the case in which (BC) may bind.<sup>18</sup> Given risk-neutrality and MLRP it is a standard result that the principal can maximize his payoff by using

<sup>15</sup> Recall that  $A_{n,k}(\mathbf{p})$  is linear in each component of  $\mathbf{p}$ .

<sup>16</sup> Laux [13] assumes that (IR) does not bind.

<sup>17</sup> Specifically,  $\underline{u} \leq -n\varepsilon$ , where recall that  $-n\varepsilon$  is the agent's utility if he shirks on all tasks and receives no compensation.

<sup>18</sup> Section 9 deals with the case in which (BC) is replaced by a monotonicity constraint.

an incentive scheme with the following pair of properties<sup>19</sup>: (1) if the number of successes is strictly less than some critical level,  $m$  say, then the agent receives nothing, and (2) if the number of successes is strictly above  $m$ , then the agent's compensation is such that (BC) holds with equality. That is, the principal picks an incentive scheme of the form

$$w_k = \begin{cases} 0 & \text{if } k < m, \\ (1 - \mu)\bar{w} & \text{if } k = m, \\ \bar{w} & \text{if } k > m \end{cases} \quad (7)$$

for some integer  $m$  and  $\mu \in [0, 1]$ . We denote this incentive scheme by  $\mathbf{w}^{m,\mu}$ , and designate any scheme of this type a *cutoff* incentive scheme. Note that any cutoff incentive scheme is completely identified by the pair of parameters  $m$  and  $\mu$ .

**Proposition 1.** *Suppose the principal does not use a cutoff incentive scheme. Then there is a cutoff incentive scheme that gives the principal a (weakly) higher payoff.*

The fact that we can restrict attention to cutoff compensation schemes plays an important role in our analysis, because for this class of compensation schemes Proposition 2 below fully characterizes the curvature properties of the agent's indirect utility function  $u$ .

#### 4.4. The first-order approach (FOA)

Given Lemma 1 and Proposition 1, the principal's contracting problem reduces to:

$$\max_{s, \mathbf{w} \text{ a cutoff incentive scheme}} F(\mathbf{p}_s) - W(\mathbf{p}_s; \mathbf{w})$$

such that

$$u(\mathbf{p}_s; \mathbf{w}) \geq \max \left\{ \max_{\tilde{s}} u(\mathbf{p}_{\tilde{s}}; \mathbf{w}), \underline{u} \right\}.$$

Relative to the initial problem, this represents a substantial simplification. The number of incentive constraints that we need to consider has been reduced to  $n$  constraints, and we can restrict attention to cutoff incentive schemes. Nonetheless, even  $n$  constraints is still a significant number, and is tempting to apply the so-called FOA and replace these  $n$  constraints by the local or first-order condition.

The FOA is the standard method for tackling principal–agent problems with a large number of incentive constraints, and consists of imposing only the condition that the agent's action is preferred to “local” deviations. Although widely used, it has long been realized that this approach is not always valid (see Mirrlees [17]). The most widely used sufficient conditions are those established by Rogerson [20], namely MLRP and the convexity of the distribution function condition (CDFC). In our context, CDFC requires that

$$\sum_{k=0}^K (A_{n,k}(\mathbf{p}_s) - A_{n,k}(\mathbf{p}_{s-1}))$$

is increasing in  $s$  for all  $K$ . Substituting in (5), the CDFC in our setting is thus equivalent to the requirement that

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<sup>19</sup> See Innes [11].

$$\sum_{k=0}^K (\pi - \varepsilon)(A_{n-1,k-1}(\mathbf{p}_{s-1}^{n-1}) - A_{n-1,k}(\mathbf{p}_{s-1}^{n-1})) = -(\pi - \varepsilon)A_{n-1,K}(\mathbf{p}_{s-1}^{n-1})$$

be increasing in  $s$  for all  $K$ . However, this condition is *never* satisfied.<sup>20</sup>

#### 4.5. The shape of the agent's indirect utility function

To circumvent the limitations of the FOA, we take advantage of the specific properties of the environment to directly characterize the curvature properties of the agent's indirect utility function. As noted, the agent chooses an effort allocation of the form  $\mathbf{p}_s$ , i.e., the agent exerts effort  $\pi$  on  $s$  tasks, and exerts effort  $\varepsilon$  on the remaining  $n - s$ . To economize on notation we will regularly write  $u(s; \mathbf{w})$  in place of  $u(\mathbf{p}_s; \mathbf{w})$ .

The key result we establish in this section is that the agent's utility function  $u(s; \mathbf{w})$  is convex then concave. In other words, there is a number of tasks  $\bar{s}$  such that the second-difference of  $u$  is negative if and only if  $s \geq \bar{s}$ , i.e.,

$$(u(s; \mathbf{w}) - u(s-1; \mathbf{w})) - (u(s-1; \mathbf{w}) - u(s-2; \mathbf{w})) \leq 0 \quad \text{if and only if } s \geq \bar{s}. \quad (8)$$

**Proposition 2.** *The agent's indirect utility function  $u(s; \mathbf{w})$  is convex then concave for any cutoff incentive scheme  $\mathbf{w}$ .*

To establish Proposition 2, first substitute the cutoff rule into expression (6) to obtain<sup>21</sup>:

$$\begin{aligned} \frac{u(s; \mathbf{w}) - u(s-1; \mathbf{w})}{\pi - \varepsilon} &= \frac{\partial}{\partial p_s} W(\mathbf{p}_s; \mathbf{w}) - \frac{\partial}{\partial p_s} e(\mathbf{p}_s) \\ &= \sum_{k=0}^{n-1} (w_{k+1} - w_k) A_{n-1,k}(\mathbf{p}_{s-1}^{n-1}) - 1 \\ &= (1 - \mu)\bar{w} A_{n-1,m-1}(\mathbf{p}_{s-1}^{n-1}) + (\bar{w} - (1 - \mu)\bar{w}) A_{n-1,m}(\mathbf{p}_{s-1}^{n-1}) - 1 \\ &= \bar{w} A_{n,m}(\mathbf{p}_{s-1}^{n-1}, 1 - \mu) - 1. \end{aligned} \quad (9)$$

(The final equality follows from (2).) From (5) the second-difference expression (8) equals

$$\begin{aligned} \bar{w}(\pi - \varepsilon)(A_{n,m}(\mathbf{p}_{s-1}^{n-1}, 1 - \mu) - A_{n,m}(\mathbf{p}_{s-2}^{n-1}, 1 - \mu)) \\ = \bar{w}(\pi - \varepsilon)^2(A_{n-1,m-1}(\mathbf{p}_{s-2}^{n-2}, 1 - \mu) - A_{n-1,m}(\mathbf{p}_{s-2}^{n-2}, 1 - \mu)). \end{aligned} \quad (10)$$

This expression is negative if  $m$  is less than the modal number of successes in  $n - 1$  tasks given effort  $(\mathbf{p}_{s-2}^{n-2}, 1 - \mu)$ , and is positive otherwise. From Lemma 2, the mode is increasing in each component of effort. As such, if  $u(s; \mathbf{w})$  is concave at some  $s$  (i.e., the second-difference expression (8) is negative) the same is true for all  $s' > s$ . In other words,  $u(s; \mathbf{w})$  is convex then concave in  $s$ , completing the proof of Proposition 2.

Proposition 2 says that the agent's utility function  $u(s; \mathbf{w})$  cannot be concave then convex then concave again. It is, however, entirely possible that it is globally convex. In particular, Laux's [13]

<sup>20</sup> Jewitt [12] gives an alternate set of sufficient conditions, which are not satisfied in our environment either. In particular, it is easily numerically verified that his condition (2.11)—concavity of the likelihood ratio—is not satisfied here.

<sup>21</sup> Recall that  $W$  and  $e$  are linear in each component of effort  $\mathbf{p}$ .

analysis of the problem without (BC) implies that if this constraint does not bind then  $u(s; \mathbf{w})$  is globally convex in  $s$ .

The main value of Proposition 2 is that its characterization of the shape of the agent's indirect utility function  $u(s; \mathbf{w})$  allows us to derive a simple sufficient condition for an allocation  $\mathbf{p}_s$  to be incentive compatible. Specifically, since for any cutoff incentive scheme  $\mathbf{w}$  the function  $u(s; \mathbf{w})$  is convex then concave, an effort choice  $\mathbf{p}_s$  is incentive compatible if and only if:

- (i) *Local deviation constraint*: the allocation  $\mathbf{p}_s$  is a local maximum, that is the agent prefers the effort allocation  $\mathbf{p}_s$  to both  $\mathbf{p}_{s-1}$  and  $\mathbf{p}_{s+1}$ , and;
- (ii) *Extremal deviation constraint*: the agent prefers  $\mathbf{p}_s$  to  $\mathbf{p}_0$  (shirking on all tasks).

**Proposition 3.** *The principal's maximum attainable utility coincides with the solution to*

$$\max_{s, \mathbf{w} \text{ a cutoff incentive scheme}} F(\mathbf{p}_s) - W(\mathbf{p}_s; \mathbf{w})$$

such that

$$\begin{aligned} u(s; \mathbf{w}) &\geq \max\{u(s-1; \mathbf{w}), u(s+1; \mathbf{w})\} && (\text{Local Deviation}), \\ u(s; \mathbf{w}) &\geq u(0; \mathbf{w}) && (\text{Extremal Deviation}), \\ u(s; \mathbf{w}) &\geq \underline{u} && (\text{IR}). \end{aligned}$$

That is, a pair of sufficient conditions for the incentive compatibility constraint to hold are that the *local deviation* constraint and the *extremal deviation* constraint are satisfied. In place of the original  $[\varepsilon, \pi]^n$  incentive constraints, we now have only four: the two local constraints that are the focus of much of the principal–agent literature, a third constraint that corresponds to the most extreme deviation the agent can possibly make, and the participation constraint.

Summarizing, our observation that the local incentive constraints are not by themselves sufficient to guarantee incentive compatibility is not new.<sup>22</sup> Instead, the novel aspect of Proposition 3 is that it provides a simple and tractable set of sufficient conditions. These conditions depend critically on the probability distribution of the principal's signals, which emerge naturally in our multitask framework.

## 5. The optimal contract

### 5.1. The effect of changing the incentive scheme on the agent's effort

From Proposition 1, we know that the principal is always able to achieve the constrained optimum by using a cutoff incentive scheme. The class of cutoff incentive schemes is easily ordered. We say a cutoff incentive scheme  $\mathbf{w}^{m_2, \mu_2}$  is *stricter* (softer) than the scheme  $\mathbf{w}^{m_1, \mu_1}$  if  $m_2 + \mu_2 > (<) m_1 + \mu_1$ .<sup>23</sup> Our next result characterizes how the agent's effort choice responds to exogenous changes in the cutoff incentive scheme. In particular, we ask: as the cutoff rule is made stricter, does the agent exert more or less effort? This comparative static is both of independent interest, and will prove useful in characterizing the optimal contract below.

<sup>22</sup> For previous work on the FOA, see also Sinclair-Desgagné [21] for an extension to the multi-signal case; Araujo and Moreira [2] for a generalization of the Lagrangian; and LiCalzi and Spaeter [14] for a class of distributions that satisfy both MLRP and CDFC.

<sup>23</sup> That is, either  $m_2 > m_1$ , or  $m_2 = m_1$  and  $\mu_2 > \mu_1$ .

**Proposition 4.** Suppose that for some cutoff rule  $\mathbf{w}^{m_1, \mu_1}$  the agent works on  $s_1$  tasks and shirks on the remainder. Let  $\mathbf{w}^{m_2, \mu_2}$  be a stricter cutoff rule, with  $m_2 + \mu_2 \geq m_1 + \mu_1 + 1$ . If the incentive scheme  $\mathbf{w}^{m_1, \mu_1}$  is replaced with  $\mathbf{w}^{m_2, \mu_2}$  then the agent either works on  $s_2 \geq s_1$  tasks, or else shirks on all tasks.

Proposition 4 says that stricter incentive schemes induce more effort—provided, that is, that they are not so strict as to completely discourage all effort. The key step in proving this result is to show that when the cutoff rule is tightened from  $\mathbf{w}^{m_2-1, \mu_2}$  to  $\mathbf{w}^{m_2, \mu_2}$ , the slope of the agent's utility function increases, i.e.,

$$u(s_2; \mathbf{w}^{m_2, \mu_2}) - u(s_2 - 1; \mathbf{w}^{m_2, \mu_2}) \geq u(s_2; \mathbf{w}^{m_2-1, \mu_2}) - u(s_2 - 1; \mathbf{w}^{m_2-1, \mu_2}). \quad (11)$$

This is established using the unimodality of the density function  $A_{n,k}(\mathbf{p})$ , Lemma 2's implication that an increase in effort increases the modal number of successes, and the fact that  $u(\cdot; \mathbf{w}^{m_2, \mu_2})$  must be concave at the agent's effort choice  $s_2$ , i.e.,

$$u(s_2; \mathbf{w}^{m_2, \mu_2}) - u(s_2 - 1; \mathbf{w}^{m_2, \mu_2}) \geq 0 > u(s_2 + 1; \mathbf{w}^{m_2, \mu_2}) - u(s_2; \mathbf{w}^{m_2, \mu_2}). \quad (12)$$

Once established, inequality (11) combines with the convex-then-concave property of the utility function (see Proposition 2) to imply that the local maximum of the utility  $u(\cdot; \mathbf{w}^{m, \mu})$  shifts to the right as the incentive scheme becomes stricter.

## 5.2. The solution to the contracting problem

From Proposition 4, when the incentive scheme is made stricter the agent's effort either weakly increases, or else collapses to the minimum possible effort level. Moreover, holding the agent's effort fixed, making the incentive scheme stricter clearly reduces the principal's expected payment to the agent. From these observations it follows that the principal should make the incentive scheme stricter until either the (IR) constraint holds, or the agent's effort level collapses.

Proceeding more formally, suppose that the agent chooses to exert some effort under at least *some* incentive scheme. (If this condition does not hold, the problem is degenerate.) Let  $\mathbf{w}^{m^*, \mu^*}$  be the strictest cutoff incentive scheme under which the agent chooses to exert effort, i.e.,

$$\begin{aligned} \mathbf{w}^{m^*, \mu^*} &\equiv \{\max \mathbf{w}: \mathbf{w} \text{ a cutoff rule such that} \\ &u(s; \mathbf{w}) \geq \max \{u(s-1; \mathbf{w}), u(s+1; \mathbf{w}), u(0; \mathbf{w}), \underline{u}\} \text{ for some } s \geq 1\} \end{aligned}$$

The cutoff incentive scheme  $\mathbf{w}^{m^*, \mu^*}$  is significant because the incentive scheme that maximizes the principal's utility lies close by:

**Proposition 5.** The principal obtains his maximal level of utility either with a cutoff incentive scheme that is (weakly) stricter than  $\mathbf{w}^{m^*-1, \mu^*}$  and (weakly) softer than  $\mathbf{w}^{m^*, \mu^*}$ ; or else with an incentive scheme that is completely flat, in which case the agent exerts no effort.

Proposition 5 establishes that the optimal contract is essentially the strictest cutoff rule that satisfies the IR, the extremal incentive constraint, and the local incentive conditions.

Proposition 5 is easily proved. Note first that by the definition of  $\mathbf{w}^{m^*, \mu^*}$ , if the principal chooses a stricter scheme the agent exerts no effort. Given this, the principal is at least as well off using a completely flat incentive scheme. On the other hand, suppose the principal chooses

a cutoff incentive scheme  $\mathbf{w}$  strictly softer than  $\mathbf{w}^{m^*-1,\mu^*}$ . Then by Proposition 4 there is a cutoff incentive scheme  $\tilde{\mathbf{w}}$  (weakly) stricter than  $\mathbf{w}^{m^*-1,\mu^*}$  and (weakly) softer than  $\mathbf{w}^{m^*,\mu^*}$  under which the agent exerts (weakly) greater effort. Moreover, it is straightforward to establish that the principal's utility must be strictly higher under  $\tilde{\mathbf{w}}$  (see Appendices A–C).

### 5.3. Binding extremal deviation constraint

From Proposition 5, the optimal contract is the cutoff incentive scheme  $\mathbf{w}^{m^*,\mu^*}$  (or else is very close by). When the outside option  $\underline{u}$  is low, this means that the extremal deviation constraint binds, since by definition

$$u(s; \mathbf{w}^{m^*,\mu^*}) = \max\{u(0; \mathbf{w}^{m^*,\mu^*}), \underline{u}\}$$

(where  $s$  is the number of tasks the agent works under  $\mathbf{w}^{m^*,\mu^*}$ ).<sup>24</sup>

An economic consequence of a binding extremal deviation constraint is that small changes in economic fundamentals can have a big impact on the agent's work level. For example, suppose that the cost of effort  $\mathbf{p}$  unexpectedly rises by a small amount (i.e., from  $e(\mathbf{p})$  to  $(1 + \delta)e(\mathbf{p})$ ). After the change, the agent prefers  $\mathbf{p}_0$  to all alternative effort allocations, and so switches to exerting no effort. In this sense, the agent's effort level is "fragile." It is important to note that this fragility of the agent's effort level does not arise in problems where the FOA is valid and only the local incentive constraints bind. In such environments, a small change in the cost of effort leads only to a local change in the agent's effort level.<sup>25</sup>

**Remark 1.** It is natural to ask how the fragility implication would be affected if the principal is aware that with some small probability  $\gamma$  economic fundamentals will change—for example, that the cost of effort will increase by a multiplicative constant  $1 + \delta$ . If the principal modifies the contract at all, he will clearly do so by enough to ensure that effort does not collapse when the shock hits. However, doing so has a discrete cost. As such, if the probability  $\gamma$  is small enough, the principal will use the same contract as when the shock is completely unanticipated.

**Remark 2.** For general parameter values we are able to show analytically only that the optimal incentive scheme is close to  $\mathbf{w}^{m^*,\mu^*}$ , in the sense stated in Proposition 5. That said, in all numerical simulations that we have conducted the optimal incentive scheme is, in fact, exactly  $\mathbf{w}^{m^*,\mu^*}$ . Moreover, one (admittedly special) case in which we can show formally that the optimal scheme is exactly  $\mathbf{w}^{m^*,\mu^*}$  is when  $\pi = 1$  (this follows from an easy adaptation of the proof of Proposition 4).

## 6. Effort misallocation

So far we have characterized the solution to the principal's contracting problem, along with the agent's effort allocation at the optimal contract. For many parameter values the agent works on only a subset of tasks  $s < n$  under the optimal contract (see the numerical example in Section 3). In such cases the agent supplies a socially suboptimal level of effort.

<sup>24</sup> To see this, observe that  $\max_{\mathbf{p} \geq \mathbf{p}_1^n} u(\mathbf{p}; \mathbf{w}^{m,\mu})$  is continuous and decreasing in  $m + \mu$ .

<sup>25</sup> Models with more than one agent and multiple equilibria also often exhibit a similar form of fragility, in the sense that large changes in outcomes can follow from small (or no) change in the economic fundamentals.

In this section we characterize a second form of inefficiency in effort provision, which arises whenever the principal's production function is convex and the agent works on  $s < n$  tasks. Under these conditions, we show that the socially optimal allocation of the agent's effort  $E = \pi s + \varepsilon(n - s)$  across the  $n$  tasks is to supply effort  $E/n$  on each task. However, the agent instead supplies effort  $\pi > E/n$  on some tasks and  $\varepsilon < E/n$  on others (see Lemma 1).

Proceeding more formally, the socially optimal allocation of a given amount of effort  $E$  is given by the solution to the problem

$$\max_{\mathbf{p}} F(\mathbf{p}) \quad \text{s.t. } e(\mathbf{p}) = E. \quad (13)$$

Problem (13) is exactly the problem studied by Hoeffding [6]. He shows if  $f$  is strictly convex then the only solution to problem (13) is to distribute effort evenly across all tasks, i.e.,  $\mathbf{p} = (E/n, \dots, E/n)$ . The reason is that the even allocation of available effort *maximizes* the variance of the number of successes, and this is desirable when the payoff function is convex.

To establish this result, Hoeffding observes that for any  $\mathbf{p} \in [\varepsilon, \pi]^n$ , the probability of  $k$  successes given  $\mathbf{p}$ ,  $A_{n,k}(\mathbf{p})$ , is linear in each component of  $\mathbf{p}$ , and symmetric across components. As such,  $F(\mathbf{p})$  has these same properties. Exploiting these properties, it follows (see Appendices A–C for details) that for any two alternate effort allocations  $\mathbf{p}$  and  $\tilde{\mathbf{p}}$  that differ only in components  $i$  and  $j$ , and such that  $p_i + p_j = \tilde{p}_i + \tilde{p}_j$ ,

$$F(\tilde{\mathbf{p}}) - F(\mathbf{p}) = (\tilde{p}_i \tilde{p}_j - p_i p_j) \frac{\partial^2}{\partial p_j \partial p_i} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon). \quad (14)$$

Eq. (14) relates the effect of a change in effort allocation on expected output  $F$  to the cross-partial derivative of expected output  $F$ . The cross-partial derivative of  $F$  can in turn be related to the convexity/concavity of  $f$  by differentiating (6) (with  $w_k$  replaced by  $f_k$ ) to obtain

$$\frac{\partial^2}{\partial p_i \partial p_j} F(\mathbf{p}) = \sum_{k=0}^{n-2} ((f_{k+2} - f_{k+1}) - (f_{k+1} - f_k)) A_{n-2,k}(\mathbf{p}^{-i,j}). \quad (15)$$

From (15), it is immediate that the cross-partial of expected output  $F$  is positive if the production function  $f$  is convex.

Finally, to see why the even allocation of effort is the only solution to problem (13) when  $f$  is convex, suppose instead that there is a solution  $\mathbf{p}$  without this property. But then simply select any two components  $p_i$  and  $p_j$  such that  $p_i < p_j$ , and increase  $p_i$  to  $\frac{p_i + p_j}{2}$  while decreasing  $p_j$  to the same amount. From (14) this leads to a strict increase in expected output.

By parallel arguments, if the production function  $f$  is concave the solution to problem (13) is to deploy an intermediate level of effort in  $(\varepsilon, \pi)$  on *at most* one task, with either minimal effort  $\varepsilon$  or maximal effort  $\pi$  on the remaining tasks. Deploying either the maximal or minimal effort on as many tasks as possible minimizes success variance, and this is desirable when the production function  $f$  is concave. Summarizing:

**Proposition 6.** *If the production function is strictly convex, the optimal allocation of effort  $E$  is to equally distribute it across all tasks,  $\mathbf{p} = (E/n, \dots, E/n)$ . If the production function is strictly concave, the optimal allocation of effort  $E$  entails either minimal effort  $\varepsilon$  or maximal effort  $\pi$  on at least  $n - 1$  tasks.*

From Lemma 1, for any incentive scheme the agent either shirks or works on each task. It follows that when the production function features increasing returns to scale, if the agent exerts

suboptimally little effort he also *misallocates* this effort across tasks. This misallocation of effort corresponds to the agent taking too little risk, since exerting either the maximal or minimal effort on all tasks minimizes the variance of the number of successful tasks. The numerical example of Section 3 features both effort underprovision and misallocation.

When the agent works on all tasks, of course, there is no scope for misallocation of effort. In this sense, underprovision of effort and misallocation of effort go hand-in-hand. Moreover, misallocation of effort only arises when the production function has increasing returns to scale. When instead the production function is concave, the agent's choice of the variance-minimizing effort allocation is socially as well as privately optimal.

### 6.1. Effect of a binding budget constraint

We close this section with a discussion of how a binding (BC) affects the contracting problem. When (BC) does not bind, our problem is close to the one studied by Laux. The key characteristics of the solution are given on page 184 above. From our analysis one can see that a binding (BC) affects the solution to the contracting problem in various ways. The agent's indirect utility function  $u(s; \mathbf{w})$  is no longer necessarily convex in  $s$ , but instead may be convex-then-concave. As a consequence, the local incentive constraints play a role in determining the agent's effort response to a contract. The economic consequence is that the agent may work on a strict subset of the tasks, instead of either working or shirking on all as he does when the budget constraint does not bind. However, in common with the case in which (BC) does not bind the extremal deviation constraint plays an important role in determining the optimal contract.

Moreover, when the agent works on a strict subset of tasks there is scope for misallocation of effort. Specifically, our analysis allows for intermediate effort levels on each individual task, and for both increasing- and decreasing-returns to scale production functions. When the production function has increasing returns to scale the agent misallocates his effort across tasks: he chooses an allocation that minimizes the variance of the number of successes, while expected output would be higher if instead he exerted the same effort on all tasks and thereby maximized output variance.

## 7. Large number of tasks: the limiting case

We next consider the limiting case in which the  $n$  tasks are subdivided into infinitely smaller subtasks. All the results of Sections 4 and 5 extend to this case. Moreover, the analysis becomes slightly simpler as we can use derivatives instead of first differences.

Start by considering a finite task version of our economy, with a total of  $n$  tasks. Consider now the economy where each of the tasks is subdivided into  $\kappa$  smaller subtasks on which the agent can exert a level of effort in the interval  $[\varepsilon_\kappa, \pi_\kappa] = [\frac{\varepsilon}{\kappa}, \frac{\pi}{\kappa}]$ . Note that the total effort in each task remains in the interval  $[\varepsilon, \pi]$  and if the agent works on a fraction  $\eta$  out of  $n\kappa$ -tasks and shirks on the remainder, his total effort and expected number of successes remains equal to  $E = (1 - \eta)n\varepsilon + \eta n\pi$ .

For the extremes in which the agent either shirks on all tasks (i.e.,  $\eta = 0$ ) or works on all tasks (i.e.,  $\eta = 1$ ), the distribution of successes in the  $n\kappa$ -task economy is binomial. For these cases, Poisson himself showed that as  $\kappa \rightarrow \infty$  the distribution of successes converges to the Poisson distribution, with parameters  $n\varepsilon$  and  $n\pi$  respectively. This result easily generalizes to:

**Lemma 4.** Fix a fraction of tasks  $\eta$  on which the agent works, so that the agent's total effort is  $E = (1 - \eta)n\varepsilon + \eta n\pi$ . Then as  $\kappa \rightarrow \infty$ , the distribution of the number of successes in the  $n\kappa$ -task economy converges to a Poisson distribution with parameter  $E$ , and the probability that the agent succeeds on  $k$  tasks is  $A_k(E) \equiv \frac{E^k e^{-E}}{k!}$ .

Since  $\frac{A_k(E)}{A_{k-1}(E)} = \frac{E}{k}$ , it is immediate that  $A_k(E)$  is log-concave in the number of successes  $k$ , and that MLRP holds. Since MLRP holds, Proposition 1 (the cutoff result) continues to hold.<sup>26</sup> As in the finite task case, the fact that a cutoff rule  $\mathbf{w}^{m,\mu}$  may pay an amount between 0 and  $\bar{w}$  when there are exactly  $m$  successes means that we are often interested in the probability distribution defined for any  $\mu \in [0, 1]$  by

$$A_k^\mu(E) \equiv (1 - \mu)A_{k-1}(E) + \mu A_k(E).$$

It is straightforward to verify that  $A_k^\mu(E)$  inherits log-concavity in  $k$  from  $A_k(E)$ , and so is unimodal in  $k$ . Moreover, similar to Lemma 2 the mode of the  $A_k(E)$  is increasing in effort  $E$ :

**Lemma 5.** If  $A_k^\mu(E) \geq A_{k-1}^\mu(E)$  and  $\tilde{E} > E > 0$  then  $A_k^\mu(\tilde{E}) \geq A_{k-1}^\mu(\tilde{E})$ , with strict inequality provided  $(k, \mu) \neq (0, 0)$ .

We write  $u(E; \mathbf{w})$  for the agent's indirect utility function. Differentiating with respect to  $E$  gives the direct analogues of Eqs. (5) and (6), namely  $A'_k(E) = A_{k-1}(E) - A_k(E)$  and  $u'(E; \mathbf{w}) = \sum_{k=0}^{\infty} A_k(E)(w_{k+1} - w_k) - 1$ . To establish convexity-then-concavity, note that under a cutoff rule  $\mathbf{w}^{m,\mu}$

$$u'(E; \mathbf{w}^{m,\mu}) = (1 - \mu)\bar{w}A_{m-1}(E) + (\bar{w} - (1 - \mu)\bar{w})A_m(E) - 1 = \bar{w}A_m^\mu(E) - 1,$$

and so we obtain the analogue of (10),  $u''(E; \mathbf{w}^{m,\mu}) = \bar{w}(A_{m-1}^\mu(E) - A_m^\mu(E))$ . Lemma 5 then implies the convex-then-concave property of  $u$ : if  $u(\cdot; \mathbf{w}^{m,\mu})$  is weakly concave at some effort level  $E$ , it is strictly concave for all higher effort levels. Moreover, there is at most a single effort level  $E > 0$  for which  $u''(\cdot; \mathbf{w}^{m,\mu}) = 0$  (except for the degenerate case of  $(m, \mu) = (0, 0)$ , in which case  $u(\cdot; \mathbf{w}^{m,\mu})$  is constant). Given this, we obtain the following analogue of Proposition 3:

**Proposition 7.** The principal's maximum attainable utility coincides with the solution to

$$\max_{E \in [n\varepsilon, n\pi], \mathbf{w} \text{ a cutoff incentive scheme}} \sum_{k=0}^{\infty} (f_k - w_k)A_k(E)$$

such that

$$\begin{aligned} u'(E; \mathbf{w}) &\leq 0 && \text{if } E = n\varepsilon \\ u'(E; \mathbf{w}) &= 0 && \text{if } E \in (n\varepsilon, n\pi) && (\text{FOC}), \\ u'(E; \mathbf{w}) &\geq 0 && \text{if } E = n\pi \\ u(E; \mathbf{w}) &\geq u(n\varepsilon; \mathbf{w}) && (\text{Extremal Deviation}), \\ u(E; \mathbf{w}) &\geq u && (\text{IR}). \end{aligned}$$

Finally, one can define a cutoff rule  $\mathbf{w}^{m^*, \mu^*}$  in exactly the same way as in the finite version of the problem. Proposition 4 holds exactly as before (the proof is easily adapted), and so the

<sup>26</sup> The sole change to the proof is that the iterative argument must be replaced with a limiting argument.

principal finds it beneficial to make the rule stricter up to the point that  $\mathbf{w}^{m^*, \mu^*}$  is reached. As such, Proposition 5 also continues to hold.

## 8. Convex costs

Thus far we have restricted attention to the case in which the agent has linear effort costs. In this section we discuss the robustness of our analysis to introducing convex effort costs. The assumption of linear effort costs is used in two main ways. First, linearity ensures that in the finite-task problem the agent exerts either effort  $\varepsilon$  or  $\pi$  on each task. This implication considerably simplifies the analysis, but would not necessarily hold if effort costs were convex. However, if one instead simply assumed that the agent's effort choice on each individual task were binary convexity of effort costs would make no difference in this regard. Moreover, in the limiting case of Section 7 this issue does not arise at all.

Second, the linearity assumption is used in establishing that the agent's indirect utility function has a convex-then-concave shape. Although linearity simplifies the proof, it is not essential for this result. Specifically, in Proposition 8 below we show that the agent's indirect utility has the same convex-then-concave shape whenever effort costs are sufficiently close to linear. We establish this claim for the limiting economy of Section 7 with effort costs given by a quadratic cost  $E + \frac{\lambda}{2}(E - n\varepsilon)^2$ , where  $\lambda \geq 0$  is a parameter indexing the convexity of effort costs and  $\lambda = 0$  represents the linear case. (Both the finite task economy and more general convex cost functions could be handled using similar methods, although with some additional complexity.) Write  $u_\lambda(E; \mathbf{w})$  for the indirect utility function given cost parameter  $\lambda$ , so that  $u_0 \equiv u$ .

The key to establishing this result is that one can characterize  $u_0$  in more detail than simply saying that it is convex-then-concave. Specifically, within the region in which  $u_0$  is convex, its second derivative is single-peaked:

**Lemma 6.** *Fix an arbitrary cutoff scheme  $\mathbf{w}^{m, \mu} \neq \mathbf{w}^{0, 0}$ . Then there exists some  $E^{m, \mu}$  such that  $u_0''(\cdot; \mathbf{w}^{m, \mu})$  is strictly positive at  $E > 0$  if and only if  $E \in (0, E^{m, \mu})$ , strictly negative if and only if  $E > E^{m, \mu}$ , and moreover,  $u_0''$  is either strictly decreasing or increasing then strictly decreasing over  $(0, E^{m, \mu})$ .*

From Lemma 6 it is easy to show:

**Proposition 8.** *For any rule  $\mathbf{w}^{\bar{m}, \bar{\mu}}$  there exists  $\bar{\lambda} > 0$  such that whenever  $\lambda \leq \bar{\lambda}$  and the cutoff rule is softer than  $\mathbf{w}^{\bar{m}, \bar{\mu}}$  the indirect utility function is convex then concave.*

Proposition 8 implies that it is possible to introduce some measure of convexity to the effort cost function without changing our main results. The agent's indirect utility function is still convex-then-concave as a function of effort, provided that the cost of effort is not *too* convex. Proposition 7's simplification of the contracting problem then still holds, provided that one can identify an incentive scheme  $\mathbf{w}^{\bar{m}, \bar{\mu}}$  such that the optimal incentive scheme is definitely softer.<sup>27</sup>

Finally, when effort costs are convex it is no longer necessarily the case that tightening the rule either leads to a complete collapse or to an increase in effort (see Proposition 4). However, this conclusion needs only minor modification: now, tightening the incentive scheme either leads to a complete collapse in effort, or to an increase in effort, or to a very slight decrease in effort:

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<sup>27</sup> The contract  $\mathbf{w}^{\bar{m}, \bar{\mu}}$  enters Proposition 8 to ensure that the set of incentive schemes under consideration is compact.

**Proposition 9.** Suppose that  $n\varepsilon \geq 1$  and that for some cutoff rule  $\mathbf{w}^{m_1, \mu_1}$  the agent exerts a total level of effort  $E_1$ . Let  $\mathbf{w}^{m_2, \mu_2}$  be a stricter cutoff rule, with  $m_2 + \mu_2 \geq m_1 + \mu_1 + 1$ . If the incentive scheme  $\mathbf{w}^{m_1, \mu_1}$  is replaced with  $\mathbf{w}^{m_2, \mu_2}$  then the agent either exerts effort  $E_2 \geq (1 - \theta(\lambda))E_1$ , where  $\theta(\lambda)$  is an increasing function of  $\lambda$  with  $\theta(0) = 0$ , or else exerts minimal effort  $E_2 = n\varepsilon$ .

From Proposition 9, the optimal contract must again lie close to the contract  $\mathbf{w}^{m^*, \mu^*}$ .

## 9. Monotonicity constraint

Sections 4–8 deal with the contracting problem given the constraint (BC). A second and distinct constraint often imposed in contracting problems is that the principal's payoff must be increasing in output. Returning to the finite task version of our environment, this constraint is

$$f_{k+1} - w_{k+1} \geq f_k - w_k \quad \text{for all } k = 0, \dots, n-1. \quad (\text{MON})$$

In this section we reanalyze the principal's problem when he is subject to constraint (MON) in place of (BC). Cutoff incentive schemes continue to play an important role, though under (MON) they take a different form. Specifically, an incentive scheme  $\mathbf{w}$  is a cutoff scheme under (MON) if it is of the form

$$w_k = \begin{cases} 0 & \text{if } k < m, \\ (1 - \mu)(f_m - f_{m-1}) & \text{if } k = m, \\ w_m + f_k - f_m & \text{if } k > m \end{cases} \quad (16)$$

for some integer  $m$  and  $\mu \in [0, 1]$ . Note that the cutoff scheme (16) is debtlike: the agent receives nothing unless there are  $m$  successes, and is the residual claimant on output above this amount. As before, we denote cutoff schemes by  $\mathbf{w}^{m, \mu}$ . As in the (BC) case, if the agent's outside option  $\underline{u}$  is low (MON) binds whenever the number of tasks is sufficiently large.

It is straightforward to show it is still optimal to use a cutoff scheme (i.e., Proposition 1), which replicates Innes [11]'s result that a debt-like contract is optimal under the monotonicity constraint. Moreover, it is also straightforward to show that for any cutoff scheme  $\mathbf{w}^{m, \mu}$  the agent's indirect utility function is still convex-then-concave (i.e., Proposition 2).<sup>28</sup> Consequently, it is still possible to simplify the principal's contracting problem to one in which only the local and extremal incentive constraints are considered. That is, Proposition 3 continues to hold.

Proposition 4 is the only result that is affected by replacing constraint (BC) with (MON). Recall that under (BC), this result says that if the incentive scheme is made stricter the agent's effort either increases or drops to its minimal level. In contrast, under (MON) this conclusion is exactly reversed: as the cutoff rule is made stricter, the agent's effort level drops:

**Proposition 10.** Suppose that the principal is subject to constraint (MON), and that for some cutoff rule  $\mathbf{w}^{m_1, \mu_1}$  the agent works on  $s_1$  tasks. Then if the incentive scheme is replaced by a stricter rule  $\mathbf{w}^{m_2, \mu_2}$  the agent works on  $s_2 \leq s_1$  tasks.

As with Proposition 4, the key to establishing this result is to characterize how the slope of the agent's utility function changes as the incentive scheme changes. We show in Appendices A–C that the exact opposite of (11) holds so that when the cutoff rule is made stricter, the slope of the agent's utility function decreases.

<sup>28</sup> Details for both these claims are given in Appendix B.

Proposition 10 implies that the optimal contract may no longer lie near  $\mathbf{w}^{m^*, \mu^*}$ . The reason is that the principal must now tradeoff two conflicting effects when he makes the incentive scheme stricter. On the one hand, holding the agent's effort fixed, the principal benefits from adopting a stricter cutoff rule. But on the other hand, under (MON) making the incentive scheme stricter reduces the agent's effort. (In contrast, under (BC) these two effects acted in the same direction.)

The balance between these two offsetting forces depends on the parameters of the model. In many numerical examples the solution is in fact  $\mathbf{w}^{m^*, \mu^*}$ : the principal picks the strictest scheme such that the agent works at all, and the extremal deviation constraint binds as before. However, examples also exist in which the solution is determined solely by the local incentive constraint.

If the production function has either globally increasing or decreasing returns to scale (i.e., is globally concave or convex) there is no scope for effort misallocation (see Section 6) under constraint (MON). The reason is that effort misallocation arises only when the production function has increasing returns. However, in this case the agent's indirect utility function  $u$  is globally convex under (MON) (this is easily seen from the proof of the convex-then-concave property for the monotonicity case). Because  $u$  is globally convex, it then follows that the agent either works on all tasks or shirks on all tasks—but for these extremes, there is no scope for effort misallocation.

(It is worth noting that effort misallocation can still arise under (MON) if no output is produced when the number of successes falls below a critical level, but the production function has decreasing returns to scale above this critical number of successes. The output of research and development processes may take this form. In this case the production function is neither globally concave nor convex. Our analysis can be extended to handle this class of production functions. Details are contained in an earlier draft of this paper.)

## 10. Distinguishable tasks

So far we have considered an environment in which the  $n$  tasks are indistinguishable from the perspective of both the principal and the agent. In this section we consider, in turn, the cases in which (I) the principal can distinguish between tasks, and (II) the agent finds some tasks easier than others.

### 10.1. Principal can distinguish between tasks

For use throughout the section, define  $N \equiv \{1, \dots, n\}$  (the set of tasks), and let  $\mathcal{P}$  denote the power set of  $N$ . In our basic model we assumed the principal was unable to distinguish between the  $n$  tasks. This is the case whenever either the tasks cannot be *ex ante* distinguished, or tasks cannot be *ex post* verified as being different. If instead the principal can distinguish between tasks, a general incentive scheme is a mapping  $w : \mathcal{P} \rightarrow \mathfrak{R}$ , where for any set  $S \subset N$ ,  $w(S)$  is the payment received by the agent if the subset of tasks  $S$  succeeds. For this case, we establish:

**Proposition 11.** *Suppose that there is an incentive scheme that induces the agent to work on the subset of tasks  $S$ . Then there is an alternate incentive scheme that depends only on the number of successes in these  $S$  tasks, induces the agent to work on the tasks  $S$ , and entails the same expected payment from the principal to the agent.*

Although the proof of Proposition 11 is long and involved, and is relegated to Appendix C, the result itself is very intuitive. It says that if there is a contract that gets the agent to work on

a subset  $S$  of tasks, then the contract can be simplified to one that rewards success on these  $S$  tasks symmetrically (see Lemma 9 in Appendix C), and does not reward success on other tasks (Lemma 7 in Appendix C).

Given Proposition 11, our analysis generalizes to the case in which the principal can distinguish between tasks. To see this, let  $w^*$  be the optimal incentive scheme, with  $S^*$  the subset of tasks on which the agent works. By Proposition 11, one can assume without loss that  $w^*$  depends only on the number of successes within the task subset  $S^*$ . But then the problem reduces to choosing a contract that depends only on the number  $k$  of successes among tasks  $S^*$ . This is the same as our original problem, with the full task set  $\{1, \dots, n\}$  simply replaced by  $S^*$ .

### 10.2. Agent finds some tasks easier than others

Next, suppose that the principal is unable to distinguish between tasks, as in the basic model, but that the agent finds some tasks easier than others. Broadly speaking, there are two possibilities.

One possibility is that the agent's minimum and maximum work effort is the same for each task, but that the success probability implied by work effort differs. That is, for each task  $i$  the probability of success is given by  $\varepsilon + e_i(\pi_i - \varepsilon)$ , where  $e_i \in [0, 1]$  denotes the agent's effort on task  $i$  and  $\{\pi_i\}$  is a set of parameters that potentially differ across tasks and determine the return to working on task  $i$ . As in the basic model, linearity of effort costs implies that the agent either exerts effort 0 or 1 on each task. Because the principal is unable to distinguish between tasks, if the agent works on  $s$  of the  $n$  tasks he naturally selects the  $s$  tasks with the highest return to effort, i.e., with the highest  $\pi_i$  values. Symmetry across tasks is used only in the misallocation results of Section 6, and so with the exception of that section our analysis applies unchanged.

The second possibility is that the agent's minimum and maximum success probability is the same for each task (i.e.,  $\varepsilon$  and  $\pi$  as in the basic model), but that the agent's cost of effort differs across tasks. That is, the agent's total effort cost is of the form  $n\varepsilon + \sum_i \zeta_i e_i(\pi - \varepsilon)$ , where  $e_i \in [0, 1]$  indexes how hard an agent works on each task and  $\{\zeta_i\}$  is a set of parameters that potentially differ across tasks. Again, if the agent works on  $s$  of the  $n$  tasks he naturally selects the  $s$  tasks with the lowest cost of effort, i.e., with the lowest  $\zeta_i$  values. In this case the agent's total cost of effort is effectively convex in the number of tasks on which he works—see Section 8.

## 11. Conclusion

In this paper we have studied a particular class of multitask principal–agent problems. Although special in important respects (in particular, task-symmetry, risk-neutrality and linear effort costs), our analysis is quite general in others: we allow for an arbitrary number of tasks, which can be both complements and substitutes, and we analyze the contracting problem incorporating commonly used contractual restrictions such as an upper bound on payments to the agent and a monotonicity constraint on the principal's payoff. Our framework can be applied to a variety of situations, including: the design of optimal compensation schemes for managers overseeing many complementary R&D projects; the design of performance incentives by a regulator who oversees the performance of professional services firms conducting many repetitions of the same basic service; the regulation of monopolistic network industries; the relation between a financial intermediary, who monitors many projects, and its investors; and the use of promotion as an incentive device in law partnerships and academic departments.

We have shown that the particular properties of the class of problems we focus make them highly amenable to study. In particular, we have been able to avoid the usual problems that plague the application of the first-order approach. Although the local incentive constraint is never a sufficient condition, the local incentive constraint combined with an extremal deviation constraint—the agent is better off than he would be if he shirked on every task—is sufficient for the agent's incentive compatibility constraints because the agent's indirect utility function exhibits a convex then concave shape.

Economically, the class of problems we analyze feature a source of misallocation not previously noted in the literature: while the agent concentrates his effort on a subset of tasks, in many circumstances the principal would prefer an effort allocation with more risk taking in which the same effort is exerted on all tasks. A second implication of our analysis is that under many circumstances the optimal contract is fragile, in the sense that small unanticipated changes in the underlying parameters cause the agent's effort level to fall to zero.

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## Appendix A. Omitted proofs

### A.1. Proof of Lemma 2

We will show the result for  $\mathbf{p} = (\mathbf{p}^{n-1}, p)$  and  $\tilde{\mathbf{p}} = (\mathbf{p}^{n-1}, \tilde{p})$  where  $\mathbf{p}^{n-1}$  is an arbitrary  $n - 1$ -vector in  $[0, 1]^{n-1}$ , and  $\tilde{p} \geq p$ . The result then follows by iteration.

For any  $m$ , by the decomposition (2)  $A_{n,m}(\tilde{\mathbf{p}}) \geq A_{n,m-1}(\tilde{\mathbf{p}})$  if and only if

$$\begin{aligned} & \tilde{p}A_{n-1,m-1}(\mathbf{p}^{n-1}) + (1 - \tilde{p})A_{n-1,m}(\mathbf{p}^{n-1}) \\ & \geq \tilde{p}A_{n-1,m-2}(\mathbf{p}^{n-1}) + (1 - \tilde{p})A_{n-1,m-1}(\mathbf{p}^{n-1}), \end{aligned}$$

or equivalently,

$$\begin{aligned} & \tilde{p}(A_{n-1,m-1}(\mathbf{p}^{n-1}) - A_{n-1,m-2}(\mathbf{p}^{n-1})) + (1 - \tilde{p})(A_{n-1,m}(\mathbf{p}^{n-1}) - A_{n-1,m-1}(\mathbf{p}^{n-1})) \\ & \geq 0. \end{aligned}$$

By unimodality, if  $A_{n-1,m-1}(\mathbf{p}^{n-1}) - A_{n-1,m-2}(\mathbf{p}^{n-1}) \leq 0$  then  $A_{n-1,m}(\mathbf{p}^{n-1}) - A_{n-1,m-1}(\mathbf{p}^{n-1}) < 0$ . Hence  $A_{n,m}(\tilde{\mathbf{p}}) \geq A_{n,m-1}(\tilde{\mathbf{p}})$  only if  $A_{n-1,m-1}(\mathbf{p}^{n-1}) - A_{n-1,m-2}(\mathbf{p}^{n-1}) > 0$ . It follows that if  $A_{n,m}(\mathbf{p}) \geq A_{n,m-1}(\mathbf{p})$  then  $A_{n,m}(\tilde{\mathbf{p}}) \geq A_{n,m-1}(\tilde{\mathbf{p}})$  also.

### A.2. Proof of Lemma 3

We must show that  $A_{n,k+1}(\tilde{\mathbf{p}})A_{n,k}(\mathbf{p}) > A_{n,k}(\tilde{\mathbf{p}})A_{n,k+1}(\mathbf{p})$ , which by decomposition (2) is equivalent to

$$\begin{aligned} & (\tilde{p}_i A_{n-1,k}(\mathbf{p}^{-i}) + (1 - \tilde{p}_i)A_{n-1,k+1}(\mathbf{p}^{-i})) (p_i A_{n-1,k-1}(\mathbf{p}^{-i}) + (1 - p_i)A_{n-1,k}(\mathbf{p}^{-i})) \\ & > (\tilde{p}_i A_{n-1,k-1}(\mathbf{p}^{-i}) + (1 - \tilde{p}_i)A_{n-1,k}(\mathbf{p}^{-i})) (p_i A_{n-1,k}(\mathbf{p}^{-i}) \\ & \quad + (1 - p_i)A_{n-1,k+1}(\mathbf{p}^{-i})). \end{aligned}$$

Collecting terms and canceling, this inequality is in turn equivalent to

$$\begin{aligned} & \tilde{p}_i(1-p_i)A_{n-1,k}(\mathbf{p}^{-i})^2 + (1-\tilde{p}_i)p_iA_{n-1,k+1}(\mathbf{p}^{-i})A_{n-1,k-1}(\mathbf{p}^{-i}) \\ & > \tilde{p}_i(1-p_i)A_{n-1,k-1}(\mathbf{p}^{-i})A_{n-1,k+1}(\mathbf{p}^{-i}) + (1-\tilde{p}_i)p_iA_{n-1,k}(\mathbf{p}^{-i})^2, \end{aligned}$$

which is itself equivalent to

$$(\tilde{p}_i - p_i)A_{n-1,k}(\mathbf{p}^{-i})^2 > (\tilde{p}_i - p_i)A_{n-1,k-1}(\mathbf{p}^{-i})A_{n-1,k+1}(\mathbf{p}^{-i}).$$

Since  $\tilde{p}_i > p_i$ , this final inequality is equivalent to the log-concave condition (1).

### A.3. Proof of Proposition 1

Take  $\mathbf{w}$  an incentive scheme that is not a cutoff rule, and suppose that the agent works on  $s$  tasks under  $\mathbf{w}$ . Since  $\mathbf{w}$  is not a cutoff rule there must exist some  $k$  and  $l > k$  such that  $w_k > 0$  and  $w_l < \bar{w}$ . Consequently, it is possible to construct a new incentive scheme  $\hat{\mathbf{w}}$  satisfying (BC) such that for some  $\delta > 0$ ,  $\hat{w}_k = w_k - \delta$ ;  $\hat{w}_l = w_l + \delta \frac{A_{n,k}(\mathbf{p}_s)}{A_{n,l}(\mathbf{p}_s)}$ ;  $\hat{w}_j = w_j$  for all  $j \neq k, l$ ; and either  $\hat{w}_k = 0$  or  $\hat{w}_l = \bar{w}$ .

Since the agent's effort choice is incentive compatible under the original incentive scheme  $\mathbf{w}$ ,

$$W(\mathbf{p}_s; \mathbf{w}) - e(\mathbf{p}_s) \geq W(\mathbf{p}_{\tilde{s}}; \mathbf{w}) - e(\mathbf{p}_{\tilde{s}}) \quad (\text{A.1})$$

for all  $\tilde{s} \neq s$ , and thus in particular for all  $\tilde{s} < s$ .

The effect on the agent's utility from strategy  $\mathbf{p}_{\tilde{s}}$  caused by changing the incentive scheme from  $\mathbf{w}$  to the alternate scheme  $\hat{\mathbf{w}}$  defined above is

$$\begin{aligned} W(\mathbf{p}_{\tilde{s}}; \hat{\mathbf{w}}) - W(\mathbf{p}_{\tilde{s}}; \mathbf{w}) &= -\delta A_{n,k}(\mathbf{p}_{\tilde{s}}) + \delta \frac{A_{n,k}(\mathbf{p}_s)}{A_{n,l}(\mathbf{p}_s)} A_{n,l}(\mathbf{p}_{\tilde{s}}) \\ &= \delta A_{n,k}(\mathbf{p}_s) \left( \frac{A_{n,l}(\mathbf{p}_{\tilde{s}})}{A_{n,l}(\mathbf{p}_s)} - \frac{A_{n,k}(\mathbf{p}_{\tilde{s}})}{A_{n,k}(\mathbf{p}_s)} \right). \end{aligned}$$

For  $\tilde{s} = s$ , the utility change is clearly zero; while for  $\tilde{s} < s$  the utility change is strictly negative by MLRP (see Lemma 3). Consequently, inequality (A.1) holds strictly for  $\tilde{s} < s$  when  $\mathbf{w}$  is replaced with  $\hat{\mathbf{w}}$ .

By iterating the above argument we obtain a cutoff incentive scheme  $\check{\mathbf{w}}$  such that the agent prefers the effort allocation  $\mathbf{p}_s$  to  $\mathbf{p}_{\tilde{s}}$  for all  $\tilde{s} < s$ , and moreover, the agent's expected compensation from the effort choice  $\mathbf{p}_s$  is the same under the two incentive schemes  $\mathbf{w}$ ,  $\check{\mathbf{w}}$ , i.e.,  $W(\mathbf{p}_s; \mathbf{w}) = W(\mathbf{p}_s; \check{\mathbf{w}})$ . Let  $\mathbf{p}_{\tilde{s}}$  be the agent's preferred effort choice under  $\check{\mathbf{w}}$ . Note that  $\tilde{s} \geq s$ . If either  $\mathbf{p}_{\tilde{s}} = \mathbf{p}_s$ , or if  $F(\mathbf{p}_{\tilde{s}}) - W(\mathbf{p}_{\tilde{s}}; \check{\mathbf{w}}) \geq F(\mathbf{p}_s) - W(\mathbf{p}_s; \mathbf{w})$  then the principal is at least weakly better off using  $\check{\mathbf{w}}$  than  $\mathbf{w}$  and the proof is complete.

For the remaining case, we proceed as follows. The agent's expected compensation from the effort choice  $\mathbf{p}_s$  is the same under the two schemes, i.e.,  $W(\mathbf{p}_s; \mathbf{w}) = W(\mathbf{p}_s; \check{\mathbf{w}})$ . As such,  $\max_{\tilde{s}} u(\mathbf{p}_{\tilde{s}}; \check{\mathbf{w}}) \geq u(\mathbf{p}_s; \mathbf{w})$ . Making the cutoff rule  $\check{\mathbf{w}}$  stricter clearly strictly reduces  $u(\mathbf{p}_{\tilde{s}}; \check{\mathbf{w}})$  for every effort allocation  $\mathbf{p}_{\tilde{s}}$ , and does so in a continuous way. Choose  $\vec{\mathbf{w}}$  as the cutoff rule such that

$$\max_{\tilde{s}} u(\mathbf{p}_{\tilde{s}}; \vec{\mathbf{w}}) = u(\mathbf{p}_s; \mathbf{w}).$$

Let  $\mathbf{p}_{\tilde{s}}$  be an effort allocation that attains  $\max_{\tilde{s}} u(\mathbf{p}_{\tilde{s}}; \vec{\mathbf{w}})$ . Since  $\vec{\mathbf{w}}$  is a stricter incentive scheme than  $\check{\mathbf{w}}$ ,

$$\max_{\tilde{s}} u(\mathbf{p}_{\tilde{s}}; \vec{\mathbf{w}}) = W(\mathbf{p}_{\tilde{s}}; \vec{\mathbf{w}}) - e(\mathbf{p}_{\tilde{s}}) \leqslant W(\mathbf{p}_{\tilde{s}}; \check{\mathbf{w}}) - e(\mathbf{p}_{\tilde{s}}) = u(\mathbf{p}_{\tilde{s}}; \check{\mathbf{w}}). \quad (\text{A.2})$$

We claim that  $\tilde{s} \geq s$ . To see this, suppose to the contrary that  $\tilde{s} < s$ . But since the agent strictly prefers  $\mathbf{p}_s$  to  $\mathbf{p}_{\tilde{s}}$  for any  $\tilde{s} < s$  when facing the incentive scheme  $\check{\mathbf{w}}$ ,

$$u(\mathbf{p}_{\tilde{s}}; \check{\mathbf{w}}) = W(\mathbf{p}_{\tilde{s}}; \check{\mathbf{w}}) - e(\mathbf{p}_{\tilde{s}}) < W(\mathbf{p}_s; \check{\mathbf{w}}) - e(\mathbf{p}_s) = u(\mathbf{p}_s; \mathbf{w}). \quad (\text{A.3})$$

Combined, inequalities (A.2) and (A.3) imply that  $\max_{\tilde{s}} u(\mathbf{p}_{\tilde{s}}; \vec{\mathbf{w}}) < u(\mathbf{p}_s; \mathbf{w})$ ; but this violates the definition of  $\vec{\mathbf{w}}$ , giving a contradiction.

Finally, we claim that the principal's payoff is higher under the cutoff incentive scheme  $\vec{\mathbf{w}}$  than the original incentive scheme  $\mathbf{w}$ . By construction, the agent's utility is the same under the two cases,  $W(\mathbf{p}_s; \mathbf{w}) - e(\mathbf{p}_s) = W(\mathbf{p}_{\tilde{s}}; \vec{\mathbf{w}}) - e(\mathbf{p}_{\tilde{s}})$ . The principal's utility change from moving from  $\mathbf{w}$  to  $\vec{\mathbf{w}}$  is thus

$$(F(\mathbf{p}_{\tilde{s}}) - W(\mathbf{p}_{\tilde{s}}; \vec{\mathbf{w}})) - (F(\mathbf{p}_s) - W(\mathbf{p}_s; \mathbf{w})) = (F(\mathbf{p}_{\tilde{s}}) - e(\mathbf{p}_{\tilde{s}})) - (F(\mathbf{p}_s) - e(\mathbf{p}_s))$$

which is positive since by assumption greater effort is always socially worthwhile. This completes the proof.

#### A.4. Proof of Proposition 4

The proof is performed by contradiction: suppose to the contrary that  $s_1 > s_2 \geq 1$ .

**Claim A.** Inequality (11) holds.

**Claim B.** If  $s_1 > s_2$ , inequality (11) implies that

$$u(s_1; \mathbf{w}^{m_2, \mu_2}) - u(s_1 - 1; \mathbf{w}^{m_2, \mu_2}) \geqslant u(s_1; \mathbf{w}^{m_1, \mu_1}) - u(s_1 - 1; \mathbf{w}^{m_1, \mu_1}). \quad (\text{A.4})$$

We prove Claims A and B below. Together they imply (A.4), which delivers a contradiction as follows. From the optimality of the agent's choice of effort under the incentive scheme  $\mathbf{w}^{m_1, \mu_1}$ ,  $u(s_1; \mathbf{w}^{m_1, \mu_1}) - u(s_1 - 1; \mathbf{w}^{m_1, \mu_1}) \geq 0$ . However, from the optimality of the agent's choice of effort under  $\mathbf{w}^{m_2, \mu_2}$  we also know that inequality (12) holds. By the convex-then-concave property of  $u(\cdot; \mathbf{w}^{m_2, \mu_2})$  (see Proposition 2), and since by hypothesis  $s_1 > s_2$ , it follows that

$$u(s_2 + 1; \mathbf{w}^{m_2, \mu_2}) - u(s_2; \mathbf{w}^{m_2, \mu_2}) \geqslant u(s_1; \mathbf{w}^{m_2, \mu_2}) - u(s_1 - 1; \mathbf{w}^{m_2, \mu_2}).$$

Thus we obtain

$$\begin{aligned} 0 &> u(s_2 + 1; \mathbf{w}^{m_2, \mu_2}) - u(s_2; \mathbf{w}^{m_2, \mu_2}) \\ &\geqslant u(s_1; \mathbf{w}^{m_2, \mu_2}) - u(s_1 - 1; \mathbf{w}^{m_2, \mu_2}) \\ &\geqslant u(s_1; \mathbf{w}^{m_1, \mu_1}) - u(s_1 - 1; \mathbf{w}^{m_1, \mu_1}) \geq 0, \end{aligned}$$

a contradiction.

**Proof of Claim A.** By (9), inequalities (11) and (12) are equivalent to

$$A_{n, m_2}(\mathbf{p}_{s_2-1}^{n-1}, 1 - \mu_2) \geqslant A_{n, m_2-1}(\mathbf{p}_{s_2-1}^{n-1}, 1 - \mu_2), \quad (\text{A.5})$$

$$A_{n, m_2}(\mathbf{p}_{s_2-1}^{n-1}, 1 - \mu_2) \geqslant A_{n, m_2}(\mathbf{p}_{s_2}^{n-1}, 1 - \mu_2). \quad (\text{A.6})$$

By (5), (A.6) is equivalent to  $A_{n-1,m_2}(\mathbf{p}_{s_2-1}^{n-2}, 1 - \mu_2) \geq A_{n-1,m_2-1}(\mathbf{p}_{s_2-1}^{n-2}, 1 - \mu_2)$ , which is in turn equivalent to  $A_{n,m_2}(\mathbf{p}_{s_2-1}^{n-2}, 1 - \mu_2, 0) \geq A_{n,m_2-1}(\mathbf{p}_{s_2-1}^{n-2}, 1 - \mu_2, 0)$  (clearly adding a task with zero success probability does not change the probability of  $m_2$  or  $m_2 - 1$  successes). By Lemma 2, this last inequality implies (A.5).  $\square$

**Proof of Claim B.** As noted, (11) is equivalent to (A.5), which by Lemma 2 implies

$$A_{n,m_2}(\mathbf{p}_{s_1-1}^{n-1}, 1 - \mu_2) \geq A_{n,m_2-1}(\mathbf{p}_{s_1-1}^{n-1}, 1 - \mu_2), \quad (\text{A.7})$$

or equivalently,

$$\begin{aligned} & (1 - \mu_2)A_{n-1,m_2-1}(\mathbf{p}_{s_1-1}^{n-1}) + \mu_2 A_{n-1,m_2}(\mathbf{p}_{s_1-1}^{n-1}) \\ & \geq (1 - \mu_2)A_{n-1,m_2-2}(\mathbf{p}_{s_1-1}^{n-1}) + \mu_2 A_{n-1,m_2-1}(\mathbf{p}_{s_1-1}^{n-1}). \end{aligned}$$

By unimodality this implies

$$A_{n-1,m}(\mathbf{p}_{s_1-1}^{n-1}) \geq A_{n-1,m-1}(\mathbf{p}_{s_1-1}^{n-1}) \quad \text{for } m \leq m_2 - 1 \quad (\text{A.8})$$

(to see this, suppose to the contrary that the inequality (A.8) were reversed at  $m = m_2 - 1$ , and apply unimodality). From (9), inequality (A.8) is in turn equivalent to

$$\frac{\partial}{\partial \mu}(u(s_1; \mathbf{w}^{m,\mu}) - u(s_1 - 1; \mathbf{w}^{m,\mu})) \geq 0 \quad \text{for } m \leq m_2 - 1. \quad (\text{A.9})$$

Since inequality (A.7) is itself equivalent to (11) with  $s_2$  replaced by  $s_1$ , we have established (A.4).  $\square$

#### A.5. Proof of Proposition 5

Let  $\mathbf{p}_s$  and  $\mathbf{p}_{\tilde{s}}$  be the effort levels associated with  $\mathbf{w}$  and  $\tilde{\mathbf{w}}$ , where  $\tilde{s} \geq s$ . Since making the rule stricter lowers the agent's utility at every point, certainly

$$W(\mathbf{p}_{\tilde{s}}; \tilde{\mathbf{w}}) - e(\mathbf{p}_{\tilde{s}}) < W(\mathbf{p}_s; \mathbf{w}) - e(\mathbf{p}_s).$$

The change in the principal's utility is thus

$$(F(\mathbf{p}_{\tilde{s}}) - W(\mathbf{p}_{\tilde{s}}; \tilde{\mathbf{w}})) - (F(\mathbf{p}_s) - W(\mathbf{p}_s; \mathbf{w})) > (F(\mathbf{p}_{\tilde{s}}) - e(\mathbf{p}_{\tilde{s}})) - (F(\mathbf{p}_s) - e(\mathbf{p}_s)),$$

which is positive since by assumption greater effort is always socially worthwhile.

#### A.6. Proof of Proposition 6

To derive Eq. (14), note first that by linearity,

$$F(\mathbf{p}) = F(\mathbf{p}^{-i}, \varepsilon) + (p_i - \varepsilon) \frac{\partial}{\partial p_i} F(\mathbf{p}^{-i}, \varepsilon),$$

where  $(\mathbf{p}^{-i}, \varepsilon)$  is the  $n$ -vector formed from  $\mathbf{p}$  by replacing the  $i$ th component with  $\varepsilon$ . Iterating, we obtain

$$F(\mathbf{p}) = F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon) + (p_j - \varepsilon) \frac{\partial}{\partial p_j} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon) + (p_i - \varepsilon) \frac{\partial}{\partial p_i} F(\mathbf{p}^{-i}, \varepsilon),$$

where  $(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon)$  is the  $n$ -vector formed from  $\mathbf{p}$  by replacing both the  $i$ th and  $j$ th components by  $\varepsilon$ . Now,  $\frac{\partial}{\partial p_i} F$  is linear in each component of  $\mathbf{p}$  also, and so

$$\frac{\partial}{\partial p_i} F(\mathbf{p}^{-i}, \varepsilon) = \frac{\partial}{\partial p_i} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon) + (p_j - \varepsilon) \frac{\partial^2}{\partial p_j \partial p_i} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon).$$

By symmetry,  $\frac{\partial}{\partial p_i} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon) = \frac{\partial}{\partial p_j} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon)$ . Combining these observations gives,

$$\begin{aligned} F(\mathbf{p}) &= F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon) + (p_i + p_j - 2\varepsilon) \frac{\partial}{\partial p_i} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon) \\ &\quad + (p_i - \varepsilon)(p_j - \varepsilon) \frac{\partial^2}{\partial p_j \partial p_i} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon). \end{aligned}$$

Eq. (14) follows.

To establish the concave half of Proposition 6, suppose to the contrary that for some strictly concave production function  $f$  problem (13) has a solution  $\mathbf{p}$  with  $\varepsilon < p_i \leq p_j < \pi$  for some pair of components  $i, j$ . Consider an alternate allocation  $\tilde{\mathbf{p}}$  such that for some  $\delta > 0$ ,  $\tilde{p}_i = p_i - \delta$ ,  $\tilde{p}_j = p_j + \delta$ ,  $\tilde{p}_k = p_k$  for  $k \neq i, j$ . From (15) the term  $\frac{\partial^2}{\partial p_j \partial p_i} F(\mathbf{p}^{-i,j}, \varepsilon, \varepsilon)$  is strictly negative. But then (14) implies that  $F(\tilde{\mathbf{p}}) - F(\mathbf{p}) > 0$ , contradicting the optimality of  $\mathbf{p}$  and completing the proof.

#### A.7. Proof of Lemma 4

The moment generating function for a Bernoulli distribution with success probability  $p$  is  $1 - p + pe^t$ . So if the agent works on a fraction  $\eta$  of  $n\kappa$  tasks, the moment generating function for the total number of successes is

$$M_n(t) \equiv (1 - \pi_\kappa + \pi_\kappa e^t)^{\eta n\kappa} (1 - \varepsilon_\kappa + \varepsilon_\kappa e^t)^{(1-\eta)n\kappa},$$

where recall that  $\varepsilon_\kappa = \varepsilon/\kappa$  and  $\pi_\kappa = \pi/\kappa$ . For any  $x$ , it is straightforward to show that  $(1 - \frac{x}{\kappa} + \frac{x}{\kappa} e^t)^\kappa \rightarrow e^{x(e^t-1)}$  as  $\kappa \rightarrow \infty$ . So  $M_n(t) \rightarrow e^{(e^t-1)(\eta n\pi + (1-\eta)n\varepsilon)}$ , which coincides with the moment generating function for a Poisson distribution with parameter  $(1 - \eta)n\varepsilon + \eta n\pi$ . Since moment generating functions of non-negative one-dimensional random variables coincide if and only if their probability measures coincide, the result follows. (See, e.g., Theorem 22.2, Billingsley [3].)

#### A.8. Proof of Lemma 5

Expanding and factoring out common terms,  $A_k^\mu(E) \geq A_{k-1}^\mu(E)$  is equivalent to

$$\begin{aligned} (1 - \mu) + \mu E - \mu &\geq 0 \quad \text{if } k = 1, \\ (1 - \mu) \frac{E}{k-1} + \mu \frac{E^2}{k(k-1)} - \mu \frac{E}{k-1} - (1 - \mu) &\geq 0 \quad \text{if } k \geq 2. \end{aligned} \tag{A.10}$$

For either  $k = 1$  or  $\mu = 0$  the result is immediate. For the remaining case, note that the LHS of (A.10) is quadratic in  $E$  with a positive coefficient on  $E^2$  and a negative value at  $E = 0$ . So if (A.10) is positive for some value of  $E$ , the same is true for all higher values.

### A.9. Proof of Lemma 6

The first half of the statement is just the convex-then-concave property. For the second half,

$$\begin{aligned} u_0''(E; \mathbf{w}^{m,\mu}) &= \bar{w}(A_{m-1}^\mu(E) - A_m^\mu(E)) \\ &= \bar{w}((1-\mu)A_{m-2}(E) + (2\mu-1)A_{m-1}(E) - \mu A_m(E)), \\ u_0'''(E; \mathbf{w}^{m,\mu}) &= \bar{w}(A_{m-2}^\mu(E) - 2A_{m-1}^\mu(E) + A_m^\mu(E)) \\ &= \bar{w}((1-\mu)A_{m-3}(E) + (3\mu-2)A_{m-2}(E) + (1-3\mu)A_{m-1}(E) \\ &\quad + \mu A_m(E)). \end{aligned}$$

Since  $\mathbf{w}^{m,1}$  and  $\mathbf{w}^{m+1,0}$  represent the same scheme, without loss we assume that  $\mu > 0$ .

Suppose first that  $m \geq 3$ . In this case,

$$\begin{aligned} u_0''(E; \mathbf{w}^{m,\mu}) &= \bar{w}A_{m-2}(E)\left((1-\mu) + (2\mu-1)\frac{E}{m-1} - \mu\frac{E^2}{m(m-1)}\right), \\ u_0'''(E; \mathbf{w}^{m,\mu}) &= \bar{w}A_{m-3}(E)\left((1-\mu) + (3\mu-2)\frac{E}{m-2}\right. \\ &\quad \left.+ (1-3\mu)\frac{E^2}{(m-1)(m-2)} + \mu\frac{E^3}{m(m-1)(m-2)}\right). \end{aligned}$$

From these expressions, one can see that  $u_0''$  is strictly positive for all  $E$  sufficiently small, strictly negative for all  $E$  sufficiently large, and has at most two inflection points over  $(0, \infty)$ . Since  $u_0''(0; \mathbf{w}^{m,\mu}) = 0$  and  $u_0''(E; \mathbf{w}^{m,\mu}) \rightarrow 0$  as  $E \rightarrow \infty$ , the result then follows.

Second, consider the case  $m = 2$ . For this case,

$$\begin{aligned} u_0''(E; \mathbf{w}^{m,\mu}) &= \bar{w}A_0(E)\left((1-\mu) + (2\mu-1)E - \mu\frac{E^2}{2}\right), \\ u_0'''(E; \mathbf{w}^{m,\mu}) &= \bar{w}A_0(E)\left((3\mu-2) + (1-3\mu)E + \mu\frac{E^2}{2}\right). \end{aligned}$$

Note that  $u_0''$  is strictly positive for  $E$  sufficiently close to 0, is strictly negative for all  $E$  sufficiently large, and  $u_0''(E; \mathbf{w}^{m,\mu}) \rightarrow 0$  as  $E \rightarrow \infty$ . If  $3\mu-2 > 0$  then  $u_0''$  is strictly increasing when  $E$  is sufficiently small and has at most two inflection points; while if  $3\mu-2 \leq 0$  then  $u_0''$  has at most one turning point over  $(0, \infty)$ . In either case the result follows. Third, if  $m = 1$  then  $u_0'''(E; \mathbf{w}^{m,\mu}) = -u_0''(E; \mathbf{w}^{m,\mu}) - \bar{w}A_0^\mu(E)$ , from which the result is immediate. Finally, if  $m = 0$  then  $u_0''(E; \mathbf{w}^{m,\mu}) = -\bar{w}A_0^\mu(E)$ , and the result holds vacuously.

### A.10. Proof of Proposition 8

Observe that  $u_\lambda'' \equiv u_0'' - \lambda$ . Clearly  $u_\lambda(\cdot; \mathbf{w}^{m,\mu})$  is globally concave if  $u_0(\cdot; \mathbf{w}^{m,\mu})$  is, which (since  $u_0$  is convex then concave) is the case if  $u_0''(n\varepsilon; \mathbf{w}^{m,\mu}) < 0$ . Moreover, from Lemma 6  $u_\lambda(\cdot; \mathbf{w}^{m,\mu})$  is either globally concave or is convex then concave if  $u_0'''(n\varepsilon; \mathbf{w}^{m,\mu}) < 0$ . As such, it remains only to deal with the cases in which  $u_0''(n\varepsilon; \mathbf{w}^{m,\mu}) \geq 0$  and  $u_0'''(n\varepsilon; \mathbf{w}^{m,\mu}) \geq 0$ . To this end, note first that from the proof of Lemma 6  $u_0''(\cdot; \mathbf{w}^{m,\mu})$  is uniformly negative for any rule softer than  $\mathbf{w}^{1,0}$ , and so  $u_\lambda(\cdot; \mathbf{w}^{m,\mu})$  is globally concave for any such rule. Then define

$$\bar{\lambda} = \min\{u_0''(n\varepsilon; \mathbf{w}^{m,\mu}): m + \mu \in [1, \bar{m} + \bar{\mu}], u_0''(n\varepsilon; \mathbf{w}^{m,\mu}) \geq 0, u_0'''(n\varepsilon; \mathbf{w}^{m,\mu}) \geq 0\}.$$

(If the set on the right-hand side is empty, the proof is complete.) To complete the proof, note that  $\bar{\lambda} > 0$ , since the existence of a rule  $(m, \mu)$  such that  $u_0''(n\varepsilon; \mathbf{w}^{m,\mu}) = 0$  and  $u_0'''(n\varepsilon; \mathbf{w}^{m,\mu}) \geq 0$  would contradict Lemma 6. The result then follows from Lemma 6.

### A.11. Proof of Proposition 9

Suppose to the contrary that  $E_2 \in (n\varepsilon, (1 - \theta(\lambda))E_1)$ , so that  $E_1 > \frac{E_2}{1-\theta(\lambda)} \geq E_2$ . We show

$$u'_\lambda(E_1; \mathbf{w}^{m_2, \mu_2}) \geq u'_\lambda(E_1; \mathbf{w}^{m_1, \mu_1}), \quad (\text{A.11})$$

which gives a contradiction as follows. From the first- and second-order conditions,  $u'_\lambda(E_2; \mathbf{w}^{m_2, \mu_2}) = 0$  and  $u''_\lambda(E_2; \mathbf{w}^{m_2, \mu_2}) \leq 0$ . By the convexity-then-concavity property,  $u'_\lambda(E_1; \mathbf{w}^{m_2, \mu_2}) < u'_\lambda(E_2; \mathbf{w}^{m_2, \mu_2})$ . But then  $u'_\lambda(E_1; \mathbf{w}^{m_1, \mu_1}) < 0$ , contradicting the optimality of effort  $E_1$  under scheme  $\mathbf{w}^{m_1, \mu_1}$ .

To establish (A.11), it is sufficient to show

$$u'_\lambda(E_1; \mathbf{w}^{m_2, \mu_2}) \geq u'_\lambda(E_1; \mathbf{w}^{m_2-1, \mu_2}), \quad (\text{A.12})$$

which is itself equivalent to

$$A_{m_2}^{\mu_2}(E_1) - A_{m_2-1}^{\mu_2}(E_1) \geq 0. \quad (\text{A.13})$$

To see this, note that by unimodality of  $A_k(E)$  inequality (A.13) implies  $A_m(E_1) - A_{m-1}(E_1) \geq 0$  for any  $m \leq m_2 - 1$ . Since  $A_m(E) - A_{m-1}(E) \geq 0$  is itself equivalent to  $\frac{\partial}{\partial \mu} u'_\lambda(E; \mathbf{w}^{m,\mu}) \geq 0$ , it follows that  $u'_\lambda(E_1; \mathbf{w}^{m_2-1, \mu_2})$  exceeds  $u'_\lambda(E_1; \mathbf{w})$  for any cutoff rule  $\mathbf{w}$  softer than  $\mathbf{w}^{m_2-1, \mu_2}$ . So (A.12) implies (A.11) as claimed.

When the effort cost is linear, i.e.,  $\lambda = 0$ , (A.12) follows immediately from the second-order condition  $\bar{w}(A_{m_2-1}^{\mu_2}(E_2) - A_{m_2}^{\mu_2}(E_2)) \leq 0$  and the fact that the mode of  $A_k^\mu(E)$  is increasing in  $E$ .

The remainder of the proof deals convex effort costs, i.e.,  $\lambda > 0$ . Denote the effort cost by  $c_\lambda(E) \equiv E + \frac{\lambda}{2}(E - n\varepsilon)^2$ . Substituting the first-order condition  $\bar{w}A_{m_2}^{\mu_2}(E_2) = c'_\lambda(E_2)$  into the second-order condition yields

$$A_{m_2-1}^{\mu_2}(E_2) - A_{m_2}^{\mu_2}(E_2) - \frac{c''_\lambda(E_2)}{c'_\lambda(E_2)} A_{m_2}^{\mu_2}(E_2) \leq 0. \quad (\text{A.14})$$

We must show that (A.14) implies (A.12), or equivalently (A.13). For  $m_2 = 0$  inequality (A.13) holds trivially. For  $m_2 = 1$  inequality (A.13) is equivalent to  $\mu_2 - (1 - \mu_2) - \mu_2 E_1 \leq 0$ , which holds since  $E_1 \geq n\varepsilon \geq 1$ . The main case is  $m_2 \geq 2$ . Define the quadratic

$$Q(x; v) \equiv 1 - \mu_2 + \mu_2 \frac{x}{m_2 - 1} - (1 + v)(1 - \mu_2) \frac{x}{m_2 - 1} - (1 + v)\mu_2 \frac{x^2}{m_2(m_2 - 1)}.$$

It is straightforward to show that inequalities (A.14) and (A.12) are equivalent to  $Q(E_2; \frac{c''_\lambda(E_2)}{c'_\lambda(E_2)}) \leq 0$  and  $Q(E_1; 1) \leq 0$  respectively. The quadratic  $Q(\cdot; v)$  has a unique strictly positive root,  $X(v)$  say, satisfying

$$\begin{aligned} 2(1 + v)\mu_2 X(v) = m_2 & ( \mu_2 - (1 + v)(1 - \mu_2) ) \\ & + \sqrt{m_2^2 (\mu_2 - (1 + v)(1 - \mu_2))^2 + 4m_2(m_2 - 1)(1 - \mu_2)(1 + v)\mu_2}. \end{aligned}$$

So we must show that  $E_2 \geq X\left(\frac{c''_\lambda(E_2)}{c'_\lambda(E_2)}\right)$  implies  $E_1 \geq X(1)$ . Since  $\frac{c''_\lambda(E_2)}{c'_\lambda(E_2)} \leq \lambda$  and  $X$  is decreasing in the parameter  $v$ ,<sup>29</sup> it suffices to show that  $E_2 \geq X(\lambda)$  implies  $E_1 \geq X(1)$ . Note that

$$\frac{X(1)}{X(\lambda)} = \frac{(1+\lambda)(\mu_2 - (1-\mu_2)) + \sqrt{(\mu_2 - (1-\mu_2))^2 + 4\frac{m_2-1}{m_2}(1-\mu_2)\mu_2}}{\mu_2 - (1+\lambda)(1-\mu_2) + \sqrt{(\mu_2 - (1+\lambda)(1-\mu_2))^2 + 4\frac{m_2-1}{m_2}(1-\mu_2)(1+\lambda)\mu_2}},$$

and that this ratio is continuous over  $\mu_2 \in (0, 1]$ , converging to  $1+\lambda$  as  $\mu_2 \rightarrow 0$  (since  $X(\lambda) \rightarrow \frac{m_2-1}{1+\lambda}$  as  $\mu_2 \rightarrow 0$ ). Since  $\frac{m_2-1}{m_2}$  lies between  $1/2$  and  $1$ , for any  $\lambda$  there exists some  $\phi(\lambda) \geq 1$  that is independent of  $\mu_2$  and  $m_2$  such that  $\frac{X(1)}{X(\lambda)} \leq \phi(\lambda)$ , and such that  $\phi(\lambda) \rightarrow 1$  as  $\lambda \rightarrow 0$ . So  $E_2 \geq X(\lambda)$  implies  $\phi(\lambda)E_2 \geq X(1)$ . Defining  $\theta(\lambda)$  by  $\frac{1}{1-\theta(\lambda)} = \phi(\lambda)$  gives the result.

## Appendix B. Analysis of monotonicity constraint

### B.1. Proof of Proposition 1

The only change relative to the (BC) case is in the very first step. If the principal is instead subject (MON) and  $\mathbf{w}$  is not a cutoff scheme, then there must exist an  $l$  such that  $f_l - w_l > f_{l-1} - w_{l-1}$  and  $w_{l-1} > 0$ . Consequently, it is again possible to construct a new incentive scheme  $\hat{\mathbf{w}}$  satisfying (MON) such that for  $k = l-1$  and some  $\delta > 0$ ,  $\hat{w}_k = w_k - \delta$ ;  $\hat{w}_l = w_l + \delta \frac{A_{n,l-1}(\mathbf{p}_s)}{A_{n,l}(\mathbf{p}_s)}$ ;  $\hat{w}_j = w_j$  for all  $j \neq k, l$ ; and either  $\hat{w}_k = 0$  or  $f_l - w_l = f_k - w_k$ . The remainder of the proof is unchanged.

### B.2. Proof of Proposition 2

Iterating (6), the second-difference of  $u$  (i.e., expression (8)) equals

$$(\pi - \varepsilon)^2 \sum_{k=0}^{n-2} ((w_{k+2} - w_{k+1}) - (w_{k+1} - w_k)) A_{n-2,k}(\mathbf{p}_{s-2}^{n-2}).$$

Substituting in the cutoff rule (16), this expression has the same sign as

$$\begin{aligned} & w_m A_{n-2,m-2}(\mathbf{p}_{s-2}^{n-2}) + (f_{m+1} - f_m - w_m) A_{n-2,m-1}(\mathbf{p}_{s-2}^{n-2}) \\ & + \sum_{k=m}^{n-2} ((f_{k+2} - f_{k+1}) - (f_{k+1} - f_k)) A_{n-2,k}(\mathbf{p}_{s-2}^{n-2}). \end{aligned} \quad (\text{B.1})$$

If the production function is weakly convex then (B.1) is clearly positive and the  $u$  is globally convex.

The remainder of the proof deals with the case in which the production function is concave. In this case, all except the first two terms of expression (B.1) are negative. There are two subcases to consider. First, suppose that  $w_m \geq f_{m+1} - f_m$ , so that only the first term is strictly positive. From MLRP, the ratio  $A_{n-2,k}(\mathbf{p}_{s-2}^{n-2})/A_{n-2,m-2}(\mathbf{p}_{s-2}^{n-2})$  is increasing in  $s$  for all  $k \geq m-1$ . Dividing (B.1) by  $A_{n-2,m-2}(\mathbf{p}_{s-2}^{n-2})$ , it is readily seen that if this expression is negative for some  $s$ , it is negative for all  $\tilde{s} > s$ .

<sup>29</sup> To see this, note that the quadratic  $Q$  is decreasing in  $v$ .

Second, suppose that  $w_m < f_{m+1} - f_m$ . For this subcase, define  $\beta = \frac{w_m}{f_{m+1} - f_m}$ . Dividing throughout by

$$A_{n-1,m-1}(\mathbf{p}_{s-2}^{n-2}, \beta) = \beta A_{n-2,m-2}(\mathbf{p}_{s-2}^{n-2}) + (1 - \beta) A_{n-2,m-1}(\mathbf{p}_{s-2}^{n-2}),$$

we can see that expression (B.1) is negative if and only if

$$(f_{m+1} - f_m) + \sum_{k=m}^{n-2} ((f_{k+2} - f_{k+1}) - (f_{k+1} - f_k)) \frac{A_{n-2,k}(\mathbf{p}_{s-2}^{n-2})}{A_{n-1,m-1}(\mathbf{p}_{s-2}^{n-2}, \beta)} < 0.$$

By MLRP

$$\frac{A_{n-1,m-1}(\mathbf{p}_{s-2}^{n-2}, \beta)}{A_{n-2,k}(\mathbf{p}_{s-2}^{n-2})} = \frac{\beta A_{n-2,m-2}(\mathbf{p}_{s-2}^{n-2}) + (1 - \beta) A_{n-2,m-1}(\mathbf{p}_{s-2}^{n-2})}{A_{n-2,k}(\mathbf{p}_{s-2}^{n-2})}$$

is decreasing in  $s$  for all  $k \geq m$ . It follows that if (B.1) is negative for some value of  $s$ , the same is true for all  $\tilde{s} > s$ .

### B.3. Proof of Proposition 10

Let  $s_1$  (respectively,  $s_2$ ) be the number of tasks on which the agent works when the incentive scheme is  $\mathbf{w}^{m_1, \mu_1}$  (respectively,  $\mathbf{w}^{m_2, \mu_2}$ ). Substituting the incentive scheme into (6) implies that for any number of tasks  $s$  on which the agent works,

$$\begin{aligned} \frac{u(s; \mathbf{w}^{m, \mu}) - u(s-1; \mathbf{w}^{m, \mu})}{\pi - \varepsilon} &= (1 - \mu)(f_m - f_{m-1}) A_{n-1,m-1}(\mathbf{p}_s^{n-1}) \\ &\quad + \sum_{k=m}^{n-1} (f_{k+1} - f_k) A_{n-1,k}(\mathbf{p}_s^{n-1}) - 1. \end{aligned}$$

It is then immediate that  $u(s; \mathbf{w}^{m, \mu}) - u(s-1; \mathbf{w}^{m, \mu})$  is strictly decreasing in  $m$  and  $\mu$  for any  $s$ . From this, the convex-then-concave property of  $u(\cdot; \mathbf{w})$  (see Proposition 2) implies that  $s_2 \leq s_1$ , as follows.

If  $s_1 = 0$  then  $u(s; \mathbf{w}^{m_1, \mu_1}) < u(0; \mathbf{w}^{m_1, \mu_1})$  for all  $s > 0$ . Since  $u(s; \mathbf{w}^{m, \mu}) - u(s-1; \mathbf{w}^{m, \mu})$  is strictly decreasing in  $m$  and  $\mu$ , then  $u(s; \mathbf{w}^{m_2, \mu_2}) < u(0; \mathbf{w}^{m_2, \mu_2})$  also, and  $s_2 = 0$ .

If  $s_1 > 0$ , suppose to the contrary that  $n \geq s_2 > s_1$ . Since  $s_2$  is the agent's optimal response to the cutoff incentive scheme  $\mathbf{w}^{m_2, \mu_2}$  then  $u(s_2; \mathbf{w}^{m_2, \mu_2}) - u(s_2-1; \mathbf{w}^{m_2, \mu_2}) \geq 0$ . Since  $\mathbf{w}^{m_2, \mu_2}$  is stricter than  $\mathbf{w}^{m_1, \mu_1}$ , then  $u(s_2; \mathbf{w}^{m_1, \mu_1}) - u(s_2-1; \mathbf{w}^{m_1, \mu_1}) > 0$ . However, since  $s_1$  is the largest number of tasks on which the agent is prepared to work maximally in response to the cutoff incentive scheme  $\mathbf{w}^{m_1, \mu_1}$ , then

$$u(s_1; \mathbf{w}^{m_1, \mu_1}) - u(s_1-1; \mathbf{w}^{m_1, \mu_1}) \geq 0 > u(s_1+1; \mathbf{w}^{m_1, \mu_1}) - u(s_1; \mathbf{w}^{m_1, \mu_1}).$$

The convex-then-concave property of  $u(\cdot; \mathbf{w})$  then gives a contradiction, and establishes Proposition 10.

## Appendix C. Distinguishable tasks

Proposition 11 makes two logically separate statements. First, it says that there is a way to incentivize the agent to work on tasks  $S$  without basing his payment in any way on the success

or failure of tasks in  $N \setminus S$ . This is established by Lemma 7. Second, it says that the agent can be induced to work on tasks  $S$  using an incentive scheme that treats these tasks symmetrically, and is based only on the number of successes in this subset. This is established by Lemma 9.

### C.1. No reward for tasks on which the agent does not work

For any set  $R \subset N$ , let  $\mathcal{P}^R$  and  $\mathcal{P}^{-R}$  denote the power sets of  $R$  and  $N \setminus R$  respectively. Let  $\Pr(T|S)$  denote the probability of success on the subset of tasks  $T$  (and failure in  $N \setminus T$ ) if the agent works on the subset  $S$  tasks. Moreover, for any  $R \subset N$  and  $Q, T \subset R$  let  $\Pr(T|Q; R)$  denote the probability that every task in  $T$  succeeds while every task in  $R \setminus T$  fails, given that the agent works on the task subset  $Q$ . Note that  $\Pr(T|S; N) = \Pr(T|S)$ . Finally, let  $W(S; w)$  denote the agent's expected payment if he works on the subset  $S$  tasks, and the incentive scheme is  $w$ . That is,

$$W(S; w) \equiv \sum_{T \in \mathcal{P}} \Pr(T|S)w(T).$$

Our first result is that when tasks are distinguishable there is nothing gained by paying the agent for tasks on which he does not work. Formally, an incentive scheme depends only on the outcomes of tasks  $S$  if

$$w(T) = w(T \cap S) \quad \text{for all } T \in \mathcal{P}.$$

**Lemma 7.** *Suppose that there is an incentive scheme that induces the agent to work on  $S \subset N$  tasks. Then there is an alternate incentive scheme that costs the same, induces the agent to work on tasks  $S$ , and that depends only on successes among these  $S$  tasks.*

The proof of Lemma 7 uses the following straightforward result:

**Lemma 8.** *Suppose that an incentive scheme  $w$  depends only on the outcomes of tasks  $S$ . Let  $S^*$  be the set of tasks on which the agent works. Then (i)  $S^* \subset S$ , and (ii) the only incentive constraints that need to be checked are those to deviations in  $S$ .*

**Proof.** Observe that whenever  $w$  depends only on the outcomes in  $S$ , then for any  $\tilde{S} \in \mathcal{P}$ ,

$$\begin{aligned} W(\tilde{S}; w) &= \sum_{T \in \mathcal{P}} \Pr(T|\tilde{S})w(T) \\ &= \sum_{T \in \mathcal{P}^S} \sum_{T' \in \mathcal{P}^{-S}} \Pr(T|\tilde{S} \cap S; S) \Pr(T'| \tilde{S} \cap (N \setminus S); N \setminus S)w(T \cup T') \\ &= \sum_{T \in \mathcal{P}^S} \sum_{T' \in \mathcal{P}^{-S}} \Pr(T|\tilde{S} \cap S; S) \Pr(T'| \tilde{S} \cap (N \setminus S); N \setminus S)w(T) \\ &= \sum_{T \in \mathcal{P}^S} \Pr(T|\tilde{S} \cap S; S)w(T). \end{aligned}$$

From this, it is immediate that for any  $\tilde{S} \in \mathcal{P}$ ,

$$W(\tilde{S}; w) = W(\tilde{S} \cap S; w).$$

Consequently the agent prefers working on tasks  $\tilde{S} \cap S$  to working on tasks  $S$ : doing so leaves his expected payoff unchanged, while decreasing his costs.  $\square$

**Proof of Lemma 7.** Let  $\hat{w}$  be the original incentive scheme. Let  $S$  be a set of tasks on which the agent works, so that

$$S \in \arg \max_{\tilde{S} \in \mathcal{P}} W(\tilde{S}; \hat{w}) - (\pi - \varepsilon)|\tilde{S}|.$$

Define a new incentive schedule by

$$w(T) = \sum_{T' \in \mathcal{P}^{-S}} \Pr(T'| \emptyset; N \setminus S) \hat{w}((T \cap S) \cup T'),$$

where  $\Pr(\cdot | \cdot; \cdot)$  is as defined above. That is,  $w$  is formed from  $\hat{w}$  by taking the average outcome over realizations outside the task subset  $S$ , conditional on the agent not working on these tasks. Observe that for any  $T \in \mathcal{P}$ ,  $w(T) = w(T \cap S)$ . That is,  $w$  depends only on the successes tasks  $S$ .

We claim that

$$W(\tilde{S}; \hat{w}) = W(\tilde{S}; w) \quad \text{for any } \tilde{S} \subset S. \quad (\text{C.1})$$

To see this, observe that for any  $\tilde{S} \subset S$

$$\begin{aligned} W(\tilde{S}; \hat{w}) &= \sum_{T \in \mathcal{P}^S} \sum_{T' \in \mathcal{P}^{-S}} \Pr(T \cup T' | \tilde{S}) \hat{w}(T \cup T') \\ &= \sum_{T \in \mathcal{P}^S} \sum_{T' \in \mathcal{P}^{-S}} \Pr(T | \tilde{S}; S) \Pr(T' | \emptyset; N \setminus S) \hat{w}(T \cup T') \\ &= \sum_{T \in \mathcal{P}^S} \Pr(T | \tilde{S}; S) \sum_{T' \in \mathcal{P}^{-S}} \Pr(T' | \emptyset; N \setminus S) \hat{w}(T \cup T') \\ &= \sum_{T \in \mathcal{P}^S} \Pr(T | \tilde{S}; S) w(T), \end{aligned}$$

which equals  $W(\tilde{S}; w)$  (see the proof of Lemma 8). These observations are enough to establish that  $S$  is incentive compatible under  $w$ , that is,

$$W(S; w) - (\pi - \varepsilon)|S| \geq W(\tilde{S}; w) - (\pi - \varepsilon)|\tilde{S}| \quad \text{for all } \tilde{S} \subset S.$$

(The restriction to deviations to  $\tilde{S} \subset S$  follows from Lemma 8.) To see this, simply note that by (C.1) the inequality is equivalent to

$$W(S; \hat{w}) - (\pi - \varepsilon)|S| \geq W(\tilde{S}; \hat{w}) - (\pi - \varepsilon)|\tilde{S}| \quad \text{for all } \tilde{S} \subset S,$$

which holds since  $S$  is incentive compatible given  $\hat{w}$ . The result then follows.  $\square$

In the following subsection, we turn to the other claim made by Proposition 11: namely that the incentive scheme whose existence is implied by Lemma 7 can be further simplified to depend only on the *number* of successes within the subset  $S$ .

### C.2. Symmetry across tasks on which the agent works

Consider any set  $R \subset N$  and initial incentive scheme  $w$ . We will define a new incentive schedule that is based on  $w$  but which treats all tasks within  $N \setminus R$  symmetrically. The reader should think of  $R$  as a set of tasks on which the agent does *not* work.

For any  $k \leq n - |R|$ , let  $\mathcal{P}_k^{-R}$  be the subset of  $\mathcal{P}^{-R}$  consisting of sets of size  $k$ . Observe that  $|\mathcal{P}_k^{-R}| = \binom{n-|R|}{k}$ . Then for any  $k \leq n - |R|$  and  $Q \in \mathcal{P}^R$  define the payment received by the agent when  $k$  tasks in  $N \setminus R$  succeed, along with the subset  $Q \subset R$ :

$$\bar{w}_k(Q; w, R) \equiv \frac{1}{|\mathcal{P}_k^{-R}|} \sum_{T \in \mathcal{P}_k^{-R}} w(T \cup Q).$$

That is,  $\bar{w}_k(Q; w, R)$  is obtained by taking an average of payments made under  $w$ . The average is taken across success realizations that match  $Q$  in  $R$ , and have  $k$  successes within  $N \setminus R$ .

Suppose the agent works on tasks  $S$  and shirks on tasks  $N \setminus S$  when the incentive scheme is  $w$ . The importance of the incentive scheme  $\bar{w}(\cdot; w, N \setminus S)$  lies in the fact that it preserves this property, while treating tasks in  $S$  symmetrically:

**Lemma 9.** *Suppose that it is incentive compatible for the agent to work on tasks  $S$  under incentive scheme  $w$ . Then the same is true under the incentive scheme  $\bar{w}(\cdot; w, N \setminus S)$ .*

Lemmas 7 and 9 together imply Proposition 11.

The proof of Lemma 9 depends on the following separate result:

**Lemma 10.** *Consider any incentive scheme  $w$  and any subset of tasks  $R$ . Then for any  $Q \subset R$ ,  $s \leq n - |R|$ , and  $S_0 \in \mathcal{P}_s^{-R}$ ,*

$$\frac{1}{|\mathcal{P}_s^{-R}|} \sum_{S \in \mathcal{P}_s^{-R}} W(S \cup Q; w) = W(S_0 \cup Q; \bar{w}). \quad (\text{C.2})$$

That is, under the new schedule  $\bar{w}$ , the agent's payoff from working on any  $s$  tasks in  $N \setminus R$  is equal to the average of the payoffs received under the old schedule, where the average is taken over all possible combinations of working on  $s$  tasks in  $N \setminus R$ .

Lemma 9 follows relatively easily from Lemma 10:

**Proof of Lemma 9.** Suppose to the contrary that there exists an  $\tilde{S}$  such that

$$W(\tilde{S}; \bar{w}) - (\pi - \varepsilon)|\tilde{S}| > W(S; \bar{w}) - (\pi - \varepsilon)|S|. \quad (\text{C.3})$$

Let  $\hat{s} = |\tilde{S} \cap S|$ . Observe that  $W(S; w) = W(S; \bar{w})$  (this follows from Lemma 10). By Lemma 10, inequality (C.3) is equivalent to

$$\frac{1}{|\mathcal{P}_{\hat{s}}^{-N \setminus S}|} \sum_{\hat{S} \in \mathcal{P}_{\hat{s}}^{-N \setminus S}} W(\hat{S} \cup (\tilde{S} \cap (N \setminus S)); w) - (\pi - \varepsilon)|\tilde{S}| > W(S; w) - (\pi - \varepsilon)|S|.$$

Thus there exists some  $\hat{S} \in \mathcal{P}_{\hat{s}}^{-N \setminus S} = \mathcal{P}_{\hat{s}}^S$  such that

$$W(\hat{S} \cup (\tilde{S} \cap (N \setminus S)); w) - (\pi - \varepsilon)|\tilde{S}| > W(S; w) - (\pi - \varepsilon)|S|.$$

But since  $|\hat{S}| + |\tilde{S} \cap (N \setminus S)| = |\tilde{S}|$ , and  $\hat{S} \cup (\tilde{S} \cap (N \setminus S))$  was a feasible deviation under the original incentive scheme  $w$ , this contradicts the hypothesis that the agent works on tasks  $S$  under  $w$ .  $\square$

It remains only to prove Lemma 10 itself:

**Proof of Lemma 10.** We start with some preliminaries. For each of  $k = 0, \dots, n - |R|$ , let  $T_k$  be a subset of  $N \setminus R$  with  $k$  elements. The agent's expected payment under  $\bar{w}$  can be written

$$\begin{aligned} W(S; \bar{w}) &= \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \bar{w}_k(T^R) \Pr(T^R \subset R \text{ successes and } k \text{ successes in } N \setminus R | S) \\ &= \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \bar{w}_k(T^R) \sum_{T \in \mathcal{P}_k^{-R}} \Pr(T \cup T^R | S). \end{aligned}$$

It is useful to rewrite  $\bar{w}_k(\cdot)$  in the following manner. Let  $\Sigma^{-R}$  be the set of permutations  $N \setminus R \rightarrow N \setminus R$ . Observe that  $|\Sigma^{-R}| = (n - |R|)!$ . Moreover, for any  $T \subset N \setminus R$ , there are  $|T|!(n - |R| - |T|)!$  permutations of  $N \setminus R$  that leave both the sets  $T$  and  $(N \setminus R) \setminus T$  unchanged. It follows that for any  $Q \subset R$ ,

$$\begin{aligned} \bar{w}_k(Q) &= \frac{1}{\binom{n-|R|}{k}} \sum_{\sigma \in \Sigma^{-R}} \frac{1}{k!(n - |R| - k)!} w(\sigma(T_k) \cup Q) \\ &= \frac{1}{(n - |R|)!} \sum_{\sigma \in \Sigma^{-R}} w(\sigma(T_k) \cup Q). \end{aligned}$$

(Dividing by  $k!(n - |R| - k)!$  avoids double counting permutations that leave members of both  $\sigma(T_k)$  and  $(N \setminus R) \setminus \sigma(T_k)$  unchanged.)

We are now ready to establish (C.2). First,

$$\begin{aligned} \frac{1}{|\mathcal{P}_s^{-R}|} \sum_{S \in \mathcal{P}_s^{-R}} W(S \cup Q; w) &= \frac{1}{\binom{n-|R|}{s}} \sum_{\sigma \in \Sigma^{-R}} \frac{1}{s!(n - |R| - s)!} W(\sigma(S_0) \cup Q; w) \\ &= \frac{1}{(n - |R|)!} \sum_{\sigma \in \Sigma^{-R}} W(\sigma(S_0) \cup Q; w). \end{aligned}$$

Second, for any  $S \in \mathcal{P}_s^{-R}$ ,

$$\begin{aligned} W(S \cup Q; w) &= \sum_{T \in \mathcal{P}} \Pr(T | S \cup Q) w(T) \\ &= \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \sum_{T \in \mathcal{P}_k^{-R}} \Pr(T \cup T^R | S \cup Q) w(T \cup T^R) \\ &= \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \sum_{\sigma \in \Sigma^{-R}} \frac{1}{k!(n - |R| - k)!} \\ &\quad \times \Pr(\sigma(T_k) \cup T^R | S \cup Q) w(\sigma(T_k) \cup T^R). \end{aligned}$$

Thus the left-hand side of (C.2) can be rewritten as

$$\frac{1}{(n - |R|)!} \sum_{\sigma' \in \Sigma^{-R}} \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \sum_{\sigma \in \Sigma^{-R}} \frac{1}{k!(n - |R| - k)!} \\ \times \Pr(\sigma(T_k) \cup T^R | \sigma'(S_0) \cup Q) w(\sigma(T_k) \cup T^R).$$

By symmetry of task success probabilities, for any permutation  $\sigma'' \in \Sigma^{-R}$ ,

$$\Pr(\sigma(T_k) \cup T^R | \sigma'(S_0) \cup Q) = \Pr(\sigma''(\sigma(T_k)) \cup T^R | \sigma''(\sigma'(S_0)) \cup Q),$$

and so in particular

$$\Pr(\sigma(T_k) \cup T^R | \sigma'(S_0) \cup Q) = \Pr((\sigma')^{-1}(\sigma(T_k)) \cup T^R | S_0 \cup Q).$$

Thus left-hand side of (C.2) becomes

$$\frac{1}{(n - |R|)!} \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \sum_{\sigma \in \Sigma^{-R}} \sum_{\sigma' \in \Sigma^{-R}} \frac{1}{k!(n - |R| - k)!} \\ \times \Pr((\sigma')^{-1}(\sigma(T_k)) \cup T^R | S_0 \cup Q) w(\sigma(T_k) \cup T^R).$$

Now, for any  $\sigma \in \Sigma^{-R}$ ,  $\{(\sigma')^{-1} \circ \sigma : \sigma' \in \Sigma^{-R}\} = \Sigma^{-R}$ . As such, the left-hand side of (C.2) rewrites once more to

$$\frac{1}{(n - |R|)!} \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \sum_{\sigma \in \Sigma^{-R}} \sum_{\sigma'' \in \Sigma^{-R}} \frac{1}{k!(n - |R| - k)!} \\ \times \Pr(\sigma''(T_k) \cup T^R | S_0 \cup Q) w(\sigma(T_k) \cup T^R) \\ = \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \sum_{\sigma'' \in \Sigma^{-R}} \frac{1}{k!(n - |R| - k)!} \Pr(\sigma''(T_k) \cup T^R | S_0 \cup Q) \frac{1}{(n - |R|)!} \\ \times \sum_{\sigma \in \Sigma^{-R}} w(\sigma(T_k) \cup T^R) \\ = \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \sum_{\sigma'' \in \Sigma^{-R}} \frac{1}{k!(n - |R| - k)!} \Pr(\sigma''(T_k) \cup T^R | S_0 \cup Q) \bar{w}_k(T^R) \\ = \sum_{T^R \in \mathcal{P}^R} \sum_{k=0}^{n-|R|} \bar{w}_k(T^R) \sum_{T \in \mathcal{P}_k^{-R}} \Pr(T \cup T^R | S_0 \cup Q) = W(S_0 \cup Q; \bar{w}). \quad \square$$

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