Multilateral Negotiations and Formation of Coalitions

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Abstract

This paper analyses properties of games modelling multilateral negotiations leading to the formation of coalitions in an environment with widespread externalities. The payoff generated by each coalition is determined by an exogenous partition function (the parameter space). We show that in almost all games, except in a set of measure zero of the parameter space, the Markov perfect equilibrium value of coalitions and the state transition probability that describe the path of coalition formation is locally unique and stable. Therefore, comparative statics analysis are well-defined and can be performed using standard calculus tools. Global uniqueness does not hold in general, but the number of equilibria is finite and odd. In addition, a sufficient condition for global uniqueness is derived, and using this sufficient condition we show that there is a globally unique equilibrium in three-player superadditive games.

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1 Introduction

We show in this paper that the Markov perfect equilibrium of multilateral bargaining games or coalitional bargaining games are generically local unique and stable, and we derive a sufficient condition for global uniqueness. The coalition formation procedure studied in this paper is similar to the coalitional bargaining game introduced in Gomes (2005). Multilateral negotiations are modelled as a dynamic game with complete information where at each stage a player becomes the proposer with exogenously given probabilities. Proposers make offers to form coalitions, followed by players who have received offers making their response whether or not to accept the offer. Similarly to Gul (1989) and Seidmann and Winter (1998), coalitions after forming do not leave the game and may continue negotiating the formation of further coalitions. Other related models are Chatterjee et al. (1993), Hart and Mas-Colell (1996), Ray and Vohra (1999), and Okada (1996) among many others.

A variety of economic problems can be addressed with coalitional bargaining games such as the formation of custom unions, merger or carter formation among firms in the same industry, legislative bargaining, and the signing of environmental agreements across regions (see, for example, Ray (2007)).

The formation of coalitions in our setting may impose externalities on other players. The externalities present in the environment are described by a set of exogenous parameters, conveniently expressed using a partition function form. The partition function form assigns a worth to each coalition depending on the coalition structure (or collection of coalitions) formed by the remaining players. This general formulation allows for the analyses of problems in which the formation of coalitions may impose positive or negative externalities (see also Ray and Vohra (1999), Bloch (1996), Jehiel and Moldovanu (1995), and Gomes (2005)).

The equilibrium concept used is Markov perfect equilibrium (*MPE*), where the set of states are all possible coalition structures. The *MPE* solutions determine, jointly, both the expected equilibrium value of coalitions and the Markov state transition probability that describes the path of coalition formation. Our goal is to develop a thorough analysis of the equilibrium properties of multilateral bargaining games.

We show that, for almost all games, except in a closed set of measure zero of the parameter space, the equilibrium is locally unique and locally stable. These properties imply that the predictions of the model about both the expected player payoffs and the path of coalition formation are sharp, in the sense that, at least locally, they are unique and robust to small perturbations of the exogenous parameters of the game. Specifically, stability and local uniqueness imply that for any small perturbation of the game parameters, a closeby unique equilibrium is guaranteed to exist and, moreover, the mapping between the game parameters and the local equilibrium points is a smooth function. These properties are important because they allow us to perform comparative statics analysis using standard calculus tools.

In this paper, we extend to multilateral bargaining models similar results that hold for other well-known economic models such as Walrasian equilibrium of competitive economies (Debreu (1970)), Nash equilibrium of n-person strategic form games (Wilson (1971) and Harsanyi (1973)), and Markov perfect equilibrium of stochastic games (Haller and Lagunoff (2000) and Herings and Peeters (2004))).

However, there is a very important difference between the results for coalitional bargaining games and competitive economies, strategic form games, and stochastic games. In all these three previous important classes of economic problems, generic local uniqueness and stability holds for the equilibrium strategies. This is not the case for coalition bargaining games. We provide a robust example of a class of three-player coalition bargaining games in which all of the three players are indifferent about which of the two pairwise coalitions they can form–thus any of the three pairwise coalitions can arise in equilibrium. We show that in this game, and for all games nearby, there are a continuum of *MPE* strategies. The generic uniqueness result applies only for the expected equilibrium value of coalitions and the probability of coalition formation. What is uniquely determined is the probability of a coalition forming, but it is indeterminate how this coalition will actually form, as a player can put more weight and another player less weight on a coalition forming, so that only the overall probability of coalition formation is uniquely determined.

This unique feature of coalition bargaining games has significant implications and, in particular, our proof methodology is different from the one used in the previous literature. The strategy profile σ belongs to a space with higher dimensionality than the transition probabilities μ, and when projecting σ into μ some important information is lost. We introduce the concept of coalitional dynamic structure, which essentially encodes the support structure of the strategy profile, which allow us to recover all the essential information about the strategy profile that is lost when projecting into the transition probabilities. We show that finding *MPE* strategies are equivalent to finding solutions, in terms of expected payoffs and transition probabilities, of games with a given coalitional dynamic structure. The key genericity local uniqueness results are obtained from the use of the transversality theorem (see Guillemin and Pollack (1974)) applied to games with a given coalitional dynamic structure.

We also show that the number of equilibrium solutions is finite and odd for almost all games. We provide an example of a game with multiple (seven) equilibrium solutions, so the equilibrium is not globally unique. Nonetheless, we derive a sufficient condition for global uniqueness, and argue that this sufficient condition is weak and is likely to be satisfied by a large class of games.

An important application of this result is to show that this sufficient condition holds for superadditive three-player coalitional bargaining games. Therefore, we establish in this paper that superadditive three-player coalition bargaining games have, generically, globally unique *MPE*.

The proof of our results about the finiteness and number of equilibrium solutions are also significantly different from similar results for competitive economies, strategic form games, and stochastic games. Again these distinctions arise because the results do not hold for the equilibrium strategies, but only to its projections to transition probabilities. We prove the results for coalitional bargaining games using a stronger version of the Lefschetz index theorem for correspondences developed by McLennan (1989). In contrast, a standard version of the Lefschetz index theorem, developed in differential calculus textbooks, suffices to develop the formula for the number of equilibria for competitive economies, strategic form games, and stochastic games.

How do the equilibrium value of players and the path of coalition formation change as a result of changes in exogenous parameters such as the partition function form and the probability of being the proposer? Knowing how to address these questions is of considerable practical interest to negotiators, as they, for example, may be able to invest in changing the likelihood of being proposers in negotiations. We show how to answer these questions using standard calculus results (the implicit function theorem), which provides a powerful tool for quickly answering comparative statics questions by simply evaluating Jacobian matrices at the solution.

We demonstrate the applicability of the results using two classic games–apex and quota games (see Shapley (1953), Davis and Maschler (1965), and Maschler (1992)). Surprisingly, in both games, a player sometimes may not benefit by investing in obtaining more initiative to propose in negotiations. Other players may adjust their strategies in such a way that lead the proposer to be worse off. The analysis also suggests several interesting regularities: when the exogenous value of a coalition increases, both the equilibrium value of the coalitional members and the likelihood that the coalition forms increase as well.

The remainder of the paper is organized as follows: Section 2 presents the coalitional bargaining game; Section 3 addresses the characterization of the equilibria; Section 4 develop the local uniqueness, stability, and genericity properties of the equilibria; Section 5 addresses the number of equilibrium solutions; and Section 6 concludes.

2 The Model

The multilateral bargaining game we study in this paper is similar to the coalitional bargaining game introduced in Gomes (2005). Coalition formation is modeled as an infinite horizon complete information game. In a nutshell, the coalition formation process is such that during any period of the game a player is chosen at random to propose to form a coalition and a payment to all coalition members. Subsequently, all coalition members respond to the offer, and the coalition is formed only if all its members agree.

Formally, let $N = \{1, 2, \cdots, n\}$ be a set of n agents. A *coalition* is a subset of agents and a *coalition structure (c.s.)* $\pi = \{i_1, ..., i_K\}$ is a partition of the set of agents N into disjoint coalitions, where each coalition i_k is a subset of N (i.e., $i_k \subset N$). We denote by Π the set of all possible coalition structures. One element of Π is $\mathcal{N} = \{\{1\}, ..., \{n\}\}$ (read as calligraphic $\mathcal{N}$), which is the c.s. in which all agents are in solo coalitions. We often represent typical coalitions in π by labels such as i and j.

In our model the players are the coalitions. Our coalition formation game closely resembles the merger and acquisition process among firms. Each coalition is a firm, and the process of coalition formation is equivalent to the merger and acquisition process among firms. For example, the firm/coalition $\{1, 2, 4\}$ is formed by the merger of 1,2, and 4. When two firms such as $\{1, 3\}$ and $\{2, 4\}$ form a coalition (or merge) they create a new firm/coalition $\{1,2,3,4\}$.[1]

The *coalition bargaining game* is the game with the following extensive form. Consider that at the beginning of a certain period of the game the c.s. is π . One of the coalitions $i \in \pi$ is randomly chosen with probability $p_i(\pi) > 0$ to be the proposer. Let the proposer probabilities be $p := (p_i(\pi))_{\substack{\pi\in\Pi \\ i\in\pi}}$. Coalition i then makes an offer (S, λ) where $S \subset \pi$ and $i \in S$ is a set of coalitions in π and λ is a vector of transfers satisfying $\sum_{j\in S} \lambda_j = 0$ (the vector $\lambda = (\lambda_j)_{j\in S}$ is such that $\lambda_i = -\sum_{j\in S\backslash i} \lambda_j$). All coalitions in S respond in a fixed sequential order whether they accept or not the offer (it turns out the the order of response is not relevant). If all coalitions in S accept the offer a new coalition $\mathcal{S} = \cup_{j\in S}\{j\} \subset N$, calligraphic $\mathcal{S}$, is formed under the control of the proposing coalition i. The coalitions $j \in S\backslash i$ ceding control receive the lump-sum payment λ_j and exit the game. The coalition structure evolves from π to $\pi S = \mathcal{S} \cup (\pi\backslash S)$. Otherwise, if any one of the coalitions receiving the offer rejects it, no new coalition is formed and the coalition structure remains equal to π. After a lapse of one period of time, the game is repeated starting with the prevailing c.s. with a new proposer being randomly chosen as just described.

Restating the game in the language of mergers and acquisitions among firms helps clarify the model. Say, for example, that the c.s. is $\pi = \{\{1,2\},\{3,4\},\{5\}\}$. So there are three firms remaining in the game (firm $\{1,2\}$, formed by the merger of 1 and 2, firm $\{3,4\}$, formed by the merger of 3 and 4, and firm $\{5\}$). Say now that firm $i = \{1,2\}$ proposes to form coalition $S = \{\{1,2\},\{3,4\}\}$, or in other words, firm i proposes to acquire (or merge with) firm $j = \{3,4\}$ subject to the payment λ_j to firm j. If firm $j = \{3,4\}$ accepts the offer then it leaves the game receiving a payment λ_j, and firm i is now renamed firm $\mathcal{S} = \{1,2,3,4\}$. Note that $\mathcal{S} = \cup_{j\in S}\{j\}$. A new coalition structure $\pi S = \{\{1,2,3,4\},\{5\}\})$ is formed after the merger. Note that $\pi S = \mathcal{S} \cup (\pi\backslash S)$. Our model of coalition formation captures in a natural way the mergers and acquisitions among firms. Note that the coalition structure becomes coarser as time elapses (that is a firm cannot divest a division in our model).

All players have the same expected intertemporal utility function, are risk-neutral and have common discount factor $\delta \in (0,1)$. The players utility over a stream of random payoffs $(x_\tau)_{\tau=0}^{\infty}$ is then $\sum_{\tau=0}^{\infty} \delta^\tau E(x_\tau)$.[2] When coalitions form they may impose externalities on

[1] For example, other ways in which the same firm $\{1,2,3,4\}$ could be formed is by the coalition (merger) between firms $\{1\}$ and $\{2,3,4\}$; the merger/coalition among $\{1\}$, $\{2\}$, and $\{3,4\}$; or the merger among the firms $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$.

[2] When a player leaves the game at time T then its payoff are $x_t = 0$ for all $t > T$.

other coalitions. This possibility is captured by a *partition function form* $v = (v_i(\pi))_{\substack{\pi\in\Pi \\ i\in\pi}}$, where coalition i's payoff flow (during a period of time), when the coalition structure is π, is equal to $(1-\delta)\, v_i(\pi)$ (so if the game stays at c.s. π forever, the value of coalition i is $v_i(\pi)$). The payoffs are distributed at the end of each period, after the coalition formation stage, with coalitions ceding control receiving a final lump-sum transfer payoff and the coalition acquiring control receiving, in addition to the lump-sum transfer, the payoff given by the partition function form (i.e., $(1-\delta)\, v_i(\pi S) + \lambda_i$ when the c.s. πS forms).

In the application to mergers and acquisitions, if the c.s. is π at the end of a period, after any mergers that may have taken place, then the profit that each firm $i \in \pi$ obtains during this period is $(1-\delta)\, v_i(\pi)$. Certainly, the profit of a firm i that makes an acquisition during a period, and now becomes a bigger firm $\mathcal{S}$, can be different than the profit without the acquisition. And since the firm merging can impose externalities on other firms not involved in the merger, their profits with and without the merger can also be different. The games we study in this paper are completely described by the game parameters (v, p).

We restrict our attention to *Markovian* strategies. Hence, the proposer's strategy only depends on the current state π, and the respondents' strategy only depends on the current state π, the current offer she receives and the responses of preceding players. A *Markov perfect equilibrium* (*MPE*) is a Markovian strategy profile where every player plays a Nash equilibrium at every stage.

In the game we are studying the players are the coalitions. However, in many applications of economic interest individual agents retain autonomy after coalitions form. Our results though are still applicable more generally because Gomes (2005) shows that there is a one-to-one mapping between the *MPE* of coalitional bargaining games (CBG) and multilateral contracting games (MCG). Multilateral contracting games are coalitional games in which the players are the original individual agents, which are randomly chosen to offer contracts to a subset of players, who then accept or decline the contract offer. Contracts are binding agreements that specify monetary transfers among signatories conditional on the coalition structures formed by players outside the contract. Contracts can be revoked or rewritten only by unanimous consent of the contract signatories, and specify the individual per-period payoffs of each player contingent on all possible coalition structures formed by the remaining players. The results in this paper will thus also allow us to have a better understanding of the equilibrium properties of MCG games.

3 Characterization of Equilibrium

Let us be given a *MPE* strategy σ. We represent by $\phi_i(\pi|\sigma)$ the *equilibrium continuation value* of coalition i which is obtained from the stochastic process induced by σ when the c.s. is at π (the value ϕ is computed at the beginning of a period before a proposer is chosen).

The equilibrium continuation value at the end of a period in which π is formed is equal to (gross of lump-sum transfers)

$$x_i(\pi|\sigma) = \delta\phi_i(\pi|\sigma) + (1-\delta)\, v_i(\pi)\,, \tag{1}$$

because coalition i receives payoff flow $(1-\delta)\, v_i(\pi)$ during the current period and, after a delay of one period, at the beginning of the next period, coalition i 's value is $\delta\phi_i(\pi|\sigma)$. Let $x = (x_i(\pi))_{\substack{\pi\in\Pi\\ i\in\pi}}$ and $\phi = (\phi_i(\pi))_{\substack{\pi\in\Pi\\ i\in\pi}}$ be the continuation values (where we sometimes omit the dependency on the strategy profile σ). So ϕ and x are, respectively, the continuation values at the beginning and end of a period.

An equilibrium σ is characterized by several properties which we now summarize. The minimum offer that coalition j receiving offer (S, λ) is willing to accept is one where $\lambda_j \geq x_j(\pi|\sigma)$. Upon rejection of any offers, no transfers are made, and the state remains at π, so the value of player j rejecting an offer is $x_j(\pi|\sigma) = \delta\phi_j(\pi|\sigma) + (1-\delta)\, v_j(\pi)$, the profit it gets this period plus its discounted value at the beginning of the next period.

In turn, coalition i proposes offers (S, λ) that maximizes the value $x_{\mathcal{S}}(\pi S|\sigma) - \sum_{j\in S\backslash i}\lambda_j$ subject to the constraint that $\lambda_j \geq x_j(\pi|\sigma)$. A new coalition $\mathcal{S}$ is formed (or coalition i is now renamed coalition $\mathcal{S}$), whose value is $x_{\mathcal{S}}(\pi S|\sigma) = \delta\phi_{\mathcal{S}}(\pi S|\sigma) + (1-\delta)\, v_{\mathcal{S}}(\pi S)$, if the offer is accepted by all players, and it costs the proposer $\sum_{j\in S\backslash i}\lambda_j$ to form this new coalition. Thus, when an offer (S, λ) is made the transfers λ are uniquely determined by $\lambda_j = x_j(\pi|\sigma)$ for $j \in S\backslash i$ and $\lambda_i = \sum_{j\in S\backslash i}\lambda_j$.

Define the *excess,* or gain from forming a coalition, by

$$\theta(\pi)(S)(x) = x_{\mathcal{S}}(\pi S|\sigma) - \sum_{j\in S} x_j(\pi|\sigma)\,, \tag{2}$$

a function of $S \subset \pi$, and x equilibrium continuation value. Proposer i randomizes across coalitions S that maximizes the excess $\max_{S\ni i, S\subset\pi}\{\theta(\pi)(S)(x)\}$.

Let $\sigma_i(\pi)(S) \in [0,1]$ represent the probability that coalition S is chosen by player i. The (behavioral) strategy $\sigma_i(\pi)$ of proposer i is a probability distribution over $\Sigma_i(\pi) =$

$\{S \subset \pi : i \in S\}$, i.e. $\sum_{S \in \Sigma_i(\pi)} \sigma_i(\pi)(S) = 1$ and $\sigma_i(\pi)(T) = 0$ for all $T \notin \Sigma_i(\pi)$. Also, we define $\Delta(\pi)$ as the set of strategy profiles when the c.s. is π, and let Δ be the set of strategy profiles (i.e., $\sigma \in \Delta$).

We use the following standard notation: $\times$ is the Cartesian product, $|A|$ is the cardinality of set A, $\mathbb{I}_A$ is the indicator function that is equal to one or zero, respectively, if statement A is true or false, and the support of $\sigma_i(\pi)$ is *supp* $\sigma_i(\pi)$, the set of all S such that $\sigma_i(\pi)(S) > 0$.

The necessary part of the following lemma, proved in the Appendix, follows directly from the above discussion and the definition of *MPE*.

Lemma 1 *A payoff structure $\phi_i(\pi)$ and a strategy profile $\sigma_i(\pi)$ is an MPE of the coalitional bargaining game if and only if the following system of equations is satisfied, where $x_i(\pi) = \delta\phi_i(\pi) + (1-\delta)v_i(\pi)$:*
1) the support of the strategy $\sigma_i(\pi)$ is

$$\mathit{supp}\ \sigma_i(\pi) \subset \underset{S \ni i, S \subset \pi}{\arg\max} \{\theta(\pi)(S)(x)\}, \tag{3}$$

2) the expected equilibrium outcome of player i conditional on player j being chosen to be the proposer $\phi_i^j(\pi)$ is equal to

$$\phi_i^j(\pi) = \begin{cases} \max\limits_{S \ni i, S \subset \pi} \{\theta(\pi)(S)(x)\} + x_i(\pi) & j = i \\ \sum_{S \subset \pi} \sigma_j(\pi)(S)\left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S)\right) & j \neq i \end{cases}, \tag{4}$$

3) the following system of equations holds

$$\phi_i(\pi) = \left(\sum_{j \in \pi} p_j(\pi)\, \phi_i^j(\pi)\right), \tag{5}$$

for all $\pi \in \Pi$, and $i, j \in \pi$.

There is a one-to-one relation between $\phi_i(\pi)$ and $x_i(\pi)$ given by equation (1), and these are the player continuation values at c.s. π. For convenience we will be solving for the vectors $x_i(\pi)$ instead of $\phi_i(\pi)$ from now on. The vector x, as well as the partition function form v, belongs to the Euclidean space R^d, where the dimension $d = \sum_{\pi \in \Pi} |\pi|$.

It follows directly from the Kakutani fixed point theorem that there always exist *MPE* solutions for all coalitional bargaining games (see for example, Gomes (2005)).

We now introduce a mixed nonlinear complementarity problem (*MNCP*) associated with coalitional bargaining games which will be used in establishing our main results about uniqueness in the following sections. See Cottle, Pang, and Stone (1992) and Harker and Pang (1990) for a comprehensive analysis of *MNCP* problems.

For all $x = (x_i(\pi))_{\substack{\pi\in\Pi \\ i\in\pi}}$ (continuation values), $e = (e_i(\pi))_{\substack{\pi\in\Pi \\ i\in\pi}}$ (excesses or gains from forming coalitions), and $\sigma \in \Delta$, consider the mapping $f(x,\sigma,e)$, where coordinate $f_i(\pi)$ is given by

$$\begin{aligned} f_i(\pi)(x,\sigma,e) = {} & x_i(\pi) - \delta p_i(\pi) e_i(\pi) - (1-\delta) v_i(\pi) \\ & - \delta \left(\sum_{S\subset\pi} \left(\sum_{j\in\pi} p_j(\pi)\sigma_j(\pi)(S) \left(\mathbb{I}_{[i\in S]} x_i(\pi) + \mathbb{I}_{[i\notin S]} x_i(\pi S) \right) \right) \right), \end{aligned} \tag{6}$$

for all $i \in \pi$ and $\pi \in \Pi$. Let the maps $h(\sigma)$ and $g(e,x)$ be defined by

$$\begin{aligned} h_i(\pi)(\sigma) &= \sum_{S\subset\pi: i\in S} \sigma_i(\pi)(S) - 1, \\ g_i(\pi)(S)(e,x) &= e_i(\pi) - \left(x_{\mathcal{S}}(\pi S) - \sum_{j\in S} x_j(\pi) \right), \end{aligned} \tag{7}$$

for all i, π, S satisfying $\pi \in \Pi$, $i \in \pi$, and $i \in S \subset \pi$. Denote the inner or scalar product of the two vectors σ and $g(e,x)$ by

$$\langle \sigma, g(e,x) \rangle = \sum_{\substack{i,\pi,S: \\ i\in S\subset\pi\in\Pi}} \sigma_i(\pi)(S) . g_i(\pi)(S)(e,x)$$

The mixed nonlinear complementarity problem is the problem of finding triples (x,σ,e) that satisfy all conditions

$$\begin{aligned} f(x,\sigma,e) &= 0, \\ h(\sigma) &= 0, \\ g(e,x) &\geq 0, \\ \sigma &\geq 0, \\ \langle \sigma, g(e,x) \rangle &= 0. \end{aligned} \tag{MNCP}$$

Note that $\langle \sigma, g(e,x) \rangle = 0$ is equivalent to $\sigma_i(\pi)(S)\, g_i(\pi)(S)(e,x) = 0$ for all i, π, S satisfying $\pi \in \Pi$, $i \in \pi$, and $i \in S \subset \pi$, given that $g(e,x) \geq 0$ and $\sigma \geq 0$.

Proposition 1 *If (x,σ) is an MPE then (x,σ,e) is a solution of the problem MNCP, where $e_i(\pi) = \max_{S \ni i, S \subset \pi} \{\theta(\pi)(S)(x)\}$. Reciprocally, if (x,σ,e) is a solution of the problem MNCP then (x,σ) is an MPE.*

PROOF: Consider the necessary part of the proposition, and say that (x,σ) is an *MPE*. Then all the conditions in items 1, 2, and 3 of Lemma 1 hold. Replacing expression (4) of $\phi_i^j(\pi)$ into equation (5), and considering that, by definition, $x_i(\pi) = \delta\phi_i(\pi) + (1-\delta)\, v_i(\pi)$, we obtain the system of equations

$$
\begin{aligned}
x_i(\pi) &= \delta p_i(\pi) \max_{S \ni i, S \subset \pi} \{\theta(\pi)(S)(x)\} + (1-\delta) v_i(\pi) + \\
&\quad + \delta \left(\sum_{S \subset \pi} \left(\sum_{j \in \pi} p_j(\pi) \sigma_j(\pi)(S) \left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S) \right) \right) \right).
\end{aligned}
$$

Let $e_i(\pi) = \max_{S \ni i, S \subset \pi} \{\theta(\pi)(S)(x)\}$. We then have that (x,σ,e) satisfies the equation $f(x,\sigma,e) = 0$. Since σ is a probability distribution then $h(\sigma) = 0$ and $\sigma \geq 0$ are automatically satisfied. Also, by definition of $e_i(\pi)$ above, $e_i(\pi) - \theta(\pi)(S)(x) \geq 0$, so that $g(e,x) \geq 0$. Finally, $\sigma_i(\pi)(S)\, g_i(\pi)(S)(e,x) = 0$ follows from definition of e and the support restriction of σ in (3). The reciprocal follows using the same arguments. Q.E.D.

4 Generic Local Uniqueness and Stability

In this section we show that almost all games, except in a set of measure zero of the parameter space (v,p), have equilibria that are locally unique and stable. These results imply that almost all games have only a finite number of equilibria and provide tools for performing comparative statics analysis in coalitional bargaining games.

Stability and local uniqueness is a property that ensures that comparative statics exercises are well defined. Roughly speaking an equilibrium point is stable if, for any small perturbation on the game parameters (v,p), there exists another equilibrium closeby and, by local uniqueness, this is the only equilibrium closeby. Moreover, the mapping between the game parameters and the local equilibrium points are a smooth function.

Consider the Markov transition probability $\mu = \mu(\sigma)$ associated with a strategy profile σ, which is defined as

$$\mu(\sigma)(\pi)(S) = \sum_{j \in \pi} p_j(\pi)\,\sigma_j(\pi)(S), \tag{8}$$

where $\mu(\sigma)(\pi)(S)$ represents the probability of moving from state π to state πS in one period.

An important difficulty in the analysis is that the equilibrium is locally unique on the transition probability μ, but it is not locally unique in the equilibrium strategies σ. The following example illustrates this point.

Example 1: Three-Player Games with Continuum of *MPE* σ

Consider a three player coalitional bargaining game with the following partition function form and proposer probabilities (v, p):

- If no coalition forms, $\pi_0 = \{\{1\},\{2\},\{3\}\}$, the values are normalized to zero $v_i(\pi_0) = 0$, and the proposer probabilities are $p_i = p_i(\pi_0) < \frac{1}{2}$, for all $i = 1, 2$, and 3;
- If the coalition $\{i, j\}$ forms, the values are $V_{ij} = v_{ij}(\{\{i,j\},\{k\}\})$ and $V_k = v_k(\{\{i,j\},\{k\}\})$, and the proposer probabilities are $P_{ij} = p_{ij}(\{\{i,j\},\{k\}\})$ and $P_k = p_k(\{\{i,j\},\{k\}\})$;
- If the grand coalition forms, the value is $V = v_N(\{N\})$ and assume the grand coalition is efficient: that is, $V > 0$ and $V > V_{ij} + V_k$ for all $k = 1, 2$, and 3.

Lemma 2 *Assume that three player coalitional bargaining game (v, p) above satisfies*

$$\Phi_{12} + \Phi_{13} + \Phi_{23} > 2V \tag{9}$$

where Φ_{ij} are defined as follows

$$\Phi_{ij} = V_{ij} + P_{ij}(V - V_k - V_{ij}).$$

Then there exists a $\bar{\delta}$ such that for all $\delta \in (\bar{\delta}, 1)$ the game (v, p) have MPE σ that is not locally unique. In fact, there are a continuum of MPE strategies σ for all games (v, p).

We show in the Appendix that the game above has *MPE* σ in which, from the initial c.s. π_0, all three players choose to form any pairwise coalition with any of the other two players (i.e., $\sigma_i(\pi_0)(\{i,j\}) > 0$ and $\sigma_i(\pi_0)(\{i,k\}) > 0$). In order for this strategy profile to be optimal, it is key that inequality (9) holds, as the excess or gain of forming a pairwise coalition is shown to be approximately equal to $\frac{1}{3}\left(\Phi_{12} + \Phi_{13} + \Phi_{23} - 2V\right)$.

The Markov transition probabilities μ and expected payoffs x are uniquely determined, but a continuum of *MPE* σ can be constructed as the solution of the system of linear equations:

$$
\begin{aligned}
p_1\sigma_1(\pi_0)(\{1,2\}) + p_2\sigma_2(\pi_0)(\{1,2\}) &= \mu(\pi_0)(\{1,2\}), \\
p_1\sigma_1(\pi_0)(\{1,3\}) + p_3\sigma_3(\pi_0)(\{1,3\}) &= \mu(\pi_0)(\{1,3\}), \\
p_2\sigma_2(\pi_0)(\{2,3\}) + p_3\sigma_3(\pi_0)(\{2,3\}) &= \mu(\pi_0)(\{2,3\}), \\
\sigma_1(\pi_0)(\{1,2\}) + \sigma_1(\pi_0)(\{1,3\}) &= 1, \\
\sigma_2(\pi_0)(\{1,2\}) + \sigma_2(\pi_0)(\{2,3\}) &= 1, \\
\sigma_3(\pi_0)(\{1,3\}) + \sigma_3(\pi_0)(\{2,3\}) &= 1.
\end{aligned}
$$

Given the transition probabilities μ, the system of 6 linear equations above on the σ's has rank 5, and thus, given that $\mu(\pi_0)(\{1,2\}) + \mu(\pi_0)(\{1,3\}) + \mu(\pi_0)(\{2,3\}) = 1$, there are infinitely many solutions σ (with dimensionality 1).[3]

Intuitively, the probability that from the initial c.s. π_0 a coalition $\{i,j\}$ will form is uniquely determined by $\mu(\pi_0)(\{i,j\})$, but how this will come about is indeterminate, and it could come from player i putting more weight on $\{i,j\}$ and player j less weight on $\{i,j\}$ as long as $p_i\sigma_i(\pi_0)(\{i,j\}) + p_j\sigma_j(\pi_0)(\{i,j\}) = \mu(\pi_0)(\{i,j\})$ is satisfied. Note that the lack of uniqueness occurs in an open set of full dimensionality of the parameter space (v, p), and is thus robust to perturbations.

The previous example shows that the local uniqueness of strategy profiles σ does not hold in general and thus the best we can hope for is to have uniqueness with respect to expected payoffs and the Markov transition probabilities μ.

Our efforts from now on are concentrated on proving uniqueness in terms of the pair (x, μ). The strategy profile σ belongs to a space with higher dimensionality than the transition probabilities μ, and when passing from σ to μ some important information is lost.

[3] Note that it is important that all $p_i < \frac{1}{2}$, for the existence of positive solutions $\sigma \geq 0$ to the system of linear equations. Also, when the inequality (9) does not hold, typically, there will not be a continuum of solutions σ.

For example, suppose that $\mu(\pi)(S) > 0$ for some $S \subset \pi$. Who are the players that choose coalition S with positive probability? It is certainly possible that the best response strategy for player $j \in S$ is to choose coalition S, but that another player $i \in S$ is strictly better off choosing a different coalition.

4.1 Coalition Dynamic Structures

We now introduce the concept of *coalitional dynamic structure* (*CDS*) which allow us to recover all the essential information about the strategy profile σ, that is not recorded in the transition probabilities μ.

Intuitively, the *CDS* represents the set of coalitions $S \subset \pi$ that are optimally chosen (i.e., the support) by any player $i \in \pi$ starting from any possible coalition structure π.

Formally, the *CDS* associated with σ is a partition of the set of players and a partition of *supp* σ (the support of σ) into the equivalence classes induced by the strong connection relation defined below. The importance of this definition to our analysis will become clear in the next subsection.

Define as follows the *strong connection equivalence relation* on the set π induced by the strategy profile σ. Given any two players $i, j \in \pi$, say that $i \rightarrow j$ if and only if there exists a coalition $S \subset \pi$ with $i, j \in S$ such that $\sigma_i(\pi)(S) > 0$. Also, say that there is a *path* from i to j if there exists a sequence of players $i_1, ..., i_k$ belonging to π such that $i \rightarrow i_1 \rightarrow ... \rightarrow i_k \rightarrow j$. Finally, we say that i and j are *strongly connected,* $i \longleftrightarrow j$, if there is a path from i to j and a path from j to i. It is straightforward to verify that strong connection is an equivalence relation (transitivity, symmetry, and reflexivity hold). Let $P_1(\pi), ..., P_{q(\pi)}(\pi)$ be the equivalence classes of this relation (the maximal strongly connected components). Let the typical element of the equivalent class be $P_r(\pi)$.

All players in $P_r(\pi)$ have the same excess or surplus from being the proposers. That is, for all i and j in $P_r(\pi)$ then

$$\max_{S \ni i, S \subset \pi} \{x_{\mathcal{S}}(\pi S|\sigma) - \sum_{k \in S} x_k(\pi|\sigma)\} = \max_{S \ni j, S \subset \pi} \{x_{\mathcal{S}}(\pi S|\sigma) - \sum_{k \in S} x_k(\pi|\sigma)\}).$$

Indeed, for any $i, j \in \pi$ such that $i \rightarrow j$ then there exists $S \subset \pi$ with $i, j \in S$ such that $\sigma_i(\pi)(S) > 0$, which implies that $\max_{S \ni i, S \subset \pi} \{x_{\mathcal{S}}(\pi S|\sigma) - \sum_{k \in S} x_k(\pi|\sigma)\} \leq \max_{S \ni j, S \subset \pi} \{x_{\mathcal{S}}(\pi S|\sigma) - \sum_{k \in S} x_k(\pi|\sigma)\})$. The same inequality holds if $i \rightarrow i_1 \rightarrow ... \rightarrow i_k \rightarrow j$. Thus for any i and j

in $P_r(\pi)$ then $i \longleftrightarrow j$, so there is a path from i to j and a path from j to i, and it must hold that the excess from being proposer for i and j must be equal.

Let now let $C_r(\pi) = \cup_{i \in P_r(\pi)} supp\ \sigma_i(\pi)$ be the union of the offers in the support of the strategy profile of players in $P_r(\pi)$. The pair $(C_r(\pi), P_r(\pi))$ thus represents the optimal coalition choices $C_r(\pi)$ for all players in $P_r(\pi)$.

Definition 1 *The coalitional dynamic structure (CDS) associated with* σ *is*
$\mathcal{C}(\sigma) = (C(\pi), P(\pi))_{\pi \in \Pi}$ *where, for each c.s.* π*:*
i) $P(\pi) = (P_1(\pi), ..., P_{q(\pi)}(\pi))$ *is a partition of* π *and* $P_r(\pi)$ *are the equivalence classes of the strong connection relation;*
ii) $C(\pi) = (C_1(\pi), ..., C_{q(\pi)}(\pi))$ *is a partition of* $supp\ \sigma(\pi) = \cup_{i \in \pi} supp\ \sigma_i(\pi)$ *and* $C_r(\pi) = \cup_{i \in P_r(\pi)} supp\ \sigma_i(\pi)$.[4]
The set of coalitional dynamic structures is $CDS = \{\mathcal{C}(\sigma) : \sigma \in \Delta\}$.

The *CDS* definition implies that the excesses of all coalitions belonging to the same equivalence class $C_r(\pi)$ are equal, that is,

$$e_r(\pi) = x_S(\pi S) - \sum_{i \in S} x_i(\pi), \tag{10}$$

for all $S \in C_r(\pi)$ and $r = 1, ..., q(\pi)$. In addition, the associated Markov transition probability $\mu = \mu(\sigma)$ satisfies

$$\sum_{S \in C_r(\pi)} \mu(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi),$$

because $supp\ \sigma_j(\pi) \subset C_r(\pi)$ for all $j \in P_r(\pi)$.

We are interested in analyzing the problem of finding an equilibrium point (x, σ) with a given *CDS* $\mathcal{C}$. If (x, σ) is an equilibrium then (x, μ, e) solves the following systems of equations, or problem $F(\mathcal{C})$ (or $F_{\mathcal{C}}$)

$$F_{\mathcal{C}}(x, \mu, e) = \begin{pmatrix} f_{\mathcal{C}}(x, \mu, e) \\ E_{\mathcal{C}}(x, e) \\ M_{\mathcal{C}}(\mu) \end{pmatrix} = 0 \tag{11}$$

[4] Note that $C_r(\pi) \cap C_{r'}(\pi) = \emptyset$ if $r \neq r'$. Otherwise, there exist $i \in P_r(\pi)$, $j \in P_{r'}(\pi)$, and $S \in C_r(\pi) \cap C_{r'}(\pi)$ such that $\sigma_i(\pi)(S) > 0$ and $\sigma_j(\pi)(S) > 0$. But this implies $i \longleftrightarrow j$ (contradiction).

where the maps $f_{\mathcal{C}}(x,\mu,e)$, $E_{\mathcal{C}}(x,e)$, and $M_{\mathcal{C}}(\mu)$ associated with *CDS* $\mathcal{C}$ are defined by:

$$\begin{aligned}
(f_{\mathcal{C}})_i(\pi)(x,\mu,e) &= x_i(\pi) - \delta p_i(\pi)\, e_r(\pi) - (1-\delta)\, v_i(\pi) \\
&\quad - \delta\left(\sum_S \mu(\pi)(S)\left(\mathbb{I}_{[i\in S]} x_i(\pi) + \mathbb{I}_{[i\notin S]} x_i(\pi S)\right)\right), \\
E_{\mathcal{C}}(\pi)(S)(x,e) &= \sum_{i\in S} x_i(\pi) + e_r(\pi) - x_S(\pi S), \qquad (12)\\
M_{\mathcal{C}}(\pi)(r)(\mu) &= \sum_{j\in P_r(\pi)} p_j(\pi) - \sum_{S\in C_r(\pi)} \mu(\pi)(S),
\end{aligned}$$

for all r, i, and S satisfying $r = 1, ..., q(\pi)$, $i \in P_r(\pi)$, $S \in C_r(\pi)$, and all $\pi \in \Pi$.

The reciprocal result also holds if we impose some additional restrictions on the solutions of $F(\mathcal{C})$. Any set of payoffs x that are candidates for equilibrium with an associated *CDS* $\mathcal{C}$ must satisfy

$$\theta(\pi)(S)(x) \geq \theta(\pi)(T)(x) \text{ for all } S \in C_r(\pi) \text{ and } T \notin C_r(\pi) \text{ with } T \cap P_r(\pi) \neq \emptyset, \quad (13)$$

because of equalities (10) and inequalities (3). Thus, the set of payoffs $\mathcal{E}_{\mathcal{C}}$ consistent with $\mathcal{C}$ is

$$\mathcal{E}_{\mathcal{C}} = \left\{x \in R^d : \text{ such that all inequalities (13) hold}\right\}.$$

Moreover, any transition probability μ that is consistent with a *CDS* $\mathcal{C}$ satisfies $\mu = \mu(\sigma)$ where σ is a strategy profile with a *CDS* $\mathcal{C}$ (i.e., $\mathcal{C}(\sigma) = \mathcal{C}$).[5] Thus, the set of transition probabilities $\mathcal{M}_{\mathcal{C}}$ consistent with $\mathcal{C}$ is

$$\mathcal{M}_{\mathcal{C}} = \{\mu = \mu(\sigma) : \text{ where } \sigma \in \Delta \text{ and } \mathcal{C}(\sigma) = \mathcal{C}\}. \quad (14)$$

The following proposition, proved in the Appendix, provides yet another useful characterization of *MPE*.

Proposition 2 *If (x,σ) is an MPE of the bargaining game then (x,μ,e) is a solution of problem $F(\mathcal{C}(\sigma))$, where $\mu = \mu(\sigma) \in \mathcal{M}_{\mathcal{C}}$, $x \in \mathcal{E}_{\mathcal{C}}$, and $e_r(\pi) = \theta(\pi)(S)(x)$ for any $S \in C_r(\pi)$. Reciprocally, if (x,μ,e) is a solution of problem $F(\mathcal{C})$ satisfying $\mu \in \mathcal{M}_{\mathcal{C}}$ and $x \in \mathcal{E}_{\mathcal{C}}$ then there exists an MPE (x,σ) of the bargaining game with $\mu = \mu(\sigma)$ and $\mathcal{C} = \mathcal{C}(\sigma)$.*

The result above allows us to transform the problem of finding equilibria into the equivalent problem of finding solutions of the system of equations $F(\mathcal{C})$.

[5] Observe that if the strategy profile σ is such that for all r and $i \in P_r(\pi)$, $supp(\sigma_i(\pi)) = C_r(\pi) \cap \{S \subset \pi : i \in S\}$ then $\mathcal{C}(\sigma) = \mathcal{C}$.

4.2 Regular and Nondegenerate Games

We seek to determine in this section conditions under which the equilibrium outcome is locally unique and stable. Note that global uniqueness does not hold in general, as Example 4 illustrates. The next best property is local uniqueness.

According to the previous section the equilibrium outcome are solutions of $F_{\mathcal{C}}(z) = 0$, where $z = (x, \mu, e)$. In order to obtain (local) uniqueness it is necessary that the problem has the same number of independent equations and of unknowns. Indeed, there are $d = \sum_{\pi \in \Pi} |\pi|$ equations $f_{\mathcal{C}}$, $m = \sum_{\pi \in \Pi} \sum_{r=1}^{q(\pi)} m_r(\pi)$, where $m_r(\pi) = |C_r(\pi)|$, equations $E_{\mathcal{C}}$, and $q = \sum_{\pi \in \Pi} q(\pi)$ equations $M_{\mathcal{C}}$ (a total of $d+m+q$ equations). Moreover, the unknowns are the d dimensional variable x, the m dimensional variable μ, and the q dimensional variable e (a total of $d+m+q$ unknowns). So the number of equations and unknowns coincide.

We now introduce the concepts of regularity and nondegeneracy. Games that satisfy both of these technical conditions are shown to have equilibrium outcome that are locally unique and stable.

Definition 2 *(Regular game) A solution z of problem $F_{\mathcal{C}}(z) = 0$ is regular if the Jacobian $d_z F_{\mathcal{C}}$ is nonsingular. A CDS $\mathcal{C}$ is regular if all the solutions of problem $F_{\mathcal{C}}$ are regular.*[6] *Finally, a game is regular if all CDSs are regular.*

The Jacobian $d_z F_{\mathcal{C}}$ is a matrix of order $d+m+q$, and is nonsingular if and only if it has full rank (equal to $d+m+q$). It has the following special structure

$$d_z F_{\mathcal{C}} = \begin{bmatrix} d_{(x,e)} f_{\mathcal{C}} & d_\mu f_{\mathcal{C}} \\ d_{(x,e)} E_{\mathcal{C}} & 0 \\ 0 & d_\mu M_{\mathcal{C}} \end{bmatrix}, \tag{15}$$

where the matrix $d_{(x,e)} E_{\mathcal{C}}$ has m rows and $d+q$ columns, and matrix $d_\mu M_{\mathcal{C}}(\mu)$ has q rows and m columns. This special structure of the Jacobian matrix will be explored later on to show that it is nonsingular almost everywhere.[7]

[6] By definition, if $F_{\mathcal{C}}$ has no solutions then the support $\mathcal{C}$ is regular.

[7] Note that if Jacobian matrix is nonsingular then the $m \times (d+q)$-matrix $dE_{\mathcal{C}}$ must have rank m. So, for example, support structures with more than $d+q$ coalitions in the support ($m > d+q$) are not candidates for a regular equilibrium point. Reciprocally, we show in the next section that if matrix $dE_{\mathcal{C}}$ has rank m then for almost all partition functions the Jacobian matrix is nonsingular.

Remark: The solution at a given c.s. π only depends on the variables evaluated at coalition structures that are coarser than π. Thus the Jacobian matrix $d_z F_{\mathcal{C}}$ can be partitioned into an upper block triangular structure with diagonal blocks equal to $d_{z(\pi)} F_{\mathcal{C}}(\pi)$ where $z(\pi) = (x(\pi), \mu(\pi), e(\pi))$ for all $\pi \in \Pi$, where all entries to the left of the diagonal blocks are zero. Therefore, the Jacobian matrix $d_z F_{\mathcal{C}}$ is nonsingular if and only if all the diagonal blocks $d_{z(\pi)} F_{\mathcal{C}}(\pi)$ are nonsingular.

Consider now the nondegeneracy technical condition, which is a property of the support that roughly means that all choices outside the support are not best response strategies (Harsanyi (1973) refer to a similar property in the context of n-person non-cooperative games as quasi-strong property). This condition is used in the next proposition to show that nearby games have equilibrium with the same support.

Definition 3 *(Nondegenerate game) A CDS $\mathcal{C}$ is nondegenerate if all solutions of $F_{\mathcal{C}}$ satisfy*

$$\theta(\pi)(S)(x) \neq \theta(\pi)(T)(x) \text{ for all } S \in C_r(\pi) \text{ and } T \notin C_r(\pi) \text{ with } T \cap P_r(\pi) \neq \emptyset. \quad (16)$$

A game is nondegenerate if all CDSs are nondegenerate.

We now introduce the formal definition of local uniqueness and stability:

Definition 4 *(Local uniqueness and stability) An equilibrium point (x^*, μ^*) of game (v^*, p^*) is locally unique and stable if and only if there exists an open neighborhood $B \subset R^d \times R^d$ of (v^*, p^*), an open neighborhood $W \subset R^d \times R^m$ of (x^*, μ^*), and a smooth mapping $(x(v,p), \mu(v,p)) : B \to W$ such that for all $(v,p) \in B$, $(x(v,p), \mu(v,p))$ is an equilibrium point of game (v,p) and $(x(v,p), \mu(v,p))$ is the only equilibrium point in the neighborhood W for game (v,p). A game is locally unique and stable if all its equilibrium points are locally unique and stable.*

The next proposition shows that regular and nondegenerate equilibrium points are locally unique and stable, a result that follows from an application of the implicit function theorem.

Proposition 3 *(Local uniqueness and stability) Regular and nondegenerate coalitional bargaining games have equilibrium points that are locally unique and stable.*

PROOF: The implicit function theorem immediately implies that, for any game (v^*, p^*) and regular solution $z^* = (x^*, \mu^*, e^*)$ there exists an open neighborhood $B \subset R^d \times R^d$ of (v^*, p^*), an open neighborhood $\widetilde{W} \subset R^d \times R^m \times R^q$ of (x^*, μ^*, e^*), and a mapping $z(v,p) = (x(v,p), \mu(v,p), e(v,p)) \in R^n \times R^m \times R^q$ such that $z(v,p)$ is the only solution of problem $F_{\mathcal{C}}$ in $\widetilde{W}$ for all games $(v,p) \in B$. Note that since $e(v,p)$ can be expressed as a function of $x(v,p)$ (see equation (10)) then $z(v,p)$ is the only solution in a cylinder $W \times R^q$ for W an open neighborhood of (x^*, μ^*).

It remains to show that $z(v,p)$ is indeed an equilibrium point. By Lemma 2, $z(v,p)$ is an equilibrium point if (i) $x(v,p) \in \mathcal{E}_{\mathcal{C}}$ and (ii) $\mu(v,p) \in \mathcal{M}_{\mathcal{C}}$. We show below that (i) and (ii) hold:

(i) holds: Because $\mathcal{C}$ has full support (nondegeneracy condition), all the inequalities in (13) are strict for x^* and thus, by continuity, the inequalities also hold for all $x(v,p)$ in an open neighborhood Q of x^*, which is equivalent to $x(v,p) \in \mathcal{E}_{\mathcal{C}}$.

(ii) holds: The result follows directly from the following lemma, proved in the Appendix.

Lemma 3 *For any CDS $\mathcal{C}$ and $\mu^* \in \mathcal{M}_{\mathcal{C}}$ there exists an open neighborhood U around μ^* such that for all μ in U that satisfies all equations* $\sum_{S \in C_r(\pi)} \mu(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi)$ *there exists σ with CDS $\mathcal{C}$ such that $\mu(\sigma) = \mu$.*

Therefore, since (i) and (ii) hold in the neighborhood $W = Q \times U$, the pair $(x(v,p), \mu(v,p))$ is an equilibrium point, which completes the proof. Q.E.D.

4.3 Genericity of Equilibria

Regularity and nondegeneracy are here shown to be generic properties. The parameter space used to establish the result is the set of all games in $(v,p) \in R^d \times \Delta^d$, where $\Delta^d = \{p \in R^d : p_i(\pi) \geq 0 \text{ and } \sum_{i \in \pi} p_i(\pi) = 1 \text{ for all } \pi \in \Pi\}$. Formally, a *generic* property is one that holds for all games, except possibly those in a subset of Lesbegue measure zero on $R^d \times \Delta^d$ (i.e., the property holds for almost all games).[8] Combining with the results of the previous section, we prove that local uniqueness and stability hold for almost all games.

This genericity result is established using the well-known transversality theorem from differential calculus (see Guillemin and Pollack (1974) and Hirsch (1976) and the Appendix

[8] We have just argued that the set of regular and nondegenerate games is an open set. Therefore the set of games that are not regular nor degenerate is a closed set.

for a restatement of the theorem). The key result of this section is the following.

Proposition 4 *(Genericity) Almost all coalitional bargaining games (v,p) in $R^d \times \Delta^d$ are regular and nondegenerate. Therefore, almost all games are locally unique and stable.*

Key for the proof is to show that, for all CDS $\mathcal{C}$, the Jacobian $d_z F_{\mathcal{C}}$, at any solutions of problem $F_{\mathcal{C}}(z) = 0$, is nonsingular, for almost every parameter (v,p) in $R^d \times \Delta^d$. That is all CDS are regular almost everywhere.

In order to illustrate the arguments involved to prove regularity, consider first that $\mathcal{C}$ is a CDS where matrix $dE_{\mathcal{C}}$ has rank m. The solutions of problem $F_{\mathcal{C}}$ can be represented as the zeros of the augmented problem $F_{\mathcal{C}}(z,v) = 0$, where we take into account the dependency with respect to the game. The Jacobian of this mapping is

$$d_{(z,v)} F_{\mathcal{C}} = \begin{bmatrix} * & * & -(1-\delta) I \\ d_{(x,e)} E_{\mathcal{C}} & 0 & 0 \\ 0 & d_{\mu} M_{\mathcal{C}} & 0 \end{bmatrix},$$

where $d_z F_{\mathcal{C}} = \begin{bmatrix} * & * \\ d_{(x,e)} E_{\mathcal{C}} & 0 \\ 0 & d_{\mu} M_{\mathcal{C}} \end{bmatrix}$ and $d_v F_C = \begin{bmatrix} -(1-\delta) I \\ 0 \\ 0 \end{bmatrix}$, and $*$ denotes arbitrary coefficients. The augmented Jacobian is a surjective matrix (with rank equal to the number of rows) because all blocks $dE_{\mathcal{C}}$, $dM_{\mathcal{C}}$, and $-(1-\delta)I$ have rank equal to the number of rows, and because of the disposition of zeros. Thus $F_{\mathcal{C}}$ is transversal to zero (i.e., $F_{\mathcal{C}} \pitchfork 0$) or 0 is a regular value of the augmented problem. By the transversality theorem, for almost every v, $F_{\mathcal{C}}(v)$ is also transversal to zero, $F_{\mathcal{C}}(v) \pitchfork 0$. Thus the square Jacobian matrix $d_z F_{\mathcal{C}}(v)$ is surjective at all solutions of $F_{\mathcal{C}}$, and thus nonsingular at all solutions for almost all $v \in R^d$ and all $p \in \Delta^d$.

Note that when the CDS $\mathcal{C}$ is such that matrix $dE_{\mathcal{C}}$ has rank smaller than m the Jacobian $d_z F_{\mathcal{C}}$ is singular for all parameters. However, in the Appendix, we show that the further augmented problem $F_{\mathcal{C}}(z,v,p) = 0$ is such that $d_{(z,v,p)} F_{\mathcal{C}}(z,v,p) = 0$ in surjective, and thus, by the transversality theorem, $d_z F_{\mathcal{C}}$ is nonsingular almost everywhere in the parameter space $R^d \times \Delta^d$.

The argument to prove that almost all games are nondegenerate is as follows. Given any support $\mathcal{C}$ consider an hyperplane H in the space R^{d+m+q} defined by equality (16), $\theta(\pi)(S)(x) = \theta(\pi)(T)(x)$ for some pair S, T (so $\mathcal{C}$ is nondegenerate if there is no solution of $F_{\mathcal{C}}(z) = 0$ such that $z \in H$). Applying the transversality theorem again (see Guillemin and Pollack (1974)) it follows that, for almost no parameters, there are solutions $F_{\mathcal{C}}(z) = 0$ such that $z \in H$: because the codimension of H in the space R^{d+m+q} is 1, the transversality theorem applied to the surjective problem $F_{\mathcal{C}}(z, v, p) = 0$ restricted to the domain $H \times R^d \times \Delta^d$ implies that for almost no parameters (v, p) there are no solutions of $F_{\mathcal{C}}(z) = 0$ such that $z \in H$. Using the fact that a finite union (there are only a finite number of pairs S, T) of sets of measure zero is a set of measure zero, we conclude that there exists a set of parameters, with complement of measure zero, where $F_{\mathcal{C}}$ is regular and nondegenerate.

4.4 Comparative Statics Analysis

Understanding how the value of coalitions and the path of coalition formation changes in response to changes in the exogenous parameters of the game v and p is a relevant comparative statics exercise. Regular and nondegenerate games are very convenient because they allow us to perform comparative statics analysis using standard calculus tools.

The following corollary is an immediate application of the implicit function theorem and Proposition 3. The sensitivity matrix $\mathcal{S}_{\mathcal{C}}$ allow us to evaluate how the equilibrium point changes $\Delta z = \mathcal{S}_{\mathcal{C}} (\Delta v, \Delta p)$ in response to local changes of the game.

Corollary 1 *(Comparative Statics) Let (v, p) be a regular and nondegenerate game and $z = (x, \mu, e)$ be an equilibrium with CDS $\mathcal{C}$. The first-order effects of a change in the exogenous parameters (v, p) on the solution z is given by the sensitivity matrix $\mathcal{S}_{\mathcal{C}} = -[d_z F_{\mathcal{C}}]^{-1} d_{(v,p)} F_{\mathcal{C}}$ (i.e., $\Delta z = \mathcal{S}_{\mathcal{C}} (\Delta v, \Delta p)$). In particular, the effect of a local change Δv of coalitional values are given by $\Delta z = \left([d_z F_{\mathcal{C}}]^{-1}\right)_{\cdot x} (1 - \delta) \Delta v$, where $\left([d_z F_{\mathcal{C}}]^{-1}\right)_{\cdot x}$ denotes the submatrix with the first d columns of the inverse Jacobian.*

The first-order effects with respect to changes in value Δv are given by the sensitivity matrix $-[d_z F_{\mathcal{C}}]^{-1} d_v F_{\mathcal{C}}$. But since

$$d_v F_{\mathcal{C}} = \begin{pmatrix} d_v f_{\mathcal{C}} \\ d_v E_{\mathcal{C}} \\ d_v M_{\mathcal{C}} \end{pmatrix} = -\begin{pmatrix} (1-\delta) I \\ 0 \\ 0 \end{pmatrix}, \tag{17}$$

the sensitivity matrix $-[d_z F_{\mathcal{C}}]^{-1} d_v F_{\mathcal{C}}$ simplifies to $\left([d_z F_{\mathcal{C}}]^{-1}\right)_{\cdot x} (1 - \delta)$. So evaluating the inverse of the Jacobian matrix at the solution yields the first-order effects of changes in value.

We illustrate with the next examples the comparative statics properties of quota and apex games.

4.5 Examples

Comparison of the equilibrium payoffs predicted by our model with established solution concepts from cooperative game theory, such as the nucleolus, bargaining set, kernel, core, and Shapley value, shows that our predictions are different than all other cooperative solution concepts.

Example 2: Quota Games

Quota games have been studied by Shapley (1953) and Maschler (1992). Consider a four-player quota game, where each pairwise coalition gets $v_{\{i,j\}} = \omega_i + \omega_j$ for all distinct pairs $i, j \in N$, where the quotas of the four players are $(\omega_1, \omega_2, \omega_3, \omega_4) = (10, 20, 30, 40)$, and all remaining coalitions get $v_S = 0$ for all $S \subset N$, $S \neq \{i, j\}$ (to simplify notation we omit the c.s. π in $v_i(\pi)$). Players are very patient (i.e., we are interested in the limit when δ converges to 1), and they all have an equal chance to be proposers.

The equilibrium point and the transition probabilities are depicted in Figure 1 (in order to simplify the notation, in the figure the c.s. $\{\{i,j\},\{k,l\}\}$ is denoted by $(ij)(kl)$; the numbers below the c.s. in parenthesis are the corresponding equilibrium values ϕ of each coalition; and the percentages above the arrows are the transition probabilities $\mu(S)$). The *CDS* at the initial state is $\mathcal{C} = (\{\{2,3\},\{2,4\},\{3,4\}\},\{\{1\}\})$, and the excesses are $e(S) = 4.938$ for $S = \{2,3\},\{2,4\}$, and $\{3,4\}$, and it can also be easily verified that this solution is a strong regular solution.

The solution $\phi = (17.41, 17.53, 27.53, 37.53)$ is different from the nucleolus (Schmeidler (1969)) and the core (both of which coincide with the quota $(10, 20, 30, 40)$), the kernel, the bargaining set (Maschler (1992)), and the Shapley value (which is equal to $(17.5, 20, 28.33, 34.16)$).[9] In our solution, player 1 gets 7.41 more than his quota and players 2, 3 and 4 get each 2.47 less than their quota values. This example illustrates that the solution proposed in the paper is different from all the other major existing solution concepts.

[9] Moreover, these classical solution concepts are not the outcome of any MPE of the quota game above. In addition, since the bargaining set contains the kernel, and the kernel contains the nucleolus, this implies that our solution is different from the kernel and bargaining set (see Maschler 1992).

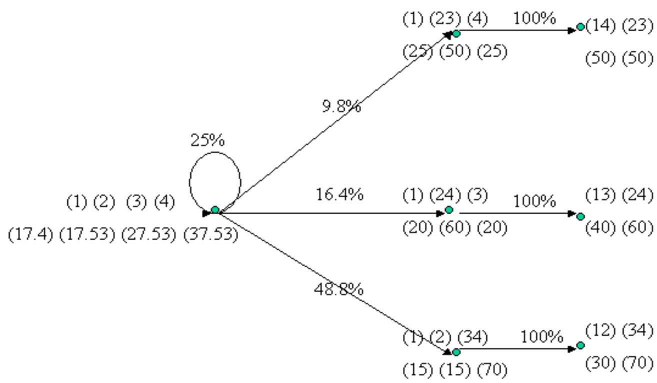

Figure 1: MPE solution of the quota game $(10, 20, 30, 40)$.

Interestingly, the equilibrium strategy of player 1 is to wait for a pairwise coalition to form, an strategy that allows player 1 to get significantly more than his quota. The solution thus makes predictions that are consistent with experimental results reported in Maschler (1992), where player 1 realized that he was weak and that his condition would improve if he waited until a pairwise coalition formed, and captures an important strategic element of the game. Indeed, player 1 is better off if the coalition $\{2, 3\}$ forms, rather than $\{2, 4\}$ or $\{3, 4\}$, because in the ensuing pairwise bargaining with 4, player 1 can get a payoff equal to 25.[10]

How do players' value change with changes in quotas and proposers' probabilities? Evaluating the value-sensitivity matrix with respect to changes in quotas, as we have seen in

[10]However, strategies considered in this paper rule out the possibility that player 1 makes side payments to players 2 and/or 3 in order to encourage them to form coalition $\{2, 3\}$.

Section 4.4, yields

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \end{bmatrix} = \begin{bmatrix} 0.366 & 0.549 & 0.062 & 0.022 \\ 0.211 & 0.816 & -0.020 & -0.007 \\ 0.211 & -0.183 & 0.979 & -0.007 \\ 0.211 & -0.183 & -0.020 & 0.992 \end{bmatrix} \begin{bmatrix} \Delta\omega_1 \\ \Delta\omega_2 \\ \Delta\omega_3 \\ \Delta\omega_4 \end{bmatrix},$$

and the coalition formation sensitivity matrix satisfies $\frac{\partial\mu(\{i,j\})}{\partial\omega_i} > 0$ and $\frac{\partial\mu(\{j,k\})}{\partial\omega_i} < 0$ for all distinct i, j, and k in {2,3,4} (for the sake of space we report only the signs of the transition probabilities).

The information contained in the value sensitivity matrix yields the following results: the value of all players increases when their quotas increase, but increases in the quota of player 1 are shared by all players, while increases in the quotas of either player 2, 3 or 4 are almost completely appropriated by them (in fact, the other two players distinct from player 1 suffer a loss). Moreover, when a player's quota goes up, all coalitions including this player become more likely to form (and coalitions not including this player are less likely to form).

The comparative statics with respect to changes in proposers' probability is described by the value-sensitivity matrix

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \end{bmatrix} = \begin{bmatrix} 0 & -5.42 & 1.96 & 3.45 \\ 0 & 1.80 & -0.65 & -1.15 \\ 0 & 1.80 & -0.65 & -1.15 \\ 0 & 1.80 & -0.65 & -1.15 \end{bmatrix} \begin{bmatrix} \Delta p'_1 \\ \Delta p'_2 \\ \Delta p'_3 \\ \Delta p'_4 \end{bmatrix},$$

where, in order to preserve the sum of probabilities equal to one, we consider $p_i = p'_i/(\sum_{j=1}^4 p'_j)$, and the coalition formation sensitivity matrix satisfies $\frac{\partial\mu(\{i,j\})}{\partial p'_i} < 0$ and $\frac{\partial\mu(\{j,k\})}{\partial p'_i} > 0$ for all distinct i, j, and k in $\{2, 3, 4\}$.

This comparative statics analysis reveals a surprising result: When player 2 has more initiative to propose, he benefits and player 1 loses from it. Interestingly, though, the opposite happens when players 3 and 4 have more initiative. Their equilibrium payoffs decrease when they have more initiative to propose![11] ■

Example 3: Apex Games

[11] This result can be rationalized as follows: when p_2 increases, coalitions {2,3} and {2,4} are less likely to form and coalition {3,4} more likely; since player 1's gains are lowest when coalition {3,4} forms he indirectly suffers when p_2 increases. By similar reasoning, when p_4 increases, coalitions {2,4} and {3,4} are less likely to form and coalition {2,3} more likely, which benefits player 1 and hurts the other players.

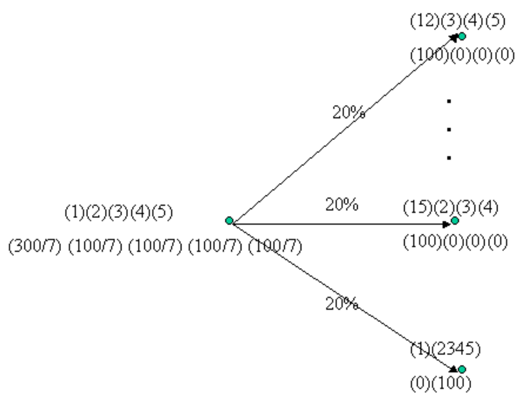

Figure 2: MPE solution of the apex game.

Apex games, introduced by Davis and Maschler (1965), are another interesting class of n-person games that have received considerable attention. In this game, only two types of coalitions create non-zero value: any coalition with the Apex player (player 1), or the coalition with all the $n-1$ remaining players (the Base players). For concreteness, consider the 5-player game $N = \{1, 2, 3, 4, 5\}$, where $v_{\{1,j\}} = 100$ for $j = 2, ..., 5$, $v_{\{2,3,4,5\}} = 100$, and $v_S = 0$ otherwise. Players are very patient (δ is infinitesimally close to 1), and all players have equal chance to be proposers. Apex games have also been studied by Montero (2002), and she shows that the expected value of this game coincides with the kernel whenever the players have equal chance to be proposers.

The solution is depicted in Figure 2 (we use the same notation as in Figure 1). The *CDS* at the initial state is $\mathcal{C} = (\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3,4,5\}\})$, and the excesses are $e(S) = \frac{300}{7}$ for $S \in \mathcal{C}$.

The solution for the game is $\phi = (42.9, 14.3, 14.3, 14.3, 14.3)$. This solution coincides with the kernel of the game and the nucleolus. However, it is different from the bargaining set, the core (which is empty), and the Shapley value (which is equal to $(60, 10, 10, 10, 10)$).

Moreover, the model also predicts that any of the four apex coalitions $\{1, j\}$, $j = 2, ..., 5$, form with 20% probability, and the base coalition $\{2, 3, 4, 5\}$ forms with 20% probability.

Comparative statics results for the apex game can also be easily obtained. The sensitivity matrix describing the changes in value is

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \\ \Delta\phi_5 \end{bmatrix} = \begin{bmatrix} 0.23 & -0.48 & 0 & 0 \\ 0.74 & 0.17 & 0 & 0 \\ -0.26 & 0.17 & 0 & 0 \\ -0.26 & 0.17 & 0 & 0 \\ -0.26 & 0.17 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v_{\{1,2\}} \\ \Delta v_{\{2,3,4,5\}} \\ \Delta p'_1 \\ \Delta p'_2 \end{bmatrix},$$

where $p_i = p'_i/(\sum_{j=1}^{5} p'_j)$, and the sensitivity matrix describing the changes in coalition formation is

$$\begin{bmatrix} \Delta\mu(\{1,2\} \\ \Delta\mu(\{1,3\} \\ \Delta\mu(\{1,4\} \\ \Delta\mu(\{1,5\} \\ \Delta\mu(\{2,3,4,5\} \end{bmatrix} = \begin{bmatrix} 0.0309 & -0.0009 & -0.2 & -2.2 \\ -0.01 & -0.0009 & -0.2 & 0.8 \\ -0.01 & -0.0009 & -0.2 & 0.8 \\ -0.01 & -0.0009 & -0.2 & 0.8 \\ -0.0009 & 0.0036 & 0.8 & -0.2 \end{bmatrix} \begin{bmatrix} \Delta v_{\{1,2\}} \\ \Delta v_{\{2,3,4,5\}} \\ \Delta p'_1 \\ \Delta p'_2 \end{bmatrix}.$$

Surprisingly, these results indicate that changes in proposer probabilities have no effect on the players' values. Also, as was the case with the previous example, whenever the value of a coalition increases then both the equilibrium value of all coalitional members and the probability that this coalition forms increase as well.

5 Uniqueness and the Global Number of Equilibria

We show in this section that even though there can be multiple equilibrium points, as illustrated by the next example, almost all games have a finite and odd number of *MPE* equilibria. Moreover, we derive a sufficient condition for the global uniqueness of equilibria. The result states that if the index of each equilibrium solution is non-negative, where the index is equal to the sign of the determinant of the Jacobian matrix $d_z F_{\mathcal{C}}$, then there is a globally unique equilibrium. We prove that the sufficient condition holds for three-player superadditive games, and thus there is only one equilibrium in this class of games.

Let us start by showing that coalitional bargaining games may have multiple equilibrium points.

Example 4: War of Attrition (Multiple Equilibria)

The following three-player symmetric example have seven *MPE* solutions. The partition function that describes this game is $v_i(\{\{1\},\{2\},\{3\}\} = 0$, $v_{\{i,j\}}(\{\{i,j\},\{k\}\}) = 1$, and $v_{\{k\}}(\{\{i,j\},\{k\}\}\}) = 3$. The three-player coalition are not allowed (or has a very low value) and we assume that proposers are chosen with equal probabilities and $\delta \in (0.5, 1)$. We describe below all the equilibria (results are derived solving equations (2), but we omit the details).

There is an equilibrium in which the expected equilibrium value is $x = (0.5, 0.5, 0.5)$; the transition probabilities are $\mu(\{i,j\}) = \frac{1-\delta}{5\delta}$, for all pairs $\{i,j\}$, and $\mu(\emptyset) = \frac{8\delta-3}{5\delta}$, where $\emptyset$ represents no proposal (or remaining at the initial state). In this equilibrium, each of the three players refrains from proposing with high probability, and only proposes with a small probability to the other two players. They all reject any proposals below 0.5, and thus players are indifferent between proposing or not.

There are three other equilibria (they are all symmetric so we just focus on one of them), in which the expected equilibrium values are $x = (0, 1, 1)$; the transition probabilities are $\mu(\{1,2\}) = \mu(\{1,3\}) = \frac{1-\delta}{2\delta}$, and $\mu(\emptyset) = \frac{2\delta-1}{\delta}$. In this equilibrium, players 2 and 3 reject any proposals lower than 1, make no proposal with high probability, and, when proposing, choose to form a coalition with player 1. Player 1 cannot afford to pay more than 1 to form a coalition and thus it makes no proposals with probability one.

Finally, there are three additional equilibria (they are also symmetric), in which the expected equilibrium values are $x = \left(\frac{6\delta}{3-\delta}, \frac{\delta}{3-\delta}, \frac{\delta}{3-\delta}\right)$, which converges to $x = (3, 0.5, 0.5)$ when $\delta \to 1$; the transition probabilities are $\mu(\emptyset) = \frac{1}{3}$, $\mu(\{2,3\}) = \frac{2}{3}$. In this equilibrium, player 1's strategy is to refrain from proposing and reject any proposal worth less than 3, and player 2's and player 3's strategies are to always "give in" and propose to form the coalition $\{2,3\}$. ■

The main result of this section establishes a formula for counting the number of equilibrium points; the formula is based on the Index theorem (see, for example, Mas-Colell et al. (1995)). The index is a number that is assigned to each equilibrium point of a regular game. Say that z is an equilibrium point with support $\mathcal{C}$; the index of z is defined as the sign of the determinant of the Jacobian $d_z F_{\mathcal{C}}$ evaluated at z (and is either $+1$ or -1). We denote *index* $z = sign \det (d_z F_{\mathcal{C}})$.

The result implies that there is an odd number of equilibria. In particular, the number of equilibrium points is not zero, so there exists at least one equilibrium. We will see next that the result can be used to obtain global uniqueness for specific classes of games.

While the Index theorem has been applied to establish similar results for competitive economies (Debreu (1970)) and normal form games (Wilson (1971) and Harsanyi (1973)), the application to coalitional bargaining games is a bit more involved due to the fact that the equilibrium is not locally unique in terms of the equilibrium strategies, but only in terms of the equilibrium points (see Section 5). Also, more recently, Acemoglu et al. (2005) establish sufficient conditions for the uniqueness of solutions of mixed nonlinear complementary problems (MNCP), but for the same reasons above, the coalition bargaining game is an MNCP that do not satisfy their necessary conditions.

To prove the result we use a stronger version of the Index theorem for correspondences developed in McLennan (1989)-see the Appendix for the restatement of the Lefschetz fixed point theorem (LFPT) which is used in the proof. For both competitive economies and normal form games a standard version of the Index theorem developed in differential calculus textbooks suffices to develop the formula for the number of equilibria.

Proposition 5 *Almost all games (all regular and nondegenerate games) have a finite and odd number of Markov perfect equilibrium points. Moreover,*

$$\sum_{\mathcal{C}} \sum_{z:\in MPE_{\mathcal{C}}} sign \det (d_z F_{\mathcal{C}}) = +1,$$

where the summation is over all CDS $\mathcal{C}$ and MPEs with equilibrium points $z = (x, \mu, e)$ and CDS $\mathcal{C}$.

Proof: Consider the correspondence $\mathcal{F} : X \to X$ where $\mathcal{F}(x) \subset R^d$ is defined by

$$\left\{ y \in R^d : \begin{array}{l} \text{where } y_i(\pi) = \delta p_i(\pi) \max_{S \ni i} \{\theta(\pi)(S)(x)\} + (1-\delta) v_i(\pi) \\ +\delta \left(\sum_{S \subset \pi} \sum_{j \in \pi} p_j(\pi) \sigma_j(\pi)(S) \left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S) \right) \right), \\ \text{and } supp(\sigma_i(\pi)) \subset \arg\max_{S \ni i} \{\theta(\pi)(S)(x)\} \end{array} \right\}. \quad (18)$$

The set X is the convex and compact set $X \subset R^d$ defined by $X = \underset{\pi \in \Pi}{\times} X(\pi)$, where

$$X(\pi) = \{x(\pi) \in R^{|\pi|} \text{ such that } \sum_{i \in \pi} x_i(\pi) \leq \overline{v} \text{ and } x_i(\pi) \geq \underline{v}_i\},$$

and $\underline{v}_i = \min_{\pi \ni i} \{v_i(\pi)\}$ and $\overline{v} = \max_{\pi \in \Pi} \left\{\sum_{i \in \pi} v_i(\pi)\right\}$. By the definition of X and $\mathcal{F}$ it is easy to verify that indeed $\mathcal{F}(X) \subset X$.

The set of fixed points of $\mathcal{F}$, $\mathcal{F}^* = \{x \in X : x \in \mathcal{F}(x)\}$, corresponds to the equilibrium points of the game (see Section 3). The set $\mathcal{F}^*$ is finite for all regular and nondegenerate games v: all the equilibrium points are, by Lemma 2, solutions of $F_{\mathcal{C}}(x, \mu, e) = 0$ for some some support $\mathcal{C}$. But since the game is regular the solutions are locally isolated (Proposition 3), and since the solution belongs to the compact X then there is only a finite number of solutions.

Moreover, it can be easily shown that $\mathcal{F} : X \to X$ is an upper hemicontinuous convex-valued correspondence (thus $\mathcal{F}(x)$ is contractible for all $x \in X$). The set $X \subset R^d$, Cartesian product of simplexes, is a simplicial complex and thus $\mathcal{F}$ satisfies the conditions of the $LFPT$.

Let U_{x^*} be an open neighborhood around each $x^* \in \mathcal{F}^*$, so that x^* is the only fixed point in $\overline{U}_{x^*}$. The Additivity Axiom of the Lefschetz index implies

$$\Lambda(\mathcal{F}, X) = \sum_{x^* \in \mathcal{F}^*} \Lambda(\mathcal{F}, U_{x^*}). \tag{19}$$

In addition, the Lefschetz index is

$$\Lambda(\mathcal{F}, X) = 1. \tag{20}$$

Indeed $\mathcal{F}$ can be approximated by a continuous map $f' : X \to X$ such that $\Lambda(\mathcal{F}, X) = \Lambda(f', X)$ (Continuity Axiom), and X is a contractible set, and thus there is an homotopy $\varphi : X \times [0,1] \to X$ where $\varphi_1 = I_X$ and $\varphi_0 = z_0 \in X$. Therefore, any continuous map $f' : X \to X$ is homotopic to the constant map so, by the Weak Normalization and Homotopy Axioms, $\Lambda(\mathcal{F}, X) = \Lambda(f', X) = 1$.

Equations (19) and (20) thus imply,

$$\Lambda(\mathcal{F}, X) = \sum_{x^* \in \mathcal{F}^*} \Lambda(\mathcal{F}, U_{x^*}) = 1. \tag{21}$$

We show in the Appendix that the Lefschetz index of a regular and nondegenerate equilibrium point $z^* = (x^*, \mu^*, e^*)$ is equal to $\Lambda(\mathcal{F}, U_{x^*}) = sign \det(d_{z^*} F_{\mathcal{C}})$. This completes the proof since from equation (21) we have

$$\sum_{\mathcal{C}} \sum_{z:\in MPE_{\mathcal{C}}} sign \det (d_z F_{\mathcal{C}}) = 1.$$

Q.E.D.

Using the result of Proposition 5 we can readily obtain a sufficient condition for global uniqueness of equilibria.

Corollary 2 *All regular and nondegenerate coalitional bargaining games have a globally unique Markov perfect equilibrium if* $\det(d_z F) \geq 0$ *where the Jacobian is evaluated at any solution* $z = (x, \mu, e)$ *of problem* $F_{\mathcal{C}}$ *for all CDS* $\mathcal{C}$.

We conjecture that a sufficient condition Corollary 2, which guarantees global uniqueness, is that there are no positive externalities imposed in any players. Formally, that the inequalities $x_i(\pi) - x_i(\pi S) \geq 0$ hold for all players i and coalitions S that can be chosen in equilibrium not including player i.

The inequalities $x_i(\pi) - x_i(\pi S) \geq 0$, where $i \notin S$, has a natural economic interpretation: all coalitions S that form, excluding player i and leading to a transition from c.s. π to πS, impose a negative externality on player i (i.e., $x_i(\pi S) \leq x_i(\pi)$, the new value of player i, $x_i(\pi S)$, is less than player's i status quo value, $x_i(\pi)$).

We show in the next proposition that these inequalities hold for all three-player games where the grand coalition is efficient, and using this property, we are able to show that all Jacobian evaluated at any solution $z = (x, \mu, e)$ of problem $F_{\mathcal{C}}$ are positive, $\det(d_z F) \geq 0$.

Proposition 6 *Almost all three-player games with externalities where the grand coalition is efficient, i.e.* $v(\{N\}) \geq \sum_{S \in \pi} v_S(\pi)$ *for all* π, *in particular superadditive games, have a globally unique Markov perfect equilibrium.*

We remark that in the war of attrition Example 4, a strongly regular game with seven equilibria, the grand coalition was not efficient, and therefore it is not in contradiction with Proposition 6.

Therefore, the equilibrium point computed explicitly in Gomes (2004) is the unique equilibrium point for almost all games.

6 Conclusion

This paper studied the equilibrium properties of n-player coalitional bargaining games in an environment with widespread externalities (where the exogenous parameters are expressed in a partition function form). The coalitional bargaining problem is modeled as a dynamic non-cooperative game in which contracts forming coalitions may be renegotiated. The equilibrium concept used is Markov perfect equilibrium, where the set of states is all possible coalition structures.

A comprehensive analysis of the equilibrium properties is developed. We show that for almost all games (except in a closed set of measure zero) the equilibrium is locally unique and stable to small perturbations of the exogenous parameters, and the number of equilibria is finite and odd. Global uniqueness does not hold in general, but a sufficient condition for global uniqueness is derived, and this sufficient condition is shown to prevail in three-player superadditive games.

Comparative statics analysis can be easily performed using standard calculus tools, allowing us to understand how the value of players and the path of coalition formation changes in response to changes in the exogenous parameters. Being able to answer comparative statics questions is valuable to negotiators, because they may be able, for example, to invest in changing the likelihood of being proposers. Applications of the technique are illustrated using the apex and quota games, and some interesting insights emerge: surprisingly, a player may not benefit from having more initiative to propose (other players may adjust their strategies in such a way that lead the proposer to be worse off). The analysis also suggests several interesting regularities: when the exogenous value of a coalition increases, both the equilibrium value of the coalitional members and the likelihood that the coalition forms increase.

It is important to note that in coalitional bargaining games the players are the coalitions. However, in many applications of economic interest individual agents retain autonomy after coalitions form, so the model studied in this paper may seem of potentially limited application. However, this is not the case because Gomes (2005) shows that there is a one-to-one mapping between the *MPE* of coalitional bargaining games and multilateral contracting games. Multilateral contracting games are also coalitional games, but one in which the players are the original individual agents who offer contracts specifying monetary transfers among signatories, conditional on the coalition structures formed by players outside the contract. Therefore, the results in this paper also allow us to have a better understanding of the equilibrium properties of the related multilateral contracting model.

Finally, the results in this paper and the methodology developed herein are likely to also be applicable to establish that the Markov perfect equilibrium of the model of coalition formation proposed by Gomes and Jehiel (2005) is also generically locally unique and stable.

Appendix

Proof of Lemma 1: The necessary part follows directly from the discussion before the statement of the result and the definition of *MPE* solution. Let us prove the sufficient part of the proposition. Suppose that we are given payoffs and strategy profiles $\left(\phi, \phi^j, \sigma\right)$ satisfying all the conditions of the lemma. We use the one-stage deviation principle for infinite-horizon games. This result states that in any infinite-horizon game with observed actions that is continuous at infinity, a strategy profile σ is subgame perfect if and only if there is no player i and strategy σ_i' that agrees with σ_i except at a single stage t of the game and history h^t, such that σ_i' is a better response to σ_{-i} than σ_i conditional on history h^t being reached (see Fudenberg and Tirole (1991)).

Note first that the coalitional bargaining game is continuous at infinity: for each player i his utility function is such that, for any two histories h and h' such that the restrictions of the histories to the first t periods coincides, then the payoff of player i, $|u_i(h) - u_i(h')|$, converges to zero as t converge to infinity. It is immediately clear that the negotiation game is continuous at infinity because $|u_i(h) - u_i(h')| \leq M\left(\delta^{t+1} + \delta^{t+2} + \cdots\right) = \frac{M}{1-\delta}\delta^{t+1}$, for M large enough.

But the strategy profile σ_i is such that, by construction, no single deviation σ_i' at both the proposal and response stage can lead to a better response than σ_i. Therefore, by the one-stage deviation principle, the stationary strategy profile σ is a subgame perfect Nash equilibrium. Q.E.D.

Proof of Lemma 2: We explicitly construct a *MPE* of the three player coalitional bargaining game. First note that, after the coalition $\{i, j\}$ forms, the subgame starting at c.s. $\pi = \{\{i, j\}, \{k\}\}$ is a standard two-player bargaining game with continuation values $\phi_{ij}(\pi)$ and $\phi_k(\pi)$ for coalition $\{i, j\}$ and player k equal to

$$\Phi_{ij} = \phi_{ij}(\pi) = V_{ij} + P_{ij}\left(V - V_k - V_{ij}\right) \text{ and } \Phi_k = \phi_k(\pi) = V_k + P_k\left(V - V_k - V_{ij}\right),$$

and the values $x_{ij}(\pi)$ and $x_k(\pi)$ are equal to

$$X_{ij}^{\delta} = x_{ij}(\pi) = \delta\Phi_{ij} + (1-\delta)V_{ij} \text{ and } X_k^{\delta} = x_k(\pi) = \delta\Phi_k + (1-\delta)V_k.$$

We denote throughout this proof by the superscript δ the dependency on the discount factor δ.

Consider the following strategy at the initial stage $\pi_0 = \{\{1\},\{2\},\{3\}\}$ (we omit references to π_0 henceforth): Player i chooses $\{i,j\}$ or $\{i,k\}$, respectively, with probabilities $\sigma_i(\pi_0)(\{i,j\}) > 0$ and $\sigma_i(\pi_0)(\{i,k\}) > 0$ such that $\sigma_i(\pi_0)(\{i,j\}) + \sigma_i(\pi_0)(\{i,k\}) = 1$. This support structure corresponds to the *CDS* $\mathcal{C} = (C,P)$ with $C = \{\{1,2\},\{1,3\},\{2,3\}\}$ and $P = \{1,2,3\}$).

The transition probabilities are

$$\mu(\pi_0)(\{i,j\}) = \mu_{ij}^{\delta} = p_i\sigma_i(\pi_0)(\{i,j\}) + p_j\sigma_j(\pi_0)(\{i,j\}),$$

and the triple $(x^{\delta},\mu^{\delta},e^{\delta})$ that solves problem $F(\mathcal{C})$, corresponding to the solution of the system of non-linear equations below are:

$$\begin{aligned}
x_1^{\delta} &= \delta p_1 e^{\delta} + \delta\left(\left(\mu_{12}^{\delta}+\mu_{13}^{\delta}\right)x_1^{\delta} + \mu_{23}^{\delta}X_1^{\delta}\right)\\
x_2^{\delta} &= \delta p_2 e^{\delta} + \delta\left(\left(\mu_{12}^{\delta}+\mu_{23}^{\delta}\right)x_2^{\delta} + \mu_{13}^{\delta}X_2^{\delta}\right)\\
x_3^{\delta} &= \delta p_3 e^{\delta} + \delta\left(\left(\mu_{13}^{\delta}+\mu_{23}^{\delta}\right)x_3^{\delta} + \mu_{12}^{\delta}X_3^{\delta}\right)\\
e^{\delta} &= X_{12}^{\delta} - x_1^{\delta} - x_2^{\delta}\\
e^{\delta} &= X_{13}^{\delta} - x_1^{\delta} - x_3^{\delta}\\
e^{\delta} &= X_{23}^{\delta} - x_2^{\delta} - x_3^{\delta}\\
1 &= \mu_{12}^{\delta} + \mu_{13}^{\delta} + \mu_{23}^{\delta},
\end{aligned}$$

where we took into account that $v_i(\pi_0) = 0$. The solution of the system of equations above is, for $\delta = 1$, equal to

$$\begin{aligned}
x_i &= \frac{1}{3}\left(V + \Phi_{ij} + \Phi_{ik} - 2\Phi_{jk}\right),\\
\mu_{ij} &= p_k,\\
e &= \frac{1}{3}\left(\Phi_{12} + \Phi_{13} + \Phi_{23} - 2V\right),
\end{aligned}$$

which can be easily verified by direct substitution.

By the implicit function theorem (IFT) a solution of the system for all $\delta \in [\overline{\delta},1)$, for some $\overline{\delta} < 1$, is also guaranteed. Indeed, the Jacobian evaluated at a fixed δ is equal to

$$\begin{bmatrix} 1-\delta\left(\mu_{12}^{\delta}+\mu_{13}^{\delta}\right) & 0 & 0 & -\delta p_1 & -\delta x_1^{\delta} & -\delta x_1^{\delta} & -\delta X_1^{\delta} \\ 0 & 1-\delta\left(\mu_{12}^{\delta}+\mu_{23}^{\delta}\right) & 0 & -\delta p_2 & -\delta x_2^{\delta} & -\delta X_2^{\delta} & -\delta x_2^{\delta} \\ 0 & 0 & 1-\delta\left(\mu_{13}^{\delta}+\mu_{23}^{\delta}\right) & -\delta p_3 & -\delta X_3^{\delta} & -\delta x_3^{\delta} & -\delta x_3^{\delta} \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

which, at the solution point and $\delta = 1$, is a non-singular matrix with determinant equal to

$$\frac{1}{3}\left(\Phi_{12}+\Phi_{13}+\Phi_{23}-2V\right)^2 \neq 0$$

Thus the problem of finding solutions for δ in a neighborhood of $\delta = 1$ satisfies all conditions of the IFT, and there exits a solution $(x^{\delta}, \mu^{\delta}, e^{\delta})$ that solves problem $F(\mathcal{C})$ for all $\delta \in [\overline{\delta}, 1)$.

The solution above is also optimal. First, choosing the grand coalition is not optimal since the excess satisfies $\lim_{\delta\to1} e_N^{\delta} = \lim_{\delta\to1}\left(\delta V - x_1^{\delta} - x_2^{\delta} - x_3^{\delta}\right) = 0$ and $\lim_{\delta\to1} e^{\delta} = \frac{1}{3}\left(\Phi_{12}+\Phi_{13}+\Phi_{23}-2V\right) > 0$ (due to inequality (9)). Therefore, we can guarantee that there exists $\overline{\delta} < 1$ such that $e^{\delta} \geq e_N^{\delta}$ for all $\delta \in [\overline{\delta}, 1)$. Second, choosing to form no coalitions is also not optimal since the excess of making this choice is zero and $e^{\delta} > 0$.

Finally, by solving the system of 6 linear equations with rank 5,

$$\begin{aligned} p_1\sigma_1(12) + p_2\sigma_2(12) &= \mu_{12}^{\delta} \\ p_1\sigma_1(13) + p_3\sigma_3(13) &= \mu_{13}^{\delta} \\ p_2\sigma_1(23) + p_3\sigma_3(23) &= \mu_{23}^{\delta} \\ \sigma_1(12) + \sigma_1(13) &= 1 \\ \sigma_2(12) + \sigma_2(23) &= 1 \\ \sigma_3(23) + \sigma_3(23) &= 1, \end{aligned}$$

where $\mu_{12}^{\delta} + \mu_{13}^{\delta} + \mu_{23}^{\delta} = 1$, a continuum of *MPE* $\sigma^{\delta} \geq 0$ for all $\delta \in [\overline{\delta}, 1)$ can be obtained, given that $p_i < \frac{1}{2}$ for all $i = 1, 2, 3$. Q.E.D.

Proof of Proposition 2: For all σ satisfying $\mathcal{C} = \mathcal{C}(\sigma)$ then $\mu = \mu(\sigma)$ satisfy

$$\begin{aligned} \sum_{S \in C_r(\pi)} \mu(\pi)(S) &= \sum_{S \in C_r(\pi)} \sum_{j \in \pi} p_j(\pi)\, \sigma_j(\pi)(S) = \\ &= \sum_{j \in \pi} p_j(\pi) \sum_{S \in C_r(\pi)} \sigma_j(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi) \text{ for all } r, \end{aligned}$$

because if $j \in P_r(\pi)$ then $supp\,(\sigma_j(\pi)) \subset C_r(\pi)$, which corresponds to the last set of equations in $F(\mathcal{C})$.

Now, if $i \to j$ then there exist a coalition S such that $i, j \in S$ and $\sigma_i(\pi)(S) > 0$. But because $supp\,(\sigma_i(\pi)) \subset \underset{\{S \subset \pi : i \in S\}}{\arg\max} \{e(\pi)(S)\, x\}$ then

$$e_i := \max_{\{S \subset \pi : i \in S\}} \left\{ x_S(\pi S) - \sum_{j \in S} x_j(\pi) \right\} \le e_j.$$

Repeating the same argument, if there is a path from i to j then $e_i \le e_j$, and if i is strongly connected to j then both have the same excess $e_i = e_j$. Thus, $e_r(\pi) = e_i = x_S(\pi S) - \sum_{j \in S} x_j$, for all $S \in \mathcal{C}_r$ and all $i \in P_r(\pi)$. Substituting the expressions for the excesses into equation (7) finishes the if part of the proof. The reciprocal follows directly from the construction of the polyhedral sets $\mathcal{M}_{\mathcal{C}}$ and $\mathcal{E}_{\mathcal{C}}$. Q.E.D.

Proof of lemma 3: The same steps of the proof applies to each c.s. π separately, so to simplify notation we eliminate explicit references to π below. The following claim implies the lemma, as shown below.

Claim: Let $\mathcal{C} = (C, P)$ be a *CDS*, σ^* a strategy profile with $\mathcal{C} = \mathcal{C}(\sigma^*)$, and let $\Sigma_i = supp\ \sigma_i^*$. *Given any* $\mu = (\mu(S))_{S \in C_r}$ *close to zero satisfying* $\sum_{S \in C_r} \mu(S) = 0$ *there exists* $\sigma = (\sigma_i(S))_{\substack{S \in \Sigma_i \\ i \in P_r}}$ *close to zero satisfying* $\sum_{S \in \Sigma_i} \sigma_i(S) = 0$ *for all* $i \in P_r$ *such that* $\mu(\sigma) = \mu$.

Suppose that the claim holds. Let $\Delta_{\mathcal{C}} := \{\sigma \in \Delta : \mathcal{C} = \mathcal{C}(\sigma)\}$. There exists $\sigma^* \in \Delta_{\mathcal{C}}$ such that $\mu(\sigma^*) = \mu^*$ (as $\mu^* \in \mathcal{M}_{\mathcal{C}}$). Let $\Sigma_i = supp\ \sigma_i^*$. Given any μ close to μ^* define $\Delta\mu = \mu - \mu^*$ (which is close to zero). Consider a $\Delta\sigma$ given by the claim (related to $\Delta\mu$) and let $\sigma = \sigma^* + \Delta\sigma$. Such σ satisfies $\sigma_i(S) > 0$ for all $S \in \Sigma_i$ (because $\sigma_i^*(S) > 0$ for all $S \in \Sigma_i$ and $\Delta\sigma(S)$ are close to zero) and $\sum_{S \in \Sigma_i} \sigma(S) = \sum_{S \in \Sigma_i} \sigma^*(S) + \sum_{S \in \Sigma_i} \Delta\sigma_i(S) = 1$. So $\sigma \in \Delta_{\mathcal{C}}$ and, by linearity of $\mu(\cdot)$, $\mu(\sigma) = \mu$. Therefore, it is sufficient to prove the claim.

Proof of claim: It is enough to analyze each component $r = 1, ..., q$ separately so we drop the subscript r (so $P = P_r$ and $C = C_r$, and say that $\#P = p$). By construction (definition of P and C), there exists an ordering $(i_1, ..., i_k, ..., i_p)$ of P where $P^k = \{i_1, ..., i_k\}$ and $C^k = \cup_{i \in P^k} \Sigma_i$ satisfy $C^k \cap \Sigma_{i_{k+1}} \neq \varnothing$ for all $k = 1, ..., p-1$ (starting from any element in P, each new element in the order is chosen so that it has a support connected to some of the previous elements chosen).

The proof now proceeds by induction. The induction hypothesis is: Suppose that the following statement holds for P^k and C^k: Given any $\mu^k = \left(\mu^k(S)\right)_{S \in C^k}$ close to zero satisfying $\sum_{S \in C^k} \mu^k(S) = 0$ there exists $\sigma^k = \left(\sigma_i^k(S)\right)_{\substack{S \in \Sigma_i \\ i \in P^k}}$ close to zero satisfying $\sum_{S \in \Sigma_i} \sigma_i^k(S) = 0$ for all $i \in P^k$ such that $\mu\left(\sigma^k\right) = \mu^k$. The statement also holds for $k+1$: Consider any $\mu = (\mu(S))_{S \in C^{k+1}}$ close to zero satisfying $\sum_{S \in C^{k+1}} \mu(S) = 0$. Let $\hat{S} \in C^k \cap \Sigma_{i_{k+1}}$. Define $\sigma_{i_{k+1}}(S) = \mu(S)$ for all $S \in \Sigma_{i_{k+1}} \backslash \{\hat{S}\}$ and let $\sigma_{i_{k+1}}\left(\hat{S}\right) = -\sum_{S \in \Sigma_{i_{k+1}} \backslash \{\hat{S}\}} \mu(S)$. Also define $\mu^k(S) = \mu(S)$ for all $S \in C^k \backslash \Sigma_{i_{k+1}}$, $\mu^k(S) = 0$ for all $S \in C^k \cap \Sigma_{i_{k+1}} \backslash \{\hat{S}\}$, and let $\mu^k\left(\hat{S}\right) = -\sum_{S \in C^k \backslash \{\hat{S}\}} \mu^k(S)$. Thus, using the induction hypothesis, there exists $\sigma = (\sigma_i(S))_{\substack{S \in \Sigma_i \\ i \in P^{k+1}}}$ which is close to zero, satisfies $\sum_{S \in \Sigma_i} \sigma_i(S) = 0$ for all $i \in P^{k+1}$ and is such that $\mu(\sigma) = \mu$. Since the statement is true for $k = 1$ (just let $\sigma = \mu$) and, by induction, the statement is also true for $k = p$, which is exactly the claim (as $P^k = P$ and $C^k = C$), this completes the proof. Q.E.D.

Transversality theorem: Suppose $f : U \times V \to R^n$ is continuously differentiable where $U \subset R^n$ and $V \subset R^m$ are open sets. If the $n \times (n+m)$ Jacobian $d_{(x,q)} f$ has rank n whenever $f(x,q) = 0$ (i.e., $f \pitchfork 0$) then the system of n equation and n unknowns $f(\cdot, q^*) = 0$ is regular for almost every $q^* \in V$.

Proof of proposition 4: The proof is by induction on the number of players and the induction hypothesis is: for games with less than n players, almost all parameters (v, p) in $R^d \times \Delta^d$ are regular and nondegenerate *and* all such games have local solution mappings $x(v, p)$ that are surjective.

The hypothesis holds for one player games: the only support is $\mathcal{C} = \{\{1\}\}$ and the Jacobian matrix of problem $F_{\mathcal{C}}$ is obviously nonsingular. Now, let π be a c.s. with n players, and let us represent by a subscript 0 the references to the c.s. π and by the subscript -0 the references to all its proper subgames. Let $V_0 \times \Delta_0$ represent the set of all

$(v_i(\pi), p_i(\pi))$ and $V_{-0} \times \Delta_{-0}$ the set of all $(v_i(\pi'), p_i(\pi'))_{\substack{\pi' \in \Pi \\ \pi' \neq \pi}}$.

Let $R_{-0} \subset V_{-0} \times \Delta_{-0}$ be the set of games that are regular and nondegenerate and the local mappings $x_{-0}(v,p)$ are surjective. According to the induction hypothesis almost all games of $V_{-0} \times \Delta_{-0}$ belong to R_{-0}. Consider the solutions of the augmented problem $F_{\mathcal{C}_0}(z_0, v_0, p_0, v_{-0}, p_{-0}) = 0$ where $z_0 = (x_0, \mu_0, e_0)$ and we consider that $x_{-0}(v_{-0}, p_{-0})$ changes with v_{-0}, p_{-0} (even though expressions $\sum_{i \in S} x_i(\pi) + e_r(\pi) - x_S(\pi S)$ do not depend directly on the parameters v, p, the term $x_S(\pi S)$ is a function of v_{-0}, p_{-0}). The Jacobian matrix at the solution, $d_{(z_0, v_0, p_0, v_{-0}, p_{-0})} F_{\mathcal{C}_0}$, is

$$\left[\begin{array}{ccccc} * & * & -(1-\delta) I_0 & 0 & * \\ * & 0 & 0 & 0 & -d_{(v_{-0}, p_{-0})} g \circ x_{-0}(v_{-0}, p_{-0}) \\ 0 & d_\mu M_{\mathcal{C}_0} & 0 & d_{p_0} M_{\mathcal{C}_0} & 0 \end{array} \right], \tag{22}$$

where $g : V_{-0} \to R^m$ is the linear map $g(x_{-0})(S) = x_S(\pi S)$ for all the sets in the support $\mathcal{C}_0 = (\Sigma_i(\pi))$, and $*$ denotes arbitrary coefficients. Note that the linear map g is surjective, and thus the composition $g \circ x_{-0}(v_{-0}, p_{-0})$ is surjective (the composition of surjective maps is surjective). But then we have that $F_{\mathcal{C}_0}(z_0, v_0, p_0, v_{-0}, p_{-0}) \overline{\pitchfork} 0$ because all blocks $-(1-\delta) I_0$, $d_\mu M_{\mathcal{C}_0}$, and $-d_{(v_{-0}, p_{-0})} g \circ x_{-0}(v_{-0}, p_{-0})$ are surjective. Therefore, by the transversality theorem, for almost every $(v,p) \in R^d \times \Delta^d$, $F_{\mathcal{C}_0}(z_0) \overline{\pitchfork} 0$. Because of the block triangular structure of the Jacobian matrix $d_z F_{\mathcal{C}}(z)$ (see remark on Section 4.2) this shows that $d_z F_{\mathcal{C}}(z)$ is nonsingular ($\mathcal{C}$ regular) almost everywhere.

The argument to show that $\mathcal{C}$ is nondegenerate almost everywhere is the same one discussed in Section 4.3.

To complete the proof, it still remains to show that the local solution mappings $x(v,p)$ of problem $F_{\mathcal{C}}$ are surjective. But

$$d_{(v,p)} x = \left[\begin{array}{cc} d_{(v_0, p_0)} x_0 & d_{(v_{-0}, p_{-0})} x_0 \\ 0 & d_{(v_{-0}, p_{-0})} x_{-0} \end{array} \right],$$

because x_{-0} does not depend on (v_{-0}, p_{-0}), and it is thus enough to prove that $x_0(v,p)$ is surjective (by the induction hypothesis $d_{(v_{-0}, p_{-0})} x_{-0}$ is surjective). The implicit function theorem gives us the expression of the derivative of the local mappings (refer to (12)) as, $d_{(v,p)} x_0(v,p) = -\left[d_{z_0} F_{\mathcal{C}_0}\right]_n^{-1} d_{(v,p)} F_{\mathcal{C}_0}$, where $\left[d_{z_0} F_{\mathcal{C}_0}\right]_n^{-1}$ is the submatrix of $\left[d_{z_0} F_{\mathcal{C}_0}\right]^{-1}$ restricted to the first n rows, and $d_{(v,p)} F_{\mathcal{C}_0}$ is given by (22). But both $\left[d_{z_0} F_{\mathcal{C}_0}\right]_n^{-1}$ and $d_{(v,p)} F_{\mathcal{C}_0}$ are surjective so $d_{(v,p)} x_0(v,p)$ is surjective. Thus we conclude that $x_0(v,p)$ is surjective. Q.E.D.

Lefschetz Fixed Point Theorem (LFPT) (McLennan 1989): Let $\mathcal{T}$ be the collection of admissible triples (X, F, U) where $X \subset R^m$ is a finite simplicial complex, $F : X \to X$ is a upper hemicontinuous contractible valued correspondence (u.h.c.c.v.), $U \subset X$ is open, and there are no fixed points of F in $\overline{U} - U$. Then there is a unique Lefschetz fixed point index $\Lambda(X, F, U)$ that satisfying the following axioms (when X is implicitly given we just say $\Lambda(F, U)$):

(Localization axiom): If $F_0, F_1 : X \to X$ are u.h.c.c.v. correspondences that agree on $\overline{U}$, and (X, F_1, U), $(X, F_0, U) \in \mathcal{T}$, then $\Lambda(X, F_1, U) = \Lambda(X, F_0, U)$.

(Continuity axiom): If $(X, F, U) \in \mathcal{T}$, then there is a neighborhood W of $Gr(F)$ such that $\Lambda(X, F', U) = \Lambda(X, F, U)$ for all u.h.c.c.v. correspondences $F' : X \to X$ with $Gr(F') \in W$.

(Homotopy axiom): If $h : [0, 1] \times X \to X$ is a homotopy with $(X, h_\tau, U) \in \mathcal{T}$, for all τ, then $\Lambda(X, h_0, U) = \Lambda(X, h_1, U)$.

(Additivity axiom): If $(X, F, U) \in \mathcal{T}$ and $U_1, ..., U_r$ is a collection of pairwise disjoint open subsets of U such that there are no fixed points of F in $U - (\cup_{k=1}^r U_k)$ then $\Lambda(X, F, U) = \sum_{k=1}^r \Lambda(X, F, U_k)$.

(Weak Normalization axiom): For $y \in X$, let c_y be the constant correspondence $c_y(x) = \{y\}$. If $y \in U$ then $\Lambda(X, c_y, U) = 1$.

(Commutativity axiom): If $X \subset R^m$ and $Y \subset R^n$ are finite simplicial complexes, $f : X \to Y$ and $g : Y \to X$ are continuous functions, and $\Lambda(X, g \circ f, U) = \Lambda(X, f \circ g, g^{-1}(U))$.

Proof of Proposition 5: Define the correspondence $\mathbf{F}(x) = x - \mathcal{F}(x)$, where $\mathcal{F}(x)$ is the correspondence defined in (18). The Lefschetz index of $\mathcal{F}$ and the degree of $\mathbf{F}$ are related by $\Lambda(\mathcal{F}, U) = \deg(\mathbf{F}, U, 0)$ (see McLennan (1989)), and, for convenience, we work in the remainder of the proof with the concept of degree.

The mixed nonlinear complementarity problem is the problem of finding triples (x, σ, e) that are the solution of problem (MNCP) in Section 3.

For each point x consider the mixed linear complementarity problem $MLCP(0)$,

$$\begin{aligned} h(\sigma) &= 0, \\ g(e, x) &\geq 0, \\ e \text{ free variable}, \sigma &\geq 0 \text{ and } \sigma^T g(e, x) = 0, \end{aligned} \tag{23}$$

Let $z(x) = (e(x), \sigma(x))$ be a solution of the $MLCP(0)$ (there can be multiple solutions).

Note that $\mathbf{F}(x) = \{f(x, z(x)) : \ z(x) \text{ is a solution of } MLCP(0)\}$.

Let (x^*, e^*, μ^*) be any regular and nondegenerate *MPE* with an associated *CDS* $\mathcal{C} = (C, P)$, with $C = (C_1, ..., C_q)$ and $P = (P_1, ..., P_q)$. By Lemma 2, there exists $\sigma^* \in \Delta_C$ such that $\mu^* = \mu^*(\sigma^*)$, and (x^*, σ^*) is *MPE*. Furthermore, because all points in P_r are connected, we can choose a strategy profile σ^* satisfying $supp(\sigma_i^*) = \mathcal{C}_r \cap \{S \subset \pi : i \in S\}$ for all $i \in P_r$.

Consider now the perturbed mixed linear complementarity problem, or $MLCP(\varepsilon)$

$$\begin{array}{c} h(\varepsilon)(e,\sigma) = h(\sigma) + \varepsilon (e - e^*) = 0, \\ g(\varepsilon)(x,\sigma,e) = g(x,e) + \varepsilon(\sigma - \sigma^*) \geq 0, \\ e \text{ free variable, } \sigma \geq 0, \ \sigma^T g(\varepsilon) = 0, \end{array} \tag{24}$$

where $\varepsilon > 0$. The Jacobian matrix $M(\varepsilon)$ of $MLCP(\varepsilon)$ is a P-matrix (i.e., a matrix with all its principal minors positive). This is so because (see Cottle et al. (1992, pg. 154)), $M(\varepsilon) = M + \varepsilon I$, where M is the Jacobian of $MLCP(0)$, is a P_0-matrix (i.e., a matrix with all its principal minors nonnegative). Let us prove that M is a P_0-matrix: Consider the principal matrix $M_{\beta\beta}$ associated with a subset β of lines (or columns).[12] We now show that either $\det(M_{\beta\beta})$ is equal to zero or one. Note first that $\det(M_{\beta\beta}) = \prod_{i\in\pi} \det(M_{\beta_i\beta_i})$ where $\beta = \cup_i \beta_i$ and β_i are the elements of β with entry i (either e_i or $\sigma_i(S)$ for some $S \ni i$). But $\det(M_{\beta_i\beta_i}) = 1$ if $\beta_i = \{e_i, \sigma_i(S)\}$ and is zero otherwise. Therefore, we conclude that all principal minors of M are nonnegative, and thus M is a P_0-matrix.

Given that $MLCP(\varepsilon)$ has a P-matrix then there is a unique solution $z_\varepsilon(x)$ (Cottle et al. (1992, pg. 150)) for all x: $MLCP(\varepsilon)$ can be transformed into a standard *LCP* eliminating the variable e and the equation $h(\varepsilon) = 0$ (this is possible because $M_{ee}(\varepsilon) = \varepsilon I$ is nonsingular), and the transformed *LCP* also has a P-matrix (the Schur complement of $M_{ee}(\varepsilon)$ in $M(\varepsilon)$). Note that, in addition, we have that $z_\varepsilon(x^*) = (e^*, \sigma^*)$, and that $z_\varepsilon(x)$ converge to a solution of $MLCP(0)$ when $\varepsilon \to 0$ (Cottle et al. (1992, pg. 442)), and that $z_\varepsilon(x)$ is piecewise linear in x.

We now show that, because x^* is a strong solution, there exists an $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ there exists an open neighborhood U_ε of x^* such that $z_\varepsilon(x)$ is smooth in

[12] We refer to the lines corresponding to ∂h_i and $\partial g_i(S)$ as lines λ_i and $\sigma_i(S)$, and the columns corresponding to $\frac{\partial \cdot}{\partial \lambda_i}$ and $\frac{\partial \cdot}{\partial \sigma_i(S)}$ as columns λ_i and $\sigma_i(S)$. Also, we use the standard notation that $A_{\alpha\alpha}$, $A_{\cdot\alpha}$, and $A_{\alpha\cdot}$ represent the submatrix of A with, respectively, rows and columns, columns, and rows extracted from the index set α. Also, $\overline{\alpha}$ denotes the complementary set of α.

U_ε. Moreover, if we let α represent the index set

$$\alpha = \{\sigma_i(S) : \text{for all } S \in \mathcal{C}_r \text{ and } i \in S\}, \tag{25}$$

then all $\sigma_i(S)$-coordinates of the solution $z_\varepsilon(x)$ that do not belong to α are zero, and $z_\varepsilon(x)$ are explicitly given by $(M_{\alpha\alpha}(\varepsilon))^{-1} q_\alpha(x)$, where $M_{\alpha\alpha}(\varepsilon)$ is

$$M_{\alpha\alpha}(\varepsilon) = \begin{bmatrix} \varepsilon I_{\alpha\alpha} & (d_e g)_\alpha \\ d_\alpha h & \varepsilon I_{ee} \end{bmatrix},$$

and the vector $q_\alpha(x)$ has e_i-coordinate equal to $(\varepsilon e_i^* - 1)$, and $\sigma_i(S)$-coordinate in α equal to $\varepsilon\sigma_i(S)^* + e(S)(x)$, for all $0 < \varepsilon < \overline{\varepsilon}$ and $x \in U_\varepsilon$.

In order to prove the above claim consider the function

$$\varphi(x) = \min \cup_{r=1}^{q} \{e(S)(x) - e(T)(x) : S \in C_r, T \cap P_r \neq \emptyset, \text{ and } T \notin C_r\}.$$

Naturally, the function φ is continuous in x and, because x^* is a strong solution, $\varphi(x^*) > 0$. Therefore, there exists an $\overline{\varepsilon} > 0$ and an open neighborhood $U \subset U_{x^*}$ of x^*, such that all $x \in U$ satisfy $\varphi(x) > 2\overline{\varepsilon}$. Now suppose that the solution $z_\varepsilon(x)$ for $x \in U$ is such that a $\sigma_i(T)$-coordinate is non-zero for $T \notin C_r$ and $i \in P_r$. Then $g_i(\varepsilon)(T) = 0$ which is equivalent to $e_i + \varepsilon(\sigma_i(T) - \sigma_i^*(T)) - e(T)(x) = 0$, and implies $e(T)(x) \geq e_i - \varepsilon$. Also, $g_i(\varepsilon)(S) \geq 0$ for all S, and thus $e_i + \varepsilon(\sigma_i(S) - \sigma_i^*(S)) - e(S)(x) \geq 0$, which implies that $e(S)(x) \leq e_i + \varepsilon$. Therefore, $e(S)(x) - e(T)(x) \leq 2\varepsilon \leq 2\overline{\varepsilon}$ for $x \in U$, in contradiction with $\varphi(x) > 2\overline{\varepsilon}$ for all $x \in U$. Now, since $z_\varepsilon(x^*) = (e^*, \sigma^*)$, and $supp(\sigma_i^*) = \mathcal{C}_r \cap \{S \subset \pi : i \in S\}$, and $z_\varepsilon(x)$ is continuous, then there exists an open neighborhood $U_\varepsilon \subset U_{x^*}$ of x^* where all $\sigma_i(S)$-coordinates of the solution belonging to α are non-zero. This implies that $g_i(\varepsilon)(S) = 0$ holds for all $\sigma_i(S)$ in α, and thus $z_\varepsilon(x) = (M_{\alpha\alpha}(\varepsilon))^{-1} q_\alpha(x)$.

Define the mapping $\mathbf{F}_\varepsilon(x) = f(x, z_\varepsilon(x))$ (this mapping is well-defined due to the uniqueness of $z_\varepsilon(x)$), where $\mathbf{F}_\varepsilon(x^*) = 0$. Since f is smooth and $z_\varepsilon(x) \to z(x)$ then $\mathbf{F}_\varepsilon(x) \to \mathbf{F}(x)$. Therefore, for every $\delta > 0$ there exists $\overline{\varepsilon}$ such that $\underset{x \in \overline{U}}{dist}(\mathbf{F}_\varepsilon(x), \mathbf{F}(x)) < \delta$, for all $0 < \varepsilon \leq \overline{\varepsilon}$. But since $\mathbf{F}(x)$ has no zeros in the boundary of ∂U then $\mathbf{F}_\varepsilon(x)$ also does not have any zeros in ∂U. By the homotopy and continuity property of the degree, $\deg(\mathbf{F}, U, 0) = \deg(\mathbf{F}_\varepsilon, U, 0)$, for ε close to zero.

Therefore, it only remains to show that $\deg(\mathbf{F}_\varepsilon, U, 0) = sgn(\det(dF_{\mathcal{C}}(z^*)))$ for ε close to zero, where $z^* = (x^*, e^*, \mu^*)$. This result follows from $sgn(\det(d_x\mathbf{F}_\varepsilon(x^*))) = sgn(\det(dF_{\mathcal{C}}(z^*))) \neq$

0, as we will show. Indeed, this implies that $\mathbf{F}_\varepsilon$ is nonsingular at x^*, and thus there exists an open neighborhood $V \subset U$ of x^* where x^* is the only zero of $\mathbf{F}_\varepsilon$. But since the point x^* is the only zero of $\mathbf{F}(x)$ in $U \subset U_{x^*}$, and $\mathbf{F}_\varepsilon(x) \to \mathbf{F}(x)$ then there are no zeros of $\mathbf{F}_\varepsilon$ in the compact region $\overline{U} \backslash V$, for ε small enough, and thus x^* is the only zero of $\mathbf{F}_\varepsilon$ in U. A well-known property of the degree then implies that $\Lambda(\mathcal{F}, U) = \deg(\mathbf{F}_\varepsilon, U, 0) = sgn\left(\det\left(d_x \mathbf{F}_\varepsilon(x^*)\right)\right) = sgn\left(\det(dF_{\mathcal{C}}(z^*))\right).$

We now show that $sgn\left(\det(d_x \mathbf{F}_\varepsilon(x^*))\right) = sgn\left(\det(dF_{\mathcal{C}}(z^*))\right)$, for ε small enough. Consider $F(x, \sigma, e)(\varepsilon)$,

$$F(x,\sigma,e)(\varepsilon) = \begin{pmatrix} f(x,\sigma,e) \\ h(\sigma) + \varepsilon(e - e^*) \\ g(e,x) + \varepsilon(\sigma - \sigma^*) \end{pmatrix}.$$

Simple linear algebra shows that the Jacobian $d_x \mathbf{F}_\varepsilon(x^*)$ is the Schur complement of $M_{\alpha\alpha}(\varepsilon)$ in $dF_{\alpha\alpha}(\varepsilon)$ ($d_x \mathbf{F}_\varepsilon(x^*) = dF_{\alpha\alpha}(\varepsilon) / M_{\alpha\alpha}$), where

$$dF_{\alpha\alpha}(\varepsilon) = \begin{bmatrix} (d_x f) & d_\alpha f & d_e f \\ (d_x g)_\alpha & \varepsilon I_{\alpha\alpha} & (d_e g)_\alpha \\ 0 & d_\alpha h & \varepsilon I_{ee} \end{bmatrix}, \tag{26}$$

is evaluated at point (x^*, e^*, σ^*). Therefore, $\det(d_x \mathbf{F}_\varepsilon(x^*)) = \det(dF_{\alpha\alpha}(\varepsilon)) / \det(M_{\alpha\alpha})$ (see Cottle et al. (1992, pg. 75)). But since $\det(M_{\alpha\alpha}) > 0$ (M is a P-matrix) then $sgn\left(\det(d_x \mathbf{F}_\varepsilon(x^*))\right) = sgn\left(\det(dF_{\alpha\alpha}(\varepsilon))\right).$

We claim that $sgn\left(\det(dF_{\alpha\alpha}(\varepsilon))\right) = sgn\left(\det(dF_{\mathcal{C}}(z^*))\right)$. In order to prove the claim we use the following formula for the determinant (Cottle et al. (1992), pg. 60): for an arbitrary diagonal matrix D, $\det(A + D) = \sum_\gamma \det D_{\overline{\gamma\gamma}} \det A_{\gamma\gamma}$ where the summation ranges over all subsets γ of lines. Observe that matrix $dF_{\alpha\alpha}(\varepsilon) = A + D$, where $A = dF_{\alpha\alpha}(0)$ and D is the diagonal matrix,

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon I_{\alpha\alpha} & 0 \\ 0 & 0 & \varepsilon I_{ee} \end{bmatrix}.$$

Developing the expression for $\det(dF_{\alpha\alpha}(\varepsilon))$ using the formula above we get a polynomial in ε ($\det D_{\overline{\gamma\gamma}}$ is a power of ε). We are only interested in the non-zero coefficient with lowest order because, when ε converges to zero, this is the coefficient that determines the sign of $\det(dF_{\alpha\alpha}(\varepsilon))$.

The rows and columns of matrix $A = dF_{\alpha\alpha}(0)$ corresponding to $\sigma_i(S)$ and e_i are

$$\begin{aligned}
R(\sigma_i(S)) &= \sum_j \mathbb{I}_{[j\in S]} e(x_j) + e(e_i), \\
R(e_i) &= -\sum_{S\in C_r} e(\sigma_i(S)), \\
C(\sigma_i(S)) &= \sum_j \mathbb{I}_{[j\notin S]} x_j(S) e(x_j) - e(e_i), \\
C(e_i) &= -p_i e(x_i) + \sum_{S\in C_r} e(\sigma_i(S)),
\end{aligned}$$

where vectors $e(x_i)$, $e(e_i)$, and $e(\sigma_i(S))$ are the unit vectors at, respectively, coordinates x_i, e_i, and $\sigma_i(S)$.

Consider $A_{\alpha\cdot}$ the submatrix of A corresponding to the rows α of A. Let β be a maximal subset of α such that $rank(A_{\beta\cdot})$ is different from zero ($|\beta| = rank(A_{\alpha\cdot})$ and $rank(A_{\beta\cdot}) = rank(A_{\alpha\cdot})$). Note that $A_{\gamma\gamma}$ where γ is the set of lines $\gamma = \beta \cup \{e_i : i \in \pi\} \cup \{x_i : i \in \pi\}$ is equal to $A_{\gamma\gamma} = dF_{\beta\beta}(0)$, according to the definition (26). Also, $\det A_{\gamma'\gamma'} = 0$ for set of lines γ' that strictly contains γ because β is a maximal subset of α such that $rank(A_{\beta\cdot}) \neq 0$.

We now show that $\det(dF_{\beta\beta}(0)) = \det(dF_{\mathcal{C}}(z^*)) \neq 0$, which proves that the lowest-order non-zero coefficient is equal to a positive integer (the number of maximal subsets $\beta \subset \alpha$) multiplied by $\det(dF_{\mathcal{C}}(z^*))$, and thus $sgn(\det(dF_{\alpha\alpha}(\varepsilon))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$, for ε small enough.

We now propose an algorithm replaces all rows and columns $\sigma_i(S)$'s with the same S by only one row and column $\sigma_i(S)$ for all $S \in C_r$, and also replaces all rows and columns e_i for all $i \in P_r$ by only one row and column e_r for each $r = 1, ..., q$.

Algorithm: Start with matrix $A = dF_{\beta\beta}(0)$.

Step 1: Choose an element r, that have not yet been chosen, from the set $\{1, 2, ..., q\}$ and proceed to the next step, or else, stop if the choice is not possible.

Step 2: Choose two distinct rows $\sigma_i(S)$ and $\sigma_j(S)$ of A with $j \neq i$ and $S \in C_r$ and proceed to the next step, or else return to step 1 if the choice is not possible.

Step 3: Subtract row $\sigma_i(S)$ from row $\sigma_j(S)$ (i.e., $R(\sigma_j(S)) = R(\sigma_j(S)) - R(\sigma_i(S))$), and add column e_j to column e_i (i.e., $C(e_i) = C(e_i) + C(e_j)$). The matrix that is obtained after the two operations have the same determinant as matrix A. Let this matrix be the new matrix A. After these two operations, row $\sigma_j(S)$ of A has only one non-zero entry at column e_j, with a value equal to 1. The determinant of A can be computed by a co-factor expansion along row $\sigma_j(S)$, and $|A| = (-1)^{(\#\sigma_j(S)+\#e_j)}|A'|$, where A' is the submatrix obtained after deleting row $\sigma_j(S)$ and column e_j of matrix A.

Now, perform the following symmetric transformations on the submatrix A': Subtract column $\sigma_i(S)$ from column $\sigma_j(S)$ (i.e., $C(\sigma_j(S)) = C(\sigma_j(S)) - C(\sigma_i(S))$) and add row e_j to row e_i (i.e., $R(e_i) = R(e_i) + R(e_j)$). The matrix that is obtained after the two operations have the same determinant as A'. Let this matrix be the new matrix A'. After these two operations, column $\sigma_j(S)$ of A' has only one non-zero entry at row e_j, with a value equal to -1. The determinant of A' can be computed by a co-factor expansion along column $\sigma_j(S)$, and $|A'| = (-1) \times (-1)^{(\#\sigma_j(S)+\#e_j-1)}|A''|$, where A'' is the submatrix of A' obtained after deleting column $\sigma_j(S)$ and row e_j: observe that the column $\sigma_j(S)$ of A' is in the same location as row $\sigma_j(S)$ of A', but row e_j appears one entry before column e_j of A (because the row $\sigma_j(S)$ that has been removed appears before row e_j). Putting together the expressions for the determinant yields $|A| = |A''|$. Let matrix A'' be the new matrix A, and return to step 2.

Because β is a maximal subset of α with $rank(A_{\beta\cdot}) \neq 0$ and $rank(E_C) \neq 0$, the algorithm starts with matrix $A = dF_{\beta\beta}(0)$ and ends with matrix $A = \det(dF_{\mathcal{C}}(z^*))$ (maintaining the same determinant in all steps).

Therefore, $\det(dF_{\beta\beta}(0)) = \det(dF_{\mathcal{C}}(z^*))$, as we claimed. Q.E.D.

Proof of Proposition 6: We focus on the c.s. $\pi = \{\{1\},\{2\},\{3\}\}$ because we already know that two-player games have a unique equilibrium (Rubinstein (1982)).

We first show that $X(i,S) = x_i(\pi) - x_i(\pi S) \geq 0$, where $i \notin S$, if there is a positive probability that S is chosen in equilibrium.

Note that if $S = \emptyset$ (i.e., the no proposal case) then $x_i(\pi S) = x_i$ and if $S = N = \{1,2,3\}$ then there are no elements $i \notin S$.

Say that $S = \{j,k\}$, where $j \neq i$ and $k \neq i$. In order to simplify the notation, let $x_i = x_i(\pi)$, $x_i(\pi S) = x_i(jk)$, $x_S(\pi S) = x_{jk}(jk)$, and $V = v_N(\{N\})$. Suppose that S is chosen in equilibrium with positive probability. Then $e(S)(x) \geq e(N)(x)$, which is equivalent to,

$$x_{jk}(jk) - x_j - x_k \geq V - x_i - x_j - x_k, \tag{27}$$

and

$$x_{jk}(jk) + x_i(jk) + x_i - x_i(jk) \geq V. \tag{28}$$

But since there is no delay in the formation of the grand coalition when the game is at the c.s. $\{\{jk\}, \{i\}\}$, we have that

$$x_{jk}(jk) + x_i(jk) = \delta V + (1-\delta)(v_{jk}(jk) + v_i(jk)).$$

Replacing this expression into (28) yields

$$X(i, jk) = x_i - x_i(jk) \geq (1-\delta)(V - (v_{jk}(jk) + v_i(jk))) \geq 0.$$

We now compute $\det(dF_{\mathcal{C}})$ for all admissible *CDS* $\mathcal{C} = (C, P)$, and show that $\det(dF_{\mathcal{C}}) \geq 0$. From the definition of *CDS*s it follows that $P = (P_1, ..., P_q)$ is a partition of N and $C = (C_1, ..., C_q)$ is an ordered disjoint collection of subsets $S \subset N$ satisfying: for all $S \in C_r$ then $S \cap P_r \neq \emptyset$ and $S \subset \cup_{s=1}^{r} P_s$, and also $\cup_{S \in C_r} S \supset P_r$. Moreover, there is no $S = \{i\}$ that is chosen in equilibrium, and thus $C_r \subset \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. A list of all admissible *CDS*s (except for permutations of the players) follows with the corresponding value for $\det(dF_{\mathcal{C}})$ (i, j, and k are distinct elements of N, and $d_i = 1 - \delta \sum_S \mu(S) \mathbb{I}_{[i \in S]}$, $z(i, jk) = \delta X(i, jk)$, and $w_i = \delta p_i$):

CDS $\mathcal{C}$	$\det(dF_{\mathcal{C}})$
$(\{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\})$	$z(2,13)z(1,23)z(3,12)$
$(\{\{1,2\}, \{1,3\}, \{1,2,3\}\})$	$z(3,12)z(2,13)(w_1 + d_1)$
$(\{\{1,2\}, \{1,2,3\}\})$	$z(3,12)(d_2 w_1 + d_1 d_2 + w_2 d_1)$
$(\{\{1,2,3\}\})$	$d_1 w_3 d_2 + d_1 d_3 d_2 + d_1 d_3 w_2 + d_2 d_3 w_1$
$(\{\{1,2\}, \{2,3\}, \{1,3\}\})$	$\sum_{i,j,k} (d_i + 2w_i)\, z(k, ij) z(j, ik)$
$(\{\{1,2\}, \{1,3\}\})$	$\sum_{i,j \neq 1} (d_i w_1 + d_i d_1 + w_i d_1) z(j, 1i)$
$(\{\{1,2\}\}, \{\{1,2,3\}, \{1,3\}, \{2,3\}\})$	$z(2,13)z(1,23)(w_3 + d_3)$
$(\{\{1,2\}\}, \{\{1,2,3\}, \{1,3\}\})$	$z(2,13)(w_1 + d_1)(w_3 + d_3)$
$(\{\{1,2\}\}, \{\{1,2,3\}\})$	$(d_2 w_1 + d_2 d_1 + w_2 d_1)(w_3 + d_3)$
$(\{\{1,2\}\}, \{\{1,3\}, \{2,3\}\})$	$\left(\sum_{i,j \neq 3} (2w_i + d_i) z(j, i3)\right)(w_3 + d_3)$
$(\{\{1,2\}\}, \{\{1,3\}\})$	$(d_2 w_1 + d_2 d_1 + w_2 d_1)(w_3 + d_3)$

Note that the first 6 entries of the table corresponds to *CDS*s with $P = (\{1,2,3\})$ and the remaining entries to *CDS*s with $P = (\{1,2\}, \{3\})$.

The determinant for all *CDS*s are nonnegative because it is a sum of nonnegative terms. Corollary 2 implies that there is a unique global *MPE* solution. Q.E.D.

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