Multilateral Negotiations and Formation of Coalitions

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Abstract

This paper analyses properties of games modelling multilateral negotiations leading to the formation of coalitions in an environment with widespread externalities. The payoff generated by each coalition is determined by a set of exogenous parameters. We show that in almost all games, except in a set of measure zero of the parameter space, the Markov perfect equilibrium is locally unique and stable, and comparative statics analysis are well-defined and can be performed using standard calculus tools. Global uniqueness does not hold in general, but the number of equilibria is finite and odd. In addition, a sufficient condition for global uniqueness is derived, and using this sufficient condition we show that there is a globally unique equilibrium in three-player superadditive games.

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1 Introduction

We study multilateral negotiations where contracts forming coalitions can be written and renegotiated, and where the formation of coalitions may impose externalities on other players. What is the path of coalition formation? What is the value of players? This paper develops a non-cooperative model of coalitional bargaining that provides answers to these two questions.

The externalities present in the environment are described by a set of exogenous parameters, conveniently expressed using a partition function form. The partition function form assigns a worth to each coalition depending on the coalition structure (or collection of coalitions) formed by the remaining players. This general formulation allows for the analyses of problems in which the formation of coalitions may impose positive or negative externalities (see also Ray and Vohra (1999), Bloch (1996), Jehiel and Moldovanu (1995), and Gomes (2005)).

Our multilateral negotiation procedure follows the traditional approach of using a dynamic game with complete information where at each stage a player becomes the proposer (such as Chatterjee et al. (1993), Hart and Mas-Colell (1996), Ray and Vohra (1999), and Okada (1996) among many others). Proposers make offers to form coalitions, followed by players who have received offers making their response whether or not to accept the offer. Similarly to Gul (1989) and Seidmann and Winter (1998), coalitions after forming do not leave the game and may continue negotiating the formation of further coalitions. A variety of situations can be addressed with our game such as the formation of custom unions, merger or carter formation among firms in the same industry, legislative bargaining, and the signing of environmental agreements across regions (see, for example, Ray (2007)).

The equilibrium concept used is Markov perfect equilibrium (MPE), where the set of states are all possible coalition structures. The MPE solutions characterize, jointly, both the expected equilibrium value of coalitions, and the Markov state transition probability that describes the path of coalition formation. Thus, the equilibria provide answers to both questions we posed initially, and our goal is to develop a thorough analysis of the equilibrium properties.

We establish the existence of equilibria for all games, and show that, for almost all
games, except in a closed set of measure zero of the parameter space, the equilibrium is locally unique and locally stable. These properties imply that the predictions of the model about both the expected payoffs of players and the path of coalition formation are sharp, in the sense that, at least locally, they are unique and robust to small perturbations of the exogenous parameters of the game.

Therefore, we extend to multilateral bargaining models similar results that hold for other well-known economic models such as Walrasian equilibrium of competitive economies (Debreu (1970)), Nash equilibrium of \( n \)-person strategic form games (Wilson (1971) and Harsanyi (1973)), and Markov perfect equilibrium of stochastic games (Holler and Lagunoff (2000)).

The problem of finding equilibria is equivalent to finding solutions of a mixed nonlinear complementarity problem (MNCP). Such problems have been extensively studied in the mathematical programing literature (see Harker and Pang (1990) and Cottle, Pang, and Stone (1992)), and several numerical algorithms have been developed. Hence, the computation of equilibria is a task that can be undertaken using several proven numerical algorithms.

The number of equilibrium solutions is finite and odd for almost all games. We provide an example of a game with multiple (seven) equilibrium solutions, so the equilibrium is not globally unique. Nonetheless, we derive a sufficient condition for global uniqueness, and argue that this sufficient condition is weak and is likely to be satisfied by a large class of games. In particular, we prove that the sufficient condition holds for superadditive three-player coalitional bargaining games.

How do the equilibrium value of players and the path of coalition formation change as a result of changes in exogenous parameters such as the partition function form and the probability of being the proposer? Knowing how to address these questions is of considerable practical interest to negotiators, as they, for example, may be able to invest in changing the likelihood of being proposers in negotiations. We show how to answer these questions using standard calculus results (the implicit function theorem), which provides a powerful tool for quickly answering comparative statics questions by simply evaluating Jacobian matrices at the solution.

We demonstrate the applicability of the results using two classic games–apex and quota games (see Shapley (1953), Davis and Maschler (1965), and Maschler (1992)).
Surprisingly, in both games, a player sometimes may not benefit by investing in obtaining more initiative to propose in negotiations. Other players may adjust their strategies in such a way that lead the proposer to be worse off. The analysis also suggests several interesting regularities: when the exogenous value of a coalition increases, both the equilibrium value of the coalitional members and the likelihood that the coalition forms increase as well.

The remainder of the paper is organized as follows: Section 2 presents the coalitional bargaining game; Section 3 addresses the existence and characterization of the equilibria; Section 4 shows how to compute the equilibria; Section 5 and 6 develop the local uniqueness, stability, and genericity properties of the equilibria; Section 7 addresses the number of equilibrium solutions; and Section 8 concludes.

2 The Model

Coalition formation in this paper is modeled as an infinite horizon complete information game. In a nutshell, the coalition formation process is such that during any period of the game a player is chosen at random to propose to form a coalition and a payment to all coalition members. Subsequently, all coalition members respond to the offer, and the coalition is formed only if all its members agree.

Formally, let $N = \{1, 2, \ldots, n\}$ be a set of $n$ agents. A coalition is a subset of agents and a coalition structure (c.s.) $\pi = \{C_1, ..., C_K\}$, where $C_k \subseteq N$, is a partition of the set of agents into disjoint coalitions (denote by $\Pi$ the set of all coalition structures). The game starts at the state where all agents are in solo coalitions (the initial c.s. is the calligraphic $\mathcal{N} = \{\{1\}, ..., \{n\}\}$).

The coalition bargaining game is the game with the following extensive form. Consider that at the beginning of a certain period of the game the c.s. is $\pi$ (typical coalitions in $\pi$ are often represented by labels $i, j$). One of the coalitions $i \in \pi$ is randomly chosen with probability $p_i(\pi) > 0$ to be the proposer (let $p = (p_i(\pi))_{i \in \pi}$). Coalition $i$ then makes an offer $(S, t)$ where $S \subseteq \pi$ is a set of coalitions in $\pi$ and $t$ is a vector of transfers satisfying $\sum_{j \in S} t_j = 0$. All coalitions in $S$ respond in a fixed sequential order whether they accept or not the offer (it turns out the the order of response is not relevant). If all coalitions in $S$ accept the offer a new coalition
\( S = \bigcup_{j \in S} \{j\} \subset N \), calligraphic \( S \), is formed under the control of the proposing coalition \( i \). The coalitions \( j \in S \setminus i \) ceding control receive the lump-sum payment \( t_j \) and exit the game. The coalition structure evolves from \( \pi \) to \( \pi S = S \cup (\pi \setminus S) \).\(^1\) Otherwise, if any one of the coalitions receiving the offer rejects it, no new coalition is formed and the coalition structure remains equal to \( \pi \). After a lapse of one period of time, the game is repeated starting with the prevailing c.s. with a new proposer being randomly chosen as just described.

All agents have the same expected intertemporal utility function, are risk-neutral and have common discount factor \( \delta \in (0, 1) \).\(^2\) When coalitions form they may impose externalities on other coalitions. This possibility is captured by a partition function form \( v = (v_i(\pi))_{\pi \in \Pi} \), where coalition \( i \)'s payoff flow (during a period of time), when the coalition structure is \( \pi \), is equal to \( (1 - \delta) v_i(\pi) \) (so if the game stays at c.s. \( \pi \) forever, the value of coalition \( i \) is \( v_i(\pi) \)). The payoffs are distributed at the end of each period, after the coalition formation stage, with coalitions ceding control receiving a final lump-sum transfer payoff and the coalition acquiring control receiving, in addition to the lump-sum transfer, the payoff given by the partition function form (i.e., \( (1 - \delta) v_i(\pi S) + t_i \) when the c.s. \( \pi S \) forms).

We restrict our attention to Markovian strategies. Hence, the proposer’s strategy only depends on the current state \( \pi \), and the respondents’ strategy only depends on the current state \( \pi \), the current offer she receives and the responses of preceding players. A Markov perfect equilibrium (MPE) is a Markovian strategy profile where every player plays a Nash equilibrium at every stage.

### 3 Characterization of Equilibrium

Let us be given a MPE strategy \( \sigma \). We represent by \( \phi_i(\pi|\sigma) \) the equilibrium continuation value of coalition \( i \) which is obtained from the stochastic process induced by \( \sigma \) when the c.s. is at \( \pi \) (the value \( \phi \) is computed at the beginning of a period before a proposer is chosen).

\(^1\)So, for example, if \( \pi = \{\{1, 2\}, \{3, 4\}, \{5\}\} \) and coalition \( i = \{1, 2\} \) proposes to form coalition \( S = \{\{1, 2\}, \{3, 4\}\} \), then \( \pi S = \{\{1, 2, 3, 4\}, \{5\}\} \) and \( S = \{1, 2, 3, 4\} \). Note that the coalition structure becomes coarser as time elapses.

\(^2\)Thus their utility over a stream of random payoffs \( (x_t)_{t=0}^\infty \) is \( \sum_{t=0}^\infty \delta^t E(x_t) \).
The equilibrium continuation value at the end of a period in which \( \pi \) is formed is equal to (gross of lump-sum transfers)

\[
x_i(\pi|\sigma) = \delta \phi_i(\pi|\sigma) + (1 - \delta) v_i(\pi),
\]

because coalition \( i \) receives payoff flow \((1 - \delta) v_i(\pi)\) during the current period and, after a delay of one period, at the beginning of the next period, coalition \( i \)'s value is \( \delta \phi_i(\pi|\sigma) \). Let \( x = (x_i(\pi))_{\pi \in \Pi} \) and \( \phi = (\phi_i(\pi))_{\pi \in \Pi} \) be the continuation values (where we sometimes omit the dependency on the strategy profile \( \sigma \)).

An equilibrium \( \sigma \) is characterized by several properties which we now summarize. The minimum offer that coalition \( j \) receiving offer \((S,t)\) is willing to accept is one where \( t_j \geq x_j(\pi|\sigma) \) (because upon rejection no transfers are made and the state remains at \( \pi \)). In turn, coalition \( i \) proposes offers \((S,t)\) that maximizes the value \( x_S(\pi S|\sigma) - \sum_{j \in S \setminus i} t_j \) subject to the constraint that \( t_j \geq x_j(\pi|\sigma) \), (because a new coalition \( S \) is formed whose value is \( x_S(\pi S|\sigma) \) if the offer is accepted by all players). Thus, when an offer \((S,t)\) is made the transfers \( t \) are uniquely determined by \( t_j = x_j(\pi|\sigma) \) for \( j \in S \setminus i \) and \( t_i = \sum_{j \in S \setminus i} t_j \).

Define the excess

\[
\theta(\pi)(S)(x) = x_S(\pi S|\sigma) - \sum_{j \in S} x_j(\pi|\sigma),
\]

a function of \( S \subset \pi \), and \( x \) equilibrium continuation value. Proposer \( i \) randomizes across coalitions \( S \) that maximizes the excess \( \max_{S \ni i, S \subset \pi} \{ \theta(\pi)(S)(x) \} \).

Let \( \sigma_i(\pi)(S) \in [0,1] \) represent the probability that coalition \( S \) is chosen by player \( i \). The (behavioral) strategy \( \sigma_i(\pi) \) of proposer \( i \) is a probability distribution over \( \Sigma_i(\pi) = \{ S \subset \pi : i \in S \} \), i.e., \( \sum_{S \in \Sigma_i(\pi)} \sigma_i(\pi)(S) = 1 \) and \( \sigma_i(\pi)(T) = 0 \) for all \( T \notin \Sigma_i(\pi) \). Also, we define \( \Delta(\pi) \) as the set of strategy profiles when the c.s. is \( \pi \), and let \( \Delta \) be the set of strategy profiles (i.e., \( \sigma \in \Delta \)).

We use the following standard notation: \( \times \) is the Cartesian product, \( |A| \) is the cardinality of set \( A \), \( 1_A \) is the indicator function that is equal to one or zero, respectively, if statement \( A \) is true or false, and the support of \( \sigma_i(\pi) \) is \( \text{supp} \sigma_i(\pi) \), the set of all \( S \) such that \( \sigma_i(\pi)(S) > 0 \).

The necessary part of the following lemma, proved in the Appendix, follows directly from the above discussion and the definition of MPE.
Lemma 1 A payoff structure $\phi_i(\pi)$ and a strategy profile $\sigma_i(\pi)$ is an MPE of the coalitional bargaining game if and only if the following system of equations is satisfied, where $x_i(\pi) = \delta \phi_i(\pi) + (1 - \delta) v_i(\pi)$:

1) the support of the strategy $\sigma_i(\pi)$ is

$$\text{supp} \sigma_i(\pi) \subset \arg \max_{S \ni i, S \subset \pi} \{\theta(\pi)(S)(x)\},$$

2) the expected equilibrium outcome of player $i$ conditional on player $j$ being chosen to be the proposer $\phi_i^j(\pi)$ is equal to

$$\phi_i^j(\pi) = \begin{cases} \max_{S \ni i, S \subset \pi} \{\theta(\pi)(S)(x)\} + x_i(\pi) & j = i \\ \sum_{S \ni \pi} \sigma_j(\pi)(S)(\mathbb{I}_{i \in S} x_i(\pi) + \mathbb{I}_{i \notin S} x_i(\pi S)) & j \neq i \end{cases},$$

3) the following system of equations holds

$$\phi_i(\pi) = \left(\sum_{j \in \pi} p_j(\pi) \phi_i^j(\pi)\right),$$

for all $\pi \in \Pi$, and $i, j \in \pi$.

There is a one-to-one relation between $\phi_i(\pi)$ and $x_i(\pi)$ given by equation (1). For convenience we will be solving for the vectors $x_i(\pi)$ instead of $\phi_i(\pi)$ from now on. The vector $x$, as well as the partition function form $v$, belongs to the Euclidean space $R^d$, where the dimension $d = \sum_{\pi \in \Pi} |\pi|$.

Our next result shows that there always exist MPE solutions for all coalitional bargaining games.

Proposition 1 There exist Markov perfect equilibrium for all coalitional bargaining games.

We define in the Appendix a correspondence $\mathcal{F} : X \to R^d$ such that the fixed points of $\mathcal{F}$ are the MPE payoffs of the game. The proposition follows from the Kakutani fixed point theorem because we show that $\mathcal{F}(X) \subset X$, $\mathcal{F}(x)$ is convex and non-empty, and $\mathcal{F}$ is an upper hemicontinuous correspondence.
4 Computing the Equilibria

The problem of finding equilibria can be restated as the solution of a certain mixed nonlinear complementarity problem (see Harker and Pang (1990) for a survey about complementarity problems). Mixed nonlinear complementarity problems (MNCP) are the subject of numerous studies in the mathematical programming literature (see Harker and Pang (1990)), and a large number of algorithms have been proposed for solving MNCP problems.

We now introduce the MNCP problem associated with coalitional bargaining games. For all \( x = (x_i(\pi))_{\pi \in \Pi} \), \( e = (e_i(\pi))_{\pi \in \Pi} \), and \( \sigma \in \Delta \), consider the mapping \( f(x, \sigma, e) \), where coordinate \( f_i(\pi) \) is given by

\[
f_i(\pi)(x, \sigma, e) = x_i(\pi) - \delta p_i(\pi)e_i(\pi) - (1 - \delta)v_i(\pi)
\]

for all \( i \in \pi \) and \( \pi \in \Pi \). Let the maps \( h(\sigma) \) and \( g(e, x) \) be defined by

\[
h_i(\pi)(\sigma) = \sum_{S \subset \pi : i \in S} \sigma_i(\pi)(S) - 1,
\]

\[
g_i(\pi)(S)(e, x) = e_i(\pi) - \left( x_S(\pi S) - \sum_{j \in S} x_j(\pi) \right),
\]

for all \( i, \pi, S \) satisfying \( \pi \in \Pi, i \in \pi, \) and \( i \in S \subset \pi \).

The mixed nonlinear complementarity problem is the problem of finding triples \( (x, \sigma, e) \) that satisfy all conditions

\[
f(x, \sigma, e) = 0,
\]

\[
h(\sigma) = 0,
\]

\[
g(e, x) \geq 0,
\]

\[
\sigma \geq 0 \text{ and } \sigma^T g(e, x) = 0,
\]

where \( \sigma^T g(e, x) = 0 \) is equivalent to \( \sigma_i(\pi)(S).g_i(\pi)(S)(e, x) = 0 \) for all \( i, \pi, S \) satisfying \( \pi \in \Pi, i \in \pi, \) and \( i \in S \subset \pi \).
Proposition 2  If \((x, \sigma)\) is an MPE then \((x, \sigma, e)\) is a solution of the problem MNCP, where \(e_i(\pi) = \max_{S \ni i, S \subseteq \pi} \{\theta(\pi)(S)(x)\}\). Reciprocally, if \((x, \sigma, e)\) is a solution of the problem MNCP then \((x, \sigma)\) is an MPE.

Proof: Consider the necessary part of the proposition, and say that \((x, \sigma)\) is an MPE. Then all the conditions in items 1, 2, and 3 of Lemma 1 hold. Replacing expression (4) of \(\phi_i^j(\pi)\) into equation (5), and considering that, by definition, \(x_i(\pi) = \delta \phi_i(\pi) + (1 - \delta) v_i(\pi)\), we obtain the system of equations

\[
x_i(\pi) = \delta p_i(\pi) \max_{S \ni i, S \subseteq \pi} \{\theta(\pi)(S)(x)\} + (1 - \delta)v_i(\pi) + \\
+ \delta \left( \sum_{S \subseteq \pi} \left( \sum_{j \in \pi} p_j(\pi)\sigma_j(\pi)(S) \left( I_{[i \in S]} x_i(\pi) + I_{[i \notin S]} x_i(\pi S) \right) \right) \right) .
\]

Let \(e_i(\pi) = \max_{S \ni i, S \subseteq \pi} \{\theta(\pi)(S)(x)\}\). We then have that \((x, \sigma, e)\) satisfies the equation

\(f(x, \sigma, e) = 0\). Since \(\sigma\) is a probability distribution then \(h(\sigma) = 0\) and \(\sigma \geq 0\) are automatically satisfied. Also, by definition of \(e_i(\pi)\) above, \(e_i(\pi) - \theta(\pi)(S)(x) \geq 0\), so that \(g(e, x) \geq 0\). Finally, \(\sigma_i(\pi)(S) g_i(\pi)(S)(e, x) = 0\) follows from definition of \(e\) and the support restriction of \(\sigma\) in (3). The reciprocal follows using the same arguments. Q.E.D.

Proposition 2 is useful because there are several numerical algorithms for solving MNCP problems. These algorithms are typically based on both Newton’s method for solving a system of equations and on methods for solving linear complementarity problems (LCP), such as Lemke-Howson’s algorithm (see Cottle et al. (1992) for a comprehensive treatment of the LCP). When solving a system of nonlinear equations, starting from a given initial condition, Newton’s algorithm solves the linear approximation of the system of equations to obtain the updating direction. In the case of MNCP problems, instead of just solving for a linear system of equations, one solves a linear complementarity problem to obtain the updating direction, so as to satisfy the inequalities in the system.

The results of this section thus imply that the computation of equilibrium points can be accomplished using well-known numerical methods.
5 Coalition Dynamic Structures

Instead of focusing on the strategy profile \( \sigma \in \Delta \), it is more convenient to focus on the associated Markov transition probability \( \mu = \mu(\sigma) \), which is defined as

\[
\mu(\sigma)(\pi)(S) = \sum_{j \in \pi} p_j(\pi) \sigma_j(\pi)(S),
\]

where \( \mu(\sigma)(\pi)(S) \) represents the probability of moving from state \( \pi \) to state \( \pi S \).

The following result shows that uniqueness of strategy profiles do not hold in general and thus the best we can hope is to have uniqueness with respect to expected payoffs and the Markov transition probabilities.

**Lemma 2** If \( (x, \sigma) \) is an MPE then \( (x, \sigma') \) is an MPE for any \( \sigma' \in \Delta \), with \( \mu(\sigma) = \mu(\sigma') \) and \( \text{supp}_i'(\pi) \subset \arg \max_{S \supset i, \pi \subset S} \{ \theta(\pi)(S)(x) \} \).

**Proof:** By Proposition 2, \( (x, \sigma, e) \) solves the MNCP problem, and thus \( f(x, \sigma, e) = 0 \). Inverting the order of summation in the expression of \( f(x, \sigma, e) \) in equation 6), we get,

\[
f_i(\pi)(x, \sigma, e) = x_i(\pi) - \delta p_i(\pi)e_i(\pi) - (1 - \delta)v_i(\pi)
\]

\[
- \delta \left( \sum_{S \subset \pi} \mu(\sigma)(\pi)(S) \left( \mathbb{I}_{i \in S} x_i(\pi) + \mathbb{I}_{i \notin S} x_i(\pi S) \right) \right)
\]

where \( \mu(\sigma) \) is as in equation (9). Therefore, for any \( \sigma' \) as above, \( f(x, \sigma', e) = 0 \), and by the reciprocal of Proposition 2 this implies that \( (x, \sigma') \) is an MPE. Q.E.D.

An important difficulty in the analysis is that the equilibrium is not locally unique in the equilibrium strategies \( \sigma \). There exist a continuum of equilibrium strategies whenever there is one equilibrium in which coalition \( S \) is the best response choice of more than one player, say \( i \) and \( j \): a continuum of strategies can be constructed by having player \( i \) reducing the weight it chooses \( S \), with player \( j \) increasing the weight on \( S \), so that the overall transition probability \( \mu \) is unchanged.

Our efforts from now on are concentrated on proving uniqueness in terms of the pair \( (x, \mu) \). The strategy profile \( \sigma \) belongs to a space with higher dimensionality
than the transition probabilities $\mu$, and when passing from $\sigma$ to $\mu$ some important information is lost. For example, suppose that $\mu(\pi)(S) > 0$ for some $S \subset \pi$. Who are the players that choose coalition $S$ with positive probability? It is certainly possible that the best response strategy for player $j \in S$ is to choose coalition $S$, but that another player $i \in S$ is strictly better off choosing a different coalition.

We now introduce the concept of coalitional dynamic structure (CDS) which allow us to recover all the essential information about the strategy profile $\sigma$, that is not recorded in the transition probabilities $\mu$.

Start by defining an equivalence relation on the set $\pi$ induced by the strategy profile $\sigma$. Given any two players $i, j \in \pi$, say that $i \rightarrow j$ if and only if there exists a coalition $S \subset \pi$ with $i, j \in S$ such that $\sigma_i(\pi)(S) > 0$. Also, say that there is a path from $i$ to $j$ if there exists a sequence of players $i_1, \ldots, i_k$ belonging to $\pi$ such that $i \rightarrow i_1 \rightarrow \ldots \rightarrow i_k \rightarrow j$. Finally, we say that $i$ and $j$ is strongly connected, $i \leftrightarrow j$, if there is a path from $i$ to $j$ and a path from $j$ to $i$. It is straightforward to verify that strong connection is an equivalence relation (transitivity, symmetry, and reflexivity hold). Let $P_r(\pi)$ be the equivalence classes of this relation (the maximal strongly connected components). Also, let $C_r(\pi) = \bigcup_{i \in P_r(\pi)} \text{supp} \sigma_i(\pi)$ be the union of the offers in the support of the strategy profile of players in $P_r(\pi)$.

The coalitional dynamic structure associated with $\sigma$ is a partition of the set of players and a partition of $\text{supp} \sigma$ into the equivalence classes induced by the strong connection relation.

**Definition 1** The coalitional dynamic structure (CDS) associated with $\sigma$ is $C(\sigma) = (C(\pi), P(\pi))_{\pi \in \Pi}$ where, for each c.s. $\pi$:

i) $P(\pi) = (P_1(\pi), \ldots, P_q(\pi))$ is a partition of $\pi$ and $P_r(\pi)$ are the equivalence classes of the strong connection relation;

ii) $C(\pi) = (C_1(\pi), \ldots, C_q(\pi))$ is a partition of $\text{supp} \sigma(\pi) = \bigcup_{i \in \pi} \text{supp} \sigma_i(\pi)$ and $C_r(\pi) = \bigcup_{i \in P_r(\pi)} \text{supp} \sigma_i(\pi)$.

The set of coalitional dynamic structures is $\text{CDS} = \{C(\sigma) : \sigma \in \Delta\}$.

We are interested in analyzing the problem of finding an equilibrium point $(x, \sigma)$ with a given CDS $C$. By definition of CDS, this implies that the excesses of all

\[3\text{Note that } C_r(\pi) \cap C_{r'}(\pi) = \emptyset \text{ if } r \neq r'. \text{ Otherwise, there exist } i \in P_r(\pi), j \in P_{r'}(\pi), \text{ and } S \in C_r(\pi) \cap C_{r'}(\pi) \text{ such that } \sigma_i(\pi)(S) > 0 \text{ and } \sigma_j(\pi)(S) > 0. \text{ But this implies } i \leftrightarrow j \text{ (contradiction).} \]
coalitions belonging to the same equivalence class \( C_r(\pi) \) are equal, that is,
\[
e_r(\pi) = x_S(\pi S) - \sum_{i \in S} x_i(\pi),
\]
for all \( S \in C_r(\pi) \) and \( r = 1, \ldots, q(\pi) \). In addition, the associated Markov transition probability \( \mu = \mu(\sigma) \) satisfies
\[
\sum_{S \in C_r(\pi)} \mu(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi),
\]
because \( \text{supp}_j(\pi) \subset C_r(\pi) \) for all \( j \in P_r(\pi) \).

Therefore, if \( (x, \sigma) \) is an equilibrium then \( (x, \mu, e) \) solves the following systems of equations, or problem \( F(\mathcal{C}) \) (or \( F_C \))
\[
F_C(x, \mu, e) = \begin{pmatrix}
f_C(x, \mu, e) \\
E_C(x, e) \\
M_C(\mu)
\end{pmatrix} = 0
\]
where the maps \( f_C(x, \mu, e), E_C(x, e), \) and \( M_C(\mu) \) associated with \( CDS \mathcal{C} \) are defined by
\[
(f_C)_i(\pi)(x, \mu, e) = x_i(\pi) - \delta p_i(\pi) e_r(\pi) - (1 - \delta) v_i(\pi)
- \delta \left( \sum_S \mu(\pi)(S) \left( \mathbb{1}_{[i \in S]} x_i(\pi) + \mathbb{1}_{[i \notin S]} x_i(\pi S) \right) \right),
\]
\[
E_C(\pi)(S)(x, e) = \sum_{i \in S} x_i(\pi) + e_r(\pi) - x_S(\pi S),
\]
\[
M_C(\pi)(r)(\mu) = \sum_{j \in P_r(\pi)} p_j(\pi) - \sum_{S \in C_r(\pi)} \mu(\pi)(S),
\]
for all \( r, i, \) and \( S \) satisfying \( r = 1, \ldots, q(\pi), i \in P_r(\pi), S \in C_r(\pi), \) and all \( \pi \in \Pi \).

The reciprocal result also holds if we impose some additional restrictions on the solutions of \( F(\mathcal{C}) \). Any set of payoffs \( x \) that are candidates for equilibrium with an associated \( CDS \mathcal{C} \) must satisfy
\[
\theta(\pi)(S)(x) \geq \theta(\pi)(T)(x) \quad \text{for all } S \in C_r(\pi) \quad \text{and } T \notin C_r(\pi) \quad \text{with } T \cap P_r(\pi) \neq \emptyset,
\]
for all \( r, \pi \).
because of equalities (10) and inequalities (3). Thus, the set of payoffs $E_C$ consistent with $C$ is

$$E_C = \{ x \in \mathbb{R}^d : \text{ such that all inequalities (13) hold} \}.$$ 

Moreover, any transition probability $\mu$ that is consistent with a CDS $C$ satisfies $\mu = \mu(\sigma)$ where $\sigma$ is a strategy profile with a CDS $C$ (i.e., $C(\sigma) = C$). Thus, the set of transition probabilities $M_C$ consistent with $C$ is

$$M_C = \{ \mu = \mu(\sigma) : \text{ where } \sigma \in \Delta \text{ and } C(\sigma) = C \}. \quad (14)$$

The following lemma, proved in the Appendix, provides yet another useful characterization of MPE.

**Lemma 3** If $(x, \sigma)$ is an MPE of the bargaining game then $(x, \mu, e)$ is a solution of problem $F(C(\sigma))$, where $\mu = \mu(\sigma) \in M_C$, $x \in E_C$, and $e_r(\pi) = \theta(\pi)(S)(x)$ for any $S \in C_r(\pi)$. Reciprocally, if $(x, \mu, e)$ is a solution of problem $F(C)$ satisfying $\mu \in M_C$ and $x \in E_C$ then there exists an MPE $(x, \sigma)$ of the bargaining game with $\mu = \mu(\sigma)$ and $C = C(\sigma)$.

This result allows us to transform the problem of finding equilibria into a lower-dimensional equivalent problem of finding solutions of the system of equations $F(C)$.

### 6  Generic Local Uniqueness and Stability

In this section we show that almost all games, except in a set of measure zero of the parameter space $(v, p)$, have equilibria that are locally unique and stable. These results imply that almost all games have only a finite number of equilibria, and provide tools for performing comparative statics analysis in coalitional bargaining games.

#### 6.1  Regular and Nondegenerate Games

We seek to determine in this section conditions under which the equilibrium outcome is locally unique and stable. Note that global uniqueness does not hold in general,

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4Observe that if the strategy profile $\sigma$ is such that for all $r$ and $i \in P_r(\pi)$, $\text{supp}(\sigma_i(\pi)) = C_r(\pi) \cap \{ S \subset \pi : i \in S \}$ then $C(\sigma) = C$. 

---

12
as Example 3 illustrates. The next best property is local uniqueness. An equilibrium outcome is locally unique if we cannot obtain another equilibrium outcome arbitrarily close to it. Stability is a property that ensures that comparative statics exercises are well defined. Roughly speaking an equilibrium point is stable if it changes smoothly for any small changes of the parameters of the game.

According to the previous section the equilibrium outcome are solutions of $F_C(z) = 0$, where $z = (x, \mu, e)$. In order to obtain (local) uniqueness it is necessary that the problem has the same number of equations and of unknowns. Indeed, there are $d = \sum_{\pi \in \Pi} |\pi|$ equations $f_C$, $m = \sum_{\pi \in \Pi} \sum_{r=1}^{q(\pi)} m_r(\pi)$, where $m_r(\pi) = |C_r(\pi)|$, equations $E_C$, and $q = \sum_{\pi \in \Pi} q(\pi)$ equations $M_C$ (a total of $d + m + q$ equations). Moreover, the unknowns are the $d$ dimensional variable $x$, the $m$ dimensional variable $\mu$, and the $q$ dimensional variable $e$ (a total of $d + m + q$ unknowns). So the number of equations and unknowns coincide.

We now introduce the concepts of regularity and nondegeneracy. Games that satisfy both of these technical conditions are shown to have equilibrium outcome that are locally unique and stable.

**Definition 2 (Regular game)** A solution $z$ of problem $F_C(z) = 0$ is regular if the Jacobian $d_z F_C$ is nonsingular. A CDS $C$ is regular if all the solutions of problem $F_C$ are regular. Finally, a game is regular if all CDSs are regular.

The Jacobian $d_z F_C$ is a matrix of order $d + m + q$, and is nonsingular if and only if it has full rank (equal to $d + m + q$). It has the following special structure

$$d_z F_C = \begin{bmatrix} d_{(x,e)f_C} & d_{\mu f_C} \\ d_{(x,e)E_C} & 0 \\ 0 & d_{\mu M_C} \end{bmatrix},$$

where the matrix $d_{(x,e)} E_C$ has $m$ rows and $d + q$ columns, and matrix $d_{\mu} M_C (\mu)$ has $q$ rows and $m$ columns. This special structure of the Jacobian matrix will be explored later on to show that it is nonsingular almost everywhere.\(^5\)

---

\(^5\)By definition, if $F_C$ has no solutions then the support $C$ is regular.

\(^6\)Note that if Jacobian matrix is nonsingular then the $m \times (d + q)$-matrix $dE_C$ must have rank $m$. So, for example, support structures with more than $d + q$ coalitions in the support ($m > d + q$) are not candidates for a regular equilibrium point. Reciprocally, we show in the next section that if matrix $dE_C$ has rank $m$ then for almost all partition functions the Jacobian matrix is nonsingular.
Remark: The solution at a given c.s. $\pi$ only depends on the variables evaluated at coalition structures that are coarser than $\pi$. Thus the Jacobian matrix $d_z F_C$ can be partitioned into an upper block triangular structure with diagonal blocks equal to $d_z F_C (\pi)$ where $z (\pi) = (x (\pi), \mu (\pi), e (\pi))$ for all $\pi \in \Pi$, where all entries to the left of the diagonal blocks are zero. Therefore, the Jacobian matrix $d_z F_C$ is nonsingular if and only if all the diagonal blocks $d_z F_C (\pi)$ are nonsingular.

Consider now the nondegeneracy technical condition, which is a property of the support that roughly means that all choices outside the support are not best response strategies (Harsanyi (1973) refer to a similar property in the context of $n$-person non-cooperative games as quasi-strong property). This condition is used in the next proposition to show that nearby games have equilibrium with the same support.

**Definition 3 (Nondegenerate game)** A CDS $C$ is nondegenerate if all solutions of $F_C$ satisfy

$$
\theta (\pi) (S) (x) \neq \theta (\pi) (T) (x) \text{ for all } S \in C_r (\pi) \text{ and } T \notin C_r (\pi) \text{ with } T \cap P_r (\pi) \neq \emptyset.
$$

(16)

A game is nondegenerate if all CDSs are nondegenerate.

We now show that the implicit function theorem implies that regular and nondegenerate equilibrium points are locally unique and stable. Formally, we show that for any game $(v^*, p^*)$ and regular and nondegenerate equilibrium point $(x^*, \mu^*)$ associated with an equilibrium with support $C$, there exists an open neighborhood $B \subset R^d \times R^d$ of $(v^*, p^*)$, an open neighborhood $W \subset R^d \times R^m$ of $(x^*, \mu^*)$ and a local mapping $(x(v, p), \mu(v, p)) \in R^d \times R^m$ such that $(x(v, p), \mu(v, p))$ is the only equilibrium point in the neighborhood $W$ for all games $(v, p) \in B$.

**Proposition 3 (Local uniqueness and stability)** Regular and nondegenerate coalition bargaining games have equilibrium points that are locally unique and stable.

**Proof**: The implicit function theorem immediately implies that, for any game $(v^*, p^*)$ and regular solution $z^* = (x^*, \mu^*, e^*)$ there exists an open neighborhood $B \subset R^d \times R^d$ of $(v^*, p^*)$, an open neighborhood $\tilde{W} \subset R^d \times R^m \times R^d$ of $(x^*, \mu^*, e^*)$, and a mapping $z(v, p) = (x(v, p), \mu(v, p), e(v, p)) \in R^m \times R^m \times R^d$ such that $z(v, p)$ is the
only solution of problem $F_C$ in $\tilde{W}$ for all games $(v,p) \in B$. Note that since $e(v,p)$ can be expressed as a function of $x(v,p)$ (see equation (10)) then $z(v,p)$ is the only solution in a cylinder $W \times R^d$ for $W$ an open neighborhood of $(x^*,\mu^*)$.

It remains to show that $z(v,p)$ is indeed an equilibrium point. By Lemma 3, $z(v,p)$ is an equilibrium point if (i) $x(v,p) \in E_C$ and (ii) $\mu(v,p) \in M_C$. We show below that (i) and (ii) hold:

(i) holds: Because $C$ has full support (nondegeneracy condition), all the inequalities in (13) are strict for $x^*$ and thus, by continuity, the inequalities also hold for all $x(v,p)$ in an open neighborhood $Q$ of $x^*$, which is equivalent to $x(v,p) \in E_C$.

(ii) holds: The result follows directly from the following lemma, proved in the Appendix.

**Lemma 4** For any CDS $C$ and $\mu^* \in M_C$ there exists an open neighborhood $U$ around $\mu^*$ such that for all $\mu$ in $U$ that satisfies all equations

$$\sum_{S \in C, \pi} \mu_\pi(S) = \sum_{j \in P, \pi} p_j(\pi)$$

there exists $\sigma$ with CDS $C$ such that $\mu(\sigma) = \mu$.

Therefore, since (i) and (ii) hold in the neighborhood $W = Q \times U$, the pair $(x(v,p), \mu(v,p))$ is an equilibrium point, which completes the proof. Q.E.D.

### 6.2 Genericity of Equilibria

Regularity and nondegeneracy are here shown to be generic properties. The parameter space used to establish the result is the set of all games in $(v,p) \in R^d \times \Delta^d$, where $\Delta^d = \{ p \in R^d : p_i(\pi) \geq 0$ and $\sum_{i \in \pi} p_i(\pi) = 1$ for all $\pi \in \Pi \}$. Formally, a generic property is one that holds for all games, except possibly those in a subset of Lesbegue measure zero on $R^d \times \Delta^d$ (i.e., the property holds for almost all games). Combining with the results of the previous section, we prove that local uniqueness and stability hold for almost all games.

This genericity result is established using the well-known transversality theorem from differential calculus (see Guillemin and Pollack (1974) and Hirsch (1976) and the Appendix for a restatement of the theorem). The key result of this section is the following.

---

7We have just argued that the set of regular and nondegenerate games is an open set. Therefore the set of games that are not regular nor degenerate is a closed set.
Proposition 4 (Genericity) Almost all coalitional bargaining games \((v, p)\) in \(R^d \times \Delta^d\) are regular and nondegenerate. Therefore, almost all games are locally unique and stable.

Key for the proof is to show that, for all CDS \(C\), the Jacobian \(d_z F_C\), at any solutions of problem \(F_C(z) = 0\), is nonsingular, for almost every parameter \((v, p)\) in \(R^d \times \Delta^d\). That is all CDS are regular almost everywhere.

In order to illustrate the arguments involved to prove regularity, consider first that \(C\) is a CDS where matrix \(dE_C\) has rank \(m\). The solutions of problem \(F_C(z) = 0\) can be represented as the zeros of the augmented problem \(F_C(z, v) = 0\), where we take into account the dependency with respect to the game. The Jacobian of this mapping is

\[
d_{(z,v)}F_C = \begin{bmatrix}
* & * & -(1-\delta)I \\
\ast & 0 & 0 \\
0 & 0 & d_{x}M_{C}
\end{bmatrix},
\]

where \(d_z F_C = \begin{bmatrix}
* & * \\
\ast & 0 \\
0 & d_{x}M_{C}
\end{bmatrix}\) and \(d_v F_C = \begin{bmatrix}
-(1-\delta)I \\
0 \\
0
\end{bmatrix}\), and * denotes arbitrary coefficients. The augmented Jacobian is a surjective matrix (with rank equal to the number of rows) because all blocks \(dE_C\), \(dM_C\), and \(-(1-\delta)I\) have rank equal to the number of rows, and because of the disposition of zeros. Thus \(F_C\) is transversal to zero (i.e., \(F_C \cap 0\)) or 0 is a regular value of the augmented problem. By the transversality theorem, for almost every \(v\), \(F_C(v)\) is also transversal to zero, \(F_C(v) \cap 0\). Thus the square Jacobian matrix \(d_z F_C(v)\) is surjective at all solutions of \(F_C\), and thus nonsingular at all solutions for almost all \(v \in R^d\) and all \(p \in \Delta^d\).

Note that when the CDS \(C\) is such that matrix \(dE_C\) has rank smaller than \(m\) the Jacobian \(d_z F_C\) is singular for all parameters. However, in the Appendix, we show that the further augmented problem \(F_C(z, v, p) = 0\) is such that \(d_{(z,v,p)}F_C(z, v, p) = 0\) in surjective, and thus, by the transversality theorem, \(d_z F_C\) is nonsingular almost everywhere in the parameter space \(R^d \times \Delta^d\).

The argument to prove that almost all games are nondegenerate is as follows. Given any support \(C\) consider an hyperplane \(H\) in the space \(R^{d+m+q}\) defined by equality \((16)\), \(\theta (\pi) (S) (x) = \theta (\pi) (T) (x)\) for some pair \(S, T\) (so \(C\) is nondegenerate if there is no solution of \(F_C(z) = 0\) such that \(z \in H\)). Applying the transversality
Theorem again (see Guillemin and Pollack (1974)) it follows that, for almost no parameters, there are solutions $F_C(z) = 0$ such that $z \in H$: because the codimension of $H$ in the space $R^{d+m+q}$ is 1, the transversality theorem applied to the surjective problem $F_C(z, v, p) = 0$ restricted to the domain $H \times R^d \times \Delta^d$ implies that for almost no parameters $(v, p)$ there are no solutions of $F_C(z) = 0$ such that $z \in H$. Using the fact that a finite union (there are only a finite number of pairs $S, T$) of sets of measure zero is a set of measure zero, we conclude that there exists a set of parameters, with complement of measure zero, where $F_C$ is regular and nondegenerate.

6.3 Comparative Statics Analysis

Understanding how the value of coalitions and the path of coalition formation changes in response to changes in the exogenous parameters of the game $v$ and $p$ is a relevant comparative statics exercise. Regular and nondegenerate games are very convenient because they allow us to perform comparative statics analysis using standard calculus tools.

The following corollary is an immediate application of the implicit function theorem and Proposition 3. The sensitivity matrix $S_C$ allow us to evaluate how the equilibrium point changes $\Delta z = S_C (\Delta v, \Delta p)$ in response to local changes of the game.

**Corollary 1 (Comparative Statics)** Let $(v, p)$ be a regular and nondegenerate game and $z = (x, \mu, e)$ be an equilibrium with CDS $C$. The first-order effects of a change in the exogenous parameters $(v, p)$ on the solution $z$ is given by the sensitivity matrix $S_C = - [d_z F_C]^{-1} d_{(v, p)} F_C$ (i.e., $\Delta z = S_C (\Delta v, \Delta p)$). In particular, the effect of a local change $\Delta v$ of coalitional values are given by $\Delta z = ([d_z F_C]^{-1})_x (1 - \delta) \Delta v$, where $([d_z F_C]^{-1})_x$ denotes the submatrix with the first $d$ columns of the inverse Jacobian.

The first-order effects with respect to changes in value $\Delta v$ are given by the sensitivity matrix $- [d_z F_C]^{-1} d_v F_C$. But since

$$d_v F_C = \begin{pmatrix} d_v f_C \\ d_v E_C \\ d_v M_C \end{pmatrix} = - \begin{pmatrix} (1 - \delta) I \\ 0 \\ 0 \end{pmatrix},$$

(17)
the sensitivity matrix \(- [dxF_C]^{-1} dF_C\) simplifies to \(((dxF_C)^{-1})_{xx} (1 - \delta)\). So evaluating
the inverse of the Jacobian matrix at the solution yields the first-order effects of changes in value.

We illustrate with the next examples the comparative statics properties of quota and apex games.

### 6.4 Examples

Comparison of the equilibrium payoffs predicted by our model with established solution concepts from cooperative game theory, such as the nucleolus, bargaining set, kernel, core, and Shapley value, shows that our predictions are different than all other cooperative solution concepts.

**Example 1: Quota Games**

Quota games have been studied by Shapley (1953) and Maschler (1992). Consider a four-player quota game, where each pairwise coalition gets \(v_{ij} = \omega_i + \omega_j\) for all distinct pairs \(i, j \in N\), where the quotas of the four players are \((\omega_1, \omega_2, \omega_3, \omega_4) = (10, 20, 30, 40)\), and all remaining coalitions get \(v_S = 0\) for all \(S \subset N, S \neq \{i, j\}\) (to simplify notation we omit the c.s. \(\pi\) in \(v_i(\pi)\)). Players are very patient (i.e., we are interested in the limit when \(\delta\) converges to 1), and they all have an equal chance to be proposers.

The equilibrium point and the transition probabilities are depicted in Figure 1 (in order to simplify the notation, in the figure the c.s. \(\{i, j\}, \{k, l\}\) is denoted by \((ij)(kl)\); the numbers below the c.s. in parenthesis are the corresponding equilibrium values \(\phi\) of each coalition; and the percentages above the arrows are the transition probabilities \(\mu(S)\)). The CDS at the initial state is \(C = (\{\{2, 3\}, \{2, 4\}, \{3, 4\}\}, \{\{1\}\})\), and the excesses are \(e(S) = 4.938\) for \(S = \{2, 3\}, \{2, 4\}, \{3, 4\}\), and \(\{3, 4\}\), and it can also be easily verified that this solution is a strong regular solution.

The solution \(\phi = (17.41, 17.53, 27.53, 37.53)\) is different from the nucleolus (Schmeidler (1969)) and the core (both of which coincide with the quota \((10, 20, 30, 40)\)), the kernel, the bargaining set (Maschler (1992)), and the Shapley value (which is equal to
In our solution, player 1 gets 7.41 more than his quota and players 2, 3 and 4 get each 2.47 less than their quota values. This example illustrates that the solution proposed in the paper is different from all the other major existing solution concepts.

Interestingly, the equilibrium strategy of player 1 is to wait for a pairwise coalition to form, an strategy that allows player 1 to get significantly more than his quota. The solution thus makes predictions that are consistent with experimental results reported in Maschler (1992), where player 1 realized that he was weak and that his condition would improve if he waited until a pairwise coalition formed, and captures an important strategic element of the game. Indeed, player 1 is better off if the coalition \{2, 3\} forms, rather than \{2, 4\} or \{3, 4\}, because in the ensuing pairwise bargaining with 4, player 1 can get a payoff equal to 25.\footnote{Since the bargaining set contains the kernel, and the kernel contains the nucleolus, this implies that our solution is different from the kernel and bargaining set (see Maschler 1992).}

\footnote{However, strategies considered in this paper rule out the possibility that player 1 makes side payments to players 2 and/or 3 in order to encourage them to form coalition \{2, 3\}.}
How do players’ value change with changes in quotas and proposers’ probabilities? Evaluating the value-sensitivity matrix with respect to changes in quotas, as we have seen in Section 6.3, yields

\[
\begin{bmatrix}
\Delta \phi_1 \\
\Delta \phi_2 \\
\Delta \phi_3 \\
\Delta \phi_4
\end{bmatrix}
= \begin{bmatrix}
0.366 & 0.549 & 0.062 & 0.022 \\
0.211 & 0.816 & -0.020 & -0.007 \\
0.211 & -0.183 & 0.979 & -0.007 \\
0.211 & -0.183 & -0.020 & 0.992
\end{bmatrix}
\begin{bmatrix}
\Delta \omega_1 \\
\Delta \omega_2 \\
\Delta \omega_3 \\
\Delta \omega_4
\end{bmatrix},
\]

and the coalition formation sensitivity matrix satisfies \(\frac{\partial \mu((i,j))}{\partial \omega_i} > 0\) and \(\frac{\partial \mu((j,k))}{\partial \omega_i} < 0\) for all distinct \(i, j,\) and \(k\) in \(\{2,3,4\}\) (for the sake of space we report only the signs of the transition probabilities).

The information contained in the value sensitivity matrix yields the following results: the value of all players increases when their quotas increase, but increases in the quota of player 1 are shared by all players, while increases in the quotas of either player 2, 3 or 4 are almost completely appropriated by them (in fact, the other two players distinct from player 1 suffer a loss). Moreover, when a player’s quota goes up, all coalitions including this player become more likely to form (and coalitions not including this player are less likely to form).

The comparative statics with respect to changes in proposers’ probability is described by the value-sensitivity matrix

\[
\begin{bmatrix}
\Delta \phi_1 \\
\Delta \phi_2 \\
\Delta \phi_3 \\
\Delta \phi_4
\end{bmatrix}
= \begin{bmatrix}
0 & -5.42 & 1.96 & 3.45 \\
0 & 1.80 & -0.65 & -1.15 \\
0 & 1.80 & -0.65 & -1.15 \\
0 & 1.80 & -0.65 & -1.15
\end{bmatrix}
\begin{bmatrix}
\Delta p'_1 \\
\Delta p'_2 \\
\Delta p'_3 \\
\Delta p'_4
\end{bmatrix},
\]

where, in order to preserve the sum of probabilities equal to one, we consider \(p_i = p'_i / (\sum_{j=1}^{4} p'_j)\), and the coalition formation sensitivity matrix satisfies \(\frac{\partial \mu((i,j))}{\partial p'_i} < 0\) and \(\frac{\partial \mu((j,k))}{\partial p'_i} > 0\) for all distinct \(i, j,\) and \(k\) in \(\{2,3,4\}\).

This comparative statics analysis reveals a surprising result: When player 2 has more initiative to propose, he benefits and player 1 loses from it. Interestingly, though, the opposite happens when players 3 and 4 have more initiative. Their equilibrium payoffs decrease when they have more initiative to propose!\(^{10}\)
Example 2: Apex Games

Apex games, introduced by Davis and Maschler (1965), are another interesting class of $n$-person games that have received considerable attention. In this game, only two types of coalitions create non-zero value: any coalition with the Apex player (player 1), or the coalition with all the $n-1$ remaining players (the Base players). For concreteness, consider the 5-player game $N = \{1, 2, 3, 4, 5\}$, where $v_{\{1,j\}} = 100$ for $j = 2, \ldots, 5$, $v_{\{2,3,4,5\}} = 100$, and $v_{S} = 0$ otherwise. Players are very patient ($\delta$ is infinitesimally close to 1), and all players have equal chance to be proposers.

The solution is depicted in Figure 2 (we use the same notation as in Figure 1). The CDS at the initial state is $C = (\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3,4,5\})$, and the excesses are $e(S) = \frac{300}{7}$ for $S \in C$. 

likely to form and coalition $\{3,4\}$ more likely; since player 1’s gains are lowest when coalition $\{3,4\}$ forms he indirectly suffers when $p_2$ increases. By similar reasoning, when $p_4$ increases, coalitions $\{2,4\}$ and $\{3,4\}$ are less likely to form and coalition $\{2,3\}$ more likely, which benefits player 1 and hurts the other players.
The solution for the game is $\phi = (42.9, 14.3, 14.3, 14.3, 14.3)$. This solution coincides with the kernel of the game and the nucleolus. However, it is different from the bargaining set, the core (which is empty), and the Shapley value (which is equal to $(60, 10, 10, 10, 10)$).

Moreover, the model also predicts that any of the four apex coalitions \{1, j\}, $j = 2, \ldots, 5$, form with 20% probability, and the base coalition \{2, 3, 4, 5\} forms with 20% probability.

Comparative statics results for the apex game can also be easily obtained. The sensitivity matrix describing the changes in value is

$$
\begin{bmatrix}
\Delta \phi_1 \\
\Delta \phi_2 \\
\Delta \phi_3 \\
\Delta \phi_4 \\
\Delta \phi_5
\end{bmatrix}
= 
\begin{bmatrix}
0.23 & -0.48 & 0 & 0 \\
0.74 & 0.17 & 0 & 0 \\
-0.26 & 0.17 & 0 & 0 \\
-0.26 & 0.17 & 0 & 0 \\
-0.26 & 0.17 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta v_{\{1,2\}} \\
\Delta v_{\{2,3,4,5\}} \\
p'_{1} \\
p'_{2}
\end{bmatrix},
$$

where $p_i = p'_i / (\sum_{j=1}^{5} p'_j)$, and the sensitivity matrix describing the changes in coalition formation is

$$
\begin{bmatrix}
\Delta \mu(\{1, 2\}) \\
\Delta \mu(\{1, 3\}) \\
\Delta \mu(\{1, 4\}) \\
\Delta \mu(\{1, 5\}) \\
\Delta \mu(\{2, 3, 4, 5\})
\end{bmatrix}
= 
\begin{bmatrix}
0.0309 & -0.0009 & -0.2 & -2.2 \\
-0.01 & -0.0009 & -0.2 & 0.8 \\
-0.01 & -0.0009 & -0.2 & 0.8 \\
-0.01 & -0.0009 & -0.2 & 0.8 \\
-0.0009 & 0.0036 & 0.8 & -0.2
\end{bmatrix}
\begin{bmatrix}
\Delta v_{\{1,2\}} \\
\Delta v_{\{2,3,4,5\}} \\
p'_{1} \\
p'_{2}
\end{bmatrix}.
$$

Surprisingly, these results indicate that changes in proposer probabilities have no effect on the players’ values. Also, as was the case with the previous example, whenever the value of a coalition increases then both the equilibrium value of all coalitional members and the probability that this coalition forms increase as well.

7 **Uniqueness and the Global Number of Equilibria**

We show in this section that even though there can be multiple equilibrium points, as illustrated by the next example, almost all games have a finite and odd number
of MPE equilibria. Moreover, we derive a sufficient condition for the global uniqueness of equilibria. The result states that if the index of each equilibrium solution is non-negative, where the index is equal to the sign of the determinant of the Jacobian matrix $d_zF_C$, then there is a globally unique equilibrium. We prove that the sufficient condition holds for three-player superadditive games, and thus there is only one equilibrium in this class of games.

Let us start by showing that coalitional bargaining games may have multiple equilibrium points.

**Example 3: War of Attrition (Multiple Equilibria)**

The following three-player symmetric example have seven MPE solutions. The partition function that describes this game is $v_i(\{\{1\}, \{2\}, \{3\}\}) = 0$, $v_{ij}(\{\{i\}, \{j\}, \{k\}\}) = 1$, and $v_{jk}(\{\{i\}, \{j\}, \{k\}\}) = 3$. The three-player coalition are not allowed (or has a very low value) and we assume that proposers are chosen with equal probabilities and $\delta \in (0.5, 1)$. We describe below all the equilibria (results are derived solving equations (3), but we omit the details).

There is an equilibrium in which the expected equilibrium value is $x = (0.5, 0.5, 0.5)$; the transition probabilities are $\mu(\{i,j\}) = \frac{1-\delta}{3\delta}$, for all pairs $\{i,j\}$, and $\mu(\emptyset) = \frac{8\delta-3}{5\delta}$, where $\emptyset$ represents no proposal (or remaining at the initial state). In this equilibrium, each of the three players refrains from proposing with high probability, and only proposes with a small probability to the other two players. They all reject any proposals below 0.5, and thus players are indifferent between proposing or not.

There are three other equilibria (they are all symmetric so we just focus on one of them), in which the expected equilibrium values are $x = (0, 1, 1)$; the transition probabilities are $\mu(\{1,2\}) = \mu(\{1,3\}) = \frac{1-\delta}{2\delta}$, and $\mu(\emptyset) = \frac{2\delta-1}{\delta}$. In this equilibrium, players 2 and 3 reject any proposals lower than 1, make no proposal with high probability, and, when proposing, choose to form a coalition with player 1. Player 1 cannot afford to pay more than 1 to form a coalition and thus it makes no proposals with probability one.

Finally, there are three additional equilibria (they are also symmetric), in which the expected equilibrium values are $x = (\frac{6\delta}{3-\delta}, \frac{\delta}{3-\delta}, \frac{\delta}{3-\delta})$, which converges to $x = (3, 0.5, 0.5)$ when $\delta \to 1$; the transition probabilities are $\mu(\emptyset) = \frac{1}{3}$, $\mu(\{2,3\}) = \frac{2}{3}$.  

23
In this equilibrium, player 1’s strategy is to refrain from proposing and reject any proposal worth less than 3, and player 2’s and player 3’s strategies are to always “give in” and propose to form the coalition \{2, 3\}. ■

The main result of this section establishes a formula for counting the number of equilibrium points; the formula is based on the Index theorem (see, for example, Mas-Colell et al. (1995)). The index is a number that is assigned to each equilibrium point of a regular game. Say that \( z \) is an equilibrium point with support \( C \); the index of \( z \) is defined as the sign of the determinant of the Jacobian \( d_z F_C \) evaluated at \( z \) (and is either +1 or −1). We denote \( \text{index } z = \text{sign } \det (d_z F_C) \).

The result is useful because it implies that there is an odd number of equilibria. In particular, the number of equilibrium points is not zero, so there exists at least one equilibrium. We will see next that the result can be used to obtain global uniqueness for specific classes of games.

While the Index theorem has been applied to establish similar results for competitive economies (Debreu (1970)) and normal form games (Wilson (1971) and Harsanyi (1973)), the application to coalitional bargaining games is a bit more involved due to the fact that the equilibrium is not locally unique in terms of the equilibrium strategies, but only in terms of the equilibrium points (see Section 5). To prove the result we use a stronger version of the Index theorem for correspondences developed in McLennan (1989)-see the Appendix for the restatement of the Lefschetz fixed point theorem (LFPT) which is used in the proof. On the other hand, for both competitive economies and normal form games a standard version of the Index theorem developed in differential calculus textbooks suffices to develop the formula for the number of equilibria.

**Proposition 5** Almost all games (all regular and nondegenerate games) have a finite and odd number of equilibrium points. Moreover,

\[
\sum_C \sum_{z \in \text{MPE}_C} \text{sign } \det (d_z F_C) = +1,
\]

where the summation is over all CDS \( C \) and MPEs with equilibrium points \( z \) and CDS \( C \).

24
Proof: Consider the correspondence $F : X \rightarrow X$ where $F(x) \subset R^d$ is defined by

$$ y \in R^d : \quad y_i = \delta \left( \frac{1}{\sum_{j \in \pi} v_j} \cdot \max \{ \theta(\pi)(S)(x) \} \right) \left( \sum_{j \in \pi} p_j(\pi) \sigma_{(S)(x)}(x) I_j \right), $$

and $\text{supp} (\sigma_{(\pi)}) \subset \arg \max \{ \theta(\pi)(S)(x) \}$.

The set $X$ is the convex and compact set $X \subset R^d$ defined by $X = \times_\pi X(\pi)$, where

$$ X(\pi) = \{ x(\pi) \in R^{|\pi|} \text{ such that } \sum_{i \in \pi} x_i(\pi) \leq \bar{v} \text{ and } x_i(\pi) \geq v_i \}, $$

and $v_i = \min_{\pi \in \pi} \{ v_i(\pi) \}$ and $\bar{v} = \max_{\pi \in \Pi} \{ \sum_{i \in \pi} v_i(\pi) \}$. By the definition of $X$ and $F$ it is easy to verify that indeed $F(X) \subset X$.

The set of fixed points of $F$, $F^* = \{ x \in X : x \in F(x) \}$, corresponds to the equilibrium points of the game (see Section 3). The set $F^*$ is finite for all regular and nondegenerate games $v$: all the equilibrium points are, by Lemma 3, solutions of $F_C(x, \mu, e) = 0$ for some some support $C$. But since the game is regular the solutions are locally isolated (Proposition 3), and since the solution belongs to the compact $X$ then there is only a finite number of solutions.

Moreover, it can be easily shown that $F : X \rightarrow X$ is an upper hemicontinuous convex-valued correspondence (thus $F(x)$ is contractible for all $x \in X$). The set $X \subset R^d$, Cartesian product of simplexes, is a simplicial complex and thus $F$ satisfies the conditions of the LFPT.

Let $U_{x^*}$ be an open neighborhood around each $x^* \in F^*$, so that $x^*$ is the only fixed point in $U_{x^*}$. The Additivity Axiom of the Lefschetz index implies

$$ \Lambda(F, X) = \sum_{x^* \in F^*} \Lambda(F, U_{x^*}). $$

In addition, the Lefschetz index is

$$ \Lambda(F, X) = 1, $$

25
because \( \mathcal{F} \) can be approximated by a continuous map \( f : X \to X \) such that 
\[ \Lambda(\mathcal{F}, X) = \Lambda(f', X) \] (Continuity Axiom), and \( X \) is a contractible set, and thus there is an homotopy \( \varphi : X \times [0, 1] \to X \) where \( \varphi_1 = I_X \) and \( \varphi_0 = z_0 \in X \). Therefore, any continuous map \( f' : X \to X \) is homotopic to the constant map \( \text{so, by the Weak Normalization and Homotopy Axioms, } \Lambda(\mathcal{F}, X) = \Lambda(f', X) = 1. \) Equations (19) and (20) imply,

\[ \Lambda(\mathcal{F}, X) = \sum_{x^* \in \mathcal{F}^*} \Lambda(\mathcal{F}, U_{x^*}) = 1. \]

The next lemma, proved in the Appendix, establishes a formula for computing the Lefschetz index \( \Lambda(\mathcal{F}, U_{x^*}) \), which shows that it is equal to \( \text{sign} \det(d_{x^*}F_C) \) where \( z^* = (x^*, \mu^*, e^*) \) is the equilibrium point.

**Lemma 5** The Lefschetz index of a regular and nondegenerate equilibrium point \( z^* = (x^*, \mu^*, e^*) \) is equal to \( \Lambda(\mathcal{F}, U_{x^*}) = \text{sign} \det(d_{x^*}F_C) \), and is equal to either \(+1\) or \(-1\).

Q.E.D.

Proposition 5 implies a sufficient condition for global uniqueness of equilibria.

**Corollary 2** All regular and nondegenerate coalitional bargaining games have a globally unique equilibrium if \( \det(d_z F_C) \geq 0 \) where the Jacobian is evaluated at any solution of problem \( F_C \).

Because of the special structure of the Jacobian \( d_z F_C \), we conjecture that a sufficient condition for \( \det(dF_C) \geq 0 \) at all solutions of problem \( F_C \) is that the inequalities \( x_i(\pi) - x_i(\pi S) \geq 0 \), for all \( S \in C_r(\pi) \) and \( i \not\in S \), hold. Direct computation of determinants reveals, for all games with three players and all \( CDSs \) \( C \), that the inequalities imply that \( \det(dF_C) \geq 0 \) (see Proposition 6).

The inequality \( x_i(\pi) - x_i(\pi S) \geq 0 \), where \( i \not\in S \), is a weak condition that has a natural economic interpretation: player \( i \) is excluded from the offer \( S \) if and only if moving from c.s. \( \pi \) to \( \pi S \) imposes a negative externality on player \( i \) (i.e., \( x_i(\pi S) \leq x_i(\pi) \)).

We show in the next proposition that these inequalities hold for all three-player games where the grand coalition is efficient. Therefore, the equilibrium point computed explicitly in Gomes (2004) is the unique equilibrium point for almost all games.
Proposition 6 Almost all three-player games with externalities where the grand coalition is efficient, i.e. \( v(\{N\}) \geq \sum_{S \in \pi} v_S(\pi) \) for all \( \pi \), in particular superadditive games, have a globally unique equilibrium.

We remark that in the war of attrition Example 3, a strongly regular game with seven equilibria, the grand coalition was not efficient, and therefore it is not in contradiction with Proposition 6.

8 Conclusion

This paper studied the equilibrium properties of \( n \)-player coalitional bargaining games in an environment with widespread externalities (where the exogenous parameters are expressed in a partition function form). The coalitional bargaining problem is modeled as a dynamic non-cooperative game in which contracts forming coalitions may be renegotiated. The equilibrium concept used is Markov perfect equilibrium, where the set of states is all possible coalition structures.

A comprehensive analysis of the equilibrium properties is developed. We show that for almost all games (except in a closed set of measure zero) the equilibrium is locally unique and stable to small perturbations of the exogenous parameters, and the number of equilibria is finite and odd. Global uniqueness does not hold in general, but a sufficient condition for global uniqueness is derived, and this sufficient condition is shown to prevail in three-player superadditive games.

Comparative statics analysis can be easily performed using standard calculus tools, allowing us to understand how the value of players and the path of coalition formation changes in response to changes in the exogenous parameters. Being able to answer comparative statics questions is valuable to negotiators, because they may be able, for example, to invest in changing the likelihood of being proposers. Applications of the technique are illustrated using the apex and quota games, and some interesting insights emerge: surprisingly, a player may not benefit from having more initiative to propose (other players may adjust their strategies in such a way that lead the proposer to be worse off). The analysis also suggests several interesting regularities: when the exogenous value of a coalition increases, both the equilibrium value of the coalitional members and the likelihood that the coalition forms increase.
Appendix

**Proof of Lemma 1:** The necessary part follows directly from the discussion before the statement of the result and the definition of MPE solution. Let us prove the sufficient part of the proposition. Suppose that we are given payoffs and strategy profiles \((\phi, \phi', \sigma)\) satisfying all the conditions of the lemma. We use the one-stage deviation principle for infinite-horizon games. This result states that in any infinite-horizon game with observed actions that is continuous at infinity, a strategy profile \(\sigma\) is subgame perfect if and only if there is no player \(i\) and strategy \(\sigma_i'\) that agrees with \(\sigma_i\) except at a single stage \(t\) of the game and history \(h^t\), such that \(\sigma_i'\) is a better response to \(\sigma_{-i}\) than \(\sigma_i\) conditional on history \(h^t\) being reached (see Fudenberg and Tirole (1991)).

Note first that the coalitional bargaining game is continuous at infinity: for each player \(i\) his utility function is such that, for any two histories \(h\) and \(h'\) such that the restrictions of the histories to the first \(t\) periods coincides, then the payoff of player \(i\), \(|u_i(h) - u_i(h')|\), converges to zero as \(t\) converge to infinity. It is immediately clear that the negotiation game is continuous at infinity because \(|u_i(h) - u_i(h')| \leq M (\delta^{t+1} + \delta^{t+2} + \cdots) = \frac{M}{1-\delta} \delta^{t+1}\), for \(M\) large enough.

But the strategy profile \(\sigma_i\) is such that, by construction, no single deviation \(\sigma_i'\) at both the proposal and response stage can lead to a better response than \(\sigma_i\). Therefore, by the one-stage deviation principle, the stationary strategy profile \(\sigma\) is a subgame perfect Nash equilibrium. Q.E.D.

**Proof of Proposition 1:** Any payoff \(x\) candidate for equilibria must satisfy some obvious restrictions. For example, the payoff \(x\) satisfies \(x_i(\pi) \geq v_i\), where the lower bound is \(v_i = \min_{\pi \ni i} \{v_i(\pi)\}\), because player \(i\) can get at least \(v_i\) by refusing to participate in any coalitions. It must also be the case, due to a feasibility constraint, that \(\sum_{i \in \pi} x_i(\pi) \leq \overline{v}\), where the upper bound is \(\overline{v} = \max_{\pi \in \Pi} \{\sum_{i \in \pi} v_i(\pi)\}\). Therefore any MPE payoff \(x\) must belong to the convex and compact set \(X \subset R^d\) defined by \(X = \times_{\pi \in \Pi} X(\pi)\), where

\[
X(\pi) = \{x(\pi) \in R^{|\pi|} \text{ such that } \sum_{i \in \pi} x_i(\pi) \leq \overline{v} \text{ and } x_i(\pi) \geq v_i\}.
\]
Consider the mapping \( f(x, \sigma, e) \) into \( R^d \), where for all \( i \in \pi \) and \( \pi \in \Pi \), the coordinate \( f_i(\pi) \) is given by

\[
f_i(\pi)(x, \sigma, e) = x_i(\pi) - \delta p_i(\pi)e_i(\pi) - (1 - \delta)v_i(\pi)
- \delta \left( \sum_{S \subseteq \pi} \left( \sum_{j \in \pi} p_j(\pi)\sigma_j(S) \right) (\mathbb{I}_{[i \in S]}x_i(\pi) + \mathbb{I}_{[i \notin S]}x_i(\pi S)) \right),
\]

and also the correspondence \( \mathcal{F} : X \to R^d \) where \( y \in \mathcal{F}(x) \) if and only if \( y = x - f(x, e, \sigma) \) and \( \sigma \) and \( e \) are such that \( \text{supp}(\sigma_i(\pi)) \subseteq \arg \max_{S \subseteq \pi} \{\theta(\pi)(S)(x)\} \) and \( e_i(\pi) = \max_{S \subseteq \pi} \{\theta(\pi)(S)(x)\} \).

We show that the correspondence \( \mathcal{F} \) satisfies all the conditions of the Kakutani fixed point theorem.

We find convenient to use the map \( f_i^j(\pi)(x, \sigma_j) \) where

\[
f_i^j(\pi)(x, \sigma_j) = \begin{cases} 
\max_{S \ni j} \left\{ x_S(\pi S) - \sum_{j \in S \setminus i} x_j(\pi) \right\} & j = i \\
\sum_{S \ni \pi} \sigma_j(S)(I(i \in S)x_i(\pi) + I(i \notin S)x_i(\pi S)) & j \neq i
\end{cases}
\]

Note that,

\[
f_i(\pi)(x, \sigma) = \delta \left( \sum_{j \in \pi} p_j(\pi) f_i^j(\pi)(x, \sigma(j)) \right) + (1 - \delta)v_i(\pi).
\]

First, by definition, \( X \) is non-empty subset that is compact and convex. In addition,

1. \( \mathcal{F}(X) \subset X \): Take any \( x \in X \) and \( y = f(x, \sigma) \) with \( \sigma \in \Delta(x) := \{\sigma \in \Delta : x = x(\sigma)\} \). We first show that \( \sum_{i \in \pi} y_i(\pi) \leq \overline{v} \) and then \( y_i(\pi) \geq \underline{v} \), which implies that \( y \in X \). First,

\[
\sum_{i \in \pi} y_i(\pi) = \sum_{i \in \pi} f_i(\pi)(x, \sigma) = \\
= \delta \left( \sum_{j \in \pi} p_j(\pi) \sum_{i \in \pi} f_i^j(\pi)(x, \sigma(j)) \right) + (1 - \delta) \sum_{i \in \pi} v_i(\pi)
\]

where the order of the first summation has been inverted. But

\[
\sum_{i \in \pi} f_i^j(\pi)(x, \sigma) = \sum_{S \subseteq \pi} \sigma_j(S)(x_S(\pi S) + \sum_{i \in S} x_i(\pi S)) \leq \overline{v}
\]
because \( \sum_S \sigma_j(\pi)(S) = 1, \sigma_j(\pi)(S) \geq 0 \), and \( x_S(\pi S) + \sum_{i \notin S} x_i(\pi S) \leq \bar{v} \), for \( x \in X \). Therefore
\[
\sum_{i \in I} y_i(\pi) \leq \delta \sum_{j \in I} p_j(\pi) \bar{v}(\pi) + (1 - \delta)\bar{v}(\pi) = \bar{v}.
\]
Also, we have that \( y_i(\pi) \geq v_i \). First note that \( f_i^j(\pi)(x, \sigma) \geq v_i \) for all \( j \), because \( f_i^j(\pi)(x, \sigma) \geq x_j(\pi) \geq v_i \) as player \( i \) can always choose not to make any offer \((S = \{i\})\) and \( x \in X \). Also, \( f_i^j(\pi)(x, \sigma) = \sum_{S \subset \pi} \sigma_j(\pi)(S)(I(i \in S)x_i(\pi) + I(i \notin S)x_i(\pi S)) \geq v_i \) because \( x \in X \). Therefore,
\[
y_i(\pi)(x, \sigma) = \delta \left( \sum_{j \in I} p_j(\pi) f_i^j(\pi)(x, \sigma_j) \right) + (1 - \delta)v_i(\pi) \geq v_i.
\]

(2) \( \mathcal{F}(x) \) is a convex (and non-empty) set for all \( x \in X \): Say that \( y, y' \in \mathcal{F}(x) \) with \( y = f(x, \sigma) \) and \( y' = f(x, \sigma') \) where \( \sigma, \sigma' \in \Delta(x) \). Then, for any \( \lambda \in [0, 1] \), \( \lambda y + (1 - \lambda) y' = f(x, \lambda \sigma + (1 - \lambda) \sigma') \in \mathcal{F}(x) \) because \( \lambda \sigma + (1 - \lambda) \sigma' \in \Delta(x) \) (\( \Delta(x) \) is convex).

(3) \( \mathcal{F} \) is u.h.c., that is, for any sequence \((x^n, f(x^n, \sigma^n)) \rightarrow (x, y)\) with \( \sigma^n \in \Delta(x^n) \) then \( y \in \mathcal{F}(x) \) (i.e., there exists an \( \sigma \in \Delta(x) \) such that \( f(x, \sigma) = y \)). The sequence \( (\sigma^n) \) belongs to \( \Delta \) a compact subset of a finite-dimension Euclidean space. Therefore, there exists a subsequence of \( (\sigma^{n_k}) \) that converges to \( \sigma \in \Delta \). Rename this subsequence as \( (\sigma^n) \) for notational simplicity. We have that \( \sigma^n(\pi)(S) \rightarrow \sigma_i(\pi)(S) \) for all \( S \subset \pi \) and \( i \in \pi \in \Pi \), and that \( f(x^n, \sigma^n) \rightarrow f(x, \sigma) \), due to the continuity of \( f \), and thus \( y = f(x, \sigma) \).

It is sufficient to show that \( \sigma \in \Delta(x) \). By the definition of \( \Delta(x) \), \( \sigma \in \Delta(x) \) if and only if \( \sigma \in \Delta \) and \( \sigma_i(\pi)(S) = 0 \) for all \( S \subset \pi \) and \( i \in \pi \in \Pi \) such that \( x_S(\pi S) - \sum_{j \in S} x_j(\pi) < \max_{\delta \in \Pi} \left( x_S(\pi S) - \sum_{j \in S} x_j(\pi) \right) \). Consider any \( S \subset \pi \) for which the inequality above holds. By continuity, we have that there exists a large enough \( n_0 \) such that for all \( n \geq n_0 \), \( x^n_S(\pi S) - \sum_{j \in S} x^n_j(\pi) < \max_{\delta \in \Pi} \left( x^n_S(\pi S) - \sum_{j \in S} x^n_j(\pi) \right) \). But since \( \sigma^n \in \Delta(x^n) \), this implies that \( \sigma^n_i(S) = 0 \), and \( \sigma_i(S) = 0 \). Q.E.D.
Proof of Lemma 3: For all \( \sigma \) satisfying \( C = C(\sigma) \) then \( \mu = \mu(\sigma) \) satisfy
\[
\sum_{S \in C_r(\pi)} \mu(\pi)(S) = \sum_{S \in C_r(\pi)} \sum_{j \in \pi} p_j(\pi) \sigma_j(\pi)(S) = \sum_{j \in \pi} p_j(\pi) \sum_{S \in C_r(\pi)} \sigma_j(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi) \text{ for all } r,
\]
because if \( j \in P_r(\pi) \) then \( \text{supp}(\sigma_j(\pi)) \subseteq C_r(\pi) \), which corresponds to the last set of equations in \( F(C) \).

Now, if \( i \to j \) then there exist a coalition \( S \) such that \( i, j \in S \) and \( \sigma_i(\pi)(S) > 0 \). But because \( \text{supp}(\sigma_i(\pi)) \subseteq \arg \max \{ e(\pi)(S) x \} \) then
\[
e_i := \max_{\{S \subseteq \pi : i \in S\}} \left\{ x_S(\pi S) - \sum_{j \in S} x_j(\pi) \right\} \leq e_j.
\]
Repeating the same argument, if there is a path from \( i \) to \( j \) then \( e_i \leq e_j \), and if \( i \) is strongly connected to \( j \) then both have the same excess \( e_i = e_j \). Thus, \( e_r(\pi) = \sum_{j \in S} x_j, \) for all \( S \in C_r \) and all \( i \in P_r(\pi) \). Substituting the expressions for the excesses into equation (7) finishes the if part of the proof. The reciprocal follows directly from the construction of the polyhedral sets \( \mathcal{M}_C \) and \( \mathcal{E}_C \). Q.E.D.

Proof of Lemma 4: The same steps of the proof applies to each c.s. \( \pi \) separately, so to simplify notation we eliminate explicit references to \( \pi \) below. The following claim implies the lemma, as shown below.

Claim: Let \( C = (C, P) \) be a CDS, \( \sigma^* \) a strategy profile with \( C = C(\sigma^*) \), and let \( \Sigma_i = \text{supp} \sigma_i^* \). Given any \( \mu = (\mu(S))_{S \in C} \) close to zero satisfying \( \sum_{S \in C} \mu(S) = 0 \) there exists \( \sigma = (\sigma_i(S))_{S \in \Sigma_i, i \in P} \) close to zero satisfying \( \sum_{S \in \Sigma_i} \sigma_i(S) = 0 \) for all \( i \in P_r \) such that \( \mu(\sigma) = \mu \).

Suppose that the claim holds. Let \( \Delta C := \{ \sigma \in \Delta : C = C(\sigma) \} \). There exists \( \sigma^* \in \Delta C \) such that \( \mu(\sigma^*) = \mu^* \) (as \( \mu^* \in \mathcal{M}_C \)). Let \( \Sigma_i = \text{supp} \sigma_i^* \). Given any \( \mu \) close to \( \mu^* \) define \( \Delta \mu = \mu - \mu^* \) (which is close to zero). Consider a \( \Delta \sigma \) given by the claim (related to \( \Delta \mu \)) and let \( \sigma = \sigma^* + \Delta \sigma \). Such \( \sigma \) satisfies \( \sigma_i(S) > 0 \) for all \( S \in \Sigma_i \) (because \( \sigma_i^*(S) > 0 \) for all \( S \in \Sigma_i \) and \( \Delta \sigma(S) \) are close to zero) and \( \sum_{S \in \Sigma_i} \sigma_i(S) = \).
The hypothesis holds for one player games: the only support is $\mathcal{C} = \{\{1\}\}$ and the Jacobian matrix of problem $F_\mathcal{C}$ is obviously nonsingular. Now, let $\pi$ be a c.s. with $n$
players, and let us represent by a subscript 0 the references to the c.s. \( \pi \) and by the subscript \(-0\) the references to all its proper subgames. Let \( V_0 \times \Delta_0 \) represent the set of all \((v_i(\pi), p_i(\pi))\) and \( V_{-0} \times \Delta_{-0} \) the set of all \((v_i(\pi'), p_i(\pi'))_{\pi' \in \Pi, \pi' \neq \pi}\).

Let \( R_{-0} \subset V_{-0} \times \Delta_{-0} \) be the set of games that are regular and nondegenerate and the local mappings \( x_{-0}(v, p) \) are surjective. According to the induction hypothesis almost all games of \( V_{-0} \times \Delta_{-0} \) belong to \( R_{-0} \). Consider the solutions of the augmented problem \( F_{C_0}(z_0, v_0, p_0, v_{-0}, p_{-0}) = 0 \) where \( z_0 = (x_0, \mu_0, e_0) \) and we consider that \( x_{-0}(v_{-0}, p_{-0}) \) changes with \( v_{-0}, p_{-0} \) (even though expressions \( \sum_{i \in S} x_i(\pi) + e_r(\pi) - x_S(\pi S) \) do not depend directly on the parameters \( v, p \), the term \( x_S(\pi S) \) is a function of \( v_{-0}, p_{-0} \). The Jacobian matrix at the solution, \( d_{(z_0, v_0, p_0, v_{-0}, p_{-0})} F_{C_0} \), is

\[
\begin{bmatrix}
  * & * & -(1 - \delta)I_0 & 0 & * \\
  * & 0 & 0 & 0 & -d_{(v_{-0}, p_{-0})} g \circ x_{-0}(v_{-0}, p_{-0}) \\
  0 & d_\mu M_{C_0} & 0 & 0 & 0 \\
  0 & 0 & 0 & d_p M_{C_0} & 0
\end{bmatrix},
\]

where \( g : V_{-0} \rightarrow R^m \) is the linear map \( g(x_{-0})(S) = x_S(\pi S) \) for all the sets in the support \( C_0 = (\Sigma_i(\pi)) \), and * denotes arbitrary coefficients. Note that the linear map \( g \) is surjective, and thus the composition \( g \circ x_{-0}(v_{-0}, p_{-0}) \) is surjective (the composition of surjective maps is surjective). But then we have that \( F_{C_0}(z_0, v_0, p_0, v_{-0}, p_{-0})|_{0} \) because all blocks \(-(1 - \delta)I_0, d_\mu M_{C_0}, \) and \(-d_{(v_{-0}, p_{-0})} g \circ x_{-0}(v_{-0}, p_{-0}) \) are surjective. Therefore, by the transversality theorem, for almost every \((v, p) \in R^d \times \Delta^d, F_{C_0}(z_0)\|0 \). Because of the block triangular structure of the Jacobian matrix \( d_z F_C(z) \) (see remark on Section 6.1) this shows that \( d_z F_C(z) \) is nonsingular (\( C \) regular) almost everywhere.

The argument to show that \( C \) is nondegenerate almost everywhere is the same one discussed in Section 6.2.

To complete the proof, it still remains to show that the local solution mappings \( x(v, p) \) of problem \( F_C \) are surjective. But

\[
d_{(v, p)} x = \begin{bmatrix}
d_{(v_0, p_0)} x_0 & d_{(v_{-0}, p_{-0})} x_0 \\
0 & d_{(v_{-0}, p_{-0})} x_{-0}
\end{bmatrix},
\]

because \( x_{-0} \) does not depend on \((v_{-0}, p_{-0})\), and it is thus enough to prove that \( x_0(v, p) \) is surjective (by the induction hypothesis \( d_{(v_{-0}, p_{-0})} x_{-0} \) is surjective). The implicit function theorem gives us the expression of the derivative of the local mappings (refer to (12)) as, \( d_{(v, p)} x_0(v, p) = -[d_{z_0} F_{C_0}]_n^{-1} d_{(v, p)} F_{C_0} \), where \([d_{z_0} F_{C_0}]_n^{-1} \) is the submatrix.
of \([d_{z_0}F_{c_0}]^{-1}\) restricted to the first \(n\) rows, and \(d_{(v,p)}F_{c_0}\) is given by (22). But both \([d_{z_0}F_{c_0}]^{-1}\) and \(d_{(v,p)}F_{c_0}\) are surjective so \(d_{(v,p)}x_0(v,p)\) is surjective. Thus we conclude that \(x_0(v,p)\) is surjective. Q.E.D.

**Lefschetz Fixed Point Theorem (LFPT) (McLennan 1989):** Let \(T\) be the collection of admissible triples \((X, F, U)\) where \(X \subset R^m\) is a finite simplicial complex, \(F : X \rightarrow X\) is a upper hemi-continuous contractible valued correspondence (u.h.c.v.), \(U \subset X\) is open, and there are no fixed points of \(F\) in \(U\). Then there is a unique Lefschetz fixed point index \(\Lambda(X, F, U)\) that satisfies the following axioms (when \(X\) is implicitly given we just say \((F, U)\)):

(1) **Localization axiom:** If \(F_0, F_1 : X \rightarrow X\) are u.h.c.v. correspondences that agree on \(U\), and \((X, F_1, U), (X, F_0, U) \in T\), then \(\Lambda(X, F_1, U) = \Lambda(X, F_0, U)\).

(2) **Continuity axiom:** If \((X, F, U) \in T\), then there is a neighborhood \(W\) of \(Gr(F)\) such that \(\Lambda(X, F', U) = \Lambda(X, F, U)\) for all u.h.c.v. correspondences \(F' : X \rightarrow X\) with \(Gr(F') \in W\).

(3) **Homotopy axiom:** If \(h : [0, 1] \times X \rightarrow X\) is a homotopy with \((X, h_t, U) \in T\), for all \(t\), then \(\Lambda(X, h_0, U) = \Lambda(X, h_1, U)\).

(4) **Additivity axiom:** If \((X, F, U) \in T\) and \(U_1, ..., U_r\) is a collection of pairwise disjoint open subsets of \(U\) such that there are no fixed points of \(F\) in \(U - (\bigcup_{k=1}^r U_k)\) then \(\Lambda(X, F, U) = \sum_{k=1}^r \Lambda(X, F, U_k)\).

(5) **Weak Normalization axiom:** For \(y \in X\), let \(c_y\) be the constant correspondence \(c_y(x) = \{y\}\). If \(y \in U\) then \(\Lambda(X, c_y, U) = 1\).

(6) **Commutativity axiom:** If \(X \subset R^m\) and \(Y \subset R^n\) are finite simplicial complexes, \(f : X \rightarrow Y\) and \(g : Y \rightarrow X\) are continuous functions, and \(\Lambda(X, g \circ f, U) = \Lambda(X, f \circ g, g^{-1}(U))\).

**Proof of Lemma 5:** Define the correspondence \(\mathbf{F}(x) = x - \mathcal{F}(x)\), where \(\mathcal{F}(x)\) is the correspondence defined in (18). The Lefschetz index of \(\mathcal{F}\) and the degree of \(\mathbf{F}\) are related by \(\Lambda(\mathcal{F}, U) = \deg(\mathbf{F}, U, 0)\) (see McLennan (1989)), and, for convenience, we work in the remainder of the proof with the concept of degree.

The mixed nonlinear complementarity problem is the problem of finding triples \((x, \sigma, e)\) that are the solution of problem (MNCP) in Section 4.
For each point $x$ consider the mixed linear complementarity problem $MLCP(0)$,

\[
\begin{align*}
  h(\sigma) &= 0, \\
  g(e, x) &\geq 0, \\
  e &\text{ free variable, } \sigma \geq 0 \text{ and } \sigma^T g(e, x) = 0,
\end{align*}
\]

Let $z(x) = (e(x), \sigma(x))$ be a solution of the $MLCP(0)$ (there can be multiple solutions). Note that $F(x) = \{f(x, z(x)) : z(x) \text{ is a solution of } MLCP(0)\}$.

Let $(x^*, e^*, \mu^*)$ be any regular and nondegenerate MPE with an associated CDS $C = (C, P)$, with $C = (C_1, \ldots, C_q)$ and $P = (P_1, \ldots, P_q)$. By Lemma 3, there exists $\sigma^* \in \Delta_C$ such that $\mu^* = \mu^*(\sigma^*)$, and $(x^*, \sigma^*)$ is MPE. Furthermore, because all points in $P_r$ are connected, we can choose a strategy profile $\sigma^*$ satisfying $\text{supp}(\sigma^*_i) = C_r \cap \{S \subset \pi : i \in S\}$ for all $i \in P_r$.

Consider now the perturbed mixed linear complementarity problem, or $MLCP(\varepsilon)$

\[
\begin{align*}
  h(\varepsilon(e, \sigma)) &= h(\sigma) + \varepsilon(e - e^*) = 0, \\
  g(\varepsilon(x, \sigma, e)) &= g(x, e) + \varepsilon(\sigma - \sigma^*) \geq 0, \\
  e &\text{ free variable, } \sigma \geq 0, \sigma^T g(\varepsilon) = 0,
\end{align*}
\]

where $\varepsilon > 0$. The Jacobian matrix $M(\varepsilon)$ of $MLCP(\varepsilon)$ is a $P$-matrix (i.e., a matrix with all its principal minors positive). This is so because (see Cottle et al. (1992, pg. 154)), $M(\varepsilon) = M + \varepsilon I$, where $M$ is the Jacobian of $MLCP(0)$, is a $P_0$-matrix (i.e., a matrix with all its principal minors nonnegative). Let us prove that $M$ is a $P_0$-matrix: Consider the principal matrix $M_{\beta\beta}$ associated with a subset $\beta$ of lines (or columns).\textsuperscript{11} We now show that either $\det(M_{\beta\beta})$ is equal to zero or one. Note first that $\det(M_{\beta\beta}) = \prod_{i \in \pi} \det(M_{\beta_i\beta_i})$ where $\beta = \cup_i \beta_i$ and $\beta_i$ are the elements of $\beta$ with entry $i$ (either $e_i$ or $\sigma_i(S)$ for some $S \ni i$). But $\det(M_{\beta_i\beta_i}) = 1$ if $\beta_i = \{e_i, \sigma_i(S)\}$ and is zero otherwise. Therefore, we conclude that all principal minors of $M$ are nonnegative, and thus $M$ is a $P_0$-matrix.

Given that $MLCP(\varepsilon)$ has a $P$-matrix then there is a unique solution $z_\varepsilon(x)$ (Cottle et al. (1992, pg. 150)) for all $x$: $MLCP(\varepsilon)$ can be transformed into a standard

\textsuperscript{11}We refer to the lines corresponding to $\partial h_i$ and $\partial g_i(S)$ as lines $\lambda_i$ and $\sigma_i(S)$, and the columns corresponding to $\frac{\partial}{\partial e_i}$ and $\frac{\partial}{\partial \sigma_i(S)}$ as columns $\lambda_i$ and $\sigma_i(S)$. Also, we use the standard notation that $A_{\alpha \alpha}$, $A_{\alpha}$, and $A_{\alpha}$, represent the submatrix of $A$ with, respectively, rows and columns, columns, and rows extracted from the index set $\alpha$. Also, $\overline{\alpha}$ denotes the complementary set of $\alpha$.}
eliminating the variable $e$ and the equation $h(\varepsilon) = 0$ (this is possible because $M_{ee}(\varepsilon) = \varepsilon I$ is nonsingular), and the transformed LCP also has a $P$-matrix (the Schur complement of $M_{ee}(\varepsilon)$ in $M(\varepsilon)$). Note that, in addition, we have that $z_\varepsilon(x^*) = (e^*, \sigma^*)$, and that $z_\varepsilon(x)$ converge to a solution of MLCP(0) when $\varepsilon \to 0$ (Cottle et al. (1992, pg. 442)), and that $z_\varepsilon(x)$ is piecewise linear in $x$.

We now show that, because $x^*$ is a strong solution, there exists an $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ there exists an open neighborhood $U_\varepsilon$ of $x^*$ such that $z_\varepsilon(x)$ is smooth in $U_\varepsilon$. Moreover, if we let $\alpha$ represent the index set

$$\alpha = \{\sigma_i(S) : \text{for all } S \in C_r \text{ and } i \in S\},$$

then all $\sigma_i(S)$-coordinates of the solution $z_\varepsilon(x)$ that do not belong to $\alpha$ are zero, and $z_\varepsilon(x)$ are explicitly given by $(M_{aa}(\varepsilon))^{-1} q_\alpha(x)$, where $M_{aa}(\varepsilon)$ is

$$M_{aa}(\varepsilon) = \begin{bmatrix} \varepsilon I_{aa} & (d_e g)_a \\ d_a h & \varepsilon I_{ee} \end{bmatrix},$$

and the vector $q_\alpha(x)$ has $e_i$-coordinate equal to $(\varepsilon e^*_i - 1)$, and $\sigma_i(S)$-coordinate in $\alpha$ equal to $\varepsilon \sigma_i(S)^* + e(S)(x)$, for all $0 < \varepsilon < \overline{\varepsilon}$ and $x \in U_\varepsilon$.

In order to prove the above claim consider the function

$$\varphi(x) = \min \cup_{r=1}^n \{e(S)(x) - e(T)(x) : S \in C_r, T \cap P_r \neq \emptyset, \text{ and } T \notin C_r\}.$$ 

Naturally, the function $\varphi$ is continuous in $x$ and, because $x^*$ is a strong solution, $\varphi(x^*) > 0$. Therefore, there exists an $\overline{\varepsilon} > 0$ and an open neighborhood $U \subset U_{x^*}$ of $x^*$, such that all $x \in U$ satisfy $\varphi(x) > 2\overline{\varepsilon}$. Now suppose that the solution $z_\varepsilon(x)$ for $x \in U$ is such that a $\sigma_i(T)$-coordinate is non-zero for $T \notin C_r$ and $i \in P_r$. Then $g_i(\varepsilon)(T) = 0$ which is equivalent to $e_i + \varepsilon (\sigma_i(T) - \sigma_i^*(T)) - e(T)(x) = 0$, and implies $e(T)(x) \geq e_i - \varepsilon$. Also, $g_i(\varepsilon)(S) \geq 0$ for all $S$, and thus $e_i + \varepsilon (\sigma_i(S) - \sigma_i^*(S)) - e(S)(x) \geq 0$, which implies that $e(S)(x) \leq e_i + \varepsilon$. Therefore, $e(S)(x) - e(T)(x) \leq 2\varepsilon \leq 2\overline{\varepsilon}$ for $x \in U$, in contradiction with $\varphi(x) > 2\overline{\varepsilon}$ for all $x \in U$. Now, since $z_\varepsilon(x^*) = (e^*, \sigma^*)$, and $\text{supp}(\sigma_i^*) = C_r \cap \{S \subset \pi : i \in S\}$, and $z_\varepsilon(x)$ is continuous, then there exists an open neighborhood $U_\varepsilon \subset U_{x^*}$ of $x^*$ where all $\sigma_i(S)$-coordinates of the solution belonging to $\alpha$ are non-zero. This implies that $g_i(\varepsilon)(S) = 0$ holds for all $\sigma_i(S)$ in $\alpha$, and thus $z_\varepsilon(x) = (M_{aa}(\varepsilon))^{-1} q_\alpha(x)$.
Define the mapping $F_\varepsilon (x) = f(x, z_\varepsilon (x))$ (this mapping is well-defined due to the uniqueness of $z_\varepsilon (x)$), where $F_\varepsilon (x^*) = 0$. Since $f$ is smooth and $z_\varepsilon (x) \to z (x)$ then $F_\varepsilon (x) \to F (x)$. Therefore, for every $\delta > 0$ there exists $\varepsilon$ such that $\text{dist} \left( F_\varepsilon (x) , F (x) \right) < \delta$, for all $0 < \varepsilon \leq \varepsilon$. But since $F (x)$ has no zeros in the boundary of $\partial U$ then $F_\varepsilon (x)$ also does not have any zeros in $\partial U$. By the homotopy and continuity property of the degree, $\deg (F, U, 0) = \deg (F_\varepsilon , U, 0)$, for $\varepsilon$ close to zero.

Therefore, it only remains to show that $\deg (F_\varepsilon , U, 0) = sgn (\det (dF_C (z^*)))$ for $\varepsilon$ close to zero, where $z^* = (x^*, e^*, \mu^*)$. This result follows from $sgn (\det (d_x F_\varepsilon (x^*))) = sgn (\det (dF_C (z^*))) \neq 0$, as we will show. Indeed, this implies that $F_\varepsilon$ is nonsingular at $x^*$, and thus there exists an open neighborhood $V \subset U$ of $x^*$ where $x^*$ is the only zero of $F_\varepsilon$. But since the point $x^*$ is the only zero of $F (x)$ in $U \subset U_{x^*}$, and $F_\varepsilon (x) \to F (x)$ then there are no zeros of $F_\varepsilon$ in the compact region $\overline{U} \setminus V$, for $\varepsilon$ small enough, and thus $x^*$ is the only zero of $F_\varepsilon$ in $U$. A well-known property of the degree then implies that $\Lambda(F, U) = \deg (F_\varepsilon , U, 0) = sgn (\det (d_x F_\varepsilon (x^*))) = sgn (\det (dF_C (z^*)))$.

We now show that $sgn (\det (d_x F_\varepsilon (x^*))) = sgn (\det (dF_C (z^*)))$, for $\varepsilon$ small enough. Consider $F(x, \sigma, e) (\varepsilon)$,

$$F(x, \sigma, e) (\varepsilon) = \begin{pmatrix} f(x, \sigma, e) \\ h(\sigma) + \varepsilon (e - e^*) \\ g(e, x) + \varepsilon (\sigma - \sigma^*) \end{pmatrix}.$$  

Simple linear algebra shows that the Jacobian $d_x F_\varepsilon (x^*)$ is the Schur complement of $M_{aa} (\varepsilon)$ in $dF_{aa} (\varepsilon) (d_x F_\varepsilon (x^*)) = dF_{aa} (\varepsilon) / M_{aa}$, where

$$dF_{aa} (\varepsilon) = \begin{bmatrix} (d_x f)_{\alpha} & d_{\alpha} f & d_{\varepsilon} f \\ (d_x g)_{\alpha} & \varepsilon I_{aa} & (d_{\varepsilon} g)_{\alpha} \\ 0 & d_{\alpha} h & \varepsilon I_{ee} \end{bmatrix},$$  \hspace{1cm} (26)

is evaluated at point $(x^*, e^*, \sigma^*)$. Therefore, $\det (d_x F_\varepsilon (x^*)) = \det (dF_{aa} (\varepsilon)) / \det (M_{aa})$ (see Cottle et al. (1992, pg. 75)). But since $\det (M_{aa}) > 0$ ($M$ is a $P$-matrix) then $sgn (\det (d_x F_\varepsilon (x^*)) = sgn (\det (dF_{aa} (\varepsilon)))$.

We claim that $sgn (\det (dF_{aa} (\varepsilon))) = sgn (\det (dF_C (z^*)))$. In order to prove the claim we use the following formula for the determinant (Cottle et al. (1992), pg. 60): for an arbitrary diagonal matrix $D$, $\det (A + D) = \sum_{\gamma} \det D_{\gamma \gamma} \det A_{\gamma \gamma}$ where the summation ranges over all subsets $\gamma$ of lines. Observe that matrix $dF_{aa} (\varepsilon) = A + D$,  

37
where \( A = dF_{\alpha \alpha} (0) \) and \( D \) is the diagonal matrix,

\[
D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon I_{\alpha \alpha} & 0 \\ 0 & 0 & \varepsilon I_{ee} \end{bmatrix}.
\]

Developing the expression for \( \det (dF_{\alpha \alpha} (\varepsilon)) \) using the formula above we get a polynomial in \( \varepsilon \) (\( \det D \) is a power of \( \varepsilon \)). We are only interested in the non-zero coefficient with lowest order because, when \( \varepsilon \) converges to zero, this is the coefficient that determines the sign of \( \det (dF_{\alpha \alpha} (\varepsilon)) \).

The rows and columns of matrix \( A = dF_{\alpha \alpha} (0) \) corresponding to \( \sigma_i (S) \) and \( e_i \) are

\[
R (\sigma_i (S)) = \sum_j I_{j \in S} e(x_j) + e (e_i),
\]

\[
R (e_i) = - \sum_{S \in C_r} e (\sigma_i (S)),
\]

\[
C (\sigma_i (S)) = \sum_j I_{j \in S} x_j e(x_j) - e (e_i),
\]

\[
C (e_i) = -p_i e(x_i) + \sum_{S \in C_r} e (\sigma_i (S)),
\]

where vectors \( e(x_i), e (e_i), \) and \( e (\sigma_i (S)) \) are the unit vectors at, respectively, coordinates \( x_i, e_i, \) and \( \sigma_i (S) \).

Consider \( A_\alpha \), the submatrix of \( A \) corresponding to the rows \( \alpha \) of \( A \). Let \( \beta \) be a maximal subset of \( \alpha \) such that \( \text{rank} (A_\beta) \) is different from zero (\( |\beta| = \text{rank} (A_\alpha) \) and \( \text{rank} (A_\beta) = \text{rank} (A_\alpha) \)). Note that \( A_{\gamma'} \), where \( \gamma ' \) is the set of lines \( \gamma = \beta \cup \{e_i : i \in \pi\} \cup \{x_i : i \in \pi\} \) is equal to \( A_{\gamma'} = dF_{\beta \beta} (0) \), according to the definition (26).

Also, \( \det A_{\gamma' \gamma} = 0 \) for set of lines \( \gamma' \) that strictly contains \( \gamma \) because \( \beta \) is a maximal subset of \( \alpha \) such that \( \text{rank} (A_\beta) \neq 0 \).

We now show that \( \det (dF_{\beta \beta} (0)) = \det (dF_C (z^*)) \neq 0 \), which proves that the lowest-order non-zero coefficient is equal to a positive integer (the number of maximal subsets \( \beta \subset \alpha \)) multiplied by \( \det (dF_C (z^*)) \), and thus \( \text{sgn} (\det (dF_{\alpha \alpha} (\varepsilon))) = \text{sgn} (\det (dF_C (z^*))) \), for \( \varepsilon \) small enough.

We now propose an algorithm replaces all rows and columns \( \sigma_i (S) \)'s with the same \( S \) by only one row and column \( \sigma_i (S) \) for all \( S \in C_r \), and also replaces all rows and columns \( e_i \) for all \( i \in P_r \) by only one row and column \( e_r \) for each \( r = 1, \ldots, q \).
Algorithm: Start with matrix $A = dF_{\beta \beta}(0)$.

Step 1: Choose an element $r$, that have not yet been chosen, from the set \{1, 2, ..., $q$\} and proceed to the next step, or else, stop if the choice is not possible.

Step 2: Choose two distinct rows $\sigma_i(S)$ and $\sigma_j(S)$ of $A$ with $j \neq i$ and $S \in C_r$ and proceed to the next step, or else return to step 1 if the choice is not possible.

Step 3: Subtract row $\sigma_i(S)$ from row $\sigma_j(S)$ (i.e., $R(\sigma_j(S)) = R(\sigma_j(S)) - R(\sigma_i(S))$), and add column $e_j$ to column $e_i$ (i.e., $C(e_i) = C(e_i) + C(e_j)$). The matrix that is obtained after the two operations have the same determinant as matrix $A$. Let this matrix be the new matrix $A$. After these two operations, row $\sigma_j(S)$ of $A$ has only one non-zero entry at column $e_j$, with a value equal to 1. The determinant of $A$ can be computed by a co-factor expansion along row $\sigma_j(S)$, and 

$$|A| = (-1)^{(#\sigma_j(S)+#e_j)}|A'|,$$

where $A'$ is the submatrix obtained after deleting row $\sigma_j(S)$ and column $e_j$ of matrix $A$.

Now, perform the following symmetric transformations on the submatrix $A'$: Subtract column $\sigma_i(S)$ from column $\sigma_j(S)$ (i.e., $C(\sigma_j(S)) = C(\sigma_j(S)) - C(\sigma_i(S))$) and add row $e_j$ to row $e_i$ (i.e., $R(e_i) = R(e_i) + R(e_j)$). The matrix that is obtained after the two operations have the same determinant as $A'$. Let this matrix be the new matrix $A''$. After these two operations, column $\sigma_j(S)$ of $A''$ has only one non-zero entry at row $e_j$, with a value equal to $-1$. The determinant of $A'$ can be computed by a co-factor expansion along column $\sigma_j(S)$, and 

$$|A'| = (-1) \times (-1)^{(#\sigma_j(S)+#e_j-1)}|A''|,$$

where $A''$ is the submatrix of $A'$ obtained after deleting column $\sigma_j(S)$ and row $e_j$; observe that the column $\sigma_j(S)$ of $A'$ is in the same location as row $\sigma_j(S)$ of $A'$, but row $e_j$ appears one entry before column $e_j$ of $A$ (because the row $\sigma_j(S)$ that has been removed appears before row $e_j$). Putting together the expressions for the determinant yields $|A| = |A''|$. Let matrix $A''$ be the new matrix $A$, and return to step 2.

Because $\beta$ is a maximal subset of $\alpha$ with $\text{rank}(A_{\beta \beta}) \neq 0$ and $\text{rank}(E_C) \neq 0$, the algorithm starts with matrix $A = dF_{\beta \beta}(0)$ and ends with matrix $A = \det(dF_C(z^*))$ (maintaining the same determinant in all steps).

Therefore, $\det(dF_{\beta \beta}(0)) = \det(dF_C(z^*))$, as we claimed. Q.E.D.

Proof of Proposition 6: We focus on the c.s. $\pi = \{1\}, \{2\}, \{3\}$ because we already know that two-player games have a unique equilibrium (Rubinstein (1982)).

We first show that $X(i, S) = x_i(\pi) - x_i(\pi S) \geq 0$, where $i \notin S$, if there is a positive
probability that $S$ is chosen in equilibrium. Say that $S = \{j, k\}$ (if $S = \emptyset$ (no proposal case) then $x_i(\pi S) = x_i$ and if $S = N = \{1, 2, 3\}$ then there are no elements $i \notin S$). In order to simplify the notation, let $x_i = x_i(\pi), x_i(\pi S) = x_i(jk), x_S(\pi S) = x_{jk}(jk)$, and $V = v_N(\{N\})$. Suppose that $S$ is chosen in equilibrium with positive probability. Then $e(S)(x) \geq e(N)(x)$, which is equivalent to,

$$x_{jk}(jk) - x_j - x_k \geq V - x_i - x_j - x_k, \quad (27)$$

and

$$x_{jk}(jk) + x_i(jk) + x_i - x_i(jk) \geq V. \quad (28)$$

But since there is no delay in the formation of the grand coalition when the game is at the c.s. $\{\{jk\}, \{i\}\}$, we have that

$$x_{jk}(jk) + x_i(jk) = \delta V + (1 - \delta)(v_{jk}(jk) + v_i(jk)).$$

Replacing this expression into (28) yields

$$X(i, jk) = x_i - x_i(jk) \geq (1 - \delta)(V - (v_{jk}(jk) + v_i(jk))) \geq 0.$$ 

We now compute $\det(dF_C)$ for all admissible CDS $C = (C, P)$, and show that $\det(dF_C) \geq 0$. From the definition of CDSs it follows that $P = (P_1, ..., P_q)$ is a partition of $N$ and $C = (C_1, ..., C_q)$ is an ordered disjoint collection of subsets $S \subset N$ satisfying: for all $S \in C_r$ then $S \cap P_r \neq \emptyset$ and $S \subset \cup_{s=1}^r P_s$, and also $\cup_{s \in C_r} S \supset P_r$. Moreover, there is no $S = \{i\}$ that is chosen in equilibrium, and thus $C_r \subset \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. A list of all admissible CDSs (except for permutations of the players) follows with the corresponding value for $\det(dF_C)$ ($i, j$, and $k$ are distinct elements of $N$, and $d_i = 1 - \delta \sum_S \mu(S) I_{[i \in S]}$, $z(i, jk) = \delta X(i, jk)$, and $w_i = \delta p_i$):
Note that the first 6 entries of the table corresponds to CDSs with \( P = (\{1, 2, 3\}) \) and the remaining entries to CDSs with \( P = (\{1, 2\}, \{3\}) \).

The determinant for all CDSs are nonnegative because it is a sum of nonnegative terms. Corollary 2 implies that there is a unique global MPE solution. Q.E.D.
References


