Coalitional Bargaining Games: A New Concept of Value and Coalition Formation

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Abstract

We propose a new non-cooperative coalition bargaining game suitable to capture the outcome of freestyle or unstructured negotiations. We characterize the strong and coalition-proof Markov perfect equilibrium of the game, and show that the coalitions forming in equilibrium are the ones that maximize the average gain of coalition members. The model does not rely on an exogenously given proposer selection protocol and, unlike random proposer models, dummy players have zero value and are never included in coalitions that form. We compare the new model solution with other classical solutions, for all 134 strong majority voting games with less than eight players. The comparison yields sharp differences in predictions: the values to larger (smaller) parties are significantly greater (less) than other solution concepts such as the Shapley-Shubik index, the Banzhaf index, the random proposer model solution, and the nucleolus. Conditional on being included in the winning coalition, the value/vote ratio is approximately independent of the number of party votes (Gamson's Law). However, the unconditional or ex-ante value/votes ratio of small parties are significantly less than larger parties because small parties are not included in the winning coalition proportionally to their voting weights.

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1 Introduction

Many political, economic, and social problems can be modelled as coalition bargaining games. In these games agents negotiate over the formation of coalitions and the allocation of some surplus and/or decisions among coalition members. For example, common situations that can be modelled as coalition bargaining games include political parties negotiating over government formation and the allocation of ministerial cabinets in multi-party parliamentary democracies (Baron and Ferejohn (1989)); large shareholders of a majority controlled corporation negotiating over the allocation of control benefits among controlling shareholders (Zingales (1994, 1995)); countries negotiating over decisions in organizations such as the European Union Council of Ministers or the United Nations Security Council (Felsenthal and Machover (2001)); or firms in an industry negotiating over R&D collaboration agreements or cartel formation (Goyal and Joshi (2003) and Bloch (1997)).

A large literature on sequential coalition bargaining games has developed extending the two-player alternating-offer bargaining model of Rubinstein (1982). The non-cooperative coalition bargaining models typically are based on an extensive-form game in which at every period there are two stages: a first stage in which a player (the proposer) makes a proposal to form a coalition with a certain surplus allocation among coalition members, followed by a second stage, in which the members of the coalition that received the proposal accept or reject the proposal (See the literature review at the end of this section).

The primary objective of coalition bargaining theories is to derive an equilibrium solution with the value (or power) of players and the coalition structures forming. A key exogenous feature among coalition bargaining models is a coalitional function which assigns a certain surplus to coalitions that can form. The coalitional function can often be readily obtained for the particular problem of interest. For example, in a majority voting game, the coalition that gets more than half of the votes get all the surplus.

However, most non-cooperative coalition bargaining models often also rely on a specific proposer selection protocol as an integral part of the model specification. The selection protocol most commonly adopted by non-cooperative bargaining models is the random proposer protocol: for example, in Okada (1996) and Compte and Jehiel (2010) the proposer recognition probability is assumed to be equal among all players; and in Baron and Ferejohn (1989), Montero (2006), and Eraslan and McLennan (2013) the proposer recognition probability can have an arbitrary distribution. Also, appearing in several models is the fixed order proposer protocol (e.g., Austen-Smith and Banks (1988), Chatterjee et al (1993), and Bloch (1996)).

Specific choices for the proposer recognition are not without significant consequences for the equilibrium outcome. For example, Kalandrakis (2006) shows that in the classic random proposer legislative bargaining model, any value can be achieved by varying the proposer recognition probability; and Baron (1991) shows that the equilibrium predictions depend crucially on whether the selection protocol is either random or fixed order. Motivated by the importance of the issue, several studies focus on endogeneizing the proposer selection process by adding a round of bidding for the right to be the proposer (e.g., Evans (1997), Perez-Castrillo and Wettstein (2001, 2002), Macho-Stadler et al. (2006), Yildirim (2007), and Ali (2015)).

The question we address in this paper is what coalitions should form and what are the ex-ante player values in an unstructured bargaining setting. That is, a setting in which agents are just given the coalitional function, and are not constrained by a specific bargaining proposal/rejection protocol. Specifically, this paper proposes a new non-cooperative coalition bargaining model that attempts to generate an equilibrium solution that more closely captures the outcome of unstructured negotiations by allowing all players to simultaneously propose which coalitions they want to join.

The coalition bargaining model developed here is a dynamic game over an infinite number of periods, in which every period starts with a coalition formation stage followed by a negotiation stage. All players maximize the expected discounted share of the surplus they receive, and they have the same discount rate. In the coalition formation stage all players simultaneously choose the winning coalitions they want to form (W denotes the subset of all possible winning coalitions which is assumed to be monotonic and proper): a coalition forms if and only if all of its members have chosen it. If no such coalition exists the game moves to the next period. In the negotiation stage that ensues all members of the coalition chosen in the first stage engage in a multilateral Nash bargaining to split the surplus in which unanimity is required to reach agreement. In case of disagreement, the game moves on to the next period, where the same two stages repeat.

We allow for a general formulation in which the surplus an stochastic coalitional function (see also Wilson and Merlo (1995) and Eraslan and Merlo (2002)). At the start of the each period there is a random independent and identically distributed draw for the surplus generated by all winning coalitions from an absolutely continuous distribution. Players, before making their decisions about which coalition to form, observe the realization of the surplus. Stochastic coalitional functions can represent small independent random perturbations of a deterministic coalitional function $(s_{\mathrm{C}})_{\mathrm{C} \in W}$, where s_{C} denotes the surplus of a winning coalition $\mathrm{C} \in W$. Of particular interest to us is a result similar to Harsanyi (1973)'s purification result, characterizing the limit equilibrium when the random perturbations around a deterministic coalition function converges to zero. In our model, players use pure strategy and the randomization comes from perturbations of the surplus, whereas with deterministic coalitional functions equilibrium typically exist only allowing for mixed strategies. However, our purification approach is different from Harsanyi's since there is no incomplete information, and all players know the surplus realizations drawn every period.

Our concept of equilibrium is Markov perfect equilibrium: pure strategy profiles that are stationary subgame perfect Nash equilibrium and depend only on payoff relevant aspects of the history. We require that the action profile at the coalition formation stage game be a strong Nash equilibrium or a coalition-proof Nash equilibrium, since this stage game has a plethora of Nash equilibrium due to coordination problems. Strong Nash equilibrium is a refinement of Nash equilibrium introduced by Aumann (1959), in which no coalition of players can gain by any joint (or group) deviations. The concept of coalition-proof Nash equilibrium was developed by Bernheim et al. (1987), and differs from strong Nash equilibrium concept because only group deviations that are self-enforcing are allowed.

We characterize the strong and coalition-proof Markov perfect equilibrium of the game and show that, remarkably, they always exist and coincide. This is surprising given that, as Bernheim et al. (1987) point out, strong Nash equilibrium is for many games "too strong" and almost never exists, and both concepts typically do not coincide.

One of the main insights coming from the model is that the coalitions that form in equilibrium are the ones that maximize the average gain per coalition member. The average gain is precisely defined as, $\gamma_{\mathrm{C}} = \frac{1}{|\mathrm{C}|}\left(s_{\mathrm{C}} - \delta \sum_{j \in \mathrm{C}} v_j\right)$, where v_i is the continuation value of player i, s_{C} is the surplus generated by the coalition C, and $|\mathrm{C}|$ is the number of players in the coalition.

Our main result is the characterization of the limit equilibrium outcome, which we refer as the coalition bargaining solution, when the surplus distribution converges to a deterministic coalitional function. The coalition bargaining solution, which includes the endogenous ex-ante value for the player v_i and the probability μ_{C} of coalition C forming, are given by the solution of the equations $v_i = (\delta v_i + \gamma)\left[\sum_{\mathrm{C}\in W} I(i \in \mathrm{C})\mu_{\mathrm{C}}\right]$ and $\sum_{\mathrm{C}\in W} \mu_{\mathrm{C}} = 1$, where $\mu_{\mathrm{C}} > 0$ only if $\mathrm{C} \in \mathrm{argmax}_{\mathrm{S}\in W}\, \gamma_{\mathrm{S}}$ and $\gamma = \max_{\mathrm{S}\in W} \gamma_{\mathrm{S}}$ is the maximum average gain, and δ is the discount rate. Note that the term in brackets above is the probability that player i is included in the coalition that forms. Moreover, players' values conditional on coalition C forming are $\delta v_i + \gamma$, their continuation value plus the maximum average gain.

The intuition for the equilibrium is that coalitions that do not maximize the average gain are dominated by coalitions that maximize the average gain. This occurs because once a coalition C is proposed players' conditional values in the negotiation stage are $\delta v_i + \gamma_{\mathrm{C}}$, which corresponds to an equal split the surplus using the continuation values as the status quo (the Nash bargaining solution). Thus there is an alignment in incentive among all players forming a coalition, since they all gain γ_{C} when forming coalition C. This incentive alignment is the key property that guarantees the existence and uniqueness of strong and coalition-proof Nash equilibrium in the coalition formation stage.

We apply the model to majority voting games in order to further our understanding of coalition government formation in multiparty democracies. While we do not prove a general equilibrium uniqueness result, we computationally verify that for a large class of majority voting games the equilibrium solution is indeed unique (see Section 7). The effect of institutional details on the empirical accuracy of coalition government models has been of particular interest to researchers (see Laver and Schofield (1990), Diermeier (2006), and Seidmann et al. (2007)). One important institutional element considered are the rules about the formateurs (proposers) of a coalition government. In some countries, such as Belgium and the Netherlands, there is an institutionalized system of formateurs usually appointed by the head of state (or president) in a non-partisan way. However, Laver and Schofield (1990), in an influential and comprehensive study of multiparty government in Europe, document that in thirteen out of twenty countries the head of state does not play an active role in coalition formation. Müller and Strom (2003) address in detail the institutions of coalition bargaining and politics. They find that in countries such as Germany and Ireland there are no rules for the recognition of formateurs and the coalition formation process is a "fairly unstructured process in which party leader engage (in relatively) freestyle bargaining".

We explore in this paper the predictions for majority voting games of the simultaneous proposer model (which we believe more accurately describe the outcome of unstructured negotiation processes) versus the random proposer model (which is likely to be more accurate in an institutionalized system of formateurs).

Both models, indeed, have significantly distinct predictions. For example, unlike the random proposer model, dummy players have zero value in the simultaneous proposer model, and are never included in coalitions that form. In this sense the coalition bargaining solution makes similar predictions to cooperative solution concepts. To further explore the properties of the simultaneous proposer model, we compare, for all possible 134 strong majority voting games with less than eight players (see Isbell (1959)'s enumeration), the coalition bargaining solution with the solutions of the non-cooperative random proposer model with both egalitarian and proportional proposer probabilities and also with the classic cooperative solution concepts such as the Shapley-Shubik and Banzhaf indices, and the nucleolus.[1]

This application is of practical relevance because majority voting games with less than eight players are quite common in real world voting applications, and any strong majority game with less than eight player is equivalent to one of the 134 games analyzed here. For example, in the empirical analysis of Diermeier and Merlo (2004), among the 313 government formations in 11 multi-party democracies over the period 1945-1997, the median distribution of the number of parties is seven, and the mean is 7.35.

Our analysis shows that the largest party value tend to be bigger under the coalition bargaining solution than all other solution concepts (and the opposite holds true for the smallest player value). Among all 134 majority voting games with less than eight players, the coalition bargaining value of the largest party is on average 11% higher than the Shapley-Shubik index, 13% higher than the Banzhaf index, 44% higher than the random proposer model with equal protocol, and 29% higher than the nucleolus and the random proposer model with proportional protocol.

Another important difference between the coalition bargaining solution relative to other existing solution concepts, is that the ex-ante (or unconditional) value per vote ratio tends to increase with the number of votes. In particular, the average value per vote ratio, v_w/w, for the largest player is 8.04, and the average ratio for the smallest player is 3.79. The value per

[1] See Roth (1988), Peleg and Sudhölter (2007), and Laruelle and Valenciano (2008) for a study of the main cooperative solution concepts.

vote ratio of the largest player is greater than the value per vote ratio of the smallest player in all but 16 of the 134 games analyzed. There is a simple intuitive economic reasoning motivating this property: large players tend to be included in winning coalitions more than proportionally to their voting weights because they contribute more to average gains than smaller players with the same aggregate number of votes. This property of the solution has implications in normative analysis aiming to determine appropriate voting weights that should be assigned to players in order to achieve a certain social objective (see Le Bretton et al (2012)).

A number of empirical studies in legislative bargaining have analyzed the relationship between the value of parties, measured typically by the number of ministerial cabinets, and their share of parliamentary seats (e.g., Browne and Franklin (1973) and Warwick and Druckman (2001)). These studies tend to find support for Gamson's law which states that the number of cabinet posts and votes contributed to the winning coalition are proportional.

Interestingly, Gamson's law holds fairly closely for the coalition bargaining solution. Among the 134 games, the ratios of the values, conditional on being included in the winning coalition, and the voting weights–which we show is equal to $(v_w + \gamma)/w$–is relatively constant in the voting weights w. Thus, conditional on being included in the coalition that forms, small parties get a similar share of the surplus as larger parties. However, very importantly, small parties are not included in the coalition that forms proportionally to their voting weights, and their ex-ante value/votes ratio are significantly less than the ex-ante value/vote ratio of larger parties.

Finally, the coalition bargaining solution predicts that only a few of the possible minimum winning coalitions actually form in equilibrium and the probability distribution of coalitions that form is very concentrated. For example, among the 134 games analyzed, there are on average 15.4 minimum winning coalitions, and the most likely coalition forms with an average probability of 53.5%, and the two most likely coalitions form with an average probability of 81.8%.

The concentration of coalitions forming in equilibrium is a noteworthy feature of the coalition bargaining solution concept. The underlying logic behind classical solution concepts such as the Shapley-Shubik and Banzhaf indices is that all minimum winning coalitions can form. Thus, there are also sharp differences in empirical predictions between the coalition bargaining solution and the classical solution concepts with respect to the equilibrium structure of coalition formation.

Related Literature on Coalition Bargaining

This paper belongs to the literature that examines bargaining protocols in which agreements to form coalitions are irreversible (see Ray (2007)). There is an extensive literature on coalition formation, and an excellent overview of the literature appears in Ray (2007) and the recent survey articles by Ray and Vohra (2014) and Bloch and Dutta (2011).[2] We review below only the literature focusing on models with an alternative structure to the sequential random or fixed order non-cooperative multilateral bargaining models.[3]

Our simultaneous coalition formation stage game has similarities with the link formation game developed by Dutta and Mutuswami (1997) and Dutta et. al (1998) in the context of network formation. Dutta and Mutuswami (1997) and Dutta et. al (1998) introduce a link formation game, in which players simultaneously announce a set of players with whom he or she wants to form a pairwise link, and a link is formed if both players want the link. Dutta and Mutuswami (1997) define a graph to be strongly stable (respectively weakly stable) if it corresponds to a strong Nash equilibrium (SNE) (respectively coalition-proof Nash equilibrium (CPNE)) of the link formation game. They focus on studying the tension between stability and efficiency in that framework. Dutta et. al (1998) study the endogenous formation of cooperation structures or communication between players, and also employ the concepts of strong and coalition-proof Nash equilibrium. They show that coalition-proof refinement leads to the formation of a full cooperation structure, and that SNE may not exist, but when it does it also leads to the formation of a full cooperation structure.

Hart and Kurz (1983) study the question of endogenous coalition formation using an axiomatic concept of value similar to the Shapley value (the CS-value developed by Owen (1977)) and a static model of coalition formation similar to ours. They propose the concept of stable coalition structures as the ones that are associated with a strong Nash equilibrium,

[2] The recent literature has focused on the possibility of externalities between coalitions (see for example Bloch (1996) and Ray and Vohra (1999)), but in this paper we do not allow for coalitional externalities. For more on a discussion of the Nash program, and the difficulties associated with generalizing from bilateral to multilateral bargaining, see Krishna and Serrano (1996).

[3] There is also an interesting literature that examines multilateral bargaining over policy and distributive dimensions (e.g., Banks and Duggan (2000, 2006) and Jackson and Moselle (2002)). This literature is not included in our discussion.

and show that strong equilibrium may in general not exist. Yi and Shin (2000) also study the issue of endogenous formation of research coalitions considering two static models of coalition formation: the exclusive membership game (which is similar to ours) and the open membership game (in which players simultaneously announce messages and coalitions are formed by all players who have announced the same message). The issue of multiplicity of equilibrium also arises in their setting. They use the CPNE refinement, and focus on the comparison of the relative efficiency of both coalition formation models.

Morelli (1999, 2007) (see also Frechette, Kagel and Morelli (2005) and Morelli and Montero (2003)) introduce the demand bargaining model of legislative bargaining in a majority rule environment. Agents are chosen in a random sequential order, with recognition probabilities proportional to voting weights, and each agent in turn make his or hers payoff demand until a majority coalition forms with a compatible set of demands. These models highlights the distinction of a proposer making an offer (which includes a coalition and a surplus division among coalition members) versus making a demand (leaving other agents with the choice between demanding the residual or disagreeing). One of the main motivations for the demand bargaining model is that the ex-post payoff distribution predicted by the model is proportional to the voting power within the majority coalition, without any first-mover advantage. Our model has some common modelling features with the demand bargaining model but in the coalition formation stage players do not make offers and instead only propose coalitions to join. Our model also has the same ex-post proportionality property of the demand bargaining model. However, the demand bargaining model also depends on an exogenous proposer probability and the ex-ante value predictions of both models are quite distinct.

Diermeier and Merlo (2000) and Baron and Diermeier (2001) propose a model of proto-coalition bargaining (see also Eraslan et al. (2003), Breitmoser (2012), and Montero (2015)). In these models a proposer (or formateur) is also randomly chosen with a given exogenous recognition probability, and the formateur then chooses a proto-coalition to bargain over the surplus allocation among proto-coalition members. The bargaining in the second stage varies across models. For example, in Baron and Diermeier (2001) it is by take-it-or-leave-it offers, and in Montero (2015) it is by offers and counter-offers with a risk of breakdown. Our model has a similar proto-coalition/bargain structure but the proto-coalition formation happens simultaneously (rather than with just one player proposing a coalition). The ex-ante value predictions of the models are significantly different as proposers recognized with higher probability will have a higher value in the proto-coalition/bargaining model.

Several studies focus on endogeneizing the recognition process to select proposers in multilateral bargaining settings. For example, Yildirim (2007) develops a model in which players exert costly effort and compete to be proposers. In Ali (2015) the right to propose an offer is sold to the highest bidder and all players pay their bids. Perez-Castrillo and Wettstein (2001, 2002) implement the Shapley value with a mechanism including a bidding stage for the right to be proposer (see also Xue and Zhang (2012) for a model of coalition formation with externalities with similar bidding mechanism). In Evans (1997) there is a contest for the right to make proposals and he shows that the equilibrium set coincides with the core. In contrast, in our study we focus on the simultaneous choice of proposer rather than a bidding mechanism.

Finally, Merlo and Wilson (1995) were the first to study multilateral bargaining games where the surplus follows a stochastic process. They restrict attention to games with unanimity where only the grand coalition can form. Eraslan and Merlo (2002) analyze symmetric majority rule games with stochastic surplus. They characterize the set of stationary subgame perfect equilibrium and show the equilibrium may not be unique.

The remainder of the paper is organized as follows: Section 2 presents the new coalitional bargaining model; Section 3 develops some basic intuition for the limit solution of the model and contrast it with the random proposer model solution; Section 4 develops the equilibrium strategies of the coalition formation and negotiation stage games; Section 5 analyzes the Markov perfect equilibrium (MPE) of the model; Section 6 addresses the properties of the limit MPE; Section 7 illustrates the properties of the model with an application to weighted majority games; finally, Section 8 concludes. The Appendix contains the proofs of all results.

2 The Model

Consider the problem of a finite set $N = \{1, ..., n\}$ of players who must choose which winning coalition to form, and how to divide the total surplus generated by the coalition among themselves. To avoid trivial cases, we shall assume that $n \geq 3$. Only winning coalitions of players generate surplus, and once a coalition forms, the game stops and no further coalitions can form. In our game, 0 is the payoff that each player obtains as long as no coalition has formed yet, and 0 is also the payoff a player gets when not a member of the coalition that forms (see also Compte and Jehiel (2010)).

The subset of all possible winning coalitions is denoted by W. We maintain throughout the paper the standard assumptions that W is (i) monotonic and (ii) proper: that is, (i) if $\mathrm{C} \in W$ then any coalition S containing C also belongs to W, and (ii) if coalition $\mathrm{C} \in W$ is a winning coalition then $N \backslash \mathrm{C} \notin W$ is not a winning coalition. These two properties combined yield that no two disjoint coalitions can both be winning.

The surpluses generated by coalitions are described by a vector $s = (s_{\mathrm{C}})_{\mathrm{C} \in W}$, where $s \in \mathbb{R}^W$ and $s_{\mathrm{C}} \in \mathbb{R}$ is the surplus generated by coalition C. This vector is also commonly known in the literature as a coalitional function (or characteristic) function. A simple game W is the special case where the surplus is $s_{\mathrm{C}} = 1$ if $\mathrm{C} \in W$ and $s_{\mathrm{C}} = 0$ if $\mathrm{C} \notin W$.

In our setting, the surpluses may receive shocks over time. The surplus realizations over time are described by a sequence $(s_t)_{t \geq 1}$ where $s_t \in \mathbb{R}^W$. Formally, the surpluses $(s_t)_{t \geq 1}$ are realizations of a stochastic process $(\mathbf{s}_t)_{t \geq 1}$, where the random vectors $\mathbf{s}_t$ are independent and identically distributed to the random vector $\mathbf{s} = (\mathbf{s}_{\mathrm{C}})_{\mathrm{C} \in W}$ with continuous density function $f(s)$ over $s \in \mathbb{R}^W$ with bounded support $\mathcal{S} \subset \mathbb{R}^W$, the set of all possible states.

We model the coalitional bargaining game as a dynamic game (muti-stage game with observed actions) over an infinite number of discrete periods $t = 1, 2, ..., \infty$ where every period has two stages. In the first stage, the coalition formation stage, all players choose which coalition to form. In the second stage, the negotiation stage, the members of the chosen coalition bargain over how to split the surplus generated by the coalition. All players maximize the expected discounted share of the surplus, and they have the same discount rate $\delta \in (0, 1)$. At every stage, players observe the complete history with all past actions and surplus realizations.

We now describe the extensive form of the coalition bargaining game, beginning with the coalition formation stage followed by the negotiation stage which takes place every period $t = 1, 2, ..., \infty$. We denote the coalition bargaining game by $\Gamma(\mathbf{s}, \delta)$.

Coalition Formation Stage: At the start of the t-period coalition formation stage, a random surplus $s_t = (s_{t,\mathrm{C}})_{\mathrm{C} \in W} \in \mathbb{R}^W$ is drawn. The surplus s_t is the state variable. It is observed by all players before making their choices in the coalition formation stage game. All agents simultaneously choose an action from the action set A_i where $A_i = W \cup \{\varnothing\}$, which is the set of winning coalitions augmented by the empty set. The action $a_i = \text{C} \in W$ means that player i wants to form coalition C and action $a_i = \varnothing$ means that player i does not want to form any coalition.

After all players choose their actions then a winning coalition is chosen if and only if all members of the coalition choose that coalition, otherwise no coalition is chosen. This rule uniquely determines the coalition in $W \cup \{\varnothing\}$ chosen for all possible action choices. Formally, following the action profile $\mathbf{a} = (a_i)_{i \in N}$, the coalition chosen for negotiation in the next stage is $\text{C} \in W$ only if all members $i \in \text{C}$ choose action $a_i = \text{C}$, and otherwise $\text{C} = \varnothing$.

This rule determines uniquely a coalition choice for all action profiles. It will be convenient to formally define this rule as the mapping C_f from action profiles into coalitions chosen as

$$\begin{aligned} \text{C}_f \quad &: \quad \prod_{i \in N} A_i \to W \cup \{\varnothing\}, \text{ where for all } \mathbf{a} = (a_i)_{i \in N}, \\ \text{C}_f(\mathbf{a}) \quad &= \quad \{\text{C} \in W : \text{ for all } i \in \text{C} \text{ then } a_i = \text{C}\}. \end{aligned} \tag{1}$$

This mapping is well-defined and $\text{C}_f(\mathbf{a}) \in W \cup \{\varnothing\}$ is a unique selection among $W \cup \{\varnothing\}$ because the set of winning coalitions W is monotonic and proper.[4] Once a coalition $\text{C} \in W$ is chosen, the game moves to the (s, C) −negotiation stage with the chosen coalition C and state s, and otherwise, if $\text{C} = \varnothing$, the game moves to the $t + 1$-period coalition formation stage.

Negotiation Stage: In the t-period negotiation game players in coalition C bargain on how to split the surplus s_{C} generated by the coalition. The bargaining stage is modelled in a standard way, and unanimity among the coalition members is required to split the surplus. The outcome of the negotiation is the Nash bargaining solution with an equal split of the surplus among coalition members where the status quo of the negotiation is the discounted continuation values.

The negotiation process follows Binmore, Rubinstein, and Wolinsky (1986) in which there is an exogenous risk of breakdown following a rejected offer. We propose in the

[4] Suppose by contradiction there exists an $\mathbf{a} \in \prod_{i \in N} A_i$ and two coalitions $\text{S} \in W$ and $\text{C} \in W$ with $\text{S} \neq \text{C}$ both in $\text{C}_f(\mathbf{a})$. Then $\text{S} \cap \text{C} = \varnothing$ because $a_i = \text{C}$ and $a_i = \text{S}$ cannot both be true. But since W is monotonic and proper then no two disjoint coalitions can both be winning. Note that $\text{C}_f(\mathbf{a}) = \varnothing$ if there are no coalitions $\text{C} \in W$ such that for all $i \in \text{C}$ then $a_i = \text{C}$.

Online Appendix (available at the author's website) an alternative formulation in which coalition members simultaneously make Nash demands for the surplus. Both formulations yield the Nash bargaining solution.

Formally, in the negotiation stage game (s, C), a player is chosen with equal probability among the members of coalition C.[5] The chosen player can either propose a surplus division $(x_i)_{i \in \text{C}}$, satisfying the budget constraint $\sum_{i \in \text{C}} x_i \leq s_{\text{C}}$, or terminate the current negotiation round and move the game to the next coalition formation period. Once a proposal is made the remaining coalition members, in a sequential order, respond whether they accept (yes) or decline (no) the offer. The coalition forms only if all responders accept the proposal. The order of response is arbitrarily given, but the order is irrelevant and does not affect the results.

In case of acceptance each player $i \in \text{C}$ receives a final payoff equal to x_i (and the players $i \notin \text{C}$ receive a final payoff of $x_i = 0$). Players' utility are $\delta^{t-1} x_i$ and the game ends. Otherwise, in case of any rejection, coalition C continues bargaining as above with probability ϕ, where $0 \leq \phi < 1$, with a new proposer choice, or there is a bargaining breakdown with probability $1 - \phi$, in which case the game moves to the next period with a new coalition formation round.[6]

We are interested in subgame perfect Nash equilibrium of the coalition bargaining game $\Gamma(\mathbf{s}, \delta)$ that are pure Markovian stationary strategies (see also Maskin and Tirole (2001)). A pure strategy profile is a stationary Markovian strategy if the strategies at every t-period stage game are time invariant and the strategies depend only on the current state or surplus realization, and are otherwise independent of any other aspects of the history of play, and past surplus realizations. We therefore omit references to the time period and game histories to simplify notation. We use the concept of Markov perfect equilibrium in our analysis.

Definition 1 *(Markov Perfect Equilibrium) A pure stationary Markovian strategy profile σ is a Markov perfect equilibrium (MPE) if σ is a subgame perfect Nash Equilibrium. That*

[5] The main difficulty in endogeneizing proposer probabilities arises in coalition formation games without unanimity. Note that at the negotiation stage, unanimity among the coalition members is required to split the surplus. Thus it is natural to assume equal proposing probabilities which corresponds to all players having equal bargaining power. The simultaneous move negotiation game proposed in the Online Appendix achieves a similar Nash bargaining solution without making assumptions about proposer probabilities.

[6] See Binmore, Rubinstein, and Wolinsky (1986), Stole and Zwiebel (1996), and Montero (2015) for similar negotiation models with an exogenous risk of breakdown.

is, each player i and after every history of play, σ_i is a best-response for player i when other players $-i$ play according to σ_{-i}.

Value functions. Associated with a Markovian strategy profile σ, we can define the ex-ante value v_i^σ to express the expected value of the discounted sum of future payoffs of player i, at the beginning of a period and before the surpluses are observed, assuming that all players follow strategy σ.

Similarly, we can define the values $v_i^\sigma(s, \mathrm{C})$ as the expected value of the discounted sum of future payoffs of player i at the beginning of the (s, C)–negotiation stage game induced by σ. We define $v_i^\sigma(s, \varnothing) = \delta v_i^\sigma$. And also the values $v_i^\sigma(s)$ as the expected value of the discounted sum of future payoffs of player i after the realization of state s at the beginning of the coalition formation stage.

We utilize two related refinements of the Nash equilibrium concept of the simultaneous move coalition formation model: the concept of strong Nash equilibrium, introduced by Aumann (1959), and the concept of coalition-proof Nash equilibrium, developed by Bernheim et al. (1987). In both refinements, an equilibrium profile at the coalition formation stage is immune not only to unilateral deviations but also to group deviations.

In order to formally define these concepts we use the coalition formation stage game denoted by $G^\sigma(s) \equiv \Gamma\left(N, (A_i)_{i\in N}, (u_i^\sigma)_{i\in N}\right)$, where s is the state realization. The game $G^\sigma(s)$ is the normal form game induced by strategy σ among the players N, in which players simultaneously choose actions $a_i \in A_i$, and their payoff are given by $u_i^\sigma(\mathbf{a}) \equiv v_i^\sigma(s, \mathrm{C}_f(\mathbf{a}))$.

Definition 2 *(Strong or coalition-proof Markov perfect equilibrium) A pure stationary Markovian strategy profile σ is a strong Markov perfect equilibrium (SMPE) or coalition-proof Markov perfect equilibrium (CPMPE) if it is a Markov perfect equilibrium and the restriction of σ to the coalition formation stage at any state s is, respectively, a strong Nash equilibrium or coalition-proof Nash equilibrium of the normal form game $G^\sigma(s)$ induced by σ.*

We postpone until Section 4 a more detailed discussion of the strong and coalition-proof Nash equilibrium concepts.

3 The Coalitional Bargaining Solution

In this section, we develop the basic intuition for the limit equilibrium of the simultaneous proposer coalition bargaining model and contrast it with the solutions of the random proposer coalition formation model of Baron and Ferejohn (1989).

Of particular interest to us is the limit equilibrium of the simultaneous proposer coalition bargaining game with stochastic surplus $\mathbf{s}^n$ converging in distribution to the surplus $\bar{s}$, i.e., $\mathbf{s}^n \to \bar{s}$, where $\bar{s} \in \mathbb{R}^W$. We refer to the limit coalition bargaining game as $\Gamma(\bar{s}, \delta)$.

One of the main results of this paper is that the limit Markov perfect equilibrium of $\Gamma(\mathbf{s}^n, \delta)$ when $\mathbf{s}^n \to \bar{s}$ is the coalitional bargaining solution (v, μ) defined below, where $v = (v_i)_{i \in N}$ are the players' values, and $\mu = (\mu_{\mathrm{C}})_{\mathrm{C} \in W}$ are the coalition formation probabilities (i.e., μ_{C} is equal to the probability that coalition C forms).

Definition 3 *The coalitional bargaining solution* (v, μ) *of game* $\Gamma(\bar{s}, \delta)$, *where* $\bar{s} = (\bar{s}_{\mathrm{C}})_{\mathrm{C} \in W}$, *is given by the value* $v = (v_i)_{i \in N}$ *and probability of coalition formation* $\mu = (\mu_{\mathrm{C}})_{\mathrm{C} \in W}$, *solution of the problem*

$$\begin{aligned} v_i &= \sum_{\mathrm{C} \in W} (\delta v_i + \gamma) I(i \in \mathrm{C}) \mu_{\mathrm{C}} \text{ and } \sum_{\mathrm{C} \in W} \mu_{\mathrm{C}} = 1, \\ \text{where } \mu_{\mathrm{C}} &> 0 \text{ only if } \gamma \equiv \max_{\mathrm{C} \in W} \gamma_{\mathrm{C}} \text{ and } \gamma_{\mathrm{C}} \equiv \frac{1}{|\mathrm{C}|} \left(\bar{s}_{\mathrm{C}} - \delta \sum_{j \in \mathrm{C}} v_j \right). \end{aligned} \tag{2}$$

A key property of the solution is that only coalitions that maximize the average gain per coalition member $\gamma_{\mathrm{C}} = \frac{1}{|\mathrm{C}|} \left(\bar{s}_{\mathrm{C}} - \delta \sum_{j \in \mathrm{C}} v_j \right)$ form in equilibrium. Moreover, the value of player i, conditional on coalition C forming during the coalition formation stage, is equal to $v_i(\mathrm{C}) = \delta v_i + \gamma_{\mathrm{C}}$, if $i \in \mathrm{C}$, and is otherwise equal to zero. The (unconditional) value of player i is given by the average $v_i = \sum_{\mathrm{C} \in W} (\delta v_i + \gamma_{\mathrm{C}}) I(i \in \mathrm{C}) \mu_{\mathrm{C}}$, where μ_{C} is the probability that coalition C forms in equilibrium.

The simultaneous proposer and the random proposer models have very different properties. In order to illustrate the differences we briefly describe the random proposer model and develop its solution.

In the random proposer model (for example, see Snyder et al (2005), Compte and Jehiel (2010), and Eraslan and McLennan (2010)), a player i is chosen to be the proposer with exogenously given probability p_i (and $\sum_{i \in N} p_i \leq 1$). The proposer i chooses a coalition

coalition $\text{C} \in W$ (with $i \in \text{C}$) with probability $\sigma_{i\text{C}}$ such that $\sum_{\text{C} \in W} \sigma_{i\text{C}} = 1$, and makes a proposal $\left(x_{i,j}^{\text{C}}\right)_{j \in \text{C}}$ to share the surplus satisfying the feasibility constraint $\sum_{j \in \text{C}} x_{i,j}^{\text{C}} \leq \bar{s}_{\text{C}}$. Players in C respond whether they accept or reject the proposal. If they all accept, the coalition C forms and each coalition member $j \in \text{C}$ gets a payoff equal to $x_{i,j}^{\text{C}}$ (and non-coalition members get 0). If one or more coalition members rejects the proposal, the game moves to the next period, and the steps above repeat after a time delay (which discounts payoffs by δ).

The Markov perfect equilibrium of the random proposal model is characterized by the set of equations and inequalities below, in which the players' value are given by v_i, and $\sigma_{i\text{C}}$ denotes the probability player i chooses coalition C whenever i is chosen to be the proposer (the proposer offers are $x_{i,j}^{\text{C}} = \delta v_j$ and are accepted by all responders):

$$\begin{aligned} v_i &= p_i e_i + \sum_{j \in N} p_j \sum_{\text{C} \in W} \delta v_i I(i \in \text{C}) \sigma_{j\text{C}} \text{ and } \sum_{\text{C} \in W:\, i \in C} \sigma_{i\text{C}} = 1, \\ \text{where } \sigma_{i\text{C}} &> 0 \text{ only if } e_{\text{C}} = e_i \equiv \max_{\text{S}: i \in \text{S} \in W} e_{\text{S}} \text{ where } e_{\text{C}} \equiv \bar{s}_{\text{C}} - \delta \sum_{j \in \text{C}} v_j. \end{aligned} \tag{3}$$

In the random proposer model, each player i only chooses coalitions that maximizes the gain $e_{\text{C}} \equiv \bar{s}_{\text{C}} - \delta \sum_{j \in \text{C}} v_j$ among all coalitions C such that $i \in \text{C} \in W$. The overall probability that coalition C forms is given by $\mu_{\text{C}} \equiv \sum_{i \in N} p_i \sigma_{i\text{C}}$.

In this model, the unconditional or ex-ante value of player i is v_i, and the value of player i conditional on player i being the proposer is $\delta v_i + e_i$. Player i's value is equal to zero if player i is not included in the coalition C that is proposed (which happens with probability $\sum_{\text{C} \in W} \mu_{\text{C}} I(i \notin \text{C})$), and finally, it is equal to δv_i if player i is included in the coalition C, proposed by other players $j \neq i$ (which happens with probability $\sum_{j \in N \setminus i} p_j \sigma_{j\text{C}}$).

We provide a detailed example in the Appendix illustrating the differences between both models. In particular, the example shows that the simultaneous proposer model, unlike the random proposer model, predicts that dummy players have zero value and should never be included in any coalition that forms in equilibrium. Sections 6 and 7 develop a more systematic comparison between the coalition bargaining solution and the random proposer model as well as other classical cooperative solutions, such as the Shapley value, the Banzhaf value, and the nucleolus, illustrating several other distinct properties among the solutions.

4 Stage Game Equilibrium

4.1 The Negotiation Stage

We first analyze the negotiation stage subgame. The negotiation stage is a fairly standard random proposer multilateral bargaining model in which a group of players are splitting a known surplus and unanimity is required to reach agreement. The status quo of negotiations are the players continuation values and there is an exogenous risk of breakdown, which is a friction with similarities to discounting (see Binmore et al (1986)). In this setting, it is well-known that the Nash bargaining solution with equal split of the gain from cooperation among coalition members arises as the equilibrium outcome.

Formally, consider a MPE σ with expected value v_i^σ and a negotiation stage subgame beginning at node (s, C) with state s and coalition C. Let $v_i \equiv v_i^\sigma$ and $v_\text{C} \equiv \sum_{i \in \text{C}} v_i$. Whenever the surplus is less than the coalition continuation value, $s_\text{C} < \delta v_\text{C}$, the negotiation stage terminate without an agreement and the game moves to the next coalition formation round. Certainly this is the case because there are no gains from cooperation.

On the other hand, if the surplus is greater than the coalition continuation value, agreement in the negotiation occurs immediately. The gain from cooperation among the coalition members C is $s_\text{C} - \delta v_\text{C} > 0$, and thus the average gain per coalition member is $\gamma(s, \text{C}) \equiv \frac{1}{|\text{C}|}(s_\text{C} - \delta v_\text{C})$. The expected value of players $v_i^\sigma(s,\text{C})$ is the Nash bargaining solution $v_i^\sigma(s,\text{C}) = \delta v_i + \gamma(s, \text{C})$ for all $i \in$C.

These well-known results from unanimity bargaining are formally stated in the following proposition.

Proposition 1 *(Negotiation Stage) Consider a MPE σ with expected value $v_i = v_i^\sigma$. Then the value $v_i^\sigma(s,\text{C})$ at the negotiation stage with state s and coalition C is:*
(i) If $s_\text{C} > \delta v_\text{C}$, the value is given by

$$v_i^\sigma(s, \text{C}) = \begin{cases} \delta v_i + \gamma(s, \text{C}) & \text{if } i \in \text{C} \\ 0 & \text{if } i \notin \text{C} \end{cases}, \tag{4}$$

where $v_\text{C} = \sum_{i \in \text{C}} v_i$ and the average gain is $\gamma(s, \text{C})$ defined by

$$\gamma(s, \text{C}) = \frac{1}{|\text{C}|}(s_\text{C} - \delta v_\text{C});$$

(ii) If $s_\text{C} \leq \delta v_\text{C}$, the value $v_i^\sigma(s,\text{C}) = \delta v_i$ for all $i \in$C and $v_i^\sigma(s,\text{C}) \leq \delta v_i$ for all $i \notin$C .

The equilibrium outcome is the Nash bargaining solution among the players in coalition C with status quo equal to the discounted continuation values and gain equal to $s_{\text{C}} - \delta v_{\text{C}}$. The proof essentially follows the discussion below, and since it is quite standard it is omitted.

The negotiation strategy is for each player $i \in \text{C}$, when proposing, to offer $\delta v_j + \phi\gamma(s, \text{C})$ to the other players $j \in \text{C} \backslash \{i\}$. Upon rejection of an offer the expected continuation value of player $j \in \text{C}$ is $\phi v_j^{\sigma}(s,\text{C}) + (1-\phi)\,\delta v_j = \delta v_j + \phi\gamma(s, \text{C})$. This is so because with probability ϕ, negotiations continue and player's j continuation value is $v_j^{\sigma}(s,\text{C})$, and with probability $(1-\phi)$, negotiations break-down and player's j continuation value is δv_j. Thus the response strategy–accept offer x if and only if $x \geq \delta v_j + \phi\gamma(s, \text{C}|v)$–is a best response strategy for player j.

Moreover, note that the expected value of player $i \in \text{C}$ at the beginning of the negotiation stage subgame is indeed equal to $v_i^{\sigma}(s,\text{C}) = \delta v_i + \gamma(s, \text{C})$. This value arises because with probability $\frac{|\text{C}|-1}{|\text{C}|}$ he receives an offer $\delta v_i + \phi\gamma(s, \text{C})$, and with probability $\frac{1}{|\text{C}|}$ he is the proposer, receiving the surplus net of payments to the other coalition members, $s_{\text{C}} - \sum_{j \in \text{C}\backslash\{i\}} [\delta v_j + \phi\gamma(s, \text{C})]$. The expected value is thus

$$v_i^{\sigma}(s, \text{C}) = \frac{|\text{C}|-1}{|\text{C}|}\left(\delta v_i + \phi\gamma(s, \text{C})\right) + \frac{1}{|\text{C}|}\left(s_{\text{C}} - \sum_{j \in \text{C}\backslash\{i\}} [\delta v_j + \phi\gamma(s, \text{C})]\right). \tag{5}$$

In the knife-edge case where $s_{\text{C}} = \delta v_{\text{C}}$, there is no gain from cooperation, and the negotiation may terminate with an agreement or not. In either case, the continuation values are $v_i^{\sigma}(s,\text{C}) = \delta v_i$ for all $i \in \text{C}$, but the values of the players $i \notin \text{C}$ are indeterminate, and all we can say is that $v_i^{\sigma}(s,\text{C}) \leq \delta v_i$, since the value of player $i \notin \text{C}$ is zero if there is an agreement and, otherwise, is equal to δv_i. Note that this case has Lebesgue measure zero due to the absolute continuity of the surplus distribution.

We now analyze the equilibrium of the coalition formation stage game.

4.2 The Coalition Formation Stage

The simultaneous move coalition formation game has numerous Nash equilibria. In fact any coalition $\text{C} \in W$ or $\text{C} = \varnothing$ can be supported as a Nash equilibrium. For example, consider the strategy profile $\mathbf{a} = (a_i)_{i \in N}$ where $a_i = \text{C}$ for all $i \in \text{C}$, and $a_i = \varnothing$ for all $i \in N \backslash \text{C}$. This profile is a Nash equilibrium, as no unilateral deviation by players improve their payoff, and

$\textsc{c}_f(\mathbf{a}) = \textsc{c}$.[7]

We utilize in this section two refinements of the Nash equilibrium concept–strong Nash equilibrium and coalition-proof Nash equilibrium–that are immune not only to unilateral deviations but also to group deviations. Both concepts are plausible and stable solution concepts in a setting where players are freely allowed to discuss their strategies, but cannot make any binding commitments.

Strong Nash equilibrium (SNE) is a refinement of Nash equilibrium introduced by Aumann (1959), in which no coalition of players can gain by any joint (or group) deviations. Coalition-proof Nash equilibrium ($CPNE$) was developed by Bernheim et al. (1987). This concept differs from the SNE concept in that instead of all possible group deviations, only group deviations that are self-enforcing are allowed.

Bernheim et al. (1987) claim that strong Nash equilibrium is for many games "too strong" and it almost never exists (i.e., $SNE = \varnothing$). In general, for all games, Bernheim et al. (1987) shows that $SNE \subset CPNE$ (this result follows immediately from the definitions of both concepts).

We prove below that the coalition formation game always has strong Nash equilibria. Moreover, we show that strong Nash equilibrium is essentially unique, and it always coincides with the coalition-proof Nash equilibrium. Thus, remarkably, for the coalition formation game, we are able to establish that $SNE = CPNE$.[8]

We focus the analysis on the one-shot coalition formation stage game denoted by $G^\sigma(s)$, where s is the state realization. The players' payoff in the coalition formation stage game $G^\sigma(s)$ are $u_i^\sigma(\mathbf{a}) \equiv v_i^\sigma(s, \textsc{c}_f(\mathbf{a}))$, for all $i \in N$, where $v_i^\sigma(s, \textsc{c})$ was determined in Proposition 1. We define $v_i^\sigma(s, \varnothing) = \delta v_i$ for the continuation values when no winning coalition forms. Coalitions form only on states s belonging to the set $\mathcal{A} \equiv \{s \in \mathcal{S} : s_\textsc{c} > \delta v_\textsc{c} \text{ for some } \textsc{c} \in W\}$. There are no gains from cooperation whenever $s \in \mathcal{A}^c$, and thus no coalitions form.

The formal definitions of strong Nash equilibrium and coalition-proof Nash equilibrium are provided below. We use the standard notation that given a strategy $\mathbf{a} = (a_i)_{i \in N}$ then

[7] Indeed, for all $i \in N \backslash \textsc{c}$, any unilateral deviations by player i do not change the outcome $\textsc{c}_f(\mathbf{a}) = \textsc{c}$; and, for all $i \in \textsc{c}$, any deviation from $a_i = \textsc{c}$ leads to no coalition forming which yields a payoff $\delta v_i \leq \delta v_i + \gamma(s, \textsc{c})$, since $\gamma(s, \textsc{c}) \geq 0$.

[8] This result typically does not hold for most games. However, for the class of common agency games studied by Bernheim and Whinston (1986) and Konishi et al. (1999), the equality $SNE = CPNE$ does hold under some mild additional restrictions.

$\mathbf{a}_{\mathrm{S}} = (a_i)_{i\in \mathrm{S}}$ is the strategies chosen by the subset S of players and $\mathbf{a}_{-\mathrm{S}} = (a_i)_{i\in N\backslash \mathrm{S}}$ is the strategies chosen by its complement $N\backslash$S, and $\mathbf{a} = (\mathbf{a}_{\mathrm{S}}, \mathbf{a}_{-\mathrm{S}})$.

Definition 4 *(Strong Nash equilibrium) A strategy $\mathbf{a} = (a_i)_{i\in N}$ is strong Nash equilibrium of a normal form game Γ with payoff $u_i(\mathbf{a})$ if and only if for all $\mathrm{S}\subset N$ and for all deviations $\mathbf{a}'_{\mathrm{S}}$ by the players S there exists an agent $i \in \mathrm{S}$ such that $u_i(\mathbf{a}) \geq u_i(\mathbf{a}'_{\mathrm{S}}, \mathbf{a}_{-\mathrm{S}})$.*

The coalition-proof Nash equilibrium definition is provided recursively (see Bernheim et al. (1987)).

Definition 5 *(Coalition-proof Nash equilibrium) (i) In a normal form game Γ with a single player ($n = 1$) (a_1^*) is a CPNE if and only if it is a Nash equilibrium.*
(ii) In a normal form game Γ where $n > 1$, the profile $\mathbf{a}^* = (a_i^*)_{i\in N}$ is self-enforcing if for all proper subsets $\mathrm{S}\subset N$, $\mathbf{a}^*_{\mathrm{S}}$ is a CPNE of the restriction game $\Gamma/\mathbf{a}^*_{-\mathrm{S}}$.
(iii) A profile $\mathbf{a}^*$ is CPNE if it is self-enforcing and there is no other self-enforcing profile $\mathbf{a}$ that yields higher payoff to all players.

We show below that the simultaneous coalition formation game always has strong Nash equilibrium and this equilibrium is essentially unique. Moreover, we show that the set of coalition-proof Nash equilibrium coincides with the set of strong Nash equilibrium. Below we denote by $\mathrm{C}^{*\sigma}(s)$ (or just by $\mathrm{C}^*(s)$) the coalition that forms after the surplus realization s at the end of the coalition formation stage when the strategy profile σ is being played.

Proposition 2 *(Coalition Formation Stage) Let σ be either a strong or coalition-proof Markov perfect equilibrium of $\Gamma(\mathbf{s}, \delta)$, and let $v = v^\sigma$. Then:*
(i) At the coalition formation stage with state $s \in \mathcal{A}$, the coalition that forms under strategy σ is $\mathrm{C}^{*\sigma}(s) \in \arg\max_{\mathrm{C}\in W} \gamma(s, \mathrm{C})$, which maximizes the average gain $\gamma(s,\mathrm{C}) \equiv \frac{1}{|\mathrm{C}|}(s_{\mathrm{C}} - \delta v_{\mathrm{C}})$, where $v_{\mathrm{C}} = \sum_{i\in \mathrm{C}} v_i$ and $\mathcal{A} \equiv \{s \in \mathcal{S} : s_{\mathrm{C}} > \delta v_{\mathrm{C}}$ for some $\mathrm{C} \in W\}$. Moreover, for all $s \in \mathcal{A}$, then

$$v_i^\sigma(s) = \begin{cases} \delta v_i + \gamma(s, \mathrm{C}^{*\sigma}(s)) & \text{if } i \in \mathrm{C}^{*\sigma}(s) \\ 0 & \text{if } i \notin \mathrm{C}^{*\sigma}(s) \end{cases},$$

and $v_i^\sigma(s) = \delta v_i$ for all $s \in \mathcal{A}^c$, except in a set of Lebesgue measure zero where $v_i^\sigma(s) \leq \delta v_i$.
(ii) The set of all (pure strategy) strong Nash equilibrium (SNE) and coalition-proof Nash

equilibrium (CPNE) of the normal form game $G^{\sigma}(s)$, at any state $s \in \mathcal{A}$, is identical and given by

$$SNE \equiv CPNE \equiv \left\{ \mathbf{a} = (a_i)_{i \in N} : a_i = \text{C}^* \textit{for all } i \in \text{C}^* \textit{ where } \text{C}^* \in \underset{\text{C} \in W}{\operatorname{argmax}} \, \gamma(s, \text{C}) \right\}.$$

Furthermore, for almost all states $s \in \mathcal{A}$, except in a set of Lebesgue measure zero, there is a unique maximizing coalition in $\operatorname{argmax}_{\text{C} \in W} \gamma(s, \text{C})$, given σ.

The key property of the strong and coalition-proof Nash equilibrium of the coalition formation game, is that a coalition maximizing the average gain $\gamma(s, \text{C})$ always forms. And this maximizing coalition, except in a set of Lebesgue measure zero of the state space, is uniquely determined.

Intuitively, coalitions that do not maximize the average gain are dominated by coalitions that maximize the average gain. This occurs because once a coalition C is proposed players' conditional values in the negotiation stage are $\delta v_i + \gamma(s, \text{C})$. Thus there is an alignment in incentive among all players forming a coalition, since they all gain $\gamma(s, \text{C})$ when forming coalition C, and they all want to choose coalitions that maximize the average gain. Unfortunately, for players that do not belong to any average maximizing coalition, they will never be included in any winning coalition that arises in equilibrium. This incentive alignment is the key property that guarantees the existence and essential uniqueness of strong and coalition-proof Nash equilibrium in the coalition formation stage.

5 Markov Perfect Equilibrium

We now prove the existence of a strong or coalition-proof Markov perfect equilibrium for any coalition bargaining game $\Gamma(\mathbf{s}, \delta)$ and provide a complete characterization result.

Key for the characterization proposition below are the results of Propositions 1 and 2 combined with the application of the one-stage deviation principle (Fudenberg and Tirole, (1991, Th. 4.2).

Proposition 3 *Consider a coalition bargaining game $\Gamma(\mathbf{s}, \delta)$. Let σ be either a strong or coalition-proof Markov perfect equilibrium. Then:*
(i) The value v^{σ} associated with σ (let $v = v^{\sigma}$) satisfies the system of equations, for all

$i \in N$,

$$v_i = \sum_{\mathrm{C} \in W} \int_{\mathcal{A}} \left(\delta v_i + \gamma\left(s, \mathrm{C}\right)\right) I(i \in \mathrm{C}) I\left(\mathrm{C}^*\left(s\right) = \mathrm{C}\right) f(s) ds + \delta v_i \int_{\mathcal{A}^c} f(s) ds, \tag{6}$$

where the coalition that forms, at the coalition formation stage with state $s \in \mathcal{A}$, is $\mathrm{C}^*(s) \in \arg\max_{\mathrm{C} \in W} \gamma(s, \mathrm{C})$, maximizer of the average gain $\gamma(s, \mathrm{C}) \equiv \frac{1}{|\mathrm{C}|}(s_{\mathrm{C}} - \delta v_{\mathrm{C}})$, where $v_{\mathrm{C}} = \sum_{i \in \mathrm{C}} v_i$ and $\mathcal{A} \equiv \{s \in \mathcal{S} : s_{\mathrm{C}} > \delta v_{\mathrm{C}}$ for some $\mathrm{C} \in W\}$.
Reciprocally, given any value $v \in \mathbb{R}^n_+$ satisfying (i) above, the strategy profile σ defined below by (ii) is a strong and coalition-proof Markov perfect equilibrium:
(ii.a) At the coalition formation stage with state s the strategy is: for any $s \in \mathcal{A}$, $a_i(s) = \mathrm{C}^*(s)$ for all $i \in \mathrm{C}^*(s)$ and $a_i(s) = \varnothing$ for all $i \notin \mathrm{C}^*(s)$; and for any $s \in \mathcal{A}^c$, $a_i(s) = \varnothing$ for all $i \in N$.
(ii.b) At the negotiation stage (s, C) the strategy is: for any $s_{\mathrm{C}} > \delta v_{\mathrm{C}}$, player $i \in \mathrm{C}$ when proposing offers $\delta v_j + \phi\gamma(s, \mathrm{C})$ to players $j \in \mathrm{C} \backslash \{i\}$, and each player j when responding accepts an offer if and only if it is greater than or equal to $\delta v_j + \phi\gamma(s, \mathrm{C})$; for any $s_{\mathrm{C}} \leq \delta v_{\mathrm{C}}$, all players $i \in \mathrm{C}$ propose to terminate negotiations (pass), and each player j when responding accepts an offer if and only if it is greater than or equal to δv_j.

The proposition provides a provide a complete characterization of the SMPE and CPMPE. The average gain is equal to $\gamma(s, \mathrm{C}) = \frac{1}{|\mathrm{C}|}(s_{\mathrm{C}} - \delta v_{\mathrm{C}})$, whenever $s_{\mathrm{C}} > \delta v_{\mathrm{C}}$. The key property of the equilibrium is that the coalition forming is ultimately the one that maximizes the average gain, given any surplus realization s of the stochastic surplus $\mathbf{s}$ with gains from trade.

The value at the beginning of the (s, C)–negotiation stage game is simply the Nash bargaining solution: $v_i(s, \mathrm{C}) = \delta v_i + \gamma(s, \mathrm{C})$ for all $i \in \mathrm{C}$–the equal split of the surplus among coalition members and status quo equal to the (discounted) continuation values. There is no disagreement in the negotiation stage, whenever a coalition C with an aggregate surplus larger than aggregate discounted continuation value is proposed. Alternatively, disagreement or break-down in negotiations, is guaranteed when the opposite inequality holds, i.e., $s_{\mathrm{C}} < \delta v_{\mathrm{C}}$.

The next proposition establishes the existence of strong and coalition-proof Markov perfect equilibrium. We construct a mapping $\Phi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ which takes values $v \in \mathbb{R}^n_+$ into expected values. The mapping is defined, for all $i \in N$, as

$$(\Phi(v))_i = \sum_{\text{C} \in W} \int_{\mathcal{A}_{\text{C}}(v)} (\delta v_i + \gamma(s, \text{C}|v)) I(i \in \text{C}) f(s) \, ds + \int_{\mathcal{A}^c(v)} \delta v_i f(s) \, ds,$$

where $\gamma(s, \text{C}|v) \equiv \frac{1}{|\text{C}|}(s_{\text{C}} - \delta v_{\text{C}})$ is the average gain, with the dependency with respect to the value v made explicit (and we define $\gamma(s, \varnothing|v) = 0$). The region with the states where coalition C$\in W$ is chosen is given by

$$\mathcal{A}_{\text{C}}(v) \equiv \{s \in \mathcal{S} : \gamma(s, \text{C}|v) > \gamma(s, \text{T}|v) \text{ for all T} \in W \cup \varnothing\}. \tag{7}$$

We have shown in the previous proposition that $\text{C}^{*\sigma}(s) = $C, for all $s \in \mathcal{A}_{\text{C}}(v^\sigma)$. The set $\mathcal{A}^c(v) \equiv \{s \in \mathcal{S} : s_{\text{C}} \leq \delta v_{\text{C}} \text{ for all C} \in W\}$ is the region with the states where there are delays, and no coalition is chosen. The integrands and the regions vary continuously with v, and thus clearly the mapping $\Phi(v)$ is continuous in v.[9]

We prove that the continuous mapping Φ has a fixed point (i.e., $v = \Phi(v)$) using the Brouwer fixed point theorem. The fixed points of the mapping Φ correspond to the solutions of equation (6). Thus, by reciprocal of Proposition 3, we obtain the existence result below.

Proposition 4 *There always exists a strong and a coalition-proof Markov perfect equilibrium σ for any coalition bargaining game $\Gamma(\mathbf{s}, \delta)$.*

6 Limit Markov Perfect Equilibrium: The Coalition Bargaining Solution

In non-cooperative coalition bargaining models, typically in order for equilibrium to exist, players have to be allowed to play mixed strategies when choosing coalitions. In our set-

[9] In fact, these regions can be explicitly obtained. Note that $\gamma(s, \text{C}|v) > \gamma(s, \text{T}|v)$ corresponds to $s_{\text{T}} < \frac{|\text{T}|}{|\text{C}|}(s_{\text{C}} - \delta v_{\text{C}}) + \delta v_{\text{T}}$, and thus

$$\mathcal{A}_{\text{C}}(v) = \left\{ s \in \mathcal{S} : s_{\text{C}} < \delta v_{\text{C}} \text{ and } s_{\text{T}} < \frac{|\text{T}|}{|\text{C}|}(s_{\text{C}} - \delta v_{\text{C}}) + \delta v_{\text{T}} \text{ for all T} \in W \backslash \{\text{C}\} \right\}.$$

Thus the probability that coalition C$\in W$ forms can be obtained directly from the multivariate integral over all T$\in W \backslash \{\text{C}\}$,

$$\Pr(s \in \mathcal{A}_{\text{C}}(v)) = \int_{\delta v_{\text{C}}}^{\infty} \int_{-\infty}^{\frac{|\text{T}|}{|\text{C}|}(s_{\text{C}} - \delta v_{\text{C}}) + \delta v_{\text{T}}} ... f(s_{\text{C}}, s_{\text{T}}, ...) \, ds_{\text{C}} ds_{\text{T}} ...,$$

and the probability that no coalition form can be obtained directly from the cumulative distribution function $F\left((\delta v_{\text{C}})_{\text{C} \in W}\right)$.

ting, players use pure strategy and the randomization comes from small perturbations of the surplus. This approach somewhat resembles Harsanyi (1973)'s purification results for normal form games, in which players have some minor private information about their own payoff. However, differently from Harsanyi (1973), all players in our game know the surplus realizations drawn every period, and there is no incomplete information.

We now characterize the limit equilibrium of the coalition bargaining game $\Gamma(\mathbf{s}^n, \delta)$ when $\mathbf{s}^n$ converges in distribution to a deterministic coalitional function $\bar{s} = (\bar{s}_{\mathrm{C}})_{\mathrm{C} \in W}$. Our main result is that the limit equilibrium is a coalition bargaining solution defined in Section 3.

Define by $\mu_{\mathrm{C}}^{\sigma} \equiv \Pr(\mathrm{C}^{*\sigma}(s) = \mathrm{C})$ the probability that coalition $\mathrm{C} \in W$ forms, when the strategy profile σ is being played. Note that

$$\mu_{\mathrm{C}}^{\sigma} = \Pr(s \in \mathcal{A}_{\mathrm{C}}(v^{\sigma})) = \int I(\mathrm{C}^{*\sigma}(s) = \mathrm{C}) f(s)\, ds,$$

where $\mathcal{A}_{\mathrm{C}}(v)$ is given by (7). Let $\mu^{\sigma} = (\mu_{\mathrm{C}}^{\sigma})_{\mathrm{C} \in W}$.

We assume that the limit coalitional function $\bar{s}$ satisfies $\bar{s}_{\mathrm{C}} > 0$ for some $\mathrm{C} \in W$ (otherwise, no coalition would create value and ever form in the limit).

The key result is that $(v^{\sigma_n}, \mu^{\sigma_n})$ converge to a coalition bargaining solution (v, μ) (see Definition 3), where σ_n is a SMPE or CPMPE of the coalition bargaining model $\Gamma(\mathbf{s}^n, \delta)$.

Proposition 5 *Consider a sequence of stochastic shocks $(\mathbf{s}^n)_{n \in \mathbb{N}}$ converging in distribution to a deterministic surplus $\bar{s} = (\bar{s}_{\mathrm{C}})_{\mathrm{C} \in W}$ satisfying $\bar{s}_{\mathrm{C}} > 0$ for some $\mathrm{C} \in W$: $\mathbf{s}^n \rightarrow \bar{s}$. Let σ_n be either a strong or coalition-proof Markov perfect equilibrium of the coalition bargaining model $\Gamma(\mathbf{s}^n, \delta)$. Then there exists a convergent subsequence (k_n) with limits $v = \lim_{n \to \infty} v^{\sigma_{k_n}}$ and $\mu_{\mathrm{C}} = \lim_{n \to \infty} \mu_{\mathrm{C}}^{\sigma_{k_n}}$ for all $\mathrm{C} \in W$. Moreover, any limit (v, μ) must be a solution of the following problem:*
(i) The value v satisfies the system of equations

$$v_i = \sum_{\mathrm{C} \in W} (\delta v_i + \gamma)\, I(i \in \mathrm{C}) \mu_{\mathrm{C}}, \text{ for all } i \in N, \tag{8}$$

where $\gamma = \max_{\mathrm{C} \in W} \gamma(\mathrm{C})$ and the average gain $\gamma(\mathrm{C})$ is given by

$$\gamma(\mathrm{C}) = \frac{1}{|\mathrm{C}|} \left(\bar{s}_{\mathrm{C}} - \delta \sum_{i \in \mathrm{C}} v_i \right).$$

(ii) Only the coalitions that maximize the average gain form in equilibrium, that is, $\mu_{\text{C}} > 0$ *only if* $\gamma(\text{C}) = \gamma$.
(iii) The limit continuation value $v_i(\text{C})$, *conditional on coalition* $\text{C} \in W$ *forming, is*

$$v_i(\text{C}) = \begin{cases} \delta v_i + \gamma(\text{C}) & \text{if } i \in \text{C} \\ 0 & \text{if } i \notin \text{C} \end{cases} .$$

(iv) There is no delay in coalition formation (i.e., $\sum_{\text{C}\in W} \mu_{\text{C}} = 1$*) and the maximum average gain is strictly positive* $\gamma > 0$.

The expression characterizing the limit value function simplifies to equations (8), where μ_{C} is the probability that coalition C forms in equilibrium, and the average gain γ is the maximum average gain among all coalitions ($\gamma = \max_{\text{C}\in W} \gamma(\text{C})$).

Conditional on coalition C forming, the value of player $i \in$C is equal to $\delta v_i + \gamma(\text{C})$, and the value of all players $i \notin$C are zero, since these players are excluded from the coalition that forms and the game ends. Therefore, we obtain that the unconditional expected value, before any coalition forms, must satisfy the value equation (8).

In equilibrium, only coalitions that maximize the average gain $\gamma(\text{C})$ form, and in the limit, with probability one, a coalition always forms in the first period (i.e., $\sum_{\text{C}\in W} \mu_{\text{C}} = 1$).

In the remainder of this section we discuss some basic properties of the coalition bargaining solution for games with dummy players and simple games.

Dummy Players: One important distinction between the model in this paper and the random proposer model is the prediction for the value of dummy players.

Dummy players are players that do not contribute anything to the surplus of coalitions in which they are members. Formally, we say that player i is a dummy player of a coalitional function $\bar{s} = (\bar{s}_{\text{C}})_{\text{C}\in W}$ if for any coalition $\text{C}\in W$ which includes player i, that is $i \in$C, then the coalition $\text{C}\backslash\{i\}$ excluding player i also belongs to W, and both surpluses are identical, i.e. $\bar{s}_{\text{C}} = \bar{s}_{\text{C}\backslash\{i\}}$.

The simultaneous proposer model, unlike the random proposer model, predicts that dummy players have zero value and should never be included in any coalition that forms in equilibrium.

Corollary 1 *Let* (v, μ) *be a coalitional bargaining solution of game* $\Gamma(\bar{s}, \delta)$ *where player* i *is a dummy player of the coalitional function* $\bar{s}$*. The dummy player value is always equal* *to zero, $v_i = 0$, and the dummy player is never included in any coalition that forms in equilibrium, $\mu_{\text{C}} = 0$ for all $\text{C} \in W$ such that $i \in \text{C}$.*

Interestingly, the coalition bargaining solution thus make similar predictions for the value of dummy players as classical cooperative solution concepts such as Shapley value, the Banzhaf value, and the nucleolus.

Simple Games: Consider a simple game (where $\bar{s}_{\text{C}} = 1$ if $\text{C} \in W$ and $\bar{s}_{\text{C}} = 0$ otherwise). We show below that when evaluating the coalition bargaining solution of simple games we just need to consider the subset of minimum winning coalitions $W^m \subset W$, because the support of the coalition formation probability μ is contained in the set of minimum winning coalitions W^m.

Remember that the set of minimum winning coalitions is defined as

$$W^m \equiv \{\text{C} \in W : \text{S} \subsetneqq \text{C} \Rightarrow \text{S} \notin W\}.$$

We formalize this result in the following corollary.

Corollary 2 *Let (v, μ) be a coalition bargaining solution of a simple game $\Gamma(\bar{s}, \delta)$; where $\bar{s}_{\text{C}} = 1$ if $\text{C} \in W$ and $\bar{s}_{\text{C}} = 0$ otherwise. Then the support of the coalition formation probability μ is contained in the set of minimum winning coalitions W^m (supp$\mu \subset W^m$). Moreover, the solution (v, μ) can be obtained by solving the problem constrained to W^m:*

$$\begin{aligned} v_i &= \sum_{\text{C} \in W^m} (\delta v_i + \gamma)\, I(i \in \text{C}) \mu_{\text{C}} \text{ and } \sum_{\text{C} \in W^m} \mu_{\text{C}} = 1, \\ \text{where } \mu_{\text{C}} &> 0 \text{ only if } \gamma = \max_{\text{C} \in W^m} \gamma_{\text{C}} \text{ and } \gamma_{\text{C}} = \frac{1}{|\text{C}|}(1 - \delta v_{\text{C}}). \end{aligned}$$

This corollary greatly facilitates the computation of the coalition bargaining solution, particularly in games with a large number of coalitions.

The coalition bargaining solution for simple games with veto players is particularly easy to characterize.

Veto players are those players that belong to every winning coalition. The set of veto players of a simple game is defined as $V \equiv \cap_{\text{C} \in W} \text{C}$. A simple game has veto players whenever $V \neq \varnothing$. When the set $V = \{i\}$ then player i is also said to be a dictator.

It is straightforward, from the definition of the coalition bargaining solution, to show that a simple game with $m > 0$ veto players and $\delta < 1$ has (i) maximum average gain equal to zero; (ii) all the surplus is equally split among the veto players; (iii) and non-veto players get zero value: i.e., $\gamma = 0$, $v_i = \frac{1}{m}$ for all $i \in V$, and $v_i = 0$ for all $i \notin V$.

Thus the coalitional bargaining solution for games with veto players is quite straightforward to derive. However, most simple games of interest, which we further study in the next section, do not have veto players.

7 Application: Weighted Majority Games

To illustrate the properties of the coalition bargaining solution, consider the class of weighted majority games. Weighted majority games are ubiquitous in the voting literature and are the most important type of simple game. They appear in many contexts such as multi-party legislatures, stockholders voting in a corporation, the United Nations Security Council, and the European Union Council of Ministers.

Weighted majority games are commonly represented as $[q; w_1, ..., w_n]$ where the quota is $q \geq \left(\sum_{i \in N} w_i\right)/2$ and w_i are the votes of each player $i \in N$. In such games a coalition C is winning if and only if $\sum_{i \in \text{C}} w_i > q$.[10] In a majority game the quota $q = \left(\sum_{i \in N} w_i\right)/2$, and thus the winning coalitions are those with more than half of the total votes.

Isbell (1959) provides a complete enumeration of all 134 strong weighted majority games with less than eight players without veto players (see Table 1).[11] There are a total of 114 majority games with seven players, 14 majority games with six players, 4 majority games with five players, and 1 majority game with three and four players each.

In this section, we compare the coalition bargaining solution for all possible 134 majority games with less than eight players with the most widely used alternative cooperative and non-cooperative solution concepts. In particular, we consider comparisons with classic cooperative solution concepts such as the Shapley-Shubik index, the Banzhaf index, and the nucleolus, and with the non-cooperative random proposer model with egalitarian and proportional proposer probabilities.

The list of all majority games with less than eight players in Table 1 is very useful. Any majority game with less than eight players is isomorphic–have the same exact set of winning coalitions–to one of the games listed in Table 1. The representation in Table 1 is

[10] Or, alternatively, by choosing $q > \left(\sum_{i \in N} w_i\right)/2$, and winning coalitions if and only if $\sum_{i \in \text{C}} w_i \geq q$.

[11] In a strong simple game ties are not possible, such that for all coalitions, either C or $N \backslash \text{C}$ is winning.

the unique representation with minimum integer weights. A majority voting game with less than eight players may have a very different representation, with different voting weights, but they have the same values, for all solution concepts, as one of the 134 majority games listed in Table 1.

Table 1 provides the coalition bargaining solution–with the player values and the probabilities of coalition formation–for all 134 majority games. We also evaluate the Shapley-Shubik index, the Banzhaf index, the nucleolus, and the random proposer value (with equal and proportional probabilities) for all these games. The GAMS program used for computing the solutions to all games, as well as all the solutions of all games, are available in the online appendix and the authors' website. Table 2 reports the result of the comparison of the coalition bargaining solution with other solution concepts. While we do not have a general proof of the uniqueness of the CBS, using the MAPLE mathematical programming language, we computationally verify that there is no other CBS solution for all majority games with less than seven players (see online appendix and the authors' website).

The majority games in the table are represented by a list with weights in increasing order $\underbrace{w_1...w_1}_{n_1}...\underbrace{w_m...w_m}_{n_m}$ with the number of repetitions indicating the number of players with the same voting weights (or type). The coalition bargaining solutions reported in Table 1 are all symmetric: any two players of the same type have the same value, and any two coalitions S and T with the same number of players of each type have the same probability of forming. The coalition bargaining solution provided in Table 1 shows the values of a player with w votes as v_w, the maximum average gain γ, and μ_{C} denotes the probability of coalitions of type $\mathrm{C}= [\underbrace{w_1...w_1}_{s_1}...\underbrace{w_m...w_m}_{s_m}]$ forming, where s_i is the number of players with w_i votes. In Table 1 the surplus has been normalized to 100 so all values are in percentages of the surplus, and the coalitions appear in decreasing order of probability μ_{C}.

This exercise is of practical interest given that voting games with less than eight players are quite common in real world applications. For example, in the empirical analysis of Diermeier and Merlo (2004), among the 313 government formations in 11 multi-party democracies over the period 1945-1997, the median distribution of the number of parties is seven, and the mean is 7.35. Thus by focusing on the 134 games in Table 1, we are addressing the most typical voting games.

Discussion of Results

We begin our comparison by focusing on apex games which are n-player majority games with one major player with $n-2$ votes (the apex player) and $n-1$ minor players with 1 vote (the base players), where to win a coalition needs at least $n-1$ votes. Apex games have been extensively studied in the literature. Observe that the following four games in Table 1 are all apex games: the seven-player game #3, the six-player game #2, the five-player game #2, and the four player game #1.

In an apex game, the only minimal wining coalitions are the coalition including all base players, and the coalition including the apex player and one base player. We derive in the Appendix the symmetric coalition bargaining solution where v_a is the apex player value, v_b is the base player value, and μ_a is the probability of a coalition with the apex player and one base player forms, and μ_b is the probability the coalition including all base players forms. For discount rate $\delta = 1$, the coalition bargaining value is $v_a = \frac{(n-2)}{n}$ and $v_b = \frac{2}{n(n-1)}$; the maximum average gain from forming a coalition is $\gamma = \frac{(n-2)}{n(n-1)}$; and the probability of coalition formation is $\mu_a = \frac{(n-1)}{n}$ and $\mu_b = \frac{1}{n}$.

For apex games, the coalition bargaining solution coincides with the Shapley-Shubik index. However, the coalition bargaining value for the apex player (and the Shapley-Shubik index) is greater than the value prescribed by the nucleolus and the random proposer model. Indeed, Montero (2002, 2006) shows that the random proposer model with proportional proposer probabilities and the nucleolus all have (ex-ante) values proportional to the voting weights: $v_a = \frac{(n-2)}{2n-3}$ and $v_b = \frac{1}{2n-3}$.[12] The Banzhaf index for the apex player is $v_a = \frac{2^n-4}{2^n+4n-8}$ and coincides with the coalition bargaining solution for $n = 4$, but is strictly bigger than the coalition bargaining solution for $n > 4$.

Interestingly, for larger apex games the solution concepts are significantly different. As n converges to infinity the coalition bargaining value (and the Shapley-Shubik and Banzhaf indices) predict that the apex player gets all the surplus $v_a = 1$, while the random proposer model and the nucleolus predict that the apex player obtains only half of the surplus $v_a = \frac{1}{2}$. This sharp differences among models provides a robust empirical test to distinguish which model produce better predictions for unstructured negotiations.

[12] The difference for the apex player value between the coalition bargaining solution and the random proposer and the nucleolus is thus, $\frac{(n-2)}{n} - \frac{(n-2)}{2n-3} = \frac{(n-2)(n-3)}{n(2n-3)} > 0$, and increases from 0.1 to 0.5 as n goes from $n = 4$ to infinity.

There are several other noteworthy regularities in the solutions in Table 1. For example, in none of the 134 games are there dummy players. However, in several of the seven-player games, such as games #14, #16, and #20, the player with one vote has zero value and does not appear in any equilibrium coalition (thus, in equilibrium, they behave like dummy players).

In Table 2 we compare the various solution concepts (we exclude the symmetric three player game 111). The largest player value tend to be bigger under the coalition bargaining solution than all other solution concepts (and the opposite holds true for the smallest player value). From Table 2 (Panel B) among all 133 games, the coalition bargaining solution value of the largest player is never smaller than the largest player value under the random proposer with equal protocol, and is smaller than the Shapley-Shubik index in only 19 games, the Banzhaf index in only 26 games, and the nucleolus (and random proposer with proportional protocol) in only 7 games.

Moreover, Table 2 (Panel C), also shows that the coalition bargaining solution value of the largest player is on average 11% higher than the Shapley-Shubik index, 13% higher than the Banzhaf index, 44% higher than the random proposer with equal protocol, and 29% higher than the nucleolus and random proposer with proportional protocol. Alternatively, from Table 2 (Panel D), we can see that the coalition bargaining solution value of the smallest player is on average 23% less than the Shapley value, 23% less than the Banzhaf index, 63% less than the random proposer with equal protocol, and 42% less than the nucleolus and random proposer with proportional protocol.

Another important difference between the coalition bargaining solution relative to other existing solution concepts, is that the ex-ante (or unconditional) value per vote ratio tends to increase with the number of votes. In particular, as Table 2 (Panel A) shows, the average value per vote ratio, $\frac{v_w}{w}$, for the largest player is 8.04, and the average ratio for the smallest player is 3.79. The value per vote ratio of the largest player is greater than the value per vote ratio of the smallest player in all but 16 of the 134 games in the table. On the other hand, we can see from Table 2 (Panel A) that the value per vote ratio is constant for the nucleolus (and random proposer with proportional probability), and tend to be decreasing for the random proposer with equal protocol (the average ratio for the largest player is 4.76, and the average ratio for the smallest player is 8.36). To a lesser degree than the coalition bargaining solution, the value per vote ratio is also increasing for the Shapley-Shubik and

Banzhaf indices.

There is an intuitive economic reasoning motivating this property of the simultaneous proposer model. Suppose the value of a player with $w + w'$ votes was less than or equal to the sum of the values of two players with w and w' votes, i.e., $v_{w+w'} \leq v_w + v_{w'}$. Any winning coalition that replaces the two players with w and w' votes with one player with $w + w'$ votes is also winning and this deviation increases the average gain because there is one less player sharing the surplus:

$$\frac{1}{|\mathrm{C}| + 2}\left(\bar{s} - \delta \sum_{i \in \mathrm{C}} (v_i + (v_w + v_{w'}))\right) < \frac{1}{|\mathrm{C}| + 1}\left(\bar{s} - \delta \sum_{i \in \mathrm{C}} (v_i + v_{w+w'})\right).$$

Thus coalitions including the larger player $w + w'$ would increase in probability, leading to an increase in the value $v_{w+w'}$, and the ones including the smaller players would decrease in probability, leading to a decrease in the values v_w and $v_{w'}$ forcing a reversal of the inequality $v_{w+w'} > v_w + v_{w'}$.

In contrast, many models in the literature yield ex-ante values for weighted voting games that are proportional to the voting weights, that is, the ratio $\frac{v_w}{w}$ is constant, or $\frac{v_w}{v_{w'}} = \frac{w}{w'}$ for any two players with w and w' votes. Notably, Synder et al. (2005) study replicas of weighted voting games where the voting weights are chosen to be a minimum integer representation of the game. They show that the proportionality of ex-ante values hold for the random proposer model, for sufficiently large replica games, whenever the recognition probabilities are proportional to the voting weights.

A number of empirical studies in legislative bargaining have analyzed the relationship between the value of parties, measured typically by the number of ministerial cabinets, and their share of parliamentary seats (e.g., Browne and Franklin (1973) and Warwick and Druckman (2001)). These studies tend to find support for Gamson's law which states that the number of cabinet posts and votes contributed to the winning coalition are proportional. The conditional proportionality result holds for the demand bargaining model of Morelli (1999) for homogeneous weighted majority games (see also Morelli and Montero (2003)).

Gamson's law proportionality result also holds fairly closely for the coalitional bargaining solution. In order to consider this result, observe that we need to compute the ratios of the values, conditional on being included in the winning coalition, and the voting weights. According to our previous results the conditional value/vote ratio is equal to $\frac{v_w + \gamma}{w}$. For example, for the six-player game 111224 (game # 4, in Table 1), the average gain is $\gamma = 13.89$ and the values are $v_1 = 2.78$, $v_2 = 19.44$, and $v_4 = 52.78$. Thus the ratios $\frac{v_w+\gamma}{w}$ are $\frac{2.78+13.89}{1} = \frac{19.44+13.89}{2} = \frac{52.78+13.89}{4} = 16.67$. Thus, while the ex-ante values/voting ratio are clearly increasing on the voting weights, $\frac{2.78}{1} < \frac{19.44}{2} < \frac{52.78}{4}$, the conditional value/voting ratio is constant. The conditional proportionality result holds exactly for only 19 of the 134 games in Table 1. But it also holds approximately well for all 134 games. For each of the 134 games in Table 1, we compute the coefficient of variation of the ratios $\frac{v_w+\gamma}{w}$ for all different voting weights w (the coefficient of variation is the ratio of the standard deviation to the mean). The average coefficient of variation among the 134 games is only 20%.

Finally, the coalition bargaining solution predicts that only a few of the possible minimum winning coalitions actually form in equilibrium (i.e., the support of μ_{C} is a proper subset of the set of W^m). Moreover, the probability distribution of μ_{C} is very concentrated, and only a small number of coalitions carry most of the weight.

For example, from Table 1, among the 114 seven player majority games there are on average 15.4 minimum winning coalitions (and 6.8 minimum winning coalition excluding double counting of types). But there are on average only 3.2 coalitions with non-zero probability, thus there are several minimum winning coalitions not in the support of μ. Moreover, the most likely coalition forms with an average probability of 53.5% (the minimum is 28.8%), and the two most likely coalitions form with an average probability of 81.8% (the minimum is 52.1%).

The concentration of coalitions forming in equilibrium is a noteworthy feature of the coalition bargaining solution concept. The underlying logic behind classical solution concepts such as the Shapley-Shubik and Banzhaf indices is that all minimum coalitions can form. Thus, there are also sharp differences in empirical predictions between the coalition bargaining solution and the classical solution concepts with respect to the equilibrium structure of coalition formation.

8 Conclusion

We have analyzed the classical problem of coalition formation and determinist or stochastic surplus division. A new non-cooperative coalition bargaining game, meant to capture the outcome of freestyle or unstructured negotiations, is proposed to address the problem.

The model developed here is a dynamic game in which at every period there is a coalition formation stage, where players simultaneously choose the coalitions they want to join, followed by a negotiation stage. An important feature of the model is that it does not rely on any exogenously given proposer selection protocol.

We characterized the strong and coalition-proof Markov perfect equilibrium of the game, and showed that the coalitions that form in equilibrium are determined by the maximization of the average gain per coalition member.

We considered the application of the model to majority voting games, which are important in the voting literature, notably coalition government formation in multiparty democracies. Comparison of the new model solution with other classical solutions, for all 134 majority games with less than eight players, yields sharp differences in predictions. In particular, the coalition bargaining values of larger (smaller) parties are significantly greater (less) than the values predicted by other solution concepts such as the Shapley-Shubik and Banzhaf indices, the random proposer model solution, and the nucleolus. Interestingly, the outcome prediction is consistent with Gamson's law, which essentially states that the value per voting weight ratio, conditional on being in the winning coalition, is constant across parties of different size. However, very importantly the (ex-ante) value per vote ratio tends to increase with the number of votes because small parties are not included in the coalition that forms proportionally to their votes. Finally, a noteworthy feature of the new solution is that it predicts a very concentrated probability of coalition formation.

It would be interesting to explore in future empirical studies whether the simultaneous proposer model accurately describes the outcome in unstructured processes of multilateral negotiations that are so pervasive in practice. Moreover, the solution properties has implications for normative analysis that attempts to determine the players' voting weights necessary to achieve a certain social objective.

Appendix: Proofs

Example *Consider a simple game with four players $N = 1234$ with winning coalitions $W = \{12, 13, 23, 124, 134, 234, 123, 1234\}$ and surplus $s_{\text{C}} =\$1$ for all $\text{C}\in W$, and discount rate $\delta = 1$. Assume proposer probabilities equal to $p_i = \frac{1}{4}$ for all $i \in N$.*

In this example, player 4 is a dummy player, and players 1,2, and 3 are symmetric. Both models above have unique solution outcomes which are obtained by solving the problems (2) and (3). We can easily directly obtain the symmetric solution outcome of both models. For the simultaneous proposer model, the symmetric solution is $v_i = v$ for $i = 1, 2, 3$ and $v_4 = 0$, and $\mu_{\text{C}} = \frac{1}{3}$ for $\text{C}= 12, 13,$ and 23. The equations in (2) then simplify to the following system of linear equations,

$$v = \frac{2}{3}(\gamma + v) \text{ and } \gamma = \frac{1}{2}(1 - 2v),$$

with solution $v = \frac{1}{3}$ and $\gamma = \frac{1}{6}$.

For the random proposer model, the strategies are not uniquely determined but the equilibrium outcomes are uniquely determined.[13] Consider the symmetric solution $v_i = v$ for $i = 1, 2, 3$ and $\sigma_{ij} = \frac{1}{2}$ for $i, j = 1, 2, 3$ and $i \neq j$ and $\sigma_{4ij} = \frac{1}{3}$ for $i, j = 1, 2, 3$ and $i \neq j$, and let $e = e_i = 1 - 2v$ for $i = 1, 2, 3$. The equations in (3) then simplify to the following system of linear equations,

$$\begin{aligned} v &= \frac{1}{4}e + \frac{2}{3}v \text{ and } v_4 = \frac{1}{4}e_4 + \frac{1}{4}v_4, \\ e &= 1 - 2v \text{ and } e_4 = 1 - 2v - v_4, \end{aligned}$$

whose solution is $v = \frac{3}{10}$, $v_4 = \frac{1}{10}$, $e = \frac{3}{10}$, and $e_4 = \frac{2}{5}$. Note that $\mu_{\text{C}} = \frac{3}{4} \times \frac{1}{3}$ for $\text{C}= 12, 13,$ and 23 and $\mu_{\text{C}} = \frac{1}{4} \times \frac{1}{3}$ for $\text{C}= 124, 134,$ and 234.

The predictions for both players' values and coalition formation probabilities are quite different between both models. The value distributions in the random proposer model are significantly more dispersed and unequal than in the simultaneous proposer model. In the random proposer model, the dummy player receives a surplus of \$0.40 with 25% chance, and its value is on average \$0.10. While in the simultaneous proposer model, the dummy player receive zero 100% of the time. Also, in the random proposer model, players 1,2, and 3 receive \$0 with probability 33.3%, \$0.30 with probability 41.7%, and \$0.70 with 25% probability, and their value are on average \$0.30. The distribution of values is much less dispersed in the simultaneous proposer model: value of \$0 with 33.3% probability, and \$0.50 with 66.7% probability (with an average of \$0.33).

The predictions for coalition formation are also quite distinct between both models. The random proposer model predicts that the coalitions 12, 13, and 23 each form with 25% probability and the coalitions 124, 134, and 234 each form with probability 8.3%. The simultaneous proposer model predicts that only the coalitions 12, 13, and 23 should form each with probability 33.3%.

Proof of Proposition 2: The coalition formation stage subgame is a static game (one-shot game) where a coalition $\text{C}\in W$ forms if and only if all players $i \in \text{C}$ choose C, or otherwise $\text{C}= \varnothing$. Player i payoff is given by $u_i(\mathbf{a}) \equiv v_i^\sigma(s, \text{C}_f(\mathbf{a}))$ (see equation (4)).

[13] For issues related to the uniqueness of equilibrium, see Yan (2009), Cho and Duggan (2003), and Eraslan and McLennan (2013).

(i) Strong Nash Equilibrium. We first show that the proposed strategy is a strong Nash equilibrium. For all players $i \in$C* the strategy yields a payoff of $v_i(s, \text{C}^*|v) = \delta v_i + \gamma(s, \text{C}^*|v)$. Since C* maximizes $\gamma(s, \text{C}|v)$ over all C$\in W$, then $v_i(s,\text{C}^*|v) \geq v_i(s,\text{C}|v)$ for all C$\in W$; and also $v_i(s,\text{C}^*|v) \geq \delta v_i$, which is the payoff if no coalition forms since $\gamma(s, \text{C}|v) > 0$. Thus no player in C* can be made better off by any coalitional deviation S$\subset N$ such that S$\cap$C$^* \neq \varnothing$ since the most players $i \in$C* can get from any deviation is $\max_{\text{C}\in W\setminus \text{C}^*} \{v_i(s, \text{C}|v), \delta v_i\} \leq v_i(s,\text{C}^*|v)$.

Moreover, since W is a proper set, no coalition S$\subset N\backslash$C*, including only players in $N\backslash$C*, is winning because C$^* \in W$. Thus any coalitional deviation S including only players in $N\backslash$C* will not make the players in S better off since any such coalition deviation will not change the outcome and players' payoff–all players in C* are not changing their strategies, and thus coalition C* forms. This shows that the proposed strategy is a strong Nash equilibrium.

We now show that only pure strategies belonging to SNE are strong Nash equilibrium. Consider any pure strategy profile in which not all the players belonging to a maximizing coalition are choosing that coalition. The outcome of such pure strategy is either a coalition C$\notin \text{argmax}_{\text{C}\in W}\, \gamma(s, \text{C}|v)$ or no coalition forms. Thus players' payoff following this strategy is less than or equal to
$\max_{\text{C}\in W} \{v_i(s, \text{C}|v), \delta v_i\}$. This strategy cannot be a strong Nash equilibrium because there exists a coalitional deviation that makes all deviating players strictly better off. For example, choose any S$\in \text{argmax}_{\text{C}\in W}\, \gamma(s, \text{C}|v)$ and let all players $i \in$S deviate to C$_i$ =S. All players $i \in$S obtain a payoff $v_i(s,\text{S}|v) > \max_{\text{C}\in W} \{v_i(s, \text{C}|v), \delta v_i\}$.

(ii) Coalition-proof Nash equilibrium: We already know that the set of strong Nash equilibrium is a subset of the set of coalition-proof Nash equilibrium ($SNE \subset CPNE$) for any stage game (this is immediate from the fact that the CPNE concept differs from SNE because it restricts the set of feasible group deviations). We now show that reciprocal $CPNE \subset SNE$ also holds for the coalitional formation game.

The proof is by contradiction. Suppose there is a $CPNE$ $\mathbf{a}^* = (a_i^*)_{i\in N}$ that is not a SNE. Let C=C$_f(\mathbf{a}^*)$ be the coalition that forms in this equilibrium. Thus, by the result we have just proven above for SNE, we have that $\gamma(s, \text{C}|v) < \max_{\text{S}\in W} \{\gamma(s, \text{S}|v)\}$.

Consider any coalitional deviation S=C* where C$^* \in \text{argmax}_{\text{S}\in W} \{\gamma(s, \text{S}|v)\}$, and thus $\gamma(s, \text{C}^*|v) > \gamma(s, \text{C}|v)$. By the coalition-proof definition (iii) the profile $\mathbf{a}^* = (a_i^*)_{i\in N}$ is self-enforcing. Thus by item (ii) of the definition, for all proper subsets S$\subset N$, a_{S}^* is a CPNE of the restricted game $\Gamma/\mathbf{a}_{-\text{S}}^*$, where $\Gamma = G^\sigma(s)$.

We now show that $\mathbf{a}_{\text{S}}^*$ cannot be a CPNE of the restricted game $\Gamma/\mathbf{a}_{-\text{S}}^*$, because the restricted game has a self-enforcing profile $\mathbf{a}_{\text{S}} = (a_i)_{i\in\text{S}}$ that is strictly better for all players in $\Gamma/\mathbf{a}_{-\text{S}}^*$ (note that the player set for the restricted game is S not N).

Consider the profile $\mathbf{a}_{\text{S}} = (a_i)_{i\in\text{S}}$ where a_i =C* for all $i \in$C* =S. First, this profile is strictly better for all players $i \in$S because $u_i\left(\mathbf{a}_{\text{S}}, \mathbf{a}_{-\text{S}}^*\right) = \delta v_i + \gamma(s, \text{C}^*|v) > \delta v_i + \gamma(s, \text{C}|v) \geq u_i(\mathbf{a}^*)$ for all $i \in$S.

Now the profile $\mathbf{a}_{\text{S}} = (a_i)_{i\in\text{S}}$ is a strong Nash equilibrium of the restricted game $\Gamma/\mathbf{a}_{-\text{S}}^*$: this is clearly true because in the restricted game $\Gamma/\mathbf{a}_{-\text{S}}^*$ all players $i \in$S achieve the maximum possible payoff $u_i\left(\mathbf{a}_{\text{S}}, \mathbf{a}_{-\text{S}}^*\right) = \delta v_i + \gamma(s, \text{C}^*|v)$, and thus no other action can strictly improve all deviating players. But we know that $SNE\left(\Gamma/\mathbf{a}_{-\text{S}}^*\right) \subset CPNE\left(\Gamma/\mathbf{a}_{-\text{S}}^*\right)$, so $\mathbf{a}_{\text{S}}$ is a $CPNE\left(\Gamma/\mathbf{a}_{-\text{S}}^*\right)$ as well, and thus also self-enforcing, which leads to a contradiction with item (iii) of the coalition-Proof definition applied to the restricted game $\Gamma/\mathbf{a}_{-\text{S}}^*$.

For almost all states s, except in a set of Lebesgue measure zero, the maximization problem $\max_{\text{C}\in W} \gamma(s, \text{C}|v)$ has a unique solution. This holds because of the absolute continuity of $\mathbf{s}$: the set of states where $\gamma(s, \text{C}|v) = \gamma(s, \text{T}|v)$ for any C,T$\in W$, which corresponds to $\frac{1}{|\text{C}|}(s_{\text{C}} - \delta v_{\text{C}}) = \frac{1}{|\text{T}|}(s_{\text{T}} - \delta v_{\text{T}})$, has measure zero. *Q.E.D.*

Proof of Proposition 3: Let σ be either a SMPE or a CPMPE of the coalition bargaining game. The value satisfies $v_i^\sigma = \int v_i^\sigma(s) f(s)ds$, where $v_i^\sigma(s)$ is given by Proposition 2. The integral $\int v_i^\sigma(s) f(s)ds$ is the expression in the right-hand-side below

$$v_i^\sigma = \sum_{\text{C}\in W} \int_{\mathcal{A}} (\delta v_i + \gamma(s, \text{C})) I(i \in \text{C}) I(\text{C}^*(s) = \text{C}) f(s)ds + \delta v_i^\sigma \int_{\mathcal{A}^c} f(s)ds.$$

This completes the necessary part of the proposition.

In order to prove the reciprocal statement, consider any value v satisfying (i), and let σ be a strategy profile defined by (ii) (note that the strategy is a function of v).

First note that the strategy σ is such that indeed $v_i^\sigma = v_i$. At any state $s \in \mathcal{A}$, the coalition that forms in the coalition formation stage is $\text{C}^*(s)$, the coalition that maximizes the average gain, and the final payoff of the players in $\text{C}^*(s)$ are $\delta v_i + \gamma(s, \text{C}^*(s))$, and the final payoff of players $i \notin \text{C}^*(s)$ is zero. This is true because in the ensuing negotiations at node $(s, \text{C}^*(s))$ the strategies are given by (ii), equation (5) holds, and there is immediate agreement. At any state $s \in \mathcal{A}^c$, there is no coalition forming and the players' value are the discounted continuation values δv_i^σ.

Therefore the value v_i^σ satisfy the following equation

$$v_i^\sigma = \sum_{\text{C}\in W} \int_{\mathcal{A}} (\delta v_i + \gamma(s, \text{C})) I(i \in \text{C}) I(\text{C}^*(s) = \text{C}) f(s)ds + \delta v_i^\sigma \int_{\mathcal{A}^c} f(s)ds.$$

and thus

$$v_i^\sigma = \left(1 - \delta \int_{\mathcal{A}^c} f(s)ds\right)^{-1} \sum_{\text{C}\in W} \int_{\mathcal{A}} (\delta v_i + \gamma(s, \text{C})) I(i \in \text{C}) I(\text{C}^*(s) = \text{C}) f(s)ds$$

which, from equation (6), shows that $v_i^\sigma = v_i$.

From Proposition 1 and the discussion in Section 4 the strategies (ii.b) at the negotiation stage game are a subgame perfect equilibrium of the negotiation stage game (s, C) with continuation values $v_i^\sigma = v_i$. Proposition 2 shows that the strategies (ii.a) at the coalition formation stage is a strong and coalition-proof Nash equilibrium of the one-shot coalition formation stage game $G^\sigma(s)$ with continuation values $v_i^\sigma = v_i$. By the one-stage-deviation principle (Fudenberg and Tirole, (1991, Th. 4.2) then σ is an MPE. Moreover, from definition 2, σ is also an SMPE and CPMPE. *Q.E.D.*

Proof of Proposition 4: Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ where

$$(\Phi(v))_i = \sum_{\text{C}\in W} \int_{\mathcal{A}_{\text{C}}(v)} (\delta v_i + \gamma(s, \text{C}|v)) I(i \in \text{C}) f(s) ds + \int_{\mathcal{A}^c(v)} \delta v_i f(s) ds,$$

and $\gamma(s, \text{C}|v) \equiv \frac{1}{|\text{C}|}(s_{\text{C}} - \delta v_{\text{C}})$, and $\mathcal{A}_{\text{C}}(v) \equiv \{s \in \mathcal{S} : \gamma(s, \text{C}|v) > \gamma(s, \text{T}|v) \text{ for all T} \in W \cup \varnothing\}$.

The mapping Φ is clearly continuous (see footnote 9). We now show that there is a convex compact (and non-empty) set $K \subset \mathbb{R}_+^n$ such that $\Phi(K) \subset K$. Thus by the Brouwer fixed point theorem, this implies that Φ has a fixed point v such that $v = \Phi(v)$.

Consider the set $K \subset \mathbb{R}_+^n$ defined by the simplex

$$K = \{v \in \mathbb{R}^n : v_i \geq 0 \text{ and } v_N \leq \bar{v}_N\}$$

where $\bar{v}_N \geq 0$ is given by the solution of the equation

$$\bar{v}_N = \int \max\left\{\delta \bar{v}_N, \max_{\mathrm{C} \in W}\{s_{\mathrm{C}}\}\right\} f(s)\, ds. \tag{9}$$

Clearly the function $G : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$G(z) = \int \max\left\{\delta z, \max_{\mathrm{C} \in W}\{s_{\mathrm{C}}\}\right\} f(s)\, ds$$

is monotonically increasing, and $|G(z) - G(w)| \leq \delta |z - w|$ for any $z, w \in \mathbb{R}_+$, so it is a contraction mapping with modulus δ. By the contraction mapping theorem there is a (unique) $\bar{v}_N$ that satisfy the equation (9), and $\bar{v}_N > 0$. Thus the set K is non-empty and well-defined.

Certainly for all $v \geq 0$, then $\Phi(v) \geq 0$. It remains to show that if $v_N \leq \bar{v}_N$ then $\sum_{i \in N} \Phi_i(v) \leq \bar{v}_N$. But

$$\begin{aligned} \sum_{i \in N} \Phi_i(v) &= \sum_{\mathrm{C} \in W} \int_{\mathcal{A}_{\mathrm{C}}(v)} s_{\mathrm{C}} f(s)\, ds + \int_{\mathcal{A}^c(v)} \delta v_N f(s)\, ds \\ &\leq \int \max\left\{\delta \bar{v}_N, \max_{\mathrm{C} \in W}\{s_{\mathrm{C}}\}\right\} f(s)\, ds = \bar{v}_N, \end{aligned}$$

which completes the proof that $\Phi(K) \subset K$.

The fixed points of the mapping Φ corresponds to the solutions of equation (6). Thus, by reciprocal of Proposition 3, we obtain explicitly a strategy profile σ that is a strong and a coalition-proof Markov perfect equilibrium of the coalition bargaining game. *Q.E.D.*

Proof of Proposition 5: Suppose that $\mathbf{s}^n$ converges in distribution to $\bar{s}$, where $\bar{s}_{\mathrm{C}} > 0$ for some $\mathrm{C} \in W$, and let σ_n be either a SMPE or CPMPE of $\Gamma(\mathbf{s}^n, \delta)$. The sequences (v^{σ_n}) and $(\mu_{\mathrm{C}}^{\sigma_n})_{\mathrm{C} \in W}$ are bounded. Thus by the Bolzano-Weierstrass theorem there is a convergent subsequence $k_n \to \infty$, such that $v = \lim_{n \to \infty} v^{\sigma_{k_n}}$ and $(\mu_{\mathrm{C}})_{\mathrm{C} \in W} = \lim_{n \to \infty} \left(\mu_{\mathrm{C}}^{\sigma_{k_n}}\right)_{\mathrm{C} \in W}$. In the remainder of the proof, we consider the convergent subsequence k_n, which we relabel so that $v = \lim_{n \to \infty} v^{\sigma_n}$ and, for all $\mathrm{C} \in W$,

$$\mu_{\mathrm{C}} = \lim_{n \to \infty} \mu_{\mathrm{C}}^{\sigma_n} = \Pr(s \in \mathcal{A}_{\mathrm{C}}(v^{\sigma_n})) = \int_{\mathcal{A}_{\mathrm{C}}(v^{\sigma_n})} f^{(n)}(s) ds = \int I\left(\mathrm{C}^{*\sigma_n}(s) = \mathrm{C}\right) f^{(n)}(s)\, ds,$$

Consider the following definitions, where the dependency on v is explicitly shown: $\gamma(s, \mathrm{C}|v) \equiv \frac{1}{|\mathrm{C}|}(s_{\mathrm{C}} - \delta v_{\mathrm{C}})$ and $\gamma(s, \varnothing|v) = 0)$, and $\mathcal{A}_{\mathrm{C}}(v) \equiv \{s \in \mathcal{S} : \gamma(s, \mathrm{C}|v) > \gamma(s, \mathrm{T}|v) \text{ for all } \mathrm{T} \in W \cup \varnothing\}$.

Applying Proposition 4 to each SMPE or CPMPE σ_n, we obtain

$$v_i^{\sigma_n} = \sum_{\mathrm{C} \in W} \int_{\mathcal{A}_{\mathrm{C}}(v^{\sigma_n})} \left(\delta v_i^{\sigma_n} + \gamma(s, \mathrm{C}|v^{\sigma_n})\right) I(i \in \mathrm{C}) f^{(n)}(s) ds + \delta v_i^{\sigma_n} \int_{\mathcal{A}^c(v^{\sigma_n})} f^{(n)}(s) ds. \tag{10}$$

First, we show that, for all $\text{C}\in W$, then

$$\lim_{n\to\infty}\int_{\mathcal{A}_{\text{C}}(v^{\sigma_n})}\left(\delta v_i^{\sigma_n}+\gamma\left(s,\text{C}|v^{\sigma_n}\right)\right)f^{(n)}(s)ds=\left(\delta v_i+\gamma\left(\bar{s},\text{C}|v\right)\right)\mu_{\text{C}} \tag{11}$$

holds: The $\lim_{n\to\infty}\int_{\mathcal{A}_{\text{C}}(v^{\sigma_n})}\left(\delta v_i^{\sigma_n}+\gamma\left(\bar{s},\text{C}|v^{\sigma_n}\right)\right)f^{(n)}(s)ds=\delta v_i+\gamma\left(\bar{s},\text{C}|v\right)\mu_{\text{C}}$ because the integrand is a constant. Moreover, $\lim_{n\to\infty}\int_{\mathcal{A}_{\text{C}}(v^{\sigma_n})}\left(\gamma\left(s,\text{C}|v^{\sigma_n}\right)-\gamma\left(\bar{s},\text{C}|v\right)\right)f^{(n)}(s)ds=0$, because of the continuity of $\gamma\left(s,\text{C}|v\right)$ and $\mathcal{A}_{\text{C}}(v)$, the convergence of $v^{\sigma_n}\to v$, and the convergence of $\mathbf{s}^n\to\bar{s}$ in distribution.

Second, we show that the maximum limit average gain $\gamma\equiv\max_{\text{C}\in W}\gamma\left(\bar{s},\text{C}|v\right)>0$ is strictly positive. Suppose by contradiction that $\gamma\leq 0$. We have the following inequality from equation (10)

$$v_i^{\sigma_n}\leq\delta v_i^{\sigma_n}+\sum_{\text{C}\in W}\int_{\mathcal{A}_{\text{C}}(v^{\sigma_n})}\gamma\left(s,\text{C}|v^{\sigma_n}\right)f^{(n)}(s)ds\leq\delta v_i^{\sigma_n}+\int\max_{\text{C}\in W}\gamma\left(s,\text{C}|v^{\sigma_n}\right)f^{(n)}(s)ds$$

Continuity of the bounded function $\max_{\text{C}\in W}\gamma\left(s,\text{C}|v^{\sigma_n}\right)$ and the convergence of $\mathbf{s}^n\to\bar{s}$ in distribution imply that $\lim_{n\to\infty}\int\max_{\text{C}\in W}\gamma\left(s,\text{C}|v^{\sigma_n}\right)f^{(n)}(s)ds=\max_{\text{C}\in W}\gamma\left(\bar{s},\text{C}|v\right)\leq 0$. Thus taking the limit of the inequality above yields

$$v_i\leq\delta v_i\Rightarrow v_i\leq 0\text{ for all }i\in N.$$

But this implies the following inequality,

$$\max_{\text{C}\in W}\gamma\left(\bar{s},\text{C}|v\right)=\max_{\text{C}\in W}\left\{\frac{1}{|\text{C}|}\left(\bar{s}_{\text{C}}-\delta v_{\text{C}}\right)\right\}\geq\max_{\text{C}\in W}\left\{\frac{1}{|\text{C}|}\bar{s}_{\text{C}}\right\}>0,$$

which yields a contradiction.

The strict inequality $\gamma>0$ implies that there exists a $\text{C}\in W$ such that $\bar{s}_{\text{C}}>\delta v_{\text{C}}$. Thus $\mathbf{s}^n\to\bar{s}$ in distribution implies that $\Pr\left(s_{\text{C}}^n\leq\delta v_{\text{C}}^{\sigma_n}\right)\to 0$. Therefore,

$$\lim_{n\to\infty}\int_{\mathcal{A}^c(v^{\sigma_n})}f^{(n)}(s)ds=\lim_{n\to\infty}\Pr\left(\cap_{\text{C}\in W}\left\{s_{\text{C}}^{(n)}\leq\delta v_{\text{C}}^{\sigma_n}\right\}\right)=0. \tag{12}$$

Note that this implies that the limit MPE has no delay, that is, $\sum_{\text{C}\in W}\mu_{\text{C}}=1$.

Taking the limit of the left and right hand-side of equation (10), and using the results of (11) and (12), yields the desired limit result

$$v_i=\sum_{\text{C}\in W}\left(\delta v_i+\gamma\right)I(i\in\text{C})\mu_{\text{C}},\text{ for all }i\in N.$$

Q.E.D.

Proof of Corollary 1: Let player i be a dummy player and consider any coalition $\text{C}\in W$ which includes player i.

Suppose by contradiction that $\mu_{\text{C}}>0$. From Proposition 7, $\gamma\left(\text{C}\right)=\gamma=\max_{\text{C}\in W}\gamma\left(\text{C}\right)>0$. The average gain $\gamma_{\text{C}}<\gamma_{\text{C}\backslash\{i\}}$ because the average gain are

$$\gamma\left(\text{C}\right)=\frac{1}{|\text{C}|}\left(\bar{s}_{\text{C}}-\delta v_{\text{C}}\right)<\gamma_{\text{C}\backslash\{i\}}=\frac{1}{|\text{C}\backslash\{i\}|}\left(\bar{s}_{\text{C}\backslash\{i\}}-\delta v_{\text{C}\backslash\{i\}}\right)$$

since $|\text{C}| > |\text{C}\backslash\{i\}|$, $v_i \geq 0$, and $\bar{s}_{\text{C}} = \bar{s}_{\text{C}\backslash\{i\}}$, which is a contradiction. This implies that $\mu_{\text{C}} = 0$. Moreover, equation $v_i = \sum_{\text{C}\in W} (\gamma + \delta v_i)\, I(i \in\text{C})\mu_{\text{C}} = 0$, proves that $v_i = 0$. *Q.E.D.*

Proof of Corollary 2: Suppose that $\mu_{\text{C}} > 0$ and $\text{C}\notin W^m$. This implies that $\gamma_{\text{C}} = \gamma = \max_{\text{S}\in W} \gamma_{\text{S}}$ and that there exists a coalition $\text{S}\subset\text{C}$ such that $\text{S}\in W$ and $|\text{S}| < |\text{C}|$. But then, since $v_i \geq 0$ for all $i \in N$,

$$\gamma_{\text{S}} = \frac{1}{|\text{S}|}\left(1 - \delta \sum_{j\in\text{S}} v_j\right) > \gamma_{\text{C}} = \frac{1}{|\text{C}|}\left(1 - \delta \sum_{j\in\text{C}} v_j\right)$$

which is in contradiction with $\gamma_{\text{C}} = \max_{\text{S}\in W} \gamma_{\text{S}}$.

The same argument also shows that $\max_{\text{S}\in W} \gamma_{\text{S}} = \max_{\text{S}\in W^m} \gamma_{\text{S}}$ which completes the proof. *Q.E.D.*

Proof of Apex game: Define $\delta_n^* = \frac{(n-1)(n-3)}{n(n-3)+1}$. Note that for any $n \geq 4$, δ_n^* belongs to the interval $0 < \delta_n^* < 1$.

(i) Consider the case $\delta > \delta_n^*$. Then the following is a symmetric coalition bargaining solution in which both the coalition with the apex player and one base player and the coalition including all base players form: value $v_a = \frac{(n+\delta-3)}{n\delta}$ and $v_b = \frac{(3-n(1-\delta)-\delta)}{n\delta(n-1)}$, average gain $\gamma = \frac{n(2-\delta)+\delta-3}{n(n-1)}$, and probability of coalition formation $\mu_a = \frac{(n+\delta-3)(n-1)}{n\delta(n-2)}$ and $\mu_b = \frac{(n(n-3)+1)\delta-(n-1)(n-3)}{n\delta(n-2)}$.

Indeed, we can immediately verify by direct substitution that the proposed solution above satisfy the system of equations below obtained from Corollary 2:

$$\begin{aligned}
v_a &= (\delta v_a + \gamma)\, \mu_a, \\
v_b &= (\delta v_b + \gamma)\left(\frac{1}{(n-1)}\mu_a + \mu_b\right), \\
\gamma &= \frac{1}{2}\left(1 - \delta\left(v_a + v_b\right)\right), \\
\gamma &= \frac{1}{(n-1)}\left(1 - \delta\left(n-1\right) v_b\right), \\
\mu_a &+ \mu_b = 1,
\end{aligned}$$

Thus the candidate solution is in fact a symmetric coalition bargaining solution. Note that, since $n \geq 4$ then $\gamma > 0$ and $\mu_a > 0$. Moreover, $\mu_b > 0$ if and only if $\delta > \delta_n^*$.

For discount rate $\delta = 1$, the coalition bargaining value is $v_a = \frac{(n-2)}{n}$ and $v_b = \frac{2}{n(n-1)}$; the maximum average gain from forming a coalition is $\gamma = \frac{(n-2)}{n(n-1)}$; and the probability of coalition formation is $\mu_a = \frac{(n-1)}{n}$ and $\mu_b = \frac{1}{n}$.

(ii) Consider now the case $\delta \leq \delta_n^*$. In this case the coalition including all base players does not form in equilibrium. Consider the following values as a candidate for a symmetric coalition bargaining solution in which only the coalition with the apex player forms: value $v_a = \frac{(n-1-\delta)}{n(2-\delta)-2}$ and $v_b = \frac{1-\delta}{n(2-\delta)-2}$, average gain $\gamma = \frac{\left(n(1-\delta)+\delta^2-1\right)}{n(2-\delta)-2}$, and probability of coalition formation $\mu_a = 1$.

It can be immediately verified that the proposed solution above satisfy the system of equations and the inequality below obtained from Corollary 2:

$$\begin{aligned}
v_a &= (\delta v_a + \gamma)\,\mu_a \\
v_b &= (\delta v_b + \gamma)\,\frac{1}{(n-1)}\mu_a \\
\gamma &= \frac{1}{2}\left(1 - \delta\left(v_a + v_b\right)\right) \\
\mu_a &= 1 \\
\gamma &> \frac{1}{(n-1)}\left(1 - \delta\,(n-1)\,v_b\right)
\end{aligned}$$

To verify that the inequality holds, note that the expression

$$\gamma - \frac{1}{(n-1)}\left(1 - \delta\,(n-1)\,v_b\right) = \frac{2\,(n-1)\,\delta^2 - \left(n^2+n-1\right)\delta + (n-1)\,(n+1)}{(n-1)\,(n\,(2-\delta)-2)}.$$

The denominator of the expression above is always positive. The numerator is the quadratic expression in δ, $f(\delta) = 2\,(n-1)\,\delta^2 - \left(n^2+n-1\right)\delta + (n-1)\,(n+1)$. The numerator is also always positive for all $\delta \leq \delta_n^*$ and $n \geq 4$: because $f(\delta)$ is a convex function and

$$f(\delta_n^*) = \frac{2\,(n-1)^2\,(n-2)^3}{\left(n^2-3n+1\right)^2} > 0 \text{ and } f'(\delta) \leq f'(\delta_n^*) < 0$$

so $f(\delta) \geq f(\delta_n^*) > 0$.

For any fixed $\delta \in [0,1]$, when n converges to infinity, the limit value is $v_a = \frac{1}{(2-\delta)} \leq 1$, and, for $\delta = 1$, the limit value is $v_a = 1$. *Q.E.D.*.

Online Appendix: Simultaneous Negotiation Game

We propose an alternative formulation for the negotiation stage game in which all members of the coalition chosen in the first stage make simultaneous demands for the surplus: the coalition forms if the demands are feasible, and the game ends. Otherwise, the game moves on to the next period, where the same two stages repeat.

The detailed specification of the negotiation stage game contains a modelling innovation. At the end of each period there is a zero-mean shock to the surplus, only observed by the players after they submit their demands during the negotiation stage–economically these shocks represent unforeseen transaction costs. We show that whenever these shocks have a monotonically increasing hazard ratio then the negotiation stage game has a unique Nash equilibrium, instead of a plethora of equilibrium when there are no shocks. The monotonicity assumption is commonly used in the auction and contract theory literature beginning with Myerson (1981), and in particular holds for the normal distribution.

We show that in the limit, when the noise converges to zero, the Nash equilibrium of the negotiation stage game converges to the equal split of the surplus among coalition members where the status quo is their continuation values.

Simultaneous Negotiation Stage Game: In the t-period negotiation stage game coalitions bargain on how to split the surplus generated by the coalition. Consider the $(s, \textsc{c})-$negotiation stage game beginning in period t with state $s_t = s = (s_{\textsc{c}})_{\textsc{c}\in W}$ and coalition $\textsc{c}\in W$.

Assume that the surplus $s_{\textsc{c}}$ receives an end-of-period zero-mean i.i.d. shock $\varepsilon_{\textsc{c}}$ only observed by the players after they submit their demands.[14] The end-of-period noise $\varepsilon_{\textsc{c}}$ has continuous density function $g_{\textsc{c}}(\cdot)$, and cumulative distribution function $G_{\textsc{c}}(\cdot)$, with mean zero and full support, i.e., $g_{\textsc{c}}(\varepsilon_{\textsc{c}}) > 0$ for all $\varepsilon_{\textsc{c}} \in \mathbb{R}$. We impose the assumption that the noise has (inverse) hazard rate $H_{\textsc{c}}(\varepsilon_{\textsc{c}}) \equiv \frac{1-G_{\textsc{c}}(\varepsilon_{\textsc{c}})}{g_{\textsc{c}}(\varepsilon_{\textsc{c}})}$ that is differentiable and monotonically decreasing over the range $\mathbb{R}$.

Each player $i \in\textsc{c}$ choose demands $d_i \in \mathbb{R}$. The demand vector $(d_i)_{i\in\textsc{c}}$ is feasible if and only if the sum of all demands satisfy the budget balancing condition,

$$\sum_{i\in\textsc{c}} d_i \leq s_{\textsc{c}} + \varepsilon_{\textsc{c}}.$$

If the demand vector is not feasible the game moves to the $t+1$-period coalition formation stage. If the demand vector is feasible, the game ends with coalition $\textsc{c}$ being chosen and each player $i \in\textsc{c}$ receives d_i plus an equal fraction of the net surplus $s_{\textsc{c}} + \varepsilon_{\textsc{c}} - \sum_{i\in\textsc{c}} d_i \geq 0$. Therefore each player $i \in\textsc{c}$ receives a final payoff equal to

$$x_i = d_i + \frac{1}{|\textsc{c}|}\left(s_{\textsc{c}} + \varepsilon_{\textsc{c}} - \sum_{i\in\textsc{c}} d_i)\right),$$

where $|\textsc{c}|$ is the number of elements in coalition $\textsc{c}$. We denote by $x_i = 0$ the final payoff of players $i \notin\textsc{c}$. The utility of each player $i \in N$ is $\delta^{t-1}x_i$ and the game ends. This completes the definition of the negotiation stage.

[14] This noise, which will be made negligibly close to zero, is necessary because it is well-know that if the surplus $s_{\textsc{c}} + \varepsilon_{\textsc{c}}$ were known by players ex-ante, the negotiation game would have multiple Nash equilibrium (in fact, any demand satisfying $d_{\textsc{c}} = s_{\textsc{c}} + \varepsilon_{\textsc{c}}$ would be a Nash equilibrium).

Equilibrium Strategies in the Negotiation Stage: Denote by $G(s, \text{C}|v)$ the negotiation stage game beginning at node (s, C) and continuation value v. Let us say that the actions chosen simultaneously by the players in C are $\mathbf{d} = (d_i)_{i \in \text{C}}$. Denote by $d_{\text{C}} \equiv \sum_{i \in \text{C}} d_i$ and $v_{\text{C}} = \sum_{i \in \text{C}} v_i$, and let $\mathbf{d} = (d_i, d_{-i})$, where d_{-i} are the actions of players $N \backslash i$. In the stage game $G(s, \text{C}|v)$, player $i \in$C payoff is given by

$$u_i(d_i, d_{-i}|v) = \int_{\mathbb{R}} \left[\begin{array}{c} \left(d_i + \frac{1}{|\text{C}|} (s_{\text{C}} + \varepsilon_{\text{C}} - d_{\text{C}}) \right) \\ \times I(d_{\text{C}} \leq s_{\text{C}} + \varepsilon_{\text{C}}) \end{array} \right] g_{\text{C}}(\varepsilon_{\text{C}}) \, d\varepsilon_{\text{C}} + \int_{\mathbb{R}} \delta v_i I(d_{\text{C}} > s_{\text{C}} + \varepsilon_{\text{C}}) \, g_{\text{C}}(\varepsilon_{\text{C}}) \, d\varepsilon_{\text{C}}.$$

The stage game payoff is as given above, because each player $i \in$C final payoff is $x_i = d_i + \frac{1}{|\text{C}|}(s_{\text{C}} + \varepsilon_{\text{C}} - d_{\text{C}})$, given a noise realization ε_{C} that satisfies the budget constraint $d_{\text{C}} \leq s_{\text{C}} + \varepsilon_{\text{C}}$, and otherwise, if $d_{\text{C}} > s_{\text{C}} + \varepsilon_{\text{C}}$, the game moves to the next period coalition formation stage, where player i's value is equal to δv_i, the discounted continuation value.

Simple algebraic manipulation allows us to rewrite the expression above for the stage game payoff as

$$u_i(d_i, d_{-i}|v) = \delta v_i + \int_{d_{\text{C}} - s_{\text{C}}}^{\infty} \left(d_i - \delta v_i + \frac{1}{|\text{C}|} (s_{\text{C}} + \varepsilon_{\text{C}} - d_{\text{C}}) \right) g_{\text{C}}(\varepsilon_{\text{C}}) \, d\varepsilon_{\text{C}}.$$

We show in the proposition below that the negotiation stage game $G(s, \text{C}|v)$ has a unique Nash equilibrium if the inverse hazard rate $H_{\text{C}}(\varepsilon)$ is monotonically decreasing. The Nash equilibrium is obtained by applying the Leibnitz rule to yield the first order condition $\frac{\partial u_i(d_i, d_{-i}|v)}{\partial d_i} = 0$ as shown in the proof below.

Proposition *(Simultaneous Negotiation Stage) Consider the negotiation stage subgame $G(s, \text{C}|v)$ beginning at node (s, C) and continuation value $v \in \mathbb{R}^N$ with $c \in W$. The unique (pure strategy) Nash equilibrium of the negotiation stage game is the strategy profile $d(s,c) = (d_i)_{i \in \text{C}}$, uniquely given by the following equations:*

$$d_i = \delta v_i + \frac{1}{|\text{C}|} \left(d_{\text{C}} - \delta \sum_{i \in \text{C}} v_i \right), \text{ for all } i \in \text{C},$$

where $d_{\text{C}} = d_{\text{C}}(s, v)$ is the unique solution of the equation,

$$d_{\text{C}} - (|\text{C}| - 1) H_{\text{C}}(d_{\text{C}} - s_{\text{C}}) = \delta v_{\text{C}} \tag{13}$$

with $H_{\text{C}}(\varepsilon) = \frac{1 - G_{\text{C}}(\varepsilon)}{g_{\text{C}}(\varepsilon)}$ the (inverse) hazard rate.
The corresponding unique equilibrium expected value $v_i(s,c|v)$, at the beginning of the negotiation stage subgame, is given by

$$v_i(s, \text{C}|v) = \begin{cases} \delta v_i + \gamma(s, \text{C}|v) & \text{if } i \in \text{C} \\ \delta v_i P(s, \text{C}|v) & \text{if } i \notin \text{C} \end{cases}, \tag{14}$$

where, the average gain $\gamma(s, \text{C}|v)$ and the disagreement probability $P(s, \text{C}|v)$ are:

$$\gamma(s, \text{C}|v) = \frac{1}{|\text{C}|}\int_{d_{\text{C}}-s_{\text{C}}}^{+\infty} (s_{\text{C}} + \varepsilon - \delta v_{\text{C}})\, g_{\text{C}}(\varepsilon)\, d\varepsilon, \text{ and} \tag{15}$$

$$P(s, \text{C}|v) = \int_{-\infty}^{d_{\text{C}}-s_{\text{C}}} g_{\text{C}}(\varepsilon)\, d\varepsilon = G_{\text{C}}(d_{\text{C}} - s_{\text{C}}). \tag{16}$$

The equilibrium outcome resembles the Nash bargaining solution among the players in coalition C with status quo equal to the discounted continuation values and total surplus equal to d_{C}. The equilibrium demands d_i of each player i are equal to each player discounted continuation value δv_i plus the gain divided equally among all players $\frac{1}{|\text{C}|}(d_{\text{C}} - \delta v_{\text{C}})$. The aggregate demand d_{C} is endogenously given by the unique solution of equation (13), and $d_{\text{C}} \leq \delta v_{\text{C}}$.

It is easy to verify that when the noise converges to zero, the Nash equilibrium of the negotiation stage game converges to the equal split of the surplus among coalition members where the status quo is their continuation values: that is, the limit value is

$$v_i(s, \text{C}|v) \to \begin{cases} \delta v_i + \frac{1}{|\text{C}|}(s_{\text{C}} - \delta v_{\text{C}}) & \text{if } i \in \text{C} \\ 0 & \text{if } i \notin \text{C} \end{cases},$$

whenever, $s_{\text{C}} > \delta v_{\text{C}}$.

In the limit there is disagreement in the negotiation stage with probability zero, whenever a coalition C with an aggregate surplus larger than aggregate discounted continuation value is proposed, i.e. $s_{\text{C}} > \delta v_{\text{C}}$, where $v_{\text{C}} = \sum_{i \in \text{C}} v_i$. Players adjust their demands d_i accordingly, so that $d_{\text{C}} = \sum_{i \in \text{C}} d_i$ is infinitesimally below s_{C}, and the budget balancing condition $d_{\text{C}} \leq s_{\text{C}} + \varepsilon_{\text{C}}$ holds with probability close to one. Thus coalition C forms and players receive s_{C} in aggregate, rather than disagreeing and receiving the lower aggregate amount δv_{C} next period. Alternatively, disagreement is guaranteed when the opposite inequality holds, i.e., $s_{\text{C}} < \delta v_{\text{C}}$.

Proof of Proposition: Consider a vector of demands $(d_i(s, \text{C}))_{i \in \text{C}}$ submitted simultaneously by the players in the coalition C. In order to minimize notation, given that throughout the proof the state is always s and the coalition is C, we refer to the demands simply as $d = (d_i)_{i \in \text{C}}$ and let $d_{\text{C}} \equiv \sum_{i \in \text{C}} d_i$ and $v_{\text{C}} \equiv \sum_{i \in \text{C}} v_i$. Throughout the proof we consider the case where $|\text{C}| > 1$. The case $|\text{C}| = 1$ is trivial, and the unique Nash equilibrium is $d_i = \delta v_i$, and no assumption about the inverse hazard rate is required.

Player $i \in \text{C}$ payoff from choosing demand d_i given that the other agents are choosing demand d_{-i}, is

$$u_i(d_i, d_{-i}|v) = \delta v_i + \int_{d_{\text{C}}-s_{\text{C}}}^{\infty} \left(d_i - \delta v_i + \frac{1}{|\text{C}|}(s_{\text{C}} + \varepsilon_{\text{C}} - d_{\text{C}})\right) g_{\text{C}}(\varepsilon_{\text{C}})\, d\varepsilon_{\text{C}}.$$

To obtain the first order condition, note that the derivative of the integrand with respect to d_i is $\frac{|\text{C}|-1}{|\text{C}|}$, and the integrand evaluated at $\varepsilon_{\text{C}} = d_{\text{C}} - s_{\text{C}}$ is $d_i - \delta v_i$. Using the Leibnitz rule, the expression for the first order condition becomes

$$\frac{\partial u_i(d_i, d_{-i}|v)}{\partial d_i} = \left(\frac{|\mathrm{C}|-1}{|\mathrm{C}|}\right) \int\limits_{d_{\mathrm{C}}-s_{\mathrm{C}}}^{\infty} g_{\mathrm{C}}\left(\varepsilon_{\mathrm{C}}\right) d\varepsilon_{\mathrm{C}} - \left(d_i - \delta v_i\right) g_{\mathrm{C}}\left(d_{\mathrm{C}} - s_{\mathrm{C}}\right) = 0,$$

which yields for all $i \in \mathrm{C}$ the equation

$$d_i - \delta v_i = \left(\frac{|\mathrm{C}|-1}{|\mathrm{C}|}\right) \frac{\int\limits_{d_{\mathrm{C}}-s_{\mathrm{C}}}^{\infty} g_{\mathrm{C}}\left(\varepsilon_{\mathrm{C}}\right) d\varepsilon_{\mathrm{C}}}{g_{\mathrm{C}}\left(d_{\mathrm{C}} - s_{\mathrm{C}}\right)} \geq 0.$$

Observe that the equilibrium demand equalizes the gains across all players in C, as the right-hand side expression does not depend on i.

Let

$$H_{\mathrm{C}}\left(d_{\mathrm{C}} - s_{\mathrm{C}}\right) = \frac{\int\limits_{d_{\mathrm{C}}-s_{\mathrm{C}}}^{\infty} g_{\mathrm{C}}\left(\varepsilon_{\mathrm{C}}\right) d\varepsilon_{\mathrm{C}}}{g_{\mathrm{C}}\left(d_{\mathrm{C}} - s_{\mathrm{C}}\right)}.$$

Summing over all $i \in \mathrm{C}$ yields

$$d_{\mathrm{C}} - \left(|\mathrm{C}| - 1\right) H_{\mathrm{C}}\left(d_{\mathrm{C}} - s_{\mathrm{C}}\right) = \delta v_{\mathrm{C}}$$

There exists a unique solution d_{C}^{*} to the equation above because the left-hand side expression is: (i) monotonically increasing, because the inverse hazard rate is monotonically decreasing, (ii) as $d_{\mathrm{C}} \to \infty$ the expression goes to ∞, and (iii) as $d_{\mathrm{C}} \to -\infty$ the expression goes to $-\infty$.

The equilibrium demand then satisfies

$$d_i = \delta v_i + \frac{1}{|\mathrm{C}|}\left(d_{\mathrm{C}} - \delta v_{\mathrm{C}}\right),$$

since $\left(|\mathrm{C}| - 1\right) H_{\mathrm{C}}\left(d_{\mathrm{C}} - s_{\mathrm{C}}\right) = d_{\mathrm{C}} - \delta v_{\mathrm{C}}$.

Define $\gamma\left(s, \mathrm{C}|v\right)$, the average gain, and $P\left(s, \mathrm{C}|v\right)$, the disagreement probability, as

$$\begin{aligned}
\gamma\left(s, \mathrm{C}|v\right) &= \frac{1}{|\mathrm{C}|} \int\limits_{d_{\mathrm{C}}-s_{\mathrm{C}}}^{\infty} \left(s_{\mathrm{C}} + \varepsilon_{\mathrm{C}} - \delta v_{\mathrm{C}}\right) g_{\mathrm{C}}\left(\varepsilon_{\mathrm{C}}\right) d\varepsilon_{\mathrm{C}} \\
P\left(s, \mathrm{C}|v\right) &= \int\limits_{-\infty}^{d_{\mathrm{C}}-s_{\mathrm{C}}} g_{\mathrm{C}}\left(\varepsilon_{\mathrm{C}}\right) d\varepsilon_{\mathrm{C}} = G_{\mathrm{C}}\left(d_{\mathrm{C}} - s_{\mathrm{C}}\right).
\end{aligned}$$

From the results above, we obtain immediately that the value at the beginning of the negotiation stage, for all $i \in \mathrm{C}$ is

$$v_i\left(s, \mathrm{C}|v\right) = u_i(d_i, d_{-i}|v) = \delta v_i + \gamma\left(s, \mathrm{C}|v\right)$$

and the value, for all $i \notin \mathrm{C}$, since if there is an agreement, the payoff of $i \notin \mathrm{C}$ is zero and the game ends, is

$$v_i\left(s, \mathrm{C}|v\right) = \delta v_i P\left(s, \mathrm{C}|v\right).$$

We now show that $d = (d_i)_{i \in \mathrm{C}}$ is the unique pure strategy Nash equilibrium of the negotiation stage game, by showing that the first derivative $\frac{\partial u_i(d_i, d_{-i}|v)}{\partial d_i}$ is zero at $d_i = d_i^*$ and is positive for all $d_i < d_i^*$, and negative for all $d_i > d_i^*$.

To show this result note that the derivative is

$$\begin{aligned} \frac{\partial u_i(d_i, d_{-i}|v)}{\partial d_i} &= \frac{|\mathrm{C}| - 1}{|\mathrm{C}|} \int_{d_\mathrm{C} - s_\mathrm{C}}^{\infty} g_\mathrm{C}(\varepsilon_\mathrm{C})\, d\varepsilon_\mathrm{C} - (d_i - \delta v_i)\, g_\mathrm{C}(d_\mathrm{C} - s_\mathrm{C}) \\ &= \left[\frac{|\mathrm{C}| - 1}{|\mathrm{C}|} H(d_\mathrm{C} - s_\mathrm{C}) - (d_i - \delta v_i) \right] g_\mathrm{C}(d_\mathrm{C} - s_\mathrm{C}). \end{aligned}$$

But the term in brackets above is strictly decreasing in d_i because the inverse hazard rate $H(d_\mathrm{C} - s_\mathrm{C})$ is a monotonically strictly decreasing function of d_i and $g_\mathrm{C}(d_\mathrm{C} - s_\mathrm{C}) > 0$, thus d_i^* is the unique global maximum, which proves that $d = (d_i)_{i \in \mathrm{C}}$ is the unique Nash equilibrium of the negotiation stage game. *Q.E.D.*

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