Mergers and Acquisitions with Private Equity Intermediation*

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Abstract

We study a search model of investors' asset trading, intermediated by financial institutions that are at risk of selling under pressure. The selling pressure leads to the development of a secondary market, where intermediaries can sell assets to each other. We calibrate the model to the data of private equity (PE) funds in the corporate acquisition market. Interestingly, an increase in intermediaries can improve an intermediary's expected utility, because the enhanced benefits of secondary trades can prevail over utility loss from narrower buy-sell spreads due to more intense competition.

Keywords: Mergers and Acquisitions, Private Equity, Secondary Buyouts, OTC Markets, Financial Intermediation

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1 Introduction

1.1 Overview

Financial intermediaries that buy assets and hold them until resale often rely on external capital. This external funding exposes them to the risk of selling assets under pressure. One example of this is closed-ended private equity (PE) buyout funds within the corporate acquisition market.[1] A PE fund acquires a small number of assets such as corporate subdivisions, and ultimately exits by selling the assets through two methods: primary buyouts (PBOs) to corporate buyers and secondary buyouts (SBOs) to other PE funds. PE buyout funds have become significant players in the corporate acquisition market. In 2017, their investments accounted for \$538 billion out of the total \$2.1 trillion in the US acquisition market, representing 4,053 out of 10,769 deals, and over the period from 2011 to 2017, these funds experienced a compounded annual growth rate of 7.5%.[2] Typically, a buyout fund operates for 10-12 years, and as it nears the end of its life span, it may sell assets under pressure, often to other PE funds in the secondary market (Arcot, Fluck, Gaspar, and Hege, 2015).

We build a search-theoretic model of asset reallocation, where intermediaries (hereafter, *funds*) are at such risk of selling under pressure. Our model is similar in spirit to Duffie, Gârleanu, and Pedersen (2005) and Hugonnier, Lester, and Weill (2018). The search-and-bargaining features are suitable for capturing intermediaries' attempts to sell under pressure in a decentralized market. In particular, while selling portfolio assets, PE funds often take several months to find an appropriate buyer and close a transaction. We explore the impact of market characteristics, such as search friction and the number of funds, on welfare, fund valuations, trading volumes, and transaction prices. We quantify how the characteristics of the funds' ecosystem determine their performance. Funds compete for intermediation opportuni-

[1]A PE fund consists of General Partners (GPs) who have specialties in specific industries and operating expertise such as finance and marketing. The GPs raise capital from outside investors, called Limited Partners (LPs). After raising capital, a typical PE fund usually has around 10 to 12 years of life after its inception. The rationale for a finite life is that funds invest in private firms whose market value is unknown. Only after an asset is sold, GPs and LPs observe gains of the fund and can determine GPs' management compensation and LPs' share.

[2]The primary data is from the 2018 US PE middle market report by Pitchbook Data Inc. at `https://pitchbook.com/news/reports/2q-2018-ma-report`. The data includes corporate acquisition deals with transaction values between \$25 million and \$1,000 million.

ties but also provide each other with greater exit opportunities (i.e., secondary transactions). As such, funds may derive benefits from having additional funds in the ecosystem.

Our model comprises a continuum of investors and funds who hold either one or zero assets. The assets are reallocated over time through (investor-to-investor) direct trading, fund-investor trading, and (fund-to-fund) secondary trading with heterogeneous search frictions. High-type investors can generate higher flow payoffs, so assets are reallocated from low-type investors to high-type investors, sometimes through transactions involved with funds. Each investor's type changes over time by an exogenous shock. Funds buy assets from low-type investors and sell to high-type investors. While holding assets, funds may receive liquidity shocks and incur a holding cost from owing assets. Then, the funds under selling pressure would attempt to sell the assets, even to other funds, through secondary transactions.

We find a unique steady-state equilibrium in which all three kinds of transactions are active. We find that funds enhance market efficiency by alleviating search frictions and providing greater liquidity to the market. The first channel, whereby intermediaries facilitate asset transfers, is well-known. The second channel – the liquidity channel – is unique to our model and is related explicitly to funds' demand for liquidity due to their exit requirements.

In Section 4, we calibrate our model to the US mid-size corporate acquisition market with PE intermediation to quantitatively address equilibrium questions involving complex interactions among parameters. The M&A market with PE intermediation is a novel application of the OTC literature. The mid-size corporate acquisition market lends itself naturally to a search framework: this market is particularly unintegrated with approximately 100,000 corporations and 2,000 PE funds, and it takes about a year for sellers to find appropriate buyers and close transactions. A typical transaction involves a sale of all the corporation assets or a subdivision by corporate investors or private equity (PE) buyout funds.[3]

Our first set of results focuses on the (fund-to-fund) secondary market. We show that secondary transactions enhance fund value and improve overall welfare (Proposition 3). Funds

[3]Certainly, our model contains some simplifications. Private equity funds in practice can hold multiple assets and directly improve the fundamental value of the holding assets. We hope our model opens the door for future work that allows such unique features of PE funds in the corporate acquisition markets. For alternative models of corporate acquisitions, see Jovanovic and Rousseau (2002), Rhodes-Kropf, Robinson, and Viswanathan (2005), Rhodes-Kropf and Robinson (2008), Eisfeldt and Rampini (2008), David (2017), and Almeida, Campello, and Hackbarth (2011).

sourcing deals provide liquidity to funds at the exit phase, while funds at the exit phase allow fund buyers to acquire assets more quickly. Secondary transactions offer a channel through which funds can complement each other (Proposition 4). As a result, perhaps surprisingly, the value of funds can increase with the number of funds (Proposition 5). For the calibrated model parameters, an increase of PE funds' flow payoff by 1% increases fund value by 6%. Secondary transactions are sometimes criticized as opportunistic behavior among fund managers passing sub-par assets to their counterparts, all the while both sides collect management fees from the fund investors.[4] However, we find that funds generate higher returns because of (and not in spite of) secondary transactions, and the possibility of secondary transactions contributes to 36% of each new fund's expected utility. If PE funds cannot enhance the operating cash flows of acquired firms, then intermediation would not be profitable in spite of their search advantages.

We study other equilibrium properties such as welfare, trade speeds, prices, and trade volumes (Proposition 8, Proposition 9, and Proposition 10). Direct trades between low and high-type investors clearly have a positive gain, but its welfare consequence is subtle because investors' direct trading deprives funds of potential trade opportunities. Funds will find it harder to turn over their inventory quickly and intermediate the asset market efficiently. Slowing down direct trading for investors enables exit-phase funds to quickly off-load assets and purchase new ones, especially when they can search for trading opportunities fast. The equilibrium welfare is 13.4% higher than the welfare in the counterfactual situation without M&A transactions and PE intermediation. Thus, PE funds contribute to the welfare both by lowering search frictions and adding operational value.

1.2 Related Literature

We discuss only closely related papers on OTC markets with intermediation and refer others to Nosal and Rocheteau (2011) and references therein.[5]

Hugonnier, Lester, and Weill (2020) study an inter-dealer market and share some similarities with our direct and secondary trading markets. There are several important distinctions.

[4] See `https://www.economist.com/node/15580148`.

[5] Our model differs from interbank network models surveyed by Allen and Babus (2009). There, the focus is on lending and borrowing.

First, our model is tailored to capture funds experiencing selling pressure due to liquidity shocks; the shocks arrive contingent on holding assets and cease to exist when funds off-load assets. Our model is suitable for funds intermediating with external capital. Second, their inter-dealer market is singled out such that investors can trade only through dealers. In contrast, we are interested in concurrent operations of unrestricted interactions among investors and funds. The unrestricted interaction is relevant for applications such as the corporate acquisition market and the real estate market (Phillips and Zhdanov (2017)). Third, their results focus on trading patterns including intermediation chains, whereas our focus is on fund valuations and welfare.

We focus on steady-state equilibrium in which funds choose to intermediate between low and high-type investors, due to their moderate flow payoffs. This self-selection of intermediaries has been studied in Neklyudov (2019), Üslü (2019), Nosal, Wong, and Wright (2016), Shen, Wei, and Yan (2021), Yang and Zeng (2018), and Farboodi, Jarosch, and Shimer (2017). It is often the case that mid-type investors, similar to our funds, choose to intermediate with comparative advantages in search skills. Atkeson, Eisfeldt, and Weill (2015) also study an OTC market with endogenous intermediation. For derivative swap contracts, investors with risky endowments may be unable to share the risk fully, because of a size limit on bilateral trades. Buy and sell prices do not reflect the aggregate risk, and the price dispersion incentivizes some banks to act as intermediaries.

The welfare effect of secondary transactions has been studied in other contexts. Gofman (2014) shows that better-connected intermediaries in financial markets can shorten intermediation chains and improve welfare. Pagano and Volpin (2012) study the bank-loan market where banks lend to consumers (primary issuance) while other banks provide liquidity by investing in securitized loans (secondary market liquidity). High securitization activities in the secondary market yield high loan issuance in the primary market, driven by greater transparency in the secondary market. Finally, Hochberg, Ljungqvist, and Lu (2007) study the benefits of PE funds' networks in sharing information about assets.

Our paper focuses on intermediaries' selling under pressure. However, in some applications, fund managers initiate secondary transactions, either under buying pressure when the fund is near the end of its investment phase with excess capital, or under selling pressure when the fund is close to the end of the fund's life (Arcot, Fluck, Gaspar, and Hege (2015), Degeorge, Martin, and Phalippou (2016), and Wang (2012)). In either case, the secondary market offers a channel through which funds can provide liquidity to each other.

The remainder of the paper is organized as follows. Section 2 introduces the formal model; Section 3 provides equilibrium properties; Section 4 presents a calibration exercise; Section 5 discusses our main analysis of secondary transactions; Section 6 provides results on welfare, prices, and trade volumes; and Section 7 concludes.

2 Model

Time runs continuously in $t \in [0, \infty)$. Over time, two kinds of agents, **investors** and **funds** trade **assets**. Initially, a fraction of investors and funds are endowed with assets. The measures of investors k_v, funds k_f, and tradable assets k_a remain constant. All agents are risk neutral and infinitely lived, with time preferences determined by a constant discount rate r. Each agent holds one or zero assets. Hence, $k_a < k_v + k_f$. We normalize the total measure of investors as $k_v = 1$.

An investor that holds an asset generates either a high payoff flow u_h or a low payoff flow u_l ($< u_h$). An investor does not receive any payoff flow when not holding an asset. An investor's ability to create payoff flow switches from low to high with Poisson intensity ρ_u, or from high to low with intensity ρ_d. The arrival rate of this Poisson shock for each type of investor is independent of other investors.[6] The set of investor types is $\mathcal{T}_v \equiv \{ho, lo, hn, ln\}$, where the letters h and l represent each investor's ability to generate payoffs and the letters o and n denote whether an investor owns an asset or not.

A fund's life cycle consists of an investment phase, a harvesting phase, and an exit phase. A fund in the investment phase does not own assets and searches for an investor or a fund selling assets. After purchasing an asset, the fund enters the harvesting phase and creates

[6] Although beyond the scope of the current paper, our model can be adapted to accommodate systematic shocks to the economy by introducing state variables and transitions from one state to another. Using economic state variables to capture aggregate shocks appears in the M&A literature. For example, acquisitions motivated by production synergies occur more during economic expansions, whereas disinvestment-related takeovers occur more during recessions (Mason and Weeds (2010); Lambrecht (2004); Lambrecht and Myers (2007). Further, Bernile, Lyandres, and Zhdanov (2012) show that horizontal mergers in oligopolistic industries tend to occur when consumer demand increases or decreases, rather than remaining stable.

payoff flow u_f. A fund in the harvesting phase sells its assets and starts a new life cycle (i.e., goes back to the investment phase),[7] or it receives a liquidity shock with intensity ρ_e and enters the exit phase. A fund in the exit phase incurs a holding cost and generates a lower payoff flow u_e ($< u_f$). After selling the asset, the fund automatically starts a new life in the investment phase. We denote a fund in the investing phase by type fn (a fund non-owner), in the harvesting phase by type fo (a fund owner), and in the exiting phase by type fe (a fund that is exiting). The set of fund types is $\mathcal{T}_f \equiv \{fn, fo, fe\}$.

We assume that funds generate moderate payoff flows, $u_l < u_e < u_f < u_h$, such that funds play the role of intermediaries by purchasing assets from low-type investors and selling them to high-type investors. In reality, buyout funds contribute to operational efficiencies of assets (thus, $u_f > u_l$). However, assets divested by buyout funds are acquired by corporate investors, indicating that the flow payoff for certain corporate investors must be even higher than that of buyout funds (thus, $u_h > u_f$). This disparity may be attributed to additional benefits associated with holding an asset, such as synergies with the investor's existing asset portfolio. Our model accounts for this synergy by incorporating an increased flow payoff u_h over u_l.

Let $\mathcal{T} \equiv \mathcal{T}_v \cup \mathcal{T}_f$ denote the set of types with typical elements i, j, etc. The measure of type $i \in \mathcal{T}$ at time $t \in [0, \infty)$ is denoted by $\mu_i(t)$. Then,

$$\begin{aligned} \mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) &= k_v (= 1), \\ \mu_{fn}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= k_f, \\ \mu_{ho}(t) + \mu_{lo}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= k_a. \end{aligned} \tag{1}$$

Agents meet each other over time and negotiate a trade. Two investors meet each other with intensity λ_d for (investor-to-investor) **direct trading**. An investor and a fund meet each other with intensity λ_f for a **fund-investor trading**. A fund in the exit phase (fe) and a fund in the investment phase (fn) meet each other with intensity λ_s for (fund-to-fund) **secondary trading**. The meeting rate between any pair of groups is linear in each group's population. That is, for any pair of investor types $i, j \in \mathcal{T}_v$ with measures μ_i and μ_j, the total meeting rate is $\lambda_d \mu_i \mu_j$. Similarly, the total meeting rate between an investor type $i \in \mathcal{T}_v$

[7] General partners of PE funds often start a new fund around the liquidation of an existing fund.

and a fund type $j \in \mathcal{T}_f$ is $\lambda_f \mu_i \mu_j$, and the total meeting rate in the secondary market is $\lambda_s \mu_{fe} \mu_{fn}$. When two agents meet each other, they trade an asset instantaneously if and only if the gain from trade (which we explain later) is positive. The assumption of immediate trading upon meeting follows the literature on bargaining without asymmetric information.

We will find an equilibrium in which all tradings denoted by $\mathcal{M} \equiv \{lo\text{-}hn, lo\text{-}fn, fo\text{-}hn, fe\text{-}hn, fe\text{-}fn\}$ are active.[8] That is, a lo-type investor sells an asset to a hn-type investor (lo-hn trade). Similarly, either a lo-type investor sells an asset to a fn-type fund (lo-fn trade), or a fund of type either fo or fe sells an asset to a hn-type investor (either fo-hn or fe-hn trade). In the secondary market, a fe-type fund sells to a fn-type fund (fe-fn trade). After all trades, the types change from 'o' to 'n' and vice versa. Overall, assets are transferred from low-type investors toward high-type investors, with a possible chain of trades among funds through secondary trades.

Figure 1 summarizes the model. Agent types are listed on the left column for owners and the right column for non-owners. An owner changes her type to one on the right column upon selling her asset; non-owner changes her type to one on the left column upon purchasing an asset (a fund's type becomes fo after an asset purchase). The solid arrows represent asset reallocations from sellers to buyers. The vertical dashed arrows represent the exogenous type changes: high vs. low for investors, or a liquidity shock to fo-type funds.

An asset market with fund intermediation is a collection of exogenous parameters $\theta \equiv (k, r, u, \rho, \lambda)$, where $k \equiv (k_v, k_f, k_a)$, $u \equiv (u_l, u_h, u_f, u_e)$, and $\lambda \equiv (\lambda_d, \lambda_f, \lambda_s)$. All exogenously given parameters are strictly positive. We provide a summary of the model parameters in Table 5 in the Appendix, along with the equilibrium variables that will be introduced in the subsequent section.

[8] However, two buyout funds, fo and fn, do not trade. Such circumstance does not yield any gains because both funds would receive the same payoff flow and have an equal chance (Poisson arrival) of experiencing a liquidity shock while holding the asset. For the same reason, the same type of investors, either high-type or low-type, also do not trade. In the real world, a new fund that acquires an asset typically does not dispose of it to another fund before reaching the end of its life cycle. This lack of trading is likely due to insufficient gains from such trades. Our model ensures that the same property holds in equilibrium.

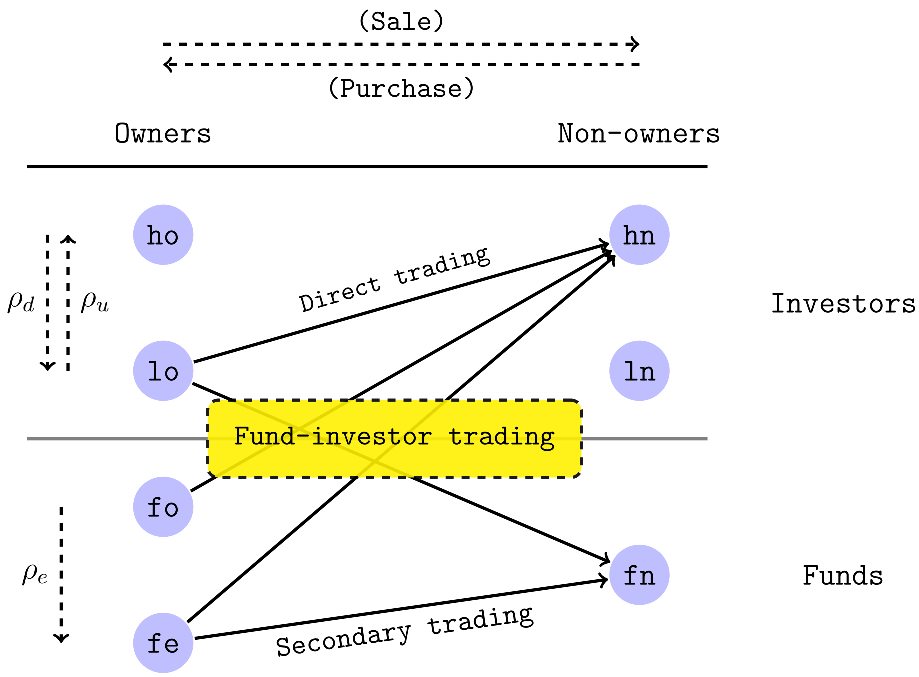

Figure 1: An asset market with fund intermediation.

3 Equilibrium

We are interested in a steady-state equilibrium in which investors trade assets amongst themselves, and funds intermediate by buying and selling assets. We first derive steady-state population measures. Let k_h and k_l denote the steady-state populations of high- and low-type investors. Since we normalized the total measure of investors as $k_v = 1$, from the rates of exogenous type changes,

$$k_h = \frac{\rho_u}{\rho_u + \rho_d} \quad \text{and} \quad k_l = \frac{\rho_d}{\rho_u + \rho_d}.$$

A hn-type investor switches its type to ho upon purchasing an asset from either a lo-type investor or a fo- or fe-type fund. As such, hn-type investors become ho-type at the rate of $(\lambda_v \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe})\mu_{hn}$. On the other hand, as a result of exogenous type changes, hn-type investors switch their types to ln at the rate of $\rho_d \mu_{hn}$; similarly, ln-type investors switch their types to hn at the rate of $\rho_u \mu_{ln}$. Thus,

$$\dot{\mu}_{hn}(t) = -(\lambda_d \mu_{lo}(t) + \lambda_f \mu_{fo}(t) + \lambda_f \mu_{fe}(t))\mu_{hn}(t) - \rho_d \mu_{hn}(t) + \rho_u \mu_{ln}(t). \tag{μ-hn}$$

The population measures for other types change over time by similar processes:

$$\dot{\mu}_{ho}(t) = (\lambda_d \mu_{lo}(t) + \lambda_f \mu_{fo}(t) + \lambda_f \mu_{fe}(t))\mu_{hn}(t) - \rho_d \mu_{ho}(t) + \rho_u \mu_{lo}(t), \tag{μ-ho}$$

$$\dot{\mu}_{ln}(t) = (\lambda_d \mu_{hn}(t) + \lambda_f \mu_{fn}(t))\mu_{lo}(t) - \rho_u \mu_{ln}(t) + \rho_d \mu_{hn}(t), \tag{μ-ln}$$

$$\dot{\mu}_{lo}(t) = -(\lambda_d \mu_{hn}(t) + \lambda_f \mu_{fn}(t))\mu_{lo}(t) - \rho_u \mu_{lo}(t) + \rho_d \mu_{ho}(t), \tag{μ-lo}$$

$$\dot{\mu}_{fn}(t) = \lambda_f \left(\mu_{hn}(t)\mu_{fo}(t) + \mu_{hn}(t)\mu_{fe}(t) - \mu_{lo}(t)\mu_{fn}(t)\right), \tag{μ-fn}$$

$$\dot{\mu}_{fo}(t) = (\lambda_f \mu_{lo}(t) + \lambda_s \mu_{fe}(t))\mu_{fn}(t) - \lambda_f \mu_{hn}(t)\mu_{fo}(t) - \rho_e \mu_{fo}(t), \tag{μ-fo}$$

$$\dot{\mu}_{fe}(t) = -(\lambda_f \mu_{hn}(t) + \lambda_s \mu_{fn}(t))\mu_{fe}(t) + \rho_e \mu_{fo}(t). \tag{μ-fe}$$

Let $P(\theta)$ denote the above system of population equations (μ-hn)-(μ-fe). A real-vector $\mu \equiv (\mu_i)_{i\in\mathcal{T}}$ with each $\mu_i \geq 0$ is a steady-state solution of $P(\theta)$ if the right-hand sides of the equations, with $\mu_i(t)$ replaced by μ_i for each $i \in \mathcal{T}$, are equal to zero.

Proposition 1. *(Steady-state Population Measures)*

1. *(Existence and Uniqueness) There exists a unique steady-state solution μ of $P(\theta)$ such that $\mu_i > 0$ for all $i \in \mathcal{T}$.*

2. *(Asymptotic Stability) Let $\mu(t)$ be a dynamic solution of $P(\theta)$ with initial condition $\mu(0)$. For any $\epsilon > 0$, there exists $\delta > 0$ such that, if $\|\mu(0)-\mu\| < \delta$, then $\|\mu(t)-\mu\| \leq \epsilon$ for all t, and $\mu(t) \to \mu$ as $t \to \infty$.*

The proof uses the Poincare-Hopf index theorem (Simsek, Ozdaglar, and Acemoglu, 2007), which generalizes the Intermediate Value Theorem and is new to the OTC market literature. To get an intuition, we set $(\mu_i)_{i\in\mathcal{T}_v} \approx 0$, while satisfying the population equations $P(\theta)$ but violating $k_v = 1$. A small increase of μ_{lo} (or μ_{hn}) increases the supply (resp., demand) of assets for investors' direct trading $\lambda_d \mu_{lo}\mu_{hn}$, and in turn the number of investors that are rightly holding or not-holding assets (μ_{ho} and μ_{ln}). The increased populations of μ_{ho} and μ_{ln} lead to more inflow $\rho_d \mu_{ho}$ of agents back to the aggregate supply and the inflow $\rho_u \mu_{ln}$ to the aggregate demand for direct trading. That is, all four investor-type populations increase. Taking into account how investor-type populations are related to fund-type populations, we find a unique supply μ_{lo} and demand μ_{hn} that yield $\sum_{i \in \mathcal{T}_v} \mu_i = k_v = 1$ by the index theorem. The second part of the proposition on stability is due to a classical result in dynamical systems.[9]

We define a steady-state equilibrium via a recursive equation of certain values (or, expected utility). The sources of value to all agents in our model are two-fold: flow payoffs while holding assets and gains from trade. Let v_{hn} denote the expected value of time-discounted future payoffs for a type-hn investor. The value is defined implicitly by

$$rv_{hn} = \lambda_d \mu_{lo} g_{lo-hn} + \lambda_f \mu_{fo} g_{fo-hn} + \lambda_f \mu_{fe} g_{fe-hn} - \rho_d (v_{hn} - v_{ln}), \tag{v-hn}$$

where each g_{lo-hn}, g_{fo-hn}, and g_{fe-hn} denotes the investor's **gain from trade** (in fact, an equal share of the gain, which we define later). The meeting rate for direct trading, taking into account the population of sellers, is $\lambda_d \mu_{lo}$, and the gain from trade is g_{lo-hn}. Two other terms are defined similarly for the cases of trading with either a fo- or fe-type fund. The investor changes its type from high to low with rate ρ_d, in which case it loses value equivalent to $v_{hn} - v_{ln}$.

The values for other types are defined similarly as follows:

$$rv_{ho} = u_h - \rho_d (v_{ho} - v_{lo}), \tag{v-ho}$$

$$rv_{ln} = \rho_u (v_{hn} - v_{ln}), \tag{v-ln}$$

$$rv_{lo} = u_l + \lambda_d \mu_{hn} g_{lo-hn} + \lambda_f \mu_{fn} g_{lo-fn} + \rho_u (v_{ho} - v_{lo}), \tag{v-lo}$$

$$rv_{fn} = \lambda_f \mu_{lo} g_{lo-fn} + \lambda_s \mu_{fe} g_{fe-fn}, \tag{v-fn}$$

$$rv_{fo} = u_f + \lambda_f \mu_{hn} g_{fo-hn} - \rho_e (v_{fo} - v_{fe}), \tag{v-fo}$$

$$rv_{fe} = u_e + \lambda_f \mu_{hn} g_{fe-hn} + \lambda_s \mu_{fn} g_{fe-fn}. \tag{v-fe}$$

A flow payoff is included for each owner type (the payoff flow is zero for non-owners).

[9] If all eigenvalues of the linearized system at the steady-state solution have negative real parts, then the solution is asymptotically stable (Hirsch and Smale, 1973).

We assume that the transaction prices (that we will characterize in the next subsection) will be set to ensure an equal division of gain from trade between a buyer and a seller.[10] Then, each agent's gain from trade is:

$$\begin{aligned} g_{lo-hn} &\equiv (1/2)(v_{ho} + v_{ln} - v_{lo} - v_{hn}), \\ g_{lo-fn} &\equiv (1/2)(v_{fo} + v_{ln} - v_{lo} - v_{fn}), \\ g_{fo-hn} &\equiv (1/2)(v_{ho} + v_{fn} - v_{fo} - v_{hn}), \\ g_{fe-hn} &\equiv (1/2)(v_{ho} + v_{fn} - v_{fe} - v_{hn}), \\ g_{fe-fn} &\equiv (1/2)(v_{fo} + v_{fn} - v_{fe} - v_{fn}) = (1/2)(v_{fo} - v_{fe}). \end{aligned}$$

Let $V(\theta)$ denote the above system of value equations (v-hn)-(v-fe), with μ being replaced by the unique steady-state solution of $P(\theta)$.

Proposition 2. *There exists a unique solution of $V(\theta)$.*

We have characterized the unique steady-state population measures μ and the values v, assuming that all tradings are active. If the unique steady-state solution (μ, v) results in positive trade gains, we call it a **steady-state equilibrium**.[11]

We provide the conditions for positive trade gains as (21) and (22) in Appendix. The trade gain g_{fe-fn} is trivially positive from $u_f > u_e$: secondary transactions bail out funds under liquidity constraints. The gains from investors' direct trading and fund-investor trading are related to each other as $g_{lo-hn} = g_{lo-fn} + g_{fo-hn}$. This implies that both direct trades and indirect trades through fund intermediation result in the same total gains. Similarly, gains in fund-investor trading and secondary trading are related as $g_{fe-hn} = g_{fo-hn} + g_{fe-fn}$. Hence, it is sufficient to ensure that g_{lo-fn} and g_{fo-hn} are positive. This can be achieved, for example, by having a significant difference between u_f and u_l (indicating a substantial contribution by buyout funds to operational efficiencies of assets), as well as a notable difference between u_h and u_f (highlighting substantial benefits from synergies for certain investors).

[10]Our qualitative results do not depend on the assumption of equal bargaining power. Ahern (2012) observes that the dollar gains of trades are often equally split between buyers and sellers.

[11]A unique steady-state equilibrium appears in Duffie, Gârleanu, and Pedersen (2005), but multiple equilibria appear more commonly with financial market applications: e.g., Vayanos and Weill (2008) and Trejos and Wright (2016).

The transaction prices are determined so that buyers and sellers equally share the gains from trades (i.e., the equal bargaining power assumption):

$$\begin{aligned}
p_{lo-hn} &\equiv (1/2)(v_{ho} + v_{lo} - v_{hn} - v_{ln}), \\
p_{lo-fn} &\equiv (1/2)(v_{fo} + v_{lo} - v_{ln} - v_{fn}), \\
p_{fo-hn} &\equiv (1/2)(v_{ho} + v_{fo} - v_{fn} - v_{hn}), \\
p_{fe-hn} &\equiv (1/2)(v_{ho} + v_{fe} - v_{fn} - v_{hn}), \\
p_{fe-fn} &\equiv (1/2)(v_{fo} + v_{fe} - 2v_{fn}).
\end{aligned}$$

Finally, we adopt as a notion of **social welfare** the sum of the utilities of investors and funds, which is a standard definition in the literature (see, for example, Duffie, Gârleanu, and Pedersen (2005) and Hugonnier, Lester, and Weill (2018). Formally the (total) welfare is

$$W \equiv \sum_{i \in \mathcal{T}} \mu_i v_i,$$

the sum of the values of all player types weighted by the populations in the economy.

4 Calibration

We calibrate our model with data from the mid-sized US corporate acquisition market to attain a quantitative sense of equilibrium. The mid-size corporate acquisition market lends itself naturally to a search framework. A typical transaction involves a sale of assets – either all the corporation assets or a subdivision – by corporate investors or by private equity (PE) buyout funds, and it takes about a year for sellers to find appropriate buyers and close transactions (Boone and Mulherin, 2011). A typical PE buyout fund acquires a small number of portfolio firms, holds the firms as inventory and adds operational value through better management, and exits by selling their portfolio firms.[12] PE funds divest the acquired

[12]We distinguish buyout funds from other PE funds such as venture capital funds, which usually invest in fractional equity stakes of start-ups and early-stage firms.

firms to corporate investors through primary buyouts (PBOs), or to other PE funds through secondary buyouts (SBOs).

The primary data is from the 2018 US PE middle market report by Pitchbook Data Inc. and the 2012 U.S. Economic Census Data. The dataset includes various model statistics listed in Table 1 related to transaction volumes, average time to sell, price multiple, and fund performance. There are approximately 102,626 mid-sized companies meeting our criteria, and 1,893 PE funds targeting the middle market (Given our normalization $k_v = 1$ for the number of mid-sized companies, $k_f = 1,893/102,626 \simeq 0.02$).[13] From 2007 to 2017, the total number of transactions is 9,626 per year, and PE funds acquired an average of 1,799 firms per year, of which 359 ($\approx 20\%$) are through SBOs and 1,440 are through PBOs. Then, we can estimate 6,387 direct transactions (η_{lo-hn}) per year.[14].

The additional statistics in Table 1 are sourced from various references. First, the average time to sell for corporate investors and PE funds is from the average of the values provided in online reports prepared by selling agents, such as business brokers or investment bankers (for a detailed list of references, please refer to the Supplemental Appendix). On average, it takes approximately 11 months (0.91 years) for PE funds and 15 months (1.25 years) for corporate investors to complete a firm sale.[15]

Second, for the fund performance, we use the Public Market Equivalent (PME) introduced by Kaplan and Schoar (2005) and Sorensen and Jagannathan (2015). We consider the PME to be the average of 1.01 from various estimates.[16] Lastly, the EV/EBITDA multiple

[13] We define mid-size companies, as those with annual revenues between \$20 million and \$1,000 million. We exclude small companies from our analysis due to the lack of reliable data on their acquisition activities. Additionally, we omit large companies to maintain homogeneity in our sample set. We apply the same normalization to the number of PE funds, assets, trading volumes, etc. Pitchbook reports the number of PE funds raised each year targeting the middle market. We cumulate these numbers for 2007-2018, as the average lifespan of PE funds is 12 years (Metrick and Yasuda, 2010).

[14] The number of direct transactions is estimated using our model. In equilibrium, the number of PE buyouts must be equal to the number of exits (i.e, $\eta_{fe-hn} + \eta_{fo-hn} = \eta_{lo-fn}$), so the total number of deals equals the sum $\eta_{lo-hn} + 2\eta_{lo-fn} + \eta_{fe-fn}$, which gives us the estimate of direct transactions (η_{lo-hn})

[15] Time to sell includes time taken in the preparation process and the listing-to-sale process. The preparation process for PE funds takes only an average of 2 months – much shorter than an average of 6 months for corporate investors. Portfolio firms of PE funds are usually in a better state of readiness to approach the market due to high-quality governance, accounting, and information systems. The listing-to-sale process takes an average of 9 months for selling agents.

[16] Public Market Equivalent (PME) is defined as the ratio of cash outflows over cash contributions, both discounted at the public market total return (e.g., S&P 500 index) after subtracting management fees paid to

Description	Data	Model Statistic (for normalized values)
Corporate (Direct) acquisitions	6387	$\eta_{lo-hn} = \lambda_d \mu_{lo} \mu_{hn}$
Primary Buyouts (PBOs)	1440	$\eta_{lo-fn} = \lambda_f \mu_{lo} \mu_{fn}$
Secondary Buyouts (SBOs)	359	$\eta_{fn-fe} = \lambda_s \mu_{fn} \mu_{fe}$
Avg time to sell for corporate investors	1.25 years	$E[\tau_{sv}]$ (eq. (4))
Avg time to sell for PE funds	0.91 years	$E[\tau_{sf}]$ (eq. (5))
Fund performance (PME)	1.01	PME (eq. (6))
Price multiple (EV/EBITDA)	9.0	P_{lo-hn}/u_l

Table 1: Key Statistics on the Corporate Acquisition Market

is set equal to the average of 9.0 from 2005 to 2017.[17]

Next, we examine the remaining model parameters listed in top panel of Table 2. Some of these parameters are directly observed in the existing literature. Low-type corporate investors' flow payoff is normalized as $u_l = 1$. We set $u_h = 1.4$ from Betton, Eckbo, and Thorburn (2008), which report an average 43% takeover premium over 4,880 acquisitions during 1980-2002, and Bargeron, Schlingemann, Stulz, and Zutter (2008), which find that the takeover premium paid by a private acquirer is 40.9%. We set $u_f = 1.2$ from Guo, Hotchkiss, and Song (2011), which estimated a median gain of 12.4% by large-market funds, with an adjustment to 20% because we focus on mid-market funds associated with higher risk and higher returns, as opposed to large-market funds.[18] For the flow payoff net of liquidity cost u_e, Nadauld, Sensoy, Vorkink, and Weisbach (2016) find that fund investors under liquidity shocks sell their PE ownership to other fund investors at a 13.8% discount.

the fund managers. Kaplan and Schoar (2005) estimate an average PME of 0.93 for PE funds in the period 1980-1994, while Phalippou and Gottschalg (2008), using a similar dataset but different methodology, report an average PME of 0.88. Harris, Jenkinson, and Kaplan (2014), on the other hand, reports significantly better performance with an average PME of 1.22 for the period 1984-2008. The estimates of PME by PitchBook Data, Inc. yield an average of 1.00 for the period 2006-15.

[17]See a recent report on EV/EBITDA by FactSet Research Systems Inc. at `https://www.factset.com/hubfs/mergerstat_em/monthly/US-Flashwire-Monthly.pdf`.

[18]In earlier version, we chose $u_f = 1.3$ from Kaplan (1989) which reported 45.5%, 72.5%, and 28.3% increases in net cash flow/sales each year for the first three years following the buyout, and Muscarella and Vetsuypens (1990), Opler (1992), and Andrade and Kaplan (1998), which estimated the increase in operating profits of target firms after fund buyouts as 23.5%, 16.5%, and 52.9%, respectively. However, the private equity industry may have undergone transformations over the years (Strömberg, 2008), and the profitability effects of large public-to-private deals have weakened (Guo, Hotchkiss, and Song, 2011).

	Parameters	Variable	Value
	No. of corporate investors	k_v	1.0
	No. of PE funds	k_f	0.02
	No. of assets	k_a	0.5
(Observed)	Flow Payoff low type	u_l	1
	Flow payoff high type	u_h	1.4
	Flow payoff PE (harvesting)	u_f	1.2
	Flow payoff PE (exiting)	u_e	1.03
	Low valuation shock	ρ_d	0.24
	High valuation shock	ρ_u	0.16
	Liquidity shock	ρ_e	0.38
(Estimated)	Match intensity (direct trading)	λ_d	46.1
	Match intensity (PBO)	λ_f	61.7
	Match intensity (SBO)	λ_s	699
	Discount rate	r	11.8%

Table 2: Fitted Parameters of Calibration

This observation motivates our choice of $u_e = (1 - 0.138) \times u_f \simeq 1.03$.[19]

We do not directly observe the number of assets, so our benchmark analysis assumes $k_a = 0.5$, but the calibration results vary insignificantly if $k_a = 0.25$ and $k_a = 0.75$.

We estimate the parameters (ρ, λ, r) in the bottom panel of Table 2 that offer the best fit to the seven key statistics in Table 1. Specifically, each choice of the remaining parameters' values, together with the directly observed parameters (k, u), defines a market $\theta = (k, r, u, \rho, \lambda)$. We compute the statistics $Y_i(\rho, \lambda, r; k, u)$ for each row $i = 1, \ldots, 7$ in Table 1, using the closed-form expressions for various metrics, such as the average time to sell (obtained in Proposition 9), PME (obtained in Proposition 11), EV/EBITDA (P_{lo-hn}/u_l), and compare them with the observed data Y_i^{obs}.

We obtain estimates for (ρ, λ, r) by minimizing the sum of squared residuals (SSR) subject to positive trade gains in the unique steady-state solution of the market $(\rho, \lambda, r; k, u)$, i.e., $\min_{\rho,\lambda,r} \sum_{i=1}^{7} \left(\frac{Y_i(\rho,\lambda,r;k,u) - Y_i^{obs}}{Y_i^{obs}} \right)^2$ subject to $g_m(\beta; k, u) \geq 0$, for each $m \in \mathcal{M}$. Our model fits the observed data with a high degree of accuracy. The minimum SSR is approximately

[19] Indeed, the equilibrium trade patterns, such as trade volumes, population distributions, and time to sell or buy, remain essentially invariant with respect to the flow-payoff parameters. We will discuss the sensitivity of other statistics such as value, welfare, and prices in later sections.

2.9×10^{-5}.

The lower section of Table 2 summarizes the parameter estimates.[20] The parameter estimates are of reasonable magnitudes. The estimated type transition rates ρ_u and ρ_d suggest that the corporate investors' type transitions take on average about $1/\rho_u = 6.30$ years from low to high, and $1/\rho_d = 4.25$ years from high to low. Moreover, a PE fund holding an asset can expect to experience a liquidity shock approximately every $1/\rho_e = 2.61$ years. The matching intensities $(\lambda_d, \lambda_f, \lambda_s)$ are not directly interpretable: for example, it takes $\frac{1}{\lambda_d \mu_{lo}}$ years for a high-type investor to meet a low type. Lastly, the estimated discount rate $r = 11.8\%$, although high, seems reasonable given that assets represent stakes in mid-size private firms.

In the following sections, we address various implications of our calibration, including: (i) Is there an oversupply or undersupply of tradable assets relative to the number of corporate buyers and PE funds? (ii) How are fund valuations affected by liquidity provision through SBOs and operational improvements made by fund managers? (iii) How does PE entry impact fund valuation and transaction prices? (iv) What are the welfare losses associated with search frictions?

5 Equilibrium Analysis of the Secondary Market

We conduct an equilibrium analysis with an emphasis on the secondary market. Our approach incorporates both qualitative and quantitative aspects, utilizing model calibration.

We find that secondary trading enhances liquidity and improves welfare as expected. Each secondary trade bails a fund out of liquidity constraints and offers fund buyers more transaction opportunities, at no cost to any other types of agents. Thus, a more liquid secondary market attenuates the effects of fund liquidity shocks and improves overall welfare.

Proposition 3. *1. While liquidity shocks reduce funds' values* $(v_{fe} < v_{fo})$*, their influence is mitigated by more liquid secondary market* $(\frac{\partial(v_{fo} - v_{fe})}{\partial \lambda_s} < 0)$ *and vanishes when the secondary market becomes completely liquid* $(\lim_{\lambda_s \to \infty}(v_{fo} - v_{fe}) = 0)$.

[20] The calibrated search rates tend to be very large due to our normalization of $k_v = 1$ and motivates us to study a fast-search market. See the fast search market analysis in Section 6 for detailed discussions.

2. The welfare increases in the secondary market liquidity ($\frac{\partial W}{\partial \lambda_s} = \frac{\partial \mu_{fo}}{\partial \lambda_s}(\frac{u_f - u_e}{r}) > 0$).

A faster secondary market enables funds under selling pressure (fe) to exit and restart their life cycle as fn swiftly, while more funds in the investment phase (fn) transition to the harvesting phase (fo). Each unit measure of secondary transactions shifts population from μ_{fe} to μ_{fo} without altering μ_{fn}, resulting in increased welfare by the value of increased payoff flow with time discount $\frac{u_f - u_e}{r}$.

Quantitatively, the influence of secondary buyouts on the overall welfare in Proposition 3 is limited due to the relatively small number of funds compared to corporate investors ($k_f = 0.02$). For instance, 1% increase in secondary market liquidity from $\lambda_s = 699$ would increase the number of secondary buyouts by at most 3.6 per year, resulting in a mere 0.03% increase in the total welfare ($\frac{\partial W}{\partial \lambda_s}\frac{\lambda_s}{W} = 0.03\%$).[21]

Self-interested funds support each other through secondary trades, allowing funds under selling pressure to exit more quickly. Additionally, fund buyers benefit from increased trade opportunities when funds look to exit. A higher search rate in the secondary market strengthens this channel, enabling additional funds to enhance their expected value at the time of the new life cycle (v_{fn}).

Proposition 4. *There exists $\overline{k}_f$ such that if $k_f < \overline{k}_f$, then there is a complementarity between the secondary market liquidity and the number of funds* ($\frac{\partial^2 v_{fn}}{\partial \lambda_s \partial k_f} > 0$).

Furthermore, the mutual benefits resulting from secondary trades among funds can be significant enough to outweigh the value reduction caused by narrower buy-sell spreads due to increased competition. Specifically, when the secondary market search rate (λ_s) is high, the expected value of each fund at the time of a new life cycle can rise in correlation with the number of funds present.

Proposition 5. *There exists $\overline{k}_f$ and a function $\overline{\lambda}_s(k_f)$ such that, if there are not too many funds ($k_f < \overline{k}_f$) and the secondary market is liquid enough ($\lambda_s > \overline{\lambda}_s(k_f)$), then funds' value*

[21] In equilibrium, the number of secondary buyouts is given by $\eta_{fn-fe} = \lambda_s \mu_{fn} \mu_{fe}$. To estimate the impact of a 1% increase in the search rate, $\lambda_s = 699$, we multiply it by $\mu_{fn} = 0.0024$, $\mu_{fe} = 0.0021$, and the number of corporate investors (102,626) to account for normalization ($k_v = 1$). This calculation provides an upper bound on the increase in the number of secondary buyouts since an increase in the search rate λ_s reduces the number of funds under selling pressure μ_{fe} and the number of fund buyers μ_{fn}.

increases in their number $(\frac{\partial v_{fn}}{\partial k_f} > 0)$.[22]

The proof of Propositions 4 and 5 obtains a closed-form expression for the marginal value $(\frac{\partial v_{fn}}{\partial k_f})$ when $k_f \approx 0$ and demonstrates its increase in the secondary-market search rate λ_s.

To gain insights into the quantitative implications of Propositions 4 and 5, we analyze the expected value of a new fund (v_{fn}) in our model's calibration. In Figure 2, we present the value of a new fund (v_{fn}) across various numbers of funds (k_f) and different levels of secondary market search rate (λ_s). The other parameters are kept constant at their calibrated values. Notably, the vertical line at k_f^* and the curve for λ_s^* correspond to the observed number of funds and the calibrated search rate for secondary trades, respectively.

Intuitively, when the market consists of a small number of funds, the addition of a new fund has a positive impact on the expected value of each fund, confirming Propositions 4 and 5. However, as the number of funds continues to rise, competition among them becomes more intense, particularly in the pursuit of intermediation opportunities. Consequently, this heightened competition gradually offsets the benefits arising from complementarities.

Our calibration indicates that the PE fund value v_{fn} increases with the number of funds k_f. This relationship is evident in Figure 2, where v_{fn} demonstrates an increasing trend for the calibrated search rate λ_s^* with respect to k_f. An increase in the number of funds by 1% leads to an increase in fund valuation by 0.07% $(\frac{\partial v_{fn}}{\partial k_f}\frac{k_f}{v_{fn}} = 0.07)$. PE fund values with SBOs are 36.4% higher than those without SBOs. Despite acknowledging criticisms against SBOs, our results show that SBOs contribute, rather than detract, to PE funds generating high returns. Additionally, a 1% improvement in firms' operation u_f leads to a significant 6% increase in a new fund's value v_{fn}.[23] However, a similar improvement in u_e has a negligible influence on fund value, with a sensitivity of only 0.68 due to the presence of a vibrant SBO

[22]The matching technology plays only a limited role in Proposition 5. Note that in our model, the rate of meetings between funds of type fe and fn is $\lambda_s \mu_{fn} \mu_{fe}$, and so scaling up the total number of funds k_f by x while holding the type distribution fixed scales up the meeting rate by x^2 and the individual fund's meeting intensities by x. The qualitative result $(\frac{\partial v_{fn}}{\partial k_f} > 0)$ holds in general. Observe that the derivative $\frac{\partial v_{fn}}{\partial k_f}$ at $k_f \approx 0$ in Figure 2 gets indefinitely large as λ_s increases. For any matching technology with increasing returns to scale, if λ_s is sufficiently large and k_f is small, then $\frac{\partial v_{fn}}{\partial k_f} > 0$.

[23]The 20% improvement in operating profits $(u_f = 1.2)$ is sourced from existing literature. However, the 6% increase in the value of new funds (v_{fn}), arising from a 1% improvement in operating profits (u_f), is an equilibrium object that must be calculated from purchase and sale prices, as well as the probability of selling an asset before encountering a liquidity shock.

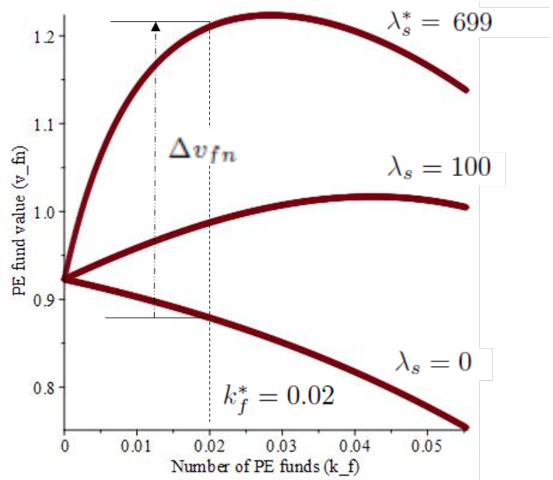

Figure 2: Each graph shows a new fund's expected value v_{fn} for various values of the number of funds k_f. The topmost curve is for the calibrated value of the secondary-market search rate λ_s while the bottom two curves are based on the counterfactual parameter values. All other parameters are fixed at calibrated values. The observed number of funds is $k_f^* = 0.02$ (normalized by the number of investors). The increase in fund valuation Δv_{fn} represents the contribution by secondary trades, i.e., an increase of λ_s from 0 to the calibrated value $\lambda_s^* = 699$.

market.

A further quantitative result is on funds exits by SBOs. When the number of funds increases, an increasing fraction of funds exit through secondary trading rather than sales of assets to investors (Figure 3). While it is obvious that the number of secondary trades increases with the growth in funds, our model explains a tandem increase in the *share* of exits through secondary trades. This pattern is indeed observed in the data. The share of firms sold by PE funds through SBOs has increased from 13% in the 1980s, 19% in 2009, to 42% in 2017.

6 Other Equilibrium Properties

Corporations, PE funds and M&A practitioners keenly focus on the current and future levels of activity in M&A markets. In this section, we develop properties for the equilibrium

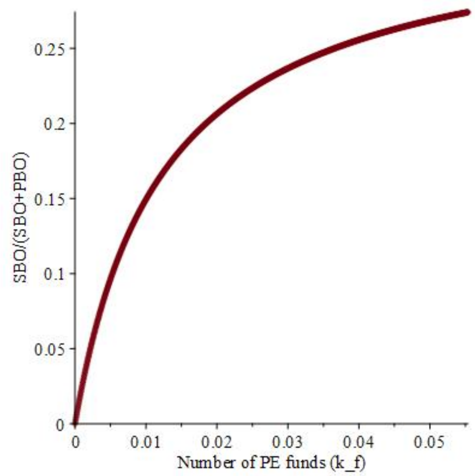

Figure 3: The share of fund exits by secondary trading $\left(\frac{\lambda_s \mu_{fe} \mu_{fn}}{\lambda_f \mu_{fe} \mu_{hn} + \lambda_s \mu_{fe} \mu_{fn}}\right)$.

prices, spreads, and trading speed and volume, and show how these key metrics respond to exogenous parameters, such as search frictions and transition rates.

We also analyze the welfare properties of the M&A market with intermediation and demonstrate that the market exhibits search externalities. Slowing down M&A activity among corporations can improve welfare by stimulating more PE activity. This allows exit-phase funds to off-load assets, reset the life cycle, and quickly purchase new assets.

6.1 Fast Search Markets

The equilibrium analysis is sometimes obtained more easily in *fast-search* markets – i.e., economies with large search rates $\lambda = (\lambda_d, \lambda_f, \lambda_s)$. We define and derive some key properties of fast search markets in this subsection.

We set up a formal fast-search market as follows. Given any exogenous parameters $\theta \equiv (k, r, u, \rho, \lambda)$, we increase meeting rates $(\lambda_d, \lambda_f, \lambda_s)$, while preserving the relative ratios. That is, we consider a sequence of markets $\theta^L \equiv (k, r, u, \rho, L\lambda)$, where $L\lambda = (L\lambda_d, L\lambda_f, L\lambda_s)$. We analyze the steady-state solution (μ^L, v^L) in the limit as L increases to infinity. To ease expositions, we assume a *regular* environment: $k_a \notin \{k_h, k_h + k_f\}$.

We show below that the fast-search market limits are well-defined as the steady-state population measures μ^L converge (the convergence of v^L, as a linear function of μ^L, follows immediately (see (16)). The speed of convergence is $O(1/L)$ which gives a precise sense of how closely a fast-search equilibrium would approximate an equilibrium of the calibrated market (in our calibration, $L \approx 103{,}000$):

Proposition 6. *(Convergence and Convergence Speed) Given a regular environment θ, for any $i \in \mathcal{T}$, the population limit $\mu_i^* \equiv \lim_{L\to\infty} \mu_i^L$ and the convergence speed $\mu_i^{**} \equiv \lim_{L\to\infty} L(\mu_i^L - \mu_i^*)$ exist.*

The results in fast-search markets approximate a market with many participants. The reason is that we normalized the total number of investors as $k_v = 1$ and proportionally re-scaled the number of funds and trade volumes. Let K_v be the total number of investors before normalization, with K_i for $i \in \mathcal{T}_v$ being the number of type-i investors. If each pair of investors meet at a Poisson rate l_d, the total number of direct trading (say, per year), with normalization, would be $(l_d K_{lo} K_{hn})/K_v = (l_d K_v)(K_{lo}/K_v)(K_{hn}/K_v) = (l_d K_v)\mu_{lo}\mu_{hn}$.

Markets for corporate acquisitions commonly have many buyers, sellers, and intermediaries, such as in our calibration. Therefore, the closed-form expressions we can obtain for fast search markets help us understand the quantitative equilibrium properties. In the remainder of this section, while some results are general, others are obtained only for fast search markets.

6.2 Welfare Analysis

The total welfare W can be decomposed into the sum of the welfare to the investors and funds in the economy: $W = W_v + W_f$, where $W_v \equiv \sum_{i\in\mathcal{T}_v} \mu_i v_i$ and $W_f \equiv \sum_{i\in\mathcal{T}_f} \mu_i v_i$. The welfare measures, which are defined based on the players values, are naturally related to the investors' and funds' payoff flows and gains from trades as follows:

Proposition 7. *(Equilibrium Welfare)*

$$
\begin{aligned}
rW &= \mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l, && (2)\\
rW_v &= \mu_{ho}u_h + \mu_{lo}u_l + \underbrace{\lambda_f \mu_{lo}\mu_{fn} p_{lo-fn}}_{\textit{sales to funds}} - \underbrace{\lambda_f \mu_{hn}(\mu_{fo}p_{fo-hn} + \mu_{fe}p_{fe-hn})}_{\textit{purchases from funds}}, \quad \textit{and} && (3)\\
rW_f &= \mu_{fo}u_f + \mu_{fe}u_e + \underbrace{\lambda_f \mu_{fo}\mu_{hn} p_{fo-hn} + \lambda_f \mu_{fe}\mu_{hn} p_{fe-hn}}_{\textit{sales to investors}} - \underbrace{\lambda_f \mu_{fn}\mu_{lo} p_{lo-fn}}_{\textit{purchases from investors}}.
\end{aligned}
$$

The first two terms for the investors' welfare W_v represent payoff flows to ho- and lo-type investors. The next term represents the inflow from selling assets to funds. Only lo-type investors sell assets to funds with the total rate $\lambda_f \mu_{lo}\mu_{fn}$ and at price p_{lo-fn}. The last term represents the hn-type investors' payments to funds: p_{fo-hn} to fo-type funds with the aggregate rate of $\lambda_f \mu_{hn}\mu_{fo}$, or p_{fe-hn} to fe-type funds with the aggregate rate of $\lambda_f \mu_{hn}\mu_{fe}$. A similar interpretation explains the funds' welfare W_f.

Inefficiency and Search Externality To study the inefficiencies created by search externalities, we turn to the fast-search equilibrium welfare $W^* \equiv \lim_{L\to\infty} W^L$. According to Proposition 7, $W^* = \mu^*_{ho}u_h + \mu^*_{fo}u_f + \mu^*_{fe}u_e + \mu^*_{lo}u_l$, where $\mu^*_i \equiv \lim_{L\to\infty} \mu^L_i$, and the welfare W^L converges to W^* at the same speed $O(1/L)$ as μ^L (Proposition 6).

We compare the fast-search equilibrium welfare W^* against two extreme situations: an autarkic economy with no functioning market or fund intermediation, and a centralized economy with a planner moving assets across agents without search friction. In the autarkic economy, a k_a fraction among $k_v(=1)$ corporations hold assets with no trades, resulting in the welfare $\underline{W}$ such that $r\underline{W} = k_a(k_h u_h + k_l u_l)$. In the centralized economy, a planner solves

$$
\begin{aligned}
r\overline{W} \equiv \max_{\mu \in \mathbb{R}^{\mathcal{T}}_+} \; & \mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l,\\
\text{subject to} \quad & \mu_{ho} + \mu_{hn} = k_h, \quad \mu_{lo} + \mu_{ln} = k_l, \quad \text{and} \quad (1).
\end{aligned}
$$

The maximum welfare $\overline{W}$ takes into account exogenous type changes ρ_u and ρ_d but ignores search frictions. The liquidity shock ρ_e imposes no restriction on the planner who can transfer assets between funds instantaneously. The planner can set aside an ϵ mass of funds as type

	A. $k_a < k_h$	B. $k_h < k_a < k_h + k_f$	C. $k_h + k_f < k_a$
$\overline{\mu}_{ho} =$	k_a	k_h	k_h
$\overline{\mu}_{fo} =$	0	$k_a - k_h$	k_f
$\overline{\mu}_{lo} =$	0	0	$k_a - k_f - k_h$
$\overline{\mu}_{fe} =$	0	0	0
$\mu^*_{ho} =$	k_a	k_h	k_h
$\mu^*_{fo} =$	0	$k_a - k_h$	$< k_f$
$\mu^*_{lo} =$	0	0	$k_a - k_f - k_h$
$\mu^*_{fe} =$	0	0	> 0

Table 3: The population under fast search (μ^*) and the efficient allocation ($\bar{\mu}$).

fn and transfer assets to them when some other funds receive a liquidity shock. The mass of fo and fe type funds remain the same. Consequently, the mass arbitrarily close to k_f of assets can be held by fo type funds.

The population measures $\overline{\mu}$ that achieves the maximum welfare is such that $\overline{\mu}_{ho} = \min\{k_a, k_h\}$, $\overline{\mu}_{fo} = \min\{(k_a - k_h)^+, k_f\}$, $\overline{\mu}_{lo} = (k_a - k_f - k_h)^+$, and $\overline{\mu}_i = 0$ for $i \neq ho, fo, lo$. In essence, assets are allocated to high-type investors up to their steady-state population k_h; any remaining assets are given to funds up to k_f; and the still-remaining assets are given to low-type investors. The maximum welfare satisfies $r\overline{W} = \overline{\mu}_{ho}u_h + \overline{\mu}_{fo}u_f + \overline{\mu}_{lo}u_l$.[24]

We compare the efficient allocation $\overline{\mu}$ (the upper part of Table 3) with the fast-search equilibrium population μ^* (the lower part of Table 3).

Proposition 8. *(Fast-search Market: Welfare) As $L \to \infty$, $W^* \equiv \lim_{L\to\infty} W^L$:*

A. *If $k_a < k_h$, then $W^* = \overline{W}$, which is independent of u_f, u_e, λ_s, and λ_d.*

B. *If $k_h < k_a < k_h + k_f$, then $W^* = \overline{W}$, which is strictly increasing in u_f and independent of u_e, λ_s, and λ_d.*

[24] The maximum welfare $\overline{W}$ is also achieved by a planner who is under the search friction, like agents, but can choose not to execute some transactions. The planner's problem is $rW_p(\lambda) \equiv \sup_{0\leq\lambda_p\leq\lambda} \mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l$, subject to μ being a solution of $P(k, r, u, \rho, \lambda_p)$. Fast search allows the planner to achieve the maximum welfare approximately: i.e., $\overline{W} = \lim_{\lambda\to\infty} W_p(\lambda)$. Intuition will become clear after Proposition 8. The planner slows down the investors' direct trading, eliminates search externalities, and increases welfare to the maximum.

C. *If $k_h + k_f < k_a$, then W^* is strictly less than $\overline{W}$, strictly increasing in u_f, u_e, and λ_s, and strictly decreasing in λ_d.*

The characterization of the fast-search equilibrium welfare depends on the number of assets (k_a) relative to the number of potential buyers (k_h, k_f). A sufficiently large number of assets ($k_a > k_h + k_f$) gives rise to an inefficient fast-search equilibrium. The calibrated ρ_u and ρ_d imply $k_h \equiv \frac{\rho_u}{\rho_u+\rho_d} = 0.40$ and an excess supply of assets $k_a > k_h + k_f$. The result, together with large meeting rates, suggests that the US corporate acquisition market is close to Case C of the fast-search market.[25]

Suppose that the fast-search market has sufficiently many potential buyers ($k_a < k_h + k_f$) as in Cases A and B (the first two columns in Table 3). Fast search allows investors and funds to quickly transfer assets from low-type investors (lo) and exiting funds (fe) to high-type investors (hn) and, in Case B, also to funds at the investment phase (fn). Accordingly, the steady-state population μ^* equals the efficient allocation $\overline{\mu}$ and achieves the maximum welfare ($W^* = \overline{W}$).

The comparative statics of the welfare becomes trivial: the maximum welfare is dependent on payoff flows (e.g., u_f) only if the corresponding type's population (resp., μ_{fo}) is non-zero. In either case, the impact of a liquidity shock is zero. Funds transfer assets without holding any inventory (Case A) just like, e.g., in Rubinstein and Wolinsky (1987); or, they hold assets, but funds under liquidity constraints transfer assets to others through speedy secondary trades (Case B), as is the case in Duffie, Gârleanu, and Pedersen (2005). Under a surplus of tradable assets relative to potential buyers ($k_h + k_f < k_a$), as in Case C (the third column in Table 3), the equilibrium is more interesting because, counter-intuitively, slowing down investors' direct trading improves the welfare ($\frac{\partial W^*}{\partial \lambda_d} < 0$). Since investors or funds on demand can quickly find sellers and purchase assets, there are negligible left-over high-type investors or fund non-owners. Hence, a significant fraction of exiting funds (fe) will find it difficult to offload their assets, and the welfare loss is $r(\overline{W} - W^*) = \mu^*_{fe}(u_f - u_e) > 0$.

In our calibration, the welfare gain by asset reallocations is 13.4%, relative to the autarkic situation welfare $\underline{W}$ (see page 23). This welfare gain ($W - \underline{W}$) attains 92.3% of the best

[25] The calibration result on the oversupply of assets is not sensitive to our choice of $k_a = 0.5$. While a choice of $k_a = 0.25$ or 0.75 results in different estimates of ρ_u and ρ_d, the oversupply remains about the same ($k_a - (k_h + k_f) \approx 0.08$).

possible gain ($\overline{W} - \underline{W}$). This fraction is lower than the gain in OTC markets for municipal bonds as described in Hugonnier, Lester, and Weill (2020), likely due to higher search frictions in the corporate acquisition market. The corporate investors' percentage share of this welfare gain is 74.7%, which leaves 25.3% to PE funds. The PE funds' welfare share is very large relative to their small number $k_f = 0.02$.

The inefficiency is a result of investors' search externalities on funds. A direct-trading by investors takes away selling opportunities from exit-phase funds and leads them to suffer from liquidity constraints for a long period of time. If the investors' direct trading were absent, an exit-phase fund could offload an asset, reset its type, and purchase another asset, all quickly under fast search. This alternative scenario results in a more efficient asset allocation.

6.3 Trading Speed, Volumes, and Prices

Average Time to Sell Statistics about the average time on the market before deal closing are widely available. We obtain below the closed-form expressions for the average time to sell for investors or funds. Those expressions are used to calibrate the model parameters.

Proposition 9. *(Time to Sell) Let τ_{sv} and τ_{sf} denote the time to sell for investors and funds. Then,*

$$E[\tau_{sv}] = \frac{1}{\lambda_d \mu_{hn} + \lambda_f \mu_{fn}}, \tag{4}$$

$$E[\tau_{sf}] = \frac{1}{\lambda_f \mu_{hn} + \rho_e} + \frac{\rho_e}{\lambda_f \mu_{hn} + \rho_e} \left(\frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}} \right). \tag{5}$$

For example, each seller-buyer meeting arrives according to a Poisson process, so the time until the first meeting by a selling investor follows an exponential distribution with parameter $\lambda_d \mu_{hn} + \lambda_f \mu_{fn}$. A similar, but more involved, calculation gives the expected time to sell for funds.

We observe from the data that a fund typically takes around 0.91 years of search to sell an asset. However, when facing selling pressure, this time is significantly reduced to approximately 0.4 years ($\frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}} = \frac{1}{(61.7 \times 0.0141) + (699 \times 0.0024)} \approx 0.4$).

Transaction Volumes The level of activity in M&A markets is an important metric followed by in M&A practitioners. We now characterize the transaction volumes and how they respond to exogenous parameters, such as search frictions and transition rates. We obtain those results focusing on fast search markets.

For each submarket $m \in \mathcal{M}$, with a seller's type s and the buyer's type b, the **trade volume** is $\eta_m^L \equiv (L\lambda_m)\mu_s^L \mu_b^L$.

Proposition 10. *(Fast-search Market: Trade Volumes) For each submarket $m \in \mathcal{M}$, trade volume in the limit $\eta_m^* \equiv \lim_{L\to\infty} \eta_m^L$ is given by:*

	A. $k_a < k_h$	*B.* $k_h < k_a < k_h + k_f$	*C.* $k_h + k_f < k_a$
$\eta^*_{lo-hn} =$	$\lambda_d \mu^{**}_{lo} \mu^*_{hn}$	0	$\lambda_d \mu^*_{lo} \mu^{**}_{hn}$
$\eta^*_{lo-fn} =$	$\lambda_f \mu^{**}_{lo} \mu^*_{fn}$	$\lambda_f \mu^{**}_{lo} \mu^*_{fn}$	$\lambda_f \mu^*_{lo} \mu^{**}_{fn}$
$\eta^*_{fo-hn} =$	$\lambda_f \mu^{**}_{fo} \mu^*_{hn}$	$\lambda_f \mu^*_{fo} \mu^{**}_{hn}$	$\lambda_f \mu^*_{fo} \mu^{**}_{hn}$
$\eta^*_{fe-hn} =$	0	0	$\lambda_f \mu^*_{fe} \mu^{**}_{hn}$
$\eta^*_{fe-fn} =$	0	$\lambda_s \mu^{**}_{fe} \mu^*_{fn}$	$\lambda_s \mu^*_{fe} \mu^{**}_{fn}$

where (i) μ^* denotes the population limit, and (ii) for type i with $\mu_i^* = 0$, $\mu_i^{**} \equiv \lim_{L\to\infty} L\mu_i^L$ denotes the convergence speed.

With a large number of high-type investors (Case A), secondary trades are unnecessary for funds; if the deficit of high-type investors is supplemented by funds (Case B), selling investors resort to fund buyers and there are no investors' direct trading; with an excess supply of tradable assets (Case C), there are transactions in all submarkets.

The trade volumes under fast search follow from the convergence and the convergence speed of population measures (Proposition 6). For each submarket $m \in \mathcal{M}$, because of fast search, the steady-state measure of either buyers or sellers vanishes: i.e., $\mu^*_{buyer} = 0$ or $\mu^*_{seller} = 0$. If $\mu^*_{seller} = 0$, then $\eta_m^L = (L\lambda_m)\mu^L_{buyer}\mu^L_{seller} = \lambda_m \mu^L_{buyer}(L\mu^L_{seller}) \to \lambda_m \mu^*_{buyer}\mu^{**}_{seller}$ as $L \to \infty$. Proposition 10 suggests that all submarkets are active under fast search only if the market has an excess supply of tradable assets ($k_a > k_h + k_f$).

The trade volumes also identify the main drivers of the convergences of certain population measures. For example, $\mu^*_{fo} = 0$ in Case A could be due to the fact that (i) funds can rarely purchase assets because of a vanishingly small number of selling investors (lo), or (ii) funds do acquire assets, but quickly re-sell to buying investors (hn). Proposition 10 implies the latter case; funds buy/sell a significant number of assets from/to investors in the fast-search market and there are no secondary transactions (like middlemen in Rubinstein and Wolinsky (1987)). Similarly, the vanishing number of selling investors (lo) and buying investors (hn) in Case B is the result of an efficient fund-investor trading rather than an efficient market for investors' direct trading – the number of investors' direct transactions (η^*_{lo-hn}) is indeed vanishingly small.

	λ_d	λ_f	λ_s	$(\rho_u, \rho_d)_{\text{(with a fixed ratio)}}$	ρ_e
η^*_{lo-hn}	$+$	$-$	0	$+$	0
η^*_{lo-fn}	$-$	$+$	0	$+$	0
η^*_{fo-hn}	$-$	$+$	$+$	$+$	$-$
η^*_{fe-hn}	$-$	$+$	$-$	$+$	$+$
η^*_{fe-fn}	$-$	$-$	$+$	$+$	$+$

Table 4: Comparative statics of trade volumes. Each $+$ (or $-$) indicates the corresponding volume to be non-decreasing (resp., non-increasing) in the parameter, and 0 indicates that the volume is independent of the parameter.

Table 4 summarizes a comparative static analysis for trade volumes relative to search frictions and transition rates.[26] Most results are intuitive. If investors' types are more volatile (i.e., larger ρ_u and ρ_d with a fixed ratio), assets will be transferred across agents frequently, ultimately from low-type investors to high-type investors, with possible fund intermediations. Fast search among investors (i.e., higher λ_d) allows them to transact directly (i.e., higher η^*_{lo-hn}), resulting in fewer intermediation opportunities for funds. The parameters for the secondary market (λ_s and ρ_e) only shift the population measures between fo and fe. Therefore, these parameters do not affect the volume of investors' direct trading (η^*_{lo-hn}) and only shift volumes between two kinds of fund-investor transactions: η^*_{fo-hn} and η^*_{fe-hn}.

The positive response of η^*_{fe-hn} to λ_f is perhaps surprising. On one hand, a fast search

[26]The results follow directly from the expressions of μ^* and μ^{**} in Table 3 and Lemmas SA.3 and SA.4 in Supplmental Appendix, so we omit. As an example, take η^*_{lo-hn} and λ_d. Note that $\eta^*_{lo-hn} = \lambda_d \mu^*_{lo} \mu^{**}_{hn}$. In Cases A and B, $\mu^*_{lo} = 0$, so η^*_{lo-hn} is independent of λ_s. In Case C, $\mu^{**}_{hn} = \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f(\mu^*_{fo} + \mu^*_{fe})} = \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f k_f}$. Note that μ^*_{lo} and μ^*_{ln} are independent of λ_d. Thus, the volume η^*_{lo-hn} is increasing in λ_d because $\frac{\partial}{\partial \lambda_d}\left(\frac{\lambda_d}{\lambda_d \mu^*_{lo} + \lambda_f k_f}\right) = \lambda_f k_f > 0$.

between funds and investors (i.e., higher $L\lambda_f$) orchestrates more transactions between exiting funds (fe) and buying investors (hn). On the other hand, as funds are able to sell assets before receiving liquidity shocks, fewer funds enter the exit phase, which could potentially reduce the trade volume between exiting funds and buying investors. It turns out that the former effect of λ_f dominates the latter.

Spreads and Prices The performance of PE funds depends on the spread between the prices at which funds buy and sell assets. A commonly used performance measure for PE funds is the Public Market Equivalent (PME) introduced by Kaplan and Schoar (2005) and Sorensen and Jagannathan (2015)). We derive the closed-form expression of PME and use it to calibrate the model parameters.

Sorensen and Jagannathan (2015)'s PME definition is for a model discrete time with a stochastic discount. Our model is in continuous time with a deterministic discount, therefore we define PME as

$$PME \equiv \frac{\text{Present value of distributions to fund investors}}{\text{Present value of capital calls made by fund investors}} = \frac{PV_{\text{dist}}}{PV_{\text{calls}}},$$

where

$$PV_{\text{dist}} \equiv E\left[e^{-r\tau_b}\int_0^{\tau_s} e^{-rt}u(t)dt + e^{-r\tau_s}P_s\right],$$
$$PV_{\text{calls}} \equiv PV_{\text{purchasing price}} + PV_{\text{management fees}} = E\left[P_b e^{-r\tau_b}\right] + E\left[(fP_b)\int_0^{\tau_b+\tau_s} e^{-rt}dt\right].$$

A fund that does not hold an asset takes τ_b period of time until purchasing an asset at a price of P_b and takes τ_s period of time (after purchasing) until selling the asset at a price P_s.

The management fees are paid retrospectively as if the flow of fees which equals a fraction of the fund size (i.e., fP_b) is paid throughout the fund's lifetime. For calibration, we set $f \approx 2\%$ based on Metrick and Yasuda (2010), which finds that management fees are usually 2% of committed capital and paid from the inception of a fund until its liquidation.

First, we obtain the closed-form expression of PV_{dist}. Since the time to purchase, τ_b , is independent of the time to sell τ_s (post-purchase) and the selling price P_s,

$$PV_{\text{dist}} = E\left[e^{-r\tau_b}\right] E\left[\int_0^{\tau_s} e^{-rt}u(t)dt + e^{-r\tau_s}P_s\right],$$

where $u(t) \in \{u_f, u_e\}$ denotes the payoff flow while holding the asset at $t \in [0, \tau_s]$.

A purchase of an asset occurs on meeting a corporate investor or a fund at the exit phase, whichever happens first ($\tau_b \equiv \min\{\tau_{lo-fn}, \tau_{fe-fn}\}$). τ_b follows an exponential distribution with parameter $\lambda_f\mu_{lo} + \lambda_s\mu_{fe}$. As such,

$$E\left[e^{-r\tau_b}\right] = \frac{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}.[27]$$

The fund can sell either (i) before receiving a liquidity shock to a corporate investor, or (ii) after receiving a liquidity shock to either a corporate investor or a fund buyer. The expected continuation payoff, upon receiving a liquidity shock before selling an asset, is

$$V_e \equiv E\left[u_e\left(\int_0^{\tau_e} e^{-rt}dt\right) + e^{-r\tau_e}P_e\right],$$

where τ_e denotes the time that the fund remains as type fe, and P_e denotes the selling price. Note that $\tau_e \equiv \min\{\tau_{fe-hn}, \tau_{fe-fn}\}$ follows an exponential distribution with parameter $\lambda_f\mu_{hn} + \lambda_s\mu_{fn}$. The probability of selling to a corporate investor $\frac{\lambda_f\mu_{hn}}{\lambda_f\mu_{hn}+\lambda_s\mu_{fn}}$ is independent of the selling time τ_e. Thus

$$\begin{aligned} V_e &= \frac{u_e}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r} + \frac{\lambda_f\mu_{hn} + \lambda_s\mu_{fn}}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}\frac{\lambda_f\mu_{hn}P_{fe-hn} + \lambda_s\mu_{fn}P_{fe-fn}}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn}} \\ &= \frac{u_e + \lambda_f\mu_{hn}P_{fe-hn} + \lambda_s\mu_{fn}P_{fe-fn}}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}. \end{aligned}$$

Similarly, an fo type fund receives a payoff flow u_f during a lifetime spanning $\tau_{fo} \equiv \min\{\tau_{fo-hn}, \tau_e\}$. Eventually, the fund either sells its asset to a buying investor at price

[27] We use (i) $\int_0^{\bar{t}} e^{-rt}dt = -\left.\frac{e^{-rt}}{r}\right|_0^{\bar{t}} = \frac{1-e^{-r\bar{t}}}{r}$, (ii) for $x \sim \exp(\alpha)$, $E[e^{-rx}] = \int_0^\infty e^{-rx}\alpha e^{-\alpha x}dx = \frac{\alpha}{\alpha+r}$, and (iii) for $x \sim \exp(\alpha)$, $\int_0^x e^{-rt}dt = E\left[\frac{1-e^{-rx}}{r}\right] = \frac{1}{\alpha+r}$.

P_{fo-hn} or receives a liquidity shock and a continuation payoff V_e. Thus,

$$E\left[\int_0^{\tau_s} e^{-rt}u(t)dt + e^{-r\tau_s}P_s\right] = \frac{u_f + \lambda_f\mu_{hn}P_{fo-hn} + \rho_e V_e}{\lambda_f\mu_{hn} + \rho_e + r}.$$

It follows that

$$PV_{\text{dist}} = \left(\frac{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}\right)\left(\frac{u_f + \lambda_f\mu_{hn}P_{fo-hn} + \rho_e\left(\frac{u_e + \lambda_f\mu_{hn}P_{fe-hn} + \lambda_s\mu_{fn}P_{fe-fn}}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}\right)}{\lambda_f\mu_{hn} + \rho_e + r}\right).$$

Following a similar analysis (see the Supplemental Appendix) we derive the closed-form expression of PV_{calls}, which yields the following closed-form expression for PME.

Proposition 11. *(PME) The PE fund performance is given by*

$$PME = \frac{PV_{dist}}{PV_{calls}}, \tag{6}$$

where

$$PV_{dist} = \left(\frac{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}\right)\left(\frac{u_f + \lambda_f\mu_{hn}P_{fo-hn} + \rho_e\left(\frac{u_e + \lambda_f\mu_{hn}P_{fe-hn} + \lambda_s\mu_{fn}P_{fe-fn}}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}\right)}{\lambda_f\mu_{hn} + \rho_e + r}\right), \text{ and}$$

$$PV_{calls} = \frac{\lambda_f\mu_{lo}P_{lo-fn} + \lambda_s\mu_{fe}P_{fe-fn}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}\left(1 + f\left(\frac{1}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}} + \frac{1 + \rho_e\left(\frac{1}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}\right)}{\lambda_f\mu_{hn} + \rho_e + r}\right)\right).$$

We conclude this section by establishing various relationships among transaction prices (see proof in the Supplemental Appendix).

Proposition 12. *(Equilibrium Prices)*

1. $p_{fo-hn} \geq p_{fe-hn} \geq p_{fe-fn}$: *funds sell at a lower price during the exit phase than in the harvesting phase, and at an even lower price in secondary trading.*

2. $p_{fo-hn} \geq p_{lo-hn} \geq p_{lo-fn}$: *funds buy assets at a lower price and sell at a higher price than investors.*

Funds that manage to sell assets before receiving liquidity shocks can generate positive profits, from payoff flows u_f and the positive spread $p_{fo-hn} - p_{lo-fn}$. However, if funds suffer liquidity shocks before selling assets, they may incur losses ex-post because the spread at the exit phase $p_{fe-hn} - p_{lo-fn}$ can be negative, as is the case in our calibration.

Our calibration predicts that doubling the number of funds (n_f) would yield only a 0.9% decrease in the direct transaction price (p_{lo-hn}). This highlights the importance of focusing on the equilibrium because more purchases by funds empower selling investors and more sales by funds increase buying investors' bargaining position, therefore the final pricing impact depends on complex equilibrium interactions illustrated by our calibration.

Finally, the sensitivity of prices and PME on flow payoffs are as follows:

$$\left[\frac{\partial x_i}{\partial \theta_j}\frac{\theta_j}{x_i}\right]_{i,j} = \left[\begin{array}{c|ccc} x_i \backslash \theta_j & u_h & u_f & u_e \\ \hline p_{lo-hn} & 0.202 & 0.010 & 0.029 \\ p_{lo-fn} & 0.180 & 0.017 & 0.038 \\ p_{fo-hn} & 0.201 & 0.017 & 0.039 \\ p_{fe-fn} & 0.180 & -0.014 & 0.081 \\ p_{fe-hn} & 0.202 & -0.021 & 0.072 \\ \text{PME} & -0.004 & 0.092 & 0.010 \end{array}\right].$$

A 1% increase in u_h would lead to transaction price increases for p_{lo-hn}, p_{lo-fn}, and p_{fo-hn} by 0.202%, 0.180%, and 0.201%, respectively. The small changes in transaction prices imply that the price spreads (i.e., the overall ratio of selling prices to purchasing prices) for funds remain largely unchanged. As a result, even after considering price changes, the PME (Equation 6) experiences an increase in the fund's flow payoffs.

7 Conclusion

We provide a search-based model of asset trading with fund intermediation. Funds in our model intermediate between buyers and sellers at risk of selling assets under pressure, possibly to other funds. Our paper offers a novel explanation of persistent intermediators' returns when the number of intermediaries increases. An increase in the number of buyout funds can lead to a reduction in the potential opportunities for each fund to trade with investors in the primary market. However, it also offers increased opportunities to buy and sell assets in the secondary market. As a result, in aggregate, the ex-ante expected value of each fund at the beginning of its life cycle increases ($\frac{\partial v_{fn}}{\partial k_f} > 0$). This finding suggests that the benefits gained from participating in the secondary market outweigh the costs arising from the potentially more intense competition in the primary market. A well-lubricated private market for corporate acquisitions can partly explain the recent shift in firm ownership from public to private, predicated by Jensen (1991). Lower liquidity costs in a market for buying and selling private firms help make PE ownership a form of governance that may indeed eclipse public companies.

Appendix

A Mathematical Symbols

	Definition	Notation
	Number of investors	k_v
	Number of funds	k_f
	Number of assets	k_a
	Flow Payoff for low type investors	u_l
	Flow payoff for high type investors	u_h
	Flow payoff for funds in the harvesting phase	u_f
	Flow payoff for funds in the exiting phase	u_e
(Model parameters)	Rate of low valuation shock	ρ_d
	Rate of high valuation shock	ρ_u
	Rate of liquidity shock	ρ_e
	Match intensity (Direct trading)	λ_d
	Match intensity (Primary buyout)	λ_f
	Match intensity (Secondary buyout)	λ_s
	Discount rate	r
	(for each type $i \in \mathcal{T} \equiv \{hn, ln, ho, lo, fn, fo, fe\}$)	
	Population of type i agents	μ_i
	Value of type i agents	v_i
	Gains from trade between $i, j \in \mathcal{T}$	g_{i-j}
(Equilibrium variables)	Price of assets when $i, j \in \mathcal{T}$ trade	p_{i-j}
	Funds' Welfare	W_f
	Investors' welfare	W_v
	(Total) Welfare	W

Table 5: Mathematical Symbols

B Proofs

B.1 Proof for Part 1 of Proposition 1

We reduce the number of variables and population equations in $P(\theta)$ by imposing some necessary conditions for a steady-state solution. Note that any steady-state solution μ must satisfy $\mu_{ho} + \mu_{hn} = k_h \equiv \frac{\rho_u}{\rho_u+\rho_d}$ and $\mu_{lo} + \mu_{ln} = k_l \equiv \frac{\rho_d}{\rho_u+\rho_d}$ (which we can obtain by adding (μ-ho) and (μ-hn), or (μ-lo) and (μ-ln), and apply $k_v = 1$). If we substitute $\mu_{ho} = k_h - \mu_{hn}$ and $\mu_{ln} = n_l - \mu_{lo}$ into (μ-ho)-(μ-fe), then we are left with the following three linearly independent equations:[28]

$$(\lambda_d \mu_{hn} + \lambda_f \mu_{fn})\mu_{lo} + \rho_u \mu_{lo} - \rho_d \mu_{ho} = 0, \qquad \text{(from } (\mu\text{-lo}))$$

$$(\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe})\mu_{hn} + \rho_d \mu_{hn} - \rho_u \mu_{ln} = 0, \qquad \text{(from } (\mu\text{-hn}))$$

$$-(\lambda_f \mu_{hn} + \lambda_s \mu_{fn})\mu_{fe} + \rho_e \mu_{fo} = 0. \qquad \text{(from } (\mu\text{-fe}))$$

We re-write the first two equations with respect to μ_{lo} and μ_{hn}.

$$\mu_{fo} + \mu_{fe} = k_a - \mu_{ho} - \mu_{lo} = k_a - (k_h - \mu_{hn}) - \mu_{lo} \quad \text{and} \tag{7}$$

$$\mu_{fn} = k_f - (\mu_{fo} + \mu_{fe}) = k_f - k_a + k_h - \mu_{hn} + \mu_{lo}, \tag{8}$$

Then,

$$(\lambda_d \mu_{hn} + \lambda_f (k_f - k_a + k_h - \mu_{hn} + \mu_{lo}))\mu_{lo} + \rho_u \mu_{lo} - \rho_d (k_h - \mu_{hn}) = 0, \tag{9}$$

$$(\lambda_d \mu_{lo} + \lambda_f (k_a - k_h + \mu_{hn} - \mu_{lo}))\mu_{hn} + \rho_d \mu_{hn} - \rho_u (k_l - \mu_{lo}) = 0. \tag{10}$$

We show below that there exists a unique solution (μ_{lo}, μ_{hn}) of (9)-(10) such that (i) $0 \le \mu_{lo} \le k_l$, (ii) $0 \le \mu_{hn} \le k_h$, and (iii) $k_a - k_f - k_h \le \mu_{lo} - \mu_{hn} \le k_a - k_h$ (for $0 \le \mu_{fn} \le k_f$).

[28] Any other equation in $P(\theta)$ is redundant, as it depends linearly on (μ-lo), (μ-hn), and (μ-fe). Each sum of the right-hand sides of (μ-ho) and (μ-hn), or (μ-lo) and (μ-ln) equals zero, which allows us to delete (μ-ho) and (μ-ln) without changing the solution set. The sum of the right-hand sides of (μ-fn), (μ-fo), and (μ-fe) equals zero, so we can delete (μ-fn). Last, the sum of the right-hand sides of (μ-ho), (μ-lo), (μ-fo), and (μ-fe) equals zero, so we can delete (μ-fo).

Other population measures will be determined by $\mu_{ho} = k_h - \mu_{hn}$, $\mu_{ln} = k_l - \mu_{lo}$, and

$$\mu_{fn} = k_f - k_a + k_h - \mu_{hn} + \mu_{lo}. \tag{11}$$

We find the last two populations (μ_{fe}, μ_{fo}) by solving

$$\begin{aligned}
\mu_{fo} &= -\mu_{fe} + (k_f - \mu_{fn}), \quad (\text{from } \mu_{fn} + \mu_{fo} + \mu_{fe} = k_f) \\
\mu_{fo} &= \frac{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}}{\rho_e} \mu_{fe}. \quad (\text{from } (\mu\text{-fe}))
\end{aligned}$$

The unique solution is

$$\mu_{fe} = \frac{\rho_e (k_f - \mu_{fn})}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + \rho_e}, \quad \text{and} \quad \mu_{fo} = \frac{(\lambda_f \mu_{hn} + \lambda_s \mu_{fn})(n_f - \mu_{fn})}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + \rho_e}. \tag{12}$$

Therefore, it remains to prove the following claim:

Claim 1. *Let* $X(\theta) \equiv \{(x_1, x_2) \in \mathbb{R}^2 : \ 0 \le x_1 \le k_l, \, 0 \le x_2 \le k_h, \, 0 \le g_{fn}(x) \le k_f\}$, *where* $g_{fn}(x) \equiv k_a - k_h + x_2 - x_1$*. Also, define* $F \equiv (F_{lo}, F_{hn}) : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$\begin{aligned}
F_{lo}(x) &\equiv (\lambda_d x_2 + \lambda_f (k_f - g_{fn}(x)))x_1 + \rho_u x_1 - \rho_d (k_h - x_2), \\
F_{hn}(x) &\equiv (\lambda_d x_1 + \lambda_f g_{fn}(x))x_2 + \rho_d x_2 - \rho_u (k_l - x_1).
\end{aligned}$$

Then, there exists a unique solution of $F(x) = 0$ *in* $X(\theta)$*.*

We apply the Poincare-Hopf index theorem, a version in Simsek, Ozdaglar, and Acemoglu (2007, p.194); see also Hirsch (2012). First, $X(\theta)$ is non-empty, compact, and convex.[29] The boundary of $X(\theta)$ is

$$\partial X(\theta) \equiv \{(x_1, x_2) \in X(\theta) : x_1 = 0, x_1 = k_l, x_2 = 0, x_2 = k_h, g_{fn}(x) = 0, \text{ or } g_{fn}(x) = k_f\}.$$

Second, the function $F(x)$ is continuously differentiable at every $x \in \mathbb{R}^2$. Third, the determinant of the Jacobian matrix of F is strictly positive for every interior point of $X(\theta)$: for

[29] The Poincare-Hopf index theorem also requires $X(\theta)$ to be a 2-dimensional smooth manifold, which a reader can easily verify by applying the identify function to the definition of a smooth manifold in Simsek, Ozdaglar, and Acemoglu (2007, p.193).

each $x \in \mathbb{R}^2$,

$$\nabla F(x) \equiv \begin{bmatrix} \frac{\partial F_{lo}}{\partial x_1} & \frac{\partial F_{lo}}{\partial x_2} \\ \frac{\partial F_{hn}}{\partial x_1} & \frac{\partial F_{hn}}{\partial x_2} \end{bmatrix}$$
$$= \begin{bmatrix} (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x))) + \lambda_f x_1 + \rho_u & (\lambda_d - \lambda_f)x_1 + \rho_d \\ (\lambda_d - \lambda_f)x_2 + \rho_u & (\lambda_d x_1 + \lambda_f g_{fn}(x)) + \lambda_f x_2 + \rho_d \end{bmatrix},$$

and for any interior point $x \in X(\theta)\backslash\partial X(\theta)$,

$$\begin{aligned} det(\nabla F(x)) \geq &(\lambda_d x_2 + \lambda_f x_1)(\lambda_d x_1 + \lambda_f x_2) + \rho_u \lambda_d x_1 + \rho_d \lambda_d x_2 \\ &- (\lambda_d - \lambda_f)^2 x_1 x_2 - (\lambda_d - \lambda_f)(\rho_d x_2 + \rho_u x_1) \\ =&\lambda_d \lambda_f (x_1^2 + x_2^2) + 2\lambda_d \lambda_f x_1 x_2 + \lambda_f(\rho_d x_2 + \rho_u x_1) > 0. \end{aligned} \tag{13}$$

Last, we show that for every boundary point $x \in \partial X(\theta)$, the vector $F(x) \in \mathbb{R}^2$ points strictly outward of $X(\theta)$. We partition the boundary $\partial X(\theta)$ into six *faces* (i.e., flat surfaces) of $X(\theta)$. For each face, we find an outward normal vector $\mathbf{n} \in \mathbb{R}^2$ and show that the angle between $\mathbf{n}$ and $F(x)$ is acute (i.e., ≤ 90) at any point x in the face:

1. ($x_1 = 0$ and $0 \leq x_2 < k_h$) $\mathbf{n} = (-1, 0)$ is an outward normal vector, and $\mathbf{n} \cdot F(x) = \rho_d(k_h - x_2) > 0$.

2. ($x_2 = 0$ and $0 \leq x_1 < k_l$) $\mathbf{n} = (0, -1)$ is an outward normal vector, and $\mathbf{n} \cdot F(x) = \rho_u(k_l - x_1) > 0$.

3. ($x_1 = k_l$ and $0 \leq x_2 \leq k_h$) $\mathbf{n} = (1, 0)$ is an outward normal vector, and

$$\begin{aligned} \mathbf{n} \cdot F(x) &= (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))k_l + \rho_u k_l - \rho_d k_h + \rho_d x_2 \\ &\geq (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))k_l \quad (\text{as } \rho_u k_l = \rho_d k_h) \\ &\geq \min\{\lambda_d x_2 k_l, \lambda_f(k_f - g_{fn}(x))k_l\}. \end{aligned}$$

As either $x_2 > 0$ or $x_2 = 0$, we have $k_f - g_{fn}(x) = k_v + k_f - k_a > 0$, and $\mathbf{n} \cdot F(x) > 0$.

4. ($x_2 = k_h$ and $0 \leq x_1 \leq k_l$) $\mathbf{n} = (0, 1)$ is an outward normal vector, and

$$\mathbf{n} \cdot F(x) = (\lambda_d x_1 + \lambda_f g_{fn}(x))k_h + \rho_d k_h - \rho_u k_l + \rho_u x_1 \geq \min\{\lambda_d x_1 k_h, \lambda_f g_{fn}(x) k_h\}.$$

   As either $x_1 > 0$ or $x_1 = 0$, we have $g_{fn}(x) = k_a > 0$, and $\mathbf{n} \cdot F(x) > 0$.

5. ($g_{fn}(x) = 0$ and $x_1 > 0$) $\mathbf{n} = (1, -1)$ is an outward normal vector, and $\mathbf{n} \cdot F(x) = F_{lo}(x) - F_{hn}(x) = \lambda_f k_f x_1 > 0$.

6. ($g_{fn}(x) = k_f$ and $x_2 > 0$) $\mathbf{n} = (-1, 1)$ is an outward normal vector, and $\mathbf{n} \cdot F(x) = F_{hn}(x) - F_{lo}(x) = \lambda_f k_f x_2 > 0$.

We are ready to apply the Poincare-Hopf index theorem in Simsek, Ozdaglar, and Acemoglu (2007, p.194). The Euler characteristic of $X(\theta)$ is 1; see their definition on p.193 for the case of non-empty and convex sets. Claim 1 follows immediately from the index theorem, which completes the proof of existence and uniqueness of the solution of $P(\theta)$.

It remains to prove:

Lemma 1. *If μ is a steady-state solution of $P(\theta)$, then $\mu_i > 0$ for all $i \in \mathcal{T}$.*

The intuition of the lemma is simple. Strictly positive rates $\lambda = (\lambda_d, \lambda_f, \lambda_s)$ and $\rho = (\rho_u, \rho_d, \rho_e)$ allow the mass of investors and funds to flow across all types. Given $0 < k_a < k_v + k_f$, some fraction of agents are owners and others are non-owners. Investors who own assets may have their types changing between high and low exogenously and non-owners have similar probabilistic type changes. As transaction rates $\lambda = (\lambda_d, \lambda_f, \lambda_s)$ are all strictly positive, some investors can buy or offload assets after their types change. However, not all investors can do so within any fixed time period. A similar idea holds for funds.

Proof. First, we show that $\mu_{ho} > 0$ and $\mu_{lo} > 0$. It is clear that $\mu_{ho} = 0$ if and only if $\mu_{lo} = 0$. If $\mu_{ho} = 0$, then $\mu_{lo} = 0$, as only *ho*-type investors flow in type *lo*; conversely, if $\mu_{lo} = 0$, then $\mu_{ho} = 0$ as the inflow to the type *lo* must be zero. Suppose, toward contradiction, that $\mu_{ho} = \mu_{lo} = 0$. That is, all investors are non-owners. The in-flow from *hn*-type investors to type *ho* must be zero, so it must be that $\mu_{fo} = \mu_{fe} = 0$. Then, $\mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = 0$, a contradiction to $k_a > 0$.

Second, we show that $\mu_{ln} > 0$ and $\mu_{hn} > 0$. As before, it is clear that $\mu_{ln} = 0$ if and only if $\mu_{hn} = 0$. If $\mu_{ln} = 0$, then $\mu_{hn} = 0$ as only ln-type investors can flow in type hn; conversely, if $\mu_{hn} = 0$, then $\mu_{ln} = 0$ as the inflow to the type hn must be zero. Suppose, toward contradiction, that $\mu_{ln} = \mu_{hn} = 0$. That is, all investors are owners, which implies that some funds are non-owners: $\mu_{fn} = k_v + k_f - k_a > 0$. Since $\lambda_f \mu_{lo} \mu_{fn} > 0$, some lo-type investors change their types and flow into type ln by trading with PE funds, a contradiction to $\mu_{ln} = 0$.

Lastly, we consider funds. Suppose that $\mu_{fn} = 0$. As the inflow to type fo becomes zero, it must be that $\mu_{fo} = 0$, which in turn leads to no inflow by liquidity shocks to type fe: i.e., $\mu_{fe} = 0$. Such a case contradicts $k_f > 0$. When $\mu_{fn} > 0$, given strictly positive population μ_{lo}, the inflow of type-fn funds to type fo is strictly positive: $\lambda_f \mu_{lo} \mu_{fn} > 0$. As such, $\mu_{fo} > 0$, which in turn creates a strictly positive inflow by liquidity shocks to type fe: $\mu_{fe} > 0$. $\square$

B.2 Proof for Part 2 of Proposition 1

We first reduce the system $P(\theta)$. For any initial condition $\mu(0)$, a dynamic solution $\mu : [0, \infty) \to \mathbb{R}^{\mathcal{T}}$ of the system $P(\theta)$ satisfies, for every $t \in [0, \infty)$,

$$\begin{aligned} \mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) &= k_v (= 1), \\ \mu_{fn}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= k_f, \quad \text{and} \\ \mu_{ho}(t) + \mu_{lo}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= k_a. \end{aligned}$$

Without changing the set of dynamic solutions, we can reduce the system $P(\theta)$ for $x(t) \equiv (\mu_{ho}(t), \mu_{hn}(t), \mu_{lo}(t), \mu_{fo}(t))$ by[30]

$$\dot{x} = F(x) \equiv (F_{ho}(x), F_{hn}(x), F_{lo}(x), F_{fo}(x)), \tag{14}$$

[30] In the proof of Part 1 of Proposition 1, we reduced $F(x; \theta) = 0$ further as a system of only two equations in Claim 1. The reduction requires $\mu_{hn} + \mu_{ho} = k_h \equiv \frac{\rho_u}{\rho_u + \rho_d}$ and $\mu_{lo} + \mu_{ln} = k_l \equiv \frac{\rho_d}{\rho_u + \rho_d}$, which hold in a steady-state but may not hold on a path of $\mu(t)$ after a perturbation.

where

$$\begin{aligned}
F_{ho}(x) &\equiv (\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe}(x))\mu_{hn} - \rho_d\mu_{ho} + \rho_u\mu_{lo}, \\
F_{hn}(x) &\equiv -(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe}(x))\mu_{hn} - \rho_d\mu_{hn} + \rho_u\mu_{ln}(x), \\
F_{lo}(x) &\equiv -(\lambda_d\mu_{hn} + \lambda_f\mu_{fn}(x))\mu_{lo} - \rho_u\mu_{lo} + \rho_d\mu_{ho}, \\
F_{fo}(x) &\equiv (\lambda_f\mu_{lo} + \lambda_s\mu_{fe}(x))\,\mu_{fn}(x) - \lambda_f\mu_{hn}\mu_{fo} - \rho_e\mu_{fo}, \quad \text{and} \\
\mu_{ln}(x) &\equiv 1 - \mu_{ho} - \mu_{hn} - \mu_{lo}, \\
\mu_{fe}(x) &\equiv k_a - \mu_{ho} - \mu_{lo} - \mu_{fo}, \\
\mu_{fn}(x) &\equiv k_f - \mu_{fo} - \mu_{fe}(x) = k_f - k_a + \mu_{ho} + \mu_{lo}. \qquad (15)
\end{aligned}$$

The reduction of the system $P(\theta)$ does not change the set of dynamic solutions. If μ is a dynamic (either steady-state or not) solution of $P(\theta)$, then $x \equiv (\mu_{ho}, \mu_{hn}, \mu_{lo}, \mu_{fo})$ solves $F(x;\theta) = 0$; conversely, for any dynamic solution x of $F(x;\theta) = 0$, we can find a dynamic solution μ of $P(\theta)$, from x and the induced μ_{ln}, μ_{fe}, and μ_{fn}. Hence, a dynamic solution μ of $P(\theta)$ is asymptotically stable if and only if $x \equiv (\mu_{ho}, \mu_{hn}, \mu_{lo}, \mu_{fo})$ is asymptotically stable.

A steady-state solution x of $F(x;\theta) = 0$ is asymptotically stable if all eigenvalues of the Jacobian matrix of $F(x;\theta)$ at the steady-state solution x have strictly negative real parts (Hirsch, 2012). The Jacobian matrix is

$$\nabla F(x) \equiv \left[\frac{\partial F_i(x)}{\partial x_j}\right]_{i,j\in\{ho,hn,lo,fo\}}$$
$$= \left[\begin{array}{ccc|c}
-\lambda_f\mu_{hn} - \rho_d & \lambda_v\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe} & (\lambda_d - \lambda_f)\mu_{hn} + \rho_u & 0 \\
\lambda_f\mu_{hn} - \rho_u & -\lambda_v\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe}) - \rho_d - \rho_u & (\lambda_f - \lambda_d)\mu_{hn} - \rho_u & 0 \\
-\lambda_f\mu_{lo} + \rho_d & -\lambda_d\mu_{lo} & -\lambda_f(\mu_{fn} + \mu_{lo}) - \lambda_d\mu_{hn} - \rho_u & 0 \\
\hline
\lambda_f\mu_{lo} + \lambda_s(\mu_{fe} - \mu_{fn}) & -\lambda_f\mu_{fo} & \lambda_f(\mu_{fn} + \mu_{lo}) + \lambda_s(\mu_{fe} - \mu_{fn}) & -\lambda_f\mu_{hn} - \lambda_s\mu_{fn} - \rho_e
\end{array}\right]$$

where we omit the dependency of μ_{fn} and μ_{fe} on x to simplify the expression.

Due to the block structure, one eigenvalue is $-\lambda_f\mu_{hn} - \lambda_s\mu_{fn} - \rho_e < 0$. The other eigenvalues are the eigenvalues of the sub-matrix with the first three rows and columns. A direct calculation shows that the other three eigenvalues are also strictly negative, which completes the proof.

B.3 Proof of Proposition 2

First, we simplify expositions:

$$
\begin{aligned}
g_1 &\equiv g_{fo-hn} = (1/2)(v_{ho} + v_{fn} - v_{fo} - v_{hn}), \\
g_2 &\equiv g_{lo-fn} = (1/2)(v_{fo} + v_{ln} - v_{lo} - v_{fn}), \\
g_3 &\equiv g_{fe-fn} = (1/2)(v_{fo} + v_{fn} - v_{fe} - v_{fn}) = (1/2)(v_{fo} - v_{fe}),
\end{aligned}
$$

so that

$$
\begin{aligned}
g_{lo-hn} &= (1/2)(v_{ho} + v_{ln} - v_{lo} - v_{hn}) = g_2 + g_1 \quad \text{and} \\
g_{fe-hn} &= (1/2)(v_{ho} + v_{fn} - v_{fe} - v_{hn}) = g_1 + g_3.
\end{aligned}
$$

The matrix representations of the value equations (v-hn)-(v-fe) are:

$$
\begin{aligned}
\begin{bmatrix} v_{ho} \\ v_{lo} \end{bmatrix} &= \begin{bmatrix} r+\rho_d & -\rho_d \\ -\rho_u & r+\rho_u \end{bmatrix}^{-1} \begin{bmatrix} u_h \\ u_l + \lambda_d \mu_{hn}(g_1+g_2) + \lambda_f \mu_{fn} g_2 \end{bmatrix}, \\
\begin{bmatrix} v_{hn} \\ v_{ln} \end{bmatrix} &= \begin{bmatrix} r+\rho_d & -\rho_d \\ -\rho_u & r+\rho_u \end{bmatrix}^{-1} \begin{bmatrix} \lambda_d \mu_{lo}(g_1+g_2) + \lambda_f \mu_{fo} g_1 + \lambda_f \mu_{fe}(g_1+g_3) \\ 0 \end{bmatrix}, \quad \text{and} \\
\begin{bmatrix} v_{fn} \\ v_{fo} \\ v_{fe} \end{bmatrix} &= \frac{1}{r} \begin{bmatrix} \lambda_f \mu_{lo} g_2 + \lambda_s \mu_{fe} g_3 \\ u_f + \lambda_f \mu_{hn} g_1 - 2\rho_e g_3 \\ u_e + \lambda_f \mu_{hn}(g_1+g_3) + \lambda_s \mu_{fn} g_3 \end{bmatrix},
\end{aligned}
\tag{16}
$$

where the inverse matrix is well-defined: i.e., $(r+\rho_d)(r+\rho_u) - \rho_d\rho_u > 0$. As in the case of $k_f = 0$, we compute the gains g_1, g_2, and g_3. Then, the solution v will be uniquely determined by the above matrix equations.

First, $2rg_3 = r(v_{fo} - v_{fe}) = (u_f - u_e) - 2\rho_e g_3 - \lambda_f \mu_{hn} g_3 - \lambda_s \mu_{fn} g_3$, which implies

$$
g_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}} > 0. \tag{17}
$$

Next,

$$
\begin{aligned}
2(g_1+g_2) &= v_{ho}+v_{ln}-v_{lo}-v_{hn} = (1,-1)\cdot(v_{ho}-v_{hn}, v_{lo}-v_{ln}) \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} r+\rho_d & -\rho_d \\ -\rho_u & r+\rho_u \end{bmatrix}^{-1} \begin{bmatrix} u_h - \lambda_d\mu_{lo}(g_1+g_2) - \lambda_f\mu_{fo}g_1 - \lambda_f\mu_{fe}(g_1+g_3) \\ u_l + \lambda_d\mu_{hn}(g_1+g_2) + \lambda_f\mu_{fn}g_2 \end{bmatrix}.
\end{aligned}
$$

Since

$$
\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} r+\rho_d & -\rho_d \\ -\rho_u & r+\rho_u \end{bmatrix}^{-1} = \frac{1}{r+\rho_u+\rho_d} \begin{bmatrix} 1 & -1 \end{bmatrix},
$$

we have

$$
\begin{aligned}
(2(r+\rho_u+\rho_d)+\lambda_d(\mu_{lo}+\mu_{hn}))\,(g_1+g_2) + \lambda_f(\mu_{fo}+\mu_{fe})g_1 + \lambda_f\mu_{fn}g_2 \\
= (u_h-u_l) - \lambda_f\mu_{fe}g_3. \qquad (18)
\end{aligned}
$$

On the other hand, by (v-lo), (v-ln), (v-fn), and (v-fo),

$$
\begin{aligned}
2rg_2 =& r(v_{fo}-v_{fn}) - r(v_{lo}-v_{ln}) \\
=& (u_f + \lambda_f\mu_{hn}g_1 - 2\rho_e g_3 - \lambda_f\mu_{lo}g_2 - \lambda_s\mu_{fe}g_3) \\
& - (u_l + \lambda_d\mu_{hn}(g_1+g_2) + \lambda_f\mu_{fn}g_2) + \rho_u(v_{ho}-v_{lo}+v_{ln}-v_{hn}).
\end{aligned}
$$

As $v_{ho}-v_{lo}+v_{ln}-v_{hn} = 2(g_1+g_2)$,

$$
\begin{aligned}
(2\rho_u+\lambda_d\mu_{hn})(g_1+g_2) - \lambda_f\mu_{hn}g_1 + (2r+\lambda_f\mu_{lo}+\lambda_f\mu_{fn})g_2 \\
= (u_f-u_l) - (2\rho_e+\lambda_s\mu_{fe})g_3. \qquad (19)
\end{aligned}
$$

The linear system of equations (18) and (19) is summarized as follows:

$$
\begin{bmatrix} c_1+\lambda_f(\mu_{fo}+\mu_{fe}) & c_1+\lambda_f\mu_{fn} \\ c_2-\lambda_f\mu_{hn} & c_2+2r+\lambda_f(\mu_{lo}+\mu_{fn}) \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} u_h-u_l-\lambda_f\mu_{fe}g_3 \\ u_f-u_l-2\rho_e g_3-\lambda_s\mu_{fe}g_3 \end{bmatrix} \qquad (20)
$$

where $c_1 \equiv 2(r+\rho_u+\rho_d)+\lambda_d(\mu_{lo}+\mu_{hn}) > 0$ and $c_2 \equiv 2\rho_u+\lambda_d\mu_{hn} > 0$.

The determinant of the coefficient matrix is bounded below by

$$2rc_1 + \lambda_f \mu_{fn}(c_1 - c_2) > 4r^2 + \lambda_f \mu_{fn}(2r + 2\rho_d + \lambda_d \mu_{lo}) > 0.$$

Thus, the above linear system has a unique solution (g_1, g_2). This solution, together with g_3, determined the unique solution v of $V(\theta)$. ■

B.4 Conditions for positive trade gains

The gains from trade are all positive if and only if

$$\begin{aligned} g_1 \equiv g_{fo-hn} \geq 0 \iff & (c_2 + 2r + \lambda_f(\mu_{lo} + \mu_{fn}))((u_h - u_l) - \lambda_f \mu_{fe} g_3) \\ & - (c_1 + \lambda_f \mu_{fn})((u_f - u_l) - (2\rho_e + \lambda_s \mu_{fe}) g_3) \geq 0. & (21) \\ g_2 \equiv g_{lo-fn} \geq 0 \iff & - (c_2 - \lambda_f \mu_{hn})((u_h - u_l) - \lambda_f \mu_{fe} g_3) \\ & + (c_1 + \lambda_f(\mu_{fo} + \mu_{fe}))((u_f - u_l) - (2\rho_e + \lambda_s \mu_{fe}) g_3) \geq 0, & (22) \end{aligned}$$

Note that both expressions depend on the steady-state population measure μ.

B.5 Proof of Proposition 3

B.5.1 Part 1

Let $(\mu(\theta), v(\theta))$ be the unique steady-state solution of population and value for each market θ. We compute the comparative static derivatives with respect to λ_s. It is intuitive that the unique steady-state measure of each investor type $(\mu_i)_{i \in \mathcal{T}_v}$ and the measure μ_{fn} are independent of λ_s. Through a secondary trade, one fund changes its type from fe to fn, replacing another fund of type changed from fn to fo.

To confirm the intuition, from the proof of Proposition 1, take the unique steady-state solution $x(\theta) \equiv (\mu_{lo}(\theta), \mu_{hn}(\theta))$ of $F(x) \equiv (F_{lo}(x), F_{hn}(x)) = 0$, where

$$\begin{aligned} F_{lo}(x) &\equiv (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))x_1 + \rho_u x_1 - \rho_d(k_h - x_2), \\ F_{hn}(x) &\equiv (\lambda_d x_1 + \lambda_f g_{fn}(x))x_2 + \rho_d x_2 - \rho_u(k_l - x_1). \end{aligned}$$

By Implicit function theorem, $x(\theta)$ is differentiable in λ_s, and

$$\frac{\partial x(\theta)}{\partial \lambda_s} = -\left[\nabla_x F(x(\theta);\theta)\right]^{-1} \frac{\partial F(x(\theta);\theta)}{\partial \lambda_s}.$$

We denoted the domain of $F(x)$ by $X(\theta)$ in the proof of Claim 1. The unique solution $x(\theta)$ of $F(x;\theta) = 0$ is an interior point of $X(\theta)$, as shown in Lemma 1 for the case of $k_f > 0$ and in the proof of Part 1 of Proposition 1 for the case of $k_f = 0$. As a result, we have shown in the proof of Claim 1, the Jacobian matrix $\nabla F(x)$ at the unique solution $x(\theta)$ is invertible.

Then,

$$\frac{\partial F(x(\theta);\theta)}{\partial \lambda_s} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \frac{\partial x(\theta)}{\partial \lambda_s} = \begin{bmatrix} \partial \mu_{lo}(\theta)/\partial \lambda_s \\ \partial \mu_{hn}(\theta)/\partial \lambda_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{23}$$

For other types, it follows from $\mu_{ho}(\theta) + \mu_{hn}(\theta) = k_h$ and $\mu_{lo}(\theta) + \mu_{ln}(\theta) = k_l$ that $\frac{\partial \mu_{ln}(\theta)}{\partial \lambda_s} = \frac{\partial \mu_{ho}(\theta)}{\partial \lambda_s} = 0$, and from $\mu_{ho}(\theta) + \mu_{lo}(\theta) + (k_f - \mu_{fn}(\theta)) = k_f$ that $\frac{\partial \mu_{fn}(\theta)}{\partial \lambda_s} = 0$. Lastly, from (12) and $\mu_{fo} + \mu_{fe} + \mu_{fn} = k_f$,

$$\frac{\partial \mu_{fe}(\theta)}{\partial \lambda_s} = \frac{-\rho_e(k_f - \mu_{fn}(\theta))\mu_{fn}(\theta)}{(\rho_e + \lambda_f \mu_{hn}(\theta) + \lambda_s \mu_{fn}(\theta))^2} = -\frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s}. \tag{24}$$

From the definition of g_3 on p. 41 and Equation 17,

$$v_{fo} - v_{fe} = 2g_{fe-fn} = 2g_3 = \frac{2(u_f - u_e)}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}}.$$

The comparative static derivatives show that μ_{hn} and μ_{fn} are independent of λ_s. Thus, $\frac{\partial (v_{fo} - v_{fe})}{\partial \lambda_s} < 0$ and $\lim_{\lambda_s \to \infty}(v_{fo} - v_{fe}) = 0$.

B.5.2 Part 2

The above comparative static derivatives with respect to λ_s show that $\frac{\partial\mu_{fo}(\theta)}{\partial\lambda_s} > 0$, $\frac{\partial\mu_{fe}(\theta)}{\partial\lambda_s} < 0$, and $\frac{\partial\mu_i(\theta)}{\partial\lambda_s} = 0$, for all $i \neq fo, fe$. Thus,

$$\begin{aligned} r\frac{\partial W(\theta)}{\partial\lambda_s} &= \frac{\partial\mu_{ho}(\theta)}{\partial\lambda_s}u_h + \frac{\partial\mu_{lo}(\theta)}{\partial\lambda_s}u_l + \frac{\partial\mu_{fo}(\theta)}{\partial\lambda_s}u_f + \frac{\partial\mu_{fe}(\theta)}{\partial\lambda_s}u_e \\ &= \frac{\partial\mu_{fo}(\theta)}{\partial\lambda_s}(u_f - u_e) + \frac{\partial(\mu_{fo} + \mu_{fe})(\theta)}{\partial\lambda_s}u_e \\ &= \frac{\partial\mu_{fo}(\theta)}{\partial\lambda_s}(u_f - u_e) - \frac{\partial\mu_{fn}(\theta)}{\partial\lambda_s}u_e = \frac{\partial\mu_{fo}(\theta)}{\partial\lambda_s}(u_f - u_e) > 0. \end{aligned}$$

B.6 Proof of Proposition 4 and Proposition 5

Recall from Claim 1 that the steady-state population is determined by a solution of $F(x; k_f) \equiv (F_{lo}(x; k_f), F_{hn}(x; k_f)) = 0$ where

$$\begin{aligned} F_{lo}(x; k_f) &\equiv (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))x_1 + \rho_u x_1 - \rho_d(k_h - x_2), \\ F_{hn}(x; k_f) &\equiv (\lambda_d x_1 + \lambda_f g_{fn}(x))x_2 + \rho_d x_2 - \rho_u(k_l - x_1), \end{aligned}$$

and $g_{fn}(x) \equiv k_a - k_h + x_2 - x_1$. We extend the system $F(x; k_f) = 0$ such that k_f can be any real number and x can be any real vector of length 2. Each solution $x = (x_1, x_2)$ defines a vector $\mu = (\mu_i)_{i\in\mathcal{T}}$ as $(\mu_{lo}, \mu_{hn}, \mu_{ln}, \mu_{ho}) = (x_1, x_2, k_l - x_1, k_h - x_2)$ and $(\mu_{fn}, \mu_{fo}, \mu_{fe})$ by (11) and (12). According to Claim 1, if $k_f > 0$, a solution exists in certain domain (denoted by $X(\theta)$ in the claim) such that the resulting vector μ is a steady-state population. In general, without any restrictions on k_f, the vector μ may not even be positive.

The proof consists of three steps. First, for $k_f = 0$, we find a population measure $\hat{\mu}$ such that $\hat{x} \equiv (\hat{\mu}_{lo}, \hat{\mu}_{hn})$ solves $F(x; k_f) = 0$. Second, by Implicit Function Theorem, we differentiate a solution function $x(k_f)$ defined in the neighborhood of $k_f = 0$ and $x = \hat{x}$, and obtain the comparative static derivative $\mu_i' \equiv \left.\frac{\partial\mu_i}{\partial k_f}\right|_{k_f=0}$ for each $i \in \mathcal{T}$. Last, we prove the following claim:

Claim 2. *There exist* $\beta_1 > 0$ *and* β_2*, each being independent of* λ_s*, such that*

$$\left.\frac{\partial v_{fn}}{\partial k_f}\right|_{k_f=0} = \beta_1\lambda_s + \beta_2.$$

Then, Proposition 4 and Proposition 5 follow immediately.

B.6.1 A benchmark model ($k_f = 0$)

We set $\hat{\mu}_i = 0$ for every fund type $i \in \mathcal{T}_f$, and impose

$$\begin{aligned}&\hat{\mu}_{ho} = k_h - \hat{\mu}_{hn}, \quad \hat{\mu}_{lo} = k_a - \hat{\mu}_{ho} = k_a - k_h + \hat{\mu}_{hn}, \quad \text{and}\\ &\hat{\mu}_{ln} = k_l - \hat{\mu}_{lo} = k_v - k_a - \hat{\mu}_{hn}.\end{aligned}$$

By substituting the above expressions of $\hat{\mu}_{lo}$ and $\hat{\mu}_{ln}$ in

$$\lambda_d\hat{\mu}_{lo}\hat{\mu}_{hn} + \rho_d\hat{\mu}_{hn} - \rho_u\hat{\mu}_{ln} = 0, \quad (\mu\text{-hn})$$

we obtain

$$\hat{\mu}_{hn} = \frac{1}{2}\left(\sqrt{(R + k_a - k_h)^2 + 4R\cdot k_h\,(1 - k_a)} - (R + k_a - k_h)\right), \tag{25}$$

where $R \equiv \frac{\rho_u+\rho_d}{\lambda_d}$. It is clear that $\hat{x} = (\hat{\mu}_{lo}, \hat{\mu}_{hn})$ solves the system $F(x; k_f) = 0$.

B.6.2 Comparative static derivatives of μ with respect to k_f

We apply Implicit Function Theorem. $F(x; k_f)$ is an infinitely differentiable function of $x \in \mathbb{R}^2$ and $k_f \in \mathbb{R}$, and the Jacobian matrix $\nabla_x F(\hat{x}; 0)$ is invertible (see Equation 13). As such, there is a differentiable function $x(k_f)$ defined in a neighborhood of $k_f = 0$ and $x = \hat{x}$ such that $F(x(k_f); k_f) = 0$. It is important to note that the derivative of $x(k_f)$ at any $k_f > 0$ is independent of the choice of the function $x(k_f)$; Claim 1 ensures that any choice of a function $x(k_f)$ gives the same value of x for each $k_f > 0$.

As explained above, the function $x(k_f)$, together with (11) and (12), defines $\mu(k_f) =$

$(\mu_i(k_f))_{i\in\mathcal{T}}$, which is also differentiable. Let $\mu_i' \equiv \frac{\partial \mu}{\partial k_f}\Big|_{k_f=0}$ for each $i \in \mathcal{T}$. Then,

$$\begin{aligned}
\begin{bmatrix} \mu_{lo}' \\ \mu_{hn}' \end{bmatrix} &= -\left[\nabla_x F(\hat{x};0)\right]^{-1} \frac{\partial F(\hat{x};0)}{\partial k_f} \\
&= -\begin{bmatrix} \lambda_d \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo} + \rho_u & (\lambda_d - \lambda_f)\hat{\mu}_{lo} + \rho_d \\ (\lambda_d - \lambda_f)\hat{\mu}_{hn} + \rho_u & \lambda_d \hat{\mu}_{lo} + \lambda_f \hat{\mu}_{hn} + \rho_d \end{bmatrix}^{-1} \begin{bmatrix} \lambda_f \hat{\mu}_{lo} \\ 0 \end{bmatrix}.
\end{aligned} \tag{26}$$

Also, $\mu_{ho}' = -\mu_{hn}'$ and $\mu_{ln}' = -\mu_{lo}'$.

From (11),

$$\begin{aligned}
\mu_{fn}' &= 1 - \mu_{hn}' + \mu_{lo}' \\
&= 1 - \frac{1}{det(\nabla_x F(\hat{x};0))} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_d \hat{\mu}_{lo} + \lambda_f \hat{\mu}_{hn} + \rho_d & * \\ -(\lambda_d - \lambda_f)\hat{\mu}_{hn} - \rho_u & * \end{bmatrix} \begin{bmatrix} \lambda_f \hat{\mu}_{lo} \\ 0 \end{bmatrix} \\
&= 1 - \frac{\lambda_f \hat{\mu}_{lo} \left(\lambda_d \hat{\mu}_{lo} + \lambda_d \hat{\mu}_{hn} + \rho_d + \rho_u\right)}{det(\nabla_x F(\hat{x};0))},
\end{aligned}$$

where

$$\begin{aligned}
det(\nabla_x F(\hat{x};0)) &= \left(\lambda_d \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo} + \rho_u\right)\left(\lambda_d \hat{\mu}_{lo} + \lambda_f \hat{\mu}_{hn} + \rho_d\right) \\
&\quad - \left((\lambda_d - \lambda_f)\hat{\mu}_{hn} + \rho_u\right)\left((\lambda_d - \lambda_f)\hat{\mu}_{lo} + \rho_d\right) \\
&= \left(\lambda_d \hat{\mu}_{hn} + \rho_u\right)\left(\lambda_f \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo}\right) + \left(\lambda_f \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo}\right)\left(\lambda_d \hat{\mu}_{lo} + \rho_d\right) \\
&= \left(\lambda_f \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo}\right)\left(\lambda_d \hat{\mu}_{lo} + \lambda_d \hat{\mu}_{hn} + \rho_d + \rho_u\right).
\end{aligned}$$

It follows that

$$\mu_{fn}' = \frac{\hat{\mu}_{hn}}{\hat{\mu}_{hn} + \hat{\mu}_{lo}} > 0. \tag{27}$$

Next, (12) implies $\mu_{fo} = \frac{(\lambda_f \mu_{hn} + \lambda_s \mu_{fn})(k_f - \mu_{fn})}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + \rho_e}$. Thus,

$$\begin{aligned}
\mu'_{fo} &= \frac{(1-\mu'_{fn})(\lambda_f \hat{\mu}_{hn})}{\rho_e + \lambda_f \hat{\mu}_{hn}} = \frac{\lambda_f \hat{\mu}_{hn} \hat{\mu}_{lo}}{(\rho_e + \lambda_f \hat{\mu}_{hn})(\hat{\mu}_{hn} + \hat{\mu}_{lo})} > 0, \\
\mu'_{fe} &= 1 - \mu'_{fn} - \mu'_{fo} = \frac{\rho_e \hat{\mu}_{lo}}{(\rho_e + \lambda_f \hat{\mu}_{hn})(\hat{\mu}_{hn} + \hat{\mu}_{lo})} > 0. \qquad (28)
\end{aligned}$$

B.6.3 Proof of Claim 2

From (16), $rv_{fn} = \lambda_f \mu_{lo} g_2 + \lambda_s \mu_{fe} g_3$, where g_2 and g_3 are determined by (17) and (20), respectively.

Let $v'_{fn} \equiv \left.\frac{\partial v_{fn}}{\partial k_f}\right|_{k_f=0}$ and $g'_m \equiv \left.\frac{\partial g_m}{\partial k_f}\right|_{k_f=0}$ for $m = 2, 3$. Then

$$rv'_{fn} = \lambda_f \left(\hat{g}_2 \mu'_{lo} + \hat{\mu}_{lo} g'_2\right) + \lambda_s \left(\hat{g}_3 \mu'_{fe} + \hat{\mu}_{fe} g'_3\right).$$

We find the value of each variable on the right-hand side of the above equation. For certain variables that we will use later, we remark whether the values are strictly positive and/or independent of λ_s.

We have observed the following properties:

1. (from (25)) the population $\hat{\mu} = (\mu_i)_{i \in \mathcal{T}}$ is strictly positive for corporate types, zero for fund types, and independent of λ_s,

2. (from (26), (27), and (28)) the derivative μ' is independent of λ_s,

3. (from (17)) As $g_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}}$, we have $\hat{g}_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}} > 0$ and $g'_3 = -\frac{(\lambda_f \mu'_{hn} + \lambda_s \mu'_{fn}) \hat{g}_3}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}}$, which are independent of λ_s.

It remains to find the values of $\hat{g}_2$ and g'_2.

To state how g_2 is determined by (20), let $c_1 \equiv 2(r + \rho_u + \rho_d) + \lambda_d(\mu_{lo} + \mu_{hn})$, $c_2 \equiv 2\rho_u + \lambda_d \mu_{hn}$, and

$$D \equiv \begin{bmatrix} c_1 + \lambda_f(\mu_{fo} + \mu_{fe}) & c_1 + \lambda_f \mu_{fn} \\ c_2 - \lambda_f \mu_{hn} & c_2 + 2r + \lambda_f(\mu_{lo} + \mu_{fn}) \end{bmatrix}.$$

Also, let $\alpha_1 \equiv \frac{-D_{21}}{det(D)}$ and $\alpha_2 \equiv \frac{D_{11}}{det(D)}$. Then,

$$g_2 = \alpha_1 \left(u_h - u_l - \lambda_f \mu_{fe} g_3\right) + \alpha_2 \left(u_f - u_l - 2\rho_e g_3 - \lambda_s \mu_{fe} g_3\right).$$

Note that, when $k_f = 0$,

$$\begin{aligned}
&\hat{c}_1 = 2(r + \rho_u + \rho_d) + \lambda_d(\hat{\mu}_{lo} + \hat{\mu}_{hn}) > 0, \\
&\hat{c}_2 = 2\rho_u + \lambda_d \hat{\mu}_{hn} > 0, \\
&\hat{D} = \begin{bmatrix} \hat{c}_1 & \hat{c}_1 \\ \hat{c}_2 - \lambda_f \hat{\mu}_{hn} & \hat{c}_2 + 2r + \lambda_f \hat{\mu}_{lo} \end{bmatrix} \quad \text{(with a strictly positive determinant)}, \\
&\hat{\alpha}_1 = \frac{-\hat{c}_2 + \lambda_f \hat{\mu}_{hn}}{det(\hat{D})} \text{ (the exact value is unnecessary for our proof)}, \quad \text{and} \\
&\hat{\alpha}_2 = \frac{\hat{c}_1}{det(\hat{D})} = \frac{1}{2r + \lambda_f(\hat{\mu}_{lo} + \hat{\mu}_{hn})} > 0,
\end{aligned} \tag{29}$$

which are all independent of λ_s. It follows that $\hat{g}_2 = \hat{\alpha}_1(u_h - u_l) + \hat{\alpha}_2(u_f - u_l - 2\rho_e \hat{g}_3)$ is independent of λ_s.

Last, let c_1', c_2', α_1', and α_2' be the corresponding variables' derivatives: e.g., $c_1' \equiv \left.\frac{\partial c_1}{\partial k_f}\right|_{k_f=0}$. The derivatives are all independent of λ_s, because $\hat{\mu}$ and μ' are independent of λ_s. Therefore,

$$\begin{aligned}
g_2' =& \alpha_1'(u_h - u_l - \lambda_f \hat{\mu}_{fe} \hat{g}_3) - \hat{\alpha}_1 \lambda_f(\mu'_{fe} \hat{g}_3 + \hat{\mu}_{fe} g_3') \\
&+ \alpha_2' \left(u_f - u_l - 2\rho_e \hat{g}_3 - \lambda_s \hat{\mu}_{fe} \hat{g}_3\right) - \hat{\alpha}_2 \left(2\rho_e g_3' + \lambda_s \mu'_{fe} \hat{g}_3 + \lambda_s \hat{\mu}_{fe} g_3'\right) \\
=& \alpha_1'(u_h - u_l) - \hat{\alpha}_1 \lambda_f \mu'_{fe} \hat{g}_3 + \alpha_2' \left(u_f - u_l - 2\rho_e \hat{g}_3\right) - \hat{\alpha}_2 \left(2\rho_e g_3' + \lambda_s \mu'_{fe} \hat{g}_3\right). \quad (\text{as } \hat{\mu}_{fe} = 0)
\end{aligned}$$

Only the last term $-\hat{\alpha}_2 \left(2\rho_e g_3' + \lambda_s \mu'_{fe} \hat{g}_3\right)$ is (affinely) dependent on λ_s, through $-\hat{\alpha}_2 \mu'_{fe} \hat{g}_3 \lambda_s$ and $g_3' = -\frac{(\lambda_f \mu'_{hn} + \lambda_s \mu'_{fn})\hat{g}_3}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}}$. As such, $g_2' = \gamma_1 \lambda_s + \gamma_2$, for $\gamma_1 = \hat{\alpha}_2 \hat{g}_3 \left(\frac{2\rho_e \mu'_{fn}}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}} - \mu'_{fe}\right)$ and some γ_2 which aggregates all remaining terms. Both γ_1 and γ_2 are independent of λ_s.

Finally,

$$
\begin{aligned}
rv'_{fn} &= \lambda_f \left(\hat{g}_2 \mu'_{lo} + \hat{\mu}_{lo} g'_2\right) + \lambda_s \left(\hat{g}_3 \mu'_{fe} + \hat{\mu}_{fe} g'_3\right) \\
&= \lambda_f \left(\hat{g}_2 \mu'_{lo} + \hat{\mu}_{lo}(\gamma_1 \lambda_s + \gamma_2)\right) + \lambda_s \hat{g}_3 \mu'_{fe} \quad (\text{as } \hat{\mu}_{fe} = 0) \\
&= (\lambda_f \hat{\mu}_{lo} \gamma_1 + \hat{g}_3 \mu'_{fe})\lambda_s + (\lambda_f \hat{g}_2 \mu'_{lo} + \lambda_f \hat{\mu}_{lo} \gamma_2),
\end{aligned}
$$

where the coefficient of λ_s and the last term are both independent of λ_s.

It remains to show that the coefficient of λ_s is strictly positive:

$$
\begin{aligned}
\lambda_f \hat{\mu}_{lo} \gamma_1 + \hat{g}_3 \mu'_{fe} &= \lambda_f \hat{\mu}_{lo} \hat{\alpha}_2 \hat{g}_3 \left(\frac{2\rho_e \mu'_{fn}}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}} - \mu'_{fe} \right) + \hat{g}_3 \mu'_{fe} \\
&> -\lambda_f \hat{\mu}_{lo} \hat{\alpha}_2 \hat{g}_3 \mu'_{fe} + \hat{g}_3 \mu'_{fe} \quad (\text{as } \hat{\mu}_{lo}, \hat{\alpha}_2, \hat{g}_3, \mu'_{fn}, \hat{\mu}_{hn} \text{ are strictly positive}) \\
&= \mu'_{fe} \hat{g}_3 \left(1 - \lambda_f \hat{\alpha}_2 \hat{\mu}_{lo}\right) \\
&= \mu'_{fe} \hat{g}_3 \left(1 - \frac{\lambda_f \hat{\mu}_{lo}}{2r + \lambda_f(\hat{\mu}_{lo} + \hat{\mu}_{hn})}\right) \quad (\text{from (29)}) \\
&> 0. \quad (\text{as } \mu'_{fe} \text{ and } \hat{g}_3 \text{ are strictly positive})
\end{aligned}
$$

B.7 Proof of Proposition 7

First, from (v-hn)-(v-ln),

$$
\begin{aligned}
rW_v \equiv & r(\mu_{ho} v_{ho} + \mu_{hn} v_{hn} + \mu_{lo} v_{lo} + \mu_{ln} v_{ln}) \\
= & \mu_{ho}(u_h + \rho_d(v_{lo} - v_{ho})) + \mu_{hn}(\lambda_d \mu_{lo} g_{lo-hn} + \lambda_f \mu_{fo} g_{fo-hn} + \lambda_f \mu_{fe} g_{fe-hn} + \rho_d (v_{ln} - v_{hn})) \\
& + \mu_{lo}(u_l + \lambda_d \mu_{hn} g_{lo-hn} + \lambda_f \mu_{fn} g_{lo-fn} + \rho_u (v_{ho} - v_{lo})) + \mu_{ln} \rho_u (v_{hn} - v_{ln}).
\end{aligned}
$$

We substitute $g_{fo-hn} = v_{ho} - v_{hn} - p_{fo-hn}$, $g_{fe-hn} = v_{ho} - v_{hn} - p_{fe-hn}$, $g_{lo-fn} = p_{lo-fn} - v_{lo} - v_{ln}$, and $g_{lo-hn} = (1/2)(v_{ho} + v_{ln} - v_{lo} + v_{hn})$. Then, (3) follows from

$$
\begin{aligned}
& rW_v - (\mu_{ho} u_h + \mu_{lo} u_l + \lambda_f \mu_{lo} \mu_{fn} p_{lo-fn} - \lambda_f \mu_{hn} (\mu_{fo} p_{fo-hn} + \mu_{fe} p_{fe-hn})) \\
& = (\rho_u \mu_{lo} - \rho_d \mu_{ho})(v_{ho} - v_{lo}) + (\rho_u \mu_{ln} - \rho_u \mu_{hn})(v_{hn} - v_{ln}) \\
& + \mu_{hn}(\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe})(v_{ho} - v_{hn}) + \mu_{lo}(\lambda_d \mu_{hn} + \lambda_f \mu_{fn})(v_{ln} - v_{lo}).
\end{aligned}
$$

The combined coefficient of v_{ho} on the right-hand side of the above equation is $-\rho_d\mu_{ho} + \rho_u\mu_{lo} + \mu_{hn}(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})$, which equals the right-hand side of the population equation (μ-ho), so it is zero. We can similarly verify that the combined coefficient of v_i for $i = hn, lo, ln$ are all equal to zero.

Second, we obtain (2) from all population equations (μ-hn)-(μ-fe) such that

$$\begin{aligned}
&rW - (\mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l) \\
&= (\rho_u\mu_{lo} - \rho_d\mu_{ho})(v_{ho} - v_{lo}) + (\rho_u\mu_{ln} - \rho_d\mu_{hn})(v_{hn} - v_{ln}) + \rho_e\mu_{fo}(v_{fe} - v_{fo}) \\
&+ \mu_{hn}(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})(v_{ho} - v_{hn}) + \mu_{lo}(\lambda_d\mu_{hn} + \lambda_f\mu_{fn})(v_{ln} - v_{lo}) \\
&+ ((\lambda_f\mu_{lo} + \lambda_s\mu_{fe})\mu_{fn} - \lambda_f\mu_{hn}\mu_{fo})v_{fo} \\
&+ \lambda_f(\mu_{hn}\mu_{fo} + \mu_{hn}\mu_{fe} - \mu_{lo}\mu_{fn})v_{fn} - (\lambda_f\mu_{hn} + \lambda_s\mu_{fn})\mu_{fe}v_{fe}.
\end{aligned}$$

As before, we can verify that the combined coefficients of v_i for each $i \in \mathcal{T}$ equal the right-hand side of the type's population equation, so it is zero.

Lastly, the expression for W_f follows from $W_f = W - W_v$.

B.8 Proof of Proposition 9

First, consider the path of a lo-type investor in a steady-state equilibrium. This investor can sell its asset upon meeting either a buying investor (hn) or a fund buyer (fn). Each kind of meeting arrives with Poisson rate $\lambda_c\mu_{hn}$ or $\lambda_f\mu_{fn}$. The time until the first meeting of each kind, denoted by τ_{lo-hn} and τ_{lo-fn}, follows the exponential distributions. Thus, the time until selling $\tau_{sc} \equiv \min\{\tau_{lo-hn}, \tau_{lo-fn}\}$ follows an exponential distribution with parameter $\lambda_c\mu_{hn} + \lambda_f\mu_{fn}$. Hence, $E[\tau_{sc}] = \frac{1}{\lambda_c\mu_{hn}+\lambda_f\mu_{fn}}$.

Second, consider the path of a fo-type fund in a steady-state equilibrium. The fund sells its asset before receiving a liquidity shock to a buying investor (hn) or receives a liquidity shock and enters the exit phase (after which it can sell to either a buying investor (hn) or a fund buyer (fn)). We denote by τ_{fo} this period for which a fund maintains its type as fo. The time τ_{fo} follows an exponential distribution with parameter $\lambda_f\mu_{hn} + \rho_e$. Hence, $E[\tau_{fo}] = \frac{1}{\lambda_f\mu_{hn}+\rho_e}$.

Finally, we evaluate the path of an fe type fund (an outcome of an fo type fund receiving a liquidity shock before meeting a buying investor with probability $\frac{\rho_e}{\lambda_f \mu_{hn}+\rho_e}$). The fe type fund maintains its type until it sells its portfolio asset either to a buying investor (hn) or a fund buyer (fn). Thus, the fund maintains its type for the time period τ_{fe}, which follows an exponential distribution with parameter $\lambda_f \mu_{hn} + \lambda_s \mu_{fn}$. Hence, $E[\tau_{fe}] = \frac{1}{\lambda_f \mu_{hn}+\lambda_s \mu_{fn}}$.

As a result, the overall expected time for a fund to sell an asset is:

$$E[\tau_{sf}] = \frac{1}{\lambda_f \mu_{hn} + \rho_e} + \frac{\rho_e}{\lambda_f \mu_{hn} + \rho_e} \left(\frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}} \right).$$

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Supplementary Appendices (Not for Publication)

In Appendix SA.1, we present the proofs of results related to the fast search market. Next, Appendix SA.2 derives a closed-form expression of Public Market Equivalent (PME). Lastly, Appendix SA.3 provides several estimates of the time it takes for buyout funds to sell assets.

SA.1 Proofs on the Fast Search Market

SA.1.1 Proof for Part 1 of Proposition 6

For any regular environment $\theta \equiv (k, r, u, \rho, \lambda)$, we consider a sequence $\theta^L \equiv (k, r, u, \rho, L\lambda)$ with $L \to \infty$. Let μ^L be the unique steady-state solution of $P(\theta^L)$ and v^L be the unique solution of $V(\theta^L)$ with $\mu(t)$ being replaced by μ^L.

In solving $P(\theta^L)$, it is more convenient to take $z \equiv 1/L$ and define another market $\psi^z \equiv (k, r, u, z\rho, \lambda)$. (i.e., low type-change rates, instead of high search rates) and solve $P(\psi^z)$. It is easy to verify that the unique steady-state solution μ^L of $P(\theta^L)$ also uniquely solves $P(\psi^z)$. Last, define $\psi^0 \equiv (k, r, u, 0, \lambda)$.

Lemma SA.1. *$\mu^0 \in \mathbb{R}^{\mathcal{T}}$ is a steady-state solution of $P(\psi^0)$ if and only if*

1. *(when $k_f + k_h > k_a$) $\mu^0_{ho} = \min\{k_a, k_h\}$, $\mu^0_{hn} = k_h - \mu^0_{ho}$, $\mu^0_{lo} = 0$, $\mu^0_{ln} = k_l$, $\mu^0_{fo} = \max\{0, k_a - k_h\}$, $\mu^0_{fe} = 0$, and $\mu^0_{fn} = k_f - \mu^0_{fo} - \mu^0_{fe}$.*

2. *(when $k_f + k_h < k_a$) $\mu^0_{ho} = k_h$, $\mu^0_{hn} = 0$, $\mu^0_{lo} = k_a - k_h - k_f$, $\mu^0_{ln} = k_l - \mu^0_{lo}$, $\mu^0_{fn} = 0$, and $\mu^0_{fo} + \mu^0_{fe} = k_f$.*

The problem $P(\psi^0)$ has multiple steady-state solutions in Case 2 ($k_f + k_h < k_a$), where many possible combinations of (μ_{fo}, μ_{fe}) satisfy $\mu_{fo} + \mu_{fe} = k_f$.

(Proof) The problem $P(\psi^0)$ consists of

$$(\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe})\mu_{hn} = 0, \qquad \text{(from (μ-ho))}$$

$$(\lambda_d \mu_{hn} + \lambda_f \mu_{fn})\mu_{lo} = 0, \qquad \text{(from (μ-ln))}$$

$$\lambda_f (\mu_{lo}\mu_{fn} - \mu_{hn}\mu_{fo}) + \lambda_s \mu_{fn}\mu_{fe} = 0, \qquad \text{(from (μ-fo))}$$

and the following four conditions that replace (μ-hn), (μ-lo), (μ-fe), and (μ-fn):

$$\mu_{ho} + \mu_{hn} = k_h, \quad \mu_{lo} + \mu_{ln} = k_l, \quad \mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = k_a, \quad \text{and} \quad \mu_{fn} + \mu_{fo} + \mu_{fe} = k_f.$$

It follows from $\lambda_d, \lambda_f, \lambda_d > 0$ that $\mu_{lo}\mu_{hn} = \mu_{fo}\mu_{hn} = \mu_{fe}\mu_{hn} = \mu_{lo}\mu_{fn} = \mu_{fn}\mu_{fe} = 0$.

Suppose that $k_f + k_h > k_a$. If $\mu_{lo} > 0$ or $\mu_{fe} > 0$, then $\mu_{hn} = 0$ and $\mu_{fn} = 0$, which results in a contradiction: $\mu_{ho} + (\mu_{fo} + \mu_{fe}) = k_h + k_f > k_a$. As $\mu_{lo} = \mu_{fe} = 0$, either $\mu_{fo} = 0$ or $\mu_{hn} = 0$. As $\mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = k_a > 0$, if $\mu_{fo} = 0$, then $\mu_{ho} = k_a$; for otherwise $\mu_{hn} = 0$ implies that $\mu_{ho} = k_h$ and $\mu_{fo} = k_a - k_h$. On the other hand, if $k_f + k_h < k_a$, then $\mu_{lo} > 0$, which implies that $\mu_{hn} = \mu_{fn} = 0$. Thus, $\mu_{ho} = k_h$, $\mu_{fo} + \mu_{fe} = k_f$, and $\mu_{lo} = k_a - k_h - k_f$. ■

The following lemma implies that $\lim_{L\to\infty} \mu^L$ exists in $\mathbb{R}_+^{\mathcal{T}}$:

Lemma SA.2. *There exists a solution μ^* of $P(\psi^0)$ such that $\mu^* \equiv \lim_{L\to\infty} \mu^L$.*

(Proof) For each $z \equiv 1/L$, let $F(\mu, \psi^z)$ denote the right-hand sides of the population equations (μ-hn)-(μ-fe) for a market $\psi^z \equiv (k, r, u, z\rho, \lambda)$. Define $f(\mu, z) \equiv -\|F(\mu, \psi^z)\|$, where $\|\cdot\|$ denotes the Euclidean norm. It is clear that μ^L with $L = 1/z$ is the unique maximizer of f with the maximum value equal to zero. Let $M(z) \equiv \{\mu^{1/z}\}$.

We similarly define $F(\mu, \psi^0)$ as the right-hand sides of the population equations for the market ψ^0 and $f(\mu, 0)$. Let $M(0)$ be the solution set of $\max_\mu f(\mu, 0)$. According to Lemma SA.1, the solution set $M(0)$ is singleton if $k_h + k_f > k_a$; for otherwise, $M(0)$ contains multiple solutions, each being different from others only in (μ_{fo}, μ_{fe}) under the constraint $\mu_{fo} + \mu_{fe} = k_f$.

The function f is continuous in μ and z because the equations F are continuous. Then, Berge's Maximum Theorem implies that $M(\cdot)$ is upper hemicontinuous at $z = 0$:

1. (when $k_h + k_f > k_a$) μ^L converges to the unique solution of $P(\psi^0)$.

2. (when $k_h + k_f < k_a$) for each type $i \neq fo, fe$, the population μ_i^L converges to μ_i^0 given in Lemma SA.1, and $\mu_{fo}^L + \mu_{fe}^L$ converges to k_f.

It remains to show that, when $k_h + k_f > k_a$, the sequence μ_{fe}^L converges. (The convergence of μ_{fo}^L follows immediately from $\lim_{L\to\infty}(\mu_{fo}^L + \mu_{fe}^L) = k_f$.)

For every $L > 0$, $L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn})\mu^L_{fe} = \rho_e(k_f - \mu^L_{fn} - \mu^L_{fe})$ (from (μ-fe)), which implies

$$\mu^L_{fe} = \frac{\rho_e(k_f - \mu^L_{fn})}{\rho_e + L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn})}. \tag{SA.1}$$

We find $\lim_{L\to\infty} L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn}$ from

$$L\mu^L_{hn}(\lambda_d \mu^L_{lo} + \lambda_f \mu^L_{fo} + \lambda_f \mu^L_{fe}) = -\rho_d \mu^L_{hn} + \rho_u \mu^L_{ln}, \qquad \text{(from (μ-hn))}$$
$$L(\lambda_v \mu^L_{hn} + \lambda_f \mu^L_{fn})\mu^L_{lo} = -\rho_u \mu^L_{lo} + \rho_d \mu^L_{ho}. \qquad \text{(from (μ-lo))}$$

By the convergence of μ^L_i for $i \neq fo, fe$, and the convergence of $\mu^L_{fe} + \mu^L_{fo}$ to n_f,

$$\lim_{L\to\infty} L\mu^L_{hn} = \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f k_f}, \quad \text{and}$$
$$\lim_{L\to\infty} L(\lambda_v \mu^L_{hn} + \lambda_f \mu^L_{fn}) = \frac{\rho_d \mu^*_{ho} - \rho_u \mu^*_{lo}}{\mu^*_{lo}} = \frac{\rho_d k_h - \rho_u \mu^*_{lo}}{\mu^*_{lo}} = \frac{\rho_u \mu^*_{ln}}{\mu^*_{lo}}.$$

It follows that

$$\begin{aligned}\lim_{L\to\infty} L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn}) &= \frac{\lambda_s \rho_u \mu^*_{ln}}{\lambda_f \mu^*_{lo}} + \left(\lambda_f - \frac{\lambda_d \lambda_s}{\lambda_f}\right) \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f k_f} \\ &= \frac{\rho_u \mu^*_{ln}}{\mu^*_{lo}} \frac{\lambda_f \mu^*_{lo} + \lambda_s k_f}{\lambda_d \mu^*_{lo} + \lambda_f k_f} > 0.\end{aligned}$$

Therefore,

$$\mu^*_{fe} \equiv \lim_{L\to\infty} \mu^L_{fe} = \frac{k_f}{1 + \frac{\rho_u \mu^*_{ln}}{\rho_e \mu^*_{lo}} \frac{\lambda_f \mu^*_{lo} + \lambda_s k_f}{\lambda_d \mu^*_{lo} + \lambda_f k_f}}, \quad \text{and} \quad \mu^*_{fo} = k_f - \mu^*_{fe}.$$

■

SA.1.2 Proof for Part 2 of Proposition 6

We divide the proof into two lemmas.

Lemma SA.3. *For every $i \in \mathcal{T}$, if $\mu^*_i = 0$, then $\mu^{**}_i \equiv \lim_{L\to\infty} L\mu^L_i$ exists in $\mathbb{R}$.*

(Proof) The following table summarizes the population limits for some types from

Lemma SA.1 and Lemma SA.2:

	A. $k_a < k_h$	B. $k_h < k_a < k_h + k_f$	C. $k_h + k_f < k_a$
$\mu^*_{ho} =$	k_a	k_h	k_h
$\mu^*_{fo} =$	0	$k_a - k_h$	$< k_f$
$\mu^*_{lo} =$	0	0	$k_a - k_f - k_h$
$\mu^*_{fe} =$	0	0	> 0

As μ^*_{ho} and μ^*_{ln} are always strictly positive, the following step considers only other types.

Suppose $\mu^*_{hn} = 0$ (Cases A, B, and C): for any L,

$$L(\lambda_c \mu^L_{lo} + \lambda_f \mu^L_{fo} + \lambda_f \mu^L_{fe})\mu^L_{hn} = -\rho_d \mu^L_{hn} + \rho_u \mu^L_{ln}. \quad \text{(from (μ-hn))}$$

By Lemma 1, $\mu^L_i > 0$ for every $i \in \mathcal{T}$,

$$L\mu^L_{hn} = \frac{\rho_u \mu^L_{ln} - \rho_d \mu^L_{hn}}{\lambda_d \mu^L_{lo} + \lambda_f \mu^L_{fo} + \lambda_f \mu^L_{fe}} \tag{SA.2}$$

$$\Longrightarrow \mu^{**}_{hn} \equiv \lim_{L\to\infty} L\mu^L_{hn} = \frac{\rho_u \mu^*_{ln} - \rho_d \mu^*_{hn}}{\lambda_d \mu^*_{lo} + \lambda_f(\mu^*_{fo} + \mu^*_{fe})} = \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f(\mu^*_{fo} + \mu^*_{fe})} > 0. \tag{SA.3}$$

Suppose $\mu^*_{lo} = 0$ (Cases A and B): for every L,

$$(\lambda_d \mu^L_{hn} + \lambda_f \mu^L_{fn})(L\mu^L_{lo}) = \rho_d \mu^L_{ho} - \rho_u \mu^L_{lo}. \quad \text{(from (μ-lo))}$$

It follows that

$$\mu^{**}_{lo} \equiv \lim_{L\to\infty} L\mu^L_{lo} = \frac{\rho_d \mu^*_{ho} - \rho_u \mu^*_{lo}}{\lambda_d \mu^*_{hn} + \lambda_f \mu^*_{fn}} = \frac{\rho_d \mu^*_{ho}}{\lambda_d \mu^*_{hn} + \lambda_f \mu^*_{fn}}.$$

Suppose $\mu^*_{fe} = 0$ (Cases A and B): for every L,

$$(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn})(L\mu^L_{fe}) = \rho_e \mu^L_{fo}. \quad \text{(from (μ-fe))}$$

It follows that

$$\mu^{**}_{fe} \equiv \lim_{L\to\infty} L\mu^L_{fe} = \frac{\rho_e \mu^*_{fo}}{\lambda_f \mu^*_{hn} + \lambda_s \mu^*_{fn}}.$$

Suppose $\mu^*_{fo} = 0$ (Case A): for every L,

$$L(\lambda_f \mu^L_{lo} + \lambda_s \mu^L_{fe})\mu^L_{fn} = L\lambda_f \mu^L_{hn}\mu^L_{fo} + \rho_e \mu^L_{fo}. \quad \text{(from (μ-fo))}$$

It follows from the convergence of $L\mu^L_{lo}$ and $L\mu^L_{fe}$ in Case A that

$$\mu^{**}_{fo} \equiv \lim_{L\to\infty} L\mu^L_{fo} = \lim_{L\to\infty} \frac{(\lambda_f(L\mu^L_{lo}) + \lambda_s(L\mu^L_{fe}))\mu^L_{fn} - \rho_e\mu^L_{fo}}{\lambda_f \mu^L_{hn}} = \frac{(\lambda_f \mu^{**}_{lo} + \lambda_s \mu^{**}_{fe})\mu^*_{fn}}{\lambda_f \mu^*_{hn}}.$$

Finally, suppose $\mu^*_{fn} = 0$ (Case C): for every L,

$$\mu^L_{hn}\mu^L_{fo} + \mu^L_{hn}\mu^L_{fe} = \mu^L_{lo}\mu^L_{fn}. \quad \text{(from (μ-fn))}$$

As $\mu^L_{lo} > 0$ (Lemma 1), we have

$$L\mu^L_{fn} = \frac{L\mu^L_{hn}(\mu^L_{fo} + \mu^L_{fe})}{\mu^L_{lo}}. \tag{SA.4}$$

It follows from the convergences of $L\mu^L_{hn}$ that

$$\mu^{**}_{fn} \equiv \lim_{L\to\infty} L\mu^L_{fn} = \lim_{L\to\infty} \frac{L\mu^L_{hn}(\mu^L_{fo} + \mu^L_{fe})}{\mu^L_{lo}} = \frac{\mu^{**}_{hn}(\mu^*_{fo} + \mu^*_{fe})}{\mu^*_{lo}} > 0. \tag{SA.5}$$

■

Lemma SA.4. *For any $i \in \mathcal{T}$, if $\mu^*_i > 0$, then $\mu^{**}_i \equiv \lim_{L\to\infty} L(\mu^L_i - \mu^*_i)$ exists in $\mathbb{R}$.*

(Proof)

First, consider Case A ($k_a < k_h$): Only $\mu^*_{ho}, \mu^*_{fn}, \mu^*_{ln}$ are strictly positive. As $L \to \infty$,

$$L(\mu^L_{ho} - \mu^*_{ho}) = L(k_a - \mu^L_{lo} - \mu^L_{fo} - \mu^L_{fe}) - L(k_a - \mu^*_{lo} - \mu^*_{fo} - \mu^*_{fe}) \to -\mu^{**}_{lo} - \mu^{**}_{fo} - \mu^{**}_{fe},$$

where the convergence of $L\mu^L_{lo}$, $L\mu^L_{fo}$, and $L\mu^L_{fe}$ holds by Lemma SA.3.

We similarly find the convergence speed for μ^L_{fn} and μ^L_{ln}:

$$
\begin{aligned}
L(\mu^L_{fn} - \mu^*_{fn}) &= L(k_f - \mu^L_{fo} - \mu^L_{fe}) - L(k_f - \mu^*_{fo} - \mu^*_{fe}) \to -\mu^{**}_{fo} - \mu^{**}_{fe}, \quad \text{and} \\
L(\mu^L_{ln} - \mu^*_{ln}) &= L(k_l - \mu^L_{ln}) - L(k_l - \mu^*_{lo}) \to -\mu^{**}_{lo}.
\end{aligned}
$$

Next, consider Case B ($k_h < k_a < k_h + k_f$):
Only $\mu^*_{ho}, \mu^*_{fo}, \mu^*_{ln}$, and μ^*_{fn} are strictly positive. As $L \to \infty$,

$$
\begin{aligned}
L(\mu^L_{ho} - \mu^*_{ho}) &= L(k_h - \mu^L_{hn}) - L(k_h - \mu^*_{hn}) \to -\mu^{**}_{hn}, \\
L(\mu^L_{ln} - \mu^*_{ln}) &= L(k_l - \mu^L_{lo}) - L(k_l - \mu^*_{lo}) \to -\mu^{**}_{lo}, \\
L(\mu^L_{fo} - \mu^*_{fo}) &= L(\mu^L_{fo} - (k_a - k_h)) = -L\mu^L_{lo} - L\mu^L_{fe} - L(\mu^L_{ho} - k_h) \\
&\to -\mu^{**}_{lo} - \mu^{**}_{fe} + \mu^{**}_{hn}, \quad \text{and} \\
L(\mu^L_{fn} - \mu^*_{fn}) &= L(\mu^L_{fn} - (k_f - k_a + k_h)) = -L\mu^L_{fe} - L(\mu^L_{fo} - (k_a - k_h)) \\
&\to -\mu^{**}_{fe} + (\mu^{**}_{lo} + \mu^{**}_{fe} - \mu^{**}_{hn}).
\end{aligned}
$$

Finally, in Case C ($k_h + k_f < k_a$), we have $\mu^*_{ho}, \mu^*_{lo}, \mu^*_{ln}, \mu^*_{fo}$, and μ^*_{fe} that are strictly positive. The proof for the first three types are similar to the previous cases: as $L \to \infty$,

$$
\begin{aligned}
L(\mu^L_{ho} - \mu^*_{ho}) &= L(k_h - \mu^L_{hn}) - L(k_h - \mu^*_{hn}) \to -\mu^{**}_{hn}, \\
L(\mu^L_{lo} - \mu^*_{lo}) &= L(\mu^L_{lo} - (k_a - k_h - k_f)) = -L(\mu^L_{fo} + \mu^L_{fe} - k_f) - L(\mu^L_{ho} - k_h) \\
&\to -\mu^{**}_{fn} + \mu^{**}_{hn}, \qquad \text{(SA.6)} \\
L(\mu^L_{ln} - \mu^*_{ln}) &= L(k_l - \mu^L_{lo}) - L(k_l - \mu^*_{lo}) \to -\mu^{**}_{lo} = \mu^{**}_{fn} - \mu^{**}_{hn}. \qquad \text{(SA.7)}
\end{aligned}
$$

It remains to show the convergence speed for μ^L_{fo} and μ^L_{fe}. On the one hand, from (μ-fe) and the convergence of μ^L_{fe}, μ^L_{fo}, $L\mu^L_{hn}$, and $L\mu^L_{fn}$, we have

$$
L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn})\mu^L_{fe} = \rho_e \mu^L_{fo} \quad \text{and} \quad (\lambda_f \mu^{**}_{hn} + \lambda_s \mu^{**}_{fn})\mu^*_{fe} = \rho_e \mu^*_{fo}.
$$

Let $\phi^L \equiv L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn})$, and $\phi^{**} \equiv \lambda_f \mu^{**}_{hn} + \lambda_s \mu^{**}_{fn}$. Then,

$$
\rho_e L(\mu^L_{fo} - \mu^*_{fo}) = \phi^L L\mu^L_{fe} - \phi^{**} L\mu^*_{fe} = L(\phi^L - \phi^{**})\mu^L_{fe} + \phi^{**} L(\mu^L_{fe} - \mu^*_{fe}). \qquad \text{(SA.8)}
$$

On the other hand, from $\mu_{fn}^L + \mu_{fo}^L + \mu_{fe}^L = k_f$ and $\mu_{fo}^* + \mu_{fe}^* = k_f$, we have

$$L(\mu_{fo}^L - \mu_{fo}^*) + L(\mu_{fe}^L - \mu_{fe}^*) = -L\mu_{fn}^L. \tag{SA.9}$$

By summarizing (SA.8) and (SA.9), for every L,

$$\begin{bmatrix} L(\mu_{fo}^L - \mu_{fo}^*) \\ L(\mu_{fe}^L - \mu_{fe}^*) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \rho_e & -\phi^{**} \end{bmatrix}^{-1} \begin{bmatrix} -L\mu_{fn}^L \\ L(\phi^L - \phi^{**})\mu_{fe}^L \end{bmatrix},$$

where the inverse matrix is well-defined because $\phi^{**} > 0$ (see (SA.3) and (SA.5)). Note that $L\mu_{fn}^L$ and μ_{fe}^L converge (see (SA.5) and Lemma SA.2). It remains to prove that

$$L(\phi^L - \phi^{**}) = \lambda_f L(L\mu_{hn}^L - \mu_{hn}^{**}) + \lambda_s L(L\mu_{fn}^L - \mu_{fn}^{**}) \quad \text{converges as } L \to \infty.$$

From (SA.2), (SA.3), $\mu_{hn}^* = 0$, and $\mu_{fo}^* + \mu_{fe}^* = k_f$,[31] we have

$$L(L\mu_{hn}^L - \mu_{hn}^{**}) = \frac{\rho_u L\mu_{ln}^L - \rho_d L\mu_{hn}^L}{\lambda_d \mu_{lo}^L + \lambda_f(\mu_{fo}^L + \mu_{fe}^L)} - \frac{\rho_u (L\mu_{ln}^*)}{\lambda_d \mu_{lo}^* + \lambda_f k_f}.$$

To ease expositions, let A^L and A^* denote the denominators in the above equation. Then,

$$\begin{aligned} L(L\mu_{hn}^L - \mu_{hn}^{**}) &= \frac{\rho_u L\mu_{ln}^L - \rho_d L\mu_{hn}^L}{A^L} - \frac{\rho_u L\mu_{ln}^*}{A^*} \\ &= \frac{\rho_u L(\mu_{ln}^L - \mu_{ln}^*) - \rho_d L\mu_{hn}^L}{A^L} + \rho_u \mu_{ln}^* L\left(\frac{1}{A^L} - \frac{1}{A^*}\right) \\ &= \frac{\rho_u L(\mu_{ln}^L - \mu_{ln}^*) - \rho_d L\mu_{hn}^L}{A^L} - \rho_u \mu_{ln}^* \frac{\lambda_d L(\mu_{lo}^L - \mu_{lo}^*) - \lambda_f L\mu_{fn}^L}{A^L A^*}, \end{aligned} \tag{SA.10}$$

which converges by (SA.3), (SA.5), (SA.6), and (SA.7).

Then, from (SA.4), (SA.5), and $\mu_{fo}^* + \mu_{fe}^* = k_f$, we have

$$L(L\mu_{fn}^L - \mu_{fn}^{**}) = \frac{L\mu_{hn}^L L(\mu_{fo}^L + \mu_{fe}^L)}{\mu_{lo}^L} - \frac{\mu_{hn}^{**} L k_f}{\mu_{lo}^*}.$$

[31] Recall that we are considering Case C ($k_h + k_f < k_a$).

Then,

$$\begin{aligned}
L(L\mu_{fn}^L - \mu_{fn}^{**}) &= \frac{L\mu_{hn}^L L(\mu_{fo}^L + \mu_{fe}^L - k_f)}{\mu_{lo}^L} + \frac{L^2\mu_{hn}^L k_f}{\mu_{lo}^L} - \frac{L\mu_{hn}^{**}k_f}{\mu_{lo}^*} \\
&= -\frac{(L\mu_{hn}^L)(L\mu_{fn}^L)}{\mu_{lo}^L} + \frac{L(L\mu_{hn}^L - \mu_{hn}^{**})k_f}{\mu_{lo}^L} + \frac{L\mu_{hn}^{**}k_f}{\mu_{lo}^L} - \frac{L\mu_{hn}^{**}k_f}{\mu_{lo}^*} \\
&= -\frac{(L\mu_{hn}^L)(L\mu_{fn}^L)}{\mu_{lo}^L} + \frac{L(L\mu_{hn}^L - \mu_{hn}^{**})k_f}{\mu_{lo}^L} - \frac{\mu_{hn}^{**}k_f L(\mu_{lo}^L - \mu_{lo}^*)}{\mu_{lo}^L\mu_{lo}^*},
\end{aligned}$$

which converges by (SA.3), (SA.5), (SA.6), and (SA.10). ■

SA.1.3 Proof of Proposition 8

By Lemma SA.1 and Lemma SA.2, if $k_a < k_h + k_f$, then $\mu_{ho}^* = \min\{k_a, k_h\}$, $\mu_{fo}^* = \max\{0, k_a - k_h\}$, $\mu_{fe}^* = 0$, and $\mu_{lo}^* = 0$. Since μ^* coincides with the efficient asset allocation $\overline{\mu}$, we have $W^* = \overline{W}$. The independence of W^* on u_f and u_e is trivial as $\mu_{fo}^* = \mu_{fe}^* = 0$. The independence of W^* on λ_d also follows from $\overline{W}$'s independence of any search friction. When $k_h < k_a < k_h + k_f$, we have $\mu_{fo} > 0$, so $W^* = \overline{W}$ is strictly increasing in u_f.

If $k_a > k_h + k_f$, then $rW^* = r\overline{W} - \mu_{fe}^*(u_f - u_e)$. We have $W^* < \overline{W}$ because

$$\mu_{fe}^* = \frac{k_f}{1 + \frac{\rho_u \mu_{ln}^*}{\rho_e \mu_{lo}^*}\frac{\lambda_f \mu_{lo}^* + \lambda_s k_f}{\lambda_d \mu_{lo}^* + \lambda_f k_f}} > 0,$$

The welfare W^* is increasing in u_f and u_e as μ_{fo}^* and μ_{fe}^* are strictly positive. Moreover, μ_{fe}^* is decreasing in λ_s and increasing in λ_d. Thus, the welfare W^* is increasing in λ_s and decreasing in λ_d.

SA.2 Proofs on Spreads and Prices

SA.2.1 Proof of Proposition 11

It remains to obtain the closed-form expression of PV_{calls}. The time taken to buy τ_b, the time taken to sell τ_s, and the event of purchasing from a low-type investor, rather than an exiting fund, are all independent from each other. Thus,

$$PV_{\text{calls}} = E\left[P_b\right] E\left[e^{-r\tau_b}\right] + E\left[(fP_b)\right] E\left[\int_0^{\tau_b+\tau_s} e^{-rt}dt\right],$$

where

$$\begin{aligned} E\left[\int_0^{\tau_b+\tau_s} e^{-rt}dt\right] &= E\left[\int_0^{\tau_b} e^{-rt}dt\right] + E\left[\int_{\tau_b}^{\tau_b+\tau_s} e^{-rt}dt\right] \\ &= E\left[\int_0^{\tau_b} e^{-rt}dt\right] + E\left[e^{-r\tau_b}\right] E\left[\int_0^{\tau_s} e^{-rt}dt\right]. \end{aligned}$$

Note that

$$\begin{aligned} E\left[P_b\right] &= \frac{\lambda_f\mu_{lo}P_{lo-fn} + \lambda_s\mu_{fe}P_{fe-fn}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}}, \\ E\left[e^{-r\tau_b}\right] &= \frac{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}, \quad \text{and} \\ E\left[\int_0^{\tau_b} e^{-rt}dt\right] &= \frac{1}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}. \end{aligned}$$

Last, recall that a fund's type remains fo or fe for the time period $\tau_{fo} \equiv \min\{\tau_{fo-hn}, \tau_e\}$ or $\tau_e \equiv \min\{\tau_{fe-hn}, \tau_{fe-fn}\}$, respectively. Then,

$$\begin{aligned} E\left[\int_0^{\tau_s} e^{-rt}dt\right] &= E\left[\int_0^{\tau_{fo}} e^{-rt}dt\right] + E\left[\mathbf{1}_{\tau_{fo}=\tau_e}\right] E\left[e^{-r\tau_{fo}}\right] E\left[\int_0^{\tau_e} e^{-rt}dt\right] \\ &= \frac{1}{\lambda_f\mu_{hn} + \rho_e + r} + \frac{\rho_e}{\lambda_f\mu_{hn} + \rho_e} \frac{\lambda_f\mu_{hn} + \rho_e}{\lambda_f\mu_{hn} + \rho_e + r} \frac{1}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}. \end{aligned}$$

It follows that

$$PV_{\text{calls}} = \frac{\lambda_f\mu_{lo}P_{lo-fn} + \lambda_s\mu_{fe}P_{fe-fn}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}\left(1 + f\left(\frac{1}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}} + \frac{1 + \rho_e\left(\frac{1}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}\right)}{\lambda_f\mu_{hn} + \rho_e + r}\right)\right).$$

SA.2.2 Proof of Proposition 12

$$\begin{aligned}
\text{For Part 1:} \quad & 2(p_{fo-hn} - p_{fe-hn}) = v_{fo} - v_{fe} = 2g_{fe-fn} \geq 0, \\
& 2(p_{fe-hn} - p_{fe-fn}) = (v_{ho} - v_{hn}) - (v_{fo} - v_{fn}) = 2g_{fo-hn} \geq 0. \\
\text{For Part 2:} \quad & 2(p_{fo-hn} - p_{lo-hn}) = (v_{ho} - v_{hn} + v_{fo} - v_{fn}) - (v_{lo} - v_{ln} + v_{fo} - v_{fn}) \\
& \qquad = (v_{ho} - v_{hn}) - (v_{lo} - v_{ln}) = 2g_{lo-hn} \geq 0, \\
& 2(p_{lo-hn} - p_{lo-fn}) = (v_{ho} - v_{hn}) - (v_{fo} - v_{fn}) = 2g_{fo-hn} \geq 0.
\end{aligned}$$

SA.3 Various Estimates of the Time to Sell

A sale of a private firm consists of two major processes: the preparation and the listing-to-sale process. The preparation takes less time if a firm already has high-quality accounting and information systems, which is the case of PE-backed firms (Kaplan and Stromberg (2009)). The preparation for PE-backed firms takes an average of 2 months, while other firms need an average of 6 months (see the upper part of Table 6). The listing-to-sale process takes about 9 months for various selling agents (see the lower part of Table 6). We set the total time for selling a firm as 11 months for PE funds and 15 months for corporate investors.

Ave. Time Taken	**Source**
For preparations	
1-6 months	https://www.highrockpartners.com/how-long-does-it-take-to-sell-a-company/
12 months	https://www.businessinsider.com/11-stages-of-selling-a-company-2011-4
From listing to sale	
6-9 months	https://www.mabusinessbrokers.com/blog/how-long-does-it-take-to-sell-a-business
9 months	https://www.exitadviser.com/seller-status.aspx?id=long-does-take-sell
9 months	https://www.allbusiness.com/how-long-does-it-take-to-sell-a-business-2-6592268-1.html
12 months	https://www.businessinsider.com/11-stages-of-selling-a-company-2011-4
9 months	https://www.moorestephens.co.uk/msuk/moore-stephens-south/news/july-2017-(1) /how-long-does-it-take-to-sell-a-small-business
9 months	https://www.tvba.co.uk/article/how-long-does-it-take-to-sell-a-company
6-9 months	https://www.simonscottcmc.co.uk/blog/long-take-sell-business/
10 months	https://www.ibgbusiness.com/tips-sell-business-long-take-sell-business/
10 months	https://www.highrockpartners.com/how-long-does-it-take-to-sell-a-company/

Table 6: Estimated time to sell a firm