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Contracting with externalities and outside options

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Abstract

This paper proposes a model of multilateral contracting where players are engaged in two parallel interactions: they dynamically form coalitions and play a repeated normal form game with temporary and permanent decisions. We show that when outside options are independent of the actions of other players all Markov perfect equilibrium without coordination failures are efficient, regardless of externalities created by interim actions. Otherwise, in the presence of externalities on outside options, all Markov perfect equilibrium may be inefficient. This formulation encompasses many economic models, and we analyze the distribution of coalitional gains and the dynamics of coalition formation in four illustrative applications.

JEL classification: C71; C72; C78; D62

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1. Introduction

Following Rubinstein [18]'s study of two-player alternative offers bargaining, a number of models have been proposed to explain the formation of coalitions and multilateral agreements. These models typically rest on two polar assumptions on the exit decisions of players. Either coalitions are forced to leave the game after they are formed, or players continuously renegotiate multilateral agreements. In the first strand of the literature [1,2,12,13,16], the outcome of the bargaining process may be inefficient, as players choose to form smaller

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coalitions and leave the game before extracting all the coalitional surplus. [1] In the second strand of the literature [6,8,14] and the reversible game in [20], the outcome of negotiations is always efficient, and players end up forming the grand coalition. [2]

In this paper, we consider a coalitional bargaining game where players *endogenously* choose whether to exit, and ask the following question: When does the game result in efficient multilateral agreements? Two forerunning studies have provided conflicting answers to that question. Perry and Reny [15] construct a game in continuous time where players endogenously choose when to exit, and prove that the outcomes of this game coincide with the core, and hence are efficient. Seidmann and Winter [20] propose a discrete-time procedure where players choose when to exit (the "irreversible game'') and provide an example where players always choose to exit before the formation of the grand coalition. By considering a model closely related to Seidmann and Winter [20], but where proposers are chosen at random, we show that their result crucially depends on the fixed protocol of offers. Applying our model with random proposers to their example, we show that *there always exists an equilibrium where players form the grand coalition.*

The previous studies of games with endogenous exit, based on games in coalitional form, assume away the presence of externalities during negotiations. However, in reality, both the actions taken by players during the course of negotiations (their "inside options'') and the actions chosen when they leave the negotiating table (their "outside options'') are likely to affect the payoff of other players. Furthermore, the outcome of the negotiations between remaining players may in turn affect the payoff of exiting players. Hence, the definition of outside options is problematic. Can players be sure that their outside option is definite, or is it likely to be affected by the choice of other players? In this paper, we distinguish between these two cases, and consider both games with *pure outside options* (which are independent of the actions of other players) and games with *externalities on outside options*. The key result of our paper is that *there always exist an efficient equilibrium outcome in games with pure outside options, regardless of externalities on inside options (or interim payoffs during negotiations). However, games with externalities on outside options may exhibit only inefficient equilibria.*

Formally, we construct a model where players are engaged in two parallel interactions: they propose to form coalitions in order to extract gains from cooperation; and coalitions participate in a repeated normal form game, where they choose actions that may either be temporary or permanent. Any period is divided into two subperiods: a contracting phase, where one of the players is chosen at random to propose to form a coalition, and an action phase where all players simultaneously choose whether to exit. Because players make simultaneous action choices, coordination failures may arise, and we select equilibria which

[1] To see this point, consider the following example, inspired by Chatterjee et al. [2]. Let $n = 3$ and the gains from cooperation be represented by a coalitional function $v(S) = 0$ if $|S| = 1$, $v(S) = 3$ if $|S| = 2$ and $v(S) = 4$ when $|S| = 3$. As δ converges to 1, the outcome of the bargaining procedure where the grand coalition forms should result in equal sharing of the coalitional surplus among the symmetric players ($\frac{4}{3}$ for every player). But clearly, players then have an incentive to deviate forming an inefficient coalition of size 2, inducing a payoff of $\frac{3}{2}$ for each coalitional member. If this coalition must leave the negotiation after its formation, the additional surplus of 1 is lost.

[2] There is yet another strand of models of coalitional bargaining, which are not directly related to Rubinstein [18]'s game, allowing for example for multiple offers to be made [11], or based on games in effectivity form [7,10].

are immune to coordination failures, by imposing that players remain in the game with a probability greater than $\varepsilon > 0$. *In games with pure outside options, we show that as players become perfectly patient, equilibria without coordination failure exist and are efficient, as long as all players have opportunities to make offers*. The intuition underlying these results is easily grasped. Early exit results in an aggregate efficiency loss. In a game with pure outside options, players are able to capture this inefficiency loss and will never choose to leave before the grand coalition is formed. By staying in the game one more period, a player is guaranteed to obtain her outside option (which remains available because outside options are pure), and is able to capture the inefficiency loss by proposing to form the grand coalition when she is recognized to make an offer. Hence, early exit will never occur in equilibrium.

The previous result depends crucially on the fact that outside options are independent of the actions of the other players. We provide an example to show that, in games with externalities on outside options, *all equilibrium outcomes may lead to the inefficient formation of partial coalitions.* The three-player example we construct displays the following features: (i) once a two-player coalition has formed, players obtain a large payoff when they are the only ones exiting the game, and the unique equilibrium of the simultaneous action game is completely mixed, so that exit occurs with a positive probability, and (ii) at the initial stage where all players are singletons, players have an incentive to form a two-player coalition, as a partial agreement results in a large asymmetry between the two-player coalition and the remaining singleton. When both features are present (incentives to exit alone and large payoff to partial agreements), in all equilibria, players initially choose to form a two-player coalition, and to exit the game with a positive probability. These equilibria are obviously inefficient, as players exit before extracting all gains from cooperation.

In the second part of the paper, we dwelve deeper in the structure of equilibrium outcomes, by considering four illustrative applications of the model. The first application is an extension of two-player games with outside options to a multilateral context. We show that the equilibrium (without coordination failures) of the multilateral game reflects the "outside option principle": either outside options are binding and the player with the largest outside option receives her outside option, or they are not binding and the outcome of bargaining is unaffected by outside options. However, when outside options are binding, the equilibrium exhibits a novel property: the equilibrium payoff of all players depend on the *entire vector of outside options.* In our second application, we analyze the principal-agent models with externalities proposed by Segal [19], where a single principal contracts with several agents, and the contract imposes externalities on non-trading agents. We focus on the dynamics of coalition formation in a model with symmetric agents, and show that when trading between the principal and agents induces positive externalities on other agents, the principal and agents contracts to form the grand coalition in one step. When trading induces negative externalities, the grand coalition is formed in two steps. The principal first chooses to contract with a subset of agents in order to reduce the outside opportunities of the remaining agents.

The last two applications consider games where the payoff of exiting players depends on the actions chosen by remaining players. We first consider a three-player pure public good game, similar to the game studied by Ray and Vohra [17]. Countries negotiate over the reduction of pollution levels, and partial contracts result in positive externalities on the other countries. We show that the grand coalition does form in equilibrium. Depending on the value of partial agreements, a global agreement is either reached immediately or in two steps. Finally, we discuss a model of market entry with synergies, where firms may merge—and benefit from synergies—before entering the market. In this model, the merger of two firms induces negative externalities on the remaining firms. Assuming that the market can only support one firm, we exhibit a range of parameter values for which the game results in the inefficient formation of a partial agreement.

The rest of the paper is organized as follows. Section 2 presents the model and Section 3 contains preliminary results on characterization and existence of equilibria. In Section 4, we discuss efficiency of the bargaining outcomes, establish our main results and discuss the relation to the previous literature—notably Seidmann and Winter [20]. Section 5 discusses our four illustrative applications, and Section 6 concludes.

2. The model

We model the formation of coalitions among $N = \{1, 2, \ldots, n\}$ agents and bargaining as an infinite horizon game, with two distinct phases at every period. Any period starts with a *contracting phase*, where a player is chosen at random to propose forming a coalition (a non-empty subset of *N*). Coalitions once formed become the decision makers (the players). The set of players after contracting is denoted by $\mathcal{N}$, which is a coalition structure or a partition of the set *N* into disjoint coalitions (typical players in $\mathcal{N}$ are represented by labels *i, j*). In the next phase, the *action phase*, each existing coalition $i \in \mathcal{N}$ chooses simultaneously an action a_i from a finite action set $A_i = E_i \cup R_i$, which may be a permanent action E_i (in which case the coalition exits the game and thereafter chooses the same action in all future periods) or a temporary action R_i ($E_i \cap R_i = \emptyset$). In many applications, the two sets only consist of one element: a single action $r \in R_i$ interpreted as "*r*emaining in the game", and a single action, $e \in E_i$, interpreted as "*e*xiting the game". The action profile determines a flow payoff for all the players, representing the underlying economic opportunities.[3] The interplay between the contracting and action phases enables us to consider simultaneously issues of coalition formation, externalities and endogenous exit decisions.

Subgames are described by a state variable $s = (\mathcal{N}, \mathcal{E}, e)$ where $i \in \mathcal{N}$ are the current coalitions/players, $\mathcal{E} \subset \mathcal{N}$ denotes the coalitions who have exited, and $e = (e_i)_{i \in \mathcal{E}}$ are their exit actions. The players in $\mathcal{N} \backslash \mathcal{E}$ are "active" players (being able to form coalitions and choose actions) while players in $\mathcal{E}$ are "inactive" with actions fixed and not being able to form new coalitions. Hence, we interpret a permanent action as an irreversible investment made by the player, which is not subject to renegotiation. Time is discrete and runs as $t = 1, 2, \ldots, \infty$, and in each period there is:

Contracting phase: Say that the state is $s = (\mathcal{N}, \mathcal{E}, e)$. One of the active players $i \in \mathcal{N} \backslash \mathcal{E}$ is chosen at random to make an offer. The probability with which player *i* is chosen to make an offer, $q_i(s)$, is exogenously given and may depend on the state. An offer is a pair $(\mathcal{S}, t)$,

[3] This formulation includes as special cases the familiar games in coalitional function and partition function forms. In games in coalitional function, a coalition/player *i* receives her worth $v(i)$ upon exiting, and a zero payoff otherwise. In games in partition function form, players only receive a positive payoff after they have all exited. Player/coalition *i* then obtains the payoff $v(\mathcal{N}, i)$ corresponding to the coalition structure $\mathcal{N}$ formed at the end of the game.

where $\mathcal{S}$ is a subset of active players $\mathcal{N}\backslash\mathcal{E}$ and t a vector of transfers satisfying $\sum_{j\in\mathcal{S}} t_j = 0$. We interpret the transfer t_j as the net present value of a flow of payments, by which player i obtains control over the resources of player j. When player i has obtained control over the resources of all the players in $\mathcal{S}$, we identify the player with the new coalition.[4] A fixed protocol determines the order in which players in $\mathcal{S}\backslash i$ respond to the offer. At the end of the contracting phase, the state changes to $s^{\mathrm{c}} = (\mathcal{N}^{\mathrm{c}}, \mathcal{E}, e)$, if all of them accept the offer, where $\mathcal{N}^{\mathrm{c}} = (\mathcal{N}\backslash\mathcal{S}) \cup \{\cup_{j\in\mathcal{S}} j\}$, or, if one of the prospective members of $\mathcal{S}$ rejects the offer, no trade takes place and $s^{\mathrm{c}} = s$.

Action phase: At the action phase starting with arbitrary state $s = (\mathcal{N}, \mathcal{E}, e)$, all active players simultaneously choose an action a_i from the action set A_i. At the end of the action phase, the state changes to $s^{\mathrm{a}} = (\mathcal{N}, \mathcal{E}^{\mathrm{a}}, e^{\mathrm{a}})$, where $\mathcal{E}^{\mathrm{a}} = \{i \in \mathcal{N} : a_i \in E_i\}$ and $e^{\mathrm{a}} = (a_i)_{i\in\mathcal{E}^{\mathrm{a}}}$.

Flow payoffs are collected by all the players at the end of the action phase. If the *action profile* is a, flow payoffs accrue to all players in $\mathcal{N}$ according to utility functions $v_i(a)$, for all $a \in \times_{i\in\mathcal{N}} A_i$ and $i \in \mathcal{N}$. (Note that payoff functions are defined, for every coalition, in all coalition structures, and any action (permanent or temporary) taken by them. The payoff flow (or single-period payoff) is $\left(1-\delta\right) v_i(a)$ with present value $v_i(a)$, where $\delta \in (0, 1)$ is the common discount rate of all players. When the grand coalition forms, we let V denote the total payoff of the grand coalition, and assume that the grand coalition is efficient, i.e. for any state different from the grand coalition

$$V > \sum_{i\in\mathcal{N}} v_i(a) \text{ for any action profile } a.$$

We restrict our attention to *Markovian* strategies. Hence, at the contracting phase, a proposer's strategy only depends on the current state s, a respondent's strategy only depends on the current state s, the current offer she receives and the responses of preceding players. At the action phase, strategies only depend on the current state s. A *Markov perfect equilibrium* (MPE) is a Markovian strategy profile, where every player acts optimally at every contracting phase of the game, and all players play a Nash equilibrium at every action phase.

For a given Markovian strategy, let $\phi_i^1(s)$ denote the continuation value of the game for player i at the contracting phase at state s (before the choice of proposer), and $\phi_i^2(s)$ denote the continuation value of player i at the action phase of state s. Finally, we denote by $\Phi_i = \phi_i^1(s_0)$ the value of the game for player i at the initial contracting stage s_0, when no coalition has formed.

3. Characterization and existence

We now provide a complete characterization of equilibrium. At any state s, we first compute the mixed strategy equilibria at the action phase, taking continuation values $\phi_i^1(.)$ as given.

[4] As active players are identified with the coalitions they control, we do not keep track of the identity of the agents who control coalitions. The new coalition is $\cup_{j\in\mathcal{S}} j$, and we sometimes, as an abuse of notation, just refer to it as $\mathcal{S}$.

The action stage starting with any state s can be represented by a standard game in strategic form $\Gamma(s) = (\mathcal{N}(s), \{A_i(s), u_i(s, \cdot)\}_{i \in \mathcal{N}(s)})$ with a set of players $\mathcal{N}(s)$, with action space $A_i(s)$, and payoff function $u_i(s, \cdot)$ corresponding to the continuation value of the game at the action phase. For any strategy profile a, players receive a flow payoff of $(1-\delta)v_i(a)$ in the current period, and the game moves to the contracting phase in the next period with state s^{a}, so that the payoff function is $u_i(s, a) = \delta\phi_i^1(s^{\mathrm{a}}) + (1-\delta)v_i(a)$. In a Markov perfect equilibrium, the strategy profile $\sigma^2(s)$ of active players at the action phase of state s must be a Nash equilibrium of game $\Gamma(s)$.

We now suppose that the continuation values at the action phases $\phi_i^2(.)$ are fixed, and compute the optimal behavior of proposers and respondents at the contracting stage of state s. If any player j rejects the contract offered, the game moves to the action phase of state s and player j receives a payoff $\phi_j^2(s)$. The minimal offer that player j accepts is $\phi_j^2(s)$, and thus any proposer optimally offers $t_j = \phi_j^2(s)$. Given an offer $(\mathcal{S}, t)$, the state obtained when the offer is accepted is s^{c} with coalition structure $(\mathcal{N} \backslash \mathcal{S}) \cup \{\cup_{j \in \mathcal{S}} j\}$. The proposer i thus selects the coalition $\mathcal{S}$ which maximizes his value $\phi_{\mathcal{S}}^2(s^{\mathrm{c}}) - \sum_{j \neq i, j \in \mathcal{S}} \phi_j^2(s)$. If different coalitions result in the same maximal payoff, player i may randomize across the coalitions formed, and we let $\sigma^1(s)$ denote the probability distribution over all subsets $\mathcal{S} \subset \mathcal{N}(s)$, $i \in \mathcal{S}$ that player i may form at state s. We have now completely characterized the Markov perfect equilibrium of the game at state s for fixed continuation values $\phi_i^1(.)$ and $\phi_i^2(.)$.

Therefore, at the action stage, the continuation value is obtained as the equilibrium payoff of a strategic game played by the active players. At the contracting stage, the continuation value is obtained as an expected value, considering three possible situations. With probability $q_i(s)$, player i is called to make an offer, and proposes to form any optimal coalition $\mathcal{S}$, obtaining a value $\phi_{\mathcal{S}}^2(s^{\mathrm{c}}) - \sum_{j \neq i, j \in \mathcal{S}} \phi_j^2(s)$. With probability $q_j(s)$, another player j is recognized to make an offer, and proposes a coalition $\mathcal{S}$. Either player i belongs to coalition $\mathcal{S}$, and she receives the offer $\phi_i^2(s)$, or otherwise she receives her continuation value $\phi_j^2(s^{\mathrm{c}})$ at the action phase (for details see Lemma A.1 in the appendix).

We now prove the existence Markov perfect equilibrium, using Kakutani's fixed point theorem.

Proposition 1. *The coalitional bargaining game admits a Markov perfect equilibrium.*

The characterization results of this section will be used to construct Markov perfect equilibria in the examples and applications of the next sections.

4. Efficiency

We start the analysis of the coalitional bargaining game by considering games with *pure outside options*, where a player's payoff after exit is independent of the actions of the other players. Formally:

Definition 1. The underlying game v is a game *with pure outside options* if and only if players' exit payoffs are independent of the actions and coalition structures chosen by the remaining players i.e. for all c.s. $\mathcal{N}, \mathcal{N}'$ such that $i \in \mathcal{N}$ and $i \in \mathcal{N}'$, and all $e_i \in E_i$, $a_{-i} = (a_j)_{j \in \mathcal{N} \setminus i}$ and $a'_{-i} \in (a'_j)_{j \in \mathcal{N}' \setminus i}$, the payoffs $v_i(a_{-i}, e_i) = v_i(a'_{-i}, e_i)$ are equal.

The class of games with pure outside option naturally includes games in coalitional form, where exiting coalition/players receive their coalitional worth. But it also encompasses a wider range of situations. For example, the condition above does not put any restriction on the interim payoff received by players in the course of negotiations (the inside options). Furthermore, in games with pure outside options, the payoff of active players may depend on the exit choice of other players. This is the case, for example, in our second illustrative application, where a principal contracts with a set of agents, and agents' payoffs depend on the exit decision of the principal.

We show that games with pure outside options always admit Markov perfect equilibria where contracting results in an approximately Pareto efficient outcome. Formally, we show that for any $\xi > 0$, $\sum \Phi_i > V - \xi$ for all $\delta \geqslant \delta(\xi)$, where we recall that Φ_i is player i value at the beginning of the game. We also show that when externalities on players' exit payoffs are negligible, the coalitional bargaining procedure results in approximately efficient contracting outcomes.

Conversely, in games where players' outside options depend on the behavior of other players, *all Markov perfect equilibria may be inefficient* (i.e., there exists a $\xi > 0$ such that $\sum \Phi_i < V - \xi$ in all equilibria and for all $\delta \geqslant \delta(\xi)$). We exhibit a three-player game where, in all equilibria, there exists an action stage where the probability of players exiting is bounded away from zero and, at the initial contracting stage, players have an incentive to reach the action stage where exit occurs with positive probability. We thus note that the critical element for the existence of efficient equilibria is the presence of externalities on *outside options*. Inside options affect the interim payoffs of the players, and hence the dynamics of coalition formation and the distribution of gains from cooperation, but play no role in the theorem and examples of this Section.

4.1. *Efficiency in games with pure outside options*

Inefficient outcomes arise in the bargaining game when players exit before the formation of the grand coalition.[5] At the action phase, as players' exit choices are simultaneous, coordination failures may lead to inefficient equilibria. Consider the following simple two-player example.

Example 1. There are two symmetric players. Each player has access to two actions: exiting (e) and remaining (r). The payoff matrix at the initial stage is:

	r	e
r	$(0, 0)$	$(0, a)$
e	$(a, 0)$	(a, a)

and the total value of coalition $\{12\}$ is $V > 2a$.

[5] As we consider situations where players' discount factors converge to 1, inefficiencies due to delay in the formation of the grand coalition become negligible.

Let ϕ^1 denote the common continuation value (at a symmetric equilibrium) of both players at the initial state. In the action phase, the game Γ played by the two players has a payoff matrix given by

	r	e
r	$(\delta\phi^1, \delta\phi^1)$	$(\delta a, a)$
e	$(a, \delta a)$	(a, a)

It is easy to see that this game admits an inefficient pure strategy equilibrium where both players simultaneously exit. Notice that this is not the only equilibrium of the game for high values of the discount factor δ. If $\delta > 2a/V$, the game also admits an efficient symmetric pure strategic equilibrium where both players remain, and $\phi^1 = V/2$.

Example 1 illustrates in the simplest way the fact that coordination failures are unavoidable in a model where exit decisions are simultaneous. However, when the discount factor δ is high enough, the game also admits efficient equilibria where both players remain in the game with positive probability. Coordination failures at the action phase can easily result in inefficient equilibria at the contracting phase as well. [6]

Hence, in our search for efficient outcomes, we focus on equilibria without coordination failures at the action phase. We define an ε–R strategy profile as one where all players put a probability at least equal to ε of remaining in the game at every action stage. Formally, we define:

Definition 2. For any $\varepsilon \in (0, 1)$, an ε–R strategy profile is a strategy profile where all players play a temporary action with probability at least equal to ε at any action phase, i.e. $\sum_{r_i \in R_i} \sigma_i^2(s)(r_i) \geqslant \varepsilon$ for all players i in any state s. An ε–R Markov perfect equilibrium is an MPE in ε–R strategies.

The class of ε–R equilibria defines a refinement of Markov perfect equilibria. This refinement is of interest because we show in Proposition 2 that when the discount factor is high enough, ε–R equilibria always exist in games with pure outside options. Moreover, in Proposition 3 we establish that all ε–R equilibria are Pareto efficient in games with pure outside options. [7] Thus ε–R equilibria defines a non-empty class of efficient equilibria which is immune to coordination failures. [8]

[6] To see this, add a third player to the game of Example 1, and suppose that a coalition of two players obtains a payoff W when it exits, with $W > 3a$, $V > W + a$ and $W > 3V/4$. In this game, for $\delta \geqslant \max\{3a/W, 3(V - W)/W\}$, there exists an equilibrium where (i) players initially contract to form a two player coalition, (ii) after a two player coalition is formed, all players simultaneously exit, and (iii) at the initial action phase when all three players are present, all players remain in the game. This equilibrium results in the formation of the inefficient two-player coalition.

[7] There may not be ε–R equilibria, even for pure outside option games, for low values of the discount factor. For example, the game analyzed in Example 1 does not admit any ε–R equilibrium for low values of δ.

[8] Preplay communication may be another reason for players avoiding coordination failures. Cooper et al. [3] find that preplay communication can be quite effective in overcoming coordination failures.

Proposition 2. *For any underlying game* v *with pure outside options and any* $1 > \varepsilon > 0$, *there exists* $\delta(\varepsilon)$ *such that for all* $\delta \geqslant \delta(\varepsilon)$, *the bargaining game admits an* ε*–*R *Markov perfect equilibrium*.

Proposition 2 is a key result of our analysis. The intuition underlying the result follows. Consider the constrained coalitional bargaining game where all players are required to play ε–R strategies. By standard arguments, this game admits a Markov perfect equilibrium. We show that any equilibrium of the ε-constrained game is also an equilibrium of the original (unconstrained) coalitional bargaining game. Suppose by contradiction that the constraint were binding for one player. Exiting would then be a strict best response at some action phase, and the game would end up with early exit with probability at least $1 - \varepsilon > 0$. This results in an aggregate inefficiency for all the players. However, if the exiting player had instead chosen to stay, her short-term losses would be negligible (as δ is close to one), and her payoff in the next period would be strictly greater than her outside option. This last statement is true for two reasons. On the one hand, if the player were not chosen to propose next period, she would always be able to get her outside option (we use here the assumption that outside options are pure, so a player cannot be prevented from getting the same outside option next period). On the other hand, if the player were to propose in the next contracting phase, she would be able to extract some of the aggregate efficiency loss, making her payoff strictly greater than her outside option. Hence remaining in the game must be a better response than exiting at the action phase, and any equilibrium of the ε-constrained game is also an equilibrium of the original game.

Note that this result depends crucially on the fact that outside options are pure. In games where outside options depend on the actions of the other players, a player may not be able to get the same payoff if she exits later in the game. Hence, as we will see in the example below, Proposition 2 does not extend to games where players' payoff upon exit depends on the behavior of other players.

Interestingly, taking ε converging to 1, Proposition 2 shows that, as δ converges to one, the bargaining game admits a Markov perfect equilibrium where the probability of exit at any action phase converges to zero. In fact, this result can be strengthened, as we can show that, as δ converges to 1, in *all* ε–R equilibria, the probability of exit of all the players at any action phase converges to zero. Furthermore, as the probability of remaining in the negotiations converges to one, the grand coalition will ultimately be formed in equilibrium, and the bargaining procedure results in an efficient outcome.

Proposition 3. *For any game with pure outside options, as* δ *converges to one, the probability of exit in all* ε*–*R *Markov perfect equilibrium converges to zero for all the players at all states, and the equilibrium outcome converges to an efficient outcome* (*i.e., for any* $\xi > 0$ *there exists a* $\delta(\xi)$ *such that* $\sum \Phi_i > V - \xi$ *for all* $\delta \geqslant \delta(\xi)$).

Propositions 2 and 3 thus show that, for games with pure outside options, as δ converges to 1, there exists a Markov perfect equilibrium resulting in an efficient outcome. We now show that these results are robust to the introduction of small external effects. When outside options are approximately pure, we can still show that ε–R Markov perfect equilibria with no coordination failure exist for high discount factors. For δ close enough to one, this guarantees that the game admits a Markov perfect equilibrium where players exit at the action phase with a probability close to zero, and the outcome of the bargaining procedure is approximately efficient. Formally:

Proposition 4. *Let* $\max_{i,e_i,a_{-i},a'_{-i}} |v(a_{-i}, e_i) - v(a'_{-i}, e_i)| = \eta(v)$. *For any* $1 > \varepsilon > 0$ *there exists* $\eta(\varepsilon) > 0$ *and* $\delta(\varepsilon)$ *such that for all* $\delta \geqslant \delta(\varepsilon)$, *and all games satisfying* $\eta(v) \leqslant \eta(\varepsilon)$, *an* ε*–R Markov perfect equilibrium exists. Furthermore, for any* $\xi > 0$ *there exits* $\eta(\varepsilon, \xi) > 0$ *and* $\delta(\varepsilon, \xi)$ *such that all* ε*–R Markov perfect equilibria result in an outcome which is approximately Pareto efficient*, i.e. $\sum \Phi_i \geqslant V - \xi$ *for all* $\delta \geqslant \delta(\varepsilon, \xi)$ *and all games satisfying* $\eta(v) \leqslant \eta(\varepsilon, \xi)$.

4.2. Externalities on outside options and inefficiencies

We now provide an example to show that when players' outside options depend on the behavior of other players *all Markov perfect equilibria may be inefficient*. As all two-player games are efficient at the contracting stage,[9] this example involves three players.

Example 2. There are three symmetric players with two actions r and e. At the initial stage (state s_1), when the players are singletons, payoff matrices are given by

	r	e
r	(0,0,0)	(0,2,0)
e	(2,0,0)	$(-1,-1,0)$

r

	r	e
r	(0,0,2)	$(0,-1,-1)$
e	$(-1,0,-1)$	$(-1,-1,-1)$

e

where player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. If two players form a coalition (state s_2), the payoff matrices are given by

	r	e
r	(0, 0)	(0, 9)
e	(2, 0)	$(-1,-1)$

where player 1 (row) is the singleton player and player 2 (column) the two-player coalition. If the grand coalition forms, the total payoff is $V = 10$. All players have equal proposer probabilities and they are very patient (δ close to one).

In this example, after the exit decision of any player, the dominant strategy of remaining players is to play r and they have no incentives to form any further coalitions. Hence, as soon as one player has exited, equilibrium behavior is easily characterized, and we focus on those states where all players are still present in the game. Our goal is to show that there are no efficient equilibria. We prove this result in two steps, first analyzing the subgame in which a two-player coalition has formed (state s_2) and then the initial contracting stage of the three-player game (state s_1).

[9] In two-player games, the sum of continuation values at the action stage is $\phi_1^2 + \phi_2^2 < V$. Thus, at the contracting phase, a proposer i always offer ϕ_{-i}^2 to form the grand coalition since its payoff is $V - \phi_{-i}^2 > \phi_i^2$, and the respondent accepts the offer.

(i) Consider state s_2 where a two-player coalition has formed. The action stage admits a unique, inefficient, equilibrium where both players employ mixed strategies.[10] The equilibrium utilities and offers are given by the solution to the nonlinear system of equations:

$$\phi_1^1 = \frac{10 - \phi_2^2 + \phi_1^2}{2},$$
$$\phi_2^1 = \frac{10 - \phi_1^2 + \phi_2^2}{2},$$
$$\phi_1^2 = \delta\sigma_2\phi_1^1 = 2\sigma_2 - (1-\sigma_2),$$
$$\phi_2^2 = \delta\sigma_1\phi_2^1 = 9\sigma_1 - (1-\sigma_1).$$

This system of equations admits a unique solution (continuous in δ) which converges to $\phi_1^2 = 0.54$ and $\phi_2^2 = 8.42$, $\sigma_1 = 0.94$, and $\sigma_2 = 0.51$, as δ converges to 1. Note that the value of the two-player coalition is $\phi_2^2 > 8$ and that there is an efficiency loss as $\sum_{i=1}^2 \phi_i^2 < 9 < V = 10$.

(ii) Now consider state s_1. We first establish that one of the players must have a continuation value at the action stage satisfying $\phi_i^2(s_1) > 2$. Suppose by contradiction that $\phi_i^2(s_1) \leqslant 2$ for all players i. At the contracting stage, by proposing to form a two-player coalition, a player will be able to obtain a payoff $\phi_2^2(s_2) - \phi_i^2(s_1) > 8 - 2 = 6$. Since every player is recognized to make an offer with probability $1/3$, we conclude that the continuation value at the contracting stage of state s_1, $\phi_i^1(s_1)$ is bounded below by $1/3(\phi_2^2(s_2) - \phi_i^2(s_1)) > 2$. But this implies that, as δ converges to 1, in equilibrium, all players remain at the action stage of state s. If at least one of the other players exits, remaining is clearly a dominant strategy. If the two other players remain, remaining must be a best response, because for δ close enough to 1, $\delta\phi_i^1(s_1) > 2$ and the outside option has value 2. As all players remain at the action stage, we thus have $\phi_i^2(s_1) = \delta\phi_i^1(s_1)$ for all players. This last statement results in a contradiction because we assumed $\phi_i^2(s_1) \leqslant 2$ and we showed $\phi_i^1(s_1) > 2$.

Now, let i be a player with continuation value $\phi_i^2(s_1) > 2$. Then, in any equilibrium, players j and k must propose to form the two-player coalition $\{j, k\}$ at the contracting stage of state s_1. This statement results from the fact that the marginal benefit of including player i in the contract $(V - \phi_2^2(s_2))$ is smaller than the minimal transfer that player i will accept to join the coalition, $\phi_i^2(s_1)$. Hence, at least in two thirds of the cases (when players j or k are chosen to make the initial offer), the equilibrium of the game results in the inefficient formation of a two-player coalition. This clearly implies that the initial values of the game satisfy $\sum \Phi_i < V$.

Example 2 is clearly robust to small perturbations in the payoff matrices. Furthermore, in the next section, we provide an economic application (market entry with synergies) giving

[10] Clearly simultaneous exit cannot be a Nash equilibrium of the game, and there cannot be equilibria where one player exits and the other one randomizes between staying and exiting. If both players choose to remain, their continuation value will be $\delta\phi_i^1$ and as $\phi_1^1 + \phi_2^1 \leqslant 10$, one of the players has an incentive to take her outside option. The strategy profiles (e, r) and (r, e) cannot be equilibria either. For example, if player 1 exits and player 2 remains, $\phi_1^2 = 2$ and $\phi_2^2 = 0$. Player 1 then has an incentive to remain, as she would obtain either her outside option (if the other player proposes next period) or the entire surplus V (if she proposes next period). A similar argument shows that there is no equilibrium where one player remains and the other one randomizes between staying and exiting.

rise to a payoff structure which is equivalent to the payoff structure of Example 2. In the absence of a complete characterization of games resulting in inefficient outcomes, we note that the example displays two crucial properties:

(i) there exists an action stage where in equilibrium, the probability of exit of all players is bounded away from zero (for all $\delta \geqslant \bar{\delta}$);
(ii) at the initial contracting stage, players have an incentive to reach the action stage where exit occurs with positive probability.

Property (i) highlights the difference between the case of pure outside options (where there always exist an equilibrium with all players remaining for δ close to 1) and general games. In our example, the outside option of a player crucially depends on the behavior of the other player. By exiting alone, both players obtain positive payoffs (of 2 or 9), but the outside options become negative when both players exit simultaneously. Hence, a player may choose to exit today, because by waiting one period, she might be unable to retain the same outside option. In games with two players and two actions, it can be checked that the structure of our example (a "game of chicken'' where (e, e) gives lower payoffs than (r, r), and the sum of the maximal outside options of the two players is greater than V) is the only structure for which all equilibria are inefficient.

Property (ii) can only be satisfied if the payoff of a two-player coalition is large with respect to the value of the grand coalition. A rapid look at the equations defining equilibrium at the action stage shows that the most favorable condition for ϕ_2^2 to be high is when the outside option of the two-player coalition is high and the outside option of the singleton is low. (In our example, the outside option of the two player coalition (9) is much higher than the outside option of the singleton player (2)). In other words, the formation of a two-player coalition must result in a large increase in the value of the outside option.

4.3. Relation to the literature

We contrast our analysis with the previous results of Seidmann and Winter [20] and Perry and Reny [15]. The extensive form considered by Seidmann and Winter [20] differs from ours in two important respects. First, at the contracting phase, their game uses a fixed protocol. After any offer is accepted, the proposer is chosen according to a fixed rule, and upon rejection of an offer, the next proposer is the player who rejected the offer. Second, players are individuals rather than coalitions, and in particular, any individual player can unilaterally choose to implement the contract and force the coalition to exit. [11] In their analysis of the irreversible game, Seidmann and Winter [20] only study three-player examples, and their objective is to "demonstrate by example that [...] the game may possess solutions in which the grand coalition does not form when players are patient. Indeed, every solution may be partial'' [20, p. 807].

[11] This leads in their model to inefficient equilibria due to a coordination failure inside coalitions. As any player can unilaterally choose to exit, there always exists an equilibrium where coalitions exit at every stage [20, Lemma (i), p. 808]. This type of coordination failure is of course reminiscent of the coordination failure across coalitions exhibited in our paper (see Example 1).

Interestingly, Seidmann and Winter [20] construct an example where *all* equilibria result in the formation of partial coalitions [20, Example 5, p. 809]. [12] As this example seemingly contradicts our results (that all games with pure outside options admit efficient equilibria when players become perfectly patient), we examine it in detail.

Example 3 (*Seidmann and Winter [20]*). $n = 3$. There are two actions, e and r. Any player exiting alone receives 0. When bilateral coalitions exit, they receive $v \in (2/3, 1)$, and when three players exit, they receive $V = 1$.

In their example, Seidmann and Winter [20] suppose that, after a two-player coalition has formed, the next proposer is always the outsider. A simple heuristic argument shows why equilibrium must result in early exit. Following the formation of the coalition, the game is an alternating-offers bargaining game among three players, where the two members of the coalition have outside options v_1 and v_2 satisfying $v_1 + v_2 = v > 2/3$. If the players do not exercise their outside options, then the initial proposer (the outsider) receives a payoff greater than $1/3$, and hence one of the two insiders has an incentive to take her outside option. If one of the players exercises her outside option, then she receives $\delta v_i < v_i$ and hence, should have exited immediately after the formation of the coalition.

Why do we obtain a different result in our game with random proposers? We suppose that all players have a positive probability to make an offer at every stage. Hence, following the formation of the two-player coalition, any insider has a positive probability to make the offer, and capture some of the additional surplus. However small the probability that this player proposes, there always exist a high enough discount factor for which the player prefers to wait one more period. Hence, there exists an equilibrium where the insiders do not exit, and the grand coalition is formed.

The parallel between our model and Perry and Reny [15]'s analysis is harder to draw. Perry and Reny [15] consider a game in continuous time, with no rigid sequence of contracting and action phases. Allowing players to make multiple rounds of offers before actions could indeed eliminate inefficiencies in games with externalities on outside options. In our Example 2, if players could make another offer following the formation of the two-player coalition, the equilibrium would result in the formation of the grand coalition. (At the contracting phase following the formation of the two-player coalition, there is in fact an equilibrium where all players make acceptable offers.) However, allowing players to make multiple rounds of offers before taking decisions would make the model closer to a game with continuous renegotiation, and decrease the role of outside options. If outside options are an important feature of the negotiations, and players cannot continuously delay their decisions, the type of externalities illustrated by Example 2 can always result in early exit.

[12] Seidmann and Winter [20] note however that this inefficiency result is only obtained for some protocol, where the first player to make an offer after a two-player coalition is formed is the outsider. If instead one of the coalition members were chosen to make the offer, there would exist an efficient equilibrium [20, Example 4, p. 810].

5. Applications

In this Section, we develop four applications of the model to economic problems. We study the Markov perfect equilibria of the bargaining game, and we analyze the distribution of the surplus induced by the equilibria and the dynamics of coalition formation.

5.1. Multilateral bargaining with outside options

Bilateral bargaining with outside options is a problem that has been widely studied (e.g., [21,22]) and has been applied to a variety of economic problems (labor negotiations, marriage, contract theory, etc.). Yet little is known about its natural extension to an arbitrary number of players. We develop this extension in this section and show that the equilibrium exhibits some novel properties.

In the multilateral bargaining game with outside options there are n players who can collectively achieve a surplus V when the grand coalition forms. Each player is characterized by an outside option v_i for all $i = 1, \dots, n$. At the action stage each player can choose between remaining or exiting. In the appendix, we obtain the following closed-form solution for the ε–R Markov perfect equilibrium as δ converges to one.

Proposition 5. *The multilateral bargaining with outside options has the following Markov perfect equilibrium outcome*:

(i) *If* $\overline{v} \geqslant \frac{1}{n} V$, *where* $\overline{v} = \max_{i=1,\dots,n} v_i$ *is the largest outside option, the equilibrium payoffs converge to*

$$\phi_i = v_i + \frac{(\overline{v} - v_i)}{\sum_{j=1}^{n} (\overline{v} - v_j)} \left(V - \sum_{j=1}^{n} v_j \right),$$

for all $i \in N$ *and only the player with largest outside option opts out*; *the opt-out probability* σ *satisfies*

$$\lim_{\delta \to 1} \frac{\sigma}{(1 - \delta)} = \frac{n\overline{v} - V}{V - \sum_{j=1}^{n} v_j}.$$

(ii) *If* $\overline{v} \leqslant \frac{1}{n} V$ *the equilibrium payoffs converges to* $\phi_i = \frac{1}{n} V$, *for all* $i \in N$ *and no player opts out at the action stage*.

This equilibrium is a generalization of the equilibrium of Example 1, when players remain in the game with positive probability. It reflects the “outside options principle”: either outside options are binding and the player with the largest outside option receives her outside option, or they are not binding and the outcome of bargaining is unaffected by the outside options (see [22]).

However, the equilibrium exhibits a novel property. The equilibrium payoff of all the players depend on the *entire* vector of outside options, including the smallest outside options. This result can easily be interpreted. In equilibrium, the player with the largest outside option randomizes between exiting and staying. Hence, if a player rejects the offer at the contracting stage, every player will obtain her outside option with positive probability. The equilibrium offer to any player i is thus a function of the entire vector of outside options, and is increasing in v_i.

The model puts forward testable empirical predictions that could be explored in experimental studies. The model has other comparative statics implications (besides the one that player's payoff are increasing on their outside option). It also predicts that an increase in the highest outside option increases the sensitivity of a player's payoff with respect to her own outside option, $\frac{\partial^2 \phi_i}{\partial \overline{v} \partial v_i} \geqslant 0$. Increasing the largest outside option $\overline{v}$, increases the probability of opting out, and the bargaining outcome becomes more sensitive to the outside options of all the players.

5.2. Contracting with externalities

Segal [19] analyzes a contracting model with externalities, in which a principal contracts with several agents. Players' utilities depend on the trades (actions) chosen by the principal and the agents he has contracted with, so that agents excluded from the contract may suffer (or benefit from) externalities. Segal [19] shows that this general structure encompasses a number of specific models, ranging from models in industrial organization (vertical contracting, exclusive dealing, network externalities, mergers to monopoly), to models in finance (debt restructuring, takeovers) or public economics (the provision of public goods and bads). Our goal here is to analyze this principal–agent problem in a dynamic setting (Segal's model is static) and explore the role of outside options, or irreversibility of trades, in the allocation of gains and the efficiency of the outcome.

Specifically, we recast Segal's principal–agent game in our model as follows. At any contracting phase, the principal proposes to form a coalition with a set S of agents. (As in [19], we suppose that the principal has all the bargaining power, and that agents cannot contract among themselves.) Hence, coalitions necessarily include the principal. At any action phase, active agents have a no-trade decision (remain) and the coalition including the principal and all agents with whom she has contracted may also choose to trade (which is an irreversible decision).

Formally, trade among the principal and agent i is described by the action $a_i \in [0, \overline{a}]$, where $i = 1, \ldots, n$. Externalities among agent's actions are captured by the following utility structure. Any agent i trading with the principal receives a payoff $a_i \alpha(A) + \beta(A)$, where A denotes aggregate trade, $A = \sum_{i=1}^{n} a_i$. If an agent does not trade with the principal, she chooses the no-trade action $a_i = 0$, and obtains a payoff $\beta(A)$. The principal's payoff is given by $F(A)$. [13] Without loss of generality, all players no-trade payoffs are normalized to zero (i.e., $F(0) = \beta(0) = 0$).

Principal agent models can be divided into two broad categories, according to the sign of the externalities that traders impose on nontraders.

[13] When agents are identical, this utility specification is equivalent to the linearity condition (condition L) proposed by Segal [19, p. 341]. Segal [19] shows that this condition is satisfied in a variety of economic models.

Definition 3. Externalities on non-traders are positive (negative) if their payoff $\beta(A)$ is increasing (decreasing) in the aggregate trade $A = \sum a_i$.

When coalition S trades (or exits) it chooses the trade A_S that maximizes its payoff, i.e. $v(S) = \max_A F(A) + A\alpha(A) + |S|\beta(A)$; the payoff received by agents outside coalition S when there is exit is thus $v_i(S) = \beta(A_S)$.

Interestingly, despite the existence of externalities, the underlying game is a pure outside option game because the aggregate payoff of principal and agents who are contracting, $v(S)$, does not depend on actions of other no-trading agents. We now characterize a Markov perfect equilibrium for δ converging to one. When externalities on nontraders are positive, it is easy to see that the multilateral bargaining procedure immediately reaches an efficient agreement. By offering to form the grand coalition, and offering to each agent his minimal payoff (the no-trade value), the principal can extract the entire surplus. As the principal has all the bargaining power and agents' outside options are minimized with no trade, these offers constitute a subgame perfect equilibrium. [14]

When externalities on nontraders are negative, the structure of equilibrium is more complex. First notice that for all coalitions $S \subset T$, $S \neq \emptyset$

$$\begin{aligned} v(S) &= F(A_S) + A_S\alpha(A_S) + |S|\beta(A_S) \\ &\geqslant F(A_T) + A_N\alpha(A_T) + |S|\beta(A_T) \\ &\geqslant F(A_T) + A_T\alpha(A_T) + |T|\beta(A_T) = v(T), \end{aligned}$$

where the first inequality is due to the fact that A_S is the optimal trade of a coalition S, and the second inequality is due to the fact that externalities are negative (because $\beta(0) = 0$, $\beta(A_T) \leqslant 0$). [15] Hence, when externalities are negative, the principal obtains a higher payoff in a subcoalition S than in the grand coalition and her exit option $v(S)$ is greater than the total surplus V.

Therefore, after a coalition S is formed, it will require a positive transfer from all remaining agents to form the grand coalition. The coalition receives a transfer $\frac{v(S)-V}{|N|-|S|}$ per agent. [16] Note that this transfer is an average of what the remaining agents get if the coalition trades, $v_i(S)$, and what they get if the coalition does not trade, 0 (the weights depend on the probabilities that the coalition chooses to exit or stay at the action stage).

The coalition that forms in the first step is the one that can extract the highest transfer from agents after formed,

$$v^* = \max_S \frac{v(S) - V}{|N| - |S|}.$$

In the appendix, we construct a Markov perfect equilibrium where the principal is able to extract a transfer v^* from all agents, and the grand coalition is formed in two steps. Summarizing our findings we have:

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[14] Segal [19, p. 368] also notes that in games with positive externalities, efficient outcomes can easily be reached by having the principal make an offer conditional on unanimous acceptance.

[15] The normalization, $\beta(0) = 0$, implies that in games with positive (negative) externalities $\beta(A) > 0$ ($\beta(A) < 0$), whenever $A > 0$.

[16] The transfer made by each agent is the solution of $v(S) + x(|N| - |S|) = V$.

Proposition 6. *The principal agent problem has the following efficient Markov perfect equilibrium outcome*:

(i) *Positive externalities*: *the principal offers to form the grand coalition offering to each agent his minimal payoff* (*the no-trade value*); *all agents get the no-trade value and the principal extracts the entire surplus*;

(ii) *Negative externalities*: *the principal proposes to form a random coalition that maximizes* $\frac{v(S)-V}{|N|-|S|}$, *asking* v^* *for each agent receiving the offer. Then, the principal forms the grand coalition with the remaining agents, also asking* v^* *per agent*, *and when at the action stage exits the game with a positive probability*, *converging to* 0 *as* δ *converges to one*.

Hence, when externalities are negative, the principal has an incentive to contract with the agents in two steps. In the first step, the principal forms a coalition which generates maximum negative external effects on the remaining players. In the second step, the principal uses his credible outside option to extract high transfers from the agents (see also [5]). [17] On the other hand, in the positive externality case, contracting takes place in only one step.

5.3. Public good provision

Ray and Vohra [17] analyze the formation of coalitions providing a pure public good. The leading illustration of their model is the formation of groups of countries deciding on abatement levels in international negotiations over transboundary pollution. They assume that each agent has a utility given by $v = Z - c(z)$, where Z is the total amount of public goods provided and z the quantity produced by the agent, with $c(.)$ strictly increasing, strictly convex and $c'(0) = 0$. When a coalition S exits, it chooses its level of public good Z_S in order to maximize

$$Z_S - c\left(\frac{Z_S}{|S|}\right).$$

We restrict our analysis to a symmetric three-player game. Suppose that every player only has access to two strategies: remaining (r), resulting in a zero contribution to the public good, and exiting (e) where she contributes her optimal level of public good. As $c'(0) = 0$, every coalition will ultimately choose to exit and provide the public good. Hence, as δ converges to 1, the only relevant payoffs are the payoffs obtained by the three players when they exit as singletons (denoted a), the payoffs obtained when a two-player coalition and a singleton exit (denoted b for the two-player coalition and c for the singleton), and the total payoff V obtained by the grand coalition when it exits. Furthermore, as the optimal level of public good for coalition S is independent of the choices of the other players, and the cost function c is strictly convex, the game is strictly superadditive, i.e. $b > 2a$ and $V > b + c$.

It is easy to check that the ε–R equilibria of this game result in the formation of the grand coalition. Once a two-player coalition has formed, the game becomes equivalent to an asymmetric version of Example 1, and in an ε–R equilibrium either both players remain

[17] Genicot and Ray [5] also consider the contracting problem among the principal and several agents (the negative externality case) using a different dynamic framework than ours. They also find that contracting will occur in several steps.

in the game (and obtain $V/2$ each), or the large player randomizes between exiting and staying, and obtains her outside option b (when $b \geqslant V/2$). Hence, either the grand coalition is formed immediately, or a two-player coalition forms, and negotiates with the remaining player to form the grand coalition. One can check that it is optimal to form the grand coalition immediately when the outside value of a partial coalition b is not too high; if this outside option is high, the grand coalition is formed in two steps, and members of the two-player coalition obtain their outside option b. Formally, we obtain the following proposition:

Proposition 7. *In the three-player public good application, there exists a unique ε–R Markov perfect equilibrium, as δ converges to* 1. *This equilibrium give rise to the formation of the grand coalition. If $b \leqslant 2V/3$, the grand coalition is formed immediately*; *if $b > 2V/3$, the grand coalition is formed in two steps*.

It is instructive to contrast the result of Proposition 7 with the analysis of Ray and Vohra [17], who use a different model of coalitional bargaining. In Ray and Vohra [17]'s model, players make offers to form coalitions according to a fixed protocol. As coalitions exit the game immediately after they are formed, exit decisions are taken sequentially. In this model, it is easy to see that the procedure may end up in an inefficient equilibrium, where some players decide to leave early, in order to free ride on the coalition formed by subsequent players. (More precisely, the first player may choose to exit, anticipating that the next two players form a coalition; this early exit decision is optimal whenever $c > V/3$.) In our model, by contrast, exit decisions are taken simultaneously. If a player makes an unacceptable offer at the initial stage (or rejects the offer), all players simultaneously choose whether to exit at the action phase, resulting in a value less than $V/3$. Hence, players cannot commit to exit and free ride on the coalitions formed by other players, and the equilibrium outcome is efficient. Finally, the dynamics of coalition formation reported in Proposition 7 is reminiscent of Seidmann and Winter [20]'s results in games without externalities. Seidmann and Winter [20] show that non-emptiness of the core is a necessary condition for the grand coalition to form immediately. If we interpret b as the value of a two-player coalition, non-emptiness of the core is equivalent to $b \leqslant 2V/3$. Hence, as in [20], but within a different model of coalitional bargaining, we observe that the grand coalition forms immediately when the worth of intermediate coalitions is small, and form gradually when the worth of intermediate coalitions becomes large.

5.4. Market entry with synergies

Suppose that three symmetric firms contemplate entering a market with fixed entry costs. By making a prior agreement, firms can benefit from synergies which will reduce their entry cost. Individual firms face an entry cost F, a coalition of two firms faces an entry cost G, with $G < F$ and the entry cost of the coalition of three firms is normalized to zero. If a single firm enters the market, it obtains a gross monopoly profit $1 > F$; if two or three firms enter the market simultaneously, price competition drives profits down to zero for all the firms which have entered the market. Obviously, the Pareto efficient outcome is for the grand coalition to form and enter the market.

We analyze this model by assuming that coalitions choose between two strategies staying out of the market (r) and entering the market (e). Clearly, once one firm has entered the market, the other firms should abstain from entering. Hence, as δ converges to 1, the payoff matrices of the game Γ played at the action phases converge to:

$$\begin{array}{c|c|c|}
 & r & e \\ \hline
r & (\phi, \phi, \phi) & (0, 1-F, 0) \\ \hline
e & (1-F, 0, 0) & (-F, -F, 0) \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|}
 & r & e \\ \hline
r & (0, 0, 1-F) & (0, -F, -F) \\ \hline
e & (-F, 0, -F) & (-F, -F, -F) \\ \hline
\end{array}$$

$$r \qquad\qquad\qquad\qquad\qquad\qquad e$$

where player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices, and ϕ denotes the continuation value of the game at the initial contracting phase.

If two players form a coalition, the payoff matrices are given by

$$\begin{array}{c|c|c|}
 & r & e \\ \hline
r & (\phi_1, \phi_2) & (0, 1-G) \\ \hline
e & (1-F, 0) & (-F, -G) \\ \hline
\end{array}$$

where player 1 is the singleton player and player 2 the two-player coalition and ϕ_1 and ϕ_2 denote the continuation value at the contracting phase of the two players.

As long as $1 > F + G$, it is easy to see that the two-player game played after a two-player coalition has formed is equivalent to the two-player game of Example 2, and the only equilibrium at the action phase involves both players employing completely mixed strategies. Now assume that there is a large different between the fixed costs of single firms and of coalitions of two firms ($F > 4/5$ and $G < 1/5$). In that case, the three-player symmetric game becomes equivalent to the game of Example 2. In the appendix, we show that the Markov perfect equilibrium of the game results in the inefficient formation of a two-player coalition, which exits the game with positive probability at the action phase.

Proposition 8. *In the three-player model of market entry with synergies, as δ converges to* 1, *the Markov perfect equilibria of the game lead to the formation of a two-player coalition in the initial contracting stage, and firms inefficiently exit the game with positive probability at the action phase*.

Proposition 8 shows that inefficient outcomes arise not only in numerical examples, but also in significant economic models. Inefficiencies may also obtain in a larger class of models in Industrial Organization where firms decide whether to invest in a project and benefit from synergies by forming coalitions. A typical example of those situations are research joint ventures (RJV) which have been analyzed, among others, in [4,9]. If spillovers in R&D are low, R&D decisions are strategic substitutes and partial RJVs create significant value because they impose negative externalities on non-participants. Our results suggest that these two ingredients combined may lead to the formation of inefficient partial RJVs. On the other hand, if there are significant R&D spillovers then R&D decisions are strategic complements, and we expect the formation of a comprehensive RJV.

6. Conclusion

This paper proposes a coalitional bargaining model in which coalitions strategically interact and endogenously choose whether or not to exit. This formulation is general enough to study the formation of coalitions and the distribution of gains from cooperation in a wide variety of economic models with externalities and outside options. We show that when outside options are independent of the actions of other players, there exists Markov perfect equilibria (the class of MPE without coordination failures) which converge to efficient outcomes when the players become perfectly patient. On the other hand, in games with general outside options, all equilibria may be inefficient.

These results highlight the difference between our model and previous models of coalitional bargaining. In a setting with externalities, Ray and Vohra [16] show that when players cannot renegotiate, the outcome of coalition formation is typically inefficient, as players have an incentive to leave the game before extracting all the surplus. On the contrary, Gomes [6] establishes that when renegotiation occurs and players cannot choose to exit, the outcome is always efficient. Our study identifies a new type of friction—externalities on players' endogenous outside options—that may lead to bargaining inefficiencies.

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Appendix. Proofs

Proof of Proposition 1. In the proof we use the following lemma, which follows from the discussion in Section 2:

Lemma A.1. *A strategy profile σ is a Markov perfect equilibrium if and only if there exists payoffs $\phi_i^1(s)$, $\phi_i^2(s)$ such that*:

(i) *at the action stage, $\sigma^2(s)$ is a Nash equilibrium of the game $\Gamma(s)$ and $\phi_i^2(s)$ is the equilibrium payoff of coalition i at this equilibrium*;

(ii) *at the contracting stage, for all offers $(t, \mathcal{S})$ in the support of $\sigma^1(s)$*:

$$t_j = \phi_j^2(s) \text{ for all } j \in \mathcal{S},$$

and

$$\mathcal{S} \in \arg \max_{\mathcal{C} \subset \mathcal{N}(s), i \in \mathcal{C}} \phi_{\mathcal{C}}^2(s^{\mathrm{c}}) - \sum_{j \in \mathcal{C}} \phi_j^2(s).$$

The continuation value at the contracting stage is given by

$$\phi_i^1(s) = q_i(s) \sum_{\mathcal{S}} \sigma_i^1(s)(\mathcal{S}) \left(\phi_{\mathcal{S}}^2(s^{\mathrm{c}}) - \sum_{j \in \mathcal{S}} \phi_j^2(s) \right) + \sum_{j \in \mathcal{N}(s)} q_j(s) \sum_{\mathcal{C}} \sigma_j^1(s)(\mathcal{C})(\mathbf{1}_{i \in \mathcal{C}} \phi_i^2(s) + \mathbf{1}_{i \notin \mathcal{C}} \phi_i^2(s^{\mathrm{c}})),$$

where $q_i(s)$ is the probability that coalition i makes an offer at state s and **1** *is the indicator function.*

We define a correspondence $F : \Phi \times \Phi \times \Sigma^1 \times \Sigma^2 \to\to \Phi \times \Phi \times \Sigma^1 \times \Sigma^2$ whose fixed points are the MPE. Φ is the set of continuation values $\phi = (\phi_i(s))$, which are bounded below by $\min\{v_i(a)\colon$ for all states s and action profiles $a\}$. Furthermore, the sum $\sum_{i \in \mathcal{N}(s)} \phi_i(s)$ is bounded above by V so Φ is a closed, convex interval of a finite-dimensional Euclidean space. Let Σ^1 be the set of proposers' strategies σ^1 at the contracting stage. Omitting the transfers (which are defined as the value of the coalition at the next action phase), $\sigma_i^1(s)$ is a probability distribution over the finite set $\{\mathcal{S} \subset \mathcal{N}(s) : i \in \mathcal{S}\}$. Let Σ^2 be the set of strategies σ^2 at the action stage. For any state s and any coalition $i \in \mathcal{N}(s)$, $\sigma_i^2(s)$ is a probability distribution over the finite set $A_i(s)$. Both Σ^1 and Σ^2 are thus convex and compact subsets of a finite-dimensional Euclidean space.

The correspondence F is defined as follows. $(\varphi^1, \varphi^2, \mu^1, \mu^2) \in F(\phi^1, \phi^2, \sigma^1, \sigma^2)$ if and only if

$$\begin{aligned}
\varphi_i^1(s) &= q_i(s) \sum_{\mathcal{S}} \sigma_i^1(s)(\mathcal{S})(\phi_{\mathcal{S}}^2(s^{\mathrm{c}}) - \sum_{j \in \mathcal{S}} \phi_j^2(s)) \\
&\quad + \sum_{j \in \mathcal{N}(s)} q_j(s) \sum_{\mathcal{C}} \sigma_j^1(s)(\mathcal{C})(\mathbf{1}_{i \in \mathcal{C}} \phi_i^2(s) + \mathbf{1}_{i \notin \mathcal{C}} \phi_i^2(s^{\mathrm{c}})), \\
\varphi_i^2(s) &= u_i(s, \sigma^2)(\phi^1, \phi^2, \sigma^1, \sigma^2), \\
\mathrm{supp}(\mu_i^1(s)) &\subset \arg \max_{\mathcal{C} \subset N(s), i \in \mathcal{C}} \phi_{\mathcal{C}}^2(s^{\mathrm{c}}) - \sum_{j \in \mathcal{C}} \phi_j^2(s), \\
\mathrm{supp}(\mu_i^2(s)) &\subset \arg \max_{a_i \in A_i(s)} \{u_S(s, a_i, \sigma_{-i}^2)(\phi^1)\},
\end{aligned}$$

where supp denotes the support of a probability distribution, and where, for any $\mu^2 \in \Sigma^2$,

$$u_i(s, \mu^2)(\phi^1) = \sum_{a=(a_j)_{j \in \mathcal{N}(s)}} \left(\prod_{j \in \mathcal{N}(s)} \mu_j^2(s)(a_j) \right) (\delta \phi_i^1(s^{\mathrm{a}}) + (1-\delta) v_i(a)).$$

According to Proposition 1, the fixed points of F are MPE. To show that F has a fixed point we apply the Kakutani fixed point theorem. We have already noted that $Z = \Phi \times \Phi \times \Sigma^1 \times \Sigma^2$ is a compact and convex subset of a finite-dimensional Euclidean space, and $F(Z) \subset Z$. Furthermore, standard arguments show that $F(z)$ is a convex (and non-empty) set for all $z \in Z$, and that F has a closed graph. Thus the Kakutani fixed point theorem applies. □

Proof of Proposition 2. The proof is by contradiction. Suppose there is a sequence δ^k converging to one such that there is no ε–R strategy that is MPE for each game with discount rate δ^k. A proof similar to the one of Proposition 1, yields that the constrained coalitional bargaining game, where all players are required to play ε–R strategies also admits a MPE (in ε–R strategies). Let σ^k be an MPE of each constrained game in the sequence. (In the remainder of the proof, we will show that σ^k is also a MPE of the unconstrained game, which leads to a contradiction.) We use a notation below where superscript k represents the game in the sequence with discount rate δ^k (so, for example, $\phi_i^{2,k}(s)$ is the value of player i in state s at the action stage of game k).

Let s be a state where no coalitions have opted out and $i \in \mathcal{N}(s)$ a coalition for which the constraint is binding,

$$v_i := \max_{a_i \in E_i} v_i(a_i, \sigma^2_{-i}) > u_i^k(s, a_i, \sigma^2_{-i}) \quad \text{for all } a_i \in R_i, \tag{1}$$

so that $\sum_{a_i \in R_i} \sigma_i^2(s)(a_i) = \varepsilon$. Let K be the minimum aggregate efficiency loss when one coalition opts out. By assumption, $K > 0$. Since coalition i opts out with probability $(1-\varepsilon)$, and, once i opts out, the aggregate payoff is $v_N(a^k) \leqslant V - K$, for all future action profiles a^k, $\phi_N^{2,k}(s) \leqslant V - (1-\varepsilon)K$. (We denote sums $\sum_{i \in \mathcal{N}(s)} v_i(a)$ by $v_N(a)$ and $\sum_{i \in \mathcal{N}(s)} \phi_i(s)$ by $\phi_N(s)$).

We now estimate the lowest payoff that coalition i can obtain in the ε-constrained game. She can guarantee for herself at least v_i by opting out, but in the ε-constrained game she can only opt with probability at most equal to $1-\varepsilon$. The strategy of opting out of the game with probability $1-\varepsilon$, and choosing a_i such that $\underline{v}_i = \min_{a_i \in R_i} \min_{a_{-i} \in A_{-i}} v_i(a_i, a_{-i})$ with probability ε at every action stage, and of not making any offers and rejecting any offers made at every contracting stage, yields coalition i at least (at either the action or contracting phases)

$$\underline{\phi}_i = \frac{\underline{v}_i \varepsilon(1-\delta) + (1-\varepsilon)v_i}{1-\delta\varepsilon}, \tag{2}$$

which converges to v_i when δ converges to 1. This formula comes from the evaluation of $\underline{\phi}_i = E[\sum_{t=0}^{\infty} \delta^t (1-\delta) v_i(a^t)]$: at $t=0$ with probability $(1-\varepsilon)$ the value is equal to v_i and with probability ε the flow at $t=0$ is $(1-\delta)\underline{v}_i$ and the continuation value is $\delta\underline{\phi}_i$. So we have that $\underline{\phi}_i = (1-\varepsilon)v_i + \varepsilon[(1-\delta)\underline{v}_i + \delta\underline{\phi}_i]$ which yields (2). Because $\phi_i^{j,k}(s)$ is the value associated with an MPE of the constrained games, and $\underline{\phi}_i$ is the lowest payoff that coalition i can obtain deviating to the ε–R strategy described above, we have $\phi_i^{j,k}(s) \geqslant \underline{\phi}_i$, for $j = 1, 2$.

In addition,

$$\begin{aligned} \phi_i^{1,k}(s) &\geqslant (1-q_i(s))\underline{\phi}_i + q_i(s)\left(V - \sum_{j \in \mathcal{N}(s)\setminus i} \phi_j^{2,k}(s)\right) \\ &= (1-q_i(s))\underline{\phi}_i + q_i(s)\phi_i^{2,k}(s) + q_i(s)\left(V - \sum_{j \in \mathcal{N}(s)} \phi_i^{2,k}(s)\right), \end{aligned} \tag{3}$$

which implies $\phi_i^{1,k}(s) \geqslant \underline{\phi}_i + q_i(s)(1-\varepsilon)K$. But coalition i's payoff $u_i^k(s, a_i, \sigma_{-i}^2)$, for any $a_i \in R_i$, is at least equal to

$$\lambda(\delta\phi_i^{1,k}(s) + (1-\delta)\underline{v}_i) + (1-\lambda)(\delta\underline{\phi}_i + (1-\delta)\underline{v}_i), \tag{4}$$

where $\lambda \geqslant \varepsilon^{\mathcal{N}(s)-1}$ is the probability that all remaining $\mathcal{N}(s)\backslash i$ coalitions choose reversible actions. Combining our findings so far, we get

$$\begin{aligned}\lim_{k\to\infty} \inf u_i^k(s, a_i, \sigma_{-i}^2) &\geqslant \lambda(v_i + q_i(s)(1-\varepsilon)K) + (1-\lambda)v_i) \\ &= v_i + \lambda \cdot q_i(s)(1-\varepsilon)K > v_i\end{aligned}$$

which is in contradiction with inequality (1). □

Proof of Proposition 3. We prove that as δ converges to one, all ε–R MPE converge to a Pareto efficient outcome. Assume by contradiction that there is a state s for which the statement is false. (In case there are multiple states for which the statement is false, choose one with the smallest number of active players.) Then there is a subsequence δ_k converging to one satisfying $\phi_N^{2,k}(s) \leqslant V - K$, where $K > 0$. The payoff of any player $i \in \mathcal{N}(s)$ also satisfies inequality (3) thus $\phi_i^{1,k}(s) \geqslant v_i + q_i(s)K > v_i$. Since player i's payoff from remaining in the game is greater than expression (4), for k large enough, it is greater than $v_i + \lambda \cdot q_i(s)K > v_i$. Thus in equilibrium no player opts out, and $\phi_i^{1,k}(s) = \phi_i^{2,k}(s)$.

Consider the players' strategies at the contracting stage of state s. Forming the grand coalition yields a strictly positive gain $V - \phi_N^{2,k}(s) \geqslant K > 0$. But then in all states s^{c}, there are fewer active players than in state s, and thus by our initial assumption, $\phi_N^{2,k}(s^{\mathrm{c}}) \to V$. In addition, the aggregate payoff is equal to

$$\phi_N^{1,k}(s) = \sum_{j\in\mathcal{N}(s)} q_j(s) \left(\sum_{\mathcal{S}} \sigma_j^1(s)(\mathcal{S})\phi_N^{2,k}(s^{\mathrm{c}}) \right),$$

which implies that $\phi_N^{1,k}(s) = \phi_N^{2,k}(s) \to V$, resulting in a contradiction. It is clear that if the probability of early exit did not converge to zero, then the MPE could not converge to the Pareto efficient outcome. Moreover, since the probability of early exit converges to zero, then the equilibrium payoffs at both stages converge to the same value. □

Proof of Proposition 4. The structure of the proof is similar to the structure of the proofs of Propositions 2 and 3 and we only outline the steps that are different.

For all i, there exist v_i such that $\max_{e_i, a_{-i}} |v_i(e_i, a_{-i}) - v_i| \leqslant \eta/2$. Inequality (1) changes to

$$v_i + \eta/2 > \max_{a_i \in E_i} v_i(a_i, \sigma_{-i}^2) > u_i^k(s, a_i, \sigma_{-i}^2) \quad \text{for all } a_i \in R_i. \tag{5}$$

The same argument as in the proof of Proposition 2 shows that $\underline{\phi}_i \geqslant v_i - \eta/2$,

$$\lim_{k\to\infty} \inf \phi_i^{1,k}(s) \geqslant v_i - \eta/2 + q_i(s)(1-\varepsilon)K,$$

and coalition i's payoff $u_i^k(s, a_i, \sigma^2_{-i})$, is at least equal to

$$\lambda(v_i - \eta/2 + q_i(s)(1-\varepsilon)K) + (1-\lambda)(v_i - \eta/2).$$

Combining the results, we get

$$\lim_{k\to\infty} \inf u_i^k(s, a_i, \sigma^2_{-i}) \geqslant v_i - \eta/2 + \lambda \cdot q_i(s)(1-\varepsilon)K,$$

which is greater than $v_i + \eta/2$ for η small enough, resulting in a contradiction.

To show that the equilibrium is approximately efficient, an adaptation of the proof of Proposition 3 shows that equilibrium payoffs satisfy $\phi_i^{1,k}(s) \geqslant v_i - \frac{1}{2}\eta + q_i(s)K > v_i + \frac{1}{2}\eta$ for k large enough and η small enough. Hence, at the action phase, no player wants to opt out, and at the contracting phase, all players want to form some coalition. This implies that the equilibrium is approximately efficient. □

Proof of Proposition 5. We construct the equilibrium. At any state s where some players have opted out, the equilibrium strategy is for all active players to opt out. At any state s where no player has opted out, let $S_1, \ldots, S_m$ be the m active coalitions (indexed by j), and let S_1 be the coalition with the highest outside option (and suppose that there is a single coalition with the highest outside option).

We propose the following strategies. At the contracting stage, every player proposes to form the grand coalition, and to offer x_j to other coalitions, resulting in expected equilibrium payoffs ϕ_j. At the action stage, player S_1 opts out of the game with probability σ and the other players continue to negotiate. Let $v_N = \Sigma_i v_i$. The variables $\sigma(s)$. $x_i(s)$ and $\phi_i(s)$ are defined by the following equations:

If $v_1 \geqslant \frac{1}{m}V$ then

$$\begin{aligned}
&\sigma(s) = (1-\delta)\frac{mv_1 - \delta V}{\delta(\delta V - \delta v_N - v_1(1-\delta))},\\
&\phi_j(s) = \frac{v_1 + \frac{\sigma}{(1-\delta)}\delta^2 v_j}{\delta(1+\delta\frac{\sigma}{(1-\delta)})} \text{ and } x_j(s) = \phi_j(s) - \frac{(1-\delta)v_1}{\delta} \quad \text{for } j = 2, \ldots, m,\\
&\phi_1(s) = \frac{v_1}{\delta} \text{ and } x_1(s) = v_1.
\end{aligned} \tag{6}$$

If $v_1 < \frac{1}{m}V$ then

$$\begin{aligned}
&\sigma(s) = 0,\\
&\phi_i(s) = \frac{1}{m}V \text{ and } x_i(s) = \frac{1}{m}\delta V \quad \text{for } i = 1, \ldots, m.
\end{aligned} \tag{7}$$

We now show that this strategy profile forms a Markov perfect equilibrium. Consider any state where all players are active. If $v_1 \geqslant \frac{1}{m}V$, at the action stage, player 1 is indifferent between opting out and continuing, as $v_1 = \delta\phi_1$. For players $j = 2, \ldots, m$,

$$\delta\phi_j - v_j = \frac{v_1 - v_j - \frac{\sigma}{(1-\delta)}\delta(1-\delta)v_j}{(1+\delta\frac{\sigma}{(1-\delta)})}.$$

For δ close enough to 1, $\delta\phi_j - v_j > 0$, so no player wants to opt out. If now $v_1 < \frac{1}{m}V$, for all players $j = 1, 2, \dots, m$, $\delta\phi_j - v_j = \frac{\delta V}{m} - v_j > 0$ for δ large enough. So no player wants to opt out either.

Consider now the contracting stage. First suppose that $v_1 \geqslant \frac{1}{m}V$. If the grand coalition is formed, the offers must satisfy

$$x_j = (1-\sigma)\delta\phi_j + \sigma\delta v_j \text{ for } j = 2, \dots, n,$$
$$\phi_i = \frac{1}{m}(V - x_N) + x_i,$$
$$x_1 = v_1 = \delta\phi_1, \tag{8}$$

Combining the last two equations,

$$\phi_1 - x_1 = \frac{(1-\delta)v_1}{\delta} = \frac{1}{m}(V - x_N),$$

so

$$x_i = \phi_i - \frac{(1-\delta)v_1}{\delta} \quad \text{for } i = 1, \dots, m,$$

Replacing the value of x_i in the first equation yields

$$\phi_j - \frac{(1-\delta)v_1}{\delta} = (1-\sigma)\delta\phi_j + \sigma\delta v_j,$$

whose unique solution is the ϕ_j given in the proposed strategy profile. Adding all equations for $j = 2, \dots, n$ results in

$$x_N - x_1 = (1-\sigma)(\delta\phi_N - \delta\phi_1) + \sigma\delta(v_N - v_1),$$

and since $\phi_N = V$, $x_N = \phi_N - m\frac{(1-\delta)v_1}{\delta}$, $\delta\phi_1 = v_1$, and $x_1 = v_1$ we can solve for σ,

$$\sigma = (1-\delta)\frac{mv_1 - \delta V}{\delta(\delta V - \delta v_N - v_1(1-\delta))},$$

as claimed. Notice that $\sigma \geqslant 0$ if and only if $\delta \geqslant \delta_0$ where $\delta_0 = \frac{v_1}{V - v_N + v_1} < 1$ because $V - v_N > 0$, and for δ close enough to 1, $\sigma \leqslant 1$.

Now suppose that $v_1 < \frac{1}{m}V$. If the grand coalition is formed,

$$x_i = \delta\phi_i \text{ for all } i,$$
$$\phi_i = \frac{1}{m}(V - x_N) + x_i \tag{9}$$

and it is straightforward to verify that the unique solution of the system of equations is given by the formula in the description of the strategy profile.

It remains to verify that no player wants to deviate by forming a subcoalition at the contracting stage. Consider a deviation by which some of the active players form a subcoalition $S \subset N$, and let ϕ'_j be the continuation value of players following the deviation. If $v_S \geqslant v_1$ then the payoff of coalition S converges to $\phi'_S = v_S$ for δ large enough. Now $v_S < \phi_S$ since $\phi_i \geqslant v_i$ with strict inequality for at least one $i \in S$. Hence, the deviation is not profitable.

Similarly, if $v_S < v_1$, all players $j \notin S \cup \{1\}$ benefit from the deviation, since their new payoff ϕ'_j is obtained by replacing m by $m - |S| + 1$ in the formula for equilibrium payoffs, and thus satisfy $\phi'_j > \phi_j$. Now, as $\sum_{j \in N} \phi'_j = \sum_{j \in N} \phi_j = V$ and $\phi'_1 = \phi_1$, players in S are better off not deviating.

Our analysis only deals with the case where there is a unique player with the highest outside value at any state. The result can be generalized to situations with multiple players with highest values as follows. Suppose that there are m players, $j = 1, \ldots, m$, such that $v_j = \max_{i \in N} v_i$. Perturb the payoffs, by adding a random vector ε to all the payoffs, and construct the equilibrium for the perturbed game, where all values are different. As ε goes to zero, because equilibrium payoffs and strategies are upper hemi continuous in the parameters of the game, one can obtain limit equilibrium payoffs and strategies for the original game. □

Proof of Proposition 6. We give an explicit construction of the Markov perfect equilibrium. Consider any state s where the principal has contracted with a set S of agents. Due to the symmetry of the problem we associate to each coalition S its cardinal, $\#S = m$, and define $v(m)$, $v_i(m)$, $\phi^k(m)$ and $\phi_i^k(m)$. Let $x^*(m)$ be the unique solution of

$$x^*(m) = \arg\min_{m' \geqslant m} \left\{ \frac{v(n) - v(m')}{n - m'} \right\}$$

and

$$v_i^*(m) = \min_{m' \geqslant m} \left\{ \frac{v(n) - v(m')}{n - m'} \right\}.$$

Consider the following strategies. At a subgame m where $x^*(m) = m$, the principal offers $\phi_i^2(m)$ to all the remaining agents, and the agents accept any offer greater than or equal to $\phi_i^2(m)$. At the action stage, the principal exits with a positive probability σ. The values σ and $\phi_i^2(m)$ are computed as solutions to the equations:

$$\begin{aligned}
&\phi^1(m) = v(n) - (n - m)\phi_i^2(m),\\
&\phi_i^1(m) = \phi_i^2(m),\\
&\phi^2(m) = \delta\phi^1(m) = v(m),\\
&\phi_i^2(m) = (1 - \sigma)\delta\phi_i^1(m) + \sigma v_i(m).
\end{aligned}$$

At a subgame m where $x^*(m) > m$, the principal offers to contract with $x^*(m) - m$ of the $n - m$ remaining agents (all agents are chosen with equal probability). She offers $\phi_i^2(m)$ to all the agents, and agents accept any offer greater than or equal to $\phi_i^2(m)$. At the action stage, the principal never exits. The value $\phi_i^2(m)$ is computed as a solution to the equations:

$$\begin{aligned}
&\phi^1(m) = \phi^2(x^*(m)) - (x^*(m) - m)\phi_i^2(m),\\
&\phi_i^1(m) = \frac{(x^*(m) - m)}{n - m}\phi_i^2(m) + \frac{(n - x^*(m))}{n - m}\phi_i^2(x^*(m)),\\
&\phi^2(m) = \delta\phi^1(m),\\
&\phi_i^2(m) = \delta\phi_i^1(m).
\end{aligned}$$

Note that, as δ converges to 1, the equilibrium offers $\phi_i^2(m)$ converge to $\frac{v(n)-v(x^*(m))}{n-x^*(m)} = v_i^*(m)$ and $\frac{\sigma}{(1-\delta)}$ converges to the positive value $\frac{v(m)-v(n)}{v(n)-v(m)-(n-m)v_i(m)}$.

To show that this strategy profile forms a subgame perfect equilibrium, we first consider subgames satisfying $x^*(m) = m$. By construction, the principal's exit decision at the action stage and the agents' responses at the contracting stage are optimal. It remains to check that the principal's offer at the contracting stage is optimal. Suppose by contradiction that the principal makes an acceptable offer to $m' < n$ agents. She would then receive a payoff $\phi^2(m') - (m' - m)\phi_i^2(m)$ instead of $v(n) - (n-m)\phi_i^2(m)$. Two cases must be distinguished. If $x^*(m') = m'$, then $\phi^2(m') = v(m')$. But because $x^*(m) = m$,

$$\phi_i^2(m) < \frac{v(n) - v(m')}{n - m'},$$

and hence

$$v(m') < v(n) - (n - m')\phi_i^2(m),$$

establishing that the deviation is unprofitable. If now $x^*(m') > m'$, in the continuation game, the principal proposes to form a coalition of size $x^*(m')$ and then moves to the grand coalition. Overall, she thus offers $v_i^*(m')$ to the remaining $(n - m')$ agents and $\phi^2(m') = v(n) - (n - m')v_i^*(m')$. But because $x^*(m) = m$, $v_i^*(m') > v_i^*(m)$ and hence,

$$v(n) - (n - m')v_i^*(m') - (m' - m)v_i^*(m) < v(n) - (n - m)v_i^*(m),$$

establishing that the deviation is unprofitable.

Consider now a subgame satisfying $x^*(m) > m$. We first show that, at the action stage, staying in the game is the optimal action of the principal. By exiting, the principal obtains a payoff of $v\,(m)$ and by staying a payoff of $\phi(m) = v\,(n) - (n - m)\,v_i^*(m)$. As $x^*(m) \neq m$,

$$v_i^*(m) < \frac{v(n) - v(m)}{n - m},$$

so that the optimal strategy is to choose a temporary action. At the contracting stage, the agents' response is optimal by construction, and by an argument similar to the argument in the case $x^*(m) = m$, the principal has no incentive to offer to form a coalition of size $m' \neq x^*(m)$. □

Proof of Proposition 7. We construct the payoff matrices of the game Γ played at the action phases when δ converges to 1. It is easy to check that, once a player has exited the game, the optimal choice of the two remaining players is to exit as a two-player coalition, and the expected payoff of each of the remaining players is equal to $b/2$. Hence, the payoff matrices are given by

$$\begin{array}{cc|c|c|} & & r & e \\ \hline & r & (\phi,\phi,\phi) & (b/2,c,b/2) \\ \hline & e & (c,b/2,b/2) & (a,a,a) \\ \hline \end{array} \qquad \begin{array}{cc|c|c|} & & r & e \\ \hline & r & (b/2,b/2,c) & (a,a,a) \\ \hline & e & (a,a,a) & (a,a,a) \\ \hline \end{array}$$

$$\qquad\qquad r \qquad\qquad\qquad\qquad\qquad\qquad e$$

where ϕ denotes the continuation value of the game at the initial contracting phase.

If two players form a coalition, the payoff matrices are given by

	r	e
r	(ϕ_1, ϕ_2)	(c, b)
e	(c, b)	(c, b)

where player 1 is the singleton player and player 2 the two-player coalition and ϕ_1 and ϕ_2 denote the continuation value at the contracting phase of the two players.

Once the two-player coalition has formed, the two-player game corresponds to an asymmetric version of Example 1, and as $V > c + b$, we can easily characterize the ε–R equilibria for δ converging to 1. If $V/2 > \max\{c, b\}$, the game admits a unique ε–R equilibrium where both players remain and $\phi_1 = \phi_2 = V/2$. If $V/2 \leqslant \max\{c, b\}$, the game admits a unique ε–R equilibrium where the player with the largest option randomizes between staying and exiting, and the player with the lowest outside option remains in the game. The probability that the player with the largest option remains in the game converges to 1 as δ converges to 1. Supposing (without loss of generality) that $c \geqslant b$, the payoffs converge to $\phi_1 = c, \phi_2 = V - c > b$.

We now consider the symmetric ε–R equilibria of the game at the initial phase, where the three players are singletons. Consider first the action stage. If $V/3 > c$, there exists an equilibrium where all players remain and $\phi = V/3$. If $V/3 < c$, the symmetric ε–R equilibrium involves all players choosing a common probability σ of remaining in the game, where σ satisfies:

$$\sigma^2 V/3 + \sigma(1-\sigma)b = \sigma^2 c + 2\sigma(1-\sigma)a.$$

Notice that, in both cases, the continuation value of players at the action stage satisfy $\phi^2 \leqslant V/3$.

Now, consider the initial contracting stage. Suppose first that $b \leqslant 2V/3$. In that case, we claim that the grand coalition is formed immediately. By forming a two player coalition, the proposer gets at most: $b - \phi^2$ whereas she would get $V - 2\phi^2$ if she proposed to form the grand coalition immediately. As $b \leqslant 2V/3$ and $\phi^2 \leqslant V/3$, $b - x \leqslant V - 2x$. If now $b > 2V/3$, we claim that the coalition is formed in two steps. As $b > 2V/3$, $c < V/3$ and hence $\phi^2 = V/3$. But then, $V - 2\phi^2 = V/3 < b - \phi^2$, and at the initial contracting stage, the proposer has an incentive to form a two-player coalition. □

Proof of Proposition 8. Consider the action stage after two players have formed a coalition. The only equilibrium of the game is a completely mixed strategy profile (σ_1, σ_2) satisfying the following equations (the notations are similar to those of Example 2),

$$x_1 = (1-\sigma_2)\phi_1 = (1-\sigma_2) - F,$$
$$x_2 = (1-\sigma_1)\phi_2 = (1-\sigma_1) - G,$$
$$\phi_1 = \tfrac{1}{2}(1 - x_2 + x_1),$$
$$\phi_2 = \tfrac{1}{2}(1 - x_1 + x_2).$$

Solving these equations we obtain

$$\begin{aligned}
&\phi_1 = 1 - t, \quad \phi_2 = t,\\
&\sigma_1 = \frac{(t-F)(1-t)}{t^2}, \quad \sigma_2 = \frac{t-F}{t},\\
&x_1 = \frac{F(1-t)}{t}, \quad x_2 = t - \frac{(t-F)(1-t)}{t},
\end{aligned} \tag{10}$$

where t is a solution of $f(t) = 2t^3 + (-3 - F + G)t^2 + (2F+1)t - F = 0$. Note that because $F + G < 1$,

$$f(F) = -F^2(1 - G - F) < 0 \quad \text{and} \quad f(1-G) = G^2(1 - G - F) > 0,$$

so that a solution $t \in (F, 1-G)$ exists.

Now consider the initial stage where no coalition has been formed. We will show that it is a weakly dominant strategy for every firm to stay. By staying, a firm obtains either $\delta\phi$ if no other firm exits, or 0 if another firm exits. By exiting, the firm either gets $1 - F$ if no other firm enters the market, or 0 otherwise. We want to show

$$\delta\phi \geqslant 1 - F. \tag{11}$$

We consider only symmetric equilibria. If players propose to form the grand coalition at the contracting stage, $\phi = 1/3$ and, as δ converges to 1 and $F > 4/5$, $\delta\phi \geqslant 1 - F$. If players propose to form a two-player coalition with equal probability of choosing any of the two other firms, and with x the continuation value at the action stage of the initial state,

$$\phi = \frac{1}{3}(x_2 - x) + \frac{1}{3}x + \frac{1}{3}x_1 = \frac{x_2 + x_1}{3}. \tag{12}$$

Solution (10) implies that inequality (11) is equivalent to

$$h(t) = 2F + Ft + 2t^2 - 4t \geqslant 0.$$

This inequality holds for all $F \geqslant 4/5$ because the quadratic expression $h(.)$ satisfies $h(F) = F(3F-2) > 0$ and $h'(F) = 5F - 4 \geqslant 0$.

Finally, we check that it is an optimal strategy for a firm to form a coalition with one of the two other firms at the initial contracting stage. By forming the grand coalition, each firm obtains a payoff $1 - 2\delta\phi$. By forming a coalition of size 2 it obtains $x_2 - \delta\phi$. Hence, we need to establish that

$$x_2 + \phi \geqslant 1,$$

which is equivalent to $g(t) = 8t^2 - (5F+7)\,t + 5F \geqslant 0$. This inequality holds for all $F \in (2/3, 1)$ because the quadratic expression $g(.)$ satisfies $g(F) = F\,(3F-2) > 0$ and $g'(F) = F\,(16 - 5F) > 0$. $\square$

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