# Coalitional bargaining games: A new concept of value and coalition formation 

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#### Abstract

We propose a new solution for coalition bargaining problems among $n$ players that can form coalitions c generating heterogenous coalitional values $s_{\mathrm{c}} \in R$. The players' values $v_{i}$ and probability of coalition formation $\mu_{\mathrm{c}}$ are given by: $$
v_{i}=\sum_{\mathrm{c} \in W}\left(\delta v_{i}+\gamma\right) I(i \in \mathrm{c}) \mu_{\mathrm{c}} \text { and } \sum_{\mathrm{c} \in W} \mu_{\mathrm{c}}=1,
$$ where coalition c is chosen only if it maximizes the average gain $\gamma_{\mathrm{c}}=\frac{1}{|\mathrm{c}|}\left(s_{\mathrm{c}}-\delta \sum_{j \in \mathrm{c}} v_{j}\right)$ and $\gamma \equiv \max _{\mathrm{C} \in W} \gamma_{\mathrm{c}}$. This solution is the strong Markov perfect equilibrium of a noncooperative coalition bargaining game where players choose simultaneously the coalition they want to join followed by negotiations to split the surplus. The solution does not rely on the specification of a proposer recognition protocol. For majority voting games, the solution exhibits more inequality among the values of large and small parties and a concentrated equilibrium coalition formation distribution.


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## 1. Introduction

Many political, economic, and social problems can be modeled as coalition bargaining games. In these games, agents negotiate over the formation of coalitions and the allocation of some surplus and/or decisions among coalition members. For example, common situations that can be modeled as coalition bargaining games include: political parties negotiating over government formation and the allocation of ministerial cabinets in multi-party parliamentary democracies (Baron and Ferejohn (1989)), large shareholders of a majority-controlled corporation negotiating over the allocation of control benefits among controlling shareholders (Zingales (1994)), or countries negotiating over decisions in organizations such as the European Union Council of Ministers or the United Nations Security Council (Felsenthal and Machover (2001)).

This paper proposes a new solution for the equilibrium value (or power) of players and the coalitions that form in coalition bargaining games. Consider a coalition bargaining problem among a set of $N=\{1, \ldots, n\}$ players that assigns to each coalition c a value or surplus $s_{\mathrm{C}} \in R$, for all coalitions in a set $W$ of coalitions. We propose as the coalition bargaining solution, for the players' values $v_{i}$ and probability of coalition formation $\mu_{\mathrm{c}}$, the solution of:

$$
v_{i}=\sum_{\mathrm{c} \in W}\left(\delta v_{i}+\gamma\right) I(i \in \mathrm{c}) \mu_{\mathrm{c}} \text { and } \sum_{\mathrm{c} \in W} \mu_{\mathrm{c}}=1, \text { where } \mu_{\mathrm{c}}>0 \text { only if } \gamma_{\mathrm{c}}=\gamma \equiv \max _{\mathrm{B} \in W} \gamma_{\mathrm{B}}
$$

[^0]where only coalitions c that maximize the average gain $\gamma_{\mathrm{c}} \equiv \frac{1}{|\mathrm{c}|}\left(s_{\mathrm{c}}-\delta \sum_{j \in \mathrm{c}} v_{j}\right)$ are chosen, $|\mathrm{c}|$ is the number of players in coalition c , and $\delta$ is the discount factor.

We show that this solution arises as the equilibrium of a non-cooperative game where players choose coalitions simultaneously, allowing them to freely discuss their coalition choices but not to make binding commitments. Specifically, the coalition bargaining model is a dynamic game over an infinite number of periods, in which every period starts with a coalition formation stage followed by a negotiation stage in which players decide how to split the coalition surplus. We rule out the possibility of two disjoint coalitions forming simultaneously as is the case in most group decision-making situations that require the consent of the majority of group members. In the coalition formation stage, all players simultaneously choose the coalitions they want to form: A coalition forms if and only if all of its members have chosen it. In the negotiation stage that ensues, all members of the coalition chosen in the first stage engage in multilateral Nash bargaining to split the surplus in which unanimity is required to reach agreement.

Our concept of equilibrium is a strong Markov perfect equilibrium (MPE): pure strategy profiles that are stationary subgame perfect Nash equilibrium and depend only on payoff-relevant aspects of the history. Moreover, since we allow players to freely discuss their coalition choices, we also require that the action profile be a strong Nash equilibrium, which is a refinement of the Nash equilibrium introduced by Aumann (1959), where no coalition of players can gain by any joint (or group) deviations. We also consider the coalition-proof MPE of the game and show that it coincides with the strong MPE.

We allow in general for the heterogenous coalition surplus $s_{C}$ to be stochastic as in Merlo and Wilson (1995) and Eraslan and Merlo (2002). At the start of each period, there is a random independent and identically distributed draw for the surplus generated by all coalitions from an absolutely continuous distribution. Before making decisions about which coalition to form, players observe the realization of the surplus.

Our main result is that the coalition bargaining solution is the limit equilibrium outcome, when the random perturbations around a deterministic coalition function converge to zero. Whereas with deterministic coalitional functions, equilibrium typically exists only allowing for mixed strategies; in our model, players use pure strategy and the randomization comes from perturbations of the surplus. One technical contribution of our paper is that our purification approach is different from Harsanyi's (1973) purification result since there is no incomplete information, and all players know the surplus realizations drawn every period.

The intuition for the equilibrium is that coalitions that do not maximize the average gain are dominated by coalitions that do maximize the average gain. This occurs because once a coalition c is proposed, players' conditional values in the negotiation stage are $\delta v_{i}+\gamma_{\mathrm{c}}$, which corresponds to an equal split of the surplus using the continuation values as the status quo (the Nash bargaining solution). Thus there is an alignment in incentive among all players forming a coalition, since they all gain $\gamma_{c}$ when forming coalition c , which provides no incentives for group deviations resulting in a strong Nash equilibrium.

The solution has several noteworthy economic properties. First, it is more parsimonious than other existing noncooperative models that rely on the additional specification of a proposer-recognition protocol. ${ }^{1}$ The significance of proposal rights is highlighted by Kalandrakis (2006), who shows that in the classic random-proposer legislative bargaining model, such as Baron and Ferejohn (1989), any value can be achieved by varying the proposer-recognition probability. While the coalitional function can be readily obtained for particular problems of interest, that is not often not the case for the proposer-recognition probability. Thus our approach is interesting to consider in coalition decision-making problems without an explicitly given proposer selection protocol.

Second, the new solution exhibits significantly more inequality among small and large parties in the important class of majority voting games than other well-known solutions such as the Shapley-Shubik and Banzhaf indices, the nucleolus, and the random-proposer model solution with equal proposer probabilities. For example, for weighted majority games with less than eight players-a total of 134 representative voting games enumerated by Isbell (1959)-the coalition bargaining solution, as $\delta \rightarrow 1$, of larger (smaller) parties is significantly greater (less) than the values predicted by the other solution concepts. Moreover, the new solution predicts a very concentrated coalition formation probability distribution with only a few of the minimum winning coalitions actually forming in equilibrium, which is something that cooperative solution concepts are silent about, because they have no explicit predictions attached to coalition formation probabilities.

Finally, the coalition bargaining solution, unlike random-proposer models, predicts that dummy players have zero value and should never be included in any coalition that forms in equilibrium. Dummy players are players that do not contribute anything to the surplus of coalitions in which they are members, and thus, naturally, should have zero value and be excluded from equilibrium coalitions. In addition, the model's predictions for seller-buyer games are consistent with the outside option principle developed in Binmore et al. (1989): The seller sells to the highest valuation buyer at a price equal to the maximum of the second-highest valuation buyer and half of the highest buyer value.

Related literature: Our simultaneous coalition formation stage game has similarities with the link formation game developed by Dutta and Mutuswami (1997) and Dutta et al. (1998) in the context of network formation. They introduce a link

[^1]formation game, in which players simultaneously announce a set of players with whom he or she wants to form a pairwise link, and a link is formed if both players want the link. They show that coalition-proof refinement leads to the formation of a full cooperation structure, and that a strong Nash equilibrium may not exist, but when it does it also leads to the formation of a full cooperation structure. ${ }^{2}$

Demand bargaining models represent a prominent class of models for legislative bargaining (e.g., Morelli (1999); Frechette et al. (2005); Montero and Vidal-Puga (2011)). In these models, agents are chosen, according to a certain recognition protocol, to make demands for a share of the surplus until a majority coalition forms with a compatible set of demands. These models highlight the distinction of a proposer making an offer (which includes a coalition and a surplus division among coalition members) versus making a demand (leaving other agents with the choice between demanding the residual or disagreeing). In these models, the proposers do not receive a large premium, and for some class of homogeneous games, the ex-post payoffs are proportional to the voting weights.

Diermeier and Merlo (2000), Breitmoser (2012), and Montero (2015) propose a model of proto-coalition bargaining. In these models, a proposer (or formateur) is also randomly chosen with a given exogenous recognition probability, and the formateur then chooses a proto-coalition to bargain over the surplus allocation among proto-coalition members. Recently, Battaglini (2021) developed a new model in which the formateur is chosen from a fixed order and picks a coalition to negotiate the allocation of the surplus; in case the negotiations break down, a new formateur is chosen according to the prescribed sequence. He shows that the model explains well-known empirical facts such as the absence of a formateur premia, the occurrence of supermajorities, and delays in reaching agreement. Our model has many similarities with these models, but a key distinction is that the proto-coalition formation happens simultaneously. The value predictions of these models are significantly different from our model, as proposers recognized with higher probability will tend to have a higher value.

## 2. The model

Consider the problem of a finite set $N=\{1, \ldots, n\}$ of players who must choose which winning coalition to form, and how to divide the total surplus generated by the coalition among themselves. To avoid trivial cases, we shall assume that $n \geq 3$. Only winning coalitions of players generate surplus, and once a coalition forms, the game stops and no further coalitions can form. Players' status quo payoffs are normalized to zero, so 0 is the payoff that each player obtains as long as no coalition has formed yet, and 0 is also the payoff a player gets when not a member of the coalition that forms (see also Compte and Jehiel (2010)).

The subset of all possible winning coalitions is denoted by $W$. We maintain throughout the paper the assumption that no two disjoint coalitions can both be winning: formally, for any $\mathrm{c}, \mathrm{c}^{\prime} \in W$ then $\mathrm{c} \cap \mathrm{c}^{\prime} \neq \varnothing$. This condition holds, for example, if $W$ is (i) monotonic and (ii) proper, ${ }^{3}$ and holds for majority voting games.

The heterogenous surpluses generated by coalitions are described by a vector $s=\left(s_{\mathrm{c}}\right)_{\mathrm{c} \in W}$, where $s \in \mathbb{R}^{W}$ and $s_{\mathrm{c}} \in \mathbb{R}$ is the surplus generated by coalition $c$. This vector is also commonly known in the literature as a coalitional function (or characteristic) function. In particular, a simple game is the special case where the surplus is homogeneous, say equal to $\$ 1$ : $s_{\mathrm{C}}=1$, for all $\mathrm{c} \in W$.

In our setting, the surpluses may receive shocks over time. The surplus realizations over time are described by a sequence $\left(s_{t}\right)_{t \geq 1}$ where $s_{t} \in \mathbb{R}^{W}$. Formally, the surpluses $\left(s_{t}\right)_{t \geq 1}$ are realizations of a stochastic process $\left(\mathbf{s}_{t}\right)_{t \geq 1}$, where the random vectors $\mathbf{s}_{t}$ are independent and identically distributed to the random vector $\mathbf{s}=\left(\mathbf{s}_{\mathrm{c}}\right)_{\mathrm{c} \in W}$ with density function $f(s)$ over $s \in \mathbb{R}^{W}$ with bounded support $\mathcal{S} \subset \mathbb{R}^{W}$, the set of all possible states.

Allowing for stochastic coalition surplus will actually simplify the analysis. In our model, players use pure strategy and the randomization comes from perturbations of the surplus. With deterministic coalitional functions, equilibrium typically exists only allowing for mixed strategies. One technical contribution of the paper is that this purification approach is different from Harsanyi's since there is no incomplete information, and all players know the surplus realizations drawn every period.

We model the coalitional bargaining game as a dynamic game (multi-stage game with observed actions) over an infinite number of discrete periods $t=1,2, \ldots, \infty$ where every period has two stages. In the first stage, the coalition formation stage, all players choose which coalition to form. In the second stage, the negotiation stage, the members of the chosen coalition bargain over how to split the surplus generated by the coalition. All players maximize the expected discounted share of the surplus, and they have the same discount rate $\delta \in(0,1)$. At every stage, players observe the complete history with all past actions and surplus realizations.

[^2]We now describe the extensive form of the coalition bargaining game, beginning with the coalition formation stage followed by the negotiation stage, which takes place every period $t=1,2, \ldots, \infty$. We denote the coalition bargaining game by $\Gamma(\mathbf{s}, \delta)$.

Coalition formation stage: At the start of the $t$-period coalition formation stage, a random surplus $s_{t}=\left(s_{t, \mathrm{c}}\right)_{\mathrm{c} \in W} \in \mathbb{R}^{W}$ is drawn. The surplus $s_{t}$ is the state variable. It is observed by all players before making their choices in the coalition formation stage game. All agents simultaneously choose an action from the action set $A_{i}$ where $A_{i}=W \cup\{\varnothing\}$, which is the set of winning coalitions augmented by the empty set. The action $a_{i}=\mathrm{c} \in W$ means that player $i$ wants to form coalition c , and action $a_{i}=\varnothing$ means player $i$ does not want to form any coalition.

After all players choose their actions, a winning coalition is chosen if and only if all members of the coalition choose that coalition; otherwise no coalition is chosen. This rule uniquely determines the coalition in $W \cup\{\varnothing\}$ chosen for all possible action choices. Formally, following the action profile $\mathbf{a}=\left(a_{i}\right)_{i \in N}$, the coalition chosen for negotiation in the next stage is $\mathrm{c} \in W$ only if all members $i \in \mathrm{c}$ choose action $a_{i}=\mathrm{c}$, and otherwise $\mathrm{c}=\varnothing$.

This rule determines uniquely a coalition choice for all action profiles. It will be convenient to formally define this rule as the mapping $\mathrm{c}_{f}$ from action profiles into coalitions chosen as

$$
\begin{align*}
\mathrm{c}_{f} & : \prod_{i \in N} A_{i} \rightarrow W \cup\{\varnothing\}, \text { where for all } \mathbf{a}=\left(a_{i}\right)_{i \in N},  \tag{1}\\
\mathrm{c}_{f}(\mathbf{a}) & =\left\{\mathrm{c} \in W: \text { for all } i \in \mathrm{c} \text { then } a_{i}=\mathrm{c}\right\} .
\end{align*}
$$

This mapping is well-defined and $c_{f}(\mathbf{a}) \in W \cup\{\varnothing\}$ is a unique selection among $W \cup\{\varnothing\}$ due to the assumption that no two disjoint coalitions can both be winning. Once a coalition $c \in W$ is chosen, the game moves to the ( $s, \mathrm{c}$ ) negotiation stage with the chosen coalition c and state $s$, and otherwise, if $\mathrm{c}=\varnothing$, the game moves to the $t+1$-period coalition formation stage.

Negotiation stage: In the $t$-period negotiation game, players in coalition c bargain on how to split the surplus $s_{\mathrm{C}}$ generated by the coalition. The bargaining stage is modeled in a standard way, and unanimity among the coalition members is required to split the surplus. The outcome of the negotiation is the Nash bargaining solution with an equal split of the surplus among coalition members where the status quo of the negotiation is the discounted continuation values.

The negotiation process follows Binmore et al. (1986) in which there is an exogenous risk of breakdown following a rejected offer. We propose in the Online Appendix (available at the author's website) an alternative formulation in which coalition members simultaneously make Nash demands for the surplus. Both formulations yield the Nash bargaining solution.

Formally, in the negotiation stage game ( $s, \mathrm{c}$ ), a player is chosen with equal probability among the members of coalition c. ${ }^{4}$ The chosen player can either propose a surplus division $\left(x_{i}\right)_{i \in \mathrm{C}}$, satisfying the budget constraint $\sum_{i \in \mathrm{C}} x_{i} \leq s_{\mathrm{C}}$, or terminate the current negotiation round and move the game to the next coalition formation period. Once a proposal is made, the remaining coalition members, in a sequential order, respond whether they accept (yes) or decline (no) the offer. The coalition forms only if all responders accept the proposal. The order of response is arbitrarily given, but the order is irrelevant and does not affect the results.

In case of acceptance, each player $i \in \mathrm{c}$ receives a final payoff equal to $x_{i}$ (and the players $i \notin \mathrm{c}$ receive a final payoff of $x_{i}=0$ ). Players' utility are $\delta^{t-1} x_{i}$ and the game ends. Otherwise, in case of any rejection, coalition c continues bargaining as above with probability $\phi$, where $0 \leq \phi<1$, with a new proposer choice, or there is a bargaining breakdown with probability $1-\phi$, in which case the game moves to the next period with a new coalition formation round. ${ }^{5}$

We are interested in a subgame perfect Nash equilibrium of the coalition bargaining game $\Gamma(\mathbf{s}, \delta)$ that is pure Markovian stationary strategies (see also Maskin and Tirole (2001)). A pure strategy profile is a stationary Markovian strategy if the strategies at every $t$-period stage game are time invariant and the strategies depend only on the current state or surplus realization, and are otherwise independent of any other aspects of the history of play and past surplus realizations. We therefore omit references to the time period and game histories to simplify notation. We use the concept of Markov perfect equilibrium in our analysis.

Definition 1 (Markov perfect equilibrium). A pure stationary Markovian strategy profile $\sigma$ is a Markov perfect equilibrium (MPE) if $\sigma$ is a subgame perfect Nash equilibrium. That is, each player $i$ and after every history of play, $\sigma_{i}$ is a best response for player $i$ when other players $-i$ play according to $\sigma_{-i}$.

Value functions. Associated with a Markovian strategy profile $\sigma$, we can define the ex-ante value $v_{i}^{\sigma}$ to express the expected value of the discounted sum of future payoffs of player $i$, at the beginning of a period and before the surpluses are observed, assuming that all players follow strategy $\sigma$.

[^3]Similarly, we can define the values $v_{i}^{\sigma}(s, \mathrm{c})$ as the expected value of the discounted sum of future payoffs of player $i$ at the beginning of the ( $s, \mathrm{c}$ )-negotiation stage game induced by $\sigma$. We define $v_{i}^{\sigma}(s, \varnothing)=\delta v_{i}^{\sigma}$. And also the values $v_{i}^{\sigma}(s)$ as the expected value of the discounted sum of future payoffs of player $i$ after the realization of state $s$ at the beginning of the coalition formation stage.

We use the concept of strong Nash equilibrium (SNE) introduced by Aumann (1959), which is immune not only to unilateral players' deviations but also to group deviations. In order to formally define the SNE concept, consider the coalition formation stage game denoted by $G^{\sigma}(s) \equiv \Gamma\left(N,\left(A_{i}\right)_{i \in N},\left(u_{i}^{\sigma}\right)_{i \in N}\right)$, where $s$ is the state realization. The game $G^{\sigma}(s)$ is the normal form game induced by strategy $\sigma$ among the players $N$, in which players simultaneously choose actions $a_{i} \in A_{i}$, and their payoff is given by $u_{i}^{\sigma}(\mathbf{a}) \equiv v_{i}^{\sigma}\left(s, \mathrm{c}_{f}(\mathbf{a})\right)$.

Definition 2 (Strong Nash equilibrium). A strategy $\mathbf{a}=\left(a_{i}\right)_{i \in N}$ is strong Nash equilibrium of a normal form game $\Gamma$ if and only if for all $\mathrm{s} \subset N$ and for all deviations $\mathbf{a}_{\mathrm{s}}^{\prime}$ by the players s there exists an agent $i \in \mathrm{~s}$ such that $u_{i}(\mathbf{a}) \geq u_{i}\left(\mathbf{a}_{\mathrm{s}}^{\prime}, \mathbf{a}_{-\mathrm{s}}\right) .{ }^{6}$

Definition 3 (Strong Markov perfect equilibrium). A pure stationary Markovian strategy profile $\sigma$ is a strong Markov perfect equilibrium (SMPE) if it is a Markov perfect equilibrium and the restriction of $\sigma$ to the coalition formation stage at any state $s$ is a strong Nash equilibrium of the normal form game $G^{\sigma}(s)$ induced by $\sigma$.

The coalition bargaining solution: Of particular interest to us is the limit equilibrium of the simultaneous-proposer coalition bargaining game with stochastic surplus $\mathbf{s}^{n}$ converging in distribution to the surplus $\bar{s}$, i.e., $\mathbf{s}^{n} \rightarrow \bar{s}$, where $\bar{s} \in \mathbb{R}^{W}$. We refer to the limit coalition bargaining game as $\Gamma(\bar{s}, \delta)$.

One of the main results of this paper is that the limit Markov perfect equilibrium of $\Gamma\left(\mathbf{s}^{n}, \delta\right)$ when $\mathbf{s}^{n} \rightarrow \bar{s}$ is the coalition bargaining solution $(v, \mu)$ defined below.

Definition 4. The coalition bargaining solution $(v, \mu)$ of game $\Gamma(\bar{s}, \delta)$, where $\bar{s}=\left(\bar{s}_{\mathrm{c}}\right)_{\mathrm{c} \in W}$, is given by the value $v=\left(v_{i}\right)_{i \in N}$ and probability of coalition formation $\mu=\left(\mu_{\mathrm{c}}\right)_{\mathrm{c} \in W}$, solution of the problem

$$
\begin{equation*}
v_{i}=\sum_{\mathrm{c} \in W}\left(\delta v_{i}+\gamma\right) I(i \in \mathrm{c}) \mu_{\mathrm{c}} \text { and } \sum_{\mathrm{c} \in W} \mu_{\mathrm{c}}=1 \tag{2}
\end{equation*}
$$

where $\mu_{\mathrm{C}}>0$ only if $\gamma_{\mathrm{C}}=\gamma \equiv \max _{\mathrm{B} \in W} \gamma_{\mathrm{B}}$ and $\gamma_{\mathrm{C}} \equiv \frac{1}{|\mathrm{C}|}\left(\bar{s}_{\mathrm{C}}-\delta \sum_{j \in \mathrm{C}} v_{j}\right)$.
A key property of the solution is that only coalitions that maximize the average gain per coalition member $\gamma_{c}$ form in equilibrium. Moreover, the value of player $i$, conditional on coalition c forming during the coalition formation stage, is equal to $v_{i}(\mathrm{c})=\delta v_{i}+\gamma_{\mathrm{c}}$, if $i \in \mathrm{c}$, and is otherwise equal to zero. The (unconditional) value of player $i$ is given by the average $v_{i}=\sum_{\mathrm{c} \in W}\left(\delta v_{i}+\gamma_{\mathrm{c}}\right) I(i \in \mathrm{c}) \mu_{\mathrm{c}}$, where $\mu_{\mathrm{c}}$ is the probability that coalition c forms in equilibrium.

We conjecture that the coalition bargaining solution is unique: The main property of the solution, contributing to its uniqueness and stability, is that as the probability $\mu_{c}$ increases, the value of the players belonging to coalition c increases, which decreases the coalition gain $\gamma_{\mathrm{c}}$, leading to a lower probability that coalition c is chosen. ${ }^{7}$

## 3. Stage game equilibrium

### 3.1. The negotiation stage

We first analyze the negotiation stage subgame. The negotiation stage is a fairly standard random-proposer multilateral bargaining model in which a group of players are splitting a known surplus and unanimity is required to reach agreement. The status quo of negotiations is the players' continuation values and there is an exogenous risk of breakdown, which is a friction with similarities to discounting (see Binmore et al. (1986)). In this setting, it is well known that the Nash bargaining solution with equal split of the gain from cooperation among coalition members arises as the equilibrium outcome.

Formally, consider an MPE $\sigma$ with expected value $v_{i}^{\sigma}$ and a negotiation stage subgame beginning at node ( $s, \mathrm{c}$ ) with state $s$ and coalition c. Let $v_{i} \equiv v_{i}^{\sigma}$ and $v_{\mathrm{C}} \equiv \sum_{i \in \mathrm{c}} v_{i}$. Whenever the surplus is less than the coalition continuation value, $s_{\mathrm{C}}<\delta v_{\mathrm{c}}$, the negotiation stage terminates without an agreement and the game moves to the next coalition formation round. Certainly this is the case because there are no gains from cooperation.

On the other hand, if the surplus is greater than the coalition continuation value, agreement in the negotiation occurs immediately. The gain from cooperation among the coalition members c is $s_{\mathrm{C}}-\delta v_{\mathrm{C}}>0$, and thus the average gain per

[^4]coalition member is $\gamma(s, \mathrm{c}) \equiv \frac{1}{|\mathrm{c}|}\left(\mathrm{s}_{\mathrm{C}}-\delta v_{\mathrm{c}}\right)$. The expected value of players $v_{i}^{\sigma}(\mathrm{s}, \mathrm{c})$ is the Nash bargaining solution $v_{i}^{\sigma}(\mathrm{s}, \mathrm{c})=$ $\delta v_{i}+\gamma(s, \mathrm{c})$ for all $i \in \mathrm{c}$.

These well-known results from unanimity bargaining are formally stated in the following proposition.
Proposition 1 (Negotiation stage). Consider an MPE $\sigma$ with expected value $v_{i}=v_{i}^{\sigma}$. Then the value $v_{i}^{\sigma}(s, \mathrm{c})$ at the negotiation stage with state $s$ and coalition c is:
(i) If $s_{\mathrm{c}}>\delta v_{\mathrm{c}}$, the value is given by

$$
v_{i}^{\sigma}(s, \mathrm{c})= \begin{cases}\delta v_{i}+\gamma(\mathrm{s}, \mathrm{c}) & \text { if } i \in \mathrm{c}  \tag{3}\\ 0 & \text { if } i \notin \mathrm{c}\end{cases}
$$

where $v_{\mathrm{c}}=\sum_{i \in \mathrm{c}} v_{i}$ and the average gain is $\gamma(\mathrm{s}, \mathrm{c})$ defined by

$$
\gamma(s, \mathrm{c})=\frac{1}{|\mathrm{c}|}\left(s_{\mathrm{c}}-\delta v_{\mathrm{c}}\right)
$$

(ii) If $s_{\mathrm{c}} \leq \delta v_{\mathrm{c}}$, the value $v_{i}^{\sigma}(\mathrm{s}, \mathrm{c})=\delta v_{i}$ for all $i \in \mathrm{c}$ and $v_{i}^{\sigma}(\mathrm{s}, \mathrm{c}) \leq \delta v_{i}$ for all $i \notin \mathrm{c}$.

The equilibrium outcome is the Nash bargaining solution among the players in coalition c with status quo equal to the discounted continuation values and gain equal to $s_{\mathrm{C}}-\delta v_{\mathrm{c}}$. The proof essentially follows the discussion below, and since it is quite standard it is omitted.

The negotiation strategy is for each player $i \in \mathrm{c}$, when proposing, to offer $\delta v_{j}+\phi \gamma(\mathrm{s}, \mathrm{c})$ to the other players $j \in \mathrm{c} \backslash\{i\}$. Upon rejection of an offer, the expected continuation value of player $j \in \mathrm{c}$ is $\phi v_{j}^{\sigma}(s, \mathrm{c})+(1-\phi) \delta v_{j}=\delta v_{j}+\phi \gamma(s, \mathrm{c})$. This is so because with probability $\phi$, negotiations continue and players' $j$ continuation value is $v_{j}^{\sigma}(s, c)$, and with probability ( $1-\phi$ ), negotiations break down and players' $j$ continuation value is $\delta v_{j}$. Thus the response strategy-accept offer $x$ if and only if $x \geq \delta v_{j}+\phi \gamma(s, \mathrm{c} \mid v)$-is a best-response strategy for player $j$.

Moreover, note that the expected value of player $i \in \mathrm{c}$ at the beginning of the negotiation stage subgame is indeed equal to $v_{i}^{\sigma}(s, c)=\delta v_{i}+\gamma(s, c)$. This value arises because with probability $\frac{|\mathrm{cc}|-1}{|\mathrm{c}|}$ he receives an offer $\delta v_{i}+\phi \gamma(s, \mathrm{c})$, and with probability $\frac{1}{|c|}$ he is the proposer, receiving the surplus net of payments to the other coalition members, $s_{\mathrm{C}}-\sum_{j \in \mathrm{C} \backslash\{i\}}\left[\delta v_{j}+\phi \gamma(s, \mathrm{c})\right]$. The expected value is thus

$$
\begin{equation*}
v_{i}^{\sigma}(s, \mathrm{c})=\frac{|\mathrm{C}|-1}{|\mathrm{c}|}\left(\delta v_{i}+\phi \gamma(s, \mathrm{c})\right)+\frac{1}{|\mathrm{C}|}\left(s_{\mathrm{C}}-\sum_{j \in \mathrm{c} \backslash\{i\}}\left[\delta v_{j}+\phi \gamma(s, \mathrm{c})\right]\right) \tag{4}
\end{equation*}
$$

In the knife-edge case where $s_{\mathrm{C}}=\delta v_{\mathrm{c}}$, there is no gain from cooperation, and the negotiation may terminate with an agreement or not. In either case, the continuation values are $v_{i}^{\sigma}(s, \mathrm{c})=\delta v_{i}$ for all $i \in \mathrm{c}$, but the values of the players $i \notin \mathrm{c}$ are indeterminate, and all we can say is that $v_{i}^{\sigma}(s, \mathrm{c}) \leq \delta v_{i}$, since the value of player $i \notin \mathrm{c}$ is zero if there is an agreement and, otherwise, is equal to $\delta v_{i}$. Note that this case has Lebesgue measure zero due to the absolute continuity of the stochastic surplus.

### 3.2. The coalition formation stage

We now characterize the strong Nash equilibrium of the coalition formation stage game. First, we show that the simultaneous move coalition formation game has numerous Nash equilibria; thus it is important to consider an equilibrium refinement. In fact, any coalition $\mathrm{c} \in W$ or $\mathrm{c}=\varnothing$ can be supported as a Nash equilibrium. For example, consider the strategy profile $\mathbf{a}=\left(a_{i}\right)_{i \in N}$ where $a_{i}=\mathrm{c}$ for all $i \in \mathrm{c}$, and $a_{i}=\varnothing$ for all $i \in N \backslash \mathrm{c}$. This profile is a Nash equilibrium, as no unilateral deviation by players improves their payoff, and $c_{f}(\mathbf{a})=c .^{8}$

Consider the strong Nash equilibrium of the coalition formation stage game $G^{\sigma}(s)$, where $s$ is the state realization (see Definition 2). The players' payoff in this game is $u_{i}^{\sigma}(\mathbf{a}) \equiv v_{i}^{\sigma}\left(s, \mathrm{c}_{f}(\mathbf{a})\right.$ ), for all $i \in N$, where $v_{i}^{\sigma}(s, \mathrm{c})$ was determined in Proposition 1. We define $v_{i}^{\sigma}(s, \varnothing)=\delta v_{i}$ for the continuation values when no winning coalition forms. Coalitions form only on states $s$ belonging to the set $\mathcal{A} \equiv\left\{s \in \mathcal{S}: s_{\mathrm{C}}>\delta v_{\mathrm{c}}\right.$ for some $\left.\mathrm{c} \in W\right\}$. There are no gains from cooperation whenever $s \in \mathcal{A}^{c} \equiv \mathcal{S} \backslash \mathcal{A}$ (the complement of $\mathcal{A}$ ) and thus no coalitions form.

The following result characterizes the strong Nash equilibrium of the simultaneous coalition formation game.
Proposition 2 (Coalition formation stage). Let $\sigma$ be a strong Markov perfect equilibrium of $\Gamma(\mathbf{s}, \delta)$, and let $v=v^{\sigma}$. Then:
(i) At the coalition formation stage with state $s \in \mathcal{A}$, the coalition that forms under strategy $\sigma$ is $\mathrm{c}^{* \sigma}(\mathrm{~s}) \in \arg \max _{\mathrm{c} \in W} \gamma(s, \mathrm{c})$, which

[^5]maximizes the average gain $\gamma(s, \mathrm{c}) \equiv \frac{1}{|\mathrm{c\mid}|}\left(s_{\mathrm{C}}-\delta v_{\mathrm{c}}\right)$, where $v_{\mathrm{c}}=\sum_{i \in \mathrm{c}} v_{i}$ and $\mathcal{A} \equiv\left\{s \in \mathcal{S}: s_{\mathrm{C}}>\delta v_{\mathrm{c}}\right.$ for some $\left.\mathrm{c} \in W\right\}$. Moreover, for all $s \in \mathcal{A}$, then
\[

v_{i}^{\sigma}(s)= $$
\begin{cases}\delta v_{i}+\gamma\left(s, \mathrm{c}^{* \sigma}(\mathrm{~s})\right) & \text { if } i \in \mathrm{c}^{* \sigma}(\mathrm{~s}) \\ 0 & \text { if } i \notin \mathrm{c}^{* \sigma}(\mathrm{~s})\end{cases}
$$
\]

and $v_{i}^{\sigma}(s)=\delta v_{i}$ for all $s \notin \mathcal{A}$, except in a set of Lebesgue measure zero where $v_{i}^{\sigma}(s) \leq \delta v_{i}$.
(ii) The set of all (pure strategy) strong Nash equilibrium of the normal form game $G^{\sigma}$ (s), at any state $s \in \mathcal{A}$, is given by $\mathbf{a}=\left(a_{i}\right)_{i \in N}$ such that $a_{i}=c^{*}$ for all $i \in c^{*}$ where $\mathrm{c}^{*} \in \operatorname{argmax}_{\mathrm{c} \in W} \gamma(\mathrm{~s}, \mathrm{c})$.
Furthermore, for almost all states $s \in \mathcal{A}$, except in a set of Lebesgue measure zero, there is a unique maximizing coalition in $\operatorname{argmax}_{\mathrm{c} \in W} \gamma(\mathrm{~s}, \mathrm{c})$, given $\sigma$.

The key property of the strong Nash equilibrium of the coalition formation game is that a coalition $\mathrm{c}^{*}$ maximizing the average gain $\max _{c \in W} \gamma(s, c)$ always forms. And this maximizing coalition, except in a set of Lebesgue measure zero of the state space, is uniquely determined.

Intuitively, coalitions that do not maximize the average gain are dominated by coalitions that maximize the average gain. Once a coalition c is proposed, players' conditional values in the negotiation stage are $\delta v_{i}+\gamma(s, \mathrm{c})$. Any deviation from $c^{*}$ to another coalition $\mathrm{c} \in W$ does not make players in $\mathrm{c} \cap \mathrm{c}^{*} \neq \varnothing$ better off, because $\gamma(\mathrm{s}, \mathrm{c}) \leq \gamma\left(\mathrm{s}, \mathrm{c}^{*}\right)$, showing that the maximizing coalition $\mathrm{c}^{*}$ is a strong Nash equilibrium.

In the Appendix we also consider, in lieu of strong Nash equilibrium (SNE), the concept of a coalition-proof Nash equilibrium (CPNE) developed by Bernheim et al. (1987). The CPNE concept differs in that instead of allowing all possible group deviations, only group deviations that are self-enforcing are allowed (see Definition 5 in the Appendix). We show that for any coalitional bargaining game $\Gamma(\mathbf{s}, \delta)$, the set of coalition-proof Markov perfect equilibria coincides with the set of strong Markov perfect equilibria. ${ }^{9}$

## 4. Markov perfect equilibrium

We now prove the existence of a strong Markov perfect equilibrium for any coalition bargaining game $\Gamma(\mathbf{s}, \delta)$ and provide a complete characterization result.

Key for the characterization proposition below are the results of Propositions 1 and 2 combined with the application of the one-stage deviation principle (Fudenberg and Tirole (1991, Th. 4.2)).

Proposition 3. Consider a coalition bargaining game $\Gamma(\mathbf{s}, \delta)$. Let $\sigma$ be a strong Markov perfect equilibrium. Then:
(i) The value $v^{\sigma}$ associated with $\sigma$ (let $v=v^{\sigma}$ ) satisfies the system of equations, for all $i \in N$,

$$
\begin{equation*}
v_{i}=\sum_{\mathrm{c} \in W_{\mathcal{A}}} \int_{\mathcal{A}}\left(\delta v_{i}+\gamma(s, \mathrm{c})\right) I(i \in \mathrm{c}) I\left(\mathrm{c}^{*}(\mathrm{~s})=\mathrm{c}\right) f(\mathrm{~s}) d s+\delta v_{\mathcal{A}^{c}} \int_{\mathcal{A}^{c}} f(s) d s, \tag{5}
\end{equation*}
$$

where the coalition that forms, at the coalition formation stage with state $s \in \mathcal{A}$, is $c^{*}(s) \in \arg \max _{c \in W} \gamma(s, c)$, maximizer of the average gain $\gamma(s, \mathrm{c}) \equiv \frac{1}{|\mathrm{c}|}\left(s_{\mathrm{c}}-\delta v_{\mathrm{c}}\right)$, where $v_{\mathrm{c}}=\sum_{i \in \mathrm{c}} v_{i}$ and $\mathcal{A} \equiv\left\{s \in \mathcal{S}: s_{\mathrm{c}}>\delta v_{\mathrm{c}}\right.$ for some $\left.\mathrm{c} \in W\right\}$ and $\mathcal{A}^{c}=S \backslash \mathcal{A}$.
Reciprocally, given any value $v \in \mathbb{R}_{+}^{n}$ satisfying (i) above, the strategy profile $\sigma$ defined below by (ii) is a strong Markov perfect equilibrium:
(ii.a) At the coalition formation stage with state $s$ the strategy is: for any $s \in \mathcal{A}, a_{i}(s)=c^{*}(s)$ for all $i \in c^{*}(s)$ and $a_{i}(s)=\varnothing$ for all $i \notin \mathrm{c}^{*}(s) ;$ and for any $s \in \mathcal{A}^{c}, a_{i}(s)=\varnothing$ for all $i \in N$.
(ii.b) At the negotiation stage ( $s, \mathrm{c}$ ) the strategy is: for any $s_{\mathrm{C}}>\delta v_{\mathrm{c}}$, player $i \in \mathrm{c}$ when proposing offers $\delta v_{j}+\phi \gamma(\mathrm{s}, \mathrm{c})$ to players $j \in \mathrm{C} \backslash\{i\}$, and each player $j$ when responding accepts an offer if and only if it is greater than or equal to $\delta v_{j}+\phi \gamma(s, \mathrm{c})$; for any $s_{\mathrm{c}} \leq \delta v_{\mathrm{c}}$, all players $i \in \mathrm{c}$ propose to terminate negotiations (pass), and each player $j$ when responding accepts an offer if and only if it is greater than or equal to $\delta v_{j}$.

The proposition provides a complete characterization of the SMPE. The average gain is equal to $\gamma(s, \mathrm{c})=\frac{1}{|c|}\left(s_{\mathrm{C}}-\delta v_{\mathrm{c}}\right)$, whenever $s_{\mathrm{C}}>\delta v_{\mathrm{c}}$. The key property of the equilibrium is that the coalition forming is ultimately the one that maximizes the average gain, given any surplus realization $s$ of the stochastic surplus $\mathbf{s}$ with gains from trade.

The value at the beginning of the ( $s, \mathrm{c}$ )-negotiation stage game is simply the Nash bargaining solution: $v_{i}(s, \mathrm{c})=$ $\delta v_{i}+\gamma(s, \mathrm{c})$ for all $i \in \mathrm{c}$-the equal split of the surplus among coalition members and status quo equal to the (discounted) continuation values. There is no disagreement in the negotiation stage whenever a coalition c with an aggregate surplus larger than the aggregate discounted continuation value is proposed. Alternatively, a disagreement or breakdown in negotiations is guaranteed when the opposite inequality holds, i.e., $s_{\mathrm{C}}<\delta v_{\mathrm{c}}$.

[^6]The next proposition establishes the existence of a strong Markov perfect equilibrium. We construct a mapping $\Phi$ : $\mathbb{R}_{+}^{n} \rightarrow \mathbb{R}_{+}^{n}$ which takes values $v \in \mathbb{R}_{+}^{n}$ into expected values. The mapping is defined, for all $i \in N$, as

$$
(\Phi(v))_{i}=\sum_{\mathrm{c} \in W_{\mathcal{A}_{\mathrm{c}}(v)}} \int_{\mathcal{A}^{(v)}}\left(\delta v_{i}+\gamma(s, \mathrm{c} \mid v)\right) I(i \in \mathrm{c}) f(s) d s+\int_{\mathcal{A}^{c}(v)} \delta v_{i} f(s) d s
$$

where $\gamma(s, \mathrm{c} \mid v) \equiv \frac{1}{|\mathrm{c}|}\left(s_{\mathrm{c}}-\delta v_{\mathrm{c}}\right)$ is the average gain, with the dependency with respect to the value $v$ made explicit (and we define $\gamma(s, \varnothing \mid v)=0)$. The region with the states where coalition $\mathrm{c} \in W$ is chosen is given by

$$
\begin{equation*}
\mathcal{A}_{\mathrm{C}}(v) \equiv\{s \in \mathcal{S}: \gamma(s, \mathrm{c} \mid v)>\gamma(s, \mathrm{~T} \mid v) \text { for all } \mathrm{T} \in W \cup \varnothing\} . \tag{6}
\end{equation*}
$$

We have shown in the previous proposition that $\mathrm{c}^{* \sigma}(s)=\mathrm{c}$, for all $s \in \mathcal{A}_{\mathrm{C}}\left(v^{\sigma}\right)$. The set $\mathcal{A}^{c}(v) \equiv\left\{s \in \mathcal{S}: s_{\mathrm{c}} \leq \delta v_{\mathrm{c}}\right.$ for all $\mathrm{c} \in$ $W\}$ is the region with the states where there are delays and no coalition is chosen. The integrands and the regions vary continuously with $v$, and thus clearly the mapping $\Phi(v)$ is continuous in $v .{ }^{10}$

We prove that the continuous mapping $\Phi$ has a fixed point (i.e., $v=\Phi(v)$ ) using the Brouwer fixed-point theorem. The fixed points of the mapping $\Phi$ correspond to the solutions of equation (5). Thus, by reciprocal of Proposition 3, we obtain the existence result below.

Proposition 4. There always exists a strong Markov perfect equilibrium $\sigma$ for any coalition bargaining game $\Gamma(\mathbf{s}, \delta)$.

## 5. Limit Markov perfect equilibrium: the coalition bargaining solution

In non-cooperative coalition bargaining models, typically in order for equilibrium to exist, players have to be allowed to play mixed strategies when choosing coalitions. In our setting, players use pure strategy and the randomization comes from small perturbations of the surplus. This approach somewhat resembles Harsanyi's (1973) purification results for normal form games, in which players have some minor private information about their own payoff. However, differently from Harsanyi (1973), all players in our game know the surplus realizations drawn every period, and there is no incomplete information.

We now characterize the limit equilibrium of the coalition bargaining game $\Gamma\left(\mathbf{s}^{n}, \delta\right)$ when $\mathbf{s}^{n}$ converges in distribution to a deterministic coalitional function $\bar{s}=\left(\bar{s}_{\mathrm{c}}\right)_{\mathrm{c} \in W}$. Our main result is that the limit equilibrium is a coalition bargaining solution defined in Section 2.

Define by $\mu_{\mathrm{c}}^{\sigma} \equiv \operatorname{Pr}\left(\mathrm{c}^{* \sigma}(\mathrm{~s})=\mathrm{c}\right)$ the probability that coalition $\mathrm{c} \in W$ forms when the strategy profile $\sigma$ is being played. Note that

$$
\begin{equation*}
\mu_{\mathrm{c}}^{\sigma}=\operatorname{Pr}\left(s \in \mathcal{A}_{\mathrm{c}}\left(v^{\sigma}\right)\right)=\int I\left(\mathrm{c}^{* \sigma}(s)=\mathrm{c}\right) f(s) d s \tag{7}
\end{equation*}
$$

where $\mathcal{A}_{\mathrm{c}}(v)$ is given by (6). Let $\mu^{\sigma}=\left(\mu_{\mathrm{c}}^{\sigma}\right)_{\mathrm{c} \in W}$.
We assume that the limit coalitional function $\bar{s}$ satisfies $\bar{s}_{\mathrm{c}}>0$ for some $\mathrm{c} \in W$ (otherwise, no coalition would create value and ever form in the limit).

The key result is that $\left(v^{\sigma_{n}}, \mu^{\sigma_{n}}\right)$ converge to a coalition bargaining solution ( $\left.v, \mu\right)$ (see Definition 4), where $\sigma_{n}$ is an SMPE of the coalition bargaining model $\Gamma\left(\mathbf{s}^{n}, \delta\right)$.

Proposition 5. Consider a sequence of stochastic shocks $\left(\mathbf{s}^{n}\right)_{n \in \mathbb{N}}$ converging in distribution to a deterministic surplus $\bar{s}=\left(\bar{s}_{c}\right)_{\mathbf{c} \in W}$ satisfying $\bar{s}_{C}>0$ for some $\mathrm{c} \in W: \mathbf{s}^{n} \rightarrow \bar{s}$. Let $\sigma_{n}$ be a strong Markov perfect equilibrium of the coalition bargaining model $\Gamma\left(\mathbf{s}^{n}, \delta\right)$. Then there exists a convergent subsequence $\left(k_{n}\right)$ with limits $v=\lim _{n \rightarrow \infty} v^{\sigma_{k_{n}}}$ and $\mu_{\mathrm{c}}=\lim _{n \rightarrow \infty} \mu_{\mathrm{c}}^{\sigma_{k_{n}}}$ for all $\mathrm{c} \in W$. Moreover, any limit $(v, \mu)$ must be a solution of the following problem:
(i) The value $v$ satisfies the system of equations

$$
\begin{equation*}
v_{i}=\sum_{\mathrm{c} \in W}\left(\delta v_{i}+\gamma\right) I(i \in \mathrm{c}) \mu_{\mathrm{c}}, \text { for all } i \in N \tag{8}
\end{equation*}
$$

$\overline{10}$ In fact, these regions can be explicitly obtained. Note that $\gamma(s, \mathrm{c} \mid v)>\gamma(s, \mathrm{~T} \mid v)$ corresponds to $s_{\mathrm{T}}<\frac{|\mathrm{T}|}{|\mathrm{C}|}\left(s_{\mathrm{C}}-\delta v_{\mathrm{C}}\right)+\delta v_{\mathrm{T}}$, and thus

$$
\mathcal{A}_{\mathrm{C}}(v)=\left\{s \in \mathcal{S}: s_{\mathrm{C}}<\delta v_{\mathrm{C}} \text { and } s_{\mathrm{T}}<\frac{|\mathrm{T}|}{|\mathrm{C}|}\left(s_{\mathrm{C}}-\delta v_{\mathrm{C}}\right)+\delta v_{\mathrm{T}} \text { for all } \mathrm{T} \in W \backslash\{\mathrm{c}\}\right\} .
$$

Thus the probability that coalition $\mathrm{c} \in W$ forms can be obtained directly from the multivariate integral over all $\mathrm{T} \in W \backslash\{\mathrm{c}\}$,

$$
\operatorname{Pr}\left(s \in \mathcal{A}_{\mathrm{C}}(v)\right)=\int_{\delta v_{\mathrm{C}}}^{\infty} \int_{-\infty}^{\frac{|\mathrm{T}|}{|c|}\left(s_{\mathrm{C}}-\delta v_{\mathrm{c}}\right)+\delta v_{\mathrm{T}}} \ldots f\left(s_{\mathrm{C}}, s_{\mathrm{T}}, \ldots\right) d s_{\mathrm{C}} d s_{\mathrm{T}} \ldots
$$

and the probability that no coalition form can be obtained directly from the cumulative distribution function $F\left(\left(\delta v_{\mathrm{c}}\right)_{\mathrm{c} \in W}\right)$.
where $\gamma=\max _{\mathrm{c} \in W} \gamma$ (c) and the average gain $\gamma$ (c) is given by

$$
\gamma(\mathrm{c})=\frac{1}{|\mathrm{C}|}\left(\bar{s}_{\mathrm{C}}-\delta \sum_{i \in \mathrm{C}} v_{i}\right)
$$

(ii) Only the coalitions that maximize the average gain form in equilibrium, that is, $\mu_{\mathrm{c}}>0$ only if $\gamma$ (c) $=\gamma$.
(iii) The limit continuation value $v_{i}(\mathrm{c})$, conditional on coalition $\mathrm{c} \in W$ forming, is

$$
v_{i}(\mathrm{c})= \begin{cases}\delta v_{i}+\gamma(\mathrm{c}) & \text { if } i \in \mathrm{c} \\ 0 & \text { if } i \notin \mathrm{c}\end{cases}
$$

(iv) There is no delay in coalition formation (i.e., $\sum_{c \in W} \mu_{c}=1$ ) and the maximum average gain is strictly positive $\gamma>0$.

The expression characterizing the limit value function simplifies to equations (8), where $\mu_{\mathrm{c}}$ is the probability that coalition c forms in equilibrium, and the average gain $\gamma$ is the maximum average gain among all coalitions $\left(\gamma=\max _{\mathrm{c} \in W} \gamma\right.$ (c)).

Conditional on coalition c forming, the value of player $i \in \mathrm{c}$ is equal to $\delta v_{i}+\gamma$ (c), and the value of all players $i \notin \mathrm{c}$ is zero, since these players are excluded from the coalition that forms and the game ends. Therefore, we obtain that the unconditional expected value, before any coalition forms, must satisfy the value equation (8).

In equilibrium, only coalitions that maximize the average gain $\gamma$ (c) form, and in the limit, with probability one, a coalition always forms in the first period (i.e., $\sum_{c \in W} \mu_{c}=1$ ).

## 6. Properties of the coalition bargaining solution

The coalition bargaining solution is more parsimonious than other existing non-cooperative models that rely on the additional specification of a proposer-recognition protocol as an integral part of the model specification. The selection protocol most commonly adopted by non-cooperative bargaining models is either the random-proposer protocol or a fixed-order proposer/formateur protocol. Kalandrakis (2006) shows that players' values are very sensitive to the proposer recognition protocol. Motivated by the importance of the issue, several studies focus on endogenizing the proposer selection process by adding a round of bidding for the right to be the proposer. ${ }^{11}$

In this paper, we propose an alternative model where players choose coalitions simultaneously, allowing them to freely discuss their coalition choices but not to make binding commitments. We believe this is an economically sensible approach to consider in the study of coalition decision-making problems without an explicitly given proposer-selection protocol.

In the remainder of this section, we develop properties of the coalition bargaining solution for majority voting games, games with dummy players, and games with outside options in order to illustrate the solutions' distinct properties.

Majority voting games: Weighted majority voting games are important in the voting literature. In a weighted majority voting game, each player $i \in N$ has $w_{i}$ votes, and any coalition c with more than half of the total votes (i.e., $\sum_{i \in \mathrm{c}} w_{i}>$ $\left.\frac{1}{2}\left(\sum_{i \in N} w_{i}\right)\right)$ gets the total value of a fixed pie.

Interestingly, Isbell (1959) shows that any weighted majority voting game with less than eight players is isomorphic-and thus has the same player values and set of winning coalitions-to one of the 134 games enumerated on Table 1 in the Online Appendix. We compare the new model solution with other classical solutions for all 134 games representing all weighted majority voting games with less than eight players, and the analysis yields sharp differences in predictions. ${ }^{12}$

The new solution exhibits significantly more inequality among small and large parties than other well-known solutions. Among all 134 majority voting games with less than eight players, the coalition bargaining value of the largest party is on average $11 \%$ higher than the Shapley-Shubik index, $13 \%$ higher than the Banzhaf index, $44 \%$ higher than the randomproposer model with equal protocol, and $29 \%$ higher than the nucleolus and the random-proposer model with proportional protocol (see Table 2 in the Online Appendix).

The coalition bargaining solution also predicts a very concentrated equilibrium coalition formation probability distribution. For example, among the 134 games analyzed, the most likely coalition forms with an average probability of $53.5 \%$, and the two most likely coalitions form with an average probability of $81.8 \%$. In contrast, the underlying logic behind classical solution concepts such as the Shapley-Shubik and Banzhaf indices is that all minimum coalitions have the same weight.

There is a simple intuitive economic reasoning motivating the inequality and concentration property of the new solution: Any winning coalition that replaces two smaller players with $w$ and $w^{\prime}$ votes with one larger player with $w+w^{\prime}$ votes is also winning, and this deviation increases the average gain because there is one less player sharing the surplus; this leads to an overall increase in value of larger players relative to smaller players and to a more concentrated outcome including larger players.

[^7]These properties of the new solution have important implications in normative analysis aiming to determine appropriate voting weights that should be assigned to players in order to achieve a certain social objective (see Le Breton et al. (2012)).

Dummy players: One important distinction between the model in this paper and the random-proposer model is the prediction for the value of dummy players.

Dummy players are players that do not contribute anything to the surplus of coalitions in which they are members. Formally, we say that player $i$ is a dummy player of a coalitional function $\bar{s}=\left(\bar{s}_{c}\right)_{c \in W}$ if for any coalition $\mathrm{c} \in W$ which includes player $i$, that is $i \in \mathrm{c}$, then the coalition $\mathrm{c} \backslash\{i\}$ excluding player $i$ also belongs to $W$, and both surpluses are identical, i.e. $\bar{s}_{\mathrm{C}}=\bar{s}_{C \backslash\{i\}}$.

The simultaneous-proposer model, unlike the random-proposer model, predicts that dummy players have zero value and should never be included in any coalition that forms in equilibrium.

Corollary 1. Let $(v, \mu)$ be a coalition bargaining solution of game $\Gamma(\bar{s}, \delta)$ where player $i$ is a dummy player of the coalitional function $\bar{s}$. The dummy player value is always equal to zero, $v_{i}=0$, and the dummy player is never included in any coalition that forms in equilibrium, $\mu_{\mathrm{c}}=0$ for all $\mathrm{c} \in W$ such that $i \in \mathrm{c}$.

Interestingly, the coalition bargaining solution thus makes similar predictions for the value of dummy players as classical cooperative solution concepts such as the Shapley value, the Banzhaf value, and the nucleolus.

Games with outside options: The new solution applied to seller-buyer games is consistent with the outside option principle developed in Binmore et al. (1989).

Consider a setting with a seller owning one good (player 1) and two buyers (players 2 and 3 ). Let the surpluses be $\bar{s}_{12}=10$ and $\bar{s}_{13}=8$, representing the net buyers' values associated with the two possible winning coalitions.

We show in the Appendix that, for each $\delta \in(0,1)$, the model has a unique coalition bargaining solution given by equation (14). ${ }^{13}$ When $\delta \rightarrow 1$ the solution converges to $v_{1}=8, v_{2}=2, v_{3}=0$ and $\mu_{12}=1, \mu_{13}=0$. Thus the model's prediction is that the seller sells to the highest valuation buyer at a price equal to the binding "outside" option of trading with the lowest valuation buyer.

Alternatively, in the example above, if say $\bar{s}_{13}$ was less than half of the highest buyer value (i.e., $\bar{s}_{13} \leq 5$ ), the "outside" option would not be binding, and the negotiations will lead the seller to sell to the highest valuation buyer at a price equal to half of the highest value: The solution is then $v_{1}=5, v_{2}=5, v_{3}=0$ and $\mu_{12}=1, \mu_{13}=0$ (see Appendix).

The new solution predictions for seller-buyer games are economically sensible and consistent with the outside option principle developed in Binmore et al. (1989). However, differently from their treatment in which the outside option has an exogenous value that can be exercised by the seller unilaterally, in our setting the outside option is derived endogenously by negotiations.

## 7. Conclusion

We propose a new coalition bargaining solution that arises as the strong Markov perfect equilibrium of a non-cooperative coalition bargaining game where players simultaneously choose the coalitions they want to join, followed by a negotiation stage. The solution has several noteworthy economic properties such as predicting an outcome with significantly more inequality than existing solution concepts for majority voting games. Moreover, the new solution is more parsimonious than other existing non-cooperative models that rely on the additional specification of a proposer-recognition protocol. It would be interesting to explore in future empirical studies how the coalition bargaining solution performs, relative to other well-known cooperative and non-cooperative solution concepts, in predicting the outcome of multilateral negotiations.

## Appendix A. Proofs

Proof of Proposition 2. The coalition formation stage subgame is a static game (one-shot game) where a coalition $\mathrm{c} \in W$ forms if and only if all players $i \in \mathrm{c}$ choose c , or otherwise $\mathrm{c}=\varnothing$. Player $i$ payoff is given by $u_{i}(\mathbf{a}) \equiv v_{i}^{\sigma}\left(s, \mathrm{c}_{f}(\mathbf{a})\right.$ ) (see equation (3)).

We first show that the proposed strategy is a strong Nash equilibrium. For all players $i \in c^{*}$ the strategy yields a payoff of $v_{i}\left(s, \mathrm{c}^{*} \mid v\right)=\delta v_{i}+\gamma\left(s, \mathrm{c}^{*} \mid v\right)$. Since $\mathrm{c}^{*}$ maximizes $\gamma(\mathrm{s}, \mathrm{c} \mid v)$ over all $\mathrm{c} \in W$, then $v_{i}\left(s, \mathrm{c}^{*} \mid v\right) \geq v_{i}(s, \mathrm{c} \mid v)$ for all $\mathrm{c} \in W$; and also $v_{i}\left(s, \mathrm{c}^{*} \mid v\right) \geq \delta v_{i}$, which is the payoff if no coalition forms since $\gamma(s, \mathrm{c} \mid v)>0$. Thus no player in $\mathrm{c}^{*}$ can be made better off by any coalitional deviation $\mathrm{s} \subset N$ such that $\mathrm{s} \cap \mathrm{c}^{*} \neq \varnothing$ since the most players $i \in \mathrm{c}^{*}$ can get from any deviation are $\max _{\mathrm{c} \in W \backslash \mathrm{c}^{*}}\left\{v_{i}(\mathrm{~s}, \mathrm{c} \mid v), \delta v_{i}\right\} \leq v_{i}\left(\mathrm{~s}, \mathrm{c}^{*} \mid v\right)$.

Moreover, no coalition $s \subset N \backslash c^{*}$ is winning because $c^{*} \in W$ and the assumption that no two disjoint coalitions can both be winning. Thus any coalitional deviation s including only players in $N \backslash \mathrm{c}^{*}$ will not make the players in s better off

[^8]since any such coalition deviation will not change the outcome and players' payoff-all players in $c^{*}$ are not changing their strategies, and thus coalition $c^{*}$ forms. This shows that the proposed strategy is a strong Nash equilibrium.

We now show that only pure strategies belonging to SNE are strong Nash equilibrium:

$$
S N E \equiv\left\{\mathbf{a}=\left(a_{i}\right)_{i \in N}: a_{i}=\mathrm{c}^{*} \text { for all } i \in \mathrm{c}^{*} \text { where } \mathrm{c}^{*} \in \operatorname{argmax}_{\mathrm{c} \in W} \gamma(\mathrm{~s}, \mathrm{c})\right\} .
$$

Consider any pure strategy profile in which not all the players belonging to a maximizing coalition are choosing that coalition. The outcome of such pure strategy is either a coalition $\mathrm{c} \notin \operatorname{argmax}_{\mathrm{c} \in W} \gamma(s, \mathrm{c} \mid v)$ or no coalition forms. Thus players' payoff following this strategy is less than or equal to $\max _{\mathrm{C} \in W}\left\{v_{i}(s, \mathrm{c} \mid v), \delta v_{i}\right\}$. This strategy cannot be a strong Nash equilibrium because there exists a coalitional deviation that makes all deviating players strictly better off. For example, choose any $\mathrm{s} \in \operatorname{argmax}_{\mathrm{c} \in W} \gamma(\mathrm{~s}, \mathrm{c} \mid v)$ and let all players $i \in \mathrm{~s}$ deviate to $\mathrm{c}_{i}=\mathrm{s}$. All players $i \in \mathrm{~s}$ obtain a payoff $v_{i}(\mathrm{~s}, \mathrm{~s} \mid v)>$ $\max _{c \in W}\left\{v_{i}(s, c \mid v), \delta v_{i}\right\}$.

For almost all states $s$, except in a set of Lebesgue measure zero, the maximization problem $\max _{\mathrm{c} \in W} \gamma(s, \mathrm{c} \mid v)$ has a unique solution. This holds because the set of states where $\gamma(s, \mathrm{c} \mid v)=\gamma(s, \mathrm{~T} \mid v)$ for any $\mathrm{c}, \mathrm{T} \in W$, which corresponds to $\frac{1}{|c|}\left(s_{\mathrm{C}}-\delta v_{\mathrm{C}}\right)=\frac{1}{|\mathrm{~T}|}\left(s_{\mathrm{T}}-\delta v_{\mathrm{T}}\right)$, has measure zero. Q.E.D.

Coalition-proof Markov perfect equilibrium: A pure stationary Markovian strategy profile $\sigma$ is a coalition-proof Markov perfect equilibrium (CPMPE) if it is a Markov perfect equilibrium and the restriction of $\sigma$ to the coalition formation stage at any state $s$ is a coalition-proof Nash equilibrium of the normal form game $G^{\sigma}(s)$ induced by $\sigma$.

The coalition-proof Nash equilibrium definition is provided recursively as follows:
Definition 5 (Coalition-proof Nash equilibrium). (i) In a normal form game $\Gamma$ with a single player ( $n=1$ ) ( $a_{1}^{*}$ ) is a CPNE if and only if it is a Nash equilibrium.
(ii) In a normal form game $\Gamma$ where $n>1$, the profile $\mathbf{a}^{*}=\left(a_{i}^{*}\right)_{i \in N}$ is self-enforcing if for all proper subsets $\mathrm{s} \subset N$, $\mathbf{a}_{\mathrm{s}}^{*}$ is a CPNE of the restriction game $\Gamma / \mathbf{a}_{-s}^{*} .{ }^{14}$
(iii) A profile $\mathbf{a}^{*}$ is CPNE if it is self-enforcing and there is no other self-enforcing profile a that yields higher payoff to all players.

Proposition 6 (CPMPE). For any coalitional bargaining game $\Gamma(\mathbf{s}, \delta)$, the set of coalition-proof Markov perfect equilibria coincides with the set of strong Markov perfect equilibria.

Proof of Proposition 6. Proposition 2 holds if we replace strong Nash by Coalition-proof Nash equilibrium. Indeed, we already know that the set of strong Nash equilibrium is a subset of the set of coalition-proof Nash equilibrium (SNE CCPNE) for any stage game (this is immediate from the fact that the CPNE concept differs from SNE because it restricts the set of feasible group deviations). Thus any strong Markov perfect equilibrium $\sigma$ is also a coalition-proof Markov perfect equilibrium.

We now show that reciprocal $C P N E \subset S N E$ also holds for the coalitional formation game. The proof is by contradiction, and the main idea is that any strategy that does not maximize the average gain cannot be CPNE because there is a joint deviation that is also self-enforcing-the SNE strategy-that leads to a higher payoff to all deviating players.

Suppose there is a CPNE $\mathbf{a}^{*}=\left(a_{i}^{*}\right)_{i \in N}$ that is not a SNE. Let $\mathbf{c}=c_{f}\left(\mathbf{a}^{*}\right)$ be the coalition that forms in this equilibrium. Thus, by the result we have just proven above for SNE, we have that $\gamma(s, \mathrm{c} \mid v)<\max _{\mathrm{s} \in W}\{\gamma(s, \mathrm{~s} \mid v)\}$.

Consider any coalitional deviation $s=c^{*}$ where $c^{*} \in \operatorname{argmax}_{s \in W}\{\gamma(s, s \mid v)\}$, and thus $\gamma\left(s, c^{*} \mid v\right)>\gamma(s, c \mid v)$. By the coalition-proof definition (iii) the profile $\mathbf{a}^{*}=\left(a_{i}^{*}\right)_{i \in N}$ is self-enforcing. Thus by item (ii) of the definition, for all proper subsets $\mathrm{s} \subset N, a_{\mathrm{s}}^{*}$ is a CPNE of the restricted game $\Gamma / \mathbf{a}_{-\mathrm{s}}^{*}$, where $\Gamma=G^{\sigma}(s)$.

But we now show that $\mathbf{a}_{s}^{*}$, with $\mathrm{s}=\mathrm{c}^{*}$, cannot be a CPNE of the restricted game $\Gamma / \mathbf{a}_{-\mathrm{s}}^{*}$, because the restricted game has a self-enforcing profile $\mathbf{a}_{\mathrm{s}}=\left(a_{i}\right)_{i \in \mathrm{~s}}$ that is strictly better for all players in $\Gamma / \mathbf{a}_{-\mathrm{s}}^{*}$ (note that the player set for the restricted game is $\mathrm{s}=\mathrm{c}^{*}$ not $N$ ).

Consider the profile $\mathbf{a}_{\mathrm{s}}=\left(a_{i}\right)_{i \in \mathrm{~s}}$ where $a_{i}=\mathrm{c}^{*}$ for all $i \in \mathrm{c}^{*}=\mathrm{s}$. First, this profile is strictly better for all players $i \in \mathrm{~s}$ because $u_{i}\left(\mathbf{a}_{\mathrm{s}}, \mathbf{a}_{-\mathrm{s}}^{*}\right)=\delta v_{i}+\gamma\left(\mathrm{s}, \mathrm{c}^{*} \mid v\right)>\delta v_{i}+\gamma(\mathrm{s}, \mathrm{c} \mid v) \geq u_{i}\left(\mathbf{a}^{*}\right)$ for all $i \in \mathrm{~s}$.

Now the profile $\mathbf{a}_{s}=\left(a_{i}\right)_{i \in s}$ is a strong Nash equilibrium of the restricted game $\Gamma / \mathbf{a}_{-s}^{*}$ : This is clearly true because in the restricted game $\Gamma / \mathbf{a}_{-\mathrm{s}}^{*}$ all players $i \in \mathrm{~s}$ achieve the maximum possible payoff $u_{i}\left(\mathbf{a}_{s}, \mathbf{a}_{-s}^{*}\right)=\delta v_{i}+\gamma\left(s, \mathrm{c}^{*} \mid v\right)$, and thus no other action can strictly improve all deviating players. But we know that $\operatorname{SNE}\left(\Gamma / \mathbf{a}_{-s}^{*}\right) \subset C P N E\left(\Gamma / \mathbf{a}_{-s}^{*}\right)$, so $\mathbf{a}_{\mathrm{s}}$ is a $C P N E\left(\Gamma / \mathbf{a}_{-s}^{*}\right)$ as well, and thus also self-enforcing, which leads to a contradiction with item (iii) of the coalition-Proof definition applied to the restricted game $\Gamma / \mathbf{a}_{-s}^{*}$. Q.E.D.

Proof of Proposition 3. Let $\sigma$ be either an SMPE of the coalition bargaining game. The value satisfies $v_{i}^{\sigma}=\int v_{i}^{\sigma}(s) f(s) d s$, where $v_{i}^{\sigma}(s)$ is given by Proposition 2. The integral $\int v_{i}^{\sigma}(s) f(s) d s$ is the expression in the right-hand-side below

[^9]$$
v_{i}^{\sigma}=\sum_{\mathrm{c} \in W} \int_{\mathcal{A}}\left(\delta v_{i}+\gamma(s, \mathrm{c})\right) I(i \in \mathrm{c}) I\left(\mathrm{c}^{*}(s)=\mathrm{c}\right) f(s) d s+\delta v_{i}^{\sigma} \int_{\mathcal{A}^{c}} f(s) d s
$$

This completes the necessary part of the proposition.
In order to prove the reciprocal statement, consider any value $v$ satisfying (i), and let $\sigma$ be a strategy profile defined by (ii) (note that the strategy is a function of $v$ ).

First note that the strategy $\sigma$ is such that indeed $v_{i}^{\sigma}=v_{i}$. At any state $s \in \mathcal{A}$, the coalition that forms in the coalition formation stage is $c^{*}(s)$, the coalition that maximizes the average gain, and the final payoff of the players in $c^{*}(s)$ are $\delta v_{i}+\gamma\left(s, \mathrm{c}^{*}(s)\right)$, and the final payoff of players $i \notin \mathrm{c}^{*}(s)$ is zero. This is true because in the ensuing negotiations at node $\left(s, c^{*}(s)\right)$ the strategies are given by (ii), equation (4) holds, and there is immediate agreement. At any state $s \in \mathcal{A}^{c}$, there is no coalition forming and the players' value are the discounted continuation values $\delta v_{i}^{\sigma}$.

Therefore the value $v_{i}^{\sigma}$ satisfies the following equation

$$
v_{i}^{\sigma}=\sum_{\mathrm{c} \in W} \int_{\mathcal{A}}\left(\delta v_{i}+\gamma(s, \mathrm{c})\right) I(i \in \mathrm{c}) I\left(\mathrm{c}^{*}(\mathrm{~s})=\mathrm{c}\right) f(s) d s+\delta v_{i}^{\sigma} \int_{\mathcal{A}^{c}} f(s) d s
$$

and thus

$$
v_{i}^{\sigma}=\left(1-\delta \int_{\mathcal{A}^{c}} f(s) d s\right)^{-1} \sum_{\mathrm{c} \in W} \int_{\mathcal{A}}\left(\delta v_{i}+\gamma(s, \mathrm{c})\right) I(i \in \mathrm{c}) I\left(\mathrm{c}^{*}(\mathrm{~s})=\mathrm{c}\right) f(\mathrm{~s}) d s
$$

which, from equation (5), shows that $v_{i}^{\sigma}=v_{i}$.
From Proposition 1 and the discussion in Section 3 the strategies (ii.b) at the negotiation stage game are a subgame perfect equilibrium of the negotiation stage game ( $s, \mathrm{c}$ ) with continuation values $v_{i}^{\sigma}=v_{i}$. Proposition 2 shows that the strategies (ii.a) at the coalition formation stage are a strong Nash equilibrium of the one-shot coalition formation stage game $G^{\sigma}(s)$ with continuation values $v_{i}^{\sigma}=v_{i}$. By the one-stage-deviation principle (Fudenberg and Tirole (1991, Th. 4.2) then $\sigma$ is an MPE. Moreover, from definition 2, $\sigma$ is also an SMPE. Q.E.D.

Proof of Proposition 4. Let $\Phi: \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}_{+}^{n}$ where

$$
(\Phi(v))_{i}=\sum_{\mathrm{c} \in W_{\mathcal{A}_{\mathrm{c}}(v)}} \int_{\mathcal{A}^{(v)}}\left(\delta v_{i}+\gamma(s, \mathrm{c} \mid v)\right) I(i \in \mathrm{c}) f(s) d s+\int_{\mathcal{A}^{c}(v)} \delta v_{i} f(s) d s
$$

and $\gamma(s, \mathrm{c} \mid v) \equiv \frac{1}{|c|}\left(s_{\mathrm{C}}-\delta v_{\mathrm{c}}\right)$, and $\mathcal{A}_{\mathrm{C}}(v) \equiv\{s \in \mathcal{S}: \gamma(s, \mathrm{c} \mid v)>\gamma(s, \mathrm{~T} \mid v)$ for all $\mathrm{T} \in W \cup \varnothing\}$.
The mapping $\Phi$ is clearly continuous (see footnote 9 ). We now show that there is a convex compact (and non-empty) set $K \subset \mathbb{R}_{+}^{n}$ such that $\Phi(K) \subset K$. Thus by the Brouwer fixed point theorem, this implies that $\Phi$ has a fixed point $v$ such that $v=\Phi(v)$.

Consider the set $K \subset \mathbb{R}_{+}^{n}$ defined by the simplex

$$
K=\left\{v \in \mathbb{R}^{n}: v_{i} \geq 0 \text { and } v_{N} \leq \bar{v}_{N}\right\}
$$

where $\bar{v}_{N} \geq 0$ is given by the solution of the equation

$$
\begin{equation*}
\bar{v}_{N}=\int \max \left\{\delta \bar{v}_{N}, \max _{\mathrm{c} \in W}\left\{s_{\mathrm{C}}\right\}\right\} f(s) d s . \tag{9}
\end{equation*}
$$

Clearly the function $G: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$defined by

$$
G(z)=\int \max \left\{\delta z, \max _{\mathrm{c} \in W}\left\{s_{\mathrm{c}}\right\}\right\} f(s) d s
$$

is monotonically increasing, and $|G(z)-G(w)| \leq \delta|z-w|$ for any $z, w \in \mathbb{R}_{+}$, so it is a contraction mapping with modulus $\delta$. By the contraction mapping theorem there is a (unique) $\bar{v}_{N}$ that satisfy the equation (9), and $\bar{v}_{N}>0$. Thus the set $K$ is non-empty and well-defined.

Certainly for all $v \geq 0$, then $\Phi(v) \geq 0$. It remains to show that if $v_{N} \leq \bar{v}_{N}$ then $\sum_{i \in N} \Phi_{i}(v) \leq \bar{v}_{N}$. But

$$
\begin{aligned}
\sum_{i \in N} \Phi_{i}(v) & =\sum_{\mathrm{c} \in W_{\mathcal{A}_{\mathrm{c}}(v)}} \int_{\mathrm{C}} f(s) d s+\int_{\mathcal{A}^{c}(v)} \delta v_{N} f(s) d s \\
& \leq \int \max \left\{\delta \bar{v}_{N}, \max _{\mathrm{c} \in W}\left\{s_{\mathrm{C}}\right\}\right\} f(s) d s=\bar{v}_{N}
\end{aligned}
$$

which completes the proof that $\Phi(K) \subset K$.
The fixed points of the mapping $\Phi$ correspond to the solutions of equation (5). Thus, by reciprocal of Proposition 3, we obtain explicitly a strategy profile $\sigma$ that is a strong Markov perfect equilibrium of the coalition bargaining game. Q.E.D.

Proof of Proposition 5. Suppose that $\mathbf{s}^{n}$ converges in distribution to $\bar{s}$, where $\bar{s}_{\mathrm{c}}>0$ for some $\mathrm{c} \in W$, and let $\sigma_{n}$ be either an SMPE of $\Gamma\left(\mathbf{s}^{n}, \delta\right)$. The sequences $\left(v^{\sigma_{n}}\right)$ and $\left(\mu_{\mathrm{c}}^{\sigma_{n}}\right)_{\mathrm{c} \in W}$ are bounded. Thus by the Bolzano-Weierstrass theorem there is a convergent subsequence $k_{n} \rightarrow \infty$, such that $v=\lim _{n \rightarrow \infty} v^{\sigma_{k_{n}}}$ and $\left(\mu_{c}\right)_{c \in W}=\lim _{n \rightarrow \infty}\left(\mu_{\mathrm{c}}^{\sigma_{k_{n}}}\right)_{\mathrm{c} \in W}$. In the remainder of the proof, we consider the convergent subsequence $k_{n}$, which we relabel so that $v=\lim _{n \rightarrow \infty} v^{\sigma_{n}}$ and, for all $\mathrm{c} \in W$,

$$
\mu_{\mathrm{c}}=\lim _{n \rightarrow \infty} \mu_{\mathrm{c}}^{\sigma_{n}}=\operatorname{Pr}\left(s \in \mathcal{A}_{\mathrm{C}}\left(v^{\sigma_{n}}\right)\right)=\int_{\mathcal{A}_{\mathrm{c}}\left(v^{\sigma_{n}}\right)} f^{(n)}(s) d s=\int I\left(\mathrm{c}^{* \sigma_{n}}(s)=\mathrm{c}\right) f^{(n)}(s) d s
$$

Consider the following definitions, where the dependency on $v$ is explicitly shown: $\gamma(s, \mathrm{c} \mid v) \equiv \frac{1}{|c|}\left(s_{\mathrm{C}}-\delta v_{\mathrm{c}}\right)$ and $\gamma(s, \varnothing \mid v)=0)$, and $\mathcal{A}_{\mathrm{C}}(v) \equiv\{s \in \mathcal{S}: \gamma(s, \mathrm{c} \mid v)>\gamma(s, \mathrm{~T} \mid v)$ for all $\mathrm{T} \in W \cup \varnothing\}$.

Applying Proposition 4 to each SMPE $\sigma_{n}$, we obtain

$$
\begin{equation*}
v_{i}^{\sigma_{n}}=\sum_{\mathrm{c} \in W_{\mathcal{A}_{\mathrm{c}}\left(v^{\sigma_{n}}\right)}} \int_{i}\left(\delta v_{i}^{\sigma_{n}}+\gamma\left(s, \mathrm{c} \mid v^{\sigma_{n}}\right)\right) I(i \in \mathrm{c}) f^{(n)}(s) d s+\delta v_{i}^{\sigma_{n}} \int_{\mathcal{A}^{c}\left(v^{\sigma_{n}}\right)} f^{(n)}(s) d s \tag{10}
\end{equation*}
$$

First, we show that, for all $\mathrm{c} \in W$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\mathcal{A}_{\mathrm{c}}\left(v^{\sigma_{n}}\right)}\left(\delta v_{i}^{\sigma_{n}}+\gamma\left(s, \mathrm{c} \mid v^{\sigma_{n}}\right)\right) f^{(n)}(s) d s=\left(\delta v_{i}+\gamma(\bar{s}, \mathrm{c} \mid v)\right) \mu_{\mathrm{c}} \tag{11}
\end{equation*}
$$

holds: The $\lim _{n \rightarrow \infty} \int_{\mathcal{A}_{\mathrm{c}}\left(v^{\sigma_{n}}\right)}\left(\delta v_{i}^{\sigma_{n}}+\gamma\left(\bar{s}, \mathrm{c} \mid v^{\sigma_{n}}\right)\right) f^{(n)}(s) d s=\delta v_{i}+\gamma(\bar{s}, \mathrm{c} \mid v) \mu_{\mathrm{c}}$ because the integrand is a constant. Moreover, $\lim _{n \rightarrow \infty} \int_{\mathcal{A}_{\mathrm{c}}\left(v^{\sigma_{n}}\right)}\left(\gamma\left(s, \mathrm{c} \mid v^{\sigma_{n}}\right)-\gamma(\bar{s}, \mathrm{c} \mid v)\right) f^{(n)}(s) d s=0$, because of the continuity of $\gamma(s, \mathrm{c} \mid v)$ and $\mathcal{A}_{\mathrm{c}}(v)$, the convergence of $v^{\sigma_{n}} \rightarrow v$, and the convergence of $\mathbf{s}^{n} \rightarrow \bar{s}$ in distribution.

Second, we show that the maximum limit average gain $\gamma \equiv \max _{\mathrm{c} \in W} \gamma(\bar{s}, \mathrm{c} \mid v)>0$ is strictly positive. Suppose by contradiction that $\gamma \leq 0$. We have the following inequality from equation (10)

$$
v_{i}^{\sigma_{n}} \leq \delta v_{i}^{\sigma_{n}}+\sum_{\mathrm{c} \in W_{\mathcal{A}_{\mathrm{c}}\left(v^{\sigma_{n}}\right)}} \int_{\mathrm{c}} \gamma\left(s, \mathrm{c} \mid v^{\sigma_{n}}\right) f^{(n)}(s) d s \leq \delta v_{i}^{\sigma_{n}}+\int \max _{\mathrm{c}} \gamma\left(s, \mathrm{c} \mid v^{\sigma_{n}}\right) f^{(n)}(s) d s
$$

Continuity of the bounded function $\max _{\mathrm{c} \in W} \gamma\left(s, \mathrm{c} \mid v^{\sigma_{n}}\right)$ and the convergence of $\mathbf{s}^{n} \rightarrow \bar{s}$ in distribution imply that $\lim _{n \rightarrow \infty} \int \max _{\mathrm{c} \in W} \gamma\left(s, \mathrm{c} \mid v^{\sigma_{n}}\right) f^{(n)}(s) d s=\max _{\mathrm{c} \in W} \gamma(\bar{s}, \mathrm{c} \mid v) \leq 0$. Thus taking the limit of the inequality above yields

$$
v_{i} \leq \delta v_{i} \Rightarrow v_{i} \leq 0 \text { for all } i \in N
$$

But this implies the following inequality,

$$
\max _{\mathrm{c} \in W} \gamma(\bar{s}, \mathrm{c} \mid v)=\max _{\mathrm{c} \in W}\left\{\frac{1}{|\mathrm{c}|}\left(\bar{s}_{\mathrm{c}}-\delta v_{\mathrm{c}}\right)\right\} \geq \max _{\mathrm{c} \in W}\left\{\frac{1}{|\mathrm{c}|} \bar{s}_{\mathrm{c}}\right\}>0,
$$

which yields a contradiction.
The strict inequality $\gamma>0$ implies that there exists a $\mathrm{c} \in W$ such that $\bar{s}_{\mathrm{C}}>\delta v_{\mathrm{C}}$. Thus $\mathbf{s}^{n} \rightarrow \bar{s}$ in distribution implies that $\operatorname{Pr}\left(s_{\mathrm{c}}^{n} \leq \delta v_{\mathrm{c}}^{\sigma_{n}}\right) \rightarrow 0$. Therefore,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\mathcal{A}^{c}\left(v^{\sigma_{n}}\right)} f^{(n)}(s) d s=\lim _{n \rightarrow \infty} \operatorname{Pr}\left(\cap_{\mathrm{c} \in W}\left\{s_{\mathrm{c}}^{(n)} \leq \delta v_{\mathrm{c}}^{\sigma_{n}}\right\}\right)=0 \tag{12}
\end{equation*}
$$

Note that this implies that the limit MPE has no delay, that is, $\sum_{\mathrm{c} \in W} \mu_{\mathrm{c}}=1$.
Taking the limit of the left and right hand side of equation (10), and using the results of (11) and (12), yields the desired limit result

$$
v_{i}=\sum_{\mathrm{c} \in W}\left(\delta v_{i}+\gamma\right) I(i \in \mathrm{c}) \mu_{\mathrm{c}}, \text { for all } i \in N . \quad \text { Q.E.D. }
$$

Proof of Corollary 1. Let player $i$ be a dummy player and consider any coalition $\mathrm{c} \in W$ which includes player $i$.
Suppose by contradiction that $\mu_{\mathrm{c}}>0$. From Proposition 5, $\gamma(\mathrm{c})=\gamma=\max _{\mathrm{c} \in W} \gamma(\mathrm{c})>0$. However, the average gain satisfies $\gamma_{\mathrm{c}}<\gamma_{\mathrm{c} \backslash\{i\}}$ because the average gain is

$$
\gamma(\mathrm{c})=\frac{1}{|\mathrm{C}|}\left(\bar{s}_{\mathrm{C}}-\delta v_{\mathrm{c}}\right)<\gamma_{\mathrm{C} \backslash\{i\}}=\frac{1}{|\mathrm{C} \backslash\{i\}|}\left(\bar{s}_{\mathrm{c} \backslash i\}}-\delta v_{\mathrm{c} \backslash\{i\}}\right)
$$

since $|\mathrm{C}|>|\mathrm{C} \backslash\{i\}|, v_{i} \geq 0$, and $\bar{s}_{\mathrm{C}}=\bar{s}_{\mathrm{c} \backslash\{i\}}$, which is a contradiction. This implies that $\mu_{\mathrm{c}}=0$. Moreover, equation $v_{i}=$ $\sum_{\mathrm{c} \in W}\left(\gamma+\delta v_{i}\right) I(i \in \mathrm{c}) \mu_{\mathrm{c}}=0$, proves that $v_{i}=0$. Q.E.D.

Seller-buyer example with outside options. The coalition bargaining solution is given by the solution of:

$$
\begin{align*}
\gamma & =\frac{1}{2}\left(10-\delta v_{1}-\delta v_{2}\right) \text { and } \gamma=\frac{1}{2}\left(8-\delta v_{1}-\delta v_{3}\right)  \tag{13}\\
v_{1} & =\delta v_{1}+\gamma, v_{2}=\left(\delta v_{2}+\gamma\right) \mu_{12}, \text { and } v_{3}=\left(\delta v_{3}+\gamma\right) \mu_{13} \\
1 & =\mu_{12}+\mu_{13} .
\end{align*}
$$

The system of equations (13) can be solved by first noting that $\gamma=(1-\delta) v_{1}$ (from the second eq.). After replacing the value of $\gamma$ in the third and fourth eqs. we obtain $\mu_{12}$ and $\mu_{13}$ as a function of $v_{1}$. Replacing these expressions into the last eq. yields a quadratic equation on $v_{1}$ and the other expressions in (14).

The steps above yield the unique solution of this system with $\mu_{12} \geq 0$ and $\mu_{13} \geq 0$ given by:

$$
\begin{align*}
v_{1} & =\frac{1}{(4-3 \delta)}\left(27-18 \delta-\sqrt{-1852 \delta+804 \delta^{2}+1049}\right), v_{2}=\frac{10-(2-\delta) v_{1}}{\delta} \text { and } v_{3}=\frac{8-(2-\delta) v_{1}}{\delta} \\
\mu_{12} & =\frac{18-(4-3 \delta) v_{1}-8 \delta}{2 \delta} \text { and } \mu_{13}=1-\mu_{12} \tag{14}
\end{align*}
$$

where $v_{1}$ is the solution of the quadratic eq. $(4-3 \delta) v_{1}^{2}-(54-36 \delta) v_{1}+80(2 \delta-1)=0$.
When $\delta \rightarrow 1$ the solution converges to:

$$
v_{1}=8, v_{2}=2, v_{3}=0, \text { and } \mu_{12}=1, \mu_{13}=0, \text { and } \gamma=0
$$

Note that there is no other solution, for any $\delta<1$, with $\mu_{12}=1$, because then $v_{3}=0$, which implies $\gamma_{13}>\gamma_{12}$, which is in contradiction with $\mu_{13}=0$. Therefore, the solution above is the unique solution.

Observe that in the case where $\bar{s}_{13} \leq 5$, then the unique coalition bargaining solution is $\mu_{12}=1, \mu_{13}=0$, and $v_{1}=$ $5, v_{2}=5, v_{3}=0$ and $\gamma=5(1-\delta)$, for all $\delta \in(0,1)$. This is an equilibrium since $\gamma_{13}=\frac{1}{2}\left(\bar{s}_{13}-\delta v_{1}-\delta v_{3}\right)=\frac{1}{2}\left(\bar{s}_{13}-5 \delta\right) \leq$ $\gamma_{12}=5(1-\delta)$ because $s_{13} \leq 5<10-5 \delta$. Q.E.D.

## Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.geb.2022.01.010.

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[^1]:    ${ }^{1}$ For example, in Okada (1996) and Compte and Jehiel (2010), the proposer recognition probability is assumed to be equal among all players, while in Baron and Ferejohn (1989) and Montero (2006) the proposer recognition probability can have an arbitrary distribution. Also, appearing in several models is the fixed-order-proposer protocol such as in Chatterjee et al. (1993).

[^2]:    2 Hart and Kurz (1983) study the question of endogenous coalition formation using an axiomatic concept of value similar to the Shapley value and a static model of coalition formation similar to ours. They propose the concept of stable coalition structures as the ones that are associated with a strong Nash equilibrium, and show that strong equilibrium may in general not exist. Another related model in the industrial organization context is the Yi and Shin (2000) study of endogenous formation of research joint ventures. They consider two static models in which joint ventures are formed simultaneously: the exclusive membership game and the open membership game. The issue of multiplicity of equilibrium also arises in their setting, and they use the coalition-proof Nash equilibrium refinement to study the relative efficiency of both models.
    ${ }^{3}$ That is, (i) if $\mathrm{c} \in W$ then any coalition s containing c also belongs to $W$, and (ii) if coalition $\mathrm{c} \in W$ is a winning coalition then $N \backslash \mathrm{c} \notin W$ is not a winning coalition.

[^3]:    4 The main difficulty in endogenizing proposer probabilities arises in coalition formation games without unanimity. Note that at the negotiation stage, unanimity among the coalition members is required to split the surplus. Thus it is natural to assume equal proposing probabilities, which corresponds to all players having equal bargaining power. The simultaneous move negotiation game proposed in the Online Appendix achieves a similar Nash bargaining solution without making assumptions about proposer probabilities.
    5 See Binmore et al. (1986); Montero (2015); and Battaglini (2021) for similar negotiation models with an exogenous risk of breakdown.

[^4]:    ${ }^{6}$ We use below the standard notation that given a strategy $\mathbf{a}=\left(a_{i}\right)_{i \in N}$ then $\mathbf{a}_{\mathrm{s}}=\left(a_{i}\right)_{i \in \mathrm{~s}}$ is the strategy chosen by the subset s of players and $\mathbf{a}_{-\mathrm{s}}=$ $\left(a_{i}\right)_{i \in N \backslash \mathrm{~s}}$ is the strategy chosen by its complement $N \backslash \mathrm{~s}$, and $\mathbf{a}=\left(\mathbf{a}_{\mathrm{s}}, \mathbf{a}_{-\mathrm{s}}\right)$.
    7 In the Online Appendix, we compare the coalition bargaining solution with the solution of random-proposer models. We also show that the solution is unique for the class of apex games, and also unique for all weighted majority games with less than seven players.

[^5]:    ${ }^{8}$ Indeed, for all $i \in N \backslash \mathrm{c}$, any unilateral deviations by player $i$ do not change the outcome $\mathrm{c}_{f}(\mathbf{a})=\mathrm{c}$; and, for all $i \in \mathrm{c}$, any deviation from $a_{i}=\mathrm{c}$ leads to no coalition forming, which yields a payoff $\delta v_{i} \leq \max \left\{\delta v_{i}, \delta v_{i}+\gamma(s, \mathrm{c})\right\}$.

[^6]:    9 The CPNE definition automatically implies that $S N E \subset C P N E$. Moreover, we also show that any CPNE is also an SNE (i.e., CPNE $\subset S N E$ ): Any coalition choice that does not maximize the average gain cannot be a CPNE because there is a self-enforcing deviation-the maximizing average gain coalition-that makes all deviating players better off.

[^7]:    11 For example, Macho-Stadler et al. (2006), Yildirim (2007), and Ali (2015).
    12 This application is of practical relevance because majority voting games with less than eight players are quite common in real world voting applications. For example, in the empirical analysis of Diermeier and Merlo (2004), among the 313 government formations in 11 multi-party democracies over the period 1945-1997, the median distribution of the number of parties is seven, and the mean is 7.35 .

[^8]:    13 In Definition 4 it is important, in general, to have $\delta \in(0,1)$ in order to obtain a unique solution. For example, with $\delta=1$, the definition yields a continuum of solutions for the seller-buyer game analyzed: it can be directly verified that any $v_{1} \in[8,10], v_{2}=10-v_{1}$ and $v_{3}=0$ and $\mu_{12}=1, \mu_{13}=0$ is a solution.

[^9]:    14 The restriction game $\Gamma / \mathbf{a}_{-\mathrm{s}}^{*}=\Gamma\left(\mathrm{s},\left(A_{i}\right)_{i \in \mathrm{~s}},\left(u_{i}^{\sigma}\left(\cdot, \mathbf{a}_{-\mathrm{s}}^{*}\right)\right)_{i \in \mathrm{~S}}\right)$ denotes the game with player set s where the strategies chosen by the complement $N \backslash \mathrm{~s}$ are fixed at $\mathbf{a}_{-s}^{*}$.

