Solving Continuous Time Affine Jump-Diffusion Models for Econometric Inference

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Abstract

This asset-pricing financial manuscript was written as a part of my dissertation to accompany our empirical working paper with S. Chib and M. Jensen on the joint Bayesian MCMC estimation of objective and risk-neutral parameters, given historical prices of the underlying S&P500 and a cross-sectional panel of options written on it, in continuous time Affine-Jump Diffusion partial equilibrium setting. All the finance-related applied technicalities of the change of measure through risk premium specification and option pricing are carefully worked out in details to avoid ambiguities and mistakes in the main paper. Such detailed treatment allows for enhanced transparency of our work, which can now focus on our contribution in terms of empirical methodology and discussion of the results, while deferring most of the asset-pricing model-specific technicalities of the implementation of our work to this manuscript.

Keywords: continuous time, asset pricing, change of measure, Girsanov theorem, fundamental theorem of asset pricing, multi-dimensional jump-diffusion Ito's lemma, Euler discretization, affine jump-diffusion, stochastic volatility, stochastic discount factor, risk premium specification, fundamental PDE for contingent claims, Feynman-Kac solution, Riccati equation, option pricing, Fourier transform

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1 Introduction and Preliminary Remarks

This manuscript is largely based on the fundamental paper in Econometrica by Duffie, Pan, Singleton (2000) to which we will refer as DPS(2000) in what follows. Our purpose is to show how in partial equilibrium setting one can start with a particular objective Affine Jump-Diffusion (AJD) dynamics of state variables, make assumptions about the dynamics of SDF followed by risk premium derivation, work out change of measure to compute the risk-neutral parameters, and finally adapt the general procedure described in DPS(2000) to option pricing for the particular economy under consideration.

We emphasize applied derivations and intuition behind the results, because the purpose of this manuscript is to provide mainly financial asset pricing details of practical implementation of the AJD model considered in the companion paper. For more rigorous mathematical treatment od some technical issues discussed here, the reader is referred to Protter (2003), Oksendal (2003) and DPS(2000).

Due to applied focus of our work, we often need to switch back and forth between continuous-time and its (Euler-)discretized analogue to fit our model to discretely observed data. Therefore, as necessary, we will attempt to make careful and explicit distinction between these two settings and contrast them from econometric and asset pricing prospectives.

2 Review of Empirical Literature

Estimation of continuous-time models has become an increasingly popular area of research over the last decade. Continuous-time is used to model state variables such as interest rates, leading to models of term-structure and credit risk [Dai and Singleton (2003)], as well as stock prices and volatility enabling us to study dynamics of equity and derivatives markets. While there are several survey papers covering these topics [Bates (2003), Garcia, Ghysels, Renault (2003), Johannes and Polson (2003)] in this section we shall focus our attention on the papers that attempt to learn about dynamics of a given underlying equity given the time series of its historical prices and a panel of options written on it.

While inference conditional on information set enriched with option prices has a strong intuitive appeal compared to estimating AJD models using only prices of the underlying equity [Eraker, Johannes, and Polson (2004), Chernov, Gallant, Ghysels, Tauchen (2003)], researchers have quickly discovered that conducting joint inference using full information methods in this setting is quite challenging and computationally expensive. Some of the pioneering work in this area has been done by Bates (1996), (2000) and Bakshi, Cao, Chen (1997), where in the context of Stochastic Volatility with Jumps (SVJ) various features were added and evaluated, such as time-varying jump intensity, state-dependent risk free rate, multi-factor volatility specification, etc. Their econometric methodology was a two-step approach: from the cross-section of option prices infer structural objective and risk premium parameters as well as latent factor realizations and then analyze them in the time-series domain given the observed evolution of the underlying security.

A number of variations of the above two-step estimation approach have followed: Chernov and Ghysels (2000) in the context of Heston (1993) Stochastic Volatility model use Gallant and Tauchen (1996) EMM to fit seminonparametric joint density of the underlying and implied volatilities, while filtering spot volatilities via reprojection. Andersen, Benzoni, Lund (2002) included the volatility factor in the mean return drift coefficient and have established "... a general correspondence between the dominant characteristics of the equity return process and option prices..." using EMM methods on SVJ model. Chernov (2003) solves for coefficients (market prices of risk) of the SDF dynamics in artificially complete multi-factor stochastic volatility model-economy. Broadie, Chernov, Johannes (2004) explicitly state that in order to "...compromise between the competing constraints of computational feasibility and statistical efficiency" they engage in a two-stage approach where they use objective "...parameter estimates obtained from prior studies using a long time-series of returns. Then, given these parameters, we use the information embedded in options to estimate volatility and risk premiums." In their effort to fit Stochastic Volatility with correlated jumps both in stock price and volatility, and investigate importance of jumps in volatility, they find it crucial to include a broad cross-section of options to learn as much as possible about the latent volatility process. However they immediately note that despite the theoretical advantages of joint estimation "...the extreme computational burdens generated when using both sources of data severely constraints how much and what type of data can be used." Bates himself is still attempting to enhance his original estimation by developing a direct filtration-based maximum likelihood methodology in transform space of characteristic functions in Bates (2004).

The data sets used in most of the papers in this area include some large index such as S&P 500, due to availability of rich panel of options written on it. These options are traded frequently for most maturities and exercise prices. However, Bakshi, Cao (2004) and Dubinsky, Johannes (2005) decided to apply similar techniques to data sets of individual stocks. As they are dealing with more than one underlying security, computational demands become even more severe and they chose to focus on the option error to maximize its likelihood or minimize squared error virtually ignoring the dynamic structure of the underlying securities in their estimation.

There were, however, successful attempts to conduct a genuine joint inference. Pan (2002) estimates SVJ model, enriched with CIR process for risk-free rate and dividends, by augmenting classical GMM method with "implied-state" volatility derived by inverting the option-pricing relation for a given set of parameters. One of the notable contributions of this paper is faster numerical integration procedure for option pricing computation based on truncation error as a function of desired precision. Among the empirical findings of this paper I would like to emphasize suggestive evidence in the support of jumps in volatility, despite the fact that she only considered SVJ model. While GMM methodology is not likelihood-based, there are several papers that employ full-information genuine likelihood-based Bayesian MCMC methods. In the simplest SV model setting Polson and Stroud (2003) illustrate how such joint inference could be conducted using a panel of three options. In a very diligently written paper Jones (2003) estimates SV with constant elasticity of variance model as well as its generalization. He uses Market Volatility Index (VIX) as a proxy for the expected future average variance, which in turn provides information about the latent volatilities via asset-pricing relationship.

One common theme runs through all these papers: trade-off between the quality of the data, richness of the model, and reliability of econometric inference. In our companion paper we will present the state-of-the-art MCMC estimation techniques that coupled with powerful computing resources would allow us to avoid compromising on all these issues.

3 Model Specification under the Objective Distribution

3.1 Continuous time security dynamics

We begin by specifying the dynamics at time t of the underlying stock prices S_t and volatility[1] V_t under the objective measure P:

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= \mu_t dt + \sqrt{V_{t-}} dW_t^s(P) + \Gamma_t^s dJ_t^s - E_t(\Gamma_t^s dJ_t^s) \\ dV_t &= \kappa_v(\theta - V_{t-})dt + \sqrt{V_{t-}}\sigma_v dW_t^v(P) + Z_t^v dJ_t^v \end{aligned} \tag{1}$$

where Brownian motions $W_t^s(P), W_t^v(P)$ are correlated: $Cov(dW_t^s(P), dW_t^v(P)) = \overline{\rho} dt$, random variables Γ_t^s and Z_t^v are respectively price and volatility jump sizes at time t. Depending on the specific distributional assumptions for these variables that we will make in section (3.5), several types of models could be specified, including stochastic volatility (SV), stochastic volatility with jumps (SVJ), and stochastic volatility with jumps in volatility (SVJV). Jump occurrence variable $J_t^{(i)}$ $(i = s, v)$ is a Poisson counter with instantaneous intensity $\overline{\lambda}$: Prob$(dJ_t = 1) = \overline{\lambda} dt$. In empirical work J_t^s is usually set equal to J_t^v, thereby assuming that the same Jump shock hits both stock price and volatility. This allows us to introduce a meaningful[2] relationship between the jump sizes Γ_t^s and Z_t^v by modelling their joint distribution leading to stochastic volatility with correlated jumps (SVCJ) model. Finally, we can allow the jump intensity parameter $\overline{\lambda}$ to be a time-varying function of the state-variable, resulting in the most general model (SVSCJ) considered in this paper.

Note that the inclusion of compensator term $E_t(\Gamma_t^s dJ_t^s)$ was necessary in order to be able to interpret the drift μ_t as $E[\frac{dS_t}{S_t}]$, which is later shown to equal the risk free rate plus risk premium in the dynamics of S_t under objective measure. However, because volatility is not a traded asset, the notion of "risk-free + risk premium" is meaningless for volatility, the compensator in

[1] In order for CIR (square-root) with Jumps dynamics of volatility to remain always positive the following technical condition is sufficient: $2\kappa\theta - \sigma_v^2 > 0$, given that volatility jump size Z_t^v is nonnegative.

[2] If $J_t^s \neq J_t^v$, the relationship between Γ_t^s and Z_t^v becomes much harder to interpret.

V_t dynamics is not introduced. The speed of volatility mean-reversion to an unknown long-run mean θ is controlled by unknown parameter κ_v. Volatility of volatility σ_v is also an unknown parameter to be estimated. We assume that our state variables $X = (S, V)$ are right-continuous (cadlag[3]) and denote the left limit $X_{t-} = \lim_{s\uparrow t} X_s$. However, for notational clarity in what follows, distinctions between the pre-jump X_{t-} and post-jump X_t values will be made only when necessary to avoid confusion and are assumed to be implicitly implied otherwise.

Equivalently, assuming $dW_t^{(1)}(P)$ and $dW_t^{(2)}(P)$ are *orthogonal* Brownian motions we have:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sqrt{V_t} dW_t^{(1)}(P) + \Gamma_t^s dJ_t^s - E_t(\Gamma_t^s dJ_t^s) \\ dV_t &= \kappa_v(\theta - V_t)dt + \sqrt{V_t}\sigma_v(\overline{\rho} dW_t^{(1)}(P) + \sqrt{1-\overline{\rho}^2} dW_t^{(2)}(P)) + Z_t^v dJ_t^v \end{aligned} \tag{2}$$

3.2 Discrete time security dynamics

For estimation purposes we need to apply stochastic discrete-time approximation to our stochastic continuous-time model because real-life databases report prices in discrete time intervals. As most papers in the field, we will use the *Euler Approximation*[4], also sometimes called Euler-Maruyama approximation, that transforms our continuous-time model (1) into its discrete-time analogue:

$$\begin{aligned} \frac{S_{t\Delta} - S_{(t-1)\Delta}}{S_{(t-1)\Delta}} &= \left(\mu_{(t-1)\Delta} - \overline{\lambda_{t\Delta}} E_{t\Delta}\left[\Gamma_{t\Delta}^s\right]\right)\Delta + \sqrt{V_{(t-1)\Delta}\Delta}\epsilon_{t\Delta}^s + \Gamma_{t\Delta}^s J_{t\Delta}^s \\ V_{t\Delta} - V_{(t-1)\Delta} &= \kappa_v(\theta - V_{(t-1)\Delta})\Delta + \sqrt{V_{(t-1)\Delta}\Delta}\epsilon_{t\Delta}^v + Z_{t\Delta}^v J_{t\Delta}^v \end{aligned} \tag{3}$$

where Δ is usually set less than or equal to the uniform time interval between market prices reported in the utilized database[5]. In discrete time, correlated Brownian motions are replaced with a bivariate normal variable:

$$(\epsilon_{t\Delta}^s, \epsilon_{t\Delta}^v)' \sim \mathcal{N}(\mathbf{0}, \Sigma), \quad \Sigma = \begin{pmatrix} 1 & \overline{\rho}\sigma_v \\ \overline{\rho}\sigma_v & \sigma_v^2 \end{pmatrix}.$$

Although jumps could occur more than once over any discrete-time interval of length Δ, empirically we wouldn't be able to distinguish "more jumps" from "bigger jumps" (jump size magnitude vs. jump arrival frequency) given no observed data points inside the time interval. Therefore, in order to preserve econometric identifiability, jump arrivals $J_{t\Delta}^s, J_{t\Delta}^v$ are modelled as Bernoulli binary random variables, in contrast to Poisson-distributed jumps in the continuous-time formulation (1). Another unfortunate side-effect of Euler discretization is that positivity of volatility can no longer be guaranteed, even if volatility jump size is modelled to be nonnegative: at any

[3]See Protter(2003), or Cont and Tankov (2004).

[4]Kloeden and Platen (1992) chapter 9 provides a detailed discussion of strong/weak convergence, consistency, numerical stability and other properties of Euler approximation. Higher order strong Taylor approximations are discussed in chapter 10.

[5]Our data in the main paper came from OptionMetrics and merged CRSP/Compustat databases, where daily closing prices are reported, therefore we used $\Delta = 1$ day. Using smaller Δ would improve approximation accuracy, but it would require data augmentation approach similar to Elerian, Chib, Shephard (2001).

time $t\Delta$ there is a non-zero probability that $\epsilon^v_{t\Delta}$ will be large enough to turn volatility negative. This probability could be greatly reduced by decreasing Δ and working with relatively small volatility of volatility parameter σ_v, but it can never be eliminated completely.

3.3 Stochastic Discount Factor dynamics and risk premiums

The cornerstone of Finance is the *Fundamental Theorem of Asset Pricing*[6] (FTAP), which states that No-Arbitrage restriction is equivalent to the existence of some positive stochastic discount factor (SDF) that prices all payoffs (even if the markets are incomplete, while complete markets imply uniqueness of SDF), which lies in the space spanned by the priced risk factors in the economy. In our case we assume that there are only 3 priced risk factors: $dW^{(1)}_t, dW^{(2)}_t, \Gamma^s_t dJ^s_t$. In addition, we assume the existence of a risk free security following $\frac{dB}{B} = r_t dt$. This implies a negative risk-free mean in the SDF dynamics, because SDF must price the risk-free security correctly as well as the stock (see the next section for more detailed explanation). Therefore, the generic dynamics of the SDF M_t could be written as:

$$\frac{dM_t}{M_t} = -r_t dt - \zeta^{(1)}_t dW^{(1)}_t(P) - \zeta^{(2)}_t dW^{(2)}_t(P) + \Gamma^M_t dJ^s_t - E_t(\Gamma^M_t dJ^s_t) \tag{4}$$

or equivalently as

$$\frac{dM_t}{M_t} = -r_t dt - \zeta^s_t dW^s_t(P) - \zeta^v_t dW^v_t(P) + \Gamma^M_t dJ^s_t - E_t(\Gamma^M_t dJ^s_t) \tag{5}$$

The above specification hinges on a simplifying assumption that volatility jump risk is not priced. By adding $Z^M_t dJ^v_t - E_t(Z^M_t dJ^v_t)$ to SDF dynamics above we can easily relax this assumption. Because to our knowledge, there is no deep GE-based economic argument behind pricing volatility jump risk, this assumption is best approached from the empirical prospective. We will argue on econometric ground that it would be hard to estimate the risk premium associated with this risk component due to the fact that we are already pricing the volatility diffusion risk, while the volatility itself is not observed.

From the asset-pricing prospective, because at every point in time we have more priced sources of risk (three) than traded securities excluding the risk free bond (one), SDF is not uniquely identified in terms of the unknown market prices of risk: $\zeta^{(1)}_t, \zeta^{(2)}_t$ and stochastic jump size Γ^M_t. Therefore, we are working with incomplete market economy setting, yet we can still appeal to the FTAP, which enables us to derive asset pricing formulas up to the unknown risk premium parameters using the intuitive No-Arbitrage assumption. Option pricing approach of DPS(2000) is entirely based on the FTAP. Having derived all the pricing results for our incomplete market economy, in the *empirical* setting inference about the unknown SDF parameters could still be made using the observed prices on a cross-section of additional traded securities such as options. Certainly, from asset-pricing prospective, simply adding two options to the

[6] See Cochrane(2000) for detailed and intuitive treatment of SDF-based asset pricing using geometric representation and linear algebra

above economy would effectively complete the market and adding even only one more for a total of three options in the cross-section would overidentify the SDF parameters leading to arbitrage opportunities in such economy. However, in our empirical model, option prices "come" with some non-priced security-specific noise, which precludes SDF parameter overidentification[7]. Intuitively, this approach could be compared to a simple multivariate linear regression, when there is only k unknown parameter to estimate, but $T > k$ data points that could potentially overidentify the system, if we try to force the exact model fit. The presence of noise comes to rescue and the issue of identifying the unknown parameters becomes one of econometric inference. That is exactly what we want to motivate here: making a transition from option pricing in incomplete market consisting only of the underlying and riskless bond using FTAP, to econometric problem of estimating the unknown parameters of SDF having added a cross-section of options.

3.4 Risk Free rate

Option pricing methodology of DPS(2000) is applicable for quite robust (although linear in state variables) specifications of the risk free rate dynamics, the most general of which would express the interest rate as a linear function of the state variables while introducing a new source of interest rate-specific uncertainty, which could add another dimension to the SDF. However, a number of empirical papers (e.g. Bakshi, Chao and Chen (1997), Pan (2002)) have found that stochastic interest rates don't influence the model fit to the data. Intuitively, this assumption is usually justified by the fact that the life-span of derivative securities is relatively too short for time-varying interest rates to significantly influence model dynamics. Therefore, as most current empirical papers on derivative pricing, we will assume that the risk free rate r is constant, while for the sake of generality we will sometimes use the time subscript r_t in some formulas.

3.5 Econometric model

All the above discussion allows us to formulate an econometric model to *make joint inference on objective and risk neutral parameters given a panel of options and a time-series of the underlying security.* Let D be the observed derivative price, Y the logarithm of the asset price and V be the latent volatility. For time $t = 0, \ldots, T-1$ and traded derivatives $j = 1, \ldots, n_t$, the time-discretization of the derivative price and the asset's and volatilities stochastic difference equation are:

$$\begin{aligned} D_{t\Delta,j} &= F(Y_{t\Delta}, V_{t\Delta}; \chi_j, \Theta, \Lambda) + \epsilon_{t\Delta,j} \qquad (6) \\ Y_{t\Delta} - Y_{(t-1)\Delta} &= \left(\mu_t - \frac{1}{2}V_{(t-1)\Delta} - \overline{\lambda} E_t\left[\exp\{Z^s_{t\Delta}\} - 1\right]\right)\Delta + \sqrt{V_{(t-1)\Delta}\Delta}\,\epsilon^s_{t\Delta} + Z^s_{t\Delta} J^s_{t\Delta} \\ V_{t\Delta} - V_{(t-1)\Delta} &= \kappa(\theta - V_{(t-1)\Delta})\Delta + \sqrt{V_{(t-1)\Delta}\Delta}\,\epsilon^v_{t\Delta} + Z^v_{t\Delta} J^v_{t\Delta} \end{aligned}$$

[7] See Bates(2000) p. 195 for a discussion of overidentifying restriction that all options be priced exactly by a parsimoniously parameterized model, and the discussion of option pricing errors.

where,

$$\begin{aligned}
\epsilon_{t\Delta,j} &\sim \mathcal{N}(\rho_j \epsilon_{(t-1)\Delta,j}, s_j^2), \qquad \epsilon_{t\Delta,j} \perp \epsilon_{t\Delta,j'},\ j \neq j',\\
(\epsilon_{t\Delta}^s, \epsilon_{t\Delta}^v)' &\sim \mathcal{N}(\mathbf{0}, \Sigma),\\
\Sigma &= \begin{pmatrix} 1 & \rho\sigma_V \\ \rho\sigma_V & \sigma_V^2 \end{pmatrix},
\end{aligned}$$

χ_j, ρ_j and s_j^2 denote respectively option j-specific parameters of contract terms, autocorrelation coefficient, and pricing error variance. The risk premium parameters contained in Λ provide *the link from objective* (Θ) *to risk neutral parameters*, which serves as the main focus of this paper, used to find the theoretical (no-arbitrage implied) option price F.

It is convenient to set: $\bar{\lambda} = \lambda^s + \lambda^v + \lambda^c$, because it incorporates the following models:

- SV:

$$J_{t\Delta}^s = J_{t\Delta}^v = 0$$

- SVJ:

$$\begin{aligned}
J_{t\Delta}^s &\sim Bernoulli(\lambda^s)\\
J_{t\Delta}^v &= 0\\
Z_{t\Delta}^s &\sim \mathcal{N}(\mu_s, \sigma_s^2)\\
\overline{\mu} &= \exp\{\mu_s + 0.5\sigma_s^2\} - 1\\
\overline{\mu}^Q &= \exp\{\mu_s^Q + 0.5\sigma_s^2\} - 1
\end{aligned}$$

- SVSJ:

$$\begin{aligned}
J_{t\Delta}^s &\sim Bernoulli(\lambda^s)\\
J_{t\Delta}^v &= 0\\
Z_{t\Delta}^s &\sim \mathcal{N}(\mu_s, \sigma_s^2)\\
\overline{\mu} &= \exp\{\mu_s + 0.5\sigma_s^2\} - 1\\
\overline{\mu}^Q &= \exp\{\mu_s^Q + 0.5\sigma_s^2\} - 1
\end{aligned}$$

  The jump intensity is time-varying, $\lambda_{t\Delta} = \lambda_0 + \lambda_1 V_{(t-1)\Delta}$.

- SVJV:

$$\begin{aligned}
J_{t\Delta}^s &= 0\\
J_{t\Delta}^v &\sim Bernoulli(\lambda^v)\\
Z_{t\Delta}^v &\sim \exp(\mu_v)\\
\overline{\mu} &= 0\\
\overline{\mu}^Q &= 0
\end{aligned}$$

- SVCJ:

$$\begin{aligned} J^s_{t\Delta} = J^v_{t\Delta} &\sim Bernoulli(\lambda^c) \\ Z^v_{t\Delta} &\sim \exp(\mu_v) \\ Z^s_{t\Delta}|Z^v_{t\Delta} &\sim \mathcal{N}(\mu_s + \rho_J Z^v_{t\Delta}, \sigma_s^2) \\ \overline{\mu} &= \exp\{\mu_s + 0.5\sigma_s^2\}/(1-\mu_v\rho_J) - 1 \\ \overline{\mu}^Q &= \exp\{\mu_s^Q + 0.5\sigma_s^2\}/(1-\mu_v\rho_J) - 1 \end{aligned}$$

- SVSCJ:

$$\begin{aligned} J^s_{t\Delta} = J^v_{t\Delta} &\sim Bernoulli(\lambda^c_{t\Delta}) \\ Z^v_{t\Delta} &\sim \exp(\mu_v) \\ Z^s_{t\Delta}|Z^v_{t\Delta} &\sim \mathcal{N}(\mu_s + \rho_J Z^v_{t\Delta}, \sigma_s^2) \\ \overline{\mu} &= \exp\{\mu_s + 0.5\sigma_s^2\}/(1-\mu_v\rho_J) - 1 \\ \overline{\mu}^Q &= \exp\{\mu_s^Q + 0.5\sigma_s^2\}/(1-\mu_v\rho_J) - 1 \end{aligned}$$

  The jump intensity is time-varying, $\lambda^c_{t\Delta} = \lambda_0 + \lambda_1 V_{(t-1)\Delta}$.

Formulas for $\overline{\mu}^Q \equiv E^Q_t\left[\exp\{Z^s_{t\Delta}\} - 1\right]$ and $\overline{\mu} \equiv E_t\left[\exp\{Z^s_{t\Delta}\} - 1\right]$ will be derived later in section (6.6).

4 Some Useful Results in Continuous Time

In derivations that follow we will make extensive use of the following important result from Ito Calculus:

Theorem 1 ***Multi-Dimensional Jump-Diffusion Ito's Formula:***[8]
Let $Y_t = F(X_t, t)$, where X_t is an N-dimensional AJD process:
$dX_t = \mu(X_{t-})dt + \sigma(X_{t-})dW_t + g(X_{t-})\Gamma_t dJ_t$. *Then,*

$$\begin{aligned} dY_t &= \frac{\partial F}{\partial t}(X_{t-}, t-)dt + \sum_i^N \frac{\partial F}{\partial X^i}(X_{t-}, t-)dX_t^i + \frac{1}{2}\sum_{i,j}^N \frac{\partial^2 F}{\partial X_i \partial X_j}(X_{t-}, t-)dX_t^i dX_t^j \\ &+ (F(X_{t-} + g(X_{t-})\Gamma_t, t-) - F(X_{t-}, t-))dJ_t \end{aligned} \tag{7}$$

This result is of such great practical use that we find helpful to also mention its special case:
One-Dimensional Jump-Diffusion Ito's Formula:
If $dS_t = \mu(S_{t-})dt + \sigma(S_{t-})dW_t + g(S_{t-})\Gamma_t dJ_t$, then for $Y_t = F(S_t, t)$:
$dY_t = (\frac{\partial F}{\partial t} + \mu(S_{t-})\frac{\partial F}{\partial S} + \frac{\sigma^2(S_{t-})}{2}\frac{\partial^2 F}{\partial^2 S})dt + \sigma(S_{t-})\frac{\partial F}{\partial S}dW_t + (F(S_{t-} + g(S_{t-})\Gamma_t, t) - F(S_{t-}, t))dJ_t$
Now, applying Ito's lemma for Jump-Diffusion case to transformations $Y_t = ln(S_t)$ and $ln(M_t)$ with dynamics of S_t and M_t given in (19) and (4) respectively we get[9]:

$$\begin{aligned} dY_t &= (\mu_t - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^{(1)}(P) + ln(1 + \Gamma_t^s)dJ_t^s - E_t(\Gamma_t^s dJ_t^s) \\ dln(M_t) &= -(r_t + \frac{1}{2}(\zeta_t^{(1)2} + \zeta_t^{(2)2}))dt - \zeta_t^{(1)}dW_t^{(1)}(P) - \zeta_t^{(2)}dW_t^{(2)}(P)) \\ &+ ln(1 + \Gamma_t^M)dJ_t^s - E_t(\Gamma_t^M dJ_t^s) \end{aligned}$$

We can provide an equivalent specification of stock price and SDF dynamics by integrating both sides of the above equations from 0 to T, which leads to the solution of the stochastic process

[8]This particular formulation was adapted here by combining Ito's formula given in Oksendal "Stochastic Differential Equations" 5th edition p.48 and Protter (2003) p. 82. See these original sources for detailed treatment, proof and specification.

[9]Note that if we model SDF dynamics using non-orthogonal specification of a high-dimensional Brownian motion dW_t^s, dW_t^v, correlation terms will have to be added to the resulting *log* SDF dynamics (as seen in the high-dimensional Ito's formula), which is written for orthogonal case in its present form for the purpose of transparency. These additional correlation-induced terms will play a role in the formulation of SDF dynamics for non-orthogonal case, when we apply Girsanov Theorem in section 6.

as a function of its current location[10]:

$$
\begin{aligned}
S_T &= S_0 \exp[\int_0^T (\mu_t - \frac{1}{2}V_t)dt - \int_0^T (E_t(\Gamma_t^s dJ_t^s))] \\
&\times \exp[\int_0^T \sqrt{V_t} dW_t^{(1)}(P)] \exp[\int_0^T ln(1+\Gamma_t^s)dJ_t^s] \\
M_T &= M_0 \exp[\int_0^T -(r_t + \frac{1}{2}(\zeta_t^{(1)2} + \zeta_t^{(2)2}))dt - \int_0^T E_t(\Gamma_t^M dJ_t^s)] \\
&\times \exp[-\int_0^T \zeta_t^{(1)} dW_t^{(1)}(P) - \int_0^T \zeta_t^{(2)} dW_t^{(2)}(P)] \exp[\int_0^T ln(1+\Gamma_t^M)dJ_t^s]
\end{aligned}
$$

where M_0 is normalized to equal 1.
It is important to note here that the above equation for M_T could also be written in terms of the *rotated* non-orthogonal Brownian motions $dW_t^s(P)$ and $dW_t^v(P)$. However, when we apply Ito's formula to the non-orthogonal case, as a result, correlation terms will be added to the above formula for M_T. This could only be seen by considering a multi-dimensional version of Ito's formula, which will also be utilized later for derivations in this manuscript.

Another useful representation of SDF dynamics under P is[11]

$$
\begin{aligned}
M_{t+dt} &= M_t \exp[-(r_t + \frac{1}{2}(\zeta_t^{(1)2} + \zeta_t^{(2)2}))dt - E_t(\Gamma_t^M dJ_t^s)] * \\
&* \exp[-\zeta_t^{(1)} dW_t^{(1)}(P) - \zeta_t^{(2)} dW_t^{(2)}(P)](1+\Gamma_t^M)^{dJ_t^s}
\end{aligned} \tag{8}
$$

In our model we will assume that size and occurrence of jump components are independent, i.e. $<\Gamma^i, J^s>=0, \quad i = S, M$. If we define the expected jump size and jump intensity to be $\overline{\Gamma^i}$ and λ_s respectively, then $E_t(\Gamma_t^i dJ_t^s) = \overline{\Gamma^i}\lambda_t dt$ for $i = S, M$. Notice, that using Taylor series expansion and the fact that $(dt)^n = 0, \quad \forall n > 1$ we have:

$$
\exp[-E_t(\Gamma_t^i dJ_t^s)] = \exp[-\overline{\Gamma^i}\lambda_t dt] = 1 - \overline{\Gamma^i}\lambda_t dt \tag{9}
$$

Although the generic form of the SDF dynamics in (4) is known, the question remains about how to model the free parameters (market prices of risks). In the next section we will show that not *any* econometric specification of risk premiums is permissible, if we want to avoid arbitrage.

[10]Possibly, volatility dynamics could also be written down for V_T as a function of V_t, (similar to Gourieroux (2000), p. 251), where O-U volatility process has a closed form solution at least without jumps. However, in our setup not only the volatility dynamics includes jumps, but also diffusion specification is of CIR (or "square-root") type. Fortunately, we don't need to use this result here because volatility is not a traded asset.

[11]Note that all results based on $(t + dt)$-type arguments are not mathematically rigorous and are provided here as a "quick and dirty" approach to finding solutions and developing intuition. For rigorous mathematical treatment see cited references.

5 No-Arbitrage Restrictions on the Free Parameters in SDF dynamics

From the FTAP we know that No-Arbitrage implies the existence of a positive stochastic discount factor M that prices all traded assets X (stock and bond) in our economy. Therefore, assuming no dividend payments, $M_t X_t$ must be a *martingale* and dynamics of all traded assets with price X must satisfy the following martingale pricing equation[12]:

$$E_t^P(d(M_t X_t)) \quad = \quad 0 \tag{10}$$

or equivalently[13]

$$0 \quad = \quad E_t^P\left(\frac{dX_t}{X_t} + \frac{dM_t}{M_t} + \frac{dM_t}{M_t}\frac{dX_t}{X_t}\right) \tag{11}$$

In the other direction (sufficiency), it can be shown that the pricing equation (10) rules out arbitrage opportunities under natural conditions of dynamic trading strategies using replication argument.

We know the dynamics under P of at least two traded assets: stock S and bond B. First, as we have already mentioned above, a risk free bond is a traded security with dynamics $\frac{dB}{B} = r_t dt$. Plugging it in the equation above restricts the mean of the SDF dynamics $E_t^P(\frac{dM_t}{M_t}) = -r_t dt$. Therefore, we can write:

$$E_t^P\left(\frac{dX_t}{X_t}\right) \quad = \quad r_t dt - E_t^P\left(\frac{dM_t}{M_t}\frac{dX_t}{X_t}\right) \tag{12}$$

Second, by plugging (4) and (2) in (12) the stock price observability imposes the following restriction:

$$\begin{aligned} 0 \quad &= \quad M_t S_t((\mu_t - r_t)dt - \sqrt{V_t}\zeta_t^{(1)}dt + E_t(\Gamma_t^s \Gamma_t^M)\lambda_t dt) \\ &= \quad (\mu_t - r_t) - \sqrt{V_t}\zeta_t^{(1)} + E_t(\Gamma_t^s \Gamma_t^M)\lambda_t \end{aligned} \tag{13}$$

or equivalently, by plugging (5) and (1) in (12):

$$0 \quad = \quad (\mu_t - r_t) - \sqrt{V_t}(\zeta_t^s + \overline{\rho}\zeta_t^v) + E_t(\Gamma_t^s \Gamma_t^M)\lambda_t \tag{14}$$

It is important to point out that the above restrictions are one and the same, because the specification of dynamics in (2) comes from Cholesky decomposition of the covariance matrix in (1) by rotating the diffusion terms in R^2. Both specifications are equivalent and simply allow for different prospective on the same stochastic process. Enforcing the restriction in terms of $\zeta^{(1)}$ automatically enforces the restriction in terms of ζ^s and ζ^v. There is still only one degree of freedom in between $\zeta^{(1)}$ and $\zeta^{(2)}$ just as between ζ^s and ζ^v.

---

[12] Cochrane (2000) "Asset Pricing" demonstrates how this relation naturally arises in a simple General Equilibrium setting

[13] Using: $d(M_t X_t) = X_{t+dt}M_{t+dt} - X_t M_t = (X_{t+dt}M_{t+dt} - X_{t+dt}M_t) + (X_{t+dt}M_t - X_t M_t) = X_{t+dt}(M_{t+dt} - M_t) + M_t(X_{t+dt} - X_t) - X_t(M_{t+dt} - M_t) + X_t(M_{t+dt} - M_t) = X_t dM_t + M_t dX_t + dM_t dX_t$

From discrete time econometric prospective, in order to avoid arbitrage, we *must* respect the restriction in (14) when we set objective parameter values in order to simulate data, or estimate parameters from the data by expressing one of the parameters as a function of the remaining ones. In discrete time the equation (14) essentially contains T restrictions, because the equality must be satisfied $\forall t \in (0, T)$. Therefore, we can not simultaneously set the mean μ_t to be exogenous *and* model prices of risk as, say, a square root of volatility times an unknown constant, because the relatively small number of free parameters cannot guarantee to satisfy T restrictions. Therefore, either the mean μ_t must be treated as endogenous, or one of the risk prices must be left unspecified and backed out from (14).

From continuous time asset pricing prospective, if we re-write the continuum of restrictions in (14) as a "risk premium map" $\mu_t = r_t + \sqrt{V_t}(\zeta_t^s + \overline{\rho}\zeta_t^v) - E_t(\Gamma_t^s \Gamma_t^M)\lambda_t$, then we gain another interpretation of this restriction, namely that the stock price mean return is equal to the risk free rate plus the adjustments (risk premiums) for the corresponding risk factors. Intuitively, the instantaneous linearity of the risk premium adjustments comes from our ability to form only linear payoff portfolios of securities used to enforce the no-arbitrage argument.

6 Risk Neutral Dynamics

One of the most powerful tools of Finance is the risk-neutral pricing. In general, an asset payoff is not a martingale under objective measure due to the presence of risk premiums. Therefore, unless risk premiums are all zero, an asset price is not equal to the expectation of its future payoffs taken under objective (also called empirical, or observed) measure. However, given an appropriate change of measure from objective to risk-neutral dynamics, the expectation taken under the risk-neutral measure yields the correct price after discounting at the risk-free rate. Radon-Nikodym theorem serves as a mathematical justification for this technique, while Feynman-Kac solution[14] demonstrates the equivalence of risk-neutral and fundamental PDE-based approaches for pricing contingent claims. Because risk-neutral valuation has proven to be an extremely convenient asset-pricing tool, we will rely on it in what follows.

6.1 Change of Measure for Diffusion processes

In order to implement the risk-neutral valuation approach we need to change the measure from objective P to risk-neutral Q so that $X_0 = \int_0^T X_t M_t dP = \int_0^T X_t dQ$ holds. Ignoring the jump component for the moment (correction for which will be introduced later), we can use the following famous result:

Theorem 2 ***Girsanov Theorem***[15]*:*
Let $X_t \in R^n$ be an Ito process of the form

$$dX_t = \mu_t dt + \theta_t dW_t(P); \qquad 0 \le t \le T \tag{15}$$

where $W_t(P) \in R^m$, $\mu_t \in R^n$ and $\theta_t \in R^{n\times m}$. Suppose there exists processes $\zeta_t \in W_H^m$ and $r_t \in W_H^n$ such that $\theta_t \zeta_t = \mu_t - r_t$ and assume that ζ_t satisfies technical Novikov's condition.
Let $M_t = \exp[-\int_0^t \zeta_s dW_s(P) - \frac{1}{2}\int_0^t \zeta_s^2 ds]$; $0 \le s \le t \le T$. Define $dQ = M_T dP$.
Then,

$$W_t(Q) = \int_0^t \zeta_s ds + W_t(P); \qquad 0 \le s \le t \le T \tag{16}$$

is a Brownian motion w.r.t. Q and in terms of $W_t(Q)$ the process S_t has the stochastic integral representation

$$dX_t = r_t dt + \theta_t dW_t(Q); \qquad 0 \le t \le T, \quad \text{where} \tag{17}$$

$$dW_t(Q) = \zeta_t dt + dW_t(P) \tag{18}$$

[14] See Duffie (2001)

[15] See Oksendal "Stochastic Differential Equations" 5th edition p.155. Note that using the change of measure framework presented here we gain a mathematical interpretation of the SDF as a Radon-Nikodym derivative

Applying Girsanov's theorem to our model we note that the above change of measure equality is for Brownian motion $dW_t(P) \in R^2$ with *orthogonal* elements $dW_t^{(1)}$ and $dW_t^{(2)}$ (otherwise the specification of M_t above would have to be modified to correspond to the dynamics of SDF in the economy where Brownian motions are not orthogonal). Therefore, for our model we can re-write (17) as:

$$\begin{aligned}
\begin{pmatrix} dS_t \\ dV_t \end{pmatrix} &= \begin{pmatrix} r_t \\ \kappa^Q(\theta^Q - V_t) \end{pmatrix} dt + \sqrt{V_t} \begin{pmatrix} 1 & 0 \\ \overline{\rho} & \sqrt{1-\overline{\rho}^2} \end{pmatrix} \begin{pmatrix} dW_t^{(1)}(Q) \\ dW_t^{(2)}(Q) \end{pmatrix} \\
\begin{pmatrix} dW_t^{(1)}(Q) \\ dW_t^{(2)}(Q) \end{pmatrix} &= \begin{pmatrix} \zeta_t^{(1)} \\ \zeta_t^{(2)} \end{pmatrix} dt + \begin{pmatrix} dW_t^{(1)}(P) \\ dW_t^{(2)}(P) \end{pmatrix}
\end{aligned} \tag{19}$$

Lemma 1 *By rotating the newly obtained orthogonal Brownian motions under Q we obtain a version of the Girsanov Theorem for non-orthogonal Brownian motions:*

$$\begin{aligned}
dW_t^s(Q) &= (\zeta_t^s + \overline{\rho}\zeta_t^v)dt + dW_t^s(P) \\
dW_t^v(Q) &= (\zeta_t^s\overline{\rho} + \zeta_t^v)dt + dW_t^v(P)
\end{aligned} \tag{20}$$

Proof:

Multiplying (18) by the rotation matrix we have:

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ \overline{\rho} & \sqrt{1-\overline{\rho}^2} \end{pmatrix} \begin{pmatrix} dW_t^{(1)}(Q) \\ dW_t^{(2)}(Q) \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ \overline{\rho} & \sqrt{1-\overline{\rho}^2} \end{pmatrix} \begin{pmatrix} \zeta_t^{(1)} \\ \zeta_t^{(2)} \end{pmatrix} dt \\
&+ \begin{pmatrix} 1 & 0 \\ \overline{\rho} & \sqrt{1-\overline{\rho}^2} \end{pmatrix} \begin{pmatrix} dW_t^{(1)}(P) \\ dW_t^{(2)}(P) \end{pmatrix}
\end{aligned} \tag{21}$$

or equivalently

$$\begin{pmatrix} dW_t^s(Q) \\ dW_t^v(Q) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \overline{\rho} & \sqrt{1-\overline{\rho}^2} \end{pmatrix} \begin{pmatrix} \zeta_t^{(1)} \\ \zeta_t^{(2)} \end{pmatrix} dt + \begin{pmatrix} dW_t^s(P) \\ dW_t^v(P) \end{pmatrix} \tag{22}$$

Now we will invoke the fact that rotation of Brownian motions is just a mathematical trick and we are still dealing with the same economy and the same SDF as a result of that. Therefore, we have:

$$\begin{pmatrix} \zeta_t^{(1)} & \zeta_t^{(2)} \end{pmatrix} \begin{pmatrix} dW_t^{(1)}(P) \\ dW_t^{(2)}(P) \end{pmatrix} = \begin{pmatrix} \zeta_t^s & \zeta_t^v \end{pmatrix} \begin{pmatrix} dW_t^s(P) \\ dW_t^v(P) \end{pmatrix} \tag{23}$$

Replacing the Brownian motions on the right-hand side with their orthogonal counterparts, we notice that the coefficients pre-multiplying the orthogonal Browniain motions on both sides could be thought of as coordinates and must be equal to each-other. Therefore, we have:

$$\begin{pmatrix} \zeta_t^{(1)} \\ \zeta_t^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & \overline{\rho} \\ 0 & \sqrt{1-\overline{\rho}^2} \end{pmatrix} \begin{pmatrix} \zeta_t^s \\ \zeta_t^v \end{pmatrix} \tag{24}$$

Plugging it in (22) we get

$$\begin{pmatrix} dW_t^s(Q) \\ dW_t^v(Q) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \overline{\rho} & \sqrt{1-\overline{\rho}^2} \end{pmatrix} \begin{pmatrix} 1 & \overline{\rho} \\ 0 & \sqrt{1-\overline{\rho}^2} \end{pmatrix} \begin{pmatrix} \zeta_t^s \\ \zeta_t^v \end{pmatrix} dt + \begin{pmatrix} dW_t^s(P) \\ dW_t^v(P) \end{pmatrix} \tag{25}$$

QED

Substituting the orthogonal $dW_t^{(1)}(P)$ and $dW_t^{(2)}(P)$ in the stock price dynamics with orthogonal risk-neutral counterparts via Girsanov's theorem, we notice that *risk premium for the source of risk i equals market price of risk i times the square root of volatility*: $\sqrt{V_t}\zeta_t^i$.

6.2 Diffusion risk premium

Consider a special case when volatility is constant (not stochastic) and no jumps are present. Then, the market is complete and the market price of risk ζ^s is determined in such a way that the risk premium doesn't depend on volatility: $\zeta^s = \eta^s\sqrt{V} = \frac{\mu-r}{V}\sqrt{V} = \frac{\mu-r}{\sqrt{V}}$ leading to the risk premium equal to $\mu - r$. However, if volatility is stochastic, *using GE framework* we can show that during a bear market the aggregate volatility tends to be higher. As a result, we expect to see higher risk premiums for holding stock risk dW_t^s associated with higher volatility values, because in a bear market agents are more poor and as a result more careful with their investments as the marginal utility cost of losing even more money becomes relatively too high. Similarly, a long position in volatility risk sensitive instrument (e.g. option) becomes an insurance against market fluctuations because options gain value when volatility increases, which means that the risk premium for volatility risk dW_t^v must be negative and decreasing in volatility. Therefore, we want to select our market prices of dW_t^s risk so that the resulting risk premiums are of the form[16] $\eta^s V_t$ and $-\eta^v V_t$ for stock and volatility risks respectively, where η^s and η^v are some unknown constants that could be determined only by observing additional traded securities (e.g. cross-section of options) in such economy. The negative sign in volatility diffusion risk premium specification allows for convenient interpretation: increase in η^v leads to higher option prices.
Given the discussion above, for the diffusions we will use the following link between the market prices of risks and risk premiums[17]:

$$\begin{pmatrix} \zeta_t^{(1)} \\ \zeta_t^{(2)} \end{pmatrix} = \begin{pmatrix} \eta^s \\ -\frac{\overline{\rho}\eta^s + \frac{\eta^v}{\sigma_v}}{\sqrt{1-\overline{\rho}^2}} \end{pmatrix} \sqrt{V_t} \tag{26}$$

Using (24) we can derive the relationship between market prices of diffusion risks from the

[16] Certainly, any other *increasing in volatility* (decreasing for dW_t^v risk) function would work - we just selected the simplest one for the purposes of tractability.
[17] The same specification was used in Pan(2002)

non-orthogonal formulation and the risk premium parameters:

$$\begin{pmatrix} \zeta_t^s \\ \zeta_t^v \end{pmatrix} = \begin{pmatrix} \eta^s + \overline{\rho}\frac{\eta^v}{\sigma_v} \\ -\overline{\rho}\eta^s - \frac{\eta^v}{\sigma_v} \end{pmatrix} \frac{\sqrt{V_t}}{1-\overline{\rho}^2} \tag{27}$$

It is easy to verify that both formulations are equivalent as they produce the same (desired) risk premiums $\eta^s V_t$ and $-\eta^v V_t$ once plugged into the *corresponding* Girsanov theorem equation (19) or (20) and stock price dynamics equation (2) or (1). This fact will be used in section 6.5 below.

6.3 Change of measure for Jump processes

In order to directly link the Girsanov Theorem with our model, we can simply move the jump-related components to the left side. Now our model matches the specification in Girsanov theorem and we can safely apply it. This method works because Jumps and Diffusions do not mix. After that, we move the jump component back to the right-hand side, but due to the change of measure that occurred, we need the distribution of the jump component under Q. First, we find out how the jump intensity is going to change. In what follows we will denote the expected jump size of the SDF by $\overline{\Gamma^M}$. Note, that although we can allow for the SDF jump size mean to change over time, we will be using the above notation because there will be no need to model the distribution of Γ_t^M explicitly as it will become clear later.
It turns out that there is a one-to-one relationship for λ_t under P and Q:

$$\lambda_t^Q = (1+\overline{\Gamma^M})\lambda_t^P, \quad \overline{\Gamma^M} \neq -1 \tag{28}$$

Proof:
Define $S_{t+dt}^* = S_{t+dt}\exp[-r_t dt]$. Using (8) and (9) we have

$$\begin{aligned}
S_t &= E_t^P(\frac{M_{t+dt}}{M_t}S_{t+dt}) = E_t^P[S_{t+dt}\exp[-(r_t+\frac{1}{2}(\zeta_t^{(1)2}+\zeta_t^{(2)2}))dt - E_t(\Gamma_t^M dJ_t^s)] * \\
&* \exp[-\zeta_t^{(1)}dW_t^{(1)}(P) - \zeta_t^{(2)}dW_t^{(2)}(P)](1+\Gamma_t^M)^{dJ_t^s}] \\
&= E_t^P[S_{t+dt}^*\exp[-\frac{1}{2}(\zeta_t^{(1)2}+\zeta_t^{(2)2})dt - \zeta_t^{(1)}dW_t^{(1)}(P) - \zeta_t^{(2)}dW_t^{(2)}(P)](1-\overline{\Gamma^M}\lambda_t^P dt)(1+\Gamma_t^M)^{dJ_t^s}] \\
&= \int_{\Omega(\zeta_t^{(1)},W_t^{(1)}(P),\zeta_t^{(2)},W_t^{(2)}(P),\Gamma_t^M,J_t^s)} S_{t+dt}^*\exp[-\frac{1}{2}(\zeta_t^{(1)2}+\zeta_t^{(2)2})dt - \zeta_t^{(1)}dW_t^{(1)}(P) - \zeta_t^{(2)}dW_t^{(2)}(P)] \\
&* (1-\overline{\Gamma^M}\lambda_t^P dt)(1+\Gamma_t^M)^{dJ_t^s}dF^P(W_t^{(1)}(P),W_t^{(2)}(P),J_t^s)dF(\zeta_t^{(1)},\zeta_t^{(2)},\Gamma_t^M) \\
&= \int_{\Omega(\zeta_t^{(1)},\zeta_t^{(2)},\Gamma_t^M)}\int_{\Omega(J_t^s)}\int_{\Omega(W_t^{(1)}(P),W_t^{(2)}(P))} S_{t+dt}^* \\
&* \exp[-\frac{1}{2}(\zeta_t^{(1)2}+\zeta_t^{(2)2})dt - \zeta_t^{(1)}dW_t^{(1)}(P) - \zeta_t^{(2)}dW_t^{(2)}(P)] \\
&* dF^P(W_t^{(1)}(P),W_t^{(2)}(P))(1-\overline{\Gamma^M}\lambda_t^P dt)(1+\Gamma_t^M)^{dJ_t^s}dF^P(J_t^s)dF(\zeta_t^{(1)},\zeta_t^{(2)},\Gamma_t^M)
\end{aligned}$$

Applying Girsanov theorem inside the integral over diffusions $\int_{\Omega(W_t^{(1)}(P),W_t^{(2)}(P))}$ and re-arranging the order of integration, using the fact that diffusions and jumps don't mix, we continue with the above derivation:

$$
\begin{aligned}
&= \int_{\Omega(\zeta_t^{(1)},\zeta_t^{(2)},\Gamma_t^M)} \int_{\Omega(W_t^{(1)}(P),W_t^{(2)}(P))} \int_{\Omega(J_t^s)} S^*_{t+dt} \\
&* \; (1-\overline{\Gamma^M}\lambda_t^P dt)(1+\Gamma_t^M)^{dJ_t^s} dF^P(J_t^s) dF^Q(W_t^{(1)}(P),W_t^{(2)}(P)) dF(\zeta_t^{(1)},\zeta_t^{(2)},\Gamma_t^M) \\
&= \int_{\Omega(W_t^{(1)}(P),W_t^{(2)}(P))} \int_{\Omega(\zeta_t^{(1)},\zeta_t^{(2)})} \int_{\Omega(\Gamma_t^M)} [S^*_{t+dt}(dJ_t^s=1)(1-\overline{\Gamma^M}\lambda_t^P dt)(1+\Gamma_t^M)\lambda_t^P dt \\
&+ \; S^*_{t+dt}(dJ_t^s=0)(1-\overline{\Gamma^M}\lambda_t^P dt)(1-\lambda_t^P dt)] dF(\zeta_t^{(1)},\zeta_t^{(2)},\Gamma_t^M) dF^Q(W_t^{(1)}(P),W_t^{(2)}(P)) \\
&= \int_{\Omega(W_t^{(1)}(P),W_t^{(2)}(P))} [S^*_{t+dt}(dJ_t^s=1)(1-\overline{\Gamma^M}\lambda_t^P dt)(1+\overline{\Gamma^M})\lambda_t^P dt \\
&+ \; S^*_{t+dt}(dJ_t^s=0)(1-\overline{\Gamma^M}\lambda_t^P dt)(1-\lambda_t^P dt)] dF^Q(W_t^{(1)}(P),W_t^{(2)}(P)) \\
&= \int_{\Omega(W_t^{(1)}(P),W_t^{(2)}(P))} [S^*_{t+dt}(dJ_t^s=1)(1+\overline{\Gamma^M})\lambda_t^P dt \\
&+ \; S^*_{t+dt}(dJ_t^s=0)(1-(1+\overline{\Gamma^M})\lambda_t^P)dt] dF^Q(W_t^{(1)}(P),W_t^{(2)}(P))
\end{aligned}
$$

Define $F^Q(J_t^s) \equiv (1+\overline{\Gamma^M})F^P(J_t^s)$. Then, $\lambda_t^Q = (1+\overline{\Gamma^M})\lambda_t^P$ and we can continue to write

$$
\begin{aligned}
&= \int_{\Omega(W_t^{(1)}(P),W_t^{(2)}(P))} [S^*_{t+dt}(dJ_t^s=1)\lambda_t^Q dt \\
&+ \; S^*_{t+dt}(dJ_t^s=0)(1-\lambda_t^Q)dt] dF^Q(W_t^{(1)}(P),W_t^{(2)}(P)) \\
&= \int_{\Omega(W_t^{(1)}(P),W_t^{(2)}(P),J_t^s)} S^*_{t+dt} dF^Q(W_t^{(1)}(P),W_t^{(2)}(P),J_t^s) \\
&= E_t^Q[S^*_{t+dt}] \qquad\qquad QED
\end{aligned}
$$

Note, that assuming that jump intensity is the same under both measures is equivalent to setting the mean jump size of the SDF to be zero. However arguable, this assumption is quite common in the recent empirical literature, including Pan (2002), Eraker (2004), and Chernov (2003).

Now we turn our attention to the distribution of the stock jump size under the risk neutral measure. The following result provides a link between the expected values of the stock jump size under both measures:

$$
E_t^Q(\Gamma_t^s) = \frac{1}{1+\overline{\Gamma^M}}(E_t^P(\Gamma_t^s) + E_t^P(\Gamma_t^M\Gamma_t^s)) \tag{29}
$$

Proof:

Using (14) we have

$$
\begin{aligned}
r_t dt &= E_t^Q[\frac{dS_t}{S_t}] = E_t^Q[\mu_t dt + \sqrt{V_t}dW_t^{(1)}(P) + (\Gamma_t^s dJ_t^s - \lambda_t^P\overline{\Gamma_t^s}dt)] \\
&= E_t^Q[(r_t + \zeta_t^{(1)}\sqrt{V_t} - E_t^P(\Gamma_t^M\Gamma_t^s))dt + \sqrt{V_t}dW_t^{(1)}(P) + (\Gamma_t^s dJ_t^s - \lambda_t^P\overline{\Gamma_t^s}dt)]
\end{aligned}
$$

Applying (28) and the fact that due to Girsanov $E_t^Q(dW_t(P)) + \zeta_t dt = E_t^Q(dW_t(Q)) = 0$ we get

$$
\begin{aligned}
(1+\overline{\Gamma^M})\lambda_t^P E_t^Q[\Gamma_t^s]dt &= \lambda_t^P(E_t^P(\Gamma_t^s) + E_t^P(\Gamma_t^M\Gamma_t^s))dt - \sqrt{V_t}E_t^Q[dW_t^{(1)}(P)] - \zeta_t^{(1)}\sqrt{V_t}dt \\
(1+\overline{\Gamma^M})E_t^Q[\Gamma_t^s] &= E_t^P(\Gamma_t^s) + E_t^P(\Gamma_t^M\Gamma_t^s) \qquad\qquad QED
\end{aligned}
$$

6.4 Jump risk premium

Note that in continuous time the *jump risk premium* equals $-E_t^P(\Gamma_t^M\Gamma_t^s)\lambda_t^P = E_t^P[\Gamma_t^s]\lambda_t^P - E_t^Q[\Gamma_t^s]\lambda_t^Q$. Now, if we set[18] $\overline{\Gamma^M} = 0$, $\lambda_t^Q = \lambda_t^P = \overline{\lambda_t} = \lambda_0 + V_{t-}\lambda_1$, and $\eta^j = \overline{\mu} - \overline{\mu}^Q = E^P[\Gamma^s] - E^Q[\Gamma^s]$, then the jump risk premium can be written as $-E_t^P(\Gamma_t^M\Gamma_t^s)\lambda_t^P = \eta^j(\lambda_0 + V_{t-}\lambda_1)$. Therefore using the above derivation and equation (27) the No-Arbitrage condition in (14) can be restated as:

$$\mu_t = r_t + V_t\eta^s + \overline{\lambda_t}\eta^j \tag{30}$$

A more general[19] result linking the joint distribution of the jump sizes under the two measures could be found in Dai and Singleton (2003):

$$f^Q(\Gamma_t^M, \Gamma_t^s) = \frac{1 + \Gamma_t^M}{1 + \overline{\Gamma^M}} f^P(\Gamma_t^M, \Gamma_t^s) \tag{31}$$

By integrating out Γ^M from the above, we can derive the distribution of jump size under risk neutral measure: $f^Q(\Gamma_t^s) = \frac{1+E_t^P[\Gamma_t^M|\Gamma_t^s]}{1+E_t^P[\Gamma_t^M]} f^P(\Gamma_t^s)$. Therefore, given the distribution of stock jump size Γ_t^s under objective measure, the risk neutral dynamics of the jump size is uniquely determined by the specification of the conditional and unconditional means of the SDF jump size, i.e. the conditional distribution of the SDF jump size given Γ_t^s under the objective measure. Now, looking in the opposite direction: given $f^P(\Gamma_t^s)$, choosing $f^Q(\Gamma_t^s)$ implies certain $E^P(\Gamma_t^M|\Gamma_t^s)$ and $E_t^P[\Gamma_t^M]$.

The Jump component can be decomposed into two random variables Γ_t^s and dJ_t^s, which could both contribute to the risk premium as a result of the change of measure. As we just saw in (28), the contribution of dJ_t^s is fully controlled by our assumption about the mean of the SDF jump size, while the contribution of Γ_t^s is determined in (31) by our assumption about the conditional $E^P(\Gamma_t^M|\Gamma_t^s)$ and unconditional $E_t^P[\Gamma_t^M]$. In most empirical literature the authors chose not to discuss the later explicitly and simply assume some form of $f^Q(\Gamma_t^s)$. As long as no explicit assumptions about $E^P(\Gamma_t^M|\Gamma_t^s)$ and $E_t^P[\Gamma_t^M]$ were made, there is no problem with such approach. The rationale is quite simple: we have no idea what these means should look like. So, we might as well just model the final product $f^Q(\Gamma_t^s)$ directly without causing any inconsistencies. Of course, in doing that we are implicitly making some assumptions about the means - no way around that!

Both Pan and Eraker argue that it is acceptable to allocate all the risk premium associated with the jump in the jump size component by setting $E(\Gamma_t^M) = 0$. But then, they are forced to make (implicit or explicit) assumptions about the conditional mean too. Notice, however, that if we

[18] Note that if we write down the discrete-time analog of this continuous-time model using Euler-discretization scheme (3), the jump intensity will depend on the last period's volatility: $\overline{\lambda_t} = \lambda_0 + V_{t-1}\lambda_1$, while the No-Arbitrage restriction (14) becomes $\mu_{t-1} = r_{t-1} + V_{t-1}\eta^s + \overline{\lambda_t}\eta^j$

[19] It is easy to verify that this result implies the relationship of means in (29), but the relationship of joint densities is much harder to prove.

allocate all the risk premium associated with the jumps in the dJ_t^s component, by assuming that Γ_t^s and Γ_t^M are orthogonal, in order to proceed with the change of measure we will *only* have to model the value of $E(\Gamma_t^M)$, which we can simply treat as a parameter. As a result, after we have allocated the risk premium, Pan's risk pricing choice will require "bigger" assumptions than putting all the premium inside $dJ_t^s(Q)$.

Certainly economic implications of these risk pricing choices must be considered in General Equilibrium setting. For example, Lucas economy, or CIR model could provide intuitive support in favor of one or the other, the same way Heston (1993) motivated his choice of the diffusion risk premium form. An empirical comparison of these jump risk premium assumptions is a potential area of further research.

6.5 Applying the change of measure

We are finally ready to write down the dynamics of S_t under risk neutral measure Q by replacing the objective parameters in (1) with their risk-neutral counterparts found in (20), (28), (29), and enforcing the no-arbitrage restriction (14):

$$\begin{aligned}\frac{dS_t}{S_t} &= (r_t + \sqrt{V_t}(\zeta_t^s + \overline{\rho}\zeta_t^v) - E_t^P(\Gamma_t^s\Gamma_t^M)\lambda_t^P)dt \\ &+ \sqrt{V_t}(dW_t^s(Q) - (\zeta_t^s + \overline{\rho}\zeta_t^v)dt) \\ &+ \Gamma_t^s(Q)dJ_t^s(Q) - ((1+\overline{\Gamma^M})E_t^Q(\Gamma_t^s) - E_t^P(\Gamma_t^s\Gamma_t^M))\lambda_t^P dt \\ dV_t &= \kappa_v(\theta - V_t)dt + \sqrt{V_t}\sigma_v(dW_t^v(Q) - (\zeta_t^s\overline{\rho} + \zeta_t^v)dt) + Z_t^v dJ_t^v\end{aligned} \tag{32}$$

and after simplification

$$\begin{aligned}\frac{dS_t}{S_t} &= r_t dt + \sqrt{V_t}dW_t^s(Q) + \Gamma_t^s(Q)dJ_t^s(Q) - E_t^Q(\Gamma_t^s)\lambda_t^Q dt \\ dV_t &= (\kappa_v(\theta - V_t) - \sqrt{V_t}\sigma_v(\zeta_t^s\overline{\rho} + \zeta_t^v))dt + \sqrt{V_t}\sigma_v dW_t^v(Q) + Z_t^v dJ_t^v\end{aligned} \tag{33}$$

where $Cov(dW_t^s(Q), dW_t^v(Q)) = \overline{\rho}dt$.

Unlike in the case of diffusions, where the risk premium (=expected change of return) happens to cancel out with the corresponding change in the mean of dW_t^i, in the case of jumps the picture is more vague. Indeed, the expected change in the mean (as a result of the change of measure) did cancel out with the risk premium associated with the corresponding jump. However, the change of measure has more profound impact on the distribution of the jump component than just changing its mean. Therefore, it is more convenient to write it as above, while imposing the condition (31).

Now, if we want to write the RN dynamics in terms of $Y_t = ln(S_t)$, we will apply Ito's Lemma for Jump-Diffusion given above. Define $ln(1+\Gamma_t^s(Q)) = Z_t^s$ and we have:

$$\begin{aligned}dY_t &= (r_t - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^s(Q) + Z_t^s dJ_t^s(Q) - E_t^Q(\Gamma_t^s)\lambda_t^Q dt \\ dV_t &= (\kappa_v(\theta - V_t) - \sqrt{V_t}\sigma_v(\zeta_t^s\overline{\rho} + \zeta_t^v))dt + \sqrt{V_t}\sigma_v dW_t^v(Q) + Z_t^v dJ_t^v\end{aligned} \tag{34}$$

Using the relationship between market prices of risks and risk premium parameters derived in (27) we get:

$$\begin{aligned} dY_t &= (r_t - E_t^Q(\Gamma_t^s)\lambda_t^Q - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^s(Q) + Z_t^s dJ_t^s(Q) \\ dV_t &= (\kappa_v(\theta - V_t) + V_t\eta^v)dt + \sqrt{V_t}\sigma_v dW_t^v(Q) + Z_t^v dJ_t^v \end{aligned} \tag{35}$$

rearranging we get

$$\begin{aligned} dY_t &= (r_t - E_t^Q(\Gamma_t^s)\lambda_t^Q - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^s(Q) + Z_t^s dJ_t^s(Q) \\ dV_t &= (\kappa_v - \eta^v)(\frac{\kappa_v\theta}{\kappa_v - \eta^v} - V_t)dt + \sqrt{V_t}\sigma_v dW_t^v(Q) + Z_t^v dJ_t^v \end{aligned} \tag{36}$$

defining $\kappa^Q \equiv \kappa_v - \eta^v$ and $\theta^Q \equiv \frac{\kappa_v\theta}{\kappa_v - \eta^v}$ we can write

$$\begin{aligned} dY_t &= (r_t - E_t^Q(\exp[Z_t^s] - 1)\lambda_t^Q - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^s(Q) + Z_t^s dJ_t^s(Q) \\ dV_t &= \kappa^Q(\theta^Q - V_t)dt + \sqrt{V_t}\sigma_v dW_t^v(Q) + Z_t^v dJ_t^v \end{aligned} \tag{37}$$

Note that stock return dynamics above is no longer jump-compensated, as the stock price dynamics was. The remnants of the stock price compensator are contained in the $\overline{\mu}^Q = E_t^Q(\exp[Z_t^s] - 1)$.

6.6 Example from DPS(2000)

The model specification under RN dynamics in (37) matches precisely (assuming zero-dividend) the set up in DPS(2000) example 4.1, because the restriction $\overline{\mu}^Q = \Theta(1,0) - 1$ translates into

$$\begin{aligned} \overline{\mu}^Q &= \int_{R^2} \exp[1Z_t^s + 0Z_t^v]\nu(Z_t)dZ_t - 1 = E_t^Q(\exp[Z_t^s]) - 1 \\ &= \int_{-\infty}^{\infty} \exp[Z_t^s]f(Z_t^s)dZ_t^s - 1 = \int_{-\infty}^{\infty} \exp[Z_t^s](\int_0^{\infty} f(Z_t^s|Z_t^v)f(Z_t^v)dZ_t^v)dZ_t^s - 1 \\ &= \int_0^{\infty}(\int_{-\infty}^{\infty} \exp[Z_t^s]f(Z_t^s|Z_t^v)dZ_t^s)f(Z_t^v)dZ_t^v - 1 \end{aligned}$$

If we make the same assumption as DPS(2000) in example 4.1 that $Z_t^s|Z_t^v \sim^Q N(\mu_s^Q + \rho_J Z_t^v, \sigma_s^2)$ and $f(Z_t^v) = \frac{\exp[-\frac{Z_t^v}{\mu_v}]}{\mu_v}$, then we can evaluate

$$\begin{aligned} \overline{\mu}^Q &= \int_0^{\infty} \exp[\mu_s^Q + \rho_J Z_t^v + 0.5\sigma_s^2]\frac{\exp[-\frac{Z_t^v}{\mu_v}]}{\mu_v}dZ_t^v - 1 \\ &= \exp[\mu_s^Q + 0.5\sigma_s^2]\frac{1}{\mu_v}\frac{1}{\frac{1}{\mu_v} - \rho_J}\int_0^{\infty}(\frac{1}{\mu_v} - \rho_J)\exp[-(\frac{1}{\mu_v} - \rho_J)Z_t^v] - 1 \\ &= \frac{\exp[\mu_s^Q + 0.5\sigma_s^2]}{1 - \mu_v\rho_J} - 1 \end{aligned}$$

This way we can derive formulas for $\overline{\mu}^Q$ and $\overline{\mu}$ in all models presented in subsection (3.5).

Note that in this example DPS(2000) doesn't even mention objective parameters, risk premiums, or SDF. Everything is modelled in terms of RN parameters. As a result, they are free to assume any RN distribution of the jump size $\nu(Z_t^s)$ without violating no-arbitrage conditions.
For our model, which goes beyond the example in DPS(2000), in that it reconciles RN and objective dynamics, it is useful to mention that assuming[20]:

$$Z_t^s|Z_t^v \sim^P N(\mu_s + \rho_J Z_t^v, \sigma_s^2) \text{ and } f(Z_t^v) = \frac{\exp[-\frac{Z_t^v}{\mu_v}]}{\mu_v}$$

we can evaluate under the objective measure:

$$E_t(\exp[Z_t^s]) = \frac{\exp[\mu_s+0.5\sigma_s^2]}{1-\mu_v\rho_J}, \text{ so } \overline{\mu}^P = \frac{\exp[\mu_s+0.5\sigma_s^2]}{1-\mu_v\rho_J} - 1$$

While discussing the jump transform $\Theta(\cdot,\cdot)$ let's extend the above derivation to establish another result that will be used in the next section for some complex numbers $u, \overline{\beta}$:

$$\begin{aligned} \Theta(u,\overline{\beta}) &= E^Q[\exp(uZ^s + \overline{\beta}Z^v)] \\ &= \int_0^\infty \exp(Z^v\overline{\beta})f(Z^v)[\int_{-\infty}^\infty \exp(Z^s u)f(Z^s|Z^v)dZ^s]dZ^v \\ &= \int_0^\infty \exp(Z^v\overline{\beta})\exp[u(\mu_s^Q + \rho_J Z^v) + 0.5u^2\sigma_s^2]\frac{\exp[-\frac{Z^v}{\mu_v}]}{\mu_v}dZ^v \\ &= \frac{\exp[\mu_s^Q u + 0.5\sigma_s^2 u^2]}{1 - \mu_v\rho_J u - \mu_v\overline{\beta}} \end{aligned} \tag{38}$$

[20]Note that such combination of RN and Objective distributions of jump size implies that only parameter μ_s changes, keeping the rest of the distribution the same after the change of measure from P to Q.

7 Option Pricing

In this section we present our derivation of the option Pricing formula in continuous time for Affine Jump-Diffusion Stochastic Volatility model (37) with time-varying (stochastic) intensity $\lambda_t^Q = \lambda_t^P = \overline{\lambda_t} = \lambda_0 + V_{t-}\lambda_1$ and jump size transform defined as in section (6.6) above. We consider the most general model (SVSCJ) as well as some special cases using Fourier transform approach. After we set up and solve Riccati equations for our model dynamics, we will obtain the time-t conditional Fourier transform of Y_T specific to our model, which we can subsequently plug in the option pricing equation on p. 1353 of DPS(2000) to get the result. We will show that, although a close-form solution to Riccati ODEs could be found for SVJ model with stochastic jump intensity (SVSJ) and SVCJ with constant intensity, only numerical solution is known for SVSCJ, which is the most general model that we discuss in this paper.

7.1 Setting up the fundamental PDEs using Martingale approach

We are looking for the time-t transform of Y_T defined as $\psi_t(u, Y_t, V_t, T-t) = \exp(-r(T-t))E_t^Q[\exp(uY_T)]$, where u is some complex number. This quantity is *not* a Martingale, because it depends on the current time t. One commonly used trick is to construct a martingale $\Psi = \exp(-r(t-0))\psi_t$ and use the martingale property that $E_t^Q[d\Psi] = 0$ to construct the identity leading to a partial differential equation (PDE).

If ψ_t was defined as $\psi_t = \exp(-r(T-t))E_t^Q[g(Y_T)]$ for some arbitrary payoff $g(Y_T)$, such method would result in the identity that is often referred to as the *fundamental PDE for contingent claims*, and $\psi_t = \exp(-r(T-t))E_t^Q[g(Y_T)]$ is called its *Feynman-Kac solution* under some technical conditions[21]. However, for most AJD models, except for the simplest ones, it would be inefficient to solve such PDE, because generally only numerical solutions are available, even for the simplest derivatives like European call with exercise price K and payoff $g(Y_T) = Max[0, \exp(Y) - K]$. Fortunately, a close form solution to the PDE for a broad range of affine dynamics of the underlying could be found when ψ_t is defined as the time-t transform of Y_T, which makes the practical option pricing application of DPS(2000) so much more efficient than solving the fundamental PDE numerically or using Monte-Carlo methods to evaluate the expectation under risk-neutral measure in the Feynman-Kac solution.

Let $X_t = (Y_t, V_t)$ denote the current state variable at time t. We use the multi-dimensional version of Ito's Lemma (7) for jump-diffusions to calculate $\forall(t < T)$

$$\begin{aligned} E_t^Q[d\Psi] &= (\frac{\partial \Psi}{\partial t} + (\mu^Q(X_t))'\Psi_X + tr[\frac{\sigma^2(X_t)}{2}\Psi_{XX}])dt \\ &+ (\lambda_0 + V_t\lambda_1)dtE_t^Q[\Psi(X_t + \Gamma_t) - \Psi(X_t)], \end{aligned} \tag{39}$$

where $\mu^Q(X_t)$ and $\sigma^2(X_t)$ are drift and diffusion coefficients of the two-dimensional state variable

[21] see Duffie (2001)

X_t under the RN measure, which in the context of our model (37) equal:

$$\begin{aligned}
\mu^Q(X_t) &= \begin{pmatrix} r_t - \lambda_0 \overline{\mu}^Q \\ \kappa^Q \theta^Q \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} - \lambda_1 \overline{\mu}^Q \\ 0 & -\kappa^Q \end{pmatrix} \begin{pmatrix} Y_t \\ V_t \end{pmatrix} \\
\sigma^2(X_t) &= \begin{pmatrix} 1 & \overline{\rho}\sigma_v \\ \overline{\rho}\sigma_v & \sigma_v^2 \end{pmatrix} V_t + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} Y_t \\
&= \begin{pmatrix} Y_t & V_t \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ \overline{\rho}\sigma_v \end{pmatrix} \\ \begin{pmatrix} 0 \\ \overline{\rho}\sigma_v \end{pmatrix} & \begin{pmatrix} 0 \\ \sigma_v^2 \end{pmatrix} \end{pmatrix}
\end{aligned} \tag{40}$$

It is obvious that $\mu^Q(X_t))'\Psi_X = \sum_{i=1}^2 \Psi_{X^i} E_t^Q(dX_t^i)$ and it is straightforward to verify that $tr[\sigma^2(X_t)\Psi_{XX}]dt = \sum_{i=1}^2 \sum_{j=1}^2 \Psi_{X^i X^j} dX^i dX^j = (\Psi_{YY} + 2\overline{\rho}\sigma_v \Psi_{YV} + \sigma_v^2 \Psi_{VV})dt$, which explains why it is possible to replace the single- and double-summations used in multi-dimensional Ito's Lemma (7) with matrix representation given above. Both notations have their advantages and with proper care could be used interchangeably.

Using the martingale property $E_t^Q[d\Psi] = 0$ we obtain the PDE that we need to solve for ψ_t subject to boundary condition $\psi_T = \exp(uY_T)$. The most common approach to solving PDEs is to guess at the solution form (with some unknown parameters $\alpha, \beta, ...$), plug it in, and back out the unknown parameters by equating the coefficients on the two sides of the equation. DPS(2000) suggests to guess that the solution to above PDE is of the following form:

$$\psi_t(u, Y_t, V_t, T-t) = \exp(\alpha(t) + \beta(t) \cdot X_t) \tag{41}$$

resulting in $\Psi = \exp(-rt + \alpha(t) + \beta(t) \cdot X_t)$, where $\alpha(t) \in C^1$ and $\beta(t) = (u, \overline{\beta}(t)) \in C^2$. Plugging it in (39) we get[22]:

$$\begin{aligned}
0 &= (-r_t\Psi + \Psi(\frac{\partial \alpha(t)}{\partial t} + (X_t)'\frac{\partial \beta(t)}{\partial t}) + (\mu^Q(X_t))'\beta(t)\Psi \\
&+ \beta'(t)\frac{\sigma^2(X_t)}{2}\beta(t)\Psi)dt + (\lambda_0 + V_t\lambda_1)dtE_t^Q[\Psi\exp(\beta(t)\Gamma_t) - 1]
\end{aligned} \tag{42}$$

or equivalently using (40)

$$\begin{aligned}
0 &= -r_t + \frac{\partial \alpha(t)}{\partial t} + (X_t)'\frac{\partial \beta(t)}{\partial t} + \begin{pmatrix} r_t - \lambda_0\overline{\mu}^Q & \kappa^Q\theta^Q \end{pmatrix} \beta(t) \\
&+ (X_t)' \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} - \lambda_1\overline{\mu}^Q & -\kappa^Q \end{pmatrix} \beta(t) \\
&+ \frac{1}{2}\beta'(t)[(X_t)' \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ \overline{\rho}\sigma_v \end{pmatrix} \\ \begin{pmatrix} 0 \\ \overline{\rho}\sigma_v \end{pmatrix} & \begin{pmatrix} 0 \\ \sigma_v^2 \end{pmatrix} \end{pmatrix}]\beta(t) \\
&+ (\lambda_0 + (X_t)' \begin{pmatrix} 0 \\ \lambda_1 \end{pmatrix})E_t^Q[\exp(\beta(t)\Gamma_t) - 1]
\end{aligned} \tag{43}$$

[22] $\beta'(t)\frac{\sigma^2(X_t)}{2}\beta(t)$ defined as in DPS(2000) to be a vector in C^2

and comparing the coefficients on X_t^1 and X_t^0 from both sides of the above identity we get the model-specific system of ODEs[23] the first of which may be regarded as Riccati Equation for some jump transform and intensity specification:

$$\begin{aligned}\frac{\partial\overline{\beta}(t)}{\partial t} &= (\frac{1}{2}+\lambda_1\overline{\mu}^Q)u+\kappa^Q\overline{\beta}(t)-\frac{1}{2}(u^2+2u\overline{\rho}\sigma_v\overline{\beta}(t)+\sigma_v^2\overline{\beta}^2(t)) \\ &- \lambda_1(\Theta(u,\overline{\beta}(t))-1)\end{aligned} \tag{44}$$

$$\frac{\partial\alpha(t)}{\partial t} = r_t-(r_t-\lambda_0\overline{\mu}^Q)u-\kappa^Q\theta^Q\overline{\beta}(t)-\lambda_0(\Theta(u,\overline{\beta}(t))-1) \tag{45}$$

with boundary conditions $\beta(T)=(u,0)$ and $\alpha(T)=0$.

7.2 Solving the Riccati equation

Given some solution for the first ODE, subject to boundary conditions the solution for the second ODE could be found by direct integration. Unfortunately, in general for SVSCJ model, no close-form solution to the first ODE equation is known and we must resort to numerical methods in order to find $\overline{\beta}(t)$, which becomes an excessive computational burden for the practical implementation of the comprehensive joint parameter estimation that we conduct in the main paper.

The problem with finding a close-form solution lies with the $\lambda_1(\Theta(u,\overline{\beta}(t))-1)$ term, aside for which the first ODE would be a Riccati equation with known analytical close-form solution. Following DPS (2000) example (4.1) and reducing SVSCJ to SVCJ model (setting $\lambda_1=0$) we can completely eliminate the difficulty-causing term. Alternatively, as in Bates (1997) and Pan (2002), we can reduce SVSCJ to SVSJ, which will result in removing $\overline{\beta}(t)$ from the jump transform $\Theta(u,\overline{\beta}(t))$. In both cases we will get a well-known Riccati equation and without formally deriving the solution using established ODE techniques, we will again guess at the solution.

For convenience we make a change of variable $\tau=T-t$ and guess that

$$\begin{aligned}\overline{\beta}(\tau) &= -\frac{a(1-\exp(-\gamma\tau))}{2\gamma-(\gamma+b)(1-\exp(-\gamma\tau))} \\ \alpha(\tau) &= r_t\tau(u-1)-\lambda_0\tau(1+\lambda_0\overline{\mu}^Qu) \\ &- \frac{\kappa^Q\theta^Q}{\sigma_v^2}((\gamma+b)\tau+2\ln[1-\frac{\gamma+b}{2\gamma}(1-\exp(-\gamma\tau))]) \\ &+ \lambda_0\int_0^\tau\Theta(u,\overline{\beta}(s,u))ds\end{aligned} \tag{46}$$

where

$$\begin{aligned}a &= u(1-u)-2\lambda_1(\Theta(u,\overline{\beta})-1-\overline{\mu}^Qu) \\ b &= \sigma_v\overline{\rho}u-\kappa^Q \\ \gamma &= \sqrt{b^2+a\sigma_v^2}\end{aligned} \tag{47}$$

[23] The equivalent system of ODEs could also be found in general form for an arbitrary AJD process in DPS(2000) equations (2.5)-(2.6).

The jump transform $\Theta(u,\overline{\beta})$ was derived in section (6.6) equation (38) and its analytic integral $\int_0^\tau \Theta(u,\overline{\beta}(s,u))ds$ could be found in DPS(2000) p.1362. Although in the case of SVSCJ model a is a function of $\overline{\beta}$, which makes the above proposed solution invalid[24], a is independent of $\overline{\beta}$ both in SVCJ and SVSJ models, because in the SVCJ case $\lambda_1 = 0$ and in SVSJ case the jump size transform $\Theta(u,\overline{\beta}) = \exp[\mu_s^Q u + 0.5\sigma_s^2 u^2]$ by setting $\mu_v = 0$ in (38).

Now we can check the proposed solution for both models simultaneously![25] by plugging it in (44). Taking the change of variable $\tau = T - t$ into account, define $e = \exp(-\gamma\tau)$ and consider the left-hand side of the first Riccati equation:

$$\begin{aligned} LHS1 &= \frac{\partial\overline{\beta}(t)}{\partial t} = -\frac{\partial\overline{\beta}(\tau)}{\partial\tau} = \frac{ae\gamma((\gamma-b)+(\gamma+b)e)+a(1-e)+(\gamma+b)\gamma e}{(\gamma-b)+(\gamma+b)e} \\ &= \frac{2a\gamma^2 e}{((\gamma-b)+(\gamma+b)e)^2} \end{aligned} \tag{48}$$

Next, consider the right-hand side (RHS) of the Riccati equation:

$$\begin{aligned} RHS1 &= -\frac{1}{2}[-(u(1-u)-2\lambda_1(\Theta(u,\overline{\beta})-1-\overline{\mu}^Q u))+2b\overline{\beta}+\sigma_v^2\overline{\beta}^2] \\ &= -\frac{1}{2}[-a+2b\overline{\beta}+\sigma_v^2\overline{\beta}^2] \end{aligned} \tag{49}$$

Notice that in this form our $RHS1$ exactly matches the corresponding RHS in DPS(2000) example. We could have stopped here, but we will continue to plug in for $\overline{\beta}$ from (46) in (49) to get:

$$RHS1 = -\frac{1}{2}\frac{[-a(\gamma-b+(\gamma+b)e)^2+2b(-a)(1-e)(\gamma-b+(\gamma+b)e)+\sigma_v^2 a^2(1-e)^2]}{((\gamma-b)+(\gamma+b)e)^2} \tag{50}$$

By scrupulously simplifying the numerator in the above expression one will notice that all e^0 and e^2 terms cancel each-other out and the remaining term in the numerator equals $-4a\gamma^2 e$.

$$\therefore LHS1 = RHS1 \quad \square \tag{51}$$

The second ODE equation is exactly identical to its counterpart in DPS(2000) with $\overline{\lambda}$ replaced by λ_0. Therefore, the same solution based on direct integration applies. Line-by-line computational verification of the solution is very similar to the first equation but too long, cumbersome, and non-essential for the purpose of this write-up and could be omitted.

Now that we have found the time-t transform of Y_T defined as $\psi_t = \exp(-r(T-t))E_t^Q[\exp(uY_T)]$, we can price calls and puts in our economy using the Fourier Transform Inversion-based formulas (2.9) and (2.12) in DPS(2000) or equivalently Black-Scholes-style formula in Pan(2002).

[24]It is easy to verify by plugging in the proposed solution in (46) that, as a is a function of $\overline{\beta}$, when we evaluate the partial derivative on the left-hand side of the Riccati equation, it quickly becomes too messy to result in the right-hand side.

[25]This beautiful fact is made possible due to the high "compartmentalization" of formulas achieved through the clever introduction of surrogate variables $(a, b, \gamma, \Theta(\cdot,\cdot))$, exploring the underlying linear structure of the problem in DPS(2000).

7.3 Fourier transform-based pricing formula

Finally, we can write-down the pricing formula for Call option and using a Put-Call parity find the Put option price. DPS (2000) has shown how $\psi_t(u, Y_t, V_t, \tau)$ that we have found above (see equations (41) and (46)) could be used to calculate option price by inverting the transform of the option payoff function[26]. Here we present a slightly modified Black-Scholes style formula of the Call option price C_t with exercise price K as a function of all risk-neutral parameters[27] Θ^* involved in the specification of risk-neutral dynamics in equation (37):

$$\begin{aligned} C_t \quad & (S_t, V_t, r_t, \Theta^*, K) = S_t \mathcal{P}_1 - K\mathcal{P}_2 \\ \mathcal{P}_1 \quad & = \frac{\psi_t(1, Y_t, V_t, \tau)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im(\psi_t(1 - iz, Y_t, V_t, \tau)\exp[iz\ln(\frac{K}{S_t})])}{z} dz \\ \mathcal{P}_2 \quad & = \frac{\psi_t(0, Y_t, V_t, \tau)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im(\psi_t(0 - iz, Y_t, V_t, \tau)\exp[iz\ln(\frac{K}{S_t})])}{z} dz \end{aligned} \tag{52}$$

where $Im(\cdot)$ denotes the imaginary component of a complex number.
Using Put-Call parity we can now find the put price P_t:

$$P_t(S_t, V_t, r_t, \Theta^*, K) = C_t(S_t, V_t, r_t, \Theta^*, K) - S_t + K\exp(-\tau r_t) \tag{53}$$

[26] See DPS (2000) Proposition 2 for derivation and detailed mathematical treatment

[27] Note that solution for $\psi_t(\cdot, \cdot, \cdot, \cdot)$ depends on risk-neutral parameters - that is how Θ^* enters the call pricing formula (52).

8 Conclusion

Finding good parameter estimates of security dynamics has been an extremely important area of research for many years. There is an enormous value, not only for market traders trying to make profit but also for macroeconomic policy-making,[28] in being able to improve estimates and forecasting ability of the future dynamics of the key market indices, in particular S&P500. In the past, due to the lack of continuous-time asset-pricing theory, econometric methods, and computing power, econometricians had to restrict the estimation information set to the prices of the underlying securities themselves. However, current advancements in the above fields and technology allow us to augment the estimation information set with generous amount of data contained in the panel of options written on the underlying securities, which should help us get a better grip on the latent factors (especially volatilities) in our continuous-time models. Although this valuable information shouldn't be discarded in analyzing the dynamics of the underlying, drawing the link between option prices and parameters of the underlying through no-arbitrage asset-pricing world happens to be quite technical for most econometricians, while researchers trained in theoretical asset-pricing often lack the expertise to conduct econometric inference at the level of sophistication that these non-linear state-space models require. This manuscript has provided the details necessary for proper implementation of asset-pricing results in the econometric models of interest.

[28] Hordahl and Vestin (2005) claim that central banks tend to underestimate the risk premia and use risk-neutral parameter estimates (obtained by analyzing a cross-section of options) as if they were objective parameters of market dynamics, which leads to significant forecasting errors.

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