

Effective Securities in Arbitrage-Free Markets with Bid-Ask Spreads at Liquidation: a Linear Programming Characterization

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Abstract

We consider a securities market with bid-ask spreads at any period, including liquidation. Although the minimum-cost super-replication problem is non-linear, we introduce an auxiliary problem that allows us to characterize no-arbitrage via linear programming techniques. We introduce the notion of effective new security and show that effectiveness restricts the no-arbitrage bid and ask prices of a new security to the interval defined by the minimum-cost problem. We discuss in details the cases in which the boundaries of this interval can be reached without violating no-arbitrage.

Keywords: arbitrage, bid-ask prices, linear programming, effective securities

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1 Introduction

The valuation of securities via super-replication in the presence of market frictions and its interplay with no-arbitrage is one of the most active research areas in finance theory. The topic has been analyzed both in discrete-time, starting from Bensaid *et al.* (1992) and Jouini and Kallal (1995), and in continuous-time, dating back to Cvitanic and Karatzas (1993).¹

This paper follows the discrete-time, event-tree approach and offers two main contributions. First, we show how to employ linear programming techniques to characterize no-arbitrage in markets with bid-ask spreads. With respect to the existing literature the contribution is that our linear programming approach *works also with bid-ask spreads at liquidation*. Second, we supply a linear programming-based proof of the fact that no-arbitrage per se imposes only an upper bound on the bid and a lower bound on the ask price of a new security. We then introduce the notion of *effective new security* and show that this notion characterizes the new securities whose bid-ask spreads are bounded.

In a seminal paper, Bensaid *et al.* (1992) incorporate bid-ask spreads in the standard binomial option pricing model and solve the super-replication problem via dynamic programming. Still in a binomial market but without bid-ask spreads at liquidation, Edirisinghe *et al.* (1993) show how to reformulate the super-replication problem as a linear programming one. Naik (1995) and Ortu (2001) analyze the general event-tree framework without bid-ask spreads at liquidation and use the linearized super-replication problem and its dual to provide alternative characterizations of no-arbitrage.

In this paper we address the general event-tree framework with bid-ask spreads also at liquidation. The presence of bid-ask spreads at liquidation arises in many practical applications. A European call option, for instance, is typically settled at maturity either with delivery of the underlying, or by cash, or at the discretion of the short position. In a world without bid-ask spreads at maturity, these different types of settlement are payoff-equivalent. In actual markets, however, bid-ask spreads are present also at maturity and different settlement provisions produce different payoff profiles. Bid-ask spreads at liquidation introduce a non-linearity in

¹Other contributions related to the present work are Dermody and Rockafeller (1991), Chen (1995), Chateaufneuf *et al.*(1996), Milne and Neave (1997), Charupat and Prisman (1997), Jouini (2000), Koehl *et al.* (2001), Zhang *et al.* (2002), Huang (2002), Delbaen *et al.* (2002). We refer to Cvitanic (1999) for a detailed survey of the literature on super-replication without and with frictions in continuous-time models.

the otherwise linear super-replication problem. Indeed, investors typically aggregate their long and short positions with the same broker. This implies that rather than the cumulative long and short positions separately, what matters at the moment of final liquidation are the net positions held in each security. With bid-ask spreads at liquidation, this makes the terminal payoff, and hence the super-replication problem, non-linear in the intertemporal trading strategies.

To deal with this non-linearity, we construct an auxiliary linear program with the same value function as the original problem, and such that any solution to the super-replication problem is a linear transformation of a solution to the auxiliary linear program. To construct this auxiliary program we first partition the set of feasible trading strategies according to the sign of the net positions at liquidation. Then, taking one strategy for each cell of this partition, we use the sum of their cashflow to super-replicate any given future payoff at the minimum cost. Our auxiliary program extends the linear programming characterization of no-arbitrage of Naik (1995) and Ortu (2001) to the case of bid-ask spreads at liquidation. Moreover, by linear duality, we are able to characterize the cases in which strict super-replication is cost-minimizing. In particular, we show that this occurs if and only if the minimum cost of super-replication strictly exceeds the value assigned to a given claim by any strictly positive linear pricing rule compatible with no-arbitrage.

Based on our linear programming approach we discuss the interplay between no-arbitrage and the notion of *effectiveness of a new security* introduced in the market. We first employ linear programming duality to formally derive a fact pointed out by Jouini and Kallal (1999), namely that no-arbitrage survives the introduction of a new security if and only if its bid price does not exceed the minimum cost to cover a short position and its ask price is greater than the maximum that can be borrowed against a liability equal to the payoff from a long position.² We then introduce the notion of *effectiveness* to characterize the new securities whose bid-ask spread will be bounded. In words, a newly traded security is long (short)-effective when it is optimal to take a long (short) position in the new security to super-replicate some future cashflow at the minimum cost. Intuitively, our notion of effectiveness conveys the idea that

²In particular, these bounds can be reached without introducing arbitrage if and only if perfect replication is the only cost optimal strategy.

the new security improves the super-hedging capabilities of the investors.

Our approach allows us to interpret the typical interval bounds for the bid and ask prices of a new security as generated by the interaction of *no-arbitrage and effectiveness*. In particular, effectiveness forces the no-arbitrage ask price of a new security to be smaller than the minimum cost incurred to super-replicate the payoff from a long position, and the no-arbitrage bid price to be larger than the maximum that can be borrowed against a liability not exceeding the one generated by a short position. Our results identify explicitly the cases in which reaching the boundaries of the interval gives rise to arbitrage opportunities. These cases occur in fact when the initial bid-ask spread vanishes, the price of the new security collapses to one of the extremes of the interval and strict super-replication is cost-optimal.

The rest of the paper is structured as follows. In the next section we introduce the basic notation and definitions. In Section 3 we supply our linear programming characterization of no-arbitrage with bid-ask spreads at liquidation. In Section 4 we introduce the notion of effective new security and compare the pure no-arbitrage bounds with those that must hold for an effective new security. We also compare our notion of effective security to Jouini and Kallal's (2001) notion of efficient trading strategy. Section 5 concludes by addressing the possibility of extending our results to more general frameworks. All proofs are in the Appendix.

2 Basic notation and definitions

We assume that J perfectly divisible securities are traded over the horizon $\mathbf{T} = \{0, 1, \dots, T\}$. Investors are price-takers and allowed to short-sell securities with full use of proceeds. Trading entails however a bid-ask spread, formalized by describing each security j as a triple $(S_j^A, S_j^B, d_j) \equiv \{S_j^A(t), S_j^B(t), d_j(t)\}_{t=0}^T$ of stochastic processes,³ where S_j^A represents the ex-dividend ask price, S_j^B the ex-dividend bid price, and d_j the dividend flow ($d_j(0) = d_j(T) = 0$ for all j without loss of generality). We assume that $S_j^A \geq S_j^B$ and in particular we allow for bid-ask spreads at liquidation. Therefore, at the terminal date T , the cost $S_j^A(T)$ of covering a unit short position may be larger than the revenue $S_j^B(T)$ from liquidating a unit long position.

³We assume that the underlying probability space (Ω, F, P) is finite, with $\Omega = \{\omega_1, \dots, \omega_{s_T}\}$, $F = 2^\Omega$ and P strictly positive on $2^\Omega \setminus \{\emptyset\}$. In this setting, the information flow shared by all investors is an event-tree \mathbb{P} , with f_k^t the generic time t node, s_t the number of time t nodes, and $L = \sum_{t=0}^T s_t$ the total number of nodes. All stochastic processes introduced hereafter are adapted to \mathbb{P} .

We model dynamic trading in each security by means of couples $\theta_j = \{\theta_j^A(t), \theta_j^B(t)\}_{t=0}^{T-1}$ of stochastic processes, where $\theta_j^A(t)$ represents the number of units of security j bought at t , and $\theta_j^B(t)$ the number of units sold at t . Then, to be feasible, a dynamic trading strategy $\theta = \{\theta_j\}_{j=1}^J$ must be non-negative. We denote by Θ the set of all feasible dynamic trading strategies.

A dynamic trading strategy $\theta \in \Theta$ generates a *cashflow process* $x_\theta = \{x_\theta(t)\}_{t=0}^T$. To describe it, we observe that $\sum_{\tau=0}^{t-1} (\theta^A(\tau) - \theta^B(\tau))$ represents the net position held on the J assets before trading at time t and with $\left[\sum_{\tau=0}^{T-1} (\theta^A(\tau) - \theta^B(\tau))\right]^+$ and $\left[\sum_{\tau=0}^{T-1} (\theta^A(\tau) - \theta^B(\tau))\right]^-$ we denote the *net long* and *short* positions held at the terminal date.⁴ Therefore we have:

$$x_\theta(t) = \begin{cases} -[\theta^A(0) \cdot S^A(0) - \theta^B(0) \cdot S^B(0)], & t = 0 \\ d(t) \cdot \sum_{\tau=0}^{t-1} (\theta^A(\tau) - \theta^B(\tau)) - [\theta^A(t) \cdot S^A(t) - \theta^B(t) \cdot S^B(t)], & t = 1, \dots, T-1 \\ S^B(T) \cdot \left[\sum_{\tau=0}^{T-1} (\theta^A(\tau) - \theta^B(\tau))\right]^+ - S^A(T) \cdot \left[\sum_{\tau=0}^{T-1} (\theta^A(\tau) - \theta^B(\tau))\right]^-, & t = T. \end{cases} \quad (1)$$

At date 0, the cashflow is just the opposite of the initial cost of θ , while at the intermediate dates $t = 1, \dots, T-1$ it is the difference of the dividends earned on the net positions held in the J assets before trading and the cost to update these positions. At the terminal date T the cashflow is the difference between the revenue from liquidating the net long positions at the bid prices and the cost of covering the net short positions at the ask prices. This definition of $x_\theta(T)$ assumes that long and short positions on each security are held in the same account, so that only the net positions matter at liquidation.

For expositional purposes, we recall the definition of arbitrage opportunity in the securities market with bid-ask spread introduced above.

Definition 1 *A feasible dynamic trading strategy θ such that $x_\theta(t) \geq 0$ for all t constitutes an arbitrage opportunity of:*⁵

1. *the first type if $x_\theta(t) > 0$ for some $t > 0$;*

⁴Given any vector $y \in \mathfrak{R}^n$, we denote with $[y]^+$, respectively $[y]^-$ the vectors with components $\max\{y_j, 0\}$ and $\max\{-y_j, 0\}$, $j = 1, \dots, n$.

⁵Hereafter $x_\theta(t) \geq 0$ means $P(x_\theta(t) \geq 0) = 1$; $x_\theta(t) > 0$ means $x_\theta(t) \geq 0$ and $P(x_\theta(t) > 0) > 0$; $x_\theta(t) \gg 0$ means $P(x_\theta(t) > 0) = 1$.

2. the second type if $x_\theta(0) > 0$.

Hereafter, we say that no-arbitrage holds if the price-dividend system (S^A, S^B, d) is free of both types of arbitrage opportunities.

The following condition is maintained throughout the paper.

Condition 1 (*The Internality Condition*) *There exists a feasible dynamic trading strategy θ such that $x_\theta(t) \gg 0$ for all $t > 0$.*

This requirement is indeed very mild, and is satisfied whenever one of the traded assets has strictly positive bid price process, and pays non-negative intermediate dividends. Since it is readily seen that the existence of second-type arbitrage opportunities implies the existence of first-type arbitrage opportunities if the Internality Condition holds, a price-dividend system (S^A, S^B, d) that satisfies the Internality Condition is arbitrage-free if and only if it is free of first-type arbitrage opportunities.

3 No-arbitrage with bid-ask spreads at liquidation

In this section, we extend the linear programming characterization of no-arbitrage with bid-ask spreads available in the literature⁶ to the case in which bid-ask spreads are present also at the date T of final liquidation. Recall that the super-replication-at-minimum-cost problem for a given future cashflow $m = \{m(t)\}_{t=1}^T$ is:

$$\begin{aligned} \inf_{\theta \in \Theta} -x_\theta(0) &= \theta^A(0) \cdot S^A(0) - \theta^B(0) \cdot S^B(0) \\ \text{s.t. } x_\theta(t) &\geq m(t), \forall t > 0. \end{aligned} \tag{\mathcal{P}[m]}$$

Recall also that, since in our setting the information structure is an event-tree \mathbb{P} ,⁷ a feasible dynamic trading strategy θ is identified by a vector in $\mathfrak{R}_+^{2J(L-s_T)}$, the cashflow x_θ by a column vector in \mathfrak{R}^L and the future cashflows m that parametrizes $\mathcal{P}[m]$ by a vector in \mathfrak{R}^{L-1} . The difference with the usual setting is that the presence of bid-ask spreads at the terminal date T makes the terminal payoff $x_\theta(T)$, and hence the constraint $x_\theta(T) \geq m(T)$, non-linear in θ . This implies that $\mathcal{P}[m]$ cannot be addressed directly by means of linear programming techniques.

⁶See in particular Naik (1995) and Ortu (2001).

⁷Recall that L is the total number of nodes, and s_T is the number of terminal nodes.

The main contribution of this section is to construct a linear program which is, in a sense to be made precise, equivalent to $\mathcal{P}[m]$. To do so, we let

$$\Theta_{\geq m} \equiv \left\{ \theta \in \mathfrak{R}_+^{2J(L-s_T)} \mid x_\theta(f_k^t) \geq m(f_k^t), \forall t > 0 \text{ and } k \right\} \quad (2)$$

denote the feasible set of $\mathcal{P}[m]$, where f_k^t denotes the generic node of the event-tree \mathbb{P} at time t . We observe that, as a consequence of the Internality Condition, $\Theta_{\geq m}$ is non-empty for any choice of m . Moreover, we denote by $\pi(m)$ the value function of $\mathcal{P}[m]$,

$$\pi(m) \equiv \inf \{ -x_\theta(0) \mid \theta \in \Theta_{\geq m} \}. \quad (3)$$

In words, $\pi(m)$ represents the minimum cost to super-replicate m . Recalling the definition of a cash-flow (1), notice that the presence of transaction costs at liquidation implies the non-linearity of the last s_T constraints of the problem $\mathcal{P}[m]$. These constraints are the ones that guarantee the super-replication of the cash-flow m in every state at the terminal date.

We are now ready to state our first Theorem. In this result we show that, although the problem $\mathcal{P}[m]$ is not linear, the linear programming approach exploited by Ortu (2001) can be extended to the case of transaction costs at liquidation.

Theorem 1 *The following facts hold:*

1. *there exist a vector $c \in \mathfrak{R}^{2J(L-s_T)}$, $K \equiv 2^{J s_T - 1}$ matrices M_k of dimension $(L-1) \times 2J(L-s_T)$, and K matrices G_k of dimension $J s_{T-1} \times 2J(L-s_T)$, such that*

$$\begin{aligned} \pi(m) = & \inf_{(\theta_1, \dots, \theta_K) \in \mathfrak{R}_+^{2KJ(L-s_T)}} \sum_{k=1}^K c \cdot \theta_k \\ & \text{s.t.} \quad \begin{cases} \sum_{k=1}^K M_k \theta_k \geq m \\ G_k \theta_k \geq 0, \quad k = 1, \dots, K. \end{cases} \end{aligned} \quad (\mathcal{LP}[m])$$

that is, problem $\mathcal{P}[m]$ and the linear program $\mathcal{LP}[m]$ have the same value function.

2. *θ^* is an optimal solution to $\mathcal{P}[m]$ if and only if $\theta^* = \sum_{k=1}^K \theta_k^*$ for some $(\theta_1^*, \dots, \theta_K^*)$ optimal solution to $\mathcal{LP}[m]$.*
3. *Denoting with $\Theta_{> m}^*$ and $\Xi_{> m}^*$ the sets of optimal solutions to $\mathcal{P}[m]$ and $\mathcal{LP}[m]$ for which the constraints are not binding, that is*

$$\Theta_{> m}^* = \{ \theta \text{ solution to } \mathcal{P}[m] \mid x_\theta(t) > m(t) \text{ for some } t > 0 \}$$

and

$$\Xi_{>m}^* = \left\{ (\theta_1, \dots, \theta_K) \text{ solution to } \mathcal{LP}[m] \mid \sum_k M_k \theta_k > m \right\}$$

then $\Theta_{>m}^* \neq \emptyset$ if and only if $\Xi_{>m}^* \neq \emptyset$.

In words, Theorem 1 states that it is possible to construct a parametric linear program $\mathcal{LP}[m]$ with the following properties. First, the value function of $\mathcal{LP}[m]$ coincides with the value function of $\mathcal{P}[m]$. Second, the set of solutions to $\mathcal{P}[m]$ is a linear transformation of the set of solutions to $\mathcal{LP}[m]$. Third, the existence of solutions to $\mathcal{P}[m]$ for which the constraints are not binding is equivalent to the existence of solutions to $\mathcal{LP}[m]$ for which the same property holds. To understand the intuition behind Theorem 1 it is important to understand the interpretation of $\mathcal{LP}[m]$. To construct it, the first step is to partition (modulo the boundaries) the set Θ of all feasible trading strategies into K convex cones Θ_k . Two feasible trading strategies belong to the same Θ_k , $k = 1, \dots, K$, if and only if at the terminal date T the sign of the net position on each asset is the same⁸ on each terminal node of the event-tree \mathbb{P} . Since the last rebalancing of the positions in the J assets occurs at $T - 1$, and since s_{T-1} denotes the number of nodes at $T - 1$, it is readily seen that the number of convex cones in which we partition Θ is indeed $K \equiv 2^{Js_{T-1}}$. Moreover, in the Appendix we show that Θ_k is constituted by the $\theta_k \in \Theta$ such that $G_k \theta_k \geq 0$, where the matrix G_k is constructed in such a way that $G_k \theta_k$ supplies the absolute values of the net positions on each asset at each of the terminal nodes of \mathbb{P} . The second step to construct $\mathcal{LP}[m]$ consists in observing that for each k the set of future cashflows generated by the trading strategies in Θ_k is the convex cone $\{M_k \theta_k \mid \theta_k \in \Theta_k\}$, with the *payoff matrix* M_k once again supplied in the Appendix. Finally, we observe that the initial cost $\theta_k^A(0) \cdot S^A(0) - \theta_k^B(0) \cdot S^B(0)$ of a generic trading strategy $\theta_k \in \Theta_k$ can be represented as the inner product $c \cdot \theta_k$, with c suitably defined. Therefore, the parametric linear program $\mathcal{LP}[m]$ consists in computing the minimum cost to super-replicate m with sums of K cashflows, where the strategies that generate the K cashflows are chosen one from each of the convex cones Θ_k that partition Θ .

Theorem 1 allows us to extend all the characterizations of no-arbitrage and no-second-type-arbitrage supplied in Ortú (2001) to the case of bid ask-spreads at liquidation. We synthesize

⁸For same sign we mean either weakly positive or weakly negative. This explains the “modulo the boundary” qualification when stating that $\Theta_1, \dots, \Theta_K$ constitute a partition of Θ .

these extensions hereafter because they constitute the buildings blocks for the discussion of the effectiveness and efficiency of a newly traded security presented in following sections. First, in Theorem 2 and Corollary 1 we present the relationships between no-arbitrage, respectively no-second-type-arbitrage, and the properties of the minimum cost function $\pi(m)$ and those of the optimal solutions to the problems $\mathcal{P}[m]$ and $\mathcal{LP}[m]$.⁹

Theorem 2 *The following statements are equivalent:*

1. *no-arbitrage holds.*
2. *$\pi(m)$ is a real-valued, strictly positive sublinear function and $\pi(0) = 0$.*
3. *$\mathcal{LP}[m]$ (equivalently, $\mathcal{P}[m]$), admits solutions $\forall m \in \mathfrak{R}^{L-1}$. Moreover, $c \cdot \sum_k \theta_k^* = 0$ and $\sum_k M_k \theta_k^* = 0$ for any $(\theta_1^*, \dots, \theta_K^*)$ solution to $\mathcal{LP}[0]$ (equivalently, $x_{\theta^*} = 0$ for any θ^* solution to $\mathcal{P}[0]$).*
4. *There exists optimal solutions $(\theta_1^*, \dots, \theta_K^*)$ to $\mathcal{LP}[0]$ (equivalently, θ^* to $\mathcal{P}[0]$) for any of which $c \cdot \sum_k \theta_k^* = 0$ and $\sum_k M_k \theta_k^* = 0$ (equivalently, $x_{\theta^*} = 0$).*

Corollary 1 *The following statements are equivalent:*

1. *no-second-type-arbitrage holds.*
2. *$\pi(m)$ is a real-valued, nonnegative sublinear function and $\pi(0) = 0$.*
3. *There exists optimal solution of $\mathcal{LP}[m]$ (equivalently, $\mathcal{P}[m]$) for all $\forall m \in \mathfrak{R}^{L-1}$.*
4. *There exists optimal solutions to $\mathcal{LP}[0]$ (equivalently, $\mathcal{P}[0]$).*

We now recall the definitions of underlying state-prices and semi-positive underlying state-prices for a securities market with bid-ask spreads. These objects are then employed to characterize no-arbitrage and no-second-type-arbitrage by means of duality techniques.

Definition 2 *We call underlying state-price (USP) any vector $\psi \in \mathfrak{R}_{++}^{L-1}$ such that $(1, \psi) \cdot x_\theta \leq 0 \forall \theta \in \Theta$. We call semi-positive USP any vector $\varphi \in \mathfrak{R}_+^{L-1}$ such that $(1, \varphi) \cdot x_\theta \leq 0 \forall \theta \in \Theta$.*

⁹The proofs of Theorem 2 and Corollary 1 are omitted because they are a straightforward consequence of the corresponding results in Ortu (2001).

As usual, the USPs represent the counterpart in the case of bid-ask spreads to the Arrow-Debreu state-prices for a frictionless securities market. We denote by Ψ and Φ the set of all the USP and semi-positive USP vectors respectively. In the next result we supply an analytic characterization of these sets based on a suitable version of the theorem of the alternatives. The characterizations of Ψ and Φ exploit the matrices M_k, G_k that constitute the building blocks of the linear program $\mathcal{LP}[m]$ discussed in Theorem 1.

Proposition 1 *We have*

$$\Psi = \left\{ (1, \psi) \in R_{++}^L \mid \psi M_k + \beta_k G_k \leq c, \beta_k \geq 0, k = 1, \dots, K \right\}$$

and

$$\Phi = \left\{ (1, \varphi) \in \mathfrak{R}_+^L \mid \varphi M_k + \beta_k G_k \leq c, \beta_k \geq 0, k = 1, \dots, K \right\}$$

We are now ready to state the result that characterizes no-arbitrage (repectively no-second-type-arbitrage) in terms of USPs (semi-positive USPs). These characterizations are readily obtained by exploiting the relationships between $\mathcal{LP}[m]$ and $\mathcal{P}[m]$ laid out in Theorem 1, together with the fundamental theorem of linear programming.

Theorem 3 *No-arbitrage (no-second-type-arbitrage) is equivalent to the existence of USPs (semi-positive USPs). Under no-arbitrage, moreover,*

$$\pi(m) = \sup_{(1, \psi) \in \Psi} \psi \cdot m \quad \forall m \in R^{L-1}$$

while under no-second-type-arbitrage

$$\pi(m) = \max_{(1, \varphi) \in \Phi} \varphi \cdot m \quad \forall m \in R^{L-1}$$

In the Appendix we provide a proof of Proposition 1 based on a Theorem of the Alternative and from this result we derive the proof of Theorem 3. The value of this approach is to offer an algebraic characterization of the set of the USPs that is exploited in several proofs of the remaining results of the paper. Notice however that a characterization of No-arbitrage by means of duality similar to the one provided in Theorem 3 can be derived in several different ways. For instance, one could apply either the generalization of Stiemke Lemma to polyhedral

cone domains provided in Huang (1998) or Tucker Theorem of Alternatives to obtain a dual formulation of problem $\mathcal{LP}[m]$.

We point out that our duality-based formulation of no-arbitrage, and the related characterization of the minimum-cost functional as the envelope of the USPs, represents an event-tree counterpart to other results of the literature as Theorem 6.4 in Cvitanic and Karatzas (1993) and Theorem 2.2 in Jouini and Kallal (1995).

We conclude this section by investigating the conditions under which the supremum in Theorem 3 is actually a maximum. As we show hereafter, this fact occurs if and only if exact replication of a future cashflow is cost-optimal.¹⁰

Theorem 4 *If no-arbitrage holds, then $\forall m \in R^{L-1}$ we have $\pi(m) = \max_{(1,\psi) \in \Psi} \psi \cdot m$ if and only if $\Theta_{>m}^* = \emptyset$.*

To further highlight the importance of Theorem 1, we remark that our proof of Theorem 4 is based on the relationships between problems $\mathcal{LP}[m]$ and $\mathcal{P}[m]$. Since the constraints of all solutions to $\mathcal{P}[m]$ are binding (that is, $\Theta_{>m}^* = \emptyset$) if and only if the same is true for $\mathcal{LP}[m]$, the necessity part in Theorem 4 follows readily from a complementary slackness argument applied to $\mathcal{LP}[m]$. To prove the converse, we suitably modify $\mathcal{LP}[m]$ in such a way that its dual has a feasible set containing only semipositive USPs that satisfy the condition $\varphi \cdot m \geq \pi(m)$. Then, we show that the modified $\mathcal{LP}[m]$ has a strictly positive value function. This allows us to construct a USP satisfying $\psi \cdot m \geq \pi(m)$, from which the result follows.

Aside from being interesting per se, Theorem 4 constitutes the pivotal result on which we base our discussion of the arbitrage bounds for effective new securities and our comparison between effectiveness and efficiency of a trading strategy. In fact, the suitably modified version of $\mathcal{LP}[m]$ used in the proof of Theorem 4 is, in essence, the linear program that, in the market extended to a new security, is equivalent to the minimum cost super-replication problem. We address these issues in the next sections.

¹⁰Therefore, our next result can also be interpreted as a preference-free counterpart to Remark 1 in Jouini and Kallal (2001).

4 Arbitrage bounds for an effective new security

In this section we discuss the pure no-arbitrage bounds on the bid-ask prices of a new security and we introduce the concept of an effective new security, that is, a security whose initial bid and ask prices represent the best way the market offers to super-replicate its future cash-flows. Then we compare the pure no-arbitrage bounds with those that must hold for a new security to be effective. The relation between our concept of effectiveness and other notions already present in the literature, is discussed in details in Section 5.

Consider a situation in which a new security (for example a derivative written on one of the original J securities) is introduced into the market. We assume that the new security trades only at time $t = 0$ and comes to maturity at the terminal date T . In other words, the only positions that the investors can take on the new security are either a long position held up to T , or a short position to be covered at T . We denote with $c^A \geq c^B$ the time 0 ask and bid prices of the new security, and with $n(t)$, $t = 1, \dots, T - 1$, the dividend flow. Moreover, we let $n^A(T)$ represent the cost of covering a short position on the new security at the final date T , and $n^B(T)$ the revenue from liquidating a long position, with $n^A(T) \geq n^B(T)$. Finally, we let $\zeta^A, \zeta^B \in \mathfrak{R}_+$ denote the units of the new security bought and shorted respectively, at time 0. Henceforth, we call *original* the securities market with securities 1 to J , and *extended market* the one in which the new security trades along with securities 1 to J . In the extended market, a trading strategy $(\theta, \zeta^A, \zeta^B)$ generates the cashflow

$$x_{(\theta, \zeta^A, \zeta^B)}(t) = \begin{cases} x_\theta(0) - \zeta^A c^A + \zeta^B c^B & t = 0 \\ x_\theta(t) + (\zeta^A - \zeta^B) n(t) & t = 1, \dots, T - 1 \\ x_\theta(T) + n^B(T) \cdot [\zeta^A - \zeta^B]^+ - n^A(T) \cdot [\zeta^A - \zeta^B]^- & t = T. \end{cases} \quad (4)$$

with $x_\theta(t)$ as defined in (1).

We first characterize no-arbitrage and no-second-type-arbitrage in the extended market. We base our characterizations on the comparison between the bid and ask prices of the new security and the minimum cost that, in the original market, one incurs to super-replicate the future cashflows $n^A = (n(1), \dots, n(T - 1), n^A(T))$ and $n^B = (n(1), \dots, n(T - 1), n^B(T))$.

Theorem 5 *If no-arbitrage holds in the original market, equivalent statements are:*

1. no-arbitrage holds in the extended market.
2. $c^B \leq \pi(n^A)$ (with strict inequality if $\Theta_{>n^A}^* \neq \emptyset$) and $c^A \geq -\pi(-n^B)$ (with strict inequality if $\Theta_{>-n^B}^* \neq \emptyset$).

Corollary 2 *If no-second-type-arbitrage holds in the original market, equivalent statements are:*

1. no-second-type-arbitrage holds in the extended market.
2. $c^B \leq \pi(n^A)$ and $c^A \geq -\pi(-n^B)$.

Theorem 5 characterizes no-arbitrage in the extended market by identifying the least upper bound and greatest lower bound for the bid and ask prices of the new security. Precisely, the least upper bound for c^B is the minimum cost to super-replicate n^A in the original market, while the greatest lower bound for c^A is the maximum amount that can be borrowed in the original market against a future liability not exceeding n^B . Corollary 2 shows that second-type arbitrage opportunities are absent from the extended market even if c^B reaches its least upper bound, or c^A its greatest lower bound.¹¹

As a consequence of Theorem 5 and Corollary 2, arbitrage considerations alone are not sufficient to restrict the bid-ask spread $c^A - c^B$ of the new security. To determine a condition that, together with no-arbitrage, bounds c^A and c^B both from above and from below, we need to impose an additional requirement on the new security, formalized in the next definition.

Definition 3 *For each $m \in \mathfrak{R}^{L-1}$ let $\mathcal{P}^{ex}[m]$ denote the super-replication problem of m in the extended market. Then we call the new security:*

1. long-effective if there exist m and $(\theta, \zeta^A, \zeta^B)$ optimal for $\mathcal{P}^{ex}[m]$ such that $\zeta^A > 0$.
2. Short-effective if there exist m and $(\theta, \zeta^A, \zeta^B)$ optimal for $\mathcal{P}^{ex}[m]$ such that $\zeta^B > 0$.

¹¹By Statement 2 in Theorem 5, in fact, c^B is allowed to attain its least upper bound without introducing an arbitrage opportunity if and only if perfect replication of the future cashflow n^A is cost-optimal in the original market. Likewise, no-arbitrage holds in the extended market even if c^A reaches its greatest lower bound if and only if the amount that can be borrowed in the original market against a future liability not exceeding n^B is maximized by shorting n^B itself.

3. *Effective if it is both long- and short-effective.*

In words, a newly traded security is long-effective when it is *held long* in some strategy that super-replicates some future cashflow at the minimum cost. Likewise, short-effectiveness means the new security is *shorted* in some strategy that super-replicates some future cashflow at the minimum cost.¹²

The following result explores the implications of effectiveness on the ask and bid prices of the new security.

Proposition 2 *Assume that no-second-type-arbitrage holds in the original market, and denote with $\pi^{ex}(\cdot)$ the value function of $\mathcal{P}^{ex}[\cdot]$. Then the new asset is:*

1. *long-effective if $\max [c^B, -\pi(-n^B)] \leq c^A \leq \pi(n^B)$, equivalently $\pi^{ex}(n^B) = c^A$.*
2. *Short-effective if $-\pi(-n^A) \leq c^B \leq \min [c^A, \pi(n^A)]$, equivalently $-\pi^{ex}(-n^A) = c^B$.*
3. *Effective if $-\pi(-n^A) \leq c^B \leq c^A \leq \pi(n^B)$, equivalently $\pi^{ex}(n^B) = c^A$, $-\pi^{ex}(-n^A) = c^B$.*

Proposition 2 characterizes effectiveness in terms of both the original market and the extended one. From the standpoint of the original market, the new security is long-effective if the ask price is bounded from above by the minimum cost incurred to super-replicate n^B , and by below by the maximum amount that can be borrowed against $-n^B$, if this amount exceeds the bid price, or otherwise by the bid price itself. From the perspective of the extended market, instead, long-effectiveness is characterized by the fact that an optimal way to super-replicate the future cash-flow n^B is to buy the new security at its ask price c^A . Part 2. of Proposition 2 characterizes short-effectiveness in a similar way, while Part 3. of Proposition 2 shows that overall effectiveness imposes $\pi(n^B) + \pi(-n^A)$ as an upper bound to the bid-ask spread $c^A - c^B$.¹³

¹²We point out that our notion of long and short-effectiveness extends to a stochastic environment the notion of *attractive* long and short security introduced by Dermody and Rockafellar (1991) in a deterministic term structure framework.

¹³Proposition 2 shows that effective new securities are characterized by a bid-ask price pair that is *consistent* in the sense of Jouini and Kallal (1999). To see this, one needs simply to replace in Definition 3.1 in Jouini and Kallal (1999) π' with our π^{ex} , and observe that viability coincides with no-arbitrage when the probability space is finite.

We are now ready to state our characterization of the arbitrage bounds that must be satisfied by a newly traded effective security.

Theorem 6 *If no-arbitrage holds in the original market, equivalent statements are:*

1. *no-arbitrage holds in the extended market and the new asset is effective.*
2. $-\pi(-n^A) \leq c^B \leq c^A \leq \pi(n^B)$, with $c^B < \pi(n^B)$ if $\pi(n^B) = \pi(n^A)$ and $\Theta_{>n^A}^* \neq \emptyset$, and $c^A > -\pi(-n^A)$ if $\pi(-n^A) = \pi(-n^B)$ and $\Theta_{>-n^B}^* \neq \emptyset$.

Corollary 3 *If no-second-type-arbitrage holds in the original market, equivalent statements are:*

1. *no-second-type-arbitrage holds in the extended market and the new security is effective.*
2. $-\pi(-n^A) \leq c^B \leq c^A \leq \pi(n^B)$.

As a consequence, the condition $-\pi(-n^A) \leq c^B \leq c^A \leq \pi(n^B)$ per se is not sufficient for no-arbitrage to be maintained after a new effective security is introduced in the market, although it guarantees that no-second-type-arbitrage will hold.¹⁴ Our results, in fact, allow us to identify explicitly the cases in which boundary bid and ask prices of a new security must be avoided in order to preserve full no-arbitrage in the extended market.¹⁵

To conclude we briefly compare our notion of effectiveness with the notion of efficient trading strategy with market frictions of Jouini and Kallal (2001).¹⁶ These authors characterize the trading strategies that are optimal for non satiated, Von Neumann-Morgenstern investors who maximize concave expected utilities given their initial wealth and uncertain future endowment. Assuming a simple one-period model with equiprobable end-of-period states, in our framework

¹⁴By Theorem 6 a new effective security introduces (first type) arbitrage opportunities when either $c^B = \pi(n^A)$ and $\Theta_{>n^A}^* \neq \emptyset$ or $c^A = -\pi(-n^B)$ and $\Theta_{>-n^B}^* \neq \emptyset$. If $c^B = \pi(n^A)$ the effectiveness of the new security forces $c^B = c^A$, so that the security can be shorted at a price equal to the minimum cost to super-replicate n^A in the original market. Moreover, $\Theta_{>n^A}^* \neq \emptyset$ means that there is a cost-minimizing strategy that strictly super-replicates n^A . Therefore an arbitrage opportunity of the first type is generated by shorting the new security and buying this strategy. Such a situation is portrayed in Example 1 in the Appendix. For the case $c^A = -\pi(-n^B)$ and $\Theta_{>-n^B}^* \neq \emptyset$ similar arguments apply.

¹⁵From this standpoint, our linear programming approach details the behavior on the boundary of the bid-ask price pairs for a new security that are consistent in the sense of Jouini and Kallal (1999).

¹⁶See Dybvig (1988) for the original notion in the frictionless case.

a long (respectively short) position in the new security is an efficient trading strategy if, given an uncertain future endowment x , there is a non-satiated and concave expected utility U and an initial wealth w such that $m = n^B$ (respectively $m = n^A$) solves $\max\{U(m+x) : \pi^{ex}(m) \leq w\}$.

Although effectiveness is clearly a necessary condition for a position in the new security to be an efficient trading strategy, it is not a sufficient one, as shown in Example 2 in the Appendix. The economic intuition that explains the difference between effectiveness and efficiency is quite simple. Investors focusing on minimum-cost super-replication are indifferent between a position in the new security and any strategy that super-replicates the payoff of that position at the same minimum cost. Non-satiated expected utility investors would instead strictly prefer to a position in the new security any strategy providing at the same cost a consumption profile exceeding the payoff of that position in at least one state. These type of situations exhaust in fact the cases in which a position in a newly traded long/short-effective security is not an efficient trading strategy.

An equivalence can however be drawn between effectiveness and the weaker notion of “zero inefficiency cost”. In words, the inefficiency cost of a position in the new security is the difference between the ask (respectively bid) price of the new security and the largest amount required by all investors with endowment x to get at least the same utility as with the payoff n^B (respectively n^A).¹⁷ It can then be shown that a new security introduced in a market free of second-type arbitrage opportunities is long (respectively short) effective if and only if a long (respectively short) position in it has zero inefficiency cost for some suitably chosen future endowment \bar{x} .¹⁸ Therefore the pricing bounds for a newly traded security discussed above can be interpreted also from the standpoint of inefficiency cost: the condition $-\pi(-n^A) \leq c^B \leq c^A \leq \pi(n^B)$ is necessary and sufficient for zero inefficiency cost of long/short positions in the

¹⁷More formally given a future endowment x the inefficiency cost of a long or short position in the new security is the quantity $c^A - \sup_U \left\{ \min_m [\pi^{ex}(m) : U(m+x) \geq U(n^B+x)] \right\}$, where $n = n^B$ or n^A and the sup is taken over all non satiated and concave expected utility investors.

¹⁸Suppose the security long-effective, so that $c^A = \pi^{ex}(n^B)$ by Proposition 2. Applying Theorem 3 to the extended market, there exists a semi-positive USP φ^{ex} such that $\pi^{ex}(n^B) = \varphi^{ex}n^B$. By suitably choosing the uncertain future endowment \bar{x} , Theorem 3 in Jouini and Kallal (2001) shows that $c^A = \pi^{ex}(n^B) = \sup_U \left\{ \min_m [\pi^{ex}(m) : U(m+\bar{x}) \geq U(n^B+\bar{x})] \right\}$. Conversely, from the definition of $\pi^{ex}(n^B)$ together with Corollary 1 in Jouini and Kallal (2001) it follows that $c^A \geq \pi^{ex}(n^B) \geq \sup_U \left\{ \min_m [\pi^{ex}(m) : U(m+x) \geq U(n^B+x)] \right\}$ for all x , so that a new security is long-effective whenever it has zero inefficiency cost for a suitably chosen future endowment \bar{x} . A specular argument delivers the prove for the case of short-effectiveness.

new security and no-second-type-arbitrage in the extended market.

5 Conclusions

To conclude, we briefly discuss two questions regarding the possibility of extending our results to more general frameworks. The first question concerns the extent to which our linear programming characterization of no-arbitrage with bid-ask spreads at liquidation carries on to an infinite set of states. In particular, the relevant case is the one with an infinite number of states at the last but one date $T - 1$. Although it is still possible to formally write our auxiliary linear program $\mathcal{LP}[m]$ by constructing the matrices M_k and G_k as in the proof of Theorem 1, in this case our procedure would generate a *continuum of matrices*¹⁹ M_k and G_k , a fact with two consequences on our results. First, the only part of the proof of Theorem 1 that works as it is is that $\pi(m)$ exceeds the value of $\mathcal{LP}[m]$. The proof of the reverse inequality, which relies on the possibility of mapping linearly any feasible strategy of $\mathcal{LP}[m]$ into a strategy with the same cost and feasible for $\mathcal{P}[m]$, may fail to hold with an infinite set of states since the linear map may be undefined. Second, in an infinite dimensional setting the equality between the value function of the primal and the dual problem may fail without additional assumptions, that is, our linear programming approach may exhibit a “duality gap”.²⁰ These and other related problems arising in an infinite dimensional setting constitute an interesting argument for future research.

A second question concerns the possibility that the new security be traded at every date, and not just at the initial date, as we assume in Section 4. In this case, the open question consists in supplying the relations that must hold between the entire sequence $\{c^A(t), c^B(t)\}$ of ask and bid prices of the new security and suitably defined super-replication problems in the original market for no-arbitrage to survive in the extended market.

These conditions are easily determined in the special case of zero bid-ask spreads on all securities. This is due to a basic property of frictionless event-tree securities markets: no-arbitrage holds on the entire tree if and only if it holds on all its one-period sub-trees (see e.g.

¹⁹Recall in fact from Theorem 1 that the set of convex cones in which we partition the feasible trading strategies has cardinality $2^{J_{s_{T-1}}}$, where s_{T-1} is the number of states at time $T - 1$.

²⁰In particular, while it is still possible to show that the value function of the dual of $\mathcal{LP}[m]$ is not greater than the value function of $\mathcal{LP}[m]$, the reverse may fail to be true (see e.g. Anderson and Nash, 1987).

Dalang et al., 1990). Therefore, in the frictionless case, the extended market is arbitrage-free if and only if in all one-period sub-trees the beginning-of-period price of the new security is bounded above by the minimum cost to super-replicate its end-of-period price, and below by the maximum that can be borrowed against a liability not exceeding its end-of-period price.²¹

With bid-ask spreads, these conditions translate into requiring that in all one-period sub-trees (a) the *beginning-of-period bid price* of the new security be bounded above by the minimum cost to super-replicate the *end-of-period ask price*, and (b) the *beginning-of-period ask price* be bounded below by the maximum that can be borrowed against a liability not exceeding the *end-of-period bid price*.²² Unfortunately these conditions, although necessary, fail to be sufficient for no-arbitrage to hold in the extended market. The reason for this lies in a fundamental discontinuity introduced by the presence of bid-ask spreads, namely, that arbitrage opportunities may very well be banned from all one-period sub-trees, and still survive in the form of multi-period arbitrage strategies (see e.g. Pham and Touzi, 1999).

In the presence of bid-ask spreads, therefore, if we want to obtain bounds on the entire sequence of ask and bid prices of the new security based on one-period super-replication problems of the original market, these problems need to be parametrized in a way that makes them intertemporally connected in a much more stringent way than in the frictionless case. We believe that this subject constitute another interesting topic for future research.

Appendix

1. Proofs of Section 3

Proof of Theorem 1

Step 1. Construction of c and of the matrices G_k .

Recall that we denote with f_h^t (for $t \geq 0$ and $h = 1, \dots, s_t$) the generic node of the event tree \mathbb{P} and with $\theta = (\theta^A(0), \theta^A(f_1^1), \dots, \theta^A(f_{s_{T-1}}^{T-1}), \theta^B(0), \theta^B(f_1^1), \dots, \theta^B(f_{s_{T-1}}^{T-1}))^T$ a generic trading strategy in $\Theta \in \mathfrak{R}_+^{2(L-s_T)J}$. Let $c = (S^A(0), 0, \dots, 0, -S^B(0), 0, \dots, 0) \in \mathfrak{R}^{2J(L-s_T)}$ be the vector such that the product $c\theta$ describes the cost of the strategy θ at time $t = 0$ as in (1). Let $k = 1, \dots, K \equiv 2^{J s_{T-1}}$ be the index associated to a generic sequence of signs of the cumulative positions on each of the J assets on f_h^{T-1} , for $h = 1, \dots, s_{T-1}$. The matrix G_k is given by $G_k \equiv (G_k^A, G_k^B)$, where both G_k^A and G_k^B have dimension $s_{T-1} \times J(L - s_T)$. For simplicity, suppose first that $J = 1$. The generic i -th row of

²¹On the one-period sub-trees over which the end-of-period price cannot be perfectly replicated by the existing securities, moreover, these upper and lower bounds must both be strict.

²²With the usual proviso that the bounds must be strict when strict super-replication is optimal.

the matrices G_k^A and G_k^B corresponds to a node f_h^{T-1} for some $h = 1, \dots, s_{T-1}$. For any $\tau < T - 1$, let p_τ denote the index of the node in $\{1, \dots, s_\tau\}$ belonging to the history of the node f_h^{T-1} . Since k is fixed, we have a sequence of signs corresponding to the cumulative positions on the asset at each final node. Suppose first that the cumulative position on the security conditional at the node f_h^{T-1} is positive. Hence the entries of the i^{th} row of G_k^A and G_k^B referring to f_h^{T-1} are all zero, except for those corresponding to the columns $1, 1 + p_1, 1 + s_1 + p_2, \dots, 1 + s_1 + \dots + s_{T-2} + h$ which are $+1$ for G_k^A and -1 for G_k^B . If the cumulative position at the node f_h^{T-1} is negative, set instead -1 at the intersection of the i^{th} row with the columns $1, 1 + p_1, 1 + s_1 + p_2, \dots, 1 + s_1 + \dots + h$ for G_k^A and $+1$ for G_k^B . In this way, for any strategy θ whose cumulative positions at time T correspond to the sequence of signs k the product of $G_k \theta$ is the absolute value of the cumulative position held on the stock at the liquidation date. In the case of $J > 1$ assets, the entries previously written are vectors of \mathfrak{R}^J .

Step 2. Construction of the matrices M_k .

Again, fix a generic sequence k of signs of cumulative final positions on each of the J assets. The matrix $M_k \equiv (M_k^A, M_k^B)$ is obtained by adjoining two $(L - 1) \times J(L - s_T)$ matrices M_k^A and M_k^B . Again, assume first that $J = 1$. The generic i^{th} row of M_k^A (and M_k^B), corresponds to a node f_h^t for some $t = 1, \dots, T$ and $h = 1, \dots, s_t$. We construct the i -th row in the following way. If $t < T$, the entries corresponding to columns $1, 1 + p_1, 1 + s_1 + p_2, \dots, 1 + s_1 + \dots + p_{t-1}$ are $d(f_h^t)$ for M_k^A and $-d(f_h^t)$ for M_k^B , respectively. The entry corresponding to the column $1 + s_1 + \dots + s_{t-1} + h$ is $-S^A(f_h^t)$ for M_k^A and $S^B(f_h^t)$ for M_k^B , respectively. All the remaining entries of the i^{th} row are 0. If $t = T$, we have to distinguish between the case of a cumulative long or short position on the asset. Recall that, since k is fixed, we have a sequence of signs corresponding to the cumulative positions on the asset at each final node. Therefore if at the node f_h^T the cumulative position on the asset is positive, the entries of columns $1, 1 + p_1, 1 + s_1 + p_2, \dots, 1 + s_1 + \dots + p_{T-1} + h$ are $S^B(f_h^T)$ for M_k^A , $-S^B(f_h^T)$ for M_k^B and 0 elsewhere. If the cumulative position at the node f_h^T is instead negative the entries of columns $1, 1 + p_1, 1 + s_1 + p_2, \dots, 1 + s_1 + \dots + p_{T-1} + h$ are $-S^A(f_h^T)$ for M_k^A , $S^A(f_h^T)$ for M_k^B and 0 elsewhere. In the case of $J > 1$ assets, the entries previously written are vectors of \mathfrak{R}^J (for example, $d(f_h^t)$ becomes $(d_1(f_h^t), \dots, d_J(f_h^t))$, and so on).

Step 3. We now show that the value functions of problems $\mathcal{P}[m]$ and $\mathcal{LP}[m]$ coincide for any $m \in \mathfrak{R}^{L-1}$, constructing a suitable correspondence between the feasible sets of the two problems. To see that $\pi(m)$ is greater or equal than the value function of $\mathcal{LP}[m]$, let $\bar{\theta}$ be feasible for problem $\mathcal{P}[m]$. Then, there exists $\bar{k} \in \{1, \dots, K\}$ such that $M_{\bar{k}} \bar{\theta}$ is the cashflow produced by $\bar{\theta}$ for $t > 0$ and $G_{\bar{k}} \bar{\theta}$ is the vector of cumulative positions on each stock held from $T - 1$ to T . Hence, $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_K)$ defined by $\hat{\theta}_k = 0$ for $k \neq \bar{k}$ and $\hat{\theta}_{\bar{k}} = \bar{\theta}$ is feasible for $\mathcal{LP}[m]$ and $-x_{\bar{\theta}}(0) = c \sum_{k=1}^K \hat{\theta}_k$. To prove the converse, let $(\bar{\theta}_1, \dots, \bar{\theta}_K)$ be feasible for problem $\mathcal{LP}[m]$ and let $\bar{\theta} = \sum_{k=1}^K \bar{\theta}_k \in \Theta$. Denoting with $x_\theta \in \mathfrak{R}^{L-1}$ the vector of future cashflow generated by a strategy θ , we have that $x_{\bar{\theta}} \geq \sum_{k=1}^K x_{\bar{\theta}_k} = \sum_{k=1}^K M_k \bar{\theta}_k \geq m$, so that $\bar{\theta}$ is feasible for problem $\mathcal{P}[m]$. Moreover, since $-x_{\bar{\theta}}(0) = c \bar{\theta} = c \sum_{k=1}^K \bar{\theta}_k$, the value function of $\mathcal{LP}[m]$ is greater or equal than $\pi(m)$. This concludes the proof of Part 1 of Theorem 1. Parts 2 and 3 follows immediately from Part 1. ■

Throughout the rest of the Appendix we use the following notation: $\tilde{c} \equiv (c, \dots, c) \in \mathfrak{R}^{2KJ(L-s_T)}$, $\sigma \equiv (\theta_1, \dots, \theta_K)^T \in \mathfrak{R}_+^{2KJ(L-s_T)}$,

$$\widetilde{M} \equiv \begin{pmatrix} M_1 & \dots & M_K \end{pmatrix}, \quad G \equiv \begin{pmatrix} G_1 & 0 & \dots & 0 \\ 0 & G_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & G_K \end{pmatrix} \quad \text{and} \quad M \equiv \begin{pmatrix} -\widetilde{c} \\ \widetilde{M} \end{pmatrix}$$

so that \widetilde{M} is a $(L-1) \times 2KJ(L-s_T)$ matrix and G is $KJ_{s_{T-1}} \times 2JK(L-s_T)$. The following fact follows easily applying this notation to Definition 1:

Fact: *A price-dividend system (S^A, S^B, d) admits arbitrage opportunities if and only if the following system is feasible*

$$\begin{cases} M\sigma > 0 \\ G\sigma \geq 0 \\ \sigma \geq 0 \end{cases} \quad (5)$$

Proof of Proposition 1. We prove the characterization for Ψ , from which the characterization of Φ follows immediately. To this end, the positivity of the USPs allows us to identify them as vectors $(1, \psi) \in \mathfrak{R}_{++}^L$ such that $(1, \psi)x \leq 0$ for all $x \leq M\sigma$ for some $\sigma \in \mathfrak{R}_+^{2KJ(L-s_T)}$. Suppose first that $\psi \in R_{++}^{L-1}$ and $\beta \in R_+^{Js_{T-1}}$ satisfy $\psi M_k + \beta G_k \leq c$ for $k = 1, \dots, K$, or equivalently $\psi \widetilde{M} + \gamma G \leq \widetilde{c}$, with $\gamma = (\beta, \beta, \dots, \beta) \in \mathfrak{R}_+^{KJs_{T-1}}$. We show then that $(1, \psi)$ is a USP. Indeed, for any $\sigma \in \mathfrak{R}_+^{2KJ(L-s_T)}$, and for any $x \leq M\sigma$, since $\gamma \geq 0$ and $G\sigma \geq 0$, we have

$$(1, \psi)x \leq (1, \psi)M\sigma = -\widetilde{c}\sigma + \psi \widetilde{M}\sigma \leq -\widetilde{c}\sigma + \psi \widetilde{M}\sigma + \gamma G\sigma \leq 0$$

where the last inequality comes from the fact that, since $\sigma \geq 0$, it preserves the inequality $(\psi \widetilde{M} + \gamma G) \leq \widetilde{c}$.

To prove the converse, let $(1, \psi)$ be a USP, and let $b \equiv -(1, \psi)M$. For any $\sigma \in \mathfrak{R}_+^{2KJ(L-s_T)}$ we have $b\sigma = -(1, \psi)(M\sigma) \geq 0$ (since for $x = M\sigma$ the definition of USP implies $(1, \psi)x \leq 0$). By Theorem 2.8 in Gale (1960), there exists then $\gamma \geq 0$ such that $\gamma G \leq b$, and since $M \equiv \begin{pmatrix} -\widetilde{c} \\ \widetilde{M} \end{pmatrix}$ then $\gamma G \leq -(1, \psi)M = \widetilde{c} - \psi \widetilde{M}$, which concludes the proof. ■

Proof of Theorem 3 Observe first that the dual of $\mathcal{LP}[m]$ can be written in compact matrix notation as follows:

$$\begin{aligned} \pi(m) &= \max_{(\varphi, \gamma) \geq 0} \varphi m \\ &\text{s.t. } \varphi \widetilde{M} + \gamma G \leq \widetilde{c} \end{aligned} \quad (\mathcal{LP}'[m])$$

Hence the supremum is taken over the set $\Phi = \{(1, \varphi) \in \mathfrak{R}_+^L \mid \varphi M_k + \beta_k G_k \leq c, \beta_k \geq 0, k \leq K\}$, i.e. the semi-positive USPs. Assuming then no-arbitrage, the existence of a USP can be established by adapting the construction of a USP in the proof of Theorem 2 in Ortu (2001). Conversely, assume that there exists a USP vector. Observe that, since the Internality Condition implies $\widetilde{M}\widetilde{\theta} \gg 0$ for some $\widetilde{\theta} = (0, \dots, \theta_k, 0, \dots, 0)^T \in \mathfrak{R}^{2JK(L-s_T)}$ (with $M_k\theta_k \gg 0$ for some k), the set Φ is bounded (to see this, notice that $0 \leq \varphi(\widetilde{M}\widetilde{\theta}) \leq \varphi(\widetilde{M}\widetilde{\theta}) + (\gamma G)\widetilde{\theta} \leq \widetilde{c}\widetilde{\theta}$, for every $(1, \varphi) \in \Phi$). Absence of arbitrage follows then by adapting the corresponding part of the proof of Theorem 2 in Ortu (2001). ■

Proof of Theorem 4 Under the Internality Condition and no-arbitrage, if $m = 0$, by Theorem 2 we have $x_{\theta^*} = 0$ for every θ^* solution of $\mathcal{P}[0]$, i.e. $\Theta_{>0}^* = \emptyset$. If $m \neq 0$ and $\pi(m) = \max_{(1, \psi) \in \Psi} \psi m$,

suppose that $\Theta_{>m}^* \neq \emptyset$ so that, by Theorem 1, $\Xi_{>m}^* \neq \emptyset$, i.e there exists σ_m^* such that $\widetilde{M}\sigma_m^* > m$. By the complementary slackness conditions, $(\widetilde{M}\sigma_m^* - m)\varphi^* = 0$, and hence for any optimal φ^* we have that $\varphi^* \notin \mathfrak{R}_{++}^{L-1}$, which contradicts our assumption.

Conversely, assume $\Theta_{>m}^* = \emptyset$. To prove that $\pi(m) = \max_{(1,\psi) \in \Psi} \psi m$ we construct explicitly in the following four steps the (strictly positive) USP vector that attains the maximum.

Step 1. Let $K > \pi(m)$, $\widehat{M} \equiv [\widetilde{M}, m, -m]$, $\widehat{G} \equiv [G, 0, 0]$, $\widehat{\sigma} \equiv [\sigma, y_1, y_2]^T$, $\widehat{c} \equiv [\widetilde{c}, K, -\pi(m)]$ and consider the program

$$\widehat{\pi}(n) = \inf \widehat{c}\widehat{\sigma} \quad \begin{cases} \widehat{M}\widehat{\sigma} \geq n \\ \widehat{G}\widehat{\sigma} \geq 0 \\ \widehat{\sigma} \geq 0 \end{cases} \quad (\widehat{LP}(n))$$

with dual

$$\widehat{\pi}(n) = \max \varphi n \quad \begin{cases} \varphi\widehat{M} + \gamma\widehat{G} \leq \widehat{c} \\ \varphi \geq 0 \\ \gamma \geq 0 \end{cases} \quad (\widehat{LP}'(n))$$

Observe that the feasible set of $\widehat{LP}'(n)$ is the set of vectors $\varphi, \gamma \geq 0$ such that $\varphi\widetilde{M} + \gamma G \leq \widetilde{c}$, $\varphi m + \gamma 0 \leq K$ and $-\varphi m + \gamma 0 \leq -\pi(m)$. Hence, we can define

$$\widehat{\Phi} \equiv \left\{ (1, \varphi) \in \mathfrak{R}_+^L \mid \varphi\widetilde{M} + \gamma G \leq \widetilde{c}, \pi(m) \leq \varphi m \leq K \right\}$$

Notice that $\widehat{\Phi} \subseteq \Phi$ and is independent from n . We use this fact in Step 3.

Step 2. Here we show that $\widehat{LP}(0)$ admits optimal solutions $\widehat{\sigma}$, all of which satisfy $\widehat{c}\widehat{\sigma} = 0$ and $\widehat{M}\widehat{\sigma} = 0$. Indeed, we have

$$\begin{aligned} \inf_{\widehat{\sigma}} \widehat{c}\widehat{\sigma} &= \inf_{(y_1, y_2) \geq 0} \left[\left(\inf_{\sigma} \left\{ \widetilde{c}\sigma \mid \widetilde{M}\sigma \geq (y_2 - y_1)m, G\sigma \geq 0 \right\} \right) + y_1 K - \pi(m)y_2 \right] \\ &= \min \left[\inf_{y_2 \geq y_1 \geq 0} (K - \pi(m))y_1, \inf_{y_1 \geq y_2 \geq 0} (y_1(K + \pi(-m)) - y_2(\pi(m) + \pi(-m))) \right] \\ &= 0 \end{aligned}$$

since both the infima are obtained for $y_1 = 0$. This implies that $\widehat{\pi}(0) = 0$ and any optimal solution $\widehat{\sigma} = [\sigma, y_1, y_2]^T$ is such that $y_1 = 0$. Now, if $y_2 = 0$, we have

$$0 = \widehat{\pi}(0) = \widehat{c}\widehat{\sigma} = \inf_{\sigma} \left\{ \widetilde{c}\sigma \mid \widetilde{M}\sigma \geq 0, G\sigma \geq 0 \right\}$$

implying that σ is optimal for $\mathcal{LP}[0]$ and, by Theorem 2 (part 4), that $\widetilde{c}\sigma = 0$ and $0 = \widetilde{M}\sigma = \widehat{M}\widehat{\sigma}$. If instead $y_2 > 0$, then we have $0 = \widehat{\pi}(0) = \inf_{\sigma} \left\{ \widetilde{c}\sigma - \pi(m)y_2 \mid \widetilde{M}\sigma \geq y_2 m, G\sigma \geq 0 \right\}$. Hence,

$$\pi(m) = \inf_{\sigma} \left\{ \widetilde{c} \cdot \frac{\sigma}{y_2} \mid \widetilde{M} \frac{\sigma}{y_2} \geq m, G \frac{\sigma}{y_2} \geq 0 \right\}$$

that is, $\frac{\sigma}{y_2}$ is optimal for $\mathcal{LP}[m]$. Since by assumption $\Xi_{>m}^* = \emptyset$, the minimum super-replication cost is achieved by perfect replication of m . Hence $\widetilde{M} \frac{\sigma}{y_2} = m$, which implies $\widehat{M}\widehat{\sigma} = \widetilde{M}\sigma + 0 - my_2 = 0$, concluding the proof of Step 2.

Step 3. Here we show that there exists an optimal solution to $\widehat{LP}(n)$ for every n and $\widehat{\pi}(n) > 0$ whenever $n > 0$. Since by Step 2 $\widehat{LP}(0)$ admits optimal solutions, the set $\widehat{\Phi}$, that is independent from n , is nonempty. Since by Step 1 $\widehat{\Phi}$ is compact, and convex, it follows that problem $\widehat{LP}(n)$ (or equivalently $\widehat{LP}(n)$) admits solution for every n . To prove the positivity of $\widehat{\pi}(n)$, suppose by contradiction that $\widehat{\pi}(n^*) \leq 0$ for some $n^* > 0$. Denoting by $\widehat{\sigma}^*$ any optimal solution to $\widehat{LP}(n^*)$, since $\widehat{M}\widehat{\sigma}^* \geq n^* > 0$, $\widehat{G}\widehat{\sigma}^* \geq 0$ and $0 \geq \widehat{\pi}(n^*) = \widehat{c}\widehat{\sigma}^*$ then $\widehat{\sigma}^*$ is also an optimal solution to $\widehat{LP}(0)$. By Step 2 it follows that $\widehat{c}\widehat{\sigma}^* = 0$ and $\widehat{M}\widehat{\sigma}^* = 0$, which contradicts $\widehat{M}\widehat{\sigma}^* > 0$.

Step 4. Finally, we show that there exists a strictly positive USP vector $(1, \widehat{\varphi}) \in \widehat{\Phi} \subseteq \Phi$ such that $\pi(m) = \max_{(1, \psi) \in \Psi} \psi m = \widehat{\varphi} m$. Let $n = I(f_h^t)$, for $t > 0$ and $h = 1, \dots, s_t$.²³ From Step 3, we know that $\widehat{\pi}(n) > 0$ and that problem $\widehat{LP}(n)$ admits an optimal solution $(1, \varphi_h^t) \in \widehat{\Phi}$ such that $\varphi_h^t I(f_h^t) = \widehat{\pi}(I(f_h^t)) > 0$. Then $\varphi = \frac{1}{L-1} \sum_{t>0} \sum_{h=1}^{s_t} \varphi_h^t \gg 0$ and $(1, \varphi) \in \widehat{\Phi}$ (since $\widehat{\Phi}$ is independent from n and convex). Hence $\pi(m) = \sup_{(1, \psi) \in \Psi} \psi m \geq \varphi m$ and recalling the structure of the set $\widehat{\Phi}$, we finally have $\varphi m \geq \pi(m)$, which concludes the proof. ■

2. Proofs of Section 4

We first write down the auxiliary linear program $\mathcal{LP}^{ex}[m]$ for the extended market. To this end we let G^{ex} be the $(KJ s_{T-1} + 2) \times (2KJ(L - s_T) + 4)$ matrix defined as follows:

$$G^{ex} = \begin{bmatrix} G & 0 \\ 0 & \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{bmatrix}$$

Moreover, we let $c^{ex} \equiv [\widetilde{c}, c^A, -c^B, c^A, -c^B]$, $\widetilde{M}^{ex} \equiv [\widetilde{M}, n^B, -n^B, n^A, -n^A]$ and $M^{ex} \equiv \begin{bmatrix} -c^{ex} \\ \widetilde{M}^{ex} \end{bmatrix}$. Denoting then by $\sigma^{ex} = (\sigma, \zeta_1^A, \zeta_1^B, \zeta_2^A, \zeta_2^B)^T$ the vectors in $\mathfrak{R}^{2KJ(L-s_T)+4}$, the linear program $\mathcal{LP}^{ex}[m]$ is

$$\pi^{ex}(m) = \inf_{\sigma^{ex}} c^{ex} \sigma^{ex} \quad \text{s.t.} \quad \begin{cases} \widetilde{M}^{ex} \sigma^{ex} \geq m \\ G^{ex} \sigma^{ex} \geq 0 \\ \sigma^{ex} \geq 0 \end{cases}$$

Applying Proposition 1 to the extended market, the set Ψ^{ex} of the USP vectors for the extended market can be written as

$$\Psi^{ex} = \left\{ (1, \psi^{ex}) \in R_{++}^L \mid \psi^{ex} \widetilde{M}^{ex} + \gamma^{ex} G^{ex} \leq c^{ex}, \gamma^{ex} = (\gamma, \gamma_1, \gamma_2) \in \mathfrak{R}_+^{KJ s_{T-1}} \times \mathfrak{R}_+^2 \right\}$$

or equivalently as

$$\Psi^{ex} = \left\{ (1, \psi^{ex}) \in \Psi \mid c^B \leq \psi^{ex} n^B + \gamma_1 \leq c^A, c^B \leq \psi^{ex} n^A - \gamma_2 \leq c^A \text{ for } \gamma_1, \gamma_2 \geq 0 \right\} \quad (6)$$

Clearly, no-arbitrage in the extended market is equivalent to $\Psi^{ex} \neq \emptyset$.

²³ $I(f_h^t) \in \mathfrak{R}^L$ is a vector of zeros except for the $(1 + s_1 + \dots + s_{t-1} + h)^{th}$ entry, which is equal to 1.

Proof of Theorem 5 Since from (6) we have that $\psi^{ex}n^B \leq \psi^{ex}n^B + \gamma_1 \leq c^A$ and $\psi^{ex}n^A \geq \psi^{ex}n^A - \gamma_2 \geq c^B$ for any $(1, \psi^{ex}) \in \Psi^{ex}$, no-arbitrage in the extended market implies the existence of a USP $(1, \psi) \in \Psi^{ex} \subseteq \Psi$ such that $\psi n^B \leq c^A, \psi n^A \geq c^B$. The rest of the proof is in two steps. In Step 1 we show that $\psi n^B \leq c^A, \psi n^A \geq c^B$ for some $(1, \psi) \in \Psi$ implies $c^B \leq \pi(n^A)$ (with strict inequality if $\Theta_{>n^A}^* \neq \emptyset$) and $c^A \geq -\pi(-n^B)$ (with strict inequality if $\Theta_{>-n^B}^* \neq \emptyset$). In Step 2 we show that this last fact implies no-arbitrage in the extended market.

Step 1. Recall that, by Theorem 3, we have $\pi(n^A) = \sup_{(1, \psi) \in \Psi} \psi n^A$. Hence, the inequality $c^B \leq \psi n^A$ for some USP implies $\pi(n^A) \geq c^B$. Similarly, we have $\pi(-n^B) = \sup_{(1, \psi) \in \Psi} \psi(-n^B)$. Since $\psi(-n^B) \geq -c^A$ for some USP, we have that $\pi(-n^B) \geq -c^A$. For the strict inequalities, suppose that $\pi(n^A) = c^B$. Since $\pi(n^A) = c^B \leq \psi n^A$ for some $(1, \psi) \in \Psi$, this can be true if and only if $\pi(n^A) = \psi n^A$ i.e. $\pi(n^A) = \max_{(1, \psi) \in \Psi} \psi n^A$. By Theorem 4 this is true if and only if $\Theta_{>n^A}^* = \emptyset$. A similar argument for $\pi(-n^B)$ establishes $c^A > -\pi(-n^B)$ if $\Theta_{>-n^B}^* \neq \emptyset$.

Step 2. To show that the extended market is arbitrage-free if $c^B \leq \pi(n^A)$ (with strict inequality if $\Theta_{>n^A}^* \neq \emptyset$) and $c^A \geq -\pi(-n^B)$ (with strict inequality if $\Theta_{>-n^B}^* \neq \emptyset$), we show that $\Psi^{ex} \neq \emptyset$. Since under our assumptions $-\pi(-n^B) = -\sup_{(1, \psi) \in \Psi} \psi(-n^B) = \inf_{(1, \psi) \in \Psi} \psi n^B \leq c^A$, there exist $\psi_1 \in \Psi$ and $\gamma_1 \geq 0$ such that

$$c^B \leq \psi_1 n^B + \gamma_1 \leq c^A \quad (7)$$

By the same token, since $\pi(n^A) = \sup_{(1, \psi) \in \Psi} \psi n^A \geq c^B$, there exist $\psi_2 \in \Psi$ and $\gamma_2 \geq 0$ such that

$$c^B \leq \psi_2 n^A - \gamma_2 \leq c^A \quad (8)$$

Multiplying (7) by $\alpha \in (0, 1)$, (8) by $(1 - \alpha) \in (0, 1)$ and summing term by term, we have

$$\begin{cases} \alpha(\psi_1 n^B + \gamma_1) + (1 - \alpha)(\psi_2 n^A - \gamma_2) \leq c^A \\ \alpha(\psi_1 n^B + \gamma_1) + (1 - \alpha)(\psi_2 n^A - \gamma_2) \geq c^B \end{cases}$$

Letting $\alpha \equiv \frac{\gamma_2}{\gamma_1 + \gamma_2}$ and $\psi \equiv \alpha\psi_1 + (1 - \alpha)\psi_2$, we find $\psi n^B \leq \alpha\psi_1 n^B + (1 - \alpha)\psi_2 n^A \leq c^A$ and $\psi n^A \geq \alpha\psi_1 n^B + (1 - \alpha)\psi_2 n^A \geq c^B$, which concludes our proof. ■

Proof of Proposition 2 We only prove Part 1, since Part 2 can be proved similarly to Part 1, and Part 3 follows from Part 1 and 2. Since long-effectiveness of the new security is clearly implied by $\pi^{ex}(n^B) = c^A$, we prove in Step 1 that long-effectiveness implies $\max[c^B, -\pi(-n^B)] \leq c^A \leq \pi(n^B)$, which in Step 2 is shown to imply $\pi^{ex}(n^B) = c^A$.

Step 1. Long-effectiveness of the new security implies that the feasible set Φ^{ex} of the dual of $\mathcal{LP}^{ex}[m]$, where

$$\Phi^{ex} = \left\{ (1, \varphi^{ex}) \in \mathfrak{R}_+^L \mid \varphi^{ex} \widetilde{M}^{ex} + \gamma^{ex} G^{ex} \leq c^{ex}, \gamma^{ex} = (\gamma^T, \gamma_1, \gamma_2) \in \mathfrak{R}_+^{KJ_{ST-1}} \times \mathfrak{R}_+^2 \right\}$$

is nonempty. Moreover, since Φ^{ex} is closed and contained in the compact set Φ , we can conclude that Φ^{ex} is compact and nonempty. This implies that the problem $\mathcal{LP}^{ex}[m]$ admits solutions for every m . This allows to construct a semi-positive USP vector for the extended market (following the

same construction described in the proof of Theorem 4). Hence, second type arbitrage opportunities are banned from the extended market and, by Corollary 2, $c^A \geq \max [c^B, -\pi(-n^B)]$. To prove that $c^A \leq \pi(n^B)$, notice that, since the new asset is long-effective, the optimal strategy²⁴ σ^{ex} for the cash flow m is represented by the vector $\sigma^{ex} = [\sigma, \zeta^A, 0, 0, 0]^T$. By the complementary slackness condition, therefore, $(\varphi^{ex} \widetilde{M}^{ex} + \gamma^{ex} G^{ex} - c^{ex}) \sigma^{ex} = 0$. Since this is a sum of nonpositive terms, it follows that $(\varphi^{ex} n^B + \gamma_1 - c^A) \zeta^A = 0$. Moreover, the complementary slackness conditions imply also that $\varphi^{ex} (\widetilde{M}^{ex} \sigma^{ex} - m) = 0$ and $(\gamma, \gamma_1, \gamma_2)(G^{ex} \sigma^{ex} - 0) = 0$. From the last one we have that $\zeta^A \gamma_1 = 0$, which implies $\gamma_1 = 0$. Since $\varphi^{ex} n^B = c^A - \gamma_1$, it follows that $\varphi^{ex} n^B = c^A$ so that $\pi(n^B) = \max_{(1, \varphi) \in \Phi} \varphi^T n^B \geq \varphi^{ex} n^B = c^A$.

Step 2: We now show that $\max [c^B, -\pi(-n^B)] \leq c^A \leq \pi(n^B)$ implies $\pi^{ex}(n^B) = c^A$. To see this, notice that since $-\pi(-n^B) \leq c^A \leq \pi(n^B)$, by continuity there exists $(1, \bar{\varphi}) \in \Phi$ such that $\bar{\varphi} n^B = c^A$. Hence by taking $\gamma_1 = 0$ and $\gamma_2 = (\bar{\varphi} n^A - c^A)^+$, we have that $(1, \bar{\varphi}) \in \Phi^{ex}$ and $\pi^{ex}(n^B) = \max_{(1, \varphi) \in \Phi^{ex}} \varphi n^B = \bar{\varphi} n^B = c^A$. ■

Proof of Theorem 6 Immediate consequence of Theorem 5 and Proposition 2. ■

Example 1 Figures 1 illustrates a simple 1-period example where the introduction of an effective new security in a (originally) no-arbitrage market generates first-type-arbitrages. The initial prices of the new security violate the strict inequalities in Statement 2 of Theorem 6.

Consider a 1-period market with only two states f_1^1 and f_2^1 at the terminal date $T = 1$. The market is constituted by a single security S , whose ask and bid prices at time $t = 0$ are $S^A(0) = 10$ and $S^B(0) = 7$. At time $t = 1$ the ask-prices are $S^A(1)(f_1^1) = 12$, $S^A(1)(f_2^1) = 6$ and the bid-prices are $S^B(1)(f_1^1) = 11$, $S^B(1)(f_2^1) = 5$. The set of (strictly) positive USPs Ψ is constituted by the elements $(1, \psi)$ with $\psi = (\psi_1, \psi_2) \in \mathfrak{R}_+^2$ such that $\psi_2 \leq -\frac{S^B(1)(f_1^1)}{S^B(1)(f_2^1)} \psi_1 + \frac{S^A}{S^B(f_2^1)}$ and $\psi_2 \geq -\frac{S^A(1)(f_1^1)}{S^A(1)(f_2^1)} \psi_1 + \frac{S^B}{S^A(f_2^1)}$. Hence the boundary of the set $\{\psi \in \mathfrak{R}_+^2 : (1, \psi) \in \Psi\}$ has four vertices: $(\frac{7}{12}, 0)$, $(\frac{10}{11}, 0)$, $(0, \frac{7}{6})$ and $(0, 2)$. In particular, since the set of (strictly) positive USPs Ψ is non-empty, the market is arbitrage-free (see Theorem 3).

Assume that a new security is introduced in the market at an initial price $c^B = c^A = 20$ and with bid-ask spreads at the terminal date: $n^A(f_2^1) = n^B(f_2^1) = 10$ and $21 = n^A(f_1^1) > n^B(f_1^1) = 20$. The set Ψ^{ex} is represented by the intersection of $\{\psi : (1, \psi) \in \Psi\}$ (lightly shaded in Figure 1) with the triangle bounded by the lines $\psi_2 = -\frac{n^B(1)(f_1^1)}{n^B(1)(f_2^1)} \psi_1 + \frac{c^A}{n^B(f_2^1)}$ and $\psi_2 = -\frac{n^A(1)(f_1^1)}{n^A(1)(f_2^1)} \psi_1 + \frac{c^B}{n^A(f_2^1)}$ (dashed in Figure 1). The intersection of the two sets is identified by $\Phi^{ex} = \{(1, \varphi^{ex})\}$, with $\varphi^{ex} = (0, \frac{c^B}{n^A(f_2^1)}) = (0, \frac{c^A}{n^B(f_2^1)}) = (0, \frac{S^A}{S^B(f_2^1)}) = (0, 2)$. By Theorem 3, the extended market is free of second-type-arbitrage, since Φ^{ex} is non-empty. However, first-type-arbitrages are possible in the extended market, since the set of *strictly positive* USPs for the extended market Ψ^{ex} is empty.

The new security is effective, since $\pi^{ex}(n^B) = \varphi^{ex} \cdot n^B = c^A$ and $-\pi^{ex}(-n^A) = -\varphi^{ex} \cdot (-n^A) = c^B$.

As guaranteed by Theorem 4, the set $\Theta_{>n^A}^*$ is non-empty since $\pi(n^A)$ is not attained by any

²⁴Due to the *static* nature of the buy-and-hold strategies allowed on the new asset, it is never optimal to take both a long and a short position on the new security. More precisely, if the strategy (ϑ, ζ) is optimal for $\mathcal{P}^{ex}[m]$, then $\zeta = (\zeta^A, \zeta^B)$ is such that $\zeta^A \zeta^B = 0$.

(strictly positive) USP. Indeed, notice that $\pi(n^A) = \sup_{(1,\psi) \in \Psi} \psi \cdot n^A = \varphi^{ex} \cdot n^A = c^B$. Moreover, the fact that $\pi(n^B) = \sup_{(1,\psi) \in \Psi} \psi \cdot n^B = \varphi^{ex} \cdot n^B = c^A$ implies that $\pi(n^A) = c^B = c^A = \pi(n^B)$ instead of $\pi(n^B) > c^B$, as required by Statement 2 of Theorem 6 when $\pi(n^A) = \pi(n^B)$. Exploiting this fact, it is easy to construct an arbitrage opportunity of the first type. Observe that n^A can be (strictly) super-replicated at the minimum cost by buying 2 units of S , since $2 \cdot S^B(1) > n^A$ and $2 \cdot S^A(0) = 20 = \pi(n^A)$. The strategy defined by short-selling 1 unit of the new security and buying 2 units of S is an arbitrage opportunity of the first type, because it has strictly positive terminal cash-flow with zero initial cost, since $c^B = \pi(n^B) = \pi(n^A)$.

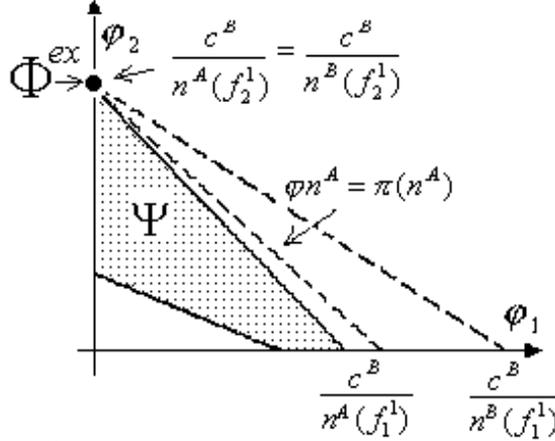


Figure 1

Example 2 Figure 2 illustrates an example of an effective security in which a long position is not an efficient trading strategy. The original 1-period market is constituted by the single security S described in Example 1. The market is arbitrage-free, since the set of (strictly positive) USPs is non-empty (compare Example 1). Then a new security is introduced in the market at the initial prices $c^B = 9 < c^A = 10$ and with bid-ask spreads at the terminal date: $n^A(f_1^1) = \frac{180}{19}$ and $n^B(f_1^1) = 2.5$ and $n^A(f_2^1) = 6$ and $n^B(f_2^1) = 5$. No-arbitrage holds in the extended market, since the set Ψ^{ex} is non-empty. In fact, the set $\{\psi : (1, \psi) \in \Psi^{ex}\}$, shaded in Figure 3, is the intersection of $\{\psi : (1, \psi) \in \Psi\}$ (lightly shaded in Figure 3) with $\{\psi \in \mathfrak{R}_+^2 : \psi_2 \leq -\frac{1}{2}\psi_1 + 2 \text{ and } \psi_2 \geq -\frac{30}{19}\psi_1 + \frac{3}{2}\}$ (bounded by the dashed lines in Figure 3). Therefore $\Psi^{ex} \neq \emptyset$ and no-arbitrage holds in the extended market by Theorem 3.

Moreover, by Proposition 2, the new security is effective, since $-\pi^{ex}(-n^A) = c^B$ and $\pi^{ex}(n^B) = c^A$. To see this, observe that $-\pi^{ex}(-n^A) = -\sup_{(1,\psi) \in \Psi^{ex}} \psi \cdot (-n^A) = \inf_{(1,\psi) \in \Psi^{ex}} \psi \cdot n^A$ is attained by every $(1, \psi) \in \Psi^{ex}$ such that $\psi_2 = -\frac{n^A(1)(f_1^1)}{n^A(1)(f_2^1)}\psi_1 + \frac{c^B}{n^A(f_2^1)}$ so that $-\pi^{ex}(-n^A) = c^B$. On the contrary, $\pi^{ex}(n^B) = \sup_{(1,\psi) \in \Psi^{ex}} \psi \cdot n^B$ is attained (only) by $\bar{\varphi}^{ex} = \left(0, \frac{c^A}{n^B(f_2^1)}\right)$ so that $\pi^{ex}(n^B) = c^A$ but n^B cannot be an efficient trading strategy.²⁵ Finally, we can verify that long/short positions

²⁵Theorem 1 in Jouini and Kallal (2001) implies that $\pi(n^B)$ must be attained by some $(1, \psi^{ex}) \in \Psi^{ex}$ in order n^B to be efficient.

in the new effective security have zero inefficiency cost, applying Corollary 1 in Jouini and Kallal (2001).²⁶

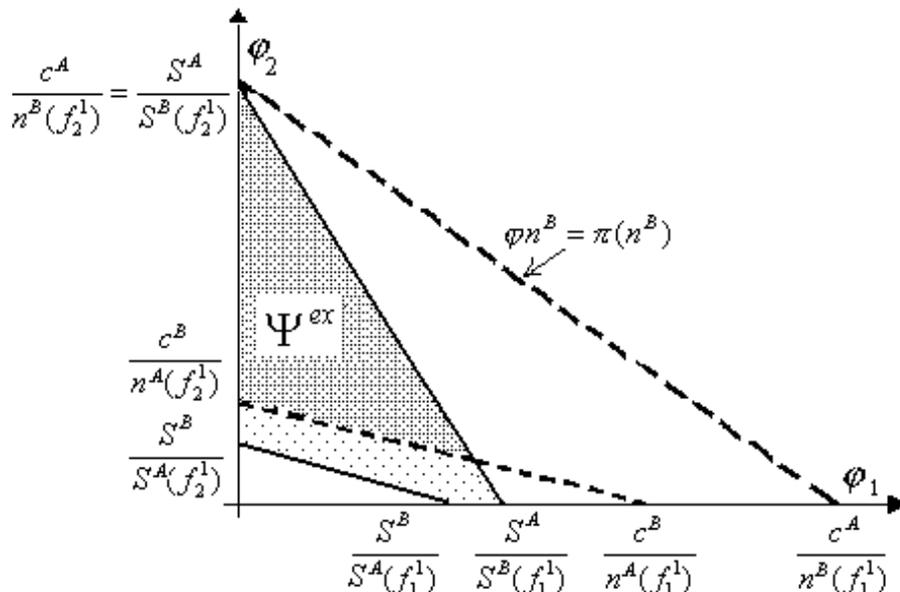


Figure 2

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²⁶The inefficiency cost of a long position in the new security is $c^A - \sup_U \left\{ \min_m [\pi^{ex}(m) : U(m+x) \geq U(n^B+x)] \right\} = c^A - \pi^{ex}(n^B) = 0$.

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