Conditional Retrospective Voting in Large Elections

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Abstract

A key feature of many elections is that voters learn the value of different alternatives (e.g., political parties) from their previously observed performance, but they can only observe the performance of an alternative in periods in which it is elected. When voters have private information, a better alternative is more likely to be elected. This non-randomized selection of alternatives implies that voters must learn from a biased sample. We study an environment with a continuum of voters in order to capture this feature in large elections. Our retrospective voting equilibrium formalizes the idea that voters’ beliefs can be systematically biased. This equilibrium notion provides a tractable framework to study several implications of retrospective voting regarding information aggregation, optimal electoral rules, the value of information, endogenous preferences, and party polarization. We also provide a game-theoretic foundation for our voting equilibrium by studying elections where the (finite) number of voters goes to infinity.

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1 Introduction

Large-scale elections constitute a popular mechanism for making economic and political decisions and have been studied from a wide range of perspectives both in economics and political science. There is one realistic feature of elections, however, that has received little attention in the literature. To illustrate this feature, consider an election between a Republican and a Democratic candidate in the United States. Voters will form beliefs about the competency of these candidates based on campaign platforms, debates, and other current information. In addition, voters are likely to use information about past performance of the parties to which these candidates belong. For example, voters who are employed in an industry with high labor turnover may favor a Democratic candidate during periods of high unemployment if they have experienced better results from previously elected Democratic administrations, compared to Republican administrations, in periods of high unemployment.¹

The tendency to learn from the past is not limited to political elections. When shareholders vote on takeover proposals, they should also learn from the outcome of previous takeovers in the same or comparable firms. A similar phenomenon occurs with legislators choosing whether to vote along party lines, union members voting to accept or reject negotiated contracts, and residents voting whether to approve additional funding for school districts.

In all of these examples, a key feature is that, while using past information to make inferences about the best alternative, voters only observe the performance of the alternatives that were elected, so that counterfactual outcomes are not observable. For example, we will never find out how McCain would have performed had he been elected President of the U.S. in 2008 instead of Obama. Similarly, shareholders will not learn the benefits of a takeover if they vote against it. Consequently, the sample from which voters learn is likely to be biased. The reason is that the selection of alternatives is not randomized: To the extent that voters have some private information, they will elect alternatives that are likely to perform better. It is reasonable to expect that voters will not be able to fully control for unobserved counterfactual outcomes (see the related literature for evidence) and will therefore end up with systematically

¹The Democratic and Republican parties have different positions on several economic and social issues that are stable over time (e.g., attitudes toward monetary and fiscal policy) while the underlying state of the world changes (e.g., recessions and booms). Bartels (2010) provides evidence that these different economic policies translate into different outcomes.
biased beliefs.

Our main objective is to understand the preference and information aggregation properties of elections when voters face the type of learning problem described above and have systematically biased beliefs. We provide a new framework for elections with a large number of players that captures the main features described above in a stylized model that is amenable to analysis. Of course, we abstract from many rich aspects of voting behavior, but we nevertheless obtain several insights that appear to be novel in the literature and contribute to our understanding of elections. Also, standard voting theory is ill-suited for our purposes because it presumes that voters have correct beliefs about the probability distribution of counterfactual outcomes. Therefore, our framework relies on a recent literature on bounded rationality.

In our setup, there is a continuum of voters and two alternatives. One of the alternatives wins the election if it receives a sufficiently high proportion of votes; otherwise the other alternative is elected. Voters have (possibly heterogeneous) payoffs that are increasing in the state of the world for one alternative and decreasing for the other. In addition, voters have some information about the state of the world. For example, consider an election between two political parties. The state of the world can represent whether the economy is overheated or in a recession, and one of the parties may be better at dealing with an overheated economy and the other better at dealing with a recession (perhaps because of their different attitudes toward monetary and fiscal policy).

We propose a new solution concept, retrospective voting equilibrium, to formalize the idea that voters form beliefs from past information in large elections. A (retrospective) voting equilibrium consists of a strategy profile and an election cutoff that satisfy two conditions. First, the election cutoff is the state at which the proportion of votes for each alternative yields a tie given the electoral rule: One alternative is elected for states higher than the election cutoff and the other is elected for lower states. Second, the strategy profile must be optimal given the election cutoff. Optimality here is defined in terms of retrospective voting: Voters’ perceptions of the benefits of each alternative derive from the observed performance of each alternative, which depends on the states in which each alternative is elected, and, therefore, on the election cutoff.

Our definition of a voting equilibrium has some parallels with the definition of a competitive equilibrium in a market economy. In our context, the role of prices is
played by the election cutoff. And the fact that the number of voters is sufficiently large means that each voter has a negligible effect on the election cutoff, which is therefore taken as given by the electorate but is endogenously determined by the strategies of all voters. The fact that we can characterize retrospective behavior in large elections in this parsimonious manner is a major advantage of our framework.

We also provide a game-theoretic foundation for our solution concept by showing that it corresponds to the behavioral equilibrium (Esponda, 2008) of a voting game when the number of players goes to infinity. As shown by Esponda and Pouzo (2012), the behavioral equilibrium of the voting game can be interpreted as the steady-state of a dynamic environment when voters learn the benefits of each alternative from past observed outcomes without controlling for the fact that counterfactuals are not observed.

We apply our framework to derive several new insights about large elections when the electorate learns from a potentially biased sample of observed performances. It is not surprising that, given that voters learn from a biased sample, full information aggregation may fail and the wrong alternative may be elected with positive probability. But understanding the tension underlying the lack of aggregation has important implications.

To understand why mistakes happen, suppose that party A is best if the underlying (unobservable) state of the economy is strong, while party B is best if the economy is weak. If equilibrium were efficient, so that party A were elected in a strong economy and party B in a weak economy, then voters would always observe party A performing better than party B (since it is easier to govern in a strong economy). Hence, all voters would prefer to vote for party A, thus contradicting the hypothesis that the right party is chosen in its corresponding state of the world. In equilibrium, party A will have to be occasionally elected into office in a weak economy; this mistake will then reduce party A’s popularity and provide incentives for voters to choose both parties in equilibrium. More generally, there are clear implications about the direction of these mistakes: The alternative that is more attractive, when judged in its best states of the world (party A in the example), will have to be occasionally elected in the wrong state of the world.

The quality of information and electoral rules play an important role in determining the amount of mistakes in equilibrium. However, a more informed electorate does not guarantee better electoral outcomes under retrospective voting. The reason
is that the source of systematically biased beliefs is not poor information but rather the impossibility of observing counterfactuals. The model can also explain why societies are conservative and require supermajorities to adopt risky alternatives. In the previous example, suppose that party B’s performance does not depend on the state of the world but that party A’s performance is risky. Since the risky alternative tends to be chosen when the economy is strong, then voters will overestimate its true performance. A supermajority rule is then needed to prevent the risky party A from being chosen too often. Thus, the biases that arise under retrospective voting provide a justification for the common use of supermajority rules for changes in the status quo (e.g., raising taxes and other legislative proposals, shareholder voting on takeovers, and constitutional changes).

Given that mistakes are often unavoidable, one concern is that elections may be undesirable if these mistakes are extremely costly. But it turns out that, in equilibrium, the probability of mistakes adjusts in order to mitigate the effect of more costly mistakes. In the previous example, suppose that party A’s performance is disastrous in a weak economy (say, because A’s typical policies are particularly damaging in that state). Then, very few mistakes are sufficient to decrease A’s popularity and induce voters to also vote for B. The fact that equilibrium retrospective voting inherently limits the expected damage that results from choosing the wrong party gives credence to the argument that elections perform relatively well despite the lack of sophistication of the electorate.

The model also warns us about drawing conclusions from the observed performance of alternatives. In particular, there is a strong tendency for observed equilibrium performance to be equalized among alternatives; otherwise, voters would deviate to the alternative that offers higher observed performance. A naïve outside observer would conclude that elections have no advantage over flipping a coin, without realizing that elections increase welfare by matching alternatives with the right state of the world. Moreover, the model explains how the alternative with the worst observed performance can actually be elected with greater probability by retrospective voters.

The fact that behavior depends endogenously on the voting behavior of other voters gives rise to composition effects that are relevant for both theoretical and empirical work. In particular, whether voters are partisans or not depends not only on their own preferences but also on the preferences of other voters. Thus, empirical work must control for aggregate effects when estimating voter preferences. One implication
is that voting in local and national elections will be different simply because the composition of the electorate differs in each type of election.

Finally, we embed the model into a larger model of political competition where parties choose their policy positions. We show that, due to retrospective voting, each party has an incentive to choose relatively extreme policies that work well in those states in which it is elected into office, since those are the states that retrospective voters use to evaluate their performance. This is true even if all voters would prefer a neutral policy that delivers a state-independent payoff. Thus, retrospective voting with private information provides an additional explanation for party polarization.

In Section 2, we introduce the framework and the solution concept. In Section 3, we discuss in more detail the implications of the framework outlined above and relate these implications to the empirical evidence. In Section 4, we provide a foundation for our solution concept. We conclude in Section 5 by mentioning some limitations of our work and possible extensions. In the remainder of this introduction, we discuss the related literature.

1.1 Related literature

This paper follows a recent literature that studies game-theoretic equilibrium concepts for boundedly rational players (e.g., Osborne and Rubinstein (1998), Jehiel (2005), Eyster and Rabin (2005), Jehiel and Samet (2007), Jehiel and Koessler (2008), and Esponda (2008))). Papers that study elections with non-Nash solution concepts include Osborne and Rubinstein (2003), Eyster and Rabin (2005), Costinot and Kartik (2007), and Martinelli (2011), although these papers capture aspects of bounded rationality that are different from the biased learning problem that motivates our work.\footnote{Callander (2009) studies a model of dynamic policy-making where voters learn the mapping between policies and outcomes.}

The idea that voters do not try to correct for unobserved counterfactuals is motivated by the empirical findings of Achen and Bartels (2004), Leigh (2009), and Wolfers (2009), who find that voters punish politicians for events that are outside of their control.\footnote{Healy and Malhotra (2010) find that punishment is related to the politician’s response to these events. Our model allows voters to condition their learning on private signals, such as campaign platforms, media reports, or economic indicators, thus allowing for a wide range of sophistication in the electorate.} This type of naivete also underlies the winner’s curse in common

2 Callander (2009) studies a model of dynamic policy-making where voters learn the mapping between policies and outcomes.

3 Healy and Malhotra (2010) find that punishment is related to the politician’s response to these events. Our model allows voters to condition their learning on private signals, such as campaign platforms, media reports, or economic indicators, thus allowing for a wide range of sophistication in the electorate.
value auctions and has received robust support in experimental settings (e.g., Thaler (1988), Kagel and Levin (2002), and Charness and Levin (2009)).\footnote{For voting experiments, Guarnaschelli et al. (2000) conclude that subjects’ votes do not deviate much from Nash equilibrium play, although Eyster and Rabin (2005) find that these deviations can be systematically attributed to naivete. Recently, Esponda and Vespa (2011) show that about half of their subjects are sophisticated. Unlike our setting, all of these experiments focus on small elections and, more importantly, tell subjects the primitives of the game.} The idea that voters base their decisions on past performance is motivated by the retrospective voting literature in political science (see below) and has also received direct empirical (Martorana and Mazza, 2012) and experimental support (Huber et al. (forthcoming), Woon (2012)).

Political scientists have long emphasized that voters are boundedly rational. A robust finding is that the electorate is poorly informed and has little understanding of ideology and policy (e.g., Delli Carpini and Keeter (1997) and Converse (2000)). The ideas and the label “retrospective” that we use to refer to our solution concept are also closely related to the notion of retrospective voting advanced by Key (1966). Under retrospective voting, voters reward or punish politicians and their parties based on their past performance. In the words of Fiorina (1981, p. 5), voters “need not know the precise economic or foreign policies of the incumbent administration in order to see or feel the results of those policies.” Retrospective voting has received empirical support and is sometimes credited for the satisfactory performance of elections despite an unsophisticated electorate (e.g., Kramer (1971), Fiorina (1978), Lewis-Beck and Stegmaier (2000)).

Our work, however, is conceptually very different to the large formal literature in retrospective voting, beginning with Barro (1973) and Ferejohn (1986), which studies elections as incentive mechanisms that hold politicians accountable. In fact, our model follows Downs’ (1957) view of retrospective voting as a way to predict how parties will perform in the future rather than as a way to simply punish or reward the party for past performance (Fiorina (1981), Chapter 1).

Bendor et al. (2010, 2011) also postulate a boundedly rational model of retrospective voting motivated by Key’s (1966) ideas.\footnote{The original literature on the political business cycle also assumed boundedly rational voters (Nordhaus, 1975).} In their model, voters follow a satisficing decision rule whereas they vote for the incumbent if it has performed well given their aspiration level. Because aspirations are endogenous, they also obtain the interesting interdependent voting behavior that is present both in our model and
under Nash equilibrium. Unlike our paper, however, voters do not have private information and, therefore, there is no role for sample bias. Moreover, their model is a non-equilibrium model and is therefore better suited to study the actual dynamics of voting behavior.

We end this section by relating our paper to the large economics literature on formal models of elections that portrays voters as sophisticated individuals who have well-defined preferences, can solve complicated signal-extraction problems, and have correct expectations about the distribution of (counterfactual) payoffs. One strand of the literature focuses on preference aggregation. An important insight is that, when voters have well-defined preferences and the capacity to understand policy positions, two-party competition provides little choice, in the sense that parties choose similar policies (Downs, 1957). In contrast, we show that there is a tendency to polarization when voters are uncertain about the quality of candidates and learn from past outcomes.

Another strand of the literature, dating back to Condorcet, points out that voters are often imperfectly informed about the best alternative, and that a major role of elections is to aggregate the information dispersed in the electorate. Several interesting insights have been obtained by postulating one of two behavioral assumptions. The first assumption is that voting is sincere, in the sense that each voter chooses the candidate that she considers to be best given her information alone (e.g., Ladha, 1992). This notion of sincere voting assumes that voters know the primitives of the environment and, therefore, implicitly assumes that voters have been able to observe the state of the world, thus learning how the unelected alternative would have performed. This notion has also been criticized for making the unrealistic prediction that voting behavior is independent of many features of the environment, such as the electoral rule or the voting behavior of other voters. Our notion of retrospective equilibrium captures the idea of sincere voting, which we believe to be natural in large elections, but in a context where voters do not know the primitives of the environment. In our context, beliefs and, therefore, voting behavior, are determined endogenously and depend on many features of the environment.

A second assumption in the information aggregation literature is that voters play a Nash equilibrium (Austen-Smith and Banks, 1996). A key insight of this literature is that a voter must utilize both her information and the information that she can infer from the hypothetical event that her vote is pivotal. An appealing property of Nash
equilibrium, also present in our model, is that voters react to their environment and their actions are determined endogenously. The main result for large elections, obtained by Feddersen and Pesendorfer (1997) under certain assumptions on the voting environment that we also make in this paper, is that the Nash equilibrium outcome is close to the outcome of an economy where everyone is perfectly informed. This result highlights the information aggregation benefits of elections when the electorate is sophisticated and also acts as if knowing the primitives and the equilibrium strategies of other players. However, this result implies that, with a large electorate, certain features of the environment, such as the precision of information or the exact (non-unanimous) electoral rule, become irrelevant. In our case with boundedly rational voters, these features are still relevant with large electorates. More generally, we depart from the previous literatures by assuming that voters do not a priori know the primitives of the environment but rather naively form beliefs from past observations.

2 Voting framework

2.1 Setup

A continuum of voters participate in an election between two alternatives (e.g., parties) A and B. A state $\omega \in \Omega = [-1,1]$ is first drawn according to a probability distribution $G$ and, conditional on the state, each player observes an independently-drawn private signal. Players then simultaneously submit a vote for either A or B. Votes are aggregated according to an electoral rule $\rho \in (0,1)$: Alternative A wins the election if the proportion of votes in favor of A is greater or equal than $\rho$; otherwise, B wins the election.

We model heterogeneity (in preferences and information) by assuming that each voter is of a particular type $\theta$, where $\phi$ is the probability distribution over the set of types $\Theta \subset \mathbb{R}$. Conditional on a state $W = \omega$, players of type $\theta$ independently draw a signal $S_\theta = s$ from a finite, nonempty set $S_\theta \subset \mathbb{R}$ with probability $q_\theta(s | \omega)$; let $s^L_\theta$ and $s^H_\theta$ denote the lowest and highest signals in $S_\theta$. The payoff of type $\theta$ is given by $u_\theta(o, \omega)$, where $o \in \{A, B\}$ is the elected alternative.

Let $\sigma_\theta : S_\theta \rightarrow [0,1]$ denote the strategy of type $\theta$, where $\sigma_\theta(s)$ is the probability that type $\theta$ votes for alternative A after observing signal $s$. A strategy $\sigma_\theta$ is nondecreasing if $\sigma_\theta(s') \geq \sigma_\theta(s)$ for all $s' > s$. A strategy profile $\sigma : \Theta \rightarrow [0,1]^S$ is
nondecreasing if \( \sigma_\theta \) is nondecreasing for each \( \theta \).

We maintain the following assumptions throughout the paper, for all \( \theta \in \Theta \):

**A1.** (i) \( u_\theta(A, \cdot) : \Omega \to \mathbb{R} \) is nondecreasing and \( u_\theta(B, \cdot) : \Omega \to \mathbb{R} \) is nonincreasing, and one of them is strictly monotone; (ii) \( u_\theta(A, \cdot) \) and \( u_\theta(B, \cdot) \) are both continuously differentiable, except possibly in a finite number of points, and \( \sup_{\theta,o,\omega} |u_\theta(o,\omega)| < K \).

**A2.** MLRP: For all \( \omega' > \omega \), and \( s' > s \):

\[
\frac{q_\theta(s'|\omega')} {q_\theta(s'|\omega)} - \frac{q_\theta(s|\omega')}{q_\theta(s|\omega)} > 0.
\]

**A3.** (i) \( G \) has a density function \( g \), where \( \inf_\Omega g(\omega) > 0 \); (ii) there exists \( d > 0 \) such that \( q_\theta(s|\omega) > d \) for all \( \theta \in \Theta \), \( s \in S \) and \( \omega \in \Omega \); (iii) \( q_\theta(s \mid \cdot) \) is continuous for all \( s \in S \).

**A4.** \( \Theta \subset \mathbb{R} \) is a compact interval (a singleton is a special case) and \( S_\theta = S \); \( u_\theta(A, \omega) \), \( u_\theta(B, \omega) \), and \( q_\theta(s \mid \omega) \) are jointly continuous in \( \Theta \times \Omega \) for all \( s \in S \).

Assumptions A1-A2 provide an ordering between states, information, and players’ preferences. Note that A2 is trivially satisfied for types with a unique signal (i.e., no private information). Assumption A3 rules out “strong signals” in the sense of (Milgrom, 1979). Assumption A4 guarantees uniqueness of the equilibrium outcome and is made only for convenience.\(^6\) Thus, the voting environment essentially coincides with the standard setup in Feddersen and Pesendorfer (1997)).\(^7\) As discussed in the introduction, two solution concepts, sincere voting and Nash equilibrium voting, have been applied to this setup. We now define a third solution concept that captures retrospective voting in large elections when voters have private information.

## 2.2 Retrospective voting equilibrium

Let

\[
\kappa(\omega; \sigma) = \int_\Theta \sum_{s \in S} q_\theta(s \mid \omega) \sigma_\theta(s) \phi(d\theta)
\]

denote the proportion of votes in favor of \( A \). Assumption A2 implies that \( \kappa(\cdot; \sigma) \) is nondecreasing if \( \sigma \) is nondecreasing. In the case where the strategy depends on private

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\(^6\)The results go through without assumption A4, but the existence of multiple equilibrium outcomes makes the statements more cumbersome.

\(^7\)A small difference is that we require \( u_\theta(A, \cdot) \) and \( u_\theta(B, \cdot) \) to be separately monotone, rather than only their difference to be increasing.
information, so that \( \sigma \) is not flat, then \( \kappa(\cdot;\sigma) \) is increasing and the outcome of the election can be characterized by a cutoff: \( A \) is elected if and only if \( \kappa(\omega;\sigma) \geq \rho \), or, equivalently, for all sufficiently high states. This observation motivates the following definition.\(^8\)

**Definition 1.** A state \( c \in \Omega \) is an election cutoff given a strategy profile \( \sigma \) if \( \kappa(\omega;\sigma) \geq \rho \) for all \( \omega > c \) and \( \kappa(\omega;\sigma) \leq \rho \) for all \( \omega < c \).

When making her decision, each voter takes the cutoff as given. A cutoff determines the set of states for which each alternative is chosen, and, consequently, each voter’s evaluation of the benefits of electing each alternative. For a given cutoff \( c \in \Omega \), the difference in benefits from electing \( A \) over \( B \) that is perceived by a voter of type \( \theta \) who observes signal \( s \) is

\[
v_\theta(s;c) \equiv E(u_\theta(A,W) \mid W \geq c, S_\theta = s) - E(u_\theta(B,W) \mid W < c, S_\theta = s). \tag{1}\]

To interpret the above expression, note that alternative \( A \) is elected whenever \( W \geq c \), so that a voter’s retrospective evaluation of \( A \) is given by the first term in the right hand side of (1). A similar interpretation holds for the second term.

The following definition captures the idea that each voter votes for the alternative that she sincerely believes to have the highest perceived benefit.

**Definition 2.** A strategy profile \( \sigma \) is optimal given an election cutoff \( c \) if

\[
\sigma_\theta(s) \in \arg \max_{a \in [0,1]} a \cdot v_\theta(s;c)
\]

for all \( \theta \in \Theta \) and \( s \in S \).

By assumptions A1-A2, \( v_\theta(\cdot;c) \) is increasing. Therefore, any strategy that is optimal given some cutoff must be nondecreasing.

\(^8\)When \( \sigma \), and, therefore, \( \kappa(\cdot;\sigma) \) are constant, this definition is motivated by the limiting case where signals satisfy MLRP but become uninformative; see Section 4.
Definition 3. A (retrospective) voting equilibrium is a strategy profile \( \sigma^* \) and an election cutoff \( c^* \) such that: (i) \( \sigma^* \) is optimal given \( c^* \), and (ii) \( c^* \) is an election cutoff given \( \sigma^* \).

A voting equilibrium requires players to optimize given an election cutoff that is endogenously determined by players’ strategies. In particular, unlike the standard notion of sincere voting, voting behavior now depends endogenously on the primitives of the environment. Moreover, the definition of a voting equilibrium is reminiscent of the definition of a competitive equilibrium in market economies. In the voting context, the role of prices is played by the election cutoff. Voters take the election cutoff as given when they optimize, and their consequent behavior yields that election cutoff. In Section 4, we provide a foundation for retrospective voting equilibrium by studying a game with a finite number of voters and letting the number of voters go to infinity.

2.3 Characterization of equilibrium

We now characterize a voting equilibrium. For each type \( \theta \) and signal \( s \), define the personal cutoffs

\[
c_\theta(s) \equiv \arg \min_{c \in \Omega} |v_\theta(s; c)|,
\]

(2)

which depend only on the primitives of the environment. Since \( \Omega \) is compact and \( v_\theta(s; \cdot) \) is continuous and increasing (by A1-A3), there exists a unique solution \( c_\theta(s) \) that is nonincreasing in \( s \). Moreover, A4 implies that \( v_\theta(s; c) \) is jointly continuous in \( (\theta, c) \) and, by the Theorem of the Maximum, \( c_\theta(s) \) is continuous in \( \theta \). Thus, we can define \( c \equiv \min_\theta c_\theta(s^H) \) and \( c \equiv \max_\theta c_\theta(s^L) \) as the lowest and highest personal cutoffs across all types.

If we knew the equilibrium election cutoff \( c^* \), then it would be straightforward to characterize the equilibrium strategy: a type \( \theta \) with signal \( s \) such that \( c_\theta(s) < c^* \) must have \( v_\theta(s; c^*) > 0 \) and, therefore, she will optimally vote for \( A \); similarly, if \( c_\theta(s) > c^* \), then she will optimally vote for \( B \). Consequently, we now characterize the set of equilibrium cutoffs. For a possible election cutoff \( \omega \in \Omega \),

\[
\bar{\kappa}(\omega) \equiv \int_\Theta \sum_{\{s : c_\theta(s) < \omega\}} q_\theta(s \mid \omega) \phi(d\theta)
\]

(3)
may be interpreted as the proportion of players that vote for A conditional on the state being the election cutoff $\omega$.\footnote{The interpretation is exact except when there is a unique type (i.e., $\Theta$ is a singleton) and $\omega$ is one of its personal cutoffs.}

**Lemma 1.** $\pi : \Omega \to [0, 1]$ is left-continuous, increasing over the subdomain $(\underline{c}, \bar{c})$, and satisfies: $\pi(\omega) = 0$ if $\omega \leq \underline{c}$ and $\pi(\omega) = 1$ if $\omega > \bar{c}$. 

**Proof.** See the Appendix. \hfill \square

**Theorem 1.** For any electoral rule $\rho \in (0, 1)$, there exists a unique equilibrium cutoff and it is given by $\bar{\kappa}^{-1}(\rho) \in [\underline{c}, \bar{c}]$.\footnote{$\bar{\kappa}^{-1} : (0, 1) \to [\underline{c}, \bar{c}]$ is defined as $\bar{\kappa}^{-1}(\rho) = \inf\{\omega \in \Omega : \bar{\kappa}(\omega) \geq \rho\}$.}

**Proof.** See the Appendix. \hfill \square

Theorem 1 says that there is a unique equilibrium cutoff and that it is essentially given by the intersection of the function $\pi$ with the electoral rule $\rho$. The following examples illustrate this result.

**Example 1.** (Homogenous informed voters) Suppose that the state is uniformly distributed in $[-1, 1]$ and that there is a unique voter type with payoffs $u(A, \omega) = \omega$, $u(B, \omega) = -\omega$, and binary signals $S = \{s^L, s^H\}$ such that $q(s^H | \omega) = 1/2 + r\omega$, where $r \in (0, 1/2]$. In particular, $c^{FB} = 0$ is the first-best election cutoff, i.e., everyone prefers $A$ in states $\omega > c^{FB}$ and $B$ in states $\omega < c^{FB}$. Simple algebra yields

\[
v(s; c) = E(W \mid W \geq c, s) - E(-W \mid W < c, s) = \frac{1}{4}(1 - c^2) + I(s)\frac{r}{3}(1 - c^3) + \frac{1}{4}(c^2 - 1) + I(s)\frac{r}{3}(c^3 + 1) + \frac{1}{2}(c + 1) + I(s)\frac{r}{2}(c^2 - 1),\]

where $I(s^H) = -I(s^L) = 1$. It easy to see that the personal cutoffs $c(s)$, which solve $v(s; c(s)) = 0$, satisfy

$c(s^H) < c^{FB} < c(s^L)$. 

In addition,

\[ \bar{\kappa}(\omega) = \begin{cases} 
0 & \text{if } \omega \leq c(s^H) \\
\frac{1}{2} + r\omega & \text{if } c(s^H) < \omega \leq c(s^L) \\
1 & \text{if } \omega > c(s^L) 
\end{cases} \]

By Theorem 1, the equilibrium cutoff as a function of the electoral rule \( \rho \) is given by

\[ c^*(\rho) = \begin{cases} 
 c(s^H) & \text{if } \rho \leq \frac{1}{2} - (-rc(s^H)) \\
\frac{1}{2}(\rho - \frac{1}{2}) & \text{if } \frac{1}{2} - (-rc(s^H)) < \rho < \frac{1}{2} + rc(s^L) \\
 c(s^L) & \text{if } \rho \geq \frac{1}{2} + rc(s^L) 
\end{cases} \]

Thus, the first-best outcome can be obtained with our boundedly rational voters if and only if the electoral rule is \( \rho = 1/2 \). In contrast, a rule that requires a supermajority to elect \( A \) (\( B \)) will inefficiently elect \( B \) (\( A \)) too often in equilibrium. This is shown in the left panel of Figure 1 for the case \( \rho > 1/2 \) and \( r = 1/4 \). In particular, the election outcome depends on the chosen electoral rule. \( \square \)

**Example 2.** *(Heterogeneous uninformed voters)* Suppose that states and types are drawn uniformly (and independently) from the interval \([-1, 1]\). Suppose also that \( S \) is a singleton (i.e., there are no signals) and that payoffs of type \( \theta \) are given by \( u_{\theta}(A, \omega) = \omega + \theta \) and \( u_{\theta}(B, \omega) = -\omega \). As explained in Section 4, equilibrium without private information should be interpreted as the limiting case of equilibrium where
the informativeness of signals vanishes. Simple algebra yields

\[ v_\theta(c) = \theta + E(W \mid W \geq c) - E(-W \mid W < c) \]

\[ = \theta + c, \]

and, therefore, type \( \theta \)'s personal cutoff is \( c_\theta = -\theta \). Then

\[ \bar{\pi}(\omega) = \int_{\{\theta > -\omega\}} \phi(\theta) d\theta = (1 + \omega)/2. \quad (4) \]

By solving \( \bar{\pi}(\omega) = \rho \), we obtain the equilibrium cutoff \( c^*(\rho) = 2\rho - 1 \). Once again, the election outcome depends on the electoral rule. The right panel of Figure 1 illustrates this example. □

### 3 Implications of Retrospective Voting

In this section, we apply the framework introduced above to study several implications of retrospective voting. First, we consider the case where everyone has the same preferences and investigate the extent to which elections aggregate information and provide accountability. Second, we consider the case where preferences are heterogeneous and study how voting behavior depends on the aggregate distribution of preferences in the electorate. Finally, we embed the voting environment in a model of political competition and show that parties have incentives to exacerbate their differences.

#### 3.1 Homogeneous preferences

In this section, we study environments with homogeneous preferences where every type has the same payoffs. Assumption A1 implies that the first-best outcome can be described by a unique cutoff \( c^{FB} \in \Omega \), where everyone prefers to elect \( A \) (\( B \)) if \( \omega > (\omega <) c^{FB} \). We assume throughout this section that the first-best cutoff is interior, i.e., \( c^{FB} \in (-1, 1) \).

Our objective is to investigate the extent to which the first-best outcome can be attained in equilibrium. One implication of Lemma 1 and Theorem 1 is that there is an electoral rule that attains the first-best outcome if \( c^{FB} \in (\underline{c}, \bar{c}) \); moreover, that
rule is essentially unique (see Example 1 in Section 2 for an illustration). Thus, at best, information can be fully aggregated by choosing a very particular voting rule, which requires the planner to have precise knowledge of the primitives. While it is not surprising that the first-best outcome may not be attained when voters are boundedly rational, there are several other questions that are hard to answer without a formal framework. In what types of environments does information aggregation fail and what is the tension underlying this failure? What are the optimal voting rules? Do elections inherently mitigate the costs associated with mistakes? Is the alternative that performs better also more likely to be elected? Does better information improve electoral outcomes?

3.1.1 Status quo environment

To illustrate the tension underlying the lack of information aggregation, we begin with a special but important case where one of the alternatives provides a constant payoff. Suppose that $A$ is a risky alternative and $B$ is a status quo alternative providing a constant payoff which is normalized to zero for convenience, $u(B, \omega) = 0$ for all $\omega \in \Omega$. Then the first-best outcome cannot be attained by any electoral rule.

**Proposition 1.** Consider any environment with homogeneous preferences where $B$ is a status quo alternative. Then $\tau < c^{FB}$ and, therefore, there exists no electoral rule that attains the first-best outcome in equilibrium.

**Proof.** By definition of first-best, $u(A, \omega) > 0$ for all $\omega > c^{FB}$. Then, for any $\theta, s$,

$$v_\theta(s_\theta; c^{FB}) = E \left( u(A, W) \mid W \geq c^{FB} \right) > 0,$$

where the inequality follows because $G$ is absolutely continuous and $c^{FB}$ is interior. Since $v_\theta(s_\theta; \cdot)$ is increasing, then $c_\theta(s) < c^{FB}$. The final implication follows from Theorem 1. 

The tension that prevents information from being fully aggregated is as follows. If the correct alternative were elected at every state, then everyone would observe $A$ in states in which it delivers a payoff greater than zero, which is the payoff observed from $B$. But then everyone would always want to vote for $A$, contradicting the hypothesis

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that the correct alternative is elected at every state. In equilibrium, alternative \( A \) must be incorrectly elected in states of the world in which it delivers negative payoffs, thus decreasing its popularity and providing incentives for voters to also vote for \( B \).

While mistakes are inevitable in equilibrium, the welfare loss can be mitigated by choosing electoral rules that bias the election against the risky alternative \( A \). Figure 2 illustrates that any rule above a threshold \( \rho^* \) is optimal in a status quo environment with a unique type.\(^\text{11}\) Intuitively, retrospective voters are biased in favor of the risky alternative. The reason is that they learn the value of the risky alternative by conditioning on all cases where the risky alternative is chosen, and these cases happen to be the ones where the risky alternative performs better than average. Electoral rules that are biased against the risky alternative mitigate this problem. This result provides a new rationale for conservatism when the electorate faces a risky option, thus complementing alternative explanations provided by Buchanan and Tullock (1967), Caplin and Nalebuff (1988), Dal Bo (2006), and Holden (2009), among others.

While this section restricted attention to the case of a status quo option, the results extend as long as one of the alternatives is sufficiently better than the other when both are compared in their respectively best states of the world. In particular, it is easy to see that a necessary and sufficient condition for \( \bar{c} < c^{FB} \) is that

\[
E(u(A, W) \mid W \geq c^{FB}, S_\theta = s^L_\theta) > E(u(B, W) \mid W < c^{FB}, S_\theta = s^L_\theta)
\]  

\(^{11}\)In environments with a continuum of types, there is no optimal rule but, instead, any rule sufficiently close to unanimity provides welfare sufficiently close to optimal.
for all $\theta \in \Theta$.

### 3.1.2 Mitigation of equilibrium mistakes

As mentioned in the introduction, a robust finding in the literature is the high level of voter ignorance. There is also evidence that voter mistakes do not cancel out in the aggregate but can rather influence election outcomes (see Bartels (2008) for a review). Consistent with the evidence, our model predicts that voters have biased beliefs and make systematic mistakes. A major concern, then, is to understand whether there is any limit to the potential underperformance of elections, particularly when the cost of these mistakes is very large (e.g., a party insists on going to war even in states where there is no real threat to its national interests). To study this issue, in this section we fix an environment with homogeneous preferences, $V = \langle g, \Theta, \{u, S_\theta, q_\theta\}_{\theta \in \Theta}, \phi \rangle$, and consider the corresponding collection of modified environments

$$
\Gamma(V) = \left\{ V' \in V : \begin{array}{l}
u'(A, \omega) = u(A, \omega) \quad \forall \omega > c^{FB} \\
u'(B, \omega) = u(B, \omega) \quad \forall \omega < c^{FB} 
\end{array} \right\},
$$

where $V$ is the set of environments that satisfy assumptions A1-A4. In words, an environment $V' \in \Gamma(V)$ differs from $V$ only in that the payoffs from mistakenly choosing $A$ or $B$ may be different. In particular, the cost of making mistakes can be set as high as desired.

For any environment $V' \in V$ and electoral rule $\rho$, let $W_{V'}(\rho)$ denote expected equilibrium welfare. Let $\overline{W}_{V'} = \sup_{\rho \in (0,1)} W_{V'}(\rho)$ and $\underline{W}_{V'} = \inf_{\rho \in (0,1)} W_{V'}(\rho)$ denote the highest and lowest possible equilibrium welfare over all possible rules.

**Proposition 2.** Consider any environment with homogeneous preferences, $V$, and first-best cutoff, $c^{FB}$. Then there exists $C > -\infty$ such that $\overline{W}_{V'} \geq C$ for all $V' \in \Gamma(V)$. In particular, for the highest possible equilibrium welfare,

$$
\overline{W}_{V'} \geq \min \left\{ E \left( u(A, W) \mid W \geq c^{FB} \right), E \left( u(B, W) \mid W < c^{FB} \right) \right\}
$$

for all $V' \in \Gamma(V)$.

**Proof.** The proof for the uniform lower bound for $\overline{W}$ is provided in the Online Appendix. Here, we prove (6). Fix any $V' \in \Gamma(V)$ and let a prime denote any of its
associated elements. First, suppose that \( c^{FB} \in (c', \bar{c}') \). Then, by Lemma 1 and Theorem 1, there is an electoral rule that achieves \( c^{FB} \), implying that welfare is a weighted average of the two terms over which the minimum is taken in (6). Second, if \( c^{FB} \) equals either \( c' \) or \( \bar{c} \), then the previous conclusion also holds given that welfare is continuous in the cutoff. Third, suppose that \( c^{FB} > \bar{c} = c_\theta(s^L) \) for some \( \theta \in \Theta \). Then

\[
E(u'(A, W) \mid W \geq \bar{c}) \geq E(u'(A, W) \mid W \geq \bar{c}, S_\theta = s^L)
\geq E(u(B, W) \mid W < \bar{c}, S_\theta = s^L)
\geq E(u(B, W) \mid W < c^{FB}, S_\theta = s^L)
\geq E(u(B, W) \mid W < c^{FB}),
\tag{7}
\]

where the first and last lines follow from A1-A2, the second line by the definition of personal cutoff and because \( u'(B, \omega) = u(B, \omega) \) for all \( \omega < c^{FB} \), and the third line by the fact that \( u(B, \cdot) \) is nonincreasing. This last condition also implies that

\[
E(u'(B, W) \mid W < \bar{c}) \geq E(u(B, W) \mid W < c^{FB}).
\tag{8}
\]

Since \( \bar{c}' < c^{FB} \), then the highest possible equilibrium welfare is given by welfare under cutoff \( \bar{c}' \), which is a weighted average of the two terms in the left hand side of (7) and (8). Therefore, a lower bound is provided by \( E(u(B, W) \mid W < c^{FB}) \). A similar argument shows that, in the case \( c' < c^{FB} \), a lower bound is provided by \( E(u(A, W) \mid W \geq c^{FB}) \).

Proposition 2 provides a uniform lower bound for welfare no matter how costly it is to choose an alternative in the wrong state of the world. In addition, the bound on the highest possible welfare, which is achieved by an appropriate choice of electoral rule, is surprisingly high. For example, in the status quo environment, optimal welfare can never fall below the payoff from the safe alternative, no matter how risky the other alternative. As argued in the previous subsection, mistakes are necessary to make an a priori more attractive option less attractive. So when the cost of mistakes increases, it is also the case that less mistakes are required to make the option less attractive. As the cost of mistakes goes to infinity, the equilibrium probability of making mistakes goes to zero and the welfare cost remains bounded.
Example 3. Let $\mathcal{V}$ be an environment with a unique type and no private information, where the state is uniform $[-1, 1]$, $u(A, \omega) = \omega$, and $u(B, \omega) = -\omega/4$. Simple algebra yields
\[
v(c) = E(u(A, W) \mid W \geq c) - E(u(B, W) \mid W < c) = \frac{5}{8}c + \frac{3}{8},
\]
so that the unique personal cutoff (hence, equilibrium cutoff for every $\rho$) is $c^* = -3/5 < c^{FB} = 0$. Now consider environments $\mathcal{V}_\beta$, indexed by $\beta \geq 1$, with a unique type, no private information,
\[
u_{\beta}(c) = E(u_{\beta}(A, W) \mid W \geq c) - E(u_{\beta}(B, W) \mid W < c) = -\left(\beta + \frac{1}{4}\right)c^2 + \frac{1}{2}c + \frac{3}{4},
\]
and $u_{\beta}(B, \cdot) = u(B, \cdot)$. In particular, $\mathcal{V}_\beta \in \Gamma(\mathcal{V})$ for all $\beta \geq 1$ and the cost of mistakenly electing $A$ goes to infinity as $\beta \to \infty$. For $c < c^{FB}$, Simple algebra yields
\[
W_{\mathcal{V}_\beta} = \int_{-1}^{c^*_\beta} -\omega g(\omega) d\omega + \int_{c^*_\beta}^{0} \beta \omega g(\omega) d\omega + \int_{0}^{1} \omega g(\omega) d\omega
= \frac{5}{16} - \left(\frac{1}{16} + \frac{\beta}{4}\right) (c^*_\beta)^2.
\]
so that the equilibrium cutoff is $c^*_{\beta} = \frac{\sqrt{\frac{1}{2} - (3\beta + 1)^{1/2}}}{2(\beta + 1/4)}$. Equilibrium welfare for environment $\mathcal{V}_\beta$ is then
\[
\lim_{\beta \to \infty} W_{\mathcal{V}_\beta} = \frac{1}{8}
= \min \left\{ E\left(u(A, W) \mid W \geq c^{FB} \right), E\left(u(B, W) \mid W < c^{FB} \right) \right\}
= \min \left\{ \frac{1}{2}, \frac{1}{8} \right\}.
\]
In particular, the bound in (6) is tight. □

3.1.3 The value of information

Our model allows the electorate to be fairly sophisticated, in the sense that voters may keep track of information that helps predict the performance of the alternatives. For example, shareholders may learn that the benefits of a takeover depend on the degree of industry concentration and voters in a political election may learn that one party is better suited to govern in periods of high inflation.

The standard view is that conditioning on more precise sources of information must be beneficial. This view is justified under sincere voting, where better information cannot decrease, and often increases, welfare. Under Nash equilibrium voting, however, the amount of information plays essentially no role in large elections: as long as there is some information, the first-best outcome nearly obtains.

One issue with the standard view is that it implicitly assumes that beliefs are not systematically biased. But this is not the case in a retrospective voting equilibrium, and, as we show next, more information may actually decrease welfare.

For simplicity, we consider an environment with a unique type and perform comparative statics with respect to the information precision by comparing \( q \) with \( q' \). Objects that correspond to the environment with \( q' \) are indexed with a prime. We follow Blackwell (1953) in saying that \( q' \) is more informative than \( q \) if the former is obtained by a garbling of the latter.

**Definition 4.** \( q' \) is more informative than \( q \) if there exists an \(|\mathbb{S}| \times |\mathbb{S}|\) matrix \( M \) with entries \( m_{s's'} \geq 0 \) that satisfy \( \sum_{s' \in \mathbb{S}'} m_{s's'} = 1 \) for all \( s' \in \mathbb{S}' \) and \( q(s' | \cdot) = \sum_{s' \in \mathbb{S}'} m_{s's'} q'(s' | \cdot) \) for all \( s \in \mathbb{S} \). If, in addition, \(|\mathbb{S}| \geq 2 \) and \( m_{s's'} > 0 \) for all \( s, s' \), then we say that \( q' \) is strictly more informative than \( q \).

By Blackwell (1953), \( q' \) is more informative than \( q \) if and only if, in an environment with any primitives \( (g, u) \), the expected utility of an individual making dictatorial decisions is (weakly) higher under \( q' \) compared to \( q \). The following result is key in understanding the effect of more informative signals on the outcome of a voting equilibrium.
Lemma 2. If \( q' \) is more informative than \( q \), then the personal cutoffs under \( q' \) are more extreme than under \( q \), i.e., \( c' \leq c \) and \( c' \geq c \). These inequalities are strict if \( q' \) is strictly more informative than \( q \) and the personal cutoffs satisfy \( c(s) \in (-1, 1) \) for all \( s \in S \).

Proof. By A1-A2, a proof similar to Milgrom (1981, Proposition 1) yields

\[
v(s^H; c) \geq v(s; c) \text{ and } v'(s^H; c) \geq v'(s; c)
\]

for all \( s \in S \) and \( c \in \Omega \). Since \( q' \) is more informative than \( q \), then

\[
\sum_{s' \in S} v'(s'; c) \alpha_{s's^H} \leq v'(s^H; c),
\]

where \( \alpha_{s's^H} = \frac{1}{\sum_{s' \in S} m_{s's^H}} \frac{1}{1 \{ \omega \geq c \}} \int q'(s' \mid \omega) g(\omega) d\omega \) and \( \sum_{s' \in S} \alpha_{s's^H} = 1 \). Therefore, by equation (2),

\[
c' = c'(s^H) \leq c(s^H) = c.
\]

Finally, if \( |S| \geq 2 \), then A1-A2 imply that \( v'(s^H; c) > v'(s'; c) \) for all \( s' \neq s^H \) and \( c \in \Omega \). Hence, if \( m_{s's} > 0 \) for all \( s, s' \), then (9) holds with strict inequality. Therefore, \( c'(s^H) < c(s^H) \) provided that \( c(s^H) \neq -1 \). A similar argument establishes that \( \sigma' \geq \sigma \), with strict inequality under the given additional conditions.

The idea behind Lemma 2 is that more informative information structures widen the difference in perceived utility between the lowest and highest signals, and, consequently, the range of personal cutoffs.

Proposition 3. Let \( q' \) be strictly more informative than \( q \) and suppose that \( c(s) \in (-1, 1) \) for all \( s \in S \). There exists an electoral rule \( \rho \) such that equilibrium welfare under \( q' \) is strictly lower than equilibrium welfare under \( q \).

Proof. First, suppose that \( c(s^H) \leq c^{FB} \). By Lemma 1 and Theorem 1, any \( \rho \leq \min \{ q(s^H \mid c(s^H)), q'(s^H \mid c'(s^H)) \} \) yields equilibrium cutoffs \( c(s^H) \) and \( c'(s^H) \) under information structures \( q \) and \( q' \). By Lemma 2, \( c(s^H) > c'(s^H) \), so that the equilibrium cutoff under \( q' \) is farther from the first best and equilibrium welfare is strictly lower. Finally, suppose that \( c(s^H) > c^{FB} \), so that \( c(s^L) > c^{FB} \). By a similar argument,
under any rule $\rho' \geq 1 - \min\{q(s^H | c(s^H)), q'(s^H | c'(s^H))\}$, the equilibrium cutoff under $q'$ is farther from the first-best cutoff and welfare is strictly lower. \hfill \Box

Proposition 3 shows that there always exist electoral rules under which more information decreases welfare. The next example illustrates this possibility.

**Example 4.** Consider an environment with a unique type where the state is uniform $[-1, 1]$, $u(A, \omega) = \omega$, and $u(B, \omega) = -\omega/4$, so that $c^{FB} = 0$. In addition, let $S = \{s^L, s^H\}$ and

$$q(s^H | \omega) = \begin{cases} d & \text{if } \omega \leq 0 \\ 1 - d & \text{if } \omega > 0 \end{cases},$$

where $d \in (0, 1/2)$.\(^{12}\) In particular, informativeness increases as the parameter $d$ decreases; henceforth we index elements by this parameter.\(^{13}\) For $c \in (-1, 0]$, simple algebra yields

$$v_d(c; s^L) = \frac{-c^2(1 - d) + d}{-2c + 2d(1 + c)} - \frac{(1 - c)}{8}$$

and

$$v_d(c; s^H) = \frac{-c^2 + 1 - d}{2 - 2d(1 + c)} - \frac{(1 - c)}{8}. \quad (11)$$

There is no need to calculate these functions for $c > 0$ because (5) is satisfied and, therefore, all personal cutoffs are nonpositive. It is easy to check that (10) decreases and (11) increases as $d$ decreases; thus, as expected from Lemma 2, the personal cutoffs $c_d(s^L)$ and $c_d(s^H)$ are increasing and decreasing in $d$, respectively. Figure 3 illustrates the cases $d' = .3$ and $d = .4$, where the former case is more informative. Under electoral rule $\rho < .2$, the corresponding equilibrium cutoffs are $c_{d'} < c_d$; hence welfare is lower under $d'$ because the cutoff is farther from the first-best cutoff $c^{FB}$. \hfill \Box

Finally, the next proposition shows that the intuition that more information increases welfare is correct as long as the electoral rule is chosen optimally. This result

\(^{12}\)The function $q(s^H | \cdot)$ is not continuous but it can be easily modified to be continuous while illustrating the same points.

\(^{13}\)Formally, for two information structures with $d' < d$, Definition 4 holds with $M$ having diagonal elements $(1 - d - d')/(1 - 2d')$ and off-diagonal elements $(d - d')/(1 - 2d')$. 22
highlights, once again, the importance of choosing the right electoral rule under retrospective voting.

**Proposition 4.** Let \( q' \) be strictly more informative than \( q \) and suppose that \( c(s) \in (0,1) \) for all \( s \in S \). There exists an electoral rule \( \rho^* \) such that equilibrium welfare under \( q' \) is weakly higher than equilibrium welfare under \( q \) with any electoral rule. Moreover, welfare is strictly higher if the first-best outcome cannot be achieved under \( q \).

**Proof.** First, suppose that \( c'(s^H) > c^{FB} \). By Lemma 2, \( c(s^H) > c'(s^H) \). Then, by Theorem 1 and the fact that welfare decreases as the equilibrium cutoff is farther from \( c^{FB} \), the highest possible equilibrium welfare under \( q \) is strictly lower than the equilibrium welfare obtained by choosing a rule that yields \( c'(s^H) \) as an equilibrium cutoff under \( q' \). A similar argument holds for the case where \( c'(s^L) < c^{FB} \). Finally, in the case where \( c^{FB} \in [c'(s^H), c'(s^L)] \), Lemma 1 and Theorem 1 imply that there exists a rule under which the first-best cutoff is obtained in equilibrium, which yields by definition the highest possible payoff. \( \square \)
3.1.4 Convergence in observed performance

Our model of retrospective voting predicts an interesting form of convergence in the performance of the alternatives. Even if one alternative is ex-ante preferred to another alternative, in equilibrium there is a tendency for both alternatives to have a similar observed performance. This phenomenon is more clearly seen in the case without private information.

**Proposition 5.** Consider any environment with homogeneous preferences and no private information where \( c^* \in (-1, 1) \) is the equilibrium cutoff. Then

\[
E (u(A, W) \mid W \geq c^*) = E (u(B, W) \mid W < c^*).
\]

**Proof.** By Theorem 1, the personal cutoff is also the equilibrium cutoff for any rule \( \rho \) in an environment with homogeneous preferences and no private information. By assumption, the equilibrium cutoff and, therefore, the personal cutoff is interior. Thus, the personal cutoff (hence, equilibrium cutoff) is the unique solution \( c^* \) to

\[
v(c^*) = E (u(A, W) \mid W \geq c^*) - E (u(B, W) \mid W < c^*) = 0.
\]

Proposition 5 shows that the election cutoff adjusts so that each party provides the same observed performance in equilibrium. In particular, a naive outside observer would incorrectly conclude that the two alternatives perform equally well and that a flip of a coin would provide the same benefits of an election.

Equilibrium analysis yields more nuanced predictions when voters have private information. As the next result shows, the alternative that is over-elected in equilibrium may actually be the one with the worst performance.

**Proposition 6.** Consider any environment with homogeneous preferences where \( c > -1 \) and \( c < 1 \). There exists an electoral rule such that, in equilibrium, the alternative that wins the election with a higher probability than the first-best probability is also the alternative with the worst observed performance.

**Proof.** The case where one type has no private information trivially follows from Proposition 5, which states that performance is equal. More generally, suppose that
every type has at least two signals. First, suppose that \( \underline{c} < c^{FB} \) and let \( \underline{\theta} \) be such that \( c_{\underline{\theta}}(s^H) = \underline{c} \). Then

\[
E(u(A, W) \mid W \geq \underline{c}) - E(u(B, W) \mid W < \underline{c}) = \sum_{s \in S} v_{\underline{\theta}}(s; \underline{c}) Pr_{\underline{\theta}}(s) < 0, \tag{12}
\]

where the inequality follows because (i) \( c_{\underline{\theta}}(s^H) = \underline{c} \) and \( \underline{c} > -1 \) imply that \( v_{\underline{\theta}}(s^H; \underline{c}) = 0 \) and (ii) A1-A2 implies that \( v_{\underline{\theta}}(\cdot; \underline{c}) \) is increasing. Moreover, by Lemma 1, Theorem 1, and continuity of welfare in the cutoff, there is a rule that results in an equilibrium welfare as close as one desires to the welfare obtained under cutoff \( \underline{c} \). Thus, (12) implies that there is a rule under which \( A \)'s observed performance is worse than \( B \)'s and, since \( \underline{c} < c^{FB} \), \( A \) is over-elected relative to the first-best outcome. Finally, the case \( \underline{c} \geq c^{FB} \) follows a similar argument by now choosing a rule that delivers an outcome close to \( \bar{c} \).

In particular, by normalizing the state to be uniformly distributed and \( c^{FB} = 0 \), Proposition 6 says that it is possible for an alternative to have a worse observed performance yet to win the election with a higher probability. Using data from U.S. presidential elections since 1948, Bartels (2010) finds that certain economic indicators have been better under Democratic presidencies. Campbell (2011) points out that this finding, when combined with the frequent success of Republican candidates, poses a challenge to theories of retrospective voting.\(^{14}\) However, our results highlight that this finding is consistent with retrospective voting provided that voters have private information.

### 3.2 Heterogeneous preferences

We now consider the case where preferences are heterogeneous and study how preferences affect voting behavior. The main insights follow from the fact that individual behavior depends, in equilibrium, on the aggregate voting behavior of the electorate.\(^{14}\)

\(^{14}\)Campbell (2011) also claims that presidential party differences with respect to main economic indicators vanish when controlling for the state of the economy when the party is in power. As shown in this section, in our model there is a tendency for such convergence in performance, and any difference in performance must be explained by heterogeneity in the preferences and information of the electorate.
3.2.1 Preferences and voting behavior

In this section we study environments with heterogeneous preferences and assume, for simplicity, that the electorate is uninformed. We assume that $u_\theta(A, \omega)$ is increasing and $u_\theta(B, \omega)$ is decreasing in $\theta$. In empirical work, the preference type $\theta$ would be linked to several demographics. We fix any such environment throughout this section and perform comparative statics with respect to the distribution over types.

**Proposition 7.** Consider any environment with heterogeneous preferences and uninformed voters with type distribution $\phi$ and electoral rule $\rho$. Then there is a threshold type $\theta_\phi(\rho)$ such that, in equilibrium, higher types vote for $A$ and lower types vote for $B$. Moreover, if $\phi_2$ first-order stochastically dominates $\phi_1$, the equilibrium probability of electing $A$ is weakly higher under $\phi_2$ compared to $\phi_1$, but $\theta_{\phi_2}(\rho) \geq \theta_{\phi_1}(\rho)$.

**Proof.** By the assumptions that $u_\theta(A, \omega)$ is increasing and $u_\theta(B, \omega)$ is decreasing in $\theta$, $v_\theta(s;c)$ is increasing in $\theta$ and, therefore, the personal cutoff $c_\theta$ is decreasing in $\theta$. By Theorem 1, $\bar{\kappa}^{-1}(\rho)$ is the equilibrium cutoff. By optimality, any type with a personal cutoff above (below) $\bar{\kappa}^{-1}(\rho)$ votes for $B$ ($A$) in equilibrium. Since $c_\theta$ is decreasing in $\theta$, there exists a threshold type $\theta_\phi(\rho)$ such that, in equilibrium, higher types vote for $A$ and lower types vote for $B$. Finally, suppose that $\phi_2$ first-order stochastically dominates $\phi_1$. By first-order stochastic dominance and the fact that, since $c_\theta$ is increasing in $\theta$, there exists $\theta_\omega \in \Theta$ such that $\{\omega : c_\theta < \omega\}$ is equivalent to $\{\theta : \theta < \theta_\omega\}$, it follows that

$$\bar{\kappa}_{\phi_1}(\omega) = \int_{\{\theta : c_\theta < \omega\}} \phi_1(d\theta) \leq \int_{\{\theta : c_\theta < \omega\}} \phi_2(d\theta) = \bar{\kappa}_{\phi_2}(\omega).$$

Then, the equilibrium cutoffs satisfy $\bar{\kappa}_{\phi_2}^{-1}(\rho) \leq \bar{\kappa}_{\phi_1}^{-1}(\rho)$, so that $A$ wins with a weakly higher probability under $\phi_2$. Moreover, optimality and the fact that $c_\theta$ is decreasing in $\theta$ imply that $\theta_{\phi_2}(\rho) \geq \theta_{\phi_1}(\rho)$. \qed

Proposition 7 shows that the equilibrium probability of electing $A$ increases with the popularity of $A$. But there is also a less obvious implication: the type threshold increases when $A$ becomes more popular. Thus, the increase in popularity for $A$ is partially mitigated by the fact that only those with a stronger preference for $A$ continue to vote for $A$. The reason is that, if $A$ is elected more often, then it must be
elected in worse states of the world. Therefore, types that were marginally willing to vote for $A$ will no longer desire to vote for $A$ when $A$ becomes more popular in the population.

One implication is that environments with greater support for an alternative will also have a greater proportion of supporters with extreme views. But this extremism, usually viewed in a negative way, is actually helping mitigate the dominance of a popular alternative, thus protecting minorities.

Example 5. Suppose that the state is uniform $[-1, 1]$, $u_\theta(A, \omega) = \omega + \theta$, $u_\theta(B, \omega) = -\omega$, and $\theta \in [-1, 1]$. Figure 4 illustrates equilibria under two type distributions: $\phi_1(\theta) = 1/2$ (i.e., uniform) and $\phi_2(\theta) = (\theta + 1)/2$. In particular, alternative $A$ is more popular (in the first-order stochastic sense) in the second case. The first case is solved in Example 2, leading to a personal cutoff $c_\theta = -\theta$, equilibrium cutoff $c^*_\phi_1(\rho) = 2\rho - 1$, and, therefore, type threshold $\theta_{\phi_1}(\rho) = 1 - 2\rho$. Under $\phi_2$, the personal cutoff is the same and, therefore,

$$\pi_{\phi_2}(\omega) = \int_{\{\theta > -\omega\}} \phi_2(\theta) d\theta = (3 + 2\omega - \omega^2)/4.$$ 

By solving $\pi_{\phi_2}(\omega) = \rho$, we obtain the equilibrium cutoff $c^*_\phi_2(\rho) = 1 - 2(1 - \rho)^{1/2}$ and the corresponding threshold $\theta_{\phi_2}(\rho) = 2(1 - \rho)^{1/2} - 1$. In particular, $c^*_\phi_2(\rho) < c^*_\phi_1(\rho)$ and $\theta_{\phi_2}(\rho) > \theta_{\phi_1}(\rho)$ for all $\rho \in (0, 1)$. Thus, an increase in popularity for $A$ leads to an increase in the probability of electing $A$, but this effect is mitigated by the fact that only those with a stronger preference for $A$ continue to vote for $A$. □

More generally, Proposition 7 shows that the model gives rise to composition effects, in the sense that the voting behavior of a particular type is influenced by the composition of the electorate. The presence of composition effects has also been documented in empirical work (e.g., Gelman et al. (2008), Leigh (2005)), although there are other explanations that can give rise to composition effects.

Proposition 7 also provides insights into the differences between local and national political elections. There is evidence that people vote differently at the local versus national level (e.g., Fiorina (1992), Chari et al. (1997), and Franck and Tavares (2008)). Of course, a reasonable explanation is that the underlying preferences differ.
Figure 4: Increase in popularity for $A$ (Example 5).
from local to national elections, perhaps because the types of candidates differ. However, we now show that differences in voting behavior at the national and local level would persist even if the underlying preferences were identical.

**Corollary 1.** Consider two districts with the same primitives except that the type distribution in district 2, \( \phi_2 \), first-order stochastically dominates the type distribution in district 1, \( \phi_1 \). Suppose that the same electoral rule \( \rho \) is used to determine the outcome of the two local district elections and of the national election (which aggregates votes from both districts). Then, in each district, a greater proportion of types vote for the alternative that is relatively favored in the district (i.e., \( B \) in district 1 and \( A \) in district 2) in national rather than in local elections.

**Proof.** By Proposition 7, the equilibrium cutoffs in the local elections satisfy \( \bar{\kappa}^{-1}_{\phi_2}(\rho) \leq \bar{\kappa}^{-1}_{\phi_1}(\rho) \). In the national election, the equilibrium cutoff \( \bar{\kappa}^{-1}_{\text{avg}}(\rho) \) is an average of the local cutoffs. In particular, in district 1, \( \bar{\kappa}^{-1}_{\text{avg}}(\rho) \leq \bar{\kappa}^{-1}_{\phi_1}(\rho) \), so that more types vote for \( B \) in the national compared to the local election. Similarly, in district 2, more types vote for \( A \) in the national compared to the local election. \( \square \)

For example, Corollary 1 says that a state like New York, which has a stronger preference for Democratic candidates relative to other states, will vote for a Democratic candidate in greater proportion during national, compared to local, elections.

### 3.2.2 Information and partisanship

When interpreted in the context of a political election, our model can be used to study endogenous partisanship.\(^{15}\) In this section, we investigate the relationship between information and partisanship.

**Definition 5.** Given a strategy profile \( \sigma \), type \( \theta \) is a partisan for party A (B) if \( \sigma_{\theta}(s) = 1 \) (0) for all \( s \in S \), and it is a non-partisan if it is not a partisan for either A or B.

We begin by investigating the case where a single (measure zero) type becomes more informed.

\(^{15}\)See Gerber and Green (1998) for an alternative model of endogenous party affiliation.
Proposition 8. Let \( q = (q_\theta)_{\theta \in \Theta} \) and \( q' = (q'_\theta)_{\theta \in \Theta} \) be such that \( q'_\theta \) is strictly more informative than \( q_\theta \) for some \( \phi \)-measure zero \( \theta^* \in \Theta \) and \( q_\theta = q'_\theta \) for all \( \theta \in \Theta \setminus \{\theta^*\} \). Suppose that the equilibrium cutoffs under \( q \) and \( q' \) are interior. If \( \theta^* \) is a partisan for one alternative [a non-partisan] in any equilibrium under \( q \), then \( \theta^* \) cannot be a partisan for the other alternative [must be a non-partisan] in any equilibrium under \( q' \).

Proof. Let \((\sigma, c)\) and \((\sigma', c')\) be equilibria under \( q \) and \( q' \), respectively. The fact that \( \theta^* \) has zero measure under \( \phi \) implies that the equilibrium cutoff is the same under \( q \) and \( q' \), i.e., \( c' = c \). First, suppose that \( \theta^* \) is partisan for B (the case for A is analogous). By optimality of \( \sigma \), it follows that \( c_{\theta^*}(s) \geq c \) for all \( s \in S \). Then Lemma 2 and the fact that \( c < 1 \) imply that \( c_{\theta^*}(s^L) > c = c' \). By optimality, \( \sigma_{\theta^*}(s^L) = 0 \), so \( \theta^* \) cannot be a partisan for A. Finally, suppose that \( \theta^* \) is a non-partisan. By optimality of \( \sigma \), it follows that \( c_{\theta^*}(s^H) \leq c \leq c_{\theta^*}(s^H) \). Then Lemma 2 and the fact that \( c \in (-1, 1) \) imply that \( c'_{\theta^*}(s^H) < c = c' < c'_{\theta^*}(s^H) \). By optimality, \( \sigma'_{\theta^*}(s^L) = 0 \) and \( \sigma'_{\theta^*}(s^H) = 1 \), so that \( \theta^* \) remains a non-partisan. \( \square \)

The simple idea behind Proposition 8 is that, for a voter who becomes more informed, her personal cutoffs will become more extreme and, therefore, she will act in a less partisan manner. A related question is whether a party would want its supporters to become more informed or not. This question cannot be answered by Proposition 8 because, once a positive measure of voters become more informed, the equilibrium cutoff might also change. The next result shows that a party cannot gain votes when its supporters become more informed.

Proposition 9. Suppose that every type in \( \Theta^* \subseteq \Theta \) is a partisan for the same specific alternative in an equilibrium with an interior cutoff under \( q = (q_\theta)_{\theta \in \Theta} \). Let \( q' = (q'_\theta)_{\theta \in \Theta} \) be such that \( q'_\theta \) is strictly more informative than \( q_\theta \) for all \( \theta^* \in \Theta^* \). Then, for any equilibrium under \( q' \), the probability of electing the specific alternative cannot be higher than in the equilibrium under \( q \).

Proof. We only show the case where every type \( \theta^* \in \Theta^* \) is a partisan for B under \( q \).
Let $c$ and $c'$ be the equilibrium cutoffs under $q$ and $q'$, respectively. First, note that

$$\bar{\kappa}'(c) = \int_{\Theta \setminus \Theta^*} \sum_{\{s : c'(s) < c\}} q'(s | \omega) \phi(d\theta) + \int_{\Theta^*} \sum_{\{s : c'(s) < c\}} q'(s | \omega) \phi(d\theta)$$

$$= \int_{\Theta \setminus \Theta^*} \sum_{\{s : c'(s) < c\}} q(s | \omega) \phi(d\theta) + \int_{\Theta^*} \sum_{\{s : c'(s) < c\}} q'(s | \omega) \phi(d\theta)$$

$$\geq \int_{\Theta \setminus \Theta^*} \sum_{\{s : c'(s) < c\}} q(s | \omega) \phi(d\theta) = \bar{\kappa}(c),$$

(13)

where the first line is by definition, the second by the fact that the personal cutoffs do not change for types in $\Theta \setminus \Theta^*$, and the last equality by the fact that every type in $\Theta^*$ is a partisan for $B$ and therefore, by optimality of $\sigma$, satisfies $c_{\theta^*}(s) \geq c$ for all $s \in S$. By Theorem 1, $c' = \inf\{\omega \in \Omega : \bar{\kappa}'(\omega) \geq \rho\}$. It then follows from (13) and the fact that $\bar{\kappa}$ is nondecreasing (Lemma 1) that $c' \leq c$. Therefore, the probability of electing $B$ is weakly lower under $q'$ compared to $q$. □

**Example 6.** The state is uniform $[-1, 1]$ and there are three types $\Theta = \{1, 2, 3\}$, with $\phi_1 = \phi_3 = .45$ and $\phi_2 = .1$. Payoffs are given by $u_{\theta}(A, \omega) = \omega - 1 + z_\theta$ and $u_{\theta}(B, \omega) = 0$ for all $\theta, \omega$, where $z_1 = 1.5$, $z_2 = 1.5$ and $z_3 = 1.6$. In particular, type

\[\text{The assumption of a finite number of types violates A4, but recall that this assumption is only needed to guarantee uniqueness of equilibrium, which we nevertheless obtain in this example.}\]
3 has a greater preference for $A$ compared to types 1 and 2. In addition, types 1 and 3 have a binary information structure with signals $S = \{s^L, s^H\}$ and precision $q_\theta(s^H | \omega) = 1/2 + r\omega$; type 2 has no private information. We let $r = .25$ and denote this information structure by $q$. For $\theta \in \{1, 3\}$,

$$v_\theta(s; c) = E(W \mid W \geq c, s) - 1 + z_\theta$$

$$= \frac{1}{2} \left( \frac{1}{2} + I(s)r \right) (c - 1) + I(s)\frac{r}{3}(c - 1)^2 \left( \frac{1}{2} + I(s)r \right) - I(s)\frac{r}{2}(c - 1) - 1 + z_\theta,$$

where $I(s^H) = -I(s^L) = 1$. The corresponding personal cutoffs are $c_1(s^H) \approx -0.08$, $c_1(s^L) \approx 0.1$, $c_3(s^H) \approx -0.32$, and $c_3(s^H) \approx -0.075$. For type 2,

$$v_2(c) = E(W \mid W \geq c) \approx 0.5$$

$$= 0.5(1 + c) - 0.5,$$

and the personal cutoff is $c_2 = 0$. Figure 5 depicts these personal cutoffs and the function $\bar{\kappa}$; the unique equilibrium under rule $\rho = .67$ is given by $c^{*} \approx -0.044$. Under this equilibrium, type 3 voters are partisans for party $A$.

Next, consider an information structure $q'$ where voters of type 3, who are partisans for $A$ in the previous equilibrium, have better information precision given by $r' = .45 > r$; all other types have the same information structure as before. The personal cutoffs for types 1 and 2 remain the same, but the personal cutoffs for type 3 are now $c'_3(s^H) \approx -0.45$ and $c'_3(s^H) \approx 0.05$. As shown in Figure 5, the new equilibrium cutoff $c^{*'} = c'_3(s^L) > c$. As predicted by Proposition 9, the probability of electing $A$ decreases when partisans for $A$ become more informed. Moreover, type 2 voters, who were originally partisans for $B$ under $q$, now become partisans for $A$ under $q'$, even though their own information structure did not change. □

The above findings have several implications. First, more informed voters have a tendency to be less partisan. In our context, being “more informed” means that a voter will learn to recognize the economic and political environments under which different parties tend to perform better. To the extent that such recognition may be more prevalent among educated voters, one implication is that more educated voters are less partisan; Shively (1979) provides empirical support. A second implication is that, given that parties have no incentives to educate voters, campaign platforms
are not likely to be very informative. Finally, Example 6 illustrates that, even if the preferences and information of a voter do not change, her voting behavior may change as a result of a change in the equilibrium cutoff driven by a change in preferences or information of other voters. Thus, when studying the demographic features that give rise to partisanship in a segment of the population, one should control for the attributes of other segments.

3.3 Party polarization

In this section, we endogenize the primitives of the voting environment by focusing on the case where $A$ and $B$ are political parties and allowing the parties to choose policies. To motivate the analysis, suppose that the economy is either in a recessionary or overheated state. Party $A$ ideologically favors expansionary fiscal policy while party $B$ favors contractionary policy. Expansionary policy does best in a recession but hurts an overheated economy, while contractionary policy does best in an overheated economy but hurts during a recession. There is also a neutral, hands-off policy that neither helps nor hurts the economy. Parties can choose to mitigate their ideological positions by choosing any combination between their ideological position and the neutral policy. Suppose that, in expectation, the neutral policy does better compared to the extreme policies. Does electoral competition mitigate or exacerbate ideological positions? Does it increase welfare?

The formal game is as follows. For simplicity, we assume that there is a unique type and that neither the electorate nor the parties have private information. Each party $j \in \{A, B\}$ simultaneously chooses a policy $p_j \in [0, 1]$ that represents a weighted average between its extreme ideological policy, $u_j(\omega)$, and a neutral policy that delivers a constant payoff (normalized to zero). Given policies $(p_A, p_B)$, the electorate’s payoff from electing party $j$ in state of the world $\omega$ is $u(j, \omega) = p_j u_j(\omega)$, where $u_j$ is continuously differentiable, $u_A(\cdot)$ is increasing, and $u_B(\cdot)$ is decreasing, so that assumption A1 is satisfied provided that $p_A + p_B > 0$. In that case, the outcome of the election is given by the voting equilibrium defined in Section 2. For the case where both parties choose the neutral policy $(p_A = p_B = 0)$, we naturally assume that, in each state of the world, the parties have an equal chance of being elected. We model electoral competition by solving for the Nash equilibrium in policies under the assumption that parties maximize their probability of being elected. We add the
following assumptions.

**B1.** \( E(u_j(W)) < 0 \) for \( j = 1, 2; u_A(1) > 0 \) and \( u_B(-1) > 0 \).

**B2.** Let \( c_A \) and \( c_B \) be defined by \( E(u_A(W) | W \geq c_A) = 0 \) and \( E(u_B(W) | W < c_B) = 0 \). We assume that \( c_A < c_B \).

Assumption B1 says that, in the case without electoral competition where either one of these parties are in charge, then the neutral policy is preferred in expectation by the electorate. However, there are some states of the world in which the extreme policies are preferred. Assumption B2 requires the parties’ extreme policies to be, in expectation, not too much worse than the neutral policy.

**Proposition 10.** Suppose that B1-B2 hold and let \( \rho \in (0, 1) \) be any electoral rule. Then, in the unique Nash equilibrium, parties choose to exacerbate their ideological positions, \( p_A = p_B = 1 \). Moreover, electoral competition leads to strictly higher welfare compared to the case where either party is put in charge and chooses the neutral policy.

**Proof.** First, note that, since there is no private information, Theorem 1 implies that, for any electoral rule \( \rho \in (0, 1) \), the voting equilibrium cut-off \( \epsilon^*(p_A, p_B) \) for \( p_A + p_B > 0 \) is the unique solution to \( v(\epsilon^*(p_A, p_B)) = 0 \), where

\[
v(c) \equiv p_A E(u_A(W) | W \geq c) - p_B E(u_B(W) | W < c)
\]

for all \( c \in \Omega \). Suppose that \( p_A > 0 \). Then, by B2, \( v(c_A) = -p_B E(u_B(W) | W < c_A) < 0 \) and \( v(c_B) = p_A E(u_A(W) | W \geq c_B) > 0 \). Therefore, the voting equilibrium cut-off \( \epsilon^*(p_A, p_B) \in (c_A, c_B) \). By the implicit function theorem, the sign of \( \frac{\partial \epsilon^*(p_A, p_B)}{\partial p_B} \) is given by the sign of \( E(u_B(W) | W < \epsilon^*(p_A, p_B)) \), which is strictly positive because \( \epsilon^*(p_A, p_B) < c_B \). Since party B maximizes the probability of being elected by choosing a policy that results in the highest possible election cutoff, her best response is to choose \( p_B = 1 \). A similar argument yields that a best response to \( p_B > 0 \) is to choose \( p_A = 1 \). Therefore, \( p_A = p_B = 1 \) is a Nash equilibrium and, since \( E(u_B(W) | W < \epsilon^*(p_A, p_B)) > 0 \) and \( E(u_A(W) | W \geq \epsilon^*(p_A, p_B)) > 0 \), equilibrium welfare is higher than under the neutral policy that yields a payoff of zero.

It remains to show that \( p_A = p_B = 0 \) is not a Nash equilibrium. Suppose that \( p_A = 0 \). Then, for any \( p_B > 0 \), the equilibrium cutoff is \( c_B \) and the probability that B

\footnote{Existence and uniqueness of \((c_A, c_B)\) follows from assumption B1.}
wins the election is $G(c_B)$. For $p_B = 0$, then party B wins the election with probability .5. Therefore $p_B = 0$ is a best response if and only if $G(c_B) \leq .5$. A similar argument shows that $p_A = 0$ is a best response to $p_B = 0$ if and only if $1 - G(c_A) \leq .5$. But then $G(c_A) \geq G(c_B)$, which contradicts assumptions B2 and A3(i).

While the tendency of both parties to polarize and go against the preference of the (median) voter seems unintuitive, the result is a natural consequence of an environment where policy outcomes are uncertain and people vote retrospectively based on observed performance. Retrospective voting implies that party A will be elected in high states and party B in low states. Therefore, each party has an incentive to choose an extreme policy that works well in those extreme states in which it is usually elected into office, since those are the payoffs that voters use to evaluate their performance. In addition, since each policy tends to be appropriately matched with the right state of the world, the coexistence of extreme policies increases welfare. McCarty et al. (2006) provides evidence of polarization in the U.S. and Gerber and Morton (1998), Alesina and Rosenthal (2000), and Kamada and Kojima (2012) provide alternative explanations.

4 Foundation for voting equilibrium

We provide a game-theoretic foundation for (retrospective) voting equilibrium by applying the notion of a naive behavioral equilibrium (Esponda, 2008) to a voting environment with a finite number of players. We begin by defining naive equilibrium for a voting environment where players’ payoffs are now perturbed to guarantee that each alternative is elected with positive probability in equilibrium; Esponda and Pouzo (2012) show that a naive equilibrium corresponds to the steady state of an explicit learning environment. We then turn to our contribution in this section, which is to study the naive equilibrium concept as the number of players goes to infinity. The main result is that the definition of (retrospective) voting equilibrium in Section 2 corresponds to the limit of naive equilibria as, first, the number of players goes to infinity and, then, the payoff perturbations vanish.

---

4.1 Voting game

The rules of the game are as described in Section 2. The difference is that there are now a finite number of players, indexed by $i = 1, ..., n$, with types $(\theta_1, ..., \theta_n)$, where we now assume that $\Theta$ is a finite set rather than a compact interval (see the end of the section for a discussion). Player $i$'s payoff when the election outcome is $o \in \{A, B\}$ is now

$$u_{\theta_i}(o, \omega) + 1 \{o = B\} \nu,$$

where $\nu \in \mathbb{R}$ is a privately-observed payoff perturbation drawn independently for each player from a probability distribution $F_{\theta_i}$. Recall that $K$ is the uniform bound on payoffs postulated in assumption A1. In addition to A1-A3, we maintain the following assumptions for all $\theta \in \Theta$:

A5. $F_{\theta}$ is absolutely continuous and satisfies $F_{\theta}(-2K) > 0$ and $F_{\theta}(2K) < 1$; its density $f$ satisfies $\inf_{x \in [-2K, 2K]} f_{\theta}(x) > 0$.

A6. $S$ has at least two elements and there exists $z > 0$ such that for all $\omega' > \omega$ and $s' > s$,

$$\frac{q_{\theta}(s'|\omega')}{q_{\theta}(s'|\omega)} - \frac{q_{\theta}(s|\omega')}{q_{\theta}(s|\omega)} \geq z(\omega' - \omega).$$

Assumption A5 guarantees that each alternative is voted with positive probability. This property implies that the probability that players are pivotal (i.e., that their vote decides the election) becomes negligible as $n \to \infty$. Assumption A6 is a strengthening of MLRP that establishes a uniform bound on the rate at which the likelihood ratio changes.

Following Harsanyi (1973), for each player there is a threshold perturbation above which the player will vote for $B$ and below which she will vote for $A$. Thus, integrating over such perturbations and noting that $F_{\theta}$ is absolutely continuous, we obtain a (mixed) strategy for each player $i$, $\alpha_i : S \to [0, 1]$, where $\alpha_i(s)$ is the probability of voting for $A$ after observing signal $s$. In addition, each strategy profile

\[\footnotesize\text{\textsuperscript{19}}\]

A5 also yields a refinement, which is standard in the literature, that rules out equilibria where everyone votes for the same alternative because a unilateral deviation cannot change the outcome. Esponda and Pouzo (2012) show that the perturbations are also important for providing a learning foundation for naive equilibrium.
\( \alpha = (\alpha_1, ..., \alpha_n) \), together with the primitives of the game, induces a distribution \( P^n(\alpha) \) over the outcomes of the game \( \{A, B\} \times S^n \times \Omega \).

To gain intuition for the notion of a naive equilibrium, suppose that player \( i \) repeatedly faces a sequence of stage games where players use strategies \( \alpha \) every period. Then, under the assumption that the payoff to alternative \( A \) is observed only whenever \( A \) is chosen, player \( i \) will come to observe that, conditional on observing signal \( s \), alternative \( A \) yields in expectation \( E_{P^n(\alpha)}(u_{\theta_i}(A, W) \mid o = A, S_i = s) \).\(^{20}\) A similar expression holds for alternative \( B \).

A naive player who observes \( \nu \) and \( s \) believes that expected utility is maximized by voting for \( A \) whenever \( \Delta_i(P^n(\alpha), s) - \nu > 0 \) and voting for \( B \) otherwise, where

\[
\Delta_i(P^n(\alpha), s) \equiv E_{P^n(\alpha)}(u_{\theta_i}(A, W) \mid o = A, S_i = s) - E_{P^n(\alpha)}(u_{\theta_i}(B, W) \mid o = B, S_i = s) \tag{14}
\]

is well-defined because of the payoff perturbations.

**Definition 6.** A strategy profile \( \alpha = (\alpha_1, ..., \alpha_n) \) is a (naive) equilibrium of the voting game if for every player \( i = 1, ..., n \) and for every \( s \in S \),

\[
\alpha_i(s) = F_{\theta_i}(\Delta_i(P^n(\alpha), s)).
\]

In equilibrium, players best respond to beliefs that are endogenously determined by both their own strategy and those of other players and that are consistent with observed equilibrium outcomes. Naive players, however, do not account for the correlation between others’ votes and the state of the world (conditional on their own private information); see Esponda and Pouzo (2012) for the proof that equilibrium exists and additional discussion.

### 4.2 Large number of players

Our technical contribution is to analyze (naive) equilibrium as the number of voters goes to infinity. We do so by studying sequences of voting games. We build such sequences by independently drawing infinite sequences of types \( \xi = (\theta_1, \theta_2, ..., \theta_n, ...) \in \)

\(^{20}\)Whenever an expectation \( E_P \) has a subscript \( P \), this means that the probabilities are taken with respect to the distribution \( P \).
Ξ according to the probability distribution \( \phi \in \Delta(\Theta) \); we denote the distribution over \( \Xi \) by \( \Phi \) and we let \( \theta_i(\xi) \) denote the type of player \( i \), i.e., the \( i \)th component of \( \xi \). We interpret each sequence of types as describing an infinite number of \( n \)-player games by letting the first \( n \) elements of \( \xi \) represent the types of the \( n \) players.

Let \( \alpha \) denote a \emph{strategy mapping} from sequences of types \( \Xi \) to sequences of strategy profiles—i.e., for all \( \xi \in \Xi \), let \( \alpha(\xi) = (\alpha^1(\xi), ..., \alpha^n(\xi), ...) \), where

\[
\alpha^n(\xi) = (\alpha^n_1(\xi), ..., \alpha^n_n(\xi))
\]

is the strategy profile that is played in the \( n \)-player game with types \( \theta_1, ..., \theta_n \). Let \( P^m(\alpha(\xi)) \) be the probability distribution over \( \{A, B\} \times S^n \times \Omega \) induced by the strategy profile \( \alpha^n(\xi) \) in the \( n \)-player game. We define two properties of strategy mappings.\(^\text{21}\)

**Definition 7.** A strategy mapping \( \alpha \) is an \( \varepsilon \)-equilibrium mapping if for a.e. \( \xi \in \Xi \) there exists \( n_{\varepsilon, \xi} \) such that for all \( n \geq n_{\varepsilon, \xi} \)

\[
\|\alpha^n_i(\xi) - F_{\theta_i(\xi)}(\Delta(P^n(\alpha(\xi)), \cdot))\| \leq \varepsilon
\]

for all \( i = 1, ..., n \). A strategy mapping \( \alpha \) is asymptotically interior if, for a.e. \( \xi \in \Xi \),

\[
\liminf_{n \to \infty} P^n(\alpha(\xi))(o = A) > 0 \quad \text{and} \quad \limsup_{n \to \infty} P^n(\alpha(\xi))(o = A) < 1.
\]

The first property in Definition 7 requires that, for large enough \( n \), players play strategies that constitute an \( \varepsilon \) equilibrium. Our notion of limit equilibrium will require this property to hold for all \( \varepsilon > 0 \); while being slightly weaker than requiring strategies to constitute an equilibrium, this condition yields a full characterization of limit equilibrium.\(^\text{22}\) The second property requires that the probabilities of choosing \( A \) and \( B \) remain bounded away from zero as the number of players increases. We will provide a full characterization of equilibria that have such a property.

\(^{\text{21}}\)The a.e. in “for a.e. \( \xi \in \Xi \)” stands for “almost every” and means that there is a set \( \Xi' \) with \( \Phi(\Xi') = 1 \) such that a condition is true for all \( \xi \in \Xi' \). The results continue to hold if we only require \( \Phi(\Xi') > 0 \).

\(^{\text{22}}\)Our result that a limit equilibrium is a fixed point of a particular correspondence remains true under the stronger requirement that strategies constitute an equilibrium. But the converse result, that any fixed point is also a limit equilibrium, relies on the notion of \( \varepsilon \) equilibrium.
In addition to characterizing the equilibrium c-cutoff, our objective is to characterize the profile of equilibrium strategies. Given a strategy mapping $\alpha$ and a sequence of types $\xi \in \Xi$, let $\sigma^n(\xi; \alpha) : \Theta \to [0, 1]^S$ represent the average strategy of each type in the $n$-player game. Formally, for all $\theta \in \Theta$ and $s \in S$,

$$
\sigma^n(\xi; \alpha)(s) = \frac{\sum_{i=1}^n 1 \{ \theta_i(\xi) = \theta \} \alpha^n_i(\xi)(s)}{\sum_{i=1}^n 1 \{ \theta_i(\xi) = \theta \}}
$$

whenever $\sum_{i=1}^n 1 \{ \theta_i(\xi) = \theta \} > 0$, and arbitrary otherwise. We call any element $\sigma : \Theta \to [0, 1]^S$ an average strategy profile and say that $\sigma$ is increasing if $s' > s$ implies $\sigma_\theta(s') > \sigma_\theta(s)$ for every type $\theta \in \Theta$.

**Definition 8.** An average strategy profile $\sigma : \Theta \to [0, 1]^S$ is a limit $\varepsilon$-equilibrium if there exists an asymptotically interior $\varepsilon$-equilibrium mapping $\alpha$ such that

$$
\lim_{n \to \infty} \| \sigma^n(\xi; \alpha) - \sigma \| = 0 \text{ for a.e. } \xi \in \Xi.
$$

An average strategy profile $\sigma$ is a limit equilibrium if it is a limit $\varepsilon$-equilibrium for all $\varepsilon > 0$.

**Lemma 3.** Let $\alpha$ be such that $\lim_{n \to \infty} \| \sigma^n(\xi; \alpha) - \sigma \| = 0$ for a.e. $\Xi$, where $\sigma$ is increasing. Then there exists $c \in \arg \min_{\omega \in \Omega} | \kappa(\omega; \sigma) - \rho |$ such that, for a.e. $\xi \in \Xi$ and all $\omega \in \Omega$,

$$
\lim_{n \to \infty} P^n(\alpha(\xi))(o = A \mid \omega) = 1 \{ \omega > c \}.
$$

Moreover, if $c \in (-1, 1)$, then for a.e. $\xi \in \Xi$ and for all $\varepsilon > 0$ there exists $n_{\xi, \varepsilon}$ such that for all $n \geq n_{\xi, \varepsilon}$,

$$
\| \Delta_i(P^n(\alpha(\xi)), \cdot) - v_{\theta_i(\xi)} (\cdot; c) \| \leq \varepsilon
$$

for all $i = 1, \ldots, n$.

**Proof.** See the Appendix.

The intuition of Lemma 3 is as follows. Since $\sigma^n(\xi; \alpha)$ converges to $\sigma$, then the probability that a randomly chosen player votes for $A$, conditional on $\omega$, converges to $\kappa(\omega; \sigma)$. By standard asymptotic arguments, the proportion of votes for $A$ becomes concentrated around $\kappa(\omega; \sigma)$. So, for states where $\kappa(\omega; \sigma) > \rho$, the probability that $A$ is elected converges to 1. Similarly, for states where $\kappa(\omega; \sigma) < \rho$, the probability that
A is elected converges to 0. Since \( \sigma \) is increasing, then there is at most one (measure zero) state such that \( \kappa(\omega; \sigma) = \rho \), so that the election outcome and, therefore, the beliefs can be characterized by a cutoff.

We now use Lemma 3 to characterize limit equilibria.

**Theorem 2.** \( \sigma \) is a limit equilibrium if and only if there exists \( c \in (-1, 1) \) such that \( \kappa(c; \sigma) = \rho \) and \( \sigma_\theta(s) = F_\theta(v_\theta(s; c)) \) for all \( \theta \in \Theta \) and \( s \in S \).

**Proof.** Only if: Let \( \sigma \) be a limit equilibrium, so that \( \sigma \) is a limit \( \epsilon \)-equilibrium for all \( \epsilon > 0 \). Lemma OA in the Online Appendix shows that \( \sigma \) must be increasing. Fix any \( \epsilon > 0 \) and let \( \alpha \) be the corresponding \( \epsilon \)-equilibrium mapping that is asymptotically interior. By Lemma 3, equation (18) holds. Moreover, because \( \alpha \) is asymptotically interior, then \( c \in (-1, 1) \) and, therefore, (19) also holds. Then, for all \( \theta \in \Theta \), there exists \( \xi \in \Xi \) and \( n' \) such that for all \( n \geq n' \),

\[
\| \sigma_\theta - F_\theta(v_\theta(\cdot; c)) \| \leq \| \sigma_\theta - \sigma_\theta^n(\xi; \alpha) \| + \left| \frac{\sum_{i=1}^n \{ \theta_i(\xi) = \theta \} \alpha_i^n(\xi)(s)}{\sum_{i=1}^n \{ \theta_i(\xi) = \theta \}} - \frac{\sum_{i=1}^n \{ \theta_i(\xi) = \theta \} F_\theta(\Delta_i(P^n(\alpha(\xi)), s))}{\sum_{i=1}^n \{ \theta_i(\xi) = \theta \}} \right| \\
+ \left| \frac{\sum_{i=1}^n \{ \theta_i(\xi) = \theta \} F_\theta(\Delta_i(P^n(\alpha(\xi)), s))}{\sum_{i=1}^n \{ \theta_i(\xi) = \theta \}} - F_\theta(v_\theta(\cdot; c)) \right| \\
\leq \epsilon + \epsilon + \epsilon,
\]

where the first inequality follows from definitions and the second inequality follows from: (i) \( \sigma \) being a limit equilibrium implies that \( \lim_{n \to \infty} \| \sigma^n(\xi; \alpha) - \sigma_\theta \| = 0 \) for a.e. \( \xi \in \Xi \); (ii) \( \alpha \) is an \( \epsilon \)-equilibrium mapping; and (iii) equation (19) and continuity of \( F_\theta \) (A5). Since the above relationship holds for every \( \epsilon > 0 \), then \( \| \sigma_\theta - F_\theta(v_\theta(\cdot; c)) \| = 0 \) for all \( \theta \).

If: Consider the strategy mapping \( \alpha \) defined by letting players of type \( \theta \) always play \( \sigma_\theta \)-i.e., for all \( \xi, s, n, \) and \( i \leq n \), \( \alpha_i^n(\xi)(s) = \sigma_{\theta_i(\xi)}(s) \). First, note that \( \sigma^n = \sigma \) converges trivially to \( \sigma \), and \( \sigma \) is increasing because \( F_\theta \) and \( v_\theta(\cdot; c) \) are increasing (by A1-A3 and A5). Moreover, \( c \in (-1, 1) \) by assumption. Then, equation (18) in Lemma 3 and the dominated convergence theorem imply that \( \alpha \) is asymptotically interior. In addition, for a.e. \( \xi \in \Xi \) and for every \( \epsilon > 0 \), there exists \( n_{\xi, \epsilon} \) such that
for all $n \geq n_{\xi, \varepsilon}$,

$$\left\| \alpha_i^n(\xi) - F_{\theta_i(\xi)}(\Delta_i(P^n(\alpha(\xi)), \cdot)) \right\| = \left\| \sigma_{\theta_i(\xi)} - F_{\theta_i(\xi)}(\Delta_i(P^n(\alpha(\xi)), \cdot)) \right\|$$

$$= \left\| F_{\theta_i(\xi)}(v_{\theta_i(\xi)}(\cdot; c)) - F_{\theta_i(\xi)}(\Delta_i(P^n(\alpha(\xi)), \cdot)) \right\| \leq \varepsilon$$

for all $i = 1, \ldots, n$, where the first line follows by construction of the strategy and the second line follows by (19) and continuity of $F_{\theta}$ (A5). Thus, $\sigma$ is a limit equilibrium.

\[ \square \]

### 4.3 Vanishing perturbations

We now consider sequences of equilibria where the perturbations vanish. We index games by a parameter $\eta$ that affects the distribution $F^\eta_{\theta}$ from which perturbations are drawn.

**Definition 9.** A family of perturbations $\{F^\eta\}_{\eta \in \mathbb{N}}$, where $F^\eta = \{F^\eta_{\theta}\}_{\theta \in \Theta}$, is **vanishing** if for all $\theta \in \Theta$ and $\eta$: assumption A5 is satisfied and

$$\lim_{\eta \to 0} F^\eta_{\theta}(\nu) = \begin{cases} 0 & \text{if } \nu < 0 \\ 1 & \text{if } \nu > 0 \end{cases}$$

The next two results provide a foundation for the notion of (retrospective) voting equilibrium introduced in Section 2.

**Theorem 3.** Suppose that there exists a vanishing family of perturbations $\{F^\eta\}_{\eta}$ and a sequence $(\sigma^\eta, c^\eta)_\eta$ such that $\lim_{\eta \to 0}(\sigma^\eta, c^\eta) = (\sigma, c)$ and where $\sigma^\eta$ is a limit equilibrium and $c^\eta$ its corresponding cutoff for all $\eta$. Then $(\sigma, c)$ is a voting equilibrium.

**Proof.** Theorem 2 implies that $\sigma_\theta(s) = \lim_{\eta \to 0} \sigma^\eta_\theta(s) = \lim_{\eta \to 0} F^\eta_{\theta}(v_\theta(s; c))$ for all $\theta \in \Theta$ and $s \in S$. Since $F^\eta$ is vanishing, then $\sigma_\theta(s) = 1$ if $v_\theta(s; c) > 0$ and $\sigma_\theta(s) = 0$ if $v_\theta(s; c) < 0$. Therefore, $\sigma$ is optimal given $c$. Next, fix any $\omega' < c$. Since $c^\eta \to c$, there exists $\bar{\eta}$ such that, for all $\eta < \bar{\eta}$, $\omega' < c^\eta$, and, by Theorem 2, $\kappa(\omega';\sigma^\eta) \leq \rho$. Since $\sigma^\eta \to \sigma$, continuity of $\kappa(\omega';\cdot)$ implies that $\kappa(\omega';\sigma) \leq \rho$. Similarly, $\kappa(\omega'';\sigma) \geq \rho$ for all $\omega'' > c$. Therefore, $c$ is an election cutoff given $\sigma$. \[ \square \]
Theorem 4. Suppose that $(\sigma, c)$ is a voting equilibrium with $c \in (-1, 1)$. Then there exists a vanishing family of perturbations $\{F^n\}_\eta$ and a sequence $(\sigma^n, c^n)_\eta$ such that $\lim_{\eta \to 0}(\sigma^n, c^n) = (\sigma, c)$ and where $\sigma^n$ is a limit equilibrium and $c^n$ its corresponding cutoff for all $\eta$.

Proof. See the Appendix.  

We conclude by making three observations. First, situations where one alternative is never chosen (i.e., $c = -1$ or $c = 1$) are easily justified: if an alternative is never chosen, then beliefs about its performance can be arbitrary. Our solution concept in Section 2 considers $c = -1$ or $c = 1$ to be an equilibrium only if it can be approached by a sequence of limit equilibria where both alternatives are chosen and, therefore, by a sequence of non-arbitrary beliefs. Second, our game-theoretic foundation uses assumption A6, which is stronger than assumption A2 in Section 2. In particular, A2 allows for the case where voters have no private information. We can provide a foundation for such a case by considering a sequence of voting games indexed by $r \in \mathbb{N}$, where $z^r > 0$ denotes the constant defined in assumption A6, and where $\lim_{r \to \infty} z^r = 0$. Therefore, the case of no information studied in Sections 2 and 3 must be viewed as the limiting case of an information structure that satisfies A6 but where informativeness vanishes. Finally, in Section 2 we assumed that $\Theta$ was a compact interval, rather than a finite set, in order to obtain uniqueness of equilibrium and facilitate the application of the framework. However, we can view the case where $\Theta$ is a compact interval as the limiting case of a sequence of environments where the finite number of elements in $\Theta$ goes to infinity.

5 Conclusion

We provided a framework that formalizes a previously ignored feature of many elections: voters learn to make decisions by observing past outcomes, but they cannot observe the consequences of outcomes that are never chosen. In our model, people hold systematically biased beliefs, but these beliefs arise endogenously from the biased sample that derives from the aggregate behavior of all voters. The framework is easy
to apply and yields several new insights about large elections, including the extent to which information can be aggregated, the benefits of using supermajority rules for risky alternatives, the capacity of elections to perform well and mitigate costly mistakes despite boundedly rational behavior, the value of information, the relationship between observed performance and probability of being elected, the influence of aggregate preferences on individual voting behavior, the difference between local and national elections, endogenous partisanship, and the incentives for polarization in the context of political elections.

The model can be generalized in several nontrivial directions. First, we could study more general boundedly rational rules that attempt to control for selection by, for example, using learning rules that also condition on vote share (see Esponda and Pouzo (2012)). Second, given a theory of why people vote, we could extend the model to allow for abstention and study turnout. Third, we could allow for more than two alternatives and let voters learn the popularity of these alternatives and, hence, their incentives to vote strategically for a less preferred candidate that has a chance of winning (Myatt, 2007). Fourth, our simple characterization of voting outcomes in large elections draws heavily on the monotonicity assumptions on preferences and information. In settings where these assumptions fail, the model would need to be extended and the analysis would be more complicated.\footnote{Bhattacharya (2008) presents examples where monotonicity fails and, using the Nash equilibrium solution concept, shows that information aggregation may fail in those cases. In our model, the lack of monotonicity will change the conditioning events from an interval such as $W \geq c$ to a subset of $\Omega$, but complications will arise if the resulting $\kappa$ function is flat when intersecting the electoral rule.} Finally, we have abstracted away from other interesting dynamics, such as the fact that the performance of the currently-elected alternative may be influenced by the performance of the previously-elected alternative. We believe that the current model can be a good starting point for these and other future extensions.

6 Appendix

Proof of Lemma 1. $\bar{\kappa}(\cdot)$ is left-continuous: By the Dominated Convergence Theorem, it suffices to show left-continuity of $\bar{q}_\theta (c_\theta(s) < \omega \mid \omega) \equiv \sum_{\{s : c_\theta(s) < \omega\}} q_\theta (s \mid \omega)$ for all $\theta \in \Theta$. Fix any $\theta \in \Theta$. Since there are a finite number of personal cutoffs for type $\theta$ (defined by equation (2)), then for each $c \in (-1, 1)$ there exists $\omega'_\theta < c$ such that all personal cutoffs of $\theta$ are outside the interval $[\omega'_\theta, c)$. Then, for all $\hat{\omega}$,
\[ q_\theta (c_\theta (s) < \omega \mid \hat{\omega}) = q_\theta (c_\theta (s) < c \mid \hat{\omega}) \] for all \( \omega \in [\omega', c] \). In addition, \( q_\theta (s \mid \cdot) \) is continuous by A3(iii). Therefore, \( \lim_{\omega \to c} q_\theta (c_\theta (s) < \omega \mid \omega) = q_\theta (c_\theta (S_\theta) < c \mid c) \).

\( \kappa(\cdot) \) is increasing over \((c, \bar{c})\): Let \( c < \omega < \omega' < \bar{c} \). Then

\[
\int_\Theta \sum_{\{s: c_\theta (s) < \omega'\}} q_\theta (s \mid \omega') \phi(d\theta) \geq \int_\Theta \sum_{\{s: c_\theta (s) < \omega\}} q_\theta (s \mid \omega) \phi(d\theta) \\
\geq \int_\Theta \sum_{\{s: c_\theta (s) < \omega\}} q_\theta (s \mid \omega) \phi(d\theta), \tag{20}
\]

where the last inequality follows because, since \( c_\theta (\cdot) \) is nondecreasing, the event \( \{c_\theta (s) < \omega\} \) is equivalent to \( \{s \leq s_\theta (\omega)\} \) for some threshold \( s_\theta (\omega) \), and, therefore, MLRP implies that \( \sum_{\{s: c_\theta (s) < \omega\}} q_\theta (s \mid \omega') \geq \sum_{\{s: c_\theta (s) < \omega\}} q_\theta (s \mid \omega) \) (see (Milgrom, 1981)).

Next, we show that the inequality in (20) holds strictly. This is trivially true if there exists a positive \( \phi \)-measure of types with personal cutoffs in \([\omega, \omega']\), so suppose that is not the case. Since, by A4, \( c_\theta (s^L) \) is continuous in \( \theta \) and \( \Theta \) is a compact interval, the union of \( c_\theta (s^L) \) over all \( \theta \in \Theta \) is a compact interval. Given that there is no positive measure of types in \([\omega, \omega']\), then, for all \( \theta \in \Theta \), \( c_\theta (s^L) \geq \omega' > \omega \) and, therefore, \( \{c_\theta (s) < \omega\} \neq \emptyset \). Then, because MLRP holds strictly (by A2), the second inequality in (20) is strict.

Finally: If \( \omega \leq \bar{c} \), then \( \{c_\theta (s) < \omega\} = \emptyset \) for all \( \theta \), so that \( \kappa(\omega) = 0 \). Similarly, if \( \omega > \bar{c} \), then \( \{c_\theta (S_\theta) < \omega\} = \emptyset \) for all \( \theta \), so that \( \kappa(\omega) = 1 \). \( \square \)

**Proof of of Theorem 1.** The proof relies on the following claim.

**Claim 1.1** Suppose that \( \sigma \) is optimal given election cutoff \( c \). Then

\[
\kappa(\omega; \sigma) = \int_\Theta \left( \sum_{\{s: c_\theta (s) < c\}} q_\theta (s \mid \omega) + \sum_{\{s: c_\theta (s) = c\}} q_\theta (s \mid \omega) \sigma_\theta (s) \right) \phi(d\theta). \tag{21}
\]

In addition, \( \kappa(\omega) \geq \kappa(\omega; \sigma) \) for \( \omega > c \) and \( \kappa(\omega) \leq \kappa(\omega; \sigma) \) for \( \omega < c \).

**Proof.** Since \( \sigma \) is optimal given \( c \), then

\[
\sigma_\theta (s) = \begin{cases} 
0 & \text{if } c_\theta (s) > c \\
1 & \text{if } c_\theta (s) < c
\end{cases} \tag{22}
\]
and equation (21) follows. In addition, for all \( \omega > c \),

\[
\kappa(\omega; \sigma) \leq \int_{\Theta} \sum_{\{s : c_\theta(s) \leq c\}} q_\theta(s \mid \omega) \phi(d\theta) \\
\leq \int_{\Theta} \sum_{\{s : c_\theta(s) < \omega\}} q_\theta(s \mid \omega) \phi(d\theta) = \bar{\kappa}(\omega).
\]

Similarly, for all \( \omega < c \), \( \kappa(\omega; \sigma) \geq \bar{\kappa}(\omega) \).

We now prove Theorem 1. Fix \( \rho \in (0, 1) \) and let

\[
c^* \equiv \kappa^{-1}(\rho) = \inf\{\omega : \bar{\kappa}(\omega) \geq \rho\}. \tag{23}
\]

Note that, by Lemma 1, \( c^* \in [c, \bar{c}] \). We begin by showing that there exists \( \sigma^* \) such that \( (\sigma^*, c^*) \) is a voting equilibrium. Let \( \sigma^* \) satisfy (22). It remains to specify \( \sigma_\rho^*(s) \) for \( (\theta, s) \) such that \( c_\theta(s) = c^* \). First, suppose that \( c^* \notin \{-1, 1\} \). If \( c^* \) is the election cutoff, then \( (\theta, s) \) such that \( c_\theta(s) = c^* \) is indifferent between \( A \) and \( B \), and, therefore, \( \sigma_\rho^*(s) = \alpha \) is optimal for any \( \alpha \in [0, 1] \). Let \( \sigma^*_\alpha \) denote the strategy profile constructed above. We now pick \( \alpha \) such that \( c^* \) is an election cutoff given \( \sigma^*_\alpha \). Let \( \hat{\kappa}(\alpha) \equiv \kappa(c^*; \sigma^*_\alpha) \).

By Claim 1.1,

\[
\hat{\kappa}(\alpha) = \int_{\Theta} \left( \sum_{\{s : c_\theta(s) < c^*\}} q_\theta(s \mid c^*) + \sum_{\{s : c_\theta(s) = c^*\}} q_\theta(s \mid c^*) \alpha \right) \phi(d\theta),
\]

which is continuous in \( \alpha \). First, we establish that \( \hat{\kappa}(0) \leq \rho \). Suppose not, so that \( \hat{\kappa}(0) = \hat{\kappa}(c^*) > \rho \). Since \( \hat{\kappa} \) is left-continuous (Lemma 1), then there exists \( \omega' < c^* \) such that \( \hat{\kappa}(\omega') > \rho \). But then (23) is contradicted. Second, we establish that \( \hat{\kappa}(1) \geq \rho \). Suppose not, so that \( \hat{\kappa}(1) = \lim_{\omega \downarrow c^*} \hat{\kappa}(\omega) < \rho \). Then, there exists \( \omega'' > c^* \) such that \( \hat{\kappa}(\omega'') < \rho \). But, since \( \hat{\kappa}(\cdot) \) is increasing (Lemma 1), then (23) is contradicted. Since \( \hat{\kappa}(0) \leq \rho \) and \( \hat{\kappa}(1) \geq \rho \), by continuity of \( \hat{\kappa} \) there exists \( \alpha^* \) such that \( \hat{\kappa}(\alpha^*) = \kappa(c^*; \sigma^*_\alpha^*) = \rho \). Since \( \kappa(\cdot ; \sigma^*_\alpha^*) \) is nondecreasing (because \( \sigma^*_\alpha^* \) is nondecreasing), then \( c^* \) is an election cutoff given \( \sigma^*_\alpha^* \). Hence, \( (\sigma^*_\alpha^*, c^*) \) is a voting equilibrium. Next, suppose that \( c^* = -1 \) (the case \( c^* = 1 \) is similar and, therefore, omitted). Now let \( \alpha^* = 1 \); in particular, \( \sigma^*_\alpha^* \) is optimal given \( c^* \) (note it would not necessarily be optimal for a different value of \( \alpha^* \)). In addition, we just established above that \( \hat{\kappa}(1) = \kappa(c^*; \sigma^*_\alpha^*) \geq \rho \). Since \( \kappa(\cdot ; \sigma^*_\alpha^*) \) is nondecreasing, it follows that \( \kappa(\omega; \sigma^*_\alpha^*) \geq \rho \)
for all $\omega$, implying that $c^* = -1$ is a cutoff given $\sigma^*_n$.

Finally, we show that, for all $c \neq c^*$, there exists no $\sigma$ such that $(\sigma, c)$ is a voting equilibrium. Suppose, in order to obtain a contradiction, that $(\sigma, c)$ is a voting equilibrium, where $c < c^*$ (the case $c > c^*$ is similar and, therefore, omitted). Let $\omega^* \in (c, c^*)$. Then $\bar{\kappa}(\omega^*) \geq \kappa(\omega^*; \sigma) \geq \rho$, where the first inequality follows from Claim 1.1 and the second from the fact that $c$ is an election cutoff given $\sigma$. But then (23) is contradicted. □

**Proof of Lemma 3.** We use the following notation. Let $x_i \in \{A, B\}$ denote the vote of player $i$, let $\kappa^n_i(\omega; \xi) \equiv P^n (x_i = A \mid \omega)$ be the probability that player $i = 1, \ldots, n$ votes for $A$ conditional on the state being $\omega$, and let $\kappa^n(\omega; \xi) \equiv \frac{1}{n} \sum^n_{i=1} \kappa^n_i(\omega; \xi)$ be the average over all players.

First, note that, for a.e. $\xi \in \Xi$, for all $\omega \in \Omega$,

$$\lim_{n \to \infty} \kappa^n(\omega; \xi) = \lim_{n \to \infty} \frac{1}{n} \sum^n_{i=1} \sum_{\theta \in \Theta} \sum_{s \in S} q_\theta(s|\omega) 1\{\theta_i(\xi) = \theta\} \alpha_i^n(\xi)(s)$$

$$= \lim_{n \to \infty} \sum_{\theta \in \Theta} \sum_{s \in S} q_\theta(s|\omega) \left\{ \frac{1}{n} \sum^n_{i=1} 1\{\theta_i(\xi) = \theta\} \alpha_i^n(\xi)(s) \right\}$$

$$= \sum_{\theta \in \Theta} \sum_{s \in S} q_\theta(s|\omega) \left\{ \lim_{n \to \infty} \sigma^n_\theta(\xi; \alpha)(s) \times \left( \lim_{n \to \infty} \frac{1}{n} \sum^n_{i=1} 1\{\theta_i(\xi) = \theta\} \right) \right\}$$

$$= \sum_{\theta \in \Theta} \sum_{s \in S} q_\theta(s|\omega) \sigma_\theta(s) \phi(\theta) = \kappa(\omega; \sigma), \quad (24)$$

where we have used the assumption that $\lim_{n \to \infty} \|\sigma^n(\xi; \alpha) - \sigma\| = 0$ a.s.-$\Xi$ and the strong law of large numbers applied to $\frac{1}{n} \sum^n_{i=1} 1\{\theta_i(\xi) = \theta\}$.

Second, let $Y^n(\omega; \xi) \equiv n^{-1/2} \sum^n_{i=1} (1\{x^n_i = A\} - \kappa^n_i(\omega; \xi))$. Fix $\omega$ such that $\rho > \kappa(\omega; \sigma)$. Then, for a.e. $\xi$ and for all $\varepsilon > 0$, there exists $n'$ such that, for all $n \geq n'$,

$$P^n(\alpha(\xi))(o = A \mid \omega) = P^n(\alpha(\xi))(Y^n(\xi \mid \omega) \geq \sqrt{n}(\rho - \kappa^n(\xi \mid \omega)) \mid \omega)$$

$$\leq P^n(\alpha(\xi))(Y^n(\omega; \xi) \geq \sqrt{n} \varepsilon \mid \omega)$$

$$\leq (n^2 \varepsilon)^{-1} \sum^n_{i=1} E \left[ (1\{x^n_i = A\} - \kappa^n_i(\omega; \xi))^2 \mid \omega \right]$$

$$\leq 4(n \varepsilon)^{-1},$$

where the second line follows from (24) and the third from the Markov inequality.
similar argument for the case \( \rho < \kappa(\omega; \sigma) \) thus implies that, for a.e. \( \xi \in \Xi \), for all \( \omega \in \Omega \),

\[
\lim_{n \to \infty} P^n(\alpha(\xi))(o = A \mid \omega) = \begin{cases} 
0 & \text{if } \rho > \kappa(\omega; \sigma) \\
n & \text{if } \rho < \kappa(\omega; \sigma)
\end{cases}
\]

Third, the facts that \( \kappa(\cdot; \sigma) \) is increasing (because \( \sigma \) is increasing) and continuous (by A3(iii)) imply that there exists \( c \in [-1, 1] \) such that \( c \in \arg \min_{\omega \in \Omega} |\kappa(\omega; \sigma) - \rho| \) and, for a.e. \( \xi \in \Xi \) and all \( \omega \in \Omega \),

\[
\lim_{n \to \infty} P^n(\alpha(\xi))(o = A \mid \omega) = 1\{\omega > c\}.
\]

Finally, suppose that \( c \in (-1, 1) \). For all \( n \) and all \( \omega \in \Omega \),

\[
P^n(\alpha(\xi))(o = A \mid \omega) = \sum_{s \in \mathbb{S}} P^n(\xi)(o = A \mid \omega, S_i = s)q_{\theta_i(\xi)}(s \mid \omega)
\]

for all \( i \leq n \). By (25), (26), and A3(ii), for a.e. \( \xi \in \Xi \) and all \( s \in \mathbb{S} \),

\[
\lim_{n \to \infty} P^n(\alpha(\xi))(o = A \mid \omega, S_i = s) = 0 \quad (= 1)
\]

for \( \omega < c \) (\( \omega > c \)), where convergence is uniform in \( i \leq n \). Therefore, for a.e. \( \xi \in \Xi \) and all \( s \in \mathbb{S} \), \( \lim_{n \to \infty} E_{P^n(\alpha(\xi))}(u_{\theta_i(\xi)}(A, W) \mid o = A, S_i = s) = \)

\[
= \lim_{n \to \infty} \frac{\int_{\Omega} P^n(\alpha(\xi))(o = A \mid W, S_i = s)q_{\theta_i(\xi)}(s \mid W)u_{\theta_i(\xi)}(A, W)G(dW)}{\int_{\Omega} P^n(\alpha(\xi))(o = A \mid W, S_i = s)q_{\theta_i(\xi)}(s \mid W)G(dW)}
\]

\[
= \frac{\int_{\Omega} \lim_{n \to \infty} P^n(\alpha(\xi))(o = A \mid W, S_i = s)q_{\theta_i(\xi)}(s \mid W)u_{\theta_i(\xi)}(A, W)G(dW)}{\int_{\Omega} \lim_{n \to \infty} P^n(\alpha(\xi))(o = A \mid W, S_i = s)q_{\theta_i(\xi)}(s \mid W)G(dW)}
\]

\[
= \frac{\int_{\Omega} 1\{W \geq c\}q_{\theta_i(\xi)}(s|W)u_{\theta_i(\xi)}(A, W)G(dW)}{\int_{\Omega} 1\{W \geq c\}q_{\theta_i(\xi)}(s|W)G(dW)}
\]

where convergence is uniform in \( i \leq n \). The first and fourth lines in (28) follow by definition, the second line follows from the dominated convergence theorem and

\footnote{Formally, suppose that \( \omega < c \). Then for all \( \varepsilon > 0 \) there exists \( n_{\xi, \omega, \varepsilon} \) such that, for all \( n \geq n_{\xi, \omega, \varepsilon} \), \( P^n(\alpha(\xi))(o = A \mid \omega, S_i = s)q_{\theta_i(\xi)}(s \mid \omega) \leq \varepsilon \) for all \( i \leq n \) and \( s \in \mathbb{S} \).}
the fact that \( u_{\theta_i} \) is bounded (and the denominator being greater than zero, as established next), and the third line follows from (27) and the fact that \( G \) is absolutely continuous, so we can ignore the case \( \{ W = c \} \) (also, note the importance of \( c < 1 \) for the denominator to be well-defined). A similar argument holds for \( E_{P^n(\alpha(\xi))}(u_{\theta_i(\xi)}(B, W) \mid o = B, S_i = s) \), thus establishing the lemma. □

**Proof of Theorem 4.** For this proof, define

\[
\tilde{\kappa}^\eta(\omega) \equiv \sum_{\theta \in \Theta} \phi(\theta) \sum_{s \in S} q_\theta(s \mid \omega) F^n_\theta(v_\theta(s; \omega)).
\]

Let \((\sigma, c)\) be a voting equilibrium with \( c \in (-1, 1) \). Because \( c \in (-1, 1) \) is an election cutoff given \( \sigma \) and \( \kappa(\cdot; \sigma) \) is continuous, then \( \kappa(c; \sigma) = \rho \). We split the proof into two cases: Either it is the case that all players vote for the same alternative (which may be different for each player) irrespective of their private information—so that \( \kappa(\cdot; \sigma) \) is a constant function—or not—so that \( \kappa(\cdot; \sigma) \) is increasing.

**Case 1 (\( \kappa(\cdot; \sigma) \) is increasing):** Rewrite \( \tilde{\kappa}^\eta \) as

\[
\tilde{\kappa}^\eta(\omega) = \sum_{\theta \in \Theta} \phi(\theta) \left\{ \sum_{s : c_\theta(s) < c} q_\theta(s \mid \omega) F^n_\theta(v_\theta(s; \omega)) + \sum_{s : c_\theta(s) = c} q_\theta(s \mid \omega) F^n_\theta(v_\theta(s; \omega)) \right\} \equiv T^n_1(\omega) + T^n_2(\omega) + T^n_3(\omega).
\]

Since \( v_\theta(s; \cdot) \) is increasing and \( c \in (-1, 1) \), then: for all \((\theta, s)\) such that \( c_\theta(s) \geq c \), \( v_\theta(s; c) \) is increasing and \( v_\theta(s; \omega) < 0 \) for all \( \omega < c \) and, for all \((\theta, s)\) such that \( c_\theta(s) \leq c \), \( v_\theta(s; \omega) > 0 \) for all \( \omega > c \). Therefore, since \( \{ F^n_{\eta} \} \) is vanishing, \( \lim_{\eta \to 0} T^n_1(\omega) + T^n_3(\omega) = 0 \) for all \( \omega < c \) and \( \lim_{\eta \to 0} T^n_1(\omega) + T^n_2(\omega) = \sum_{\theta \in \Theta} \phi(\theta) q_\theta(c_\theta(S_\theta) \leq c \mid \omega) \geq \kappa(\omega; \sigma) \) for all \( \omega > c \). In addition, \( T^n_1(\omega) \leq \kappa(\omega; \sigma) \) and \( T^n_3(\omega) \geq 0 \) for all \( \omega \). Therefore, \( \lim_{\eta \to 0} \tilde{\kappa}^\eta(\omega) \leq \kappa(\omega; \sigma) \leq \kappa(c; \sigma) = \rho \) for all \( \omega < c \) and \( \lim_{\eta \to 0} \tilde{\kappa}^\eta(\omega) \geq \kappa(\omega; \sigma) > \kappa(c; \sigma) = \rho \) for all \( \omega > c \). Consequently, by continuity of \( \kappa(\cdot; \cdot) \), there exists \( (c^n)_{\eta} \) such that \( c^n \to c \in (-1, 1) \) and \( \tilde{\kappa}^\eta(c^n) = \rho \) for all sufficiently small \( \eta \). By letting \( \sigma^n(\theta) = F^n_\theta(v_\theta(s; c^n)) \) for all \( \theta, s \), it follows that \( \kappa(c^n; \sigma^n) = \tilde{\kappa}^\eta(c^n) = \rho \) for all sufficiently small \( \eta \) and, by Theorem 2, that \( \sigma^n \) is a limit equilibrium and \( c^n \) its corresponding cutoff for all sufficiently small \( \eta \). Finally, it remains to establish that \( \sigma^n \to \sigma \). Consider a type and signal such that \( c_\theta(s) < c \), so that \( v_\theta(s; c) > 0 \). By continuity of \( v_\theta(s; \cdot) \) and the
fact that \( c^n \to c \), it follows that \( v_\theta(s; c^n) > 0 \) for all sufficiently small \( \eta \) and, therefore, because \( \{ F^n \}_n \) is vanishing, that \( \lim_{\eta \to 0} \sigma^n_\theta(s) = 1 = \sigma_\theta(s) \), where the last equality follows since \( \sigma \) is optimal given \( c \)–see equation (22). A similar argument establishes that \( \lim_{\eta \to 0} \sigma^n_\theta(s) = 0 = \sigma_\theta(s) \) for types and signals such that \( c_\theta(s) > c \). Therefore, if \( \{ s : c_\theta(s) = c \} = \emptyset \) for all \( \theta \), we have shown that, for any family of vanishing perturbations, there exists a sequence of limit equilibria that converge to a voting equilibria. In the case where \( \{ s : c_\theta(s) = c \} \neq \emptyset \) for some \( \theta \), we construct a specific family of perturbations \( \{ F^n \}_n \) with the property that \( \lim_{\eta \to 0} F^n_\theta(v_\theta(s; c^n)) = \sigma_\theta(s) \) for all \( (\theta, s) \) such that \( c_\theta(s) = c \); the details that show existence of such a family are tedious but straightforward and are available from the authors upon request.

Case 2 (\( \kappa(\omega; \sigma) = \rho \) for all \( \omega \)): Without loss of generality, suppose that \( S_\theta \subset (0, \infty) \) for all \( \theta \). Let \( T_B = \{ (\theta, s) : v_\theta(s; c) < 0 \text{ or } (v_\theta(s; c) = 0 \& \sigma_\theta(s) = 0) \} \), \( T_A = \{ (\theta, s) : v_\theta(s; c) > 0 \text{ or } (v_\theta(s; c) = 0 \& \sigma_\theta(s) = 1) \} \), and \( T_0 = \{ (\theta, s) : v_\theta(s; c) = 0 \& \sigma_\theta(s) \in (0, 1) \} \). Note that, since \((\sigma, c)\) is a voting equilibrium, then \( \sigma_\theta(s) = 0 \) if \((\theta, s) \in T_B \) and \( \sigma_\theta(s) = 1 \) if \((\theta, s) \in T_A \). Define \( X_B \equiv \sum_{(\theta, s) \in T_B} \phi(\theta) q(s \mid c) s \geq 0 \), \( X_A \equiv \sum_{(\theta, s) \in T_A} \phi(\theta) q(s \mid c) s \geq 0 \), and \( X_0 \equiv \sum_{(\theta, s) \in T_0} \phi(\theta) q(s \mid c) s \geq 0 \). The proof constructs a specific family of perturbations. For all \( \eta \) and all \( \theta \in \Theta \) and \( s \in S_\theta \) let

\[
F^n_\theta(v_\theta(s; c)) = \begin{cases} 
\zeta_B s \eta & \text{if } v_\theta(s; c) < 0 \text{ or } (v_\theta(s; c) = 0 \& \sigma_\theta(s) = 0) \\
\sigma_\theta(s) + \zeta_0 \eta & \text{if } v_\theta(s; c) = 0 \& \sigma_\theta(s) \in (0, 1) \\
1 - \frac{\zeta_A}{s} \eta & \text{if } v_\theta(s; c) > 0 \text{ or } (v_\theta(s; c) = 0 \& \sigma_\theta(s) = 1) 
\end{cases}
\]

By construction, for all \( \zeta_j \in (0, \infty), j = A, B \) and \( \zeta_0 \in [0, \infty) \), and for all \( \eta \) sufficiently low, there exists a vanishing family \( \{ F^n \}_n \) that satisfies the above restrictions; note that, by MLRP, for each \( \theta \) there is at most one signal that satisfies \( v_\theta(s; c) = 0 \). Then, since \( c \in (-1, 1) \),

\[
\bar{\kappa}^n(c) - \rho = \bar{\kappa}^n(c) - \kappa(c; \sigma) = \sum_{(\theta, s)} \phi(\theta) q(s \mid c) (F^n_\theta(v_\theta(s; c)) - \sigma_\theta(s)) \\
= \eta (\zeta_A X_A + \zeta_B X_B + \zeta_0 X_0).
\]

It is straightforward to check that we can always pick \( \zeta_A, \zeta_B, \zeta_0 \) such that \( -\zeta_A X_A + \zeta_B X_B + \zeta_0 X_0 = 0 \) and, therefore, \( \bar{\kappa}^n(c) = \rho \) for all \( \eta \) sufficiently small. As in Case 1, by letting \( \sigma^n_\theta(s) = F^n(v_\theta(s; c^n)) \) for all \((\theta, s)\), it follows that \( \sigma^n \) is a limit equilibrium.
and \( c \) its corresponding cutoff for all sufficiently small \( \eta \). The proof is completed by noting that, by construction, \( \lim_{\eta \to 0} \sigma^n = \sigma \). □

References


Online Appendix

This online appendix contains additional proofs for “Conditional Retrospective Voting in Large Elections,” by Ignacio Esponda and Demian Pouzo.

Proof. (Proof of Proposition 2) Here, we prove the lower bound for $W$. (The proof for $\overline{W}$ is already provided in the main text.) Fix any $V' \in \Gamma(V)$; we use a prime to denote any of its associated elements. We do the proof for the case $c'_\theta(s^L_\theta) < c^{FB}$, the other case is similar. Since $c'_\theta(s^L_\theta) < c^{FB}$, the lowest possible welfare is obtained by choosing a rule that yields $c'_\theta(s^H_\theta)$. Hence, in order to bound $W$, it suffices to bound from below, $E(u'(A,W) \mid W \geq c'_\theta(s^H_\theta))$ and $E(u(B,W) \mid W \leq c'_\theta(s^H_\theta))$. Since $c'_\theta(s^H_\theta) < c^{FB}$, $E(u'(B,W) \mid W \leq c'_\theta(s^H_\theta)) \geq E(u(B,W) \mid W \leq c^{FB})$. Therefore, we only need to bound $E(u'(A,W) \mid W \geq c'_\theta(s^H_\theta))$.

As in the previous proof, it suffices to bound $E(u'(A,W) \mid W \geq c'_\theta(s^H_\theta), S_\theta = s^L_\theta)$, from below. By definition of personal cutoff,

$$
\int_{c'_\theta(s^H_\theta)}^{1} u'(A,\omega)g(\omega \mid s^H_\theta) d\omega \geq \int_{-1}^{c'_\theta(s^H_\theta)} u(B,\omega)g(\omega \mid s^H_\theta) d\omega.
$$

Since $g(\omega|s) = \frac{q(s|\omega)g(\omega)}{\int q(s|\omega)g(\omega) d\omega}$, under assumption A3(ii), for any $s$ and $\omega$, it follows

$$
gg(\omega) \leq g(\omega|s) \leq q^{-1}g(\omega).
$$

Hence, by integration by parts,

$$
\int_{c'_\theta(s^H_\theta)}^{1} u'(A,\omega)gg(\omega) d\omega - \int_{c'_\theta(s^H_\theta)}^{1} u'(A,\omega)gg(\omega \mid s^H_\theta) d\omega =
$$

$$
u'(A, \omega) \left( \int_{c'_\theta(s^H_\theta)}^{\omega} (gg(v) - g(v|s^H_\theta)) dv \right) \bigg|_{c'_\theta(s^H_\theta)}^{1}
$$

$$
- \int_{c'_\theta(s^H_\theta)}^{1} \frac{du}{d\omega}(A, \omega) \left\{ \int_{c'_\theta(s^H_\theta)}^{\omega} (gg(v) - g(v|s^H_\theta)) dv \right\} d\omega.
$$

By assumption A1, $u(A, \cdot)$ is nondecreasing, also $\int_{c'_\theta(s^H_\theta)}^{\omega} (gg(v) - g(v|s^H_\theta)) dv \leq 0$;
thus, the second term in the RHS is negative. Therefore

\[
\int_{c_i^0 (s_i^H)}^{1} u'(A, \omega)q^{-1}g(\omega)d\omega \geq \frac{1}{q^2} \int_{-1}^{c_i^0 (s_i^0) - 1} u(B, \omega)g(\omega | s_i^H) d\omega + \frac{u(A, 1)}{q^2} \left( \int_{c_i^0 (s_i^H)}^{1} (g(v | s_i^0) - g(v | s_i^H)) dv \right)
\] (29)

where the RHS follows from the fact that \( u'(A, 1) = u(A, 1) \) and from the first inequality in this step.

By similar algebra and since \( \int_{c_i^0 (s_i^H)}^{\omega} (g(v | s_i^0) - q^{-1}g(v)) dv \leq 0 \), it follows that

\[
\int_{c_i^0 (s_i^H)}^{1} u'(A, \omega)g(\omega | s_i^0) d\omega - \int_{c_i^0 (s_i^H)}^{1} u'(A, \omega)q^{-1}g(\omega)d\omega \geq u(A, 1) \left( \int_{c_i^0 (s_i^H)}^{1} (g(v | s_i^0) - q^{-1}g(v)) dv \right).
\]

Hence, after some algebra, it can be shown that this inequality and the one in equation 29, imply

\[
\int_{c_i^0 (s_i^H)}^{1} u'(A, \omega)g(\omega | s_i^0) d\omega \geq u(A, 1) \left( \int_{c_i^0 (s_i^H)}^{1} (g(v | s_i^0) - q^{-2}g(v | s_i^H)) dv \right) + \frac{1}{q^2} \int_{-1}^{c_i^0 (s_i^0) - 1} u(B, \omega)g(\omega | s_i^0) d\omega.
\]

It is easy to see that the RHS is bounded from below by some finite constant. Dividing at both sides by \( \int_{c_i^0 (s_i^0)}^{1} g(\omega | s_i^0) d\omega \) (which is bounded below from below by \( q \int_{c_i^0}^{1} g(\omega) d\omega > 0 \)) we obtain the desired result. \( \square \)

**Lemma OA.** There exists \( \varepsilon \) such that for all \( \varepsilon < \varepsilon \): If \( \sigma \) is a limit \( \varepsilon \)-equilibrium, then it is increasing.

**Proof:** Throughout the proof let \( \Xi' \) be the set in Definition 8 and fix \( \xi \in \Xi' \) and a strategy mapping \( \overline{\sigma} \) such that 1.-3. in Definition 8 are satisfied. We drop \( \xi \) and \( \overline{\sigma} \) from the notation, let \( P^n \equiv P^n (\overline{\sigma} (\xi)) \) and, for each strategy \( \alpha^n_\iota \), let \( P^n_{\alpha_\iota} \equiv P^n (\alpha^n_\iota, \overline{\alpha}^n_\iota (\xi)) \). The proof relies on the following claims; the proofs of the first three claims appear at the end of this section.

**Claim OA.1:** For all \( \delta > 0 \) and \( \omega \in \Omega \), there exists \( n_{\delta, \omega} \) such that for all \( n \geq n_{\delta, \omega} \),

\[
\left| P^n_{\alpha_\iota} (o = A | \omega, s_i) - P^n_{\alpha'_\iota} (o = A | \omega, s'_i) \right| < \delta \quad \text{uniformly over } i, s_i, s'_i, \alpha^n_\iota, \alpha'^n_\iota.
\]
Claim OA.2: For all $\delta > 0$ there exist $n_\delta$ such that for all $n \geq n_\delta$, $|\Delta_i(P^n, s_i) - \Delta_i(P^n_{\alpha_i}, s_i)| < \delta$ uniformly over $i, s_i, \alpha_i^n$.

Claim OA.3: There exists $c > 0$ and $n_c$ such that for all $n \geq n_c$, $\Delta_i(P^n_{\alpha_i}, s_i') - \Delta_i(P^n_{\alpha_i}, s_i) \geq c$ for all $i$ and $s_i' > s_i$ such that $\alpha_i^n(s_i') = \alpha_i^n(s_i)$.

Claim OA: There exists $c' > 0$ and $n_{c'}$ such that for all $n \geq n_{c'}$, $\Delta_i(P^n, s_i') - \Delta_i(P^n, s_i) \geq c'$ for all $i$ and $s_i' > s_i$.

Proof of Claim OA. Fix any $\alpha_i^n$ such that $\alpha_i^n(s_i') = \alpha_i^n(s_i)$. By Claims OA.2 and OA.3, for all $n \geq \max\{n_c, n_\delta\}$

$$\Delta_i(P^n, s_i') - \Delta_i(P^n, s_i) \geq (\Delta_i(P^n_{\alpha_i}, s_i') - \delta) - (\Delta_i(P^n_{\alpha_i}, s_i) + \delta)$$

$$\geq c - 2\delta.$$  

The claim follows by setting $\delta = c/4$ and $c' = c/2 > 0$. \hfill \qed

Proof of Lemma OA. The definition of $\varepsilon$-equilibrium implies that for all $i$, $s_i' > s_i$, $n \geq n_\varepsilon$,

$$\bar{\alpha}_i^n(s_i') - \bar{\alpha}_i^n(s_i) \geq F_{\tilde{\theta}_i}(\Delta_i(P^n, s_i')) - F_{\tilde{\theta}_i}(\Delta_i(P^n, s_i)) - 2\varepsilon.$$  

$$+ F_{\tilde{\theta}_i}(\Delta_i(P^n, s_i) + c') - F_{\tilde{\theta}_i}(\Delta_i(P^n, s_i) + c'),$$  

$$\tag{30}$$

where we have added and subtracted the same term to the RHS. Let $c' > 0$ be as defined in Claim OA. Since $F_{\tilde{\theta}_i}$ is absolutely continuous, then

$$F_{\tilde{\theta}_i}(\Delta_i(P^n, s_i) + c') - F_{\tilde{\theta}_i}(\Delta_i(P^n, s_i)) = \int_{\Delta_i(P^n, s_i)}^{\Delta_i(P^n, s_i) + c'} f_{\tilde{\theta}_i}(t) \, dt \geq c'' > 0,$$

where the inequality follows from A5 and the fact that $c' > 0$. Hence, the sum of the second and fourth terms in the RHS of (30) is at least $c'' > 0$. By Claim OA, the sum of the first and last terms in the RHS of (30) is positive. Therefore, for all $i$, $s_i' > s_i$, $n \geq n_\varepsilon$,

$$\bar{\alpha}_i^n(s_i') - \bar{\alpha}_i^n(s_i) \geq c'' - 2\varepsilon > 0.$$  

Since $\sigma_\theta^n(\xi, \alpha)$ are averages of the strategies, then for all $\theta, s' > s$, and $n \geq n_\varepsilon$, it follows that $\sigma_\theta^n(s') - \sigma^n(s) \geq c'' - 2\varepsilon$. Since $\lim_{n \to \infty} \|\sigma^n - \sigma\| = 0$, then it follows that
σ_θ(s') - σ_θ(s) ≥ ε'' - 2ε > 0, thus establishing that limit ε-equilibrium are increasing as long as 0 < ε < ε ≡ ε''/2 > 0.

Proof of Claim OA.1. The proof is divided into 3 steps.

Step 1. We first show that the probability of being pivotal goes to zero; i.e., for all ω ∈ Ω, for all i, lim_{n→∞} Piv_n^{ω,i} = 0, where

\[ Piv_n^{ω,i} \equiv P_n^1(o = A | ω) - P_n^0(o = A | ω), \]

where the “1” and “0” are understood as vectors of the same dimension as α_i. The sub-index “i” indicates that agent i is the one being pivotal.

By simple algebra,

\[ Piv_n^{ω,i} = P_n(\sqrt{n}K_n^{ω} + \frac{κ_n^{ω,i} - 1}{V_n^{ω}/\sqrt{n}} + \frac{Z_n^{ω,i}}{\sqrt{n}} \leq \sum_{j=1}^{n} \frac{Z_j^{ω,i}}{\sqrt{n}} < \sqrt{n}K_n^{ω} + \frac{κ_n^{ω,i} - 1}{V_n^{ω}/\sqrt{n}} + \frac{Z_n^{ω,i}}{\sqrt{n}} | ω), \]

where \( Z_j^{ω,i} \equiv \frac{(x_j^{ω} = A) - κ_n^{ω,i} j}{V_n^{ω}/\sqrt{n}}, \) \( V_n^{ω} \equiv \sqrt{\frac{1}{n} \sum_{j=1}^{n} κ_n^{ω,i} j (1 - κ_n^{ω,i} j)}, \) and \( K_n^{ω} \equiv \frac{ρ - κ_n^{ω} j}{V_n^{ω}}. \) Note that, for a given n, \( \{Z_n^{ω,i}\}_j \) are independent, they have zero mean and unit variance. Moreover, by Step 3 below, \( \lim inf_{n→∞} V_n^{ω} > 0, \) so that

\[ \sum_{j=1}^{n} E \left[ \frac{|Z_j^{ω,i}|^3}{\sqrt{n}} \right] \leq \frac{2}{\sqrt{n} (V_n^{ω})^3} \to 0 \text{ as } n \to \infty, \]

Hence by Lindeberg-Feller CLT, it follows that, given ω, \( \sum_{j=1}^{n} \frac{Z_j^{ω,i}}{\sqrt{n}} \Rightarrow N(0,1) \) as \( n \to \infty. \)

Note also that, \( \frac{Z_n^{ω,i}}{\sqrt{n}} \to 0 \) a.s. as \( n \to \infty \) and this limit is uniform on i.

We divide the remainder of the proof in 3 cases: (a) \( \sqrt{n}K_n^{ω} \to -∞, \) (b) \( \sqrt{n}K_n^{ω} \to K \in (-∞, ∞) \) or (c) \( \sqrt{n}K_n^{ω} \to ∞ \) (if necessary, we take a subsequence that converges, which exists since \( (V_n^{ω}(ξ))_n \) and \( (κ_n^{ω}(ξ))_n \) are uniformly bounded).

We first explore case (a) (case (c) is symmetrical). Note that, since \( \lim inf_{n→∞} V_n^{ω} > 0, \) then \( \frac{κ_n^{ω,i} - 1}{V_n^{ω}/\sqrt{n}} \to 0. \) Therefore, \( \sqrt{n}K_n^{ω} + \frac{κ_n^{ω,i} - 1}{V_n^{ω}/\sqrt{n}} + \frac{Z_n^{ω,i}}{\sqrt{n}} \to -∞, \) (and this limit holds uniformly for \( i = 1, ..., n \)) so that we can take \( n \geq n_{M,ε} \) such that \( \sqrt{n}K_n^{ω} + \frac{κ_n^{ω,i} - 1}{V_n^{ω}/\sqrt{n}} + \frac{Z_n^{ω,i}}{\sqrt{n}} \leq -M, \) where \( \mathcal{L}_N(-M) < 0.5ε \) (where \( \mathcal{L}_N \) is the standard Gaussian cdf) for any \( ε. \)
Therefore, for all $\epsilon > 0$ there exists $n_{\epsilon, \omega}$ such that for all $n \geq \max\{n_{\epsilon, \omega}, n_{M, \epsilon}\}$:

$$\Pi v_{\omega, i}^n \leq P^n \left( \frac{\sum_{j=1}^{n} Z_{j\omega}^n}{\sqrt{n}} < -M \mid \omega \right) \leq 0.5\epsilon + L_N(-M) < \epsilon$$

uniformly over $i = 1, \ldots, n$, where the first inequality follows from the fact that $n \geq n_{M, \epsilon}$ and the second follows from CLT and our choice of $M$.

For case (b) (i.e., $K$ finite). Let $\delta > 0$ be such that $L_N(K+\delta) - L_N(K-\delta) < 0.5\epsilon$. Note that since $\lim_{n \to \infty} (V_{\omega}^n \sqrt{n})^{-1} = 0$, there exists a $n_{\delta, \omega}$ such that $(V_{\omega}^n \sqrt{n})^{-1} < 0.5\delta$ for all $n \geq n_{\delta, \omega}$; also since $\frac{Z^n_{\omega}}{\sqrt{n}} \to 0$ a.s. as $n \to \infty$, we can take $n_{\delta, \omega}$ such that $\left| \frac{Z^n_{\omega}}{\sqrt{n}} \right| < 0.5\delta$ (note that $n_{\delta, \omega}$ does not depend on $i$ since convergence is uniform on $i$). Then, it follows for all $\epsilon > 0$, there exists $n_{\epsilon, \omega}$ such that for all $n \geq \max\{n_{\epsilon, \omega}, n_{\delta, \epsilon}\}$:

$$\Pi v_{\omega, i}^n \leq P^n \left( \sqrt{n} K_{\omega}^n - \frac{1}{V_{\omega}^n \sqrt{n}} + \frac{Z_{\omega}^n}{\sqrt{n}} \leq \frac{\sum_{j=1}^{n} Z_{j\omega}^n}{\sqrt{n}} < \sqrt{n} K_{\omega}^n + \frac{1}{V_{\omega}^n \sqrt{n}} + \frac{Z_{\omega}^n}{\sqrt{n}} \mid \omega \right)$$

$$\leq P^n \left( K - \delta < \frac{\sum_{j=1}^{n} Z_{j\omega}^n}{\sqrt{n}} \leq K + \delta \mid \omega \right)$$

$$\leq 0.5\epsilon + L_N(K + \delta) - L_N(K - \delta) < \epsilon,$$

where the third inequality follows from the CLT. We showed that for any convergent subsequence $(K_{\omega}^n)_n$, the associated subsequences of probabilities converge to zero, thus this result must hold for the whole sequence.

**Step 2.** Note that:

$$P^n_{\alpha_i} (o = A \mid \omega, s_i) = \alpha^n_i(s_i) P^n_1 (o = A \mid \omega) + (1 - \alpha^n_i(s_i)) P^n_0 (o = A \mid \omega)$$

$$= P^n (o = A \mid \omega)$$

$$+ \alpha^n_i(s_i) (P^n_1 (o = A \mid \omega) - P^n_0 (o = A \mid \omega))$$

$$= P^n (o = A \mid \omega) + \alpha^n_i(s_i) \Pi v_{\omega, i}^n$$

Therefore

$$|P^n_{\alpha_i} (o = A \mid \omega, s_i) - P^n_{\alpha_i'} (o = A \mid \omega, s_i')| \leq |\alpha^n_i(s_i) - \alpha^n_i'(s_i)| \cdot |\Pi v_{\omega, i}^n|.$$

By step 1, it follows that for all $n \geq n_{\delta, \omega}$: $|\Pi v_{\omega, i}^n| \leq \delta$. Since $|\alpha^n_i(s_i) - \alpha^n_i'(s_i)| \leq 1$
the desired result follows.

**Step 3.** We now show that for all $\omega \in \Omega$,

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \kappa_{j,\omega}^n (1 - \kappa_{j,\omega}^n) > 0. \quad (31)$$

Fix any $n$ and $j \leq n$. By assumption, $\alpha_j^n (s_j) \in [F_j (-2K), F_j (2K)] \subset (0, 1)$ for all $s_j$. Therefore, $0 < \kappa_{j,\omega}^n < 1$ for all $\omega$, thus implying equation $(31)$.

**Proof of Claim OA.2.** We prove that

$$\lim_{n \to \infty} \left( E_{P^n} (u_{\theta_i}(A, W) \mid o = A, S = s_i) - E_{P^n_{\alpha_i}} (u_{\theta_i}(A, W) \mid o = A, S = s_i) \right) = 0;$$

the proof for $o = B$ is similar and therefore omitted. We first show that, for all $i, s_i, \alpha_i$,

$$E_{P^n_{\alpha_i}} (u_{\theta_i}(A, W) \mid o = A, S = s_i) = \int_{\Omega} P^n_{\alpha_i} (o = A \mid W, s_i) q_{\theta_i}(s_i \mid W) u_{\theta_i}(A, W) G(dW) \int_{\Omega} P^n_{\alpha_i} (o = A \mid W, s_i) q_{\theta_i}(s_i \mid W) G(dW)$$

is well-defined for sufficiently large $n$. Fix any $i$. A3(ii) and the fact that $\bar{\alpha}$ is asymptotically interior imply that there exists $\bar{n}$ such that for all $n \geq \bar{n}$, there exists $s_i^*$ such that

$$P^n (o = A, s_i^*) = \int_{\Omega} P^n (o = A \mid W, s_i^*) q_{\theta_i}(s_i^* \mid W) G(dW) \geq c > 0,$$

which implies that $\int_{\Omega} P^n (o = A \mid W, s_i^*) G(dW) \geq c > 0$. By Claim OA.1, for each $s_i, \alpha_i^n$, $P^n (o = A \mid \omega, s_i^*) - P^n_{\alpha_i} (o = A \mid \omega, s_i)$ converges to zero as $n \to \infty$. Since both probabilities are bounded by one, then the dominated convergence theorem implies that $\int_{\Omega} (P^n (o = A \mid W, s_i^*) - P^n_{\alpha_i} (o = A \mid W, s_i)) G(dW) \to 0$ as $n \to \infty$, uniformly over $\alpha_i$. Therefore, there exists $n_{.5c}$ such that $\sup_{\alpha_i} \left| \int_{\Omega} [P^n (o = A \mid W, s_i^*) - P^n_{\alpha_i} (o = A \mid W, s_i)] G(dW) \right| < .5c$ for all $n \geq n_{.5c}$. So for all $n \geq \max \bar{n}, n_{.5c} \equiv \bar{n}_c$,

$$\int_{\Omega} P^n_{\alpha_i} (o = A \mid W, s_i) q_{\theta_i}(s_i \mid W) G(dW) \geq d \int_{\Omega} P^n_{\alpha_i} (o = A \mid W, s_i) G(dW) > .5dc > 0.$$

Hence, $E_{P^n_{\alpha_i}} (u_{\theta_i}(A, W) \mid o = A, S = s_i)$ is well defined.

By simple algebra, and letting $\Delta P^n_{\alpha_i} (A, \omega, s_i) \equiv P^n (o = A \mid \omega, s_i) - P^n_{\alpha_i} (o = A \mid \omega, s_i)$,
\[ E_{P^n}(u_{\theta_i}(A,W) \mid o = A, S = s_i) - E_{P^n}(u_{\theta_i}(A,W) \mid o = A, S = s_i) \]
\[ \leq \left| \int_{\Omega} \Delta P^n_{o_i}(A,W,s_i)q_{\theta_i}(s_i \mid W)u_{\theta_i}(A,W)G(dW) \right| \]
\[ + \frac{\int_{\Omega} \Delta P^n_{o_i}(A,W,s_i)q_{\theta_i}(s_i \mid W)G(dW)\int_{\Omega} P^n(o = A \mid W)q_{\theta_i}(s_i \mid W)u_{\theta_i}(A,W)G(dW)}{\int_{\Omega} P^n(o = A \mid W)q_{\theta_i}(s_i \mid W)G(dW)} \]

To establish the desired result, it is sufficient to show that each of the two absolute value terms in the numerator of the second and third line converge to zero as \( n \to \infty \). However, this result follows by the dominated convergence theorem since \( |u_{\theta_i}(A,\omega)| < K, q_{\theta_i}(s \mid \omega) \leq 1 \), and pointwise convergence (for each \( \omega \)) is obtained by Claim OA.1.

\[ \square \]

**Proof of Claim OA.3.** For each \( O \in \{A, B\} \): Let \( g^n_{\alpha_i}(\omega \mid O, s_i) \equiv P^n_{\alpha_i}(d\omega \mid o = O, s_i) \) denote the density of \( \omega \) conditional on \( o = O \) and \( s_i \), and let \( G^n_{\alpha_i}(\omega \mid O, s_i) \equiv P^n_{\alpha_i}(\{W \leq \omega\} \mid o = O, s_i) \) denote the cdf. Let \( \Delta g^n_{\alpha_i}(\omega \mid O, s_i) \equiv g^n_{\alpha_i}(\omega \mid O, s'_i) - g^n_{\alpha_i}(\omega \mid O, s_i) \) and \( \Delta G^n_{\alpha_i}(\omega \mid O, s_i) \equiv G^n_{\alpha_i}(\omega \mid O, s'_i) - G^n_{\alpha_i}(\omega \mid O, s_i) \).

Then

\[ \Delta_i(P^n_{\alpha_i}, s'_i) - \Delta_i(P^n_{\alpha_i}, s_i) = \int_{\Omega} (u_{\theta_i}(A,W)\Delta g^n_{\alpha_i}(W \mid A, s'_i, s_i) - u_{\theta_i}(B,W)\Delta g^n_{\alpha_i}(W \mid B, s'_i, s_i)) dW \]
\[ = \int_{\Omega} \left( \frac{du_{\theta_i}(A,W)\Delta G^n_{\alpha_i}(W \mid A, s_i, s'_i)}{d\omega} - \frac{du_{\theta_i}(B,W)\Delta G^n_{\alpha_i}(W \mid B, s_i, s'_i)}{d\omega} \right) dW \]
\[ \geq \int_{\Omega^n \subset \Omega} \frac{du_{\theta_i}(A,W)\Delta G^n_{\alpha_i}(W \mid A, s_i, s'_i)}{d\omega} dW \]
\[ \geq c_M \int_{\Omega^n \subset \Omega} \frac{du_{\theta_i}(A,W)\Delta G^n_{\alpha_i}(W \mid A, s_i, s'_i)}{d\omega} dW \]
\[ \geq c_m \cdot c_M \inf_{O \in \{A, B\}, W \in \Omega} \frac{du_{\theta_i}(A,W)}{d\omega} \]
\[ \equiv c > 0 \]

for all \( n \geq n' \) (where \( \Omega^n, c_m, c_M > 0 \), and \( n' \) are all defined in Claim OA.3.1 below), where the first line follows by definition, the second by integration by parts (note how the signals are inverted), the third by Claim OA.3.1(i) (see below) and the facts that \( \frac{du_{\theta_i}(A,\omega)}{d\omega} > 0 \) and \( \frac{du_{\theta_i}(B,\omega)}{d\omega} < 0 \) for all \( \omega \), the fourth by Claim OA.3.1(ii).
Finally, for the fifth line, let $\Omega = \Omega \setminus \cup_{i=1}^{N}(\omega_i - \epsilon, \omega_i + \epsilon)$ where $(\omega_1, ..., \omega_N)$ are the discontinuity points of $\frac{du_{\theta}}{du}(A, \cdot)$; by assumption there are finitely many, so $N < \infty$ and $\epsilon > 0$ is chosen such that $\epsilon < \min_{i \neq j} |\omega_i - \omega_j|$. It is easy to see that $\Omega$ is compact and over it, $\frac{du_{\theta}}{du}(A, \cdot)$ is well-defined and continuous. Since $c_m \cdot c_M > 0$ and $\inf_{\omega \in \Omega} \frac{du_{\theta}}{du}(A, \omega) = \min_{\omega \in \Omega} \frac{du_{\theta}}{du}(A, \omega) > 0$ where (because $u_{\theta_i}$ is continuously differentiable in $\Omega$ and $\frac{du_{\theta}}{du}(A, \omega) > 0$ for all $\omega$).

**Claim OA.3.1:** For all $i$ and $s_i' > s_i$ such that $\alpha_i^n(s_i) = \alpha_i^n(s_i')$: (i) For all $n$, $\Delta G^n_{\alpha_i}(\omega \mid O, s_i, s_i') \geq 0$ for all $\omega$ and $O \in \{A, B\}$; (ii) There exists $n'$ and $(\Omega^n)_n$ with $\Omega^n = [l_n, u_n] \subseteq \Omega$ and $\lim_{n \to \infty} u_n - l_n = \beta_2 > 0$ such that for all $n \geq n'$ and all $\omega^* \in \Omega^n \setminus \{-1, 1\}$,

$$\Delta G^n_{\alpha_i}(\omega \mid A, s_i, s_i') \geq C_M > 0.$$

**Proof of Claim OA.3.1.** There exists $z > 0$ such that for all $n$ and all $\omega' > \omega$,

$$g^n_{\alpha_i}(\omega' \mid O, s_i')g^n_{\alpha_i}(\omega \mid O, s_i) - g^n_{\alpha_i}(\omega' \mid O, s_i)g^n_{\alpha_i}(\omega \mid O, s_i')
\geq z \frac{P^n_{\alpha_i}(O \mid \omega', s_i')P^n_{\alpha_i}(O \mid \omega, s_i)g(\omega)g(\omega')}{{P^n_{\alpha_i}(O, s_i')P^n_{\alpha_i}(O, s_i)}}
\geq 0$$

(32)

where the first line uses the fact that $P^n_{\alpha_i}(O \mid \hat{\omega}, s_i) = P^n_{\alpha_i}(O \mid \hat{\omega}, s_i')$ for all $\hat{\omega}$ (because of conditional independence and the fact that $\alpha_i^n(s_i) = \alpha_i^n(s_i')$), the second line follows from A6, and the third line follows because $z > 0$ and $\omega' > \omega$. Therefore, it follows from Milgrom (1981, Proposition 1) that, for all $n$, $\Delta G^n_{\alpha_i}(\omega \mid O, s_i, s_i') \geq 0$ for all $\omega$.

(ii) From the proof of Claim OA.2, there exists $n'$ and $c' > 0$ such that, for all $n \geq n'$,

$$\int_{\Omega} P^n_{\alpha_i}(o = A \mid W, s_i)G(dW) \geq c'$$

for all $i, \alpha_i, s_i$. For $a \in (0, 1)$, let

$$\omega^n_a = \min \left\{ \omega' : \int_{W \leq \omega'} P^n_{\alpha_i}(o = A \mid W, s_i)G(d\omega) \geq a \cdot c' \right\} \in \Omega.$$
Fix any $n \geq n'$. Then
\[ c'/4 = \int_{\omega_{0.25}^n \leq W \leq \omega_{0.50}^n} P^m_{\alpha_1}(o = A \mid W, s_i) G(dW) \leq G(\omega_{0.50}^n) - G(\omega_{0.25}^n). \]

Therefore the fact that $G$ has no mass points implies that $\omega_{0.50}^n - \omega_{0.25}^n = c_L > 0$. A similar argument establishes that $\omega_{0.75}^n - \omega_{0.50}^n \geq c_R > 0$.

Let $\Omega^n = [\omega_{0.50}^n - c_m/2, \omega_{0.50}^n + c_m/2]$, where $c_m \equiv \min\{c_L, c_R\} > 0$. Then, $u_n - l_n = c_m > 0$. In addition, fix any $\omega^* \in \Omega^n$. Then, by construction,
\[ \int_{\omega < \omega^* - c_m/2} P^m_{\alpha_1}(o = A \mid W, s_i) G(dW) \geq c'/4 \quad (33) \]
and
\[ \int_{\omega > \omega^* + c_m/2} P^m_{\alpha_1}(o = A \mid W, s_i) G(dW) \geq c'/4. \quad (34) \]

By integrating each side of (32) twice, first with respect to $G(d\omega)$ over $\omega \leq \omega^*$ and second with respect to $G(d\omega')$ over $\omega' > \omega^*$, we obtain
\[ \Delta E^n_{\alpha}(\omega^* \mid A, s_i, s'_i) = \]

where the first inequality follows from $P^n_{\alpha_1}(A, s'_i) P^n_{\alpha_1}(A, s_i) \leq 1$, the second from A3, and the third from (33) and (34). □