Optimal Dynamic Matching

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Abstract

We study a dynamic matching environment where individuals arrive sequentially. There is a tradeoff between waiting for a thicker market, allowing for higher quality matches, and minimizing agents' waiting costs. The optimal mechanism cumulates a stock of incongruent pairs up to a threshold and matches all others in an assortative fashion instantaneously. In discretionary settings, a similar protocol ensues in equilibrium, but expected queues are inefficiently long. We quantify the welfare gain from centralization, which can be substantial, even for low waiting costs. We also evaluate welfare improvements generated by transfer schemes, and alternative priority protocols.

Keywords: Dynamic Matching, Mechanism Design, Organ Donation, Market Design.

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1 Introduction

1.1 Overview

Many matching processes are inherently dynamic, with participants arriving and matches being created over time. For instance, in the child-adoption process, parents and children arrive steadily—data from one U.S. adoption facilitator who links adoptive parents and birth mothers willing to relinquish children for adoption indicates a rate of about 11 new potential adoptive parents and 13 new birth mothers entering the facilitator's operation each month.[1] While the overall statistics on the entry of parents and children into the U.S. adoption process are not well-documented, adoption touches upon many lives: The Census 2010 indicates that about 1.5 million or 2.4 percent of all children have been adopted. Likewise, many labor markets entail unemployed workers and job openings that become available at different periods—the U.S. Bureau of Labor Statistics reports approximately five million new job openings and slightly fewer than five million newly unemployed workers each month this year. A similar picture emerges when considering organ donation. According to the Organ Donation and Transplantation Statistics, a new patient is added to the kidney transplant list every 14 minutes and about $3,000$ patients are added to the kidney transplant list each month. A significant fraction of transplants are carried out using live donors—in 2014, about a third of approximately $17,000$ kidney transplants that took place in the U.S. involved such donors.

Nonetheless, by and large, the extant matching literature has taken a static approach to market design—participants all enter at the same time and the market's operations are restricted in their horizon (see the literature review below for several important exceptions). In the current paper, we offer techniques for extending that approach to dynamic settings.

All of the examples mentioned above share two important features. First, match quality varies and agents care about whom they match with. Second, waiting for a match is costly, be it for financial costs of keeping lawyers on retainer for potential adoptive parents, children's hardship from growing older in the care of social services, the lack of wages and needed employees in labor markets, or medical risks for organ patients and psychological waiting costs for donors. These two features introduce a crucial trade-off. On the one hand, a thick market can help generate a greater match surplus; on the other hand, a thin market allows for

[1] This adoption facilitator is one of 25 registered in its state of operation. See Baccara, Collard-Wexler, Felli, and Yariv (2014) for details.

quicker matching and cuts down on waiting costs. The goal of this paper is to characterize the resolution of this trade-off in both centralized and relatively more decentralized environments. Namely, we identify the optimal protocol by which a social planner would match agents over time. We also identify conditions under which discretionary matching processes would especially benefit from centralized intervention using the optimal protocol.

Specifically, we consider a market that evolves dynamically. There are two classes of agents, which we refer to as "squares" and "rounds." At each period, a pair consisting of a square and a round enters the market. Squares and rounds each have two types, one type more desirable than the other. For instance, if we think of squares and rounds as children relinquished for adoption and potential adoptive parents, types can stand for gender of children and wealth levels of potential adoptive parents, respectively (see Baccara, Collard-Wexler, Felli, and Yariv, 2014). Alternatively, if we think of the two classes of agents as workers and firms, worker types can correspond to skills and firm types can correspond to various benefit packages offered. We assume that preferences are super-modular so that the (market-wide) assortative matching maximizes joint welfare. We also assume that, once agents arrive at the market, waiting before being matched comes at a per-period cost.

We start by analyzing the optimal matching mechanism in such settings, the mechanism that maximizes the expected per-period payoffs for market participants. We show that the optimal mechanism takes a simple form. Whenever congruent pairs of agents—a square and round that are of the same type—are present in the market, they are matched instantaneously. When only incongruent agents are present in the market, they are held in a queue. When the stock of incongruent pairs in the queue exceeds a certain threshold, they are matched in sequence, until the queue length falls back within the threshold. Such thresholds induce a Markov process, where states correspond to the length of queues of incongruent pairs of agents. Any threshold yields a different steady-state distribution over possible queue lengths. We evaluate the expected welfare of such threshold mechanisms in the steady-state. The optimal mechanism utilizes the threshold that maximizes welfare. When waiting costs are vanishingly small, the welfare under the optimal mechanism approaches the maximum feasible, that generated by no matches of incongruent pairs. As waiting costs increase, the welfare generated by the optimal mechanism decreases.

This welfare decrease raises the question of the value of dynamic clearinghouses for non-trivial waiting costs in different environments, identified by type distributions and preferences.

We therefore study the performance of a simple discretionary matching process in our setting. As before, we consider agents arriving at the market in sequence. At each period, agents in the market declare their willingness to match with partners of either type. After these demands have been made, the maximal number of pairs of willing agents are matched in order of arrival (*first-in-first-out*, or FIFO protocol).[2] Those who prefer to stay in the market, or have to stay for lack of willing partners, form a queue.[3] In our environment, desirable individuals waiting in the market impose three types of externalities. First, they impose a longer wait and potentially missed desirable matches on those that follow them in the queue. Second, they prevent undesirable agents present in the market from matching immediately. Last, they also impose a positive externality on desirable agents on the other side of the market, who are more likely to get a quicker match with a partner they prefer. As it turns out, the negative externalities of waiting overwhelms this positive externality and leads to excessive waiting in the discretionary setting. In fact, the matching protocol induced by equilibrium in the discretionary matching process ends up resembling the protocol corresponding to the optimal mechanism, but with higher thresholds for the queues' lengths.

We evaluate the difference in welfare generated by a centralized and a discretionary process as a function of the underlying primitives of the environment, namely the agents' type distribution and the cost of waiting.

With respect to the type distribution, as the frequency of desirable types increases, the option value of waiting becomes higher and the wedge between the performance of the centralized and discretionary processes grows.

The comparative statics with respect to costs of waiting are more subtle. An increase in the cost of waiting has a direct and indirect effect. The direct effect is due to the longer expected queues in the discretionary setting. Fixing the expected queue lengths corresponding to the optimal and discretionary processes, an increase in per-period waiting costs effectively has a multiplier effect—the generated welfare differential is the difference between the expected time agents wait in queue under the two processes, multiplied by the change in costs. The indirect effect is that both the optimal threshold as well as the equilibrium threshold in the discretionary process decrease as a function of waiting costs. The difference between these

[2] This process is reminiscent of a double auction, as each agent submits a "demand function" specifying which types of agents she would be interested in matching with immediately.

[3] We provide preference restrictions that assure the process is individually rational for all participants.

two thresholds therefore narrows as costs increase, which works to mute the welfare gap between the two processes. We show that the combination of these effects leads to a welfare wedge that is locally increasing in costs (formally, it is piece-wise increasing), but exhibits a general decreasing trend. Ultimately, when costs are prohibitively high, both processes lead to instantaneous matches and identical welfare levels.

The discretionary process we focus on relies on the FIFO protocol. While this seems to approximate many real-life decentralized processes, we also analyze two alternative protocols. First, we consider a discretionary setting governed by a *last-in-first-out* (LIFO) priority protocol. We show that this alternative protocol also generates excessive waiting, but less so than the FIFO protocol. Moreover, we show that the equilibrium under the LIFO protocol is asymptotically efficient as waiting costs vanish. In the Online Appendix we also consider a discretionary process in which individuals are matched according to a uniformly random priority rule. We provide simulation results showing that this protocol, while superior to FIFO in welfare terms, produces worse outcomes than LIFO. Nonetheless, since LIFO is potentially fragile to manipulations and often thought as lacking basic fairness features, we suspect FIFO or the uniform priority protocol may be superior descriptive benchmarks.

Finally, we ask in which ways one can improve upon a discretionary setting with interventions that are *simpler* than the full-fledged optimal mechanism. Indeed, centralization may be hard to implement for two main reasons: first, it requires the central planner to be able to force matches upon individuals and, second, it requires the central planner to monitor arrivals and to possibly create matches at every period, yielding potentially high administrative costs. To address the first issue, we consider a discretionary setting in which per-period taxes are introduced for the agents that decide to wait. Our characterization of the optimal mechanism allows us to identify a budget-balanced tax scheme that implements the optimal welfare levels. Such a tax scheme can be tailored so that it does not distort agents' incentives to enter the market to begin with. Nonetheless, even such a scheme may be viewed as cumbersome administratively since it requires continuous monitoring of agents' location in the queue. To address this problem, in the Online Appendix we also analyze a simple mechanism in which all matches occur at fixed time intervals. The length of these time intervals can be chosen to balance the costs of waiting and the quality of the resulting matches. We show that such a simple procedure, while still inferior to the fully optimal mechanism, can improve welfare substantially relative to a fully discretionary market.

In the Online Appendix, we also detail extensions of the main model to asymmetric environments, richer type sets, and different arrival processes.

1.2 Related Literature

The interest surrounding dynamic matching is recent and the literature on this topic is still relatively limited. Much of this literature originally stemmed from the organ donation application. Zenios (1999) develops a queueing model to explain the differences between waiting times of different categories of patients anticipating a kidney transplant. In the context of kidney exchange, Ünver (2010) focuses on a market in which donors and recipients arrive stochastically, preferences are compatibility-based, and the goal of a central planner is to minimize total discounted waiting costs. Under some conditions, he shows that the efficient two-way matching mechanism always carries out compatible bilateral matches as soon as they become available. However, when multi-way matches are possible, some two-way matches could be withheld in order to allow future multi-way matches.[4]

Akbarpour, Li, and Oveis Gharan (2018), like us, inspect the benefits of different mechanisms in a dynamic matching environment. In their setting, however, preferences are based on compatibility, according to a network mapping the set of exchange possibilities. Agents in the system (thought of as patient-donor pairs) become "critical" at random dates, and perish immediately if they are unmatched. Therefore, when waiting costs are negligible, the goal of the planner is to minimize the number of perished agents. Market thickness is beneficial in that it guarantees the availability of immediate matches for agents who become critical. Left to their own devices, agents in that setting would match quickly and useful mechanisms induce agents to wait. In contrast, in our setting, due to the different trade-offs at play, agents in a discretionary process wait *too long* and useful mechanisms induce agents to wait shorter times on the market. In addition, while the welfare benchmark in Akbarpour, Li, and Oveis Gharan (2018) is that of an omniscient planner, our different focus allows us to fully characterize the optimal mechanism, which serves as the benchmark for welfare comparisons.[5]

[4] Some recent models in inventory control have a similar flavor to the compatibility-based matching process considered by Ünver (2010), see e.g. Gurvich and Ward (2014) and Hu and Zhou (2018).

[5] Loertcher, Muir, and Taylor (2018) follow up on the current paper and consider a setting similar to ours. They focus on the optimal mechanism when the planner and participants use the same discount factor to assess future utilities. The interpretation of the objective function in their setting is subtle. In particular, when two agents of identical types who arrived at different times are matched at date t, the agents themselves experience different discounted utilities, but the planner's utility from the two matches is identical. Similarly,

Dynamic assignments, in which only one side of the market has agency, have received some attention in the queueing literature. For instance, Naor (1969) illustrated that individuals who randomly decide whether to join a first-in-first-out queue for some service may wait excessively. In his model, waiting agents impose a purely negative externality on others who decide to join the queue by increasing their expected wait times. Hassin (1985) showed that a last-in-first-out queue would yield equilibrium behavior that emulates the socially optimal. The negative externality in these papers and the work that followed are also present in our model. However, the two-sided nature of our setting introduces additional positive externalities. Consequently, the analysis of our decentralized process is quite different. In particular, last-in-first-out protocols do not generally yield socially-optimal outcomes, unless waiting costs vanish.

Related, Leshno (2017) studies a one-sided market in which potentially-heterogeneous objects (say, public houses) need to be allocated to agents who wait in a queue. Welfare maximization always requires agents to be matched to their preferred objects. However, if agents' preferences are unknown to the planner and their preferred item is in short supply, agents may prefer a mismatched item earlier to avoid costly waiting. Leshno (2017) shows that the welfare loss from mismatches can be reduced substantially through a policy under which all agents who decline a mismatched item face the same expected wait for their preferred item.[6] Anderson, Ashlagi, Gamarnik, and Kanoria (2017) study an environment in which each agent is endowed with an item that can be exchanged with an item owned by someone else. Compatibility is stochastic, and three classes of feasible exchanges are considered: two-way

when assessing future matches, the planner cares only about *when* they are formed, rather than on how much wait they generated for participants. In other words, individual costs of delay do not enter the planner's objective function. Doval and Szentes (2018) also follow up on the current paper and consider a setting where the planner benefits from the discounted utility agents receive in steady state. In their setting, once agents have been in the market for a while, their impact on welfare becomes vanishingly small. The planner is then willing to "store" agents for a long while in the hopes of them allowing new arrivals to match quickly. While Doval and Szentes do not analyze the fully-optimal mechanism, their results suggest that, naturally, the planner may impose more waiting on individuals than they themselves would in a discretionary protocol governed by FIFO. Herbst and Schickner (2016) study markets in which agents are drawn from a unique pool (in contrast with our two-sided setting). They consider environments in which heterogeneous agents arrive randomly over time and need to be grouped in pairs. They characterize the optimal mechanism and analyze the impacts of incomplete information in such team-formation settings.

[6] Bloch and Cantala (2017) also study dynamic allocations. In their setting, a mechanism is a probability distribution over all priority orders consistent with a closed waiting list. Given a priority order, whenever a new object becomes available, agents are proposed the object in sequence and can either accept or reject. They show the benefits of a strict seniority order. Schummer (2017) considers the welfare implications of policies that impact deferral choices by impatient agents waiting in queue. He shows that individuals' level of impatience and risk aversion determines whether allowing deferral of sub-optimal options is desirable.

exchanges only, two- and three-way cycles, and any kind of chain. They find that a policy that maximizes immediate exchanges without withholding them in the interest of market thickness performs nearly optimally.[7,8]

There is also a recent theoretical literature that studies discretionary matching processes that are dynamic, considering both informational and time frictions (see, e.g., Haeringer and Wooders, 2011, Niederle and Yariv, 2009, and Pais, 2008). In that literature, the number of agents on each side of the market is fixed at the outset and agents on one side can make directed offers to agents on the other side. The main goal is the identification of market features that guarantee that an equilibrium of the induced game generates a stable matching.

Another related strand of literature is the search and matching literature (e.g., Burdett and Coles, 1997, Eeckhout, 1999, and the survey by Rogerson, Shimer, and Wright, 2005). There, each period, workers and firms randomly encounter each other, observe the resulting match utilities, and decide jointly whether to pursue the match and leave the market or to separate and wait for future periods. In markets with assortative preferences, as time frictions vanish, generated outcomes are close to a stable matching. A crucial difference with our setting is the stationarity of the market – the perceived distribution of potential partners does not change with time, and each side of the market solves an option value problem.[9]

Last, there is a rather large literature that considers dynamic matching of buyers and sellers and inspects protocols that increase efficiency or allow for Walrasian equilibrium outcomes to emerge as agents become increasingly patient (see, e.g., Satterthwaite and Shneyerov, 2007 and Taylor, 1995).[10]

[7] On the benefits of algorithms that create thicker pools in sparse dynamic allocation environments, see also Ashlagi, Jaillette, and Manshadi (2014), Ashlagi, Burq, Jaillette, and Manshadi (2018), and references therein.

[8] While most of this literature has been focusing on the design of algorithms to achieve socially desirable matchings Doval (2018) introduces a notion of stability in dynamic environments, and provides conditions under which dynamically stable allocations exist.

[9] In the context of marriage markets, Kocer (2014) considers learning over time and models the choice of temporary interactions with different potential partners as a multi-armed bandit problem. Choo (2015) develops a new model for empirically analyzing dynamic matching in the marriage market and applies that model to recent changes in the patterns of U.S. marriages.

[10] Budish, Cramton, and Shim (2015) consider financial exchanges and argue that high-frequency trading leads to inefficiencies, while frequent batch auctions, uniform-price double auctions that occur at fixed and small time intervals, can provide efficiency improvements. These results are reminiscent of our observations regarding the welfare improvements generated by matchings that occur at the end of each fixed window of time, described in the Online Appendix.

2 Setup

We study an infinite-horizon dynamic matching market. There are two kinds of agents: squares and rounds. Squares and rounds can stand for potential adoptive parents and children relinquished for adoption, workers and employers, patients and (good samaritan) donors, etc.

At each time $t \in \{1, 2, \dots\}$, one square and one round arrive at the market. Each square can be of either type "high" (H) or "low" (L) with probability p or $1-p$, respectively, and each round can be of type "high" (h) or "low" (l) with probability p or $1-p$, respectively. These types correspond to the attributes of participants—they can stand for the wealth of parents and race of children in the adoption application, level of education of employees and social benefits or promotion likelihoods for employers in labor markets[11], age or tissue types in the organ donation context[12], etc.

In our model, squares seek to match with rounds and vice versa. We denote by $U_x(y)$ the surplus for a type-x participant from matching with a type-y participant. We assume that preferences are assortative: H-squares are more desirable for all rounds and h-rounds are more desirable for all squares. That is,

$$\begin{aligned} U_H(h) &> U_H(l), \qquad U_L(h) > U_L(l), \\ U_h(H) &> U_h(L), \qquad U_l(H) > U_l(L). \end{aligned}$$

It will be convenient to denote:

$$\begin{array}{ll} U_{Hh} \equiv U_H(h) + U_h(H), & U_{Hl} \equiv U_H(l) + U_l(H), \\ U_{Lh} \equiv U_L(h) + U_h(L), & U_{Ll} \equiv U_L(l) + U_l(L), \end{array}$$

as well as

$$U \equiv U_{Hh} + U_{Ll} - U_{Hl} - U_{Lh}.$$

[11] In some markets, wages differ across individual employees and can be thought of as transfers, which this paper does not handle. However, Hall and Kruger (2012) suggest that a large fraction of jobs have posted wages. Naturally, these wages may reflect general equilibrium wages tailored to the precise composition of the market. Nonetheless, the fact that these wages are fairly constant and do not fluctuate dramatically suggests they may not respond to particular characteristics of individual employees. Our model speaks to this segment of the market.

[12] This is a simplified representation that aims at capturing heterogeneity in types and the quality of different matches it implies. For organ donation, patients and donors are often classified into coarse categories based on age. However, more than two tissue types are often considered. For example, for kidney transplantation, the medical community currently looks at six tissue types, called major histo-compatibility complex or HLA antigens. For an extension to richer type sets, see the Online Appendix.

We will further assume that $U > 0$ so that the utilitarian efficient matching in a static market creates the maximal number of (H, h) and (L, l) pairs. The value of U captures the efficiency gain from such an assortative matching relative to the anti-assortative matching. Notice that $U > 0$ is tantamount to assuming super-modular assortative preferences (a-la Becker, 1974) and U can be thought of as the degree of super-modularity preferences exhibit.

We assume that each square and round suffer a cost $c > 0$ for each period they spend on the market waiting to be matched. We also assume that agents leave the market only by matching. In the Online Appendix, we provide bounds on agents' utility from remaining unmatched that assure this assumption is consistent with individual rationality in the analyzed processes.[13]

Several assumptions merit discussion.

Super-modularity. We assume that preferences are super-modular and that waiting costs are identical for squares and rounds for presentation simplicity. These assumptions are common in the literature and, as we describe in Section 5, lead to a conservative comparison of the optimal and discretionary matching protocols.[14]

Distribution symmetry. The assumption that the distribution of types of rounds mirrors that of squares also simplifies our analysis. It implies that if we drew a large population of rounds and squares, the realized distributions of types would be approximately balanced with high probability. This may be a fairly reasonable assumption for certain applications, such as organ donation. Indeed, the distribution of tissue types of donors and patients is arguably similar. Furthermore, the age of a donor is known to have a strong impact on the expected survival of a graft (see, e.g., Gjertson, 2004 and Oien et al., 2007) and younger recipients have been suggested as the natural recipients of higher-quality organs (see Stein, 2011). Our assumptions then fit a world in which both patients and donors are classified as "young" or "mature" and patients' and donors' age distributions are similar. Our assumption also approximately holds for certain attributes in the online dating world (see Hitch, Hortacsu, and Ariely, 2010). Nonetheless, it might be a rather harsh assumption for other applications.

[13] We show that individual rationality is guaranteed when all agents are acceptable and when any l-round receives a utility lower than $U_l(L) - \frac{p}{1-p}\left[U_h(H) - U_h(L)\right]$ when leaving the market unmatched (analogously for L-squares).

[14] Our analysis carries through fully if participants are horizontally differentiated. In particular, if H-squares and h-rounds prefer one another and L-squares and l-rounds prefer one another, the utilitarian efficient static matching is assortative without further assumptions. This may be relevant in some child-adoption contexts if both adoptive parents and birth mothers display what is often termed "homophilic preferences," preferring to match with individuals of their own race. We describe our results assuming assortative preferences since they are a leading example in the extant literature and potentially tie to more applications.

As it turns out, the techniques we introduce can be used were we to relax this symmetry assumption. We replicate some of our analysis for general asymmetric settings in the Online Appendix.

Arrival process. In our setting, a pair of agents arrives at the market in each period. The analysis would remain virtually identical were we to assume that pairs arrive at random times following, say, a Poisson distribution. Moreover, our results extend directly if we assume that each period a fixed number of square-round pairs, possibly greater than one, enter the market. However, the assumption that participants arrive in pairs is important for the techniques we use. This assumption assures that the market is balanced throughout the matching process. It is a reasonable assumption for some applications. For example, in the adoption process presumably potential adoptive parents and birth mothers make important decisions (whom to match with, whether to leave the process, etc.) at spaced-out intervals. Given the limited variability in the volume of entrants on a monthly basis, the assumption of balanced arrivals provides a decent approximation of reality. Allowing for different arrival rates on both sides of the market introduces new considerations as matching participants in thin markets, while still entailing low waiting costs, imposes a loss in terms of both the quality of matches and the number of individuals matched. The Online Appendix offers a more thorough discussion of how the analysis might be extended to allow for different arrival rates of squares and rounds.

Waiting costs. In our model, agents incur a fixed cost c for every period they spend on the market unmatched. An alternative way to model waiting costs would be to consider agents' payoffs as discounted match utilities. As a first step, our criterion allows us for greater tractability. To see why, notice that, in the presence of discounting, the benefits of matching an agent would depend on the number of periods that agent already spent on the market. The relevant state space for the designer would then be vast. Also, the randomness inherent in the environment suggests that the timing of matches is potentially a random variable. Keeping track of expected exponentially discounted values then introduces non-trivial complications.

Type richness. Last, the assumption that there are only two types of squares and rounds is made for tractability. It corresponds to a coarse description of types in many applications. In the Online Appendix, we illustrate the insights our binary-type analysis allows for environments with richer type sets.

3 Optimal Dynamic Matching

3.1 The Matching Process

At any time $t \in \{1, 2, \ldots\}$, after a new square-round pair enters the market, a queue corresponds to a vector $\mathbf{s}^t = (s_H, s_L, s_h, s_l)$, where each entry is *the stock of squares or rounds* of a particular type waiting in line. We represent the profile of matches created at time t by the vector $\mathbf{m}^t = (m_{Hh}, m_{Hl}, m_{Lh}, m_{Ll})$. For every $\mathbf{s}^t \in \mathbb{Z}^4_+$, a match profile $\mathbf{m}^t \in \mathbb{Z}^4_+$ has to satisfy a feasibility condition

$$\begin{aligned} m_{xh} + m_{xl} &\leq s_x \quad \text{for } x \in \{H, L\}, \\ m_{Hy} + m_{Ly} &\leq s_y \quad \text{for } y \in \{h, l\}. \end{aligned}$$

The surplus generated by the matches is:

$$S(\mathbf{m}) \equiv \sum_{(x,y)\in\{H,L\}\times\{h,l\}} m_{xy} U_{xy}.$$

We denote the volume of remaining agents by $\mathbf{k}^t = (k_H, k_L, k_h, k_l)$, where

$$\begin{aligned} k_x &= s_x - (m_{xh} + m_{xl}) \quad \text{for } x \in \{H, L\}, \\ k_y &= s_y - (m_{Hy} + m_{Ly}) \quad \text{for } y \in \{h, l\}. \end{aligned}$$

The total waiting costs incurred by the remaining agents in period t are then:

$$C(\mathbf{s}, \mathbf{m}) \equiv c \left(\sum_{x\in\{H,L,h,l\}} k_x \right).$$

Finally, the welfare generated at time t is

$$w(\mathbf{s}, \mathbf{m}) \equiv S(\mathbf{m}) - C(\mathbf{s}, \mathbf{m}),$$

if the profile of matches $\mathbf{m}$ is feasible given the stock $\mathbf{s}$, and $w(\mathbf{s}, \mathbf{m}) = -\infty$ otherwise. At time $t + 1$, the queue $\mathbf{s}^{t+1}$, is determined by the number of remaining agents $\mathbf{k}^t$ and the types of agents arriving at $t + 1$. As an initial condition, we have $\mathbf{k}^0 = (0, 0, 0, 0)$. A *mechanism* μ is any rule governing matching profiles. We evaluate a mechanism by considering the *average welfare* it generates:

$$v(\mu) \equiv \liminf_{T\to\infty} \frac{1}{T} E_\mu \left[\sum_{t=1}^{T} w(\mathbf{s}^t, \mathbf{m}^t) \right]. \tag{1}$$

Note that for any mechanism μ, $v(\mu) \in \mathbb{R} \cup \{-\infty\}$, and the average welfare is bounded above by U_{Hh}. This criterion allows us to focus on the long-run performance of mechanisms. We say that μ^* is *optimal* if

$$v(\mu^*) = \sup_{\mu} v(\mu).$$

We will consider the class of mechanisms that satisfy the following two conditions:

Condition 1 (H, h) and (L, l) pairs are matched as soon as they become available;

Condition 2 No more than $\frac{U}{2c}$ squares (and rounds) are ever held in the market.

Condition 1 requires that congruent pairs be matched immediately. Intuitively, the only reason to hold on to, say, an (H, h) pair is to create future (H, l) or (L, h) pairs. However, super-modularity implies that this is inferior to matching immediately the (H, h) pair and the matching the future (L, l) pair.

To understand Condition 2, recall that U captures the extent of super-modularity of preferences—the utilitarian benefit of matching congruent pairs (H, h) and (L, l) over incongruent pairs (H, l) and (L, h). Suppose more than $U/2c$ squares (equivalently, rounds) are held in the market. This implies that at least one pair has been waiting for more than $U/2c$ periods. The utility benefit for that pair is at most U, while the cost per period for the pair is $2c$. It would have been more efficient to match that pair immediately.

In the Online Appendix, we show that there always exists an optimal mechanism satisfying Conditions 1 and 2. Thus, assuming these conditions is without loss of generality.

Conditions 1 and 2 guarantee that the relevant state space is finite. Standard techniques (see Ross, 2014, and details in the Appendix) allow us to focus on the set of *stationary and deterministic mechanisms* (SD-mechanisms). The matches created by a SD-mechanism $\mu^{SD} : \mathbb{Z}_+^4 \to \mathbb{Z}_+^4$ at every period depend only on the queue in place at that period.

3.2 Structure of Optimal Dynamic Mechanisms

Conditions 1 and 2 imply that, at any point in time, an optimal dynamic mechanism entails queues of only H-squares and l-rounds, or only L-squares and h-rounds. That is, the queue can take the form of either $(k, 0, 0, k)$ or $(0, k, k, 0)$, for some $k \geq 0$. The optimal dynamic mechanism is then identified by the maximal stock of H-squares (and l-rounds) and the

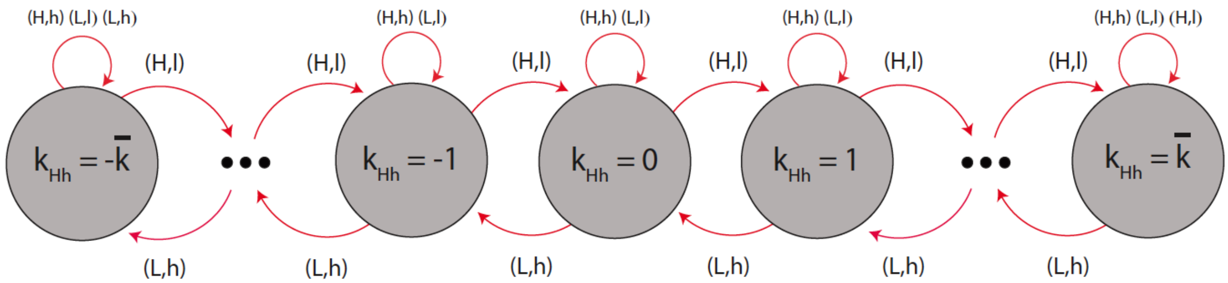

Figure 1: Structure of Optimal and Decentralized Matching Processes

maximal stock of h-rounds (and L-squares) that are kept waiting in queue. In the following proposition we characterize the structure of the optimal mechanism.[15]

Proposition 1 (Optimal Mechanisms) *An optimal dynamic mechanism is identified by a pair of thresholds* $(\bar{k}_H, \bar{k}_h) \in \mathbb{Z}_+$ *such that*

1. *whenever more than* $\bar{k}_H$ *H-squares are present, the excess* (H, l) *pairs are matched immediately, and*
2. *whenever more than* $\bar{k}_h$ *h-rounds are present, the excess* (L, h) *pairs are matched immediately.*

As will soon be stated formally, the symmetry of our environment assures that, generically, an optimal mechanism corresponds to symmetric thresholds: $\bar{k} = \bar{k}_H = \bar{k}_h$.[16] A dynamic mechanism with symmetric thresholds $(\bar{k}, \bar{k})$ as defined in Proposition 1 is depicted in Figure 1, where $k_{Hh} = k_H - k_h$ captures the difference between the length of the queue of H-squares and the length of the queue of h-rounds. We call k_{Hh} the *(signed) length of the H-h queue.*

This process induces the following Markov chain. Let k_{Hh}^t denote the number of H-squares (or l-rounds) minus the number of h-rounds (or L-squares) at the end of time t, after the arrival of that period's square-round pair and any matches imposed by the mechanism. If an

[15] In principle, there could be multiple mechanisms that are identified with the same thresholds. For instance, consider a mechanism in which an (H, l) pair is matched whenever there are $\bar{k}_H + 1$ H-squares present, or whenever there are $\bar{k}_H + 2$ H-squares present. Such a mechanism would be equivalent to a mechanism that matches (H, l) pairs only when there are precisely $\bar{k}_H + 1$ H-squares present. We focus only on the thresholds with the minimal magnitude, which identify outcomes fully, and ignore multiplicity that arises from prescriptions of the social planner over events that are never reached.

[16] As mentioned, in the Online Appendix we work out the extension to asymmetric environments, where the two thresholds may differ.

(H,h) or a (L,l) pair arrive in period $t+1$, the mechanism matches an (H,h) or a (L,l) pair immediately, and the state remains the same: $k^t_{Hh} = k^{t+1}_{Hh}$. Suppose an (H,l) pair arrives in period $t+1$. As long as $0 \leq k^t_{Hh} < \bar{k}$, the mechanism creates no matches and k^{t+1}_{Hh} becomes $k^t_{Hh}+1$. If $k^t_{Hh} < 0$, the mechanism creates one (H,h) match and one (L,l) match, and k^{t+1}_{Hh} becomes $k^t_{Hh}+1$. Finally, if $k^t_{Hh} = \bar{k}$, the mechanism creates one (H,l) pair, and k^{t+1}_{Hh} remains the same, $k^{t+1}_{Hh} = k^t_{Hh} = \bar{k}$. Analogous transitions occur with the arrival of a (L,h) pair.

Therefore, we can describe the probabilistic transition as follows. Denote by

$$\mathbf{x}^t \equiv (x^t_{-\bar{k}}, x^t_{-\bar{k}+1}, ..., x^t_{\bar{k}-1}, x^t_{\bar{k}})^{tr} \in \{0,1\}^{2\bar{k}+1}$$

the timed vector capturing the state, $x^t_i = \mathbf{1}(k^t_{Hh} = i)$. That is, x^t_i is an indicator that takes the value of 1 if the state is i and 0 otherwise. Then,

$$\mathbf{x}^{t+1} = \mathbf{T}_{\bar{k}}\mathbf{x}^t,$$

where

$$\mathbf{T}_{\bar{k}} = \begin{pmatrix} 1-p(1-p) & p(1-p) & \dots & 0 & 0 \\ p(1-p) & 1-2p(1-p) & \dots & 0 & 0 \\ 0 & p(1-p) & \dots & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & \dots & p(1-p) & 0 \\ 0 & 0 & \dots & 1-2p(1-p) & p(1-p) \\ 0 & 0 & \dots & p(1-p) & 1-p(1-p) \end{pmatrix}. \tag{2}$$

The above Markov chain is ergodic (i.e., irreducible, aperiodic, and positively recurrent). Therefore, an optimal mechanism corresponds to a matching process that reaches a steady state with a unique stationary distribution. For $T_{\bar{k}}$, the steady-state distribution is uniform so that each state $k_{Hh} = -\bar{k}, \dots, \bar{k}$ occurs with an equal probability of $\frac{1}{2\bar{k}+1}$.

3.3 Optimal Thresholds

In order to characterize the optimal threshold, we first evaluate the welfare corresponding to any arbitrary symmetric threshold. First, we compute the average total waiting costs incurred by agents waiting in line for one period of time. During the transition from time $t-1$ to time t, $2|k^{t-1}_{Hh}|$ agents wait in line. So, the total costs of waiting incurred during this one time period is $2|k^{t-1}_{Hh}|c$. Thus, a mechanism with threshold $\bar{k}$ results in expected total costs of waiting equal to

$$\frac{1}{2\bar{k}+1}\left(\sum_{k_{Hh}=-\bar{k}}^{\bar{k}} 2|k_{Hh}|\right)c = \frac{2\bar{k}(\bar{k}+1)c}{2\bar{k}+1}.$$

Next, we compute the average total surplus generated during one time period, tracking the Markov process described above. A newly-arrived pair is of type (H,h) with probability p^2, in which case the optimal mechanism generates a surplus equal to U_{Hh}. Similarly, when a new pair of type (L,l) arrives, which occurs with probability $(1-p)^2$, the optimal mechanism generates a surplus equal to U_{Ll}.

Suppose an (H,l) pair arrives at time t. If $k_{Hh}^{t-1} < 0$, the mechanism creates one (H,h) pair and one (L,l) pair, generating a surplus equal to $U_{Hh}+U_{Ll}$. If $0 \le k_{Hh}^{t-1} < \bar{k}$, the mechanism creates no matches (and no additional surplus), and if $k_{Hh}^{t-1} = \bar{k}$, the mechanism creates one (H,l) pair and generates a surplus equal to U_{Hl}. Analogous conclusions pertain to the case in which a (L,h) pair arrives. Thus, a mechanism with threshold $\bar{k}$ generates an expected total surplus equal to

$$\begin{aligned} & p^2U_{Hh}+(1-p)^2U_{Ll}+\frac{2p(1-p)}{2\bar{k}+1}\left[\bar{k}\left(U_{Hh}+U_{Ll}\right)+\frac{U_{Hl}+U_{Lh}}{2}\right] \\ = \;& pU_{Hh}+(1-p)U_{Ll}-\frac{p(1-p)U}{2\bar{k}+1}. \end{aligned}$$

Therefore, the net expected total welfare per period, accounting for waiting costs, is:

$$pU_{Hh}+(1-p)U_{Ll}-\frac{p(1-p)U}{2\bar{k}+1}-\frac{2\bar{k}(\bar{k}+1)c}{2\bar{k}+1}. \tag{3}$$

The optimal threshold $\bar{k}^{opt}$ maximizes the welfare as given in (3). The following proposition summarizes our discussion and provides the full characterization of the optimal mechanism.

Proposition 2 (Optimal Thresholds) *The threshold*

$$\bar{k}^{opt} = \left\lfloor \sqrt{\frac{p(1-p)U}{2c}} \right\rfloor$$

identifies an optimal dynamic mechanism. In this optimal mechanism, all available (H,h) and (L,l) pairs, and any number of (H,l) or (L,h) pairs exceeding $\bar{k}^{opt}$, are matched immediately. Furthermore, the optimal mechanism is generically unique.[17]

[17] Multiplicity arises only when $\sqrt{\frac{p(1-p)U}{2c}}$ is an integer.

The optimal threshold increases with the probability of any incongruent pair, $p(1-p)$, and with the degree of super-modularity U, which reflects the value of assortative matches. It decreases with waiting costs. In fact, when waiting costs are prohibitively high, namely when $c > \frac{p(1-p)U}{2}$, the maximal queue length is $\bar{k}^{opt} = 0$ and all matches are instantaneous.

3.4 Welfare

We now turn to the expected per-period welfare in the steady state under the optimal mechanism. Were we to consider no costs of waiting, the optimal mechanism would naturally entail long waits to get the maximal possible match surplus asymptotically by matching H-squares with h-rounds and L-squares with l-rounds. We denote the resulting welfare by:

$$S_\infty \equiv pU_{Hh} + (1-p)U_{Ll}.$$

The optimal threshold identified in Proposition 2 allows us to characterize the welfare achieved by the optimal mechanism through equation (3) and to get the following corollary.

Corollary 1 (Optimal Welfare) *The welfare under the optimal mechanism is given by* $W^{opt}(c) = S_\infty - \Theta(c)$, *where* $\Theta(c)$ *is continuous, increasing, and concave in* c, $\lim_{c\to 0}\Theta(c) = 0$, *and* $\Theta(c) = p(1-p)U$ *for all* $c \geq \frac{p(1-p)U}{2}$.

As waiting costs approach 0, the welfare induced by the optimal mechanism approaches S_∞. For costs large enough, the optimal mechanism matches all square-round pairs instantaneously as they arrive and the resulting welfare is $S_\infty - p(1-p)U$. For intermediate costs, the optimal mechanism generates welfare that is naturally in between these two values.[18,19] The observation that welfare under the optimal mechanism decreases as c increases is rather intuitive. Indeed, suppose $c_1 > c_2$. Were we to implement the optimal mechanism with waiting cost c_1 when the waiting cost is c_2, the distribution of matches would remain identical, while waiting costs would go down, thereby leading to greater welfare overall. This implies that the

[18] The value of S_∞ is effectively the analogue of the value generated by an "omnsicient" planner in our setting, which is used as one benchmark in Akbarpour, Li, and Oveis Gharan (2017). Corollary 1 suggests that the omniscient planner's value is a valid feasible benchmark when waiting costs vanish.

[19] In the Appendix, we provide the analytical formula for $\Theta(c)$ in terms of the fundamental parameters of our setting. In fact, simple algebraic manipulations imply that:

$$S_\infty - \sqrt{2p(1-p)Uc} - c \leq W^{opt}(c) \leq S_\infty - \sqrt{2p(1-p)Uc} + c.$$

welfare under the optimal mechanism with waiting cost c_2, which would be at least weakly higher, is greater than that corresponding to waiting cost c_1. The amount by which welfare decreases when waiting costs increase depends on the number of agents expected to wait in line in the steady state. The higher the waiting costs, the lower the number of agents waiting in line on average. Therefore, the impact of an increase in costs by a fixed increment is greater at smaller costs, which leads to the concavity of $\Theta(c)$.[20]

4 Discretionary Matching

Many dynamic matching processes are in essence discretionary, in the sense that participants have the choice of declining a match they do not wish to form: child adoption in the US and abroad, job searches in many industries, etc. It is therefore important to understand the implications of discretionary dynamics, particularly when considering centralized interventions. In this section, we provide a framework for analyzing a class of discretionary matching processes. We first focus on one that resembles what is often seen in applications. In Section 6 and in the Online Appendix we consider various alternatives.

In our discretionary matching process, we assume individuals join the market in sequence and decide when to match with a potential partner immediately and when to stay in the market and wait for a potentially superior match. While the discretionary setting we study still requires some centralized governance, as matches occur according to some order, it provides a convenient benchmark for studying dynamic matching markets that are lightly regulated.

We assume that at each period t, there are three stages. First, a square and round enter the market with random attributes as before: with probability p the square is an H-square and with probability p the round is an h-round. Second, individuals of each type are ordered by some priority rule that we describe formally below. In the third stage, each square and round declare their demands—whether a square will match only with an h-round, or is willing to match with either an h-round or a l-round, and whether a round will match only with an H-square, or with either an H-square or a L-square. Given the order and the participants' demands, the market clears sequentially according to the priority rule. Any remaining participants proceed to period $t+1$ at the additional cost of c.

[20] Continuity follows directly from concavity. Alternatively, fix any mechanism that is optimal for some waiting costs. An increase in waiting costs reduces the resulting welfare continuously, in fact linearly. Thus, we cannot get discontinuous reductions in welfare when choosing an optimal mechanism.

4.1 The Matching Process

In each period t, one square w^t and one round r^t arrive at the market, and their types are realized. Upon their arrival, a period-t stage-game begins:

$$G^t \equiv \{I^t, (D_i)_{i\in I^t}, \phi, (u_i(\cdot;\phi) : \Pi_{i\in I^t} D_i \to \mathbb{R})_{\forall i \in I^t}\}.$$

The components of G^t are defined as follows. The set of players is $I^t \equiv H^t \cup L^t \cup h^t \cup l^t$, where $H^t \subseteq \{x^{t'} : 1 \leq t' \leq t\}$ is the set of H-squares present in the market in period t, and the other sets, L^t, h^t, and l^t, are defined similarly. Each H-square, say player i, in I^t chooses an action in $D_i = \{h, l\}$, with h denoting a demand for only h-rounds and l denoting a demand for *either* type of round.[21] The action sets for the other agents' types are defined analogously. A *priority rule* ϕ assigns a linear order over each set H^t, L^t, h^t, and l^t. First, we consider a *first-in-first-out (FIFO) protocol*, which assigns a linear order $\succ$ over, say, H^t such that

$$\forall x^{t'}, x^{t''} \in H^t, \quad x^{t'} \succ x^{t''} \iff t' < t'' \leq t.$$

There is anecdotal evidence that order of arrivals affects the order of matches in many markets, and FIFO is a commonly used protocol. For instance, in the child adoption context, many countries follow a FIFO protocol to match relinquished children to adoptive parents.[22] In Section 6.2 and in the Online Appendix, we consider alternative priority protocols.

The stage-game payoffs are determined by sequential market clearing. First, we take H-squares and h-rounds in the order induced by ϕ and form as many (H, h) pairs as possible (regardless of their demands).[23] If there are remaining H-squares demanding l-rounds, they are matched with l-rounds sequentially according to ϕ and independently of the demands made by the l-rounds. Analogously, if there are remaining h-rounds demanding L-squares, they are matched with L-squares sequentially according to ϕ and independently of the demands made by the L-squares. Last, all remaining L-squares and l-rounds who are flexible in their demands form matches sequentially in the order induced by ϕ. The stage-game payoff for a type-x agent

[21] This restriction on the action space is made for simplicity of exposition. An equilibrium similar to the one we describe below arises if we allow players to demand only inferior matches on the other side of the market, or to submit a demand for no one at all.

[22] For example, see the protocol adopted by the China Center of Children's Welfare and Adoption (CCCWA) here: http://www.aacadoption.com/programs/china-program.html.

[23] This market-clearing assumption allows us to simplify some steps of the proofs, and avoid inefficient equilibria in which (H, h) pairs remain on the market unmatched.

matched with a type-y agent is $U_x(y)$. If a player remains unmatched, her stage-game payoff is $-c$.

We complete the definition of a dynamic discretionary matching game by characterizing the evolution of the stage-games G^t, and each player's dynamic-game payoff. The initial set of players is $I^0 \equiv \emptyset$. All players in I^t who remain unmatched in period t, together with new arrivals x^{t+1} and r^{t+1}, form I^{t+1}. Consider a player i, who arrives in period t and is matched at $t'' \in \mathbb{Z}_+ \cup \{\infty\}$. Such a player receives stage-game payoffs $(u_i^t, u_i^{t+1}, \dots)$, and a dynamic game payoff $\sum_{t'=t}^{\infty} u_i^{t'} (\in \mathbb{R} \cup \{-\infty\})$, where $u_i^{t''}$ is i's match utility, $u_i^{t'} = 0$ for $t' > t''$ (and $u_i^{t'} = -c$ for any $t' < t''$).

The dynamic game has complete information and arbitrary (dynamic) strategies. Each player i, say an H-square, chooses a demand every period she remains in the market, since her arrival till she matches. A (dynamic) strategy σ_i indicates the probability of demanding h in each of these periods and can depend on the complete history from $t = 0$ and on. As before, let $\mathbf{s}^t = (s_H^t, s_L^t, s_h^t, s_l^t)$ be the *state* (or stock) at period t, and let $q_i^t \in \mathbb{Z}_+$ be player i's *rank* according to ϕ in period t.[24] Let $\theta_i^t \equiv (\mathbf{s}^t, q_i)$ denote an *augmented state* for player i.

Definition 1. *A strategy σ_i is a* ***stationary and deterministic strategy (SD-strategy)*** *for an H-square i if there exists $\psi_i^H : \{(\mathbf{s}, q) \in \mathbb{Z}_+^5\} \to \{h, l\}$ such that, for any t such that $i \in H^t$ and $\theta_i^t = (\mathbf{s}^t, q_i^t)$, player i demands $\psi_i^H(\mathbf{s}^t, q_i^t)$.*

We similarly define SD-strategies for L-squares, h-rounds, and l-rounds. A *symmetric, stationary, and deterministic strategy profile*, which we name *stationary* strategy profile*, is a profile of SD-strategies, such that all players of the same type use the same strategy, i.e. $\psi_i^x = \psi^x$ for all t, $i \in x^t$, and $x = H, L, h, l$. We denote a *stationary* strategy profile* by $\Psi = (\psi^H, \psi^L, \psi^h, \psi^l)$.

Definition 2. *A stationary* strategy-profile Ψ is a* ***stationary* equilibrium*** *if it is an equilibrium of the dynamic matching game.*[25]

For simplicity, we assume from now on a symmetric setting (results pertaining to asym-

[24] That is, under FIFO, $q_i^t = 1$ if player i arrived before all other H-squares in H^t, $q_i^t = 2$ if player i arrived second among all other H-squares in H^t, and so on.

[25] In a stationary* equilibrium we allow a player's deviation to be any dynamic strategy, including history-dependent and random.

metric settings appear in the Online Appendix). That is, we assume:

$$U_H(h) - U_H(l) = U_h(H) - U_h(L) \text{ and } U_L(h) - U_L(l) = U_l(H) - U_l(L).$$

Last, we assume that the environment is regular in that

$$p(U_H(h) - U_H(l)) \neq kc$$

for all natural numbers $k \in \mathbb{N}$. Regularity assures that neither squares nor rounds are ever indifferent between waiting in queue and matching immediately with an available partner.[26]

4.2 Equilibrium Characterization with the FIFO Protocol

In this section, we present necessary conditions for a stationary* equilibrium that are sufficient to compute equilibrium welfare. We guarantee the existence of stationary* equilibria and provide their characterization in the Appendix.

By construction, at the beginning of a period, the queue cannot entail both H-squares and h-rounds. As before, we denote the (signed) length of the H-h queue after an arrival of a new pair by $s_{Hh} \equiv s_H - s_h$, and after agents form matches by $k_{Hh} \equiv k_H - k_h$. We first consider the decisions of an H-square (analogous analysis holds for an h-round). Suppose an H-square arrives at the market and an h-round is available, one that had either been waiting in the queue or one that has just arrived at the market as well. In this case, an H-square is matched immediately to an h-round, the identities of whom are prescribed by the order of arrival. In particular, if the arriving H-square is the first in line, that square is matched to an h-round. If there are H-squares already in queue, this implies that the available h-round arrived at the same time as our H-square, and that this h-round will be matched with the first H-square in the queue. The newly arrived H-square then has a choice between waiting in line and matching with a l-round. However, notice that this square's decision is equivalent to that of the last H-square who arrived and decided to wait. Therefore, in a stationary* equilibrium, the new H-square waits and the queues remain as they were.

Suppose now that an H-square enters the market and an h-round is not available. This implies that there is at least one l-round available that the square can match with. Therefore, the H-square has to decide whether to match immediately with a l-round or to wait in line

[26] The assumption of regularity is not crucial and similar analysis follows without regularity for any arbitrary tie-breaking rule. However, the presentation is far simpler for regular environments.

based on the number of H-squares already waiting. An immediate match with a l-round delivers $U_H(l)$, whereas waiting in line till eventually matching with an h-round delivers $U_H(h)$ at an uncertain cost of waiting.

Note that once an H-square decides to wait in the queue, rather than match immediately with a l-round, she will wait until matching with an h-round, rather than leave the queue by matching with a l-round at a later point. Indeed, as matches form on a FIFO basis, the H-square's position in the queue moves up over time, and the expected time until matching with an h-round becomes shorter. Therefore, if it is optimal for an H-square to wait in the queue upon entry, it is optimal for her to wait at any later period. The expected waiting time till a match with an h-round is therefore solely determined by the number of other H-squares who precede her in the queue. The following result identifies bounds on the size of the H-h queue:

Lemma 1 (FIFO Thresholds) *In all stationary* equilibria under FIFO, in all periods,* $-\bar{k}^{fifo} \leq k_{Hh} \leq \bar{k}^{fifo}$ *where*[27]

$$\bar{k}^{fifo} \equiv \left\lfloor \frac{p(U_H(h) - U_H(l))}{c} \right\rfloor = \left\lfloor \frac{p(U_h(H) - U_h(L))}{c} \right\rfloor. \quad (4)$$

Intuitively, the time till an h-round enters the market is distributed geometrically (with parameter p), so the expected time till an h-round arrives at the market is $\frac{1}{p}$. An H-square who is k-th in line in the queue will be matched when the k-th h-round arrives, which is expected to occur in $\frac{k}{p}$ periods. The expected waiting costs are therefore $\frac{kc}{p}$, which generate an increase in match utility of $U_H(h) - U_H(l)$ (relative to matching with a l-round immediately). An H-square will wait as long as the expected benefit of waiting exceeds its costs, i.e., whenever

$$\frac{kc}{p} < U_H(h) - U_H(l),$$

which is the comparison underlying the maximal size of the queue described in Lemma 1. Our regularity assumption further guarantees that an H-square or an h-round are never indifferent between waiting in line and matching immediately. Whenever there are fewer than $\bar{k}^{fifo}$ H-squares in the queue, a new H-square will wait in the market. Whenever there are $\bar{k}^{fifo}$ or more H-squares in the queue, the new H-square prefers to match with a l-round immediately.

[27] In the Appendix, we show that in all stationary* equilibria the full support of the H-h queue is $\{-\bar{k}^{fifo}, .., \bar{k}^{fifo}\}$. Therefore, the bounds described in this lemma are achieved in equilibrium.

An analogous description holds for h-rounds and our symmetry assumptions assure that the maximal queue length is identical for H-squares and h-rounds.

We now turn to the decisions of L-squares and l-rounds. A l-round (similarly, a L-square) may decide to wait, hoping to match with an H-square who will become available when the line for H-squares exceeds $\bar{k}^{fifo}$. In principle, there are two effects at work. The first is similar to that experienced by the H-squares waiting in line: the longer the queue of l-rounds waiting ahead in line, the longer the new l-round has to wait. The second effect, however, is due to H-squares' behavior in equilibrium: the longer is the queue, the closer H-squares are to the threshold $\bar{k}^{fifo}$ and to the point of accepting matches with l-rounds.

As it turns out, at least the last l-round in the queue has an incentive to match immediately with any square. To gain intuition, consider the first l-round, say player i, arriving at the market. There cannot be other H-squares waiting in the market since any such squares would have arrived with l-rounds, in contradiction to our l-round being the first in line. Suppose that player i arrives with a L-square. Observe that by Lemma 1 the first $\bar{k}^{fifo}$ H-squares wait in line until they are matched with an h-round. Thus, player i has to wait for the arrival of at least $\bar{k}^{fifo}+1$ H-squares to match with an H-square. Given this observation, consider now the following hypothetical optimal stopping problem for player i: in each period, player i can choose between matching with a L-square, or waiting for the $(\bar{k}^{fifo}+1)$-th arriving H-square, which we assume will be available for player i. The expected payoff for player i from the hypothetical optimal stopping problem is (weakly) higher than his expected payoff in the dynamic matching game, for any profile of behavior of others. It is easy to see that the expected payoff from this stopping problem is $U_l(L)$. Indeed, the expected cost of waiting until the $(\bar{k}^{fifo}+1)$-th arriving H-square is $\frac{(\bar{k}^{fifo}+1)c}{p}$, which is strictly greater than the benefit from waiting since

$$U_l(H) - U_l(L) < U_h(H) - U_h(L) = U_H(h) - U_H(l) < \frac{\left(\bar{k}^{fifo}+1\right)c}{p}.$$

It follows that our player i would therefore prefer to match with a L-square, who is available, immediately. In fact, this intuition generalizes and yields the following result.

Lemma 2 (Equilibrium under FIFO) *There exists a stationary* equilibrium such that there are never both a L-square and a l-round waiting in the market.*

Lemma 2 implies that there exists a stationary* equilibrium that follows a protocol similar to that implemented by the optimal mechanism, though the threshold governing when incongruent matches are formed, $\bar{k}^{fifo}$, may differ from the optimal threshold $\bar{k}^{opt}$.

Lemma 1 and its discussion in the Appendix guarantee that, since the behavior of H-squares and h-rounds is the same in all stationary* equilibria, so is the welfare generated by matches involving H-squares and h-rounds. Therefore, the stationary* equilibrium described by Lemma 2, in which L-squares and l-rounds do not delay matching with one another, is the one that maximizes welfare, as stated in the following corollary.

Corollary 2 *The stationary* equilibrium in which there are never both a L-square and a l-round waiting in the market is welfare-maximizing among all stationary* equilibria under FIFO.*

4.3 Steady State of Discretionary Matching

As for the optimal mechanism, the length of the H-h queue k_{Hh} in the equilibrium described in Lemma 2 and Corollary 2 is characterized by a Markov chain with a transition matrix analogous to that described in Section 3.2. Similar analysis allows the characterization of the equilibrium steady state of the discretionary process under the FIFO protocol.

Proposition 3 (Discretionary Steady State) *The welfare-maximizing stationary* equilibrium under FIFO is associated with a unique steady state distribution over queue lengths, such that the length of the H-h queue* $k_{Hh} = k_H - k_h$ *is uniformly distributed over* $\{-\bar{k}^{fifo}, \ldots, \bar{k}^{fifo}\}$ *and, in any period, the queues contain only H-squares and l-rounds or only h-rounds and L-squares.*

The threshold $\bar{k}^{fifo}$ is determined by the decisions of H-squares and h-rounds to wait, as specified in Lemma 1. The crucial difference between the discretionary and optimal mechanism is the threshold placed on the maximal stock of H-squares or h-rounds in waiting. A decision to wait in the market by, say, a square imposes a negative externality on succeeding squares, as it potentially affects their waiting time, and possibly the quality of their matches[28], as well as on the round she would otherwise match with. However, a decision to wait can also impose

[28] From a welfare perspective, the externality on the quality of the match is of less importance. As long as a social planner views identical agents as interchangeable, an immediate mismatch or a later mismatch have similar welfare consequences.

a positive externality on future desirable agents on the other side of the market, who would face a ready desirable agent upon arrival.

To glean some intuition for the relative strength of these externalities, consider the extreme case in which, in the discretionary process, matches are immediate ($\bar{k}^{fifo} = 0$), which happens when $\frac{c}{p} > U_H(h) - U_H(l)$. In such a market, consider an (H, l) pair arriving when no other agents are present. In the discretionary setting, the pair would match immediately. Might a social planner want to keep this pair waiting? Suppose the planner holds on to the (H, l) pair until an h-round arrives with a square of either type. At that point, the H-square is to be matched with the h-round, while the l-round is to be matched with the newly-arrived square. To simplify our illustration, suppose that all other participants are matched instantaneously. The H-square certainly does not benefit from this imposed wait (else, she would wait even in the discretionary setting). The cost imposed on the l-round entailed by waiting for an h-round is c/p. Now, the anticipated h-round may arrive with either an H-square, with probability p, or with a L-square, with probability $1 - p$. In the latter case, the positive externality of our H-square on this h-round comes to light—the h-round matches with an H-square instead of a L-square he would match with otherwise, generating a marginal benefit of $U_H(h) - U_H(l)$ (since match payoffs are symmetric across market sides). This positive externality is overwhelmed by the cost of waiting incurred by the l-round, even ignoring all other negative externalities on the match qualities of the original l-round as well as the square arriving with the h-round, since $\frac{c}{p} > U_H(h) - U_H(l) > (1 - p)(U_H(h) - U_H(l))$. In particular, delaying a match is sub-optimal from the social planner's perspective. This intuition extends—the negative externalities present in discretionary settings overwhelm the positive externalities. Thus, the optimal mechanism is always governed by a smaller threshold for waiting than the one selected through equilibrium in the discretionary process, as described in the following corollary.

Corollary 3 (Thresholds' Comparison) *Maximal waiting queues are longer under FIFO than they are under the optimal mechanism. That is, $\bar{k}^{opt} \leq \bar{k}^{fifo}$, with strict inequality for sufficiently small waiting costs c.*

4.4 Welfare

Since the protocols are similar except for the queues' thresholds, the expected per-period welfare in the steady state characterized in Proposition 3 can be found using an analogous derivation to that carried out for the optimal mechanism. This derivation leads to an expression mirroring equation (3), accounting for the discretionary process' threshold $\bar{k}^{fifo}$. Namely, the expected per-period net welfare is given by:

$$W^{fifo}(c) = S_{\infty} - \frac{p(1-p)U}{2\bar{k}^{fifo}+1} - \frac{2\bar{k}^{fifo}(\bar{k}^{fifo}+1)c}{2\bar{k}^{fifo}+1},$$

where $\bar{k}^{fifo}$ is defined in (4). To summarize, we have the following corollary:

Corollary 4 (Decentralized Welfare) *The maximum equilibrium welfare under FIFO is given by* $W^{fifo}(c) = S_{\infty} - \Psi(c)$*, where* $\lim_{c \to 0} \Psi(c) = p\left(U_H(h) - U_H(l)\right)$*, and* $\Psi(c) = p(1-p)U$ *for all* $c \geq p\left(U_H(h) - U_H(l)\right)$*.*

Recall Corollary 1, which characterized the welfare under the optimal mechanism. By definition, the welfare generated under the optimal mechanism is higher than that generated by the discretionary process, so that $\Theta(c) \leq \Psi(c)$ for all c. While the optimal mechanism generates welfare that is decreasing in waiting costs, this is not necessarily the case under the discretionary process. Furthermore, while the welfare under the optimal mechanism approaches S_{∞} as waiting costs diminish, this is not the case under the discretionary process. As waiting costs become very small, there is a race between two forces. For any given threshold, the overall waiting costs decline. However, in equilibrium, discretionary thresholds increase, leading to greater expected wait times. As it turns out, the balance between these two forces generates significant welfare losses, given by $p\left(U_H(h) - U_H(l)\right)$, even for vanishingly small costs. Next, we provide a detailed comparison of the two procedures in terms of welfare.

5 Welfare Comparisons

By construction, the optimal mechanism generates welfare that is at least as high as that generated by the discretionary process.[29] In this section we inspect how the welfare wedge responds to the underlying parameters of the environment, suggesting the settings in which centralized intervention might be particularly useful.

[29] It is possible to show that the optimal mechanism presents a Pareto improvement with respect to the decentralized setting. In fact, it is easy to see that L-squares and l-rounds are better off under the optimal mechanism (as it implies better matches and shorter waiting times). Moreover, one can show that, as long as $\bar{k}^{opt} \geq 2$, the expected payoff of H-squares and h-rounds as they enter the market is higher under the optimal mechanism than under the discretionary process as well.

The following proposition captures the effects on the welfare wedge $W^{opt}(c) - W^{fifo}(c)$ of the waiting costs c, the frequency p of H-squares or h-rounds, and the utility benefit of desirable types from matching with desirable types relative to less desirable ones.

Proposition 4 (Welfare Wedge – Comparative Statics)

1. *For any interval* $[\underline{c}, \overline{c})$, *where* $\underline{c} > 0$, *there is a partition* $\{[c_i, c_{i+1})\}_{i=1}^{M-1}$, *where* $\underline{c} = c_1 < c_2 < ... < c_M = \overline{c}$, *such that* $W^{opt}(c) - W^{fifo}(c)$ *is continuous and increasing over* (c_i, c_{i+1}) *and*
$$W^{opt}(c_i) - W^{fifo}(c_i) > W^{opt}(c_{i+1}) - W^{fifo}(c_{i+1})$$
for all $i = 1, ..., M - 1$.

2. *As* c *becomes vanishingly small, the welfare gap* $W^{opt}(c) - W^{fifo}(c)$ *converges to a value that is increasing in* $p \in (0, 1)$ *and in* $U_H(h) - U_H(l)$.

To see the intuition for the comparative statics corresponding to waiting costs, notice that an increase in costs has two effects on the welfare gap. Since the equilibrium threshold under the discretionary process is greater than the optimal threshold (Corollary 3), an increase in waiting costs has a direct effect of magnifying the welfare gap. Nonetheless, there is also an indirect effect of an increase in waiting costs that arises from the potential changes in the induced thresholds. Consider a slight increase in waiting costs such that the optimal threshold does not change, but the discretionary threshold decreases. The discretionary process is then "closer" to the optimal process—both the matching surplus and the waiting costs are closer and the welfare gap decreases. In fact, as costs become prohibitively high, both processes lead to instantaneous matches and identical welfare levels. As we show in the proof of Proposition 4, the indirect effect overwhelms the direct effect at precisely such transition points and acts to shrink the welfare gap. The construction of the partition is done as follows. Each atom $[c_i, c_{i+1})$ of the partition corresponds to constant thresholds under the discretionary process. Over these intervals, only the direct effect operates and the welfare gap is increasing. Each of the endpoints $\{c_i\}_i$ corresponds to a decrease of the discretionary threshold by one. Therefore, when comparing two such endpoints, the indirect effect kicks in and the decreasing trend of the welfare gap emerges. Figure 2 depicts the resulting pattern the welfare gap exhibits for $U_H(h) = U_h(H) = 3$, $U_H(l) = U_h(L) = U_L(h) = U_l(H) = 1$, $U_L(l) = U_l(L) = 0$, and $p = 0.3$.

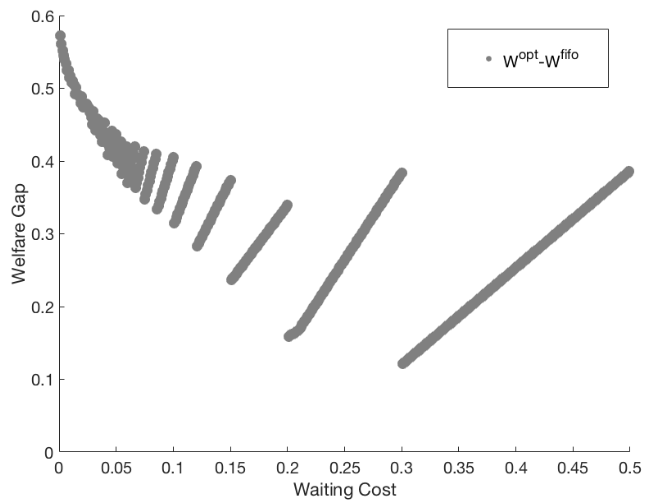

Figure 2: Welfare Gap between Optimal and Discretionary Matching (FIFO) as a Function of Costs

As suggested by the proposition, $W^{opt}(c) - W^{fifo}(c)$ is piece-wise increasing in c. Nevertheless, overall, the gap has a decreasing trend.

To glean some intuition on the comparative statics the welfare gap displays with respect to p, consider two type distributions governed by p_1 and p_2 such that $p_1 < p_2 = mp_1$, $m > 1$. Suppose further that costs are so low that the match surplus in the discretionary process is very close to the optimum, S_∞. The individual incentives to wait for H-squares and h-rounds are higher under p_2 than under p_1. In fact, in the discretionary setting, the distribution of steady-state queue length is the uniform distribution where, from (4), under p_2, roughly $1 - 1/m$ of the probability mass is allocated to queue lengths larger than those realized under p_1. For each of these large steady state queue lengths, we have more pairs of agents waiting, i.e., increased per-period waiting costs. The optimal mechanism internalizes the negative externalities, so the effect of the increased waiting costs is weaker. On the other hand, the benefit of this increase in queue length is a lower chance of producing mismatches. However, for sufficiently low c, the match surplus under p_1 is already close to its optimum of S_∞ and this effect is weak; in particular, the difference in terms of match surplus that the optimal and discretionary processes generate is similar under p_1 and p_2. Therefore, for sufficiently low c, the dominant effect is the one produced by the differential in expected waiting costs, which generates our comparative statics.[30] Note that as p approaches 0 or 1, both the optimal mechanism and the discretionary processes generate similar welfare levels as in those cases incongruent pairs arrive at a vanishing rate.

Going back to our assumption of super-modular preferences, notice that the construction of the optimal mechanism would remain essentially identical were preferences sub-modular (with an appropriate relabeling of market participants). However, in the discretionary setting, sub-modular preferences would lead to a negative welfare effect compounding the negative externalities present in our setup. Namely, individual incentives would be misaligned with market-wide ones. In that respect, our comparison of optimal and discretionary processes assuming super-modular preferences is a conservative one.

Similarly, considering waiting costs that differ across the two sides of the market would lead to a greater welfare wedge as well. Intuitively, suppose that squares experience a waiting cost of c_S and rounds experience a waiting cost of c_R, where $c_S > c_R$, with an average cost of $c = (c_S + c_R)/2$. The optimal mechanism with asymmetric costs would coincide with that corresponding to identical costs of c since per-pair costs are the same in both cases. In the discretionary process, H-squares would be willing to wait when the queue of H-squares is no longer than $\bar{k}_S^{fifo}$ and h-rounds would be willing to wait when the queue of h-rounds is no longer than $\bar{k}_R^{fifo}$, where

$$\bar{k}_S^{fifo} = \left\lfloor \frac{p\left(U_H(h) - U_H(l)\right)}{c_S} \right\rfloor \quad \text{and} \quad \bar{k}_R^{fifo} = \left\lfloor \frac{p\left(U_h(H) - U_h(L)\right)}{c_R} \right\rfloor .$$

Suppose $\frac{p(U_H(h)-U_H(l))}{c_x} \in \mathbb{N}$ for $x = S, R$ to avoid rounding issues. From convexity, it follows that the threshold $\bar{k}^{fifo}$ corresponding to identical costs of c satisfies $\bar{k}^{fifo} \leq (\bar{k}_S^{fifo} + \bar{k}_R^{fifo})/2$. Therefore, the excessive waiting discretionary processes exhibit would be even more pronounced when costs are asymmetric across market sides.[31]

6 Transfers and Alternative Protocols

So far, we have shown that intervention in dynamic matching markets can have a substantial impact on welfare, at least when centralization is carried out using the optimal dynamic

[30] In fact, we can show that for any $\Delta p > 0$, there exists $\delta > 0$ such that for every $c < \delta$ and $p \in [0, 1 - \Delta p]$, the welfare wedge under $p + \Delta p$ and c is greater than under p and c. Furthermore, $\delta \to 0$ as $\Delta p \to 0$.

[31] In fact, in such a setting, l-rounds may wait in the market even when L-squares are present, leading to another channel of inefficient waiting. We show formally how our main results carry over to this more general environment in the Online Appendix.

mechanism. However, the full-fledged optimal mechanism may be hard to implement. It requires that the formation of matches, even those of individuals who would prefer to wait in the market, be within the purview of the centralized planner. It also requires the central planner to monitor the market continuously to determine when matches should be formed, which may be administratively costly. In this section (and in the Online Appendix) we show that improvements to discretionary settings under FIFO can be achieved by mechanisms that relax one of these two requirements. We start by showing that a tax scheme imposed on those who wait in the market can yield the optimal waiting patterns, if tailored appropriately. We then analyze an alternative setting that does not require the clearinghouse to monitor the market continuously and can provide substantial welfare improvements over the discretionary matching process under FIFO: a discretionary environment in which matches are formed following a last-in-first-out (LIFO) protocol. In addition, in the Online Appendix, we analyze two alternative protocols, one in which the centralized clearinghouse matches all available agents every fixed number of periods, and another in which, in a discretionary market, agents' priorities at every period are determined at random.

6.1 Optimal Taxation

Certainly, a fixed per-period tax on waiting $\tau \geq 0$ can be set so that the resulting de-facto cost of waiting, $c + \tau$, is such that the discretionary process generates the optimal threshold (with costs c). Namely, τ can be set so that

$$\frac{p\left(U_H(h) - U_H(l)\right)}{c + \tau} = \sqrt{\frac{p(1-p)U}{2c}}. \tag{5}$$

In fact, if collected taxes in period t are given to new entrants in period $t+1$, they would have no effect on either overall welfare or individual incentives to wait once in the market. Nonetheless, there is a risk that such a policy would introduce a strong incentive for agents to enter the market to begin with, only to gather the previous generation's taxes. As it turns out, there is a tax scheme that is budget balanced such that the expected tax (or subsidy) for an entering agent who is not privy to the pattern of queues in place is nil. Under such a scheme, no agent is tempted to enter the market only for the sake of reaping the benefits of taxes. Indeed, consider a linear tax scheme—agents who are k-th in line pay $\tau^* k$ to the matching institution, regardless of whether they are a square or a round and regardless of their type. To achieve budget balance, the collected taxes are equally redistributed back to existing agents in the market in each period. When the length of queue is $\hat{k}$, the resulting net tax, which is added to the fixed cost c, for an agent who is k-th in line is

$$\tau^* k - \frac{2 \cdot \sum_{k=1}^{\hat{k}} \tau^* k}{2\hat{k}} = \tau^* \left(k - \frac{\hat{k}+1}{2} \right).$$

In particular, the net added tax on waiting for the last agent in the queue is $\frac{\tau^*(\hat{k}-1)}{2}$, which is increasing in the queue's length $\hat{k}$. Thus, using the definition of τ from equation (5), we want the tax levied on the last agents in the queue of length $\bar{k}^{opt}$ to satisfy

$$\frac{\tau^*(\bar{k}^{opt}-1)}{2} = \tau \qquad \Longleftrightarrow \qquad \tau^* = \frac{2\tau}{\bar{k}^{opt}-1}.$$

Such a taxation policy might still be difficult to administer in terms of the financial activity it would entail—time-dependent taxes and redistribution of resources at each period. In the next subsection we offer the analysis of an alternative discretionary process to that governed by FIFO, which while not optimal, can generate substantial welfare improvements.

6.2 Last-In-First-Out

While the FIFO protocol we analyze resembles discretionary processes in various applications, as we saw, it generates excessive waiting. It is then natural to consider alternative protocols in which waiting is disciplined in some manner. We consider here the often-discussed last-in-first-out (LIFO) protocol (see, e.g., Hassin, 1985, and more recently, Tornøe and Østerdal, 2017, as well as references therein). In such a protocol, waiting is disciplined as it entails a transition to a bad position in the queue and, consequently, may improve on the welfare generated by the FIFO protocol. It is important to keep in mind, however, that protocols such as LIFO face well-known implementation hurdles. In particular, they are subject to manipulation as they introduce incentives to leave and re-enter queues (see Margaria, 2017). They are also considered "unfair" in that individuals who exert no cost of waiting are catered to first, while identical others who have been waiting remain in the queue. In what follows, we describe the characterization of outcomes under the LIFO protocol. Detailed analysis and proofs appear in the Online Appendix.

Formally, we consider a discretionary setting that has the same structure described in Section 4.1 but, once every agent on the market has specified their demands, matches form according to a LIFO protocol. This protocol assigns a linear order $\succ$ over, say, H^t such that

$$\forall x^{t'}, x^{t''} \in H^t, \quad x^{t''} \succ x^{t'} \iff t' < t'' \leq t.$$

We first consider the decisions of H-squares (and we omit the analogous discussion for h-rounds). If an H-square finds an h-round upon arrival, the H-square matches with the last arrived h-round. If no h-round is available, the H-square needs to decide whether to match with the last l-round, who must have just arrived together with the H-square, or to wait in the queue. Under LIFO, this decision is independent of other H-squares who have been waiting in the queue. Rather, the decision depends on the anticipated behavior of H-squares who will arrive at the market in future periods.

We consider a SD-strategy ψ_H for H-squares relying a threshold $\bar{k}_H$. Using this strategy, an H-square stays in the market at any stage as long as the difference between her anticipated position in the queue, calculated after a new pair arrives, and the expected number of h rounds is no greater than $\bar{k}_H$.[32] In particular, if no h-round is available, an H-square, say player i, waits by demanding an h-round as long as her rank q_i according to LIFO is at most $\bar{k}_H$: i.e., when there are fewer than $\bar{k}_H$ other H-squares who arrived *after* player i.

To gain intuition for our equilibrium characterization, suppose that all H-squares, including player i, use the threshold $\bar{k}_H = 1$. If player i finds no available h-round upon arrival, then she waits by demanding an h-round. In the next period, player i continues to wait if either a pair (H, h) or a pair (L, l) arrive, because in these scenarios her rank according to LIFO remains the same: a new pair (H, h) will match immediately, and a pair (L, l) does not affect the queue of H-squares. However, if an (H, l) pair arrives, the new H-square, who also uses $\bar{k}_H = 1$, demands an h-round. According to ψ_H, player i then demands a l-round and leaves the market. Finally, if a (L, h) pair arrives, player i matches to the h-round. To summarize, player i exits the market matched with either an h-round or a l-round, with probability 1/2 each, as soon as the first incongruent pair arrives. Since the expected number of periods until the first arrival of an incongruent pair is $\frac{1}{2p(1-p)}$, the expected payoff for player i is

[32] For example, consider a market with one H-square and no h-rounds. Upon the entry of an (H, h) pair, the H-square's anticipated rank in the queue is 1 and the anticipated number of h-rounds is 0. Upon the entry of an (H, l) pair who is expected to stay (in equilibrium), her anticipated rank in the queue is 2, while the anticipated number of h-rounds is 0. In fact, the anticipated number of h-rounds will always be 0 or 1 in the equilibrium we consider.

$$\frac{U_H(h) + U_H(l)}{2} - \frac{c}{2p(1-p)}.$$

Consider a possible deviation of player i in which i demands a l-round when she finds no available h-rounds upon her arrival. This deviation is not strictly profitable if and only if

$$U_H(l) \leq \frac{U_H(h) + U_H(l)}{2} - \frac{c}{2p(1-p)},$$

which we can rewrite as

$$\frac{p(1-p)\left(U_H(h) - U_H(l)\right)}{c} \geq 1 = \frac{\bar{k}_H(\bar{k}_H + 1)}{2}. \tag{6}$$

Consider another potential deviation by player i: if one more H-square arrives after player i and no h-round is available, player i, instead of demanding a l-round, increases her threshold to $\bar{k}'_H = 2$ indefinitely and remains in the market. If player i uses the threshold $\bar{k}'_H = 2$, while all other H-squares use $\bar{k}_H = 1$, player i will match to an h-round for sure. We use an *absorbing Markov chain* to compute the expected continuation payoff for player i. In what follows, we normalize time to *event time*, denoted by τ, which increases upon each arrival of an incongruent pair (which, in expectation, occurs every $1/2p(1-p)$ periods). The state space is $\{1, 2, h\}$: the two *transient states* (1 and 2) denote the H-square's rank, and the *absorbing state* (h) denotes player i's matching to an h-round. The matrix of transition probabilities p_{ij} from state i to state j is

$$P = \begin{bmatrix} Q & R \\ \mathbf{0} & 1 \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}.$$

The matrix Q represents transition probabilities between transient states.[33] Let

$$T \equiv (I_2 - Q)^{-1} \cdot 1 = 4 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix},$$

where I_2 is the 2×2 identity matrix. If the initial state of the absorbing Markov chain is 2, it is well-known in the absorbing Markov chain literature (see, e.g., Kemeny and Snell, 1960)

[33] Take any event time τ, and suppose that the state at time τ is 2: i.e., there is an another H-square waiting, who arrived after player i. The event time τ progresses to $\tau + 1$ by an arrival of an incongruent pair. If the incongruent pair is (L, h), the rank of player i moves up to 1. This transition occurs with probability $Q_{21} = 1/2$. Otherwise, the new incongruent pair is (H, l). According to ψ_H, the H-square who has been waiting with player i demands a l-round and leaves the market, leaving player i's rank at 2. This transition occurs with probability $Q_{22} = 1/2$. If the state in period τ is 1, and (L, h) arrives, then player i matches to the h-round. This last transition occurs with probability $R_{11} = 1/2$.

that the chain will be absorbed by state h within $T_2 = 6$ expected periods of event time. Therefore, if player i deviates by increasing her threshold perpetually to $\bar{k}'_H = 2$, the expected continuation payoff is $U_H(h) - \frac{6c}{2p(1-p)}$. Such deviation is not strictly profitable if

$$U_H(l) \geq U_H(h) - \frac{6c}{2p(1-p)},$$

which is equivalent to

$$\frac{p(1-p)\left(U_H(h) - U_H(l)\right)}{c} \leq 3 = \frac{(\bar{k}_H+1)(\bar{k}_H+2)}{2}. \tag{7}$$

In the Online Appendix, we show that (6) and (7) generalize to an arbitrary threshold $\bar{k}_H \in \mathbb{Z}_+$ as follows

$$\frac{\bar{k}_H(\bar{k}_H+1)}{2} \leq \frac{p(1-p)(U_H(h) - U_H(l))}{c} \leq \frac{(\bar{k}_H+1)(\bar{k}_H+2)}{2}.$$

In particular, these inequalities are necessary conditions for any SD-strategy with threshold $\bar{k}_H$ to be a part of an equilibrium: the first inequality guarantees that an H-square with rank $\bar{k}_H$ would not deviate by demanding a l-round when no h-rounds are available in the market, and the second inequality guarantees that an H-square, whose rank would become $\bar{k}_H + 1$ by waiting, prefers to match with a l-round immediately.

Similar to the FIFO environment, we assume that the environment is regular in that

$$p(1-p)(U_H(h) - U_H(l)) \neq kc$$

for all natural numbers $k \in \mathbb{N}$.

The generalization of the analysis above then yields:

Lemma 3 (Thresholds under LIFO) *In all stationary* equilibria under LIFO in which H-squares and h-rounds use threshold strategies, in all periods, $-\bar{k}^{lifo} \leq k_{Hh} \leq \bar{k}^{lifo}$ where*[34]

$$\bar{k}^{lifo} \equiv \left\lfloor \sqrt{\frac{2p(1-p)(U_H(h) - U_H(l))}{c} + \frac{1}{4}} - \frac{1}{2} \right\rfloor.$$

We now turn to the decisions of l-rounds (or, analogously, those of L-squares). Suppose that H-squares play a SD-strategy with threshold $\bar{k}_H$. A l-round matches with an H-square only when that H-square arrives with a l-round. LIFO then prescribes the H-square to be

[34] In the Online Appendix, we show that in all stationary* equilibria, the full support of the H-h queue is $\{-\bar{k}^{lifo}, .., \bar{k}^{lifo}\}$. Therefore, the bounds described in Lemma 3 are achieved in equilibrium.

matched with the last l-round to enter the market. It follows that, if a l-round remains unmatched in the period of his arrival, he will never be matched with an H-square later. Therefore, l-rounds have an incentive to leave the market as soon as possible. The following result guarantees the existence of a stationary* equilibrium in which H-squares and h-rounds use a threshold strategy identified by $\bar{k}^{lifo}$, and L-squares and l-rounds match immediately whenever possible. Similar to the FIFO case, this equilibrium is welfare-maximizing among all stationary* equilibria.

Lemma 4 (Equilibrium under LIFO) *There exists a stationary* equilibrium in which H-squares and h-rounds use a threshold $\bar{k}^{lifo}$ and such that there can never be both L-squares and l-rounds waiting in the market. This equilibrium is welfare-maximizing among all stationary* equilibria.*

Similar arguments to those used in Section 4 then generate the characterization of the equilibrium steady state corresponding to the LIFO protocol:

Proposition 5 (Discretionary Steady State under LIFO) *The welfare-maximizing stationary* equilibrium under LIFO is associated with a unique steady state distribution over queue lengths, such that the length of the H-h queue $k_{Hh} = k_H - k_h$ is uniformly distributed over $\{-\bar{k}^{lifo}, \dots, \bar{k}^{lifo}\}$ and, in any period, the queues contain only H-squares and l-rounds or only h-rounds and L-squares.*

We can now compare this threshold, as well as consequent welfare levels, to those emerging from the other protocols discussed throughout the paper.

Corollary 5 (Thresholds and Welfare Comparisons under LIFO)

1. *For sufficiently small waiting costs c, the maximal waiting queues under LIFO are longer than under the optimal mechanism, but shorter than under FIFO. That is, $\bar{k}^{opt} < \bar{k}^{lifo} < \bar{k}^{fifo}$.*
2. *The LIFO protocol is asymptotically efficient—that is, the maximum equilibrium welfare under LIFO is given by $W^{lifo}(c) = S_\infty - \Gamma(c)$, where $\lim_{c \to 0} \Gamma(c) = 0$.*

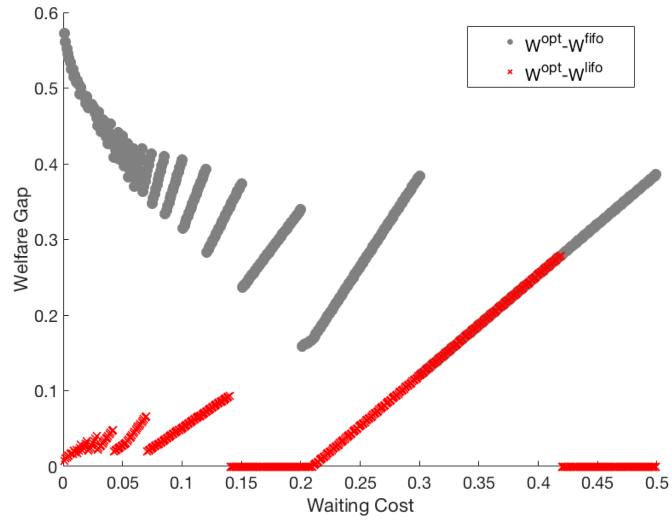

Figure 3: Welfare Gaps between Optimal and Discretionary Matching (FIFO and LIFO) as a Function of Costs

Corollary 5 suggests that the LIFO protocol could represent a substantial improvement with respect to the FIFO protocol in discretionary settings, at least for small costs. Figure 3 depicts the welfare losses generated by both the FIFO and the LIFO protocols with respect to the optimal mechanism for the parameter values used in Figure 2: $U_H(h) = U_h(H) = 3$, $U_H(l) = U_h(L) = U_L(h) = U_l(H) = 1$, $U_L(l) = U_l(L) = 0$, and $p = 0.3$. The figure illustrates that the welfare gap decreases significantly under LIFO, even for costs far away from zero.

7 Conclusions

We considered a dynamic matching environment and identified the optimal matching mechanism. The optimal mechanism always matches congruent pairs immediately and holds on to a stock of incongruent pairs up to a certain threshold. When matching follows a discretionary process, a similar matching protocol ensues *in equilibrium*, but the induced thresholds for waiting in the market are larger as individuals do not internalize the net negative externalities they impose on those who follow. This difference generates a potentially significant welfare wedge between discretionary processes and centralized clearinghouses, even when waiting costs are vanishingly small. Our results provide guidance as to the features of the economy that could make centralized intervention more appealing.

We also offer some simple interventions to discretionary markets—transfer schemes that can induce optimal outcomes, and an alternative last-in-first-out priority protocol, which are arguably less complex than the full-fledged optimal mechanism and can provide substantial welfare improvements relative to the first-in-first-out protocol.

There are several other natural interventions and extensions that we analyze in the Online Appendix: a simple centralized mechanism that matches individuals at fixed-time intervals, a discretionary setting in which agents are ranked at every period according to a random priority rule, general asymmetric environments, a richer set of participant types, and different arrival processes.

8 Appendix

8.1 Proofs Regarding the Optimal Mechanism

First, we present an auxiliary result, Lemma A1, whose proof appears in the Online Appendix. This Lemma allows us to show that the restriction to SD-mechanisms satisfying Conditions 1 and 2 we imposed in the main text is without loss of generality. Furthermore, it simplifies our analysis of the optimal mechanism.

Lemma A1 *(1) For any mechanism μ, there exists a mechanism μ', with $v(\mu') \geq v(\mu)$, which never holds H-squares and h-rounds that are both available, or L-squares and l-rounds that are both available; (2) For any mechanism μ, there exists a mechanism μ', with $v(\mu') \geq v(\mu)$, which never holds more than $\frac{U}{2c}$ squares (and rounds) in the market.*

Lemma A1 allows us to simplify our problem using the following Markov decision problem with agents arriving in incongruent pairs, a finite set of states, and a finite set of actions:

$$(MDP, s^0) \equiv \{T, S, s^0, (r(s,k), p(\cdot|k))_{s \in S, k \in H_s}\},$$

where s^0 denotes a particular initial state. Each component is defined as follows:

1. $T \equiv \{0, 1, 2, \dots\}$ is *the set of decision event times.* As described in the body of the paper, event times correspond to times at which incongruent pairs (H, l) or (L, h) arrive. Since the probability of an incongruent pair arriving at any period is $2p(1-p)$, the expected time between event times is $\frac{1}{2p(1-p)}$.

2. $S \equiv \{z \in \mathbb{Z} : -(U/2c) - 1 \leq z \leq (U/2c) + 1\}$ is *the set of possible states (or stocks).* Each state $s_{Hh} \equiv s_H - s_h \in S$ represents the (signed) number of incongruent pairs of type (H, l) or (L, h) in the market. Since we restrict our attention to mechanisms that do not hold more than $U/2c$ squares (and rounds), a state, which takes a new arriving pair into account, has to belong to the set $\{-\lfloor U/2c \rfloor - 1, ..., \lfloor U/2c \rfloor + 1\}$.

3. $s^0 = 0$ is the *the initial state.* Initially, no agent waits.

4. $K \equiv \{z \in \mathbb{Z} : -U/2c \leq z \leq U/2c\}$ is *the set of available actions.* Each $k \in K$ represents the (signed) number of incongruent pairs held in the market from one period to the next.

5. $r(s,k)$ is *the reward function*: for every $s \in S$, $k \in K$,

$$r(s,k) = \begin{cases} (s-k)U_{Hl} - \frac{kc}{2p(1-p)} & \text{if } s \geq k \geq 0 \\ (|s|-|k|)U_{Lh} - \frac{|k|c}{2p(1-p)} & \text{if } s \leq k \leq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

   The expected waiting cost incurred by any agent who waits for one event time is $\frac{c}{2p(1-p)}$. The reward function returns $-\infty$ if an action is infeasible. For all feasible actions, the values of the reward function are in the interval $\left[-\frac{U}{4p(1-p)}, (\frac{U}{2c}+1)U_{Hh}\right]$.

6. $p(s,k)$ is the *transition probability*, the probability the system is in state $s \in S$ at any time $\tau+1$, after the action k has been chosen at time τ.

$$p(s,k) = \begin{cases} 1/2 & \text{for } s = k-1, k+1, \\ 0 & \text{otherwise.} \end{cases}$$

(MDP, s^0) is stationary in the sense that the reward function $r(s,k)$ and the transition probability function $p(s,k)$ do not depend on time, or event times, explicitly. A *policy* of (MDP, s^0) is any rule, deterministic or randomized, governing the choice of actions. Such a rule may, in principle, be history-dependent. The value of a policy μ is then,

$$v(\mu) \equiv \liminf_{T\to\infty} \frac{1}{T} E_\mu \left[\sum_{\tau=1}^{T} r(s^\tau, k^\tau) \right].$$

A stationary and deterministic policy, which we call a *SD-policy*, of (MDP, s^0) applies the same deterministic decision rule $\mu^{SD} : S \to K$ regardless of the history. The value of μ^{SD} is then

$$v(\mu^{SD}) = \lim_{N\to\infty} \frac{1}{N} E \left[\sum_{\tau=1}^{T} r(s^\tau, \mu^{SD}(s^\tau)) \right].$$

The limit exists, as guaranteed, for example, by Proposition 8.1.1(b) in Puterman (2005).

As mentioned, in the Online Appendix, we show that restricting attention to SD-mechanisms in the main text is without loss of generality (see also Theorem Puterman 3 below).

Proof of Proposition 1: The proof follows several steps.

Step 1 (Existence of Thresholds $(\bar{k}_H, \bar{k}_h)$*)*

Any stationary and deterministic policy d of (MDP) is associated with two thresholds, representing the largest number of (H,l) and (L,h) pairs held in the market at any time, respectively.[35] Define

$$\bar{k}_H \equiv \min\{s \mid s > 0, d(s) < s\} - 1, \quad \text{and}$$
$$\bar{k}_h \equiv \min\{|s| \mid s < 0, d(s) > s\} + 1.$$

The thresholds $(\bar{k}_H, \bar{k}_h)$ are well-defined. Indeed, policies maintain only a bounded number of unmatched pairs in the market. We claim that the value of a policy is uniquely determined by the thresholds and decisions at the thresholds. Given policy d, define

$$d'(s) \equiv \begin{cases} d(s) & \text{if } -\bar{k}_h \leq s \leq \bar{k}_H \\ d(\bar{k}_H) & \text{if } s > \bar{k}_H \\ d(-\bar{k}_h) & \text{if } s < -\bar{k}_h. \end{cases}$$

The Markov processes induced by d and d', namely $\{(s^\tau, r(s^\tau, d(s^\tau))\}_{\tau=0}^{\infty}$ and $\{(s^\tau, r(s^\tau, d'(s^\tau)))\}_{\tau=0}^{\infty}$, are identical. Thus, $v(d) = v(d')$. We can therefore characterize any policy d by its corresponding thresholds $(\bar{k}_H, \bar{k}_h)$ and decisions at the thresholds $(d(\bar{k}_H), d(\bar{k}_h))$.[36]

Step 2 (Stationary Distribution of k_{Hh})

We characterize the unique stationary distribution of k_{Hh} corresponding to the ergodic Markov process induced by a policy d.

Claim 1 *Take $\bar{k}_H, \bar{k}_h \in [1, \frac{U}{2c}] \cap \mathbb{Z}_+$, and $z_H, z_h \in \mathbb{Z}_+$ with $z_H \leq \bar{k}_H$ and $z_h \leq \bar{k}_h$. A policy d of (MDP) defined by*

$$d(s) \equiv \begin{cases} s & \text{if } -\bar{k}_h \leq s \leq \bar{k}_H \\ \bar{k}_H - z_H & \text{if } s > \bar{k}_H \\ -\bar{k}_h + z_h & \text{if } s < -\bar{k}_h. \end{cases}$$

induces a Markov chain corresponding to k_{Hh}. The unique steady state distribution $\boldsymbol{\pi} = (\pi_{-\bar{k}_h}, \ldots, \pi_{\bar{k}_H})$ is such that:

1. *(Middle Range) for $-\bar{k}_h + z_h \leq k \leq \bar{k}_H - z_H$, $\pi_k = \pi_0 = \frac{1}{\bar{k}_H + \bar{k}_h - z_H/2 - z_h/2 + 1}$,*
2. *(Upper Range) for $z = 1, \ldots, z_H$, $\pi_{\bar{k}_H - z_H + z} = \pi_0 \left(1 - \frac{z}{z_H + 1}\right)$,*

[35] Indeed, suppose a policy dictates matches to be formed when the number of, say, (H, l) pairs exceeds k_H^1 or $k_H^2 > k_H^1$. The number of (H, l) pairs would then never surpass k_H^2, so the relevant threshold for outcomes would be the minimal threshold k_H^1.

[36] As mentioned in the body of the text, there is multiplicity regarding prescriptions for states that are never reached. With thresholds $\bar{k}_H$ and $\bar{k}_h$ the market never has more than $\bar{k}_H + 1$ H-squares or more than $\bar{k}_h + 1$ h-rounds. The specification of what happens outside of these regions therefore has no impact on outcomes.

3. *(Lower Range) for* $z = 1, \ldots, z_h$, $\pi_{-\bar{k}_h + z_h - z} = \pi_0 \left(1 - \frac{z}{z_h+1}\right)$.

That is, the stationary distribution is uniform in the middle range. The stationary probability mass decreases as k_{Hh} approaches $\bar{k}_H$ or $\bar{k}_h$.

Proof of Claim 1: Denote by

$$\mathbf{x}^\tau \equiv (x^\tau_{-\bar{k}_h}, x^\tau_{-\bar{k}_h+1}, \ldots, x^\tau_{\bar{k}_H-1}, x^\tau_{\bar{k}_H})^{tr} \in \{0,1\}^{\bar{k}_H+\bar{k}_h+1}$$

the timed vector such that $x_i^\tau = \mathbf{1}(k_{Hh} = i)$. Then, $\mathbf{x}^{\tau+1} = T_d \mathbf{x}^\tau$, where

$$\mathbf{T}_d = \begin{pmatrix} 0 & 1/2 & \ldots & 0 & 0 \\ 1/2 & 0 & \ldots & 0 & 0 \\ 0 & 1/2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1/2 & 0 \\ 0 & 0 & \ldots & 0 & 1/2 \\ 0 & 0 & \ldots & 1/2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 1/2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1/2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

The second matrix on the right hand side has two non-zero elements valued at $1/2$. Each represents two scenarios, a transition from $\bar{k}_H$ upon the arrival of an (H,l) pair to $\bar{k}_H - z_H$, and a transition from $-\bar{k}_h$ upon the arrival of a (L,h) pair to $-\bar{k}_h + z_h$. The first matrix includes all other transitions. The Markov chain is ergodic, and the unique stationary distribution of k_{Hh} exists. Then, $\boldsymbol{\pi}$ in Claim 1 is the unique stationary distribution using straightforward calculations.

Step 3 (Welfare)

We compute the average welfare (i.e., total welfare per period) for any stationary and deterministic mechanism μ. Let d be the associated policy of (MDP) with thresholds $(\bar{k}_H, \bar{k}_h)$ and the decisions at the thresholds identified by (z_H, z_h).[37]

First, we compute the average total surplus generated in one time period. A newly arrived pair is of type (H,h) with probability p^2, in which case the optimal mechanism generates a surplus equal to U_{Hh}. Similarly, a newly arrived pair is of type (L,l) with probability $(1-p)^2$, in which case the optimal mechanism generates a surplus equal to U_{Ll}.

Suppose an (H,l) pair arrives at time t when the stock is k_{Hh}^{t-1}. If $k_{Hh}^{t-1} < 0$, the mechanism creates one (H,h) and one (L,l) pair, generating a surplus equal to $U_{Hh} + U_{Ll}$. If $0 \le k_{Hh}^{t-1} <$

[37] That is, for $s > \bar{k}_H, d(s) = \bar{k}_H - z_H$ and for $s < -\bar{k}_h, d(s) = -\bar{k}_h + z_h$.

$\bar{k}_H$, the mechanism creates no matches (and no additional surplus), and if $k_{Hh}^{t-1} = \bar{k}_H$, the mechanism creates $(z_H + 1)$ matches of (H, l) pairs. Analogous conclusions pertain to the case in which a (L, h) pair arrives. The expected match surplus per period is therefore:

$$\begin{aligned}
& p^2 U_{Hh} + (1-p)^2 U_{Ll} + (1-\pi_0)p(1-p)(U_{Hh} + U_{Ll}) \\
& + \pi_{\bar{k}_H} p(1-p)(z_H+1)U_{Hl} + \pi_{\bar{k}_h} p(1-p)(z_h+1)U_{Lh} \\
= \; & pU_{Hh} + (1-p)U_{Ll} - \pi_0 p(1-p)U.
\end{aligned}$$

Next, we compute the average total waiting costs incurred by agents waiting in line for one period. During the transition from time t culminating at stock k_{Hh} to time $t+1$, $2|k_{Hh}|$ agents wait in line so the total costs of waiting incurred during this one time period are $2|k_{Hh}|c$. Thus, a mechanism with thresholds $(\bar{k}_H, \bar{k}_h)$ results in expected total costs of waiting equal to

$$\sum_{k=1}^{\bar{k}_H} 2c\pi_k |k| + \sum_{k=-1}^{-\bar{k}_h} 2c\pi_k |k|$$

The first term equals to

$$\begin{aligned}
& (2c\pi_0) \sum_{k=1}^{\bar{k}_H - z_H} k + (2c\pi_0) \sum_{z=1}^{z_H} \left(1 - \frac{1}{z_H+1}\right)(\bar{k}_H - z_H + z) \\
= \; & (2c\pi_0) \left(\frac{(\bar{k}_H - z_H)(\bar{k}_H - z_H + 1)}{2} + \frac{(\bar{k}_H - z_H) z_H}{2} + \frac{z_H(z_H+1)}{2} - \frac{z_H(2z_H+1)}{6} \right) \\
= \; & (c\pi_0) \left((\bar{k}_H - z_H)(\bar{k}_H + 1) + \frac{z_H(z_H+2)}{3} \right).
\end{aligned}$$

The second term is computed similarly. The average welfare of the mechanism μ is then

$$\begin{aligned}
& W(\bar{k}_H, \bar{k}_h, z_H, z_h) = pU_{Hh} + (1-p)U_{Ll} - \pi_0 p(1-p)U \\
& -(c\pi_0) \left((\bar{k}_H - z_H)(\bar{k}_H + 1) + (\bar{k}_h - z_h)(\bar{k}_h + 1) + \frac{z_H(z_H+2) + z_h(z_h+2)}{3} \right), \qquad (8)
\end{aligned}$$

where $\pi_0 = \frac{2}{2\bar{k}_H + 2\bar{k}_h - z_H - z_h + 2}$.

Step 4 (Matching at Most One Pair at a Time)

We now show that we can focus on mechanisms satisfying $z_H = z_h = 0$. In fact, generically this restriction is necessary for a mechanism to be optimal. The proof follows from the following claim, which completes the proof of Proposition 1.

Claim 2 *Fix any* $\bar{k}_h$ *and* $z_h(\leq \bar{k}_h)$. *For any* $\bar{k}_H \geq 1$ *and* $0 \leq z_H \leq \bar{k}_H - 1$,

$$W(\bar{k}_H, \bar{k}_h, z_H + 1, z_h) \geq W(\bar{k}_H, \bar{k}_h, z_H, z_h)$$

implies

$$W(\bar{k}_H - 1, \bar{k}_h, z_H, z_h) \geq W(\bar{k}_H, \bar{k}_h, z_H + 1, z_h).$$

That is, whenever a mechanism with a larger z_H leads to a higher average welfare, we can find a mechanism with an even higher average welfare by decreasing the threshold $\bar{k}_H$, while adhering to a smaller z_H.

Proof of Claim 2: Let

$$\begin{aligned} \phi &= 2\bar{k}_H + 2\bar{k}_h - z_H - z_h + 1 \quad \text{and} \\ \psi &= (\bar{k}_H - z_H)(\bar{k}_H + 1) + (\bar{k}_h - z_h)(\bar{k}_h + 1) + \frac{z_H(z_H + 2) + z_h(z_h + 2)}{3}. \end{aligned}$$

The first inequality in Claim 2 holds if and only if

$$\frac{p(1-p)U}{\phi} + \frac{c}{\phi}\left(\psi - \bar{k}_H + \frac{2z_H}{3}\right) \leq \frac{p(1-p)U}{\phi + 1} + \frac{c}{\phi + 1}\psi,$$

or equivalently

$$p(1-p)U + c\psi - (\phi + 1)c\left(\bar{k}_H - \frac{2z_H}{3}\right) \leq 0. \tag{9}$$

The second inequality in Claim 2 holds if and only if

$$\frac{p(1-p)U}{\phi - 1} + \frac{c}{\phi - 1}\left(\psi - 2\bar{k}_H + z_H\right) \leq \frac{p(1-p)U}{\phi} + \frac{c}{\phi}\left(\psi - \bar{k}_H + \frac{2z_H}{3}\right),$$

or equivalently

$$p(1-p)U + c\psi - c\left(\bar{k}_H - \frac{2z_H}{3}\right) - \phi c\left(\bar{k}_H - \frac{z_H}{3}\right) \leq 0. \tag{10}$$

Clearly, (9) implies (10).

Claim 2 completes the proof of Proposition 1. Furthermore, this claim illustrates that there is always an optimal mechanism identified by $z_H = z_h = 0$. From the proof, notice that if $z_H > 0$, inequality (9) implies that inequality (10) holds with a strict inequality. Therefore, in any optimal mechanism, $z_H, z_h < 2$. In fact, multiplicity can emerge only when there is multiplicity in the thresholds $\bar{k}_H, \bar{k}_h$ fixing $z_H = z_h = 0$. Indeed, suppose there is an optimal mechanism with $\bar{k}_H$ and $z_H = 1$ and some $\bar{k}_h, z_h$. From the proof of Claim 2, it follows that

$$W(\bar{k}_H, \bar{k}_h, 1, z_h) - W(\bar{k}_H, \bar{k}_h, 0, z_h) = W(\bar{k}_H - 1, \bar{k}_h, 0, z_h) - W(\bar{k}_H, \bar{k}_h, 1, z_h).$$

The optimality of $\bar{k}_H$ and $z_H = 1$ implies that, in the above equality, both sides equal to 0 (otherwise, the mechanism identified by $\bar{k}_H - 1$ and $z_H = 0$, with $\bar{k}_h, z_h$, would generate greater welfare). In particular, there are optimal mechanisms identified by both $\bar{k}_H - 1$ and $z_H = 0$ as well as $\bar{k}_H$ and $z_H = 0$. ■

Proof of Proposition 2: We find an optimal threshold pair $(\bar{k}_H, \bar{k}_h)$, assuming that $z_H = z_h = 0$. To prove Proposition 2, we write the average welfare described in the main text as

$$pU_{Hh} + (1-p)U_{Ll} - \frac{p(1-p)U}{\bar{k}_H + \bar{k}_h + 1} - \frac{(\bar{k}_H(\bar{k}_H+1) + \bar{k}_h(\bar{k}_h+1))c}{\bar{k}_H + \bar{k}_h + 1}.$$

We use the following change of variables

$$\phi \equiv \bar{k}_H + \bar{k}_h, \quad \text{and} \quad \psi \equiv \bar{k}_H - \bar{k}_h,$$

and rewrite the above expression for welfare as

$$pU_{Hh} + (1-p)U_{Ll} - \frac{p(1-p)U}{\phi+1} - \frac{(\phi^2 + 2\phi + \psi^2)c}{2(\phi+1)}.$$

The welfare is maximized when $\psi = 0$ (i.e., $\bar{k}_H = \bar{k}_h = \frac{\phi}{2}$) if ϕ is even, or $|\psi| = 1$ if ϕ is odd. We take into account this necessary condition of an optimal threshold pair and rewrite the welfare as

$$W(\phi) = \begin{cases} pU_{Hh} + (1-p)U_{Ll} - \frac{p(1-p)U}{\phi+1} - \frac{(\phi+1)c}{2} & \text{if } \phi \text{ is odd} \\ pU_{Hh} + (1-p)U_{Ll} - \frac{p(1-p)U}{\phi+1} - \frac{(\phi+1)c}{2} + \frac{c}{2(\phi+1)} & \text{if } \phi \text{ is even.} \end{cases}$$

Define the marginal increase of welfare when increasing the threshold by one as:

$$MW(\phi) \equiv W(\phi+1) - W(\phi).$$

If $\phi \in \mathbb{Z}_+$ is odd,

$$\begin{aligned} MW(\phi) &= -\frac{p(1-p)U}{\phi+2} - \frac{(\phi+2)c}{2} + \frac{c}{2(\phi+2)} + \frac{p(1-p)U}{\phi+1} + \frac{(\phi+1)c}{2} \\ &= \frac{p(1-p)U}{(\phi+1)(\phi+2)} - \frac{c}{2}\left(\frac{\phi+1}{\phi+2}\right). \end{aligned}$$

If $\phi \in \mathbb{Z}_+$ is even,

$$\begin{aligned} MW(\phi) &= -\frac{p(1-p)U}{\phi+2} - \frac{(\phi+2)c}{2} + \frac{p(1-p)U}{\phi+1} + \frac{(\phi+1)c}{2} - \frac{c}{2(\phi+1)} \\ &= \frac{p(1-p)U}{(\phi+1)(\phi+2)} - \frac{c}{2}\left(\frac{\phi+2}{\phi+1}\right). \end{aligned}$$

For non-trivial (i.e., non-zero) optimal thresholds, it is necessary that $MW(0) > 0$, or equivalently $c < \frac{p(1-p)U}{2}$. Suppose c is small enough that this is the case. A necessary condition for an optimal sum of thresholds $\phi^*(\geq 1)$ is

$$MW(\phi^*) \leq 0 \leq MW(\phi^* - 1).$$

Thus, a necessary condition for an odd ϕ^* is

$$\frac{p(1-p)U}{(\phi^*+1)(\phi^*+2)} - \frac{c}{2}\left(\frac{\phi^*+1}{\phi^*+2}\right) \leq 0 \leq \frac{p(1-p)U}{\phi^*(\phi^*+1)} - \frac{c}{2}\left(\frac{\phi^*+1}{\phi^*}\right),$$

which is equivalent to

$$\phi^* = \sqrt{\frac{2p(1-p)U}{c}} - 1.$$

Similarly, a necessary condition for an even ϕ^* is

$$\frac{p(1-p)U}{(\phi^*+1)(\phi^*+2)} - \frac{c}{2}\left(\frac{\phi^*+2}{\phi^*+1}\right) \leq 0 \leq \frac{p(1-p)U}{\phi^*(\phi^*+1)} - \frac{c}{2}\left(\frac{\phi^*}{\phi^*+1}\right),$$

which is equivalent to

$$\sqrt{\frac{2p(1-p)U}{c}} - 2 \leq \phi^* \leq \sqrt{\frac{2p(1-p)U}{c}}.$$

Therefore, an optimal thresholds sum ϕ^* must be even unless $\sqrt{\frac{2p(1-p)U}{c}}$ is an even integer. The generically unique optimal threshold is identified by

$$\bar{k}_H^{opt} = \bar{k}_h^{opt} = \frac{\phi^*}{2} = \left\lfloor \sqrt{\frac{p(1-p)U}{2c}} \right\rfloor.$$

It is easy to verify that, when $\sqrt{\frac{p(1-p)U}{2c}}$ is an integer, any combination of thresholds $(\bar{k}_H^{opt}, \bar{k}_h^{opt})$ such that

$$\bar{k}_H^{opt}, \bar{k}_h^{opt} \in \left\{ \sqrt{\frac{p(1-p)U}{2c}}, \sqrt{\frac{p(1-p)U}{2c}} - 1 \right\}$$

identifies an optimal mechanism. Furthermore, multiplicity emerges only when $\sqrt{\frac{p(1-p)U}{2c}}$ is an integer. ■

Proof of Corollary 1: Using the optimal thresholds from Proposition 2, we get that for $c \leq \frac{p(1-p)U}{2}$,

$$f(c) \equiv \frac{p(1-p)U}{2\bar{k}^{opt}+1} = \frac{p(1-p)U}{2\left\lfloor \sqrt{\frac{p(1-p)U}{2c}} \right\rfloor + 1}, \quad \text{and}$$

$$g(c) \equiv \frac{2\bar{k}^{opt}(\bar{k}^{opt}+1)}{2\bar{k}^{opt}+1}c = \frac{(2\bar{k}^{opt}+1)c}{2} - \frac{c}{2(2\bar{k}^{opt}+1)} =$$
$$= \left[\left(\left\lfloor\sqrt{\frac{p(1-p)U}{2c}}\right\rfloor + \frac{1}{2}\right) - \frac{1}{4\left\lfloor\sqrt{\frac{p(1-p)U}{2c}}\right\rfloor + 2}\right] c.$$

We can then define $\Theta(c) \equiv f(c) + g(c)$ to get the representation of $W^{opt}(c)$ in the corollary.

Take any $c < \frac{p(1-p)U}{2}$ for which $\bar{k}^{opt} \notin \mathbb{Z}_+$. There exists $\varepsilon > 0$ such that for every c' with $|c' - c| < \varepsilon$, $\bar{k}^{opt}(c') = \bar{k}^{opt}(c)$.[38] Thus, $\Theta(c)$ is differentiable at c. Moreover, for any $c < \frac{p(1-p)U}{2}$ such that $\bar{k}^{opt} \in \mathbb{Z}_+$, Θ is semi-differentiable. Hence, $\Theta(c)$ is continuous.

At any differentiable point c (around which $\bar{k}^{opt}(c)$ is constant),

$$\frac{d\Theta(c)}{dc} = \frac{\partial\Theta(\bar{k}^{opt}(c), c)}{\partial c} = \frac{2\bar{k}^{opt}(\bar{k}^{opt}+1)}{2\bar{k}^{opt}+1} > 0.$$

Furthermore, the concavity of $\Theta(c)$ follows from the fact that at any semi-differentiable but not differentiable point c,

$$\frac{d_-\Theta(c)}{dc} > \frac{d_+\Theta(c)}{dc}.$$

■

8.2 Proofs Regarding Discretionary Matching

8.2.1 Players' Markov Decision Problem

In this section we study stationary* equilibria under several priority protocols. A key first step is to formalize each player's dynamic decision problem, defined by other players' equilibrium strategies, as a Markov decision problem (MDP). We consider here an H-square's problem and omit the analogous descriptions for other player types. Our formalization applies to the case of the FIFO priority protocol, as well as to the alternative protocols studied in Section 5 and in the Online Appendix.

Fix any priority rule, and take an H-square, say player i, who arrived in period $t_0 \geq 1$. Assume that all other players follow a stationary* strategy profile Ψ_{-i}.[39] Player i solves

[38] We slightly abuse our notation and make the dependence of $\bar{k}^{opt}$ on the cost c explicit here.

[39] That is, restricting attention to all players but i, the strategy profile is stationary* and in particular symmetric.

an infinite-horizon dynamic decision problem, defined by Ψ_{-i}. For each period $t \geq t_0$, let $\theta_i^t = (\mathbf{s}^t, q_i^t)$ denote the player's augmented state, where $\mathbf{s}^t = (s_H^t, s_L^t, s_h^t, s_l^t)$ denotes the state of the market, and q_i^t denotes player i's rank among the H-squares present. We write $q_i^t = 0$ if player i is matched before period t. We denote by Θ_i the set of player i's possible augmented states. In each period $t \geq t_0$, player i chooses a demand $d_i \in \{h, l\}$, where h represents a demand for an h-round, and l represents a demand for *any* round. The stage-game payoff $u_i(d_i, \theta_i, \Psi_{-i})$ is either a match surplus ($U_H(h)$ or $U_H(l)$), waiting cost $-c$, or 0 (if $q_i = 0$). The initial augmented state is $\theta_i^{t_0} = (\mathbf{s}^{t_0}, q_i^{t_0})$ such that $q_i^{t_0} = s_H^{t_0}$ under FIFO and $q_i^{t_0} = 1$ under LIFO. The transition between augmented states is straightforward from our description of the model, hence we omit it here.

A strategy σ_i is any rule prescribing demands submitted over time. It may entail randomization and it may be history-dependent. The payoff for player i from strategy σ_i is

$$U_i(\sigma_i; \theta_i, \Psi_{-i}) \equiv E_{\sigma_i}\left[\sum_{t=t_0}^{\infty} u_i(d_i, \theta_i^t, \Psi_{-i}) : \theta_i^{t_0} = \theta_i\right].^{40}$$

We focus on player i's Markov random strategies in the sense that a choice in each period is independent of past history. This restriction is without loss of generality since

$$\sup_{\sigma_i \in \Sigma_i} U_i(\sigma_i; \theta_i, \Psi_{-i}) = \sup_{\sigma_i \in \Sigma_i^{MR}} U_i(\sigma_i; \theta_i, \Psi_{-i}),$$

where Σ_i and Σ_i^{MR} denote the set of all strategies and all Markov random strategies, respectively (Proposition 7.1.1 of Puterman, 2014). The restriction to Markov random strategies allows us to normalize player i's arrival time as $t_0 = 0$ and write

$$U_i(\sigma_i; \theta_i, \Psi_{-i}) = E_{\sigma_i}\left[\sum_{t=0}^{\infty} u_i(d_i, \theta_i^t, \Psi_{-i}) : \theta_i^0 = \theta_i\right] \quad \text{for each } \sigma_i \in \Sigma_i^{MR}.$$

We extend player i's decision problem to a Markov decision problem (MDP) with an arbitrary initial state. That is, an initial state θ_i^0 can be any element in $\Theta_i \subseteq \{(\mathbf{s}, q_i) \in \mathbb{Z}_+^5 : s_H + s_L = s_h + s_l, 1 \leq q_i \leq s_H\}$.[41] A *policy* μ_i of the (MDP) is any Markovian random rule for choosing demands. A *stationary and deterministic policy* (SD-policy, for short) applies the same decision rule in every period. We denote a SD-policy by $\psi_i : \Theta_i \to \{h, l\}$. The *value of a policy* μ_i, for each initial state θ_i, is defined by

$$v_i(\mu_i; \theta_i, \Psi_{-i}) \equiv U_i(\mu_i; \theta_i, \Psi_{-i}).$$

[41] As mentioned, the initial condition $\theta_i^0 = (\mathbf{s}^0, q_i^0)$ should satisfy $q_i^0 = s_H^0$ under FIFO and $q_i^0 = 1$ under LIFO. We remove such restrictions in the (MDP).

Last, the *value of the (MDP)*, for each initial state θ_i, is

$$v_i^*(\theta_i; \Psi_{-i}) \equiv \sup_{\mu_i} v_i(\mu_i; \theta_i, \Psi_{-i}).$$

We characterize the value of the (MDP), $v_i^*(\cdot; \Psi_{-i}) : \Theta_i \to \mathbb{R} \cup \{-\infty\}$ and find an optimal SD-policy, whose value is equal to $v_i^*(\theta_i; \Psi_{-i})$ for every initial state θ_i. An optimal SD-policy of the (MDP) defines a best-response that is a stationary and deterministic strategy for player i given any initial state. A stationary* strategy profile $\Psi = (\psi_H, \psi_L, \psi_h, \psi_l)$ is a stationary* equilibrium if, for every H-square (similarly for other types), ψ_H is an optimal SD-policy of the (MDP) defined by all other players' equilibrium strategies Ψ_{-i}.

We use the following definition and theorems from Puterman (2014) that are associated with player i's problem, but hold for general Markov decision problems.

Definition 3. *(Optimality Equation; Equation 6.2.2 with $\lambda = 1$, or Equation 7.1.8 of Puterman, 2014) We refer to the following system of equations as the* optimality equation*:*

$$v(\theta_i) = \max_{d \in \{h,l\}} \left[u_i(d_i, \theta_i, \Psi_{-i}) + \sum_{\theta_i' \in \Theta_i} p(\theta_i' | \theta_i, d_i, \Psi_{-i}) v(\theta_i') \right],$$

for all $\theta_i \in \Theta_i$.

Theorem Puterman 1 (Theorem 7.1.3 of Puterman, 2014) *The value of the (MDP), $v_i^*(\cdot; \Psi_{-i})$, is a solution of the optimality equation.*

Theorem Puterman 2 (Theorem 7.2.5 (a) of Puterman, 2014) *A policy μ_i^* is optimal if and only if the value of the policy $v_i^*(\cdot; \mu_i^*, \Psi_{-i}) : \Theta_i \to R \cup \{-\infty\}$ is a solution of the optimality equation.*

Note that the value of the (MDP) is not a unique solution of the optimality equation. For example, we can add a constant to the value and find another solution. This non-uniqueness is a consequence of there not being discounting in our model.

Finally, the state space Θ_i for player i can be finite under some stationary* strategy-profile Ψ_{-i} chosen by other players. We have

Theorem Puterman 3 (Theorem 7.1.9 of Puterman, 2014) *If Θ_i is finite, then there exists an optimal SD-policy.*

8.2.2 Proofs Regarding Stationary* Equilibria under FIFO

The following Lemmas A2 and A3 are employed in the proofs of Lemmas 1 and 2.

Lemma A2 *Under FIFO, if* $\Psi^* = (\psi_H^*, \psi_L^*, \psi_h^*, \psi_l^*)$ *is a stationary* equilibrium, then*

$$\psi_H^*(\mathbf{s}, q) = \begin{cases} h \text{ or } l & \text{if } q \leq s_h, \\ h & \text{if } 1 \leq q - s_h \leq \bar{k}^{fifo}, \\ l & \text{otherwise,} \end{cases} \tag{11}$$

where

$$\bar{k}^{fifo} \equiv \left\lfloor \frac{p(U_H(h) - U_H(l))}{c} \right\rfloor = \left\lfloor \frac{p(U_h(H) - U_h(L))}{c} \right\rfloor.$$

An analogous claim holds for h-rounds.

Proof of Lemma A2: We show that if Ψ^* is a stationary* equilibrium, for any augmented state $\theta_i = (\mathbf{s}, q_i)$ for player i, who is an H-square, we have

$$\psi_H^*(\theta_i) = \begin{cases} h & \text{if } 1 \leq q_i - s_h \leq \bar{k}^{fifo} \\ l & \text{if } q_i - s_h > \bar{k}^{fifo}. \end{cases} \tag{12}$$

The proof is by induction. Take any stationary* strategy-profile $\Psi = (\psi_H, \psi_L, \psi_h, \psi_l)$. First, we characterize the equilibrium behavior of player i, an H-square, when she finds no available h-round in a period, and is positioned in the queue so that she is to become first in line if she stays for an additional period. Formally, i's augmented state in period t_0 satisfies $q_i^{t_0} = s_h^{t_0} + 1$. Indeed, in period t_0, player i finds no available h-round (i.e., $q_i^{t_0} > s_h^{t_0}$) and, if she is not matched, she becomes the first H-square in the queue (i.e., $q_i^{t_0} - s_h^{t_0} = 1$). In finding a dynamic best-response from period t_0 onward, it is without loss of generality to restrict attention to player i's Markov random strategies, see (**??**). Once we restrict attention to Markov random strategies, we can normalize $t_0 = 0$. Player i solves the following problem:

$$v_i^*(\theta_i; \Psi_{-i}) \equiv \sup_{\sigma_i \in \Sigma_i^{MR}} E_{\sigma_i} \left[\sum_{t=0}^{\infty} u_i(d_i, \theta_i^t, \Psi_{-i}) : \theta_i^0 = \theta_i \right].$$

Since $s_l^0 \geq q_i^0 + s_L^0 - s_h^0 > 0$, we know that at least one l-round is available in period $t_0 = 0$. Also, the first h-round to arrive at the market will be available to match with player i, but it takes, in expectation, $1/p$ periods until such h-round arrives. Therefore, we have

$$v_i^*(\theta_i; \Psi_{-i}) = \max \left\{ U_H(l), U_H(h) - \frac{c}{p} \right\}.$$

Hence, if ψ_H^* is part of a stationary* equilibrium, it must be that

$$\psi_H^*(\theta_i) = \begin{cases} h & \text{if } q_i - s_h = 1 \leq \bar{k}^{fifo} \\ l & \text{if } q_i - s_h = 1 > \bar{k}^{fifo}. \end{cases}$$

That is, (12) holds for every $\theta_i = (\mathbf{s}, q_i)$ with $q_i - s_h = 1$.

Next, we complete the induction. Take any $k \in \mathbb{Z}_{++}$ and a stationary* strategy-profile Ψ such that ψ_H satisfies (12) for every augmented state $\theta = (\mathbf{s}, q)$ with $q - s_h \leq k$. Consider any H-square, say player i, whose augmented state in period 0 (normalized, as above) satisfies $q_i^0 = s_h^0 + (k+1)$. Assume that every other H-square, say player j with $q_j^0 \leq s_h^0 + k < q_i^0$, plays ψ_H. Given that each player's rank in the queue only improves over time, the first $\min\{k, \bar{k}^{fifo}\}$ arriving h-rounds in the future are not available for player i, but the next arriving h-round will be. In expectation, it takes $\frac{\min\{k, \bar{k}^{fifo}\}+1}{p}$ periods until an h-round becomes available for player i. As such,

$$\begin{aligned} v_i^*(\theta_i; \Psi_{-i}) &= \max\left\{U_H(l), U_H(h) - \frac{(\min\{k, \bar{k}^{fifo}\} + 1)c}{p}\right\} \\ &= \max\left\{U_H(l), U_H(h) - \frac{(k+1)c}{p}, U_H(h) - \frac{(\bar{k}^{fifo} + 1)c}{p}\right\} \\ &= \max\left\{U_H(l), U_H(h) - \frac{(k+1)c}{p}\right\}, \end{aligned}$$

where the last equality follows from the definition of $\bar{k}^{fifo}$.

Therefore, if ψ_H is part of a stationary* equilibrium, $\psi_H(\theta_i)$ must satisfy (12) for any augmented state $\theta_i = (\mathbf{s}, q_i)$ with $q_i - s_h = k + 1$. ■

Lemma A3 *There exists a stationary* equilibrium* $\Psi^* = (\psi_H^*, \psi_L^*, \psi_h^*, \psi_l^*)$ *such that*

- **(a)** ψ_H^* *(and* ψ_h^**) satisfies* (11) *(with an analogous condition for h-rounds), and*
- **(b)** $\psi_l^*(s, s_l) = L$ *and* $\psi_L^*(s, s_L) = l$*, whenever* $s_L > 0$ *and* $s_l > 0$.

Proof of Lemma A3: We start with the analysis of the H-squares' decisions. Take any stationary* strategy-profile $\Psi = (\psi_H, \psi_L, \psi_h, \psi_l)$ that satisfies conditions (a) and (b) in the claim. We prove that ψ_H is a best-response for an H-square, say player i, regardless of her initial augmented state. Let Θ_H be the set of all possible augmented states for player i, conditional on Ψ_{-i}. That is[42],

$$\Theta_H \equiv \{(\mathbf{s}, q) \in \mathbb{Z}_+^5 : -\bar{k}^{fifo} - 1 \leq s_{Hh} \leq \bar{k}^{fifo} + 2,\ q_i \leq s_H\}.$$

We extend player i's decision problem as a (MDP) with an arbitrary initial state (ignoring the fact that her initial state in the discretionary matching is $q_i = s_H$). That is, player i's (MDP) is

$$v_i^*(\theta_i; \Psi_{-i}) \equiv \sup_{\mu_i \in \Sigma_i^{MR}} v_i(\mu_i; \theta_i, \Psi_{-i}), \quad \text{for all } \theta_i \in \Theta_H,$$

where

$$v_i(\theta_i; \mu_i, \Psi_{-i}) \equiv U_i(\mu_i; \theta_i, \Psi_{-i}) \equiv E_{\mu_i}\left[\sum_{t=0}^{\infty} u_i(d_i, \theta_i^t, \Psi_{-i}) : \theta_i^0 = \theta_i\right].$$

If player i follows the SD-policy ψ_H that satisfies (11), then, for $\theta_i = (\mathbf{s}, q_i)$,

$$\begin{aligned} v_i(\theta_i; \psi_H, \Psi_{-i}) &= E\left[\sum_{t=0}^{\infty} u_i(\psi_H(\theta_i^t), \theta_i^t, \Psi_{-i}) : \theta_i^0 = \theta_i\right] \\ &= \begin{cases} U_H(h) & \text{if } q_i \leq s_h \\ U_H(h) - \frac{(q_i - s_h)c}{p} & \text{if } 1 \leq q_i - s_h \leq \bar{k}^{fifo} \\ U_H(l) & \text{otherwise.} \end{cases} \end{aligned}$$

From the construction of ψ_H in the proof of Lemma A2, it is easy to verify that $v_i(\cdot; \psi_H, \Psi_{-i}) : \Theta_H \to \mathbb{R} \cup \{-\infty\}$ solves the optimality equation. Thus, by Theorem Puterman 2, ψ_H is an optimal SD-policy of player i's (MDP). In particular, each H-square is best-responding by playing ψ_H, regardless of her initial augmented state.

Next, we consider l-rounds' decisions. Let Θ_l denote the set of all possible augmented states that a l-round may experience:

$$\Theta_l \equiv \{(\mathbf{s}, q) \in \mathbb{Z}_+^5 : s_H + s_L = s_h + s_l,\ q \leq s_l\},$$

where $q = 0$ represents the augmented state after the player is matched.

We take ψ_H and ψ_h satisfying condition (a) in Lemma A3. We want to construct a SD-strategy $\psi_l : \Theta_l \to \{H, L\}$ (and $\psi_L : \Theta_L \to \{h, l\}$, whose analogous construction we omit) such

[42] Recall that $q_i^t = 0$ for any period t after player i matches. If $q_i^t = 0$, player i's stage-game payoff is $u_i^t = 0$. Moreover, $s_{Hh} = \bar{k}^{fifo} + 2$ can occur, if player i deviates from ψ_H. For example, player i may arrive at the market with a rank $q_i = s_{Hh} = \bar{k}^{fifo} + 1$. If she deviates from ψ_H by demanding h, then s_{Hh} can be $\bar{k}^{fifo} + 2$ in the following period due to an additional arrival of an H-square.

that $\Psi = (\psi_H, \psi_L, \psi_h, \psi_l)$ constitutes a stationary* equilibrium. The following assumption on ψ_l (and ψ_L) will be useful for our construction:

Assumption A1 *For any $(s, s_l) \in \Theta_l$, $(s, s_L) \in \Theta_L$ with $s_l > 0$ and $s_L > 0$,*

(1) $\psi_l(s, s_l) = \psi_l(s, s_l - 1) = ... = \psi_l(s, 1 + s_{Hh}^+) = L$;

(2) $\psi_L(s, s_L) = \psi_L(s, s_L - 1) = ... = \psi_L(s, 1 + s_{Hh}^-) = l$,

where $s_{Hh}^+ = \max\{s_{Hh}, 0\}$ *and* $s_{Hh}^- = -\min\{s_{Hh}, 0\}$.

Note that Assumption A1 is consistent with condition (b) in Lemma A3. The construction of ψ_l that follows will guarantee that ψ_l is a best-response for a l-round in any period and with any initial augmented state, if all other players satisfy Assumption A1. We will then justify Assumption A1 as describing best-response strategies.

Take any l-round, say player i, and any stationary* strategy-profile Ψ_{-i} such that ψ_H and ψ_h satisfy (11) and Assumption A1 holds. As argued before, there is no period in which both H-squares and h-rounds wait at the market. Therefore, for any t, the stock $\mathbf{k}^t \equiv (k_H^t, k_L^t, k_h^t, k_l^t)$ satisfies $k_H^t k_h^t = 0$ and $-\bar{k}^{fifo} \leq k_{Hh}^t \leq \bar{k}^{fifo}$. In addition, by Assumption A1, there is no period in which at least two L-squares and two l-rounds wait by demanding h-rounds and H-squares, respectively.

We characterize the set of augmented states for player i, which we denote by $\Theta_l' \subseteq \Theta_l$. Let $K \subseteq \mathbb{Z}_+^4$ denote the set of possible states at the end of each period. That is,

$$\mathbf{k} \equiv (k_H, k_L, k_h, k_l) \in K \iff \begin{array}{ll} (i) & k_H + k_L = k_h + k_l, \\ (ii) & k_H k_h = 0, \\ (iii) & -\bar{k}^{fifo} \leq k_{Hh} \leq \bar{k}^{fifo}, \\ (iv) & k_{Hh} \geq 0 \implies k_L \leq 1, \text{ and } k_{Hh} \leq 0 \implies k_l \leq 1. \end{array}$$

Then, Θ_l' is a subset of Θ_l such that

$$(\mathbf{s}, q) \in \Theta_l' \iff \begin{array}{l} (\exists \mathbf{k} \in K) \quad s.t.\ \mathbf{s} - \mathbf{k} \in \{(1,0,1,0), (0,1,0,1), (1,0,0,1), (0,1,1,0)\} \\ \text{and } q \leq s_l. \end{array}$$

It is clear that Θ_l' is finite, and for any $(\mathbf{s}, q) \in \Theta_l'$, we have $0 \leq q \leq \bar{k}^{fifo} + 2$. It is sufficient to define ψ_l over augmented states in Θ_l' only, as an augmented state $(\mathbf{s}, q) \notin \Theta_l'$ never occurs. Under the FIFO protocol, the ranking of a l-round, such as our player i, improves as she waits in the market. Thus, player i's continuation payoff from the (MDP) after her rank becomes

1 is independent of her actions in a state with a rank lower than 1. For each possible ranking of a l-round, $q \in \{1, 2, \ldots, \bar{k}^{fifo} + 2\}$, let $\Theta'_{l,q}$ be the set of augmented states with rank q (i.e., $\Theta'_{l,q} \equiv \{\mathbf{s} \mid (\mathbf{s}, q) \in \Theta'_l\}$). We construct $\psi_{l,q} : \Theta'_{l,q} \to \{H, L\}$ sequentially from $q = 1$ to $q = \bar{k}^{fifo} + 2$, and define $\psi_l : \Theta'_l \to \{H, L\}$ as $\psi_l(\mathbf{s}, q) \equiv \psi_{l,q}(\mathbf{s}, q)$. In the construction, we will guarantee that ψ_l constitutes a best-response for a l-round, taking as given ψ_H, ψ_h, and Assumption A1 applied to all other players.

The proof is inductive with the following induction hypothesis:

Induction Hypothesis *There exists $\psi_{l,q} : \Theta'_{l,q} \to \{H, L\}$ for $q \leq \bar{k}^{fifo} + 2$ such that,*

1. $\psi_{l,\leq q} = (\psi_{l,q}, \psi_{l,q-1}, \ldots, \psi_{l,1})$ *is a optimal SD-policy for player i, and*
2. *the maximal total expected payoff for player i, given any $\theta_i = (s, q_i) \in \Theta'_{l,q}$, is*

$$v_i^*(\theta_i) \leq \max\left\{U_l(L), U_l(H) - \frac{(\bar{k}^{fifo} - s_{Hh} + q_i)c}{p}\right\}.$$

Step 1: Construction of $\psi_{l,1}$

Consider a l-round, say our player i, who is the first in the queue at some period. Player i solves a dynamic decision problem, defined by ψ_H, ψ_h, and Assumption A1 (applied to other players' strategies). We extend player i's dynamic decision problem as a (MDP) with an arbitrary initial state $\theta_i \in \Theta'_{l,1}$. Let $v^*(\theta_i)$ denote the maximal expected total payoff for player i with an initial augmented state θ_i. Theorem Puterman 3 guarantees that there exists an optimal SD-policy. Moreover, any policy whose values solve the optimality equation is optimal by Theorem Puterman 2, which allows us to choose a particular optimal SD-policy $\psi_{l,1}$ that is consistent with Assumption A1. To proceed with the construction, we show the following Claims.

Claim 1 *For any $\theta_i \in (s, 1) \in \Theta_{l,1}$,*

$$v^*(\theta_i) \leq \max\left\{U_l(L), U_l(H) - \frac{(\bar{k}^{fifo} - s_{Hh} + 1)c}{p}\right\}.$$

Proof of Claim 1: Take a l-round, say player i, who is the first in the queue for l-rounds in some period, which we normalize to be $t_0 = 0$, and augmented state $(\mathbf{s}^0, 1) \in \Theta'_{l,1}$. Given ψ_H, to match with an H-square, player i must wait for at least $\bar{k}^{fifo} - s^0_{Hh} + 1$ additional arrivals of H-squares.

Consider now the following optimal stopping problem:

> **[P]** *A boy* (l) *stands under an apple* (A) *tree and holds a banana* (B). *In each period, one apple falls from the tree with probability* p. *The first* $\bar{k}^*(\equiv \bar{k}^{fifo} - s^0_{Hh})$ *apples should be handed over to the owner of the tree. The boy can consume exactly one piece of fruit, either an apple or a banana. He prefers an apple, with payoff* $U_l(H)$, *to the banana, with payoff* $U_l(L)$. *Thus, while he can consume the banana and walk away with* $U_l(L)$ *in any period, he may want to wait for falling apples. He incurs a cost* c *for each period of waiting without consuming any fruit.*

Let $\Theta_{(P)} \equiv \{0, 1, \ldots, \bar{k}^* + 1\} \cup \{\Delta\}$ denote the state space of [P], where Δ denotes the (absorbing) state after the boy consumes a piece of fruit. In each period t and state $\theta^t_{(P)} \in \Theta_{(P)} \backslash \{\Delta\}$, the boy chooses a demand $d \in \{H, L\}$. The stage payoff from demand H is either $U_l(H)$ in state $\theta^t_{(P)} = \bar{k}^* + 1$, or $-c$ in any other state in $\Theta_{(P)} \backslash \{\Delta\}$. The stage payoff from demand L is $U_l(L)$ in any state in $\Theta_{(P)} \backslash \{\Delta\}$. In state Δ (i.e., after consuming a piece of fruit), the boy gets zero stage payoff forever. The value of [P] with an arbitrary initial state $\theta \in \Theta_{(P)}$ is[43]

$$v^*_{(P)}(\theta) \equiv \sup_{\mu} E_{\mu} \left[\sum_{t=0}^{\infty} u(d, \theta^t) : \theta^0 = \theta \right].$$

It is clear from the description of [P] that $v^*_{(P)}(0)$ constitutes an upper bound for the maximal expected total payoff of player i (i.e., $v^*(\theta_i)$). In fact, unlike player i, the boy in [P] can always consume a banana and walk away. Also, while player i must wait for *at least* $\bar{k}^* + 1$ arrivals of H-squares to match with an H-square, the boy in [P] is guaranteed to get the $(\bar{k}^* + 1)$-th falling apple. As such, to prove the claim, it is sufficient to show that

$$v^*_{(P)}(0) \leq \max \left\{ U_l(L), U_l(H) - \frac{(\bar{k}^{fifo} - s^0_{Hh} + 1)c}{p} \right\}.$$

Let

$$\bar{k}^{**} \equiv \left\lfloor \frac{p(U_l(H) - U_l(L))}{c} \right\rfloor \leq \bar{k}^{fifo}.$$

[43] The limit exists, because P is a positive bounded problem (see p.279 of Puterman, 2014).

(i) Suppose that $\bar{k}^* < \bar{k}^{**}$. Then, compared to consuming a banana immediately, it is weakly more profitable to wait until $\bar{k}^* + 1 = \bar{k}^{fifo} - s^0_{Hh} + 1 (\leq \bar{k}^{**})$ apples fall. Once the boy decides to wait, he will continue to wait until he obtains an apple. Thus,

$$v^*_{(P)}(0) = U_l(H) - \frac{(\bar{k}^* + 1)c}{p} = U_l(H) - \frac{(\bar{k}^{fifo} - s^0_{Hh} + 1)c}{p}.$$

(ii) Suppose that $\bar{k}^* = \bar{k}^{**}$. As $v^*_{(P)}(\cdot)$ solves the optimality equation, we have

$$v^*_{(P)}(0) = \max\left\{U_l(L), -c + p\left(v^*_{(P)}(1)\right) + (1-p)\left(v^*_{(P)}(0)\right)\right\}.$$

Suppose, toward a contradiction, that

$$v^*_{(P)}(0) = -c + p\left(v^*_{(P)}(1)\right) + (1-p)v^*_{(P)}(0) > U_l(L).$$

Then,

$$\begin{aligned} v^*_{(P)}(0) &= v^*_{(P)}(1) - \frac{c}{p} = \left(U_l(H) - \frac{\bar{k}^* c}{p}\right) - \frac{c}{p} \\ &= U_l(H) - \frac{(\bar{k}^{**} + 1)c}{p} > U_l(L), \end{aligned}$$

where the second equality follows from case (i) above (after the first apple falls, the boy needs to hand over only $\bar{k}^* - 1 (< \bar{k}^{**})$ additional apples to the owner). Notice that the last inequality contradicts the definition of $\bar{k}^{**}$. Therefore, $v^*_{(P)}(0) \leq U_l(L)$.

(iii) Suppose that $\bar{k}^* > \bar{k}^{**}$. More apples should be handed over to the owner than in the previous case, so $v^*_{(P)}(0) \leq U_l(L)$.

This concludes the proof of Claim 1.

Claim 2 *There exists an optimal SD-policy $\psi_{l,1} : \Theta'_{l,1} \to \{H, L\}$ of the (MDP) for player i such that*

$$\psi_{l,1}(\theta_i) = L, \quad \text{for all } \theta_i = (\mathbf{s}, 1) \in \Theta'_{l,1} \text{ with } s_{Hh} < 1.$$

Proof of Claim 2: Let $\psi_{l,1} : \Theta'_{l,1} \to \{H, L\}$ such that

$$\psi_{l,1}(\theta_i) = \begin{cases} H & \text{if } v^*(\theta_i) > U_l(L), \\ L & \text{if } v^*(\theta_i) \leq U_l(L). \end{cases}$$

Then, $v_i(\cdot; \psi_{l,1}) : \Theta'_{l,1} \to \mathbb{R} \cup \{-\infty\}$ is a solution of the optimality equation of player i's (MDP). It follows from Theorem Puterman 2 that the SD-policy $\psi_{l,1}$ is optimal.

By Claim 1, for any $\theta_i = (\mathbf{s}, 1) \in \Theta'_{l,1}$ with $s_{Hh} < 1$,

$$v_i^*(\theta_i) \le \max\left\{U_l(L), U_l(H) - \frac{(\bar{k}^{fifo}+1)c}{p}\right\} = U_l(L),$$

so that $\psi_{l,1}(\theta_i) = L$. This concludes the proof of Claim 2.

Step 2: Construction of $\psi_{l,q+1}$ *given* $(\psi_{l,1}, \psi_{l,2}, \ldots, \psi_{l,q})$

Fix $q \in \{1, 2, \ldots, \bar{k}^{fifo}+1\}$. For a l-round, say player i, who enters as q-th in line, we extend the player's dynamic decision problem as a (MDP) with an arbitrary initial augmented-state set $\Theta'_{l,\le q} \equiv \bigcup_{q' \le q} \Theta'_{l,q'}$. Note that the (MDP) for player i is defined by ψ_H, ψ_h, $\psi_{l,<q} \equiv (\psi_{l,q-1}, \psi_{l,q-2}, \ldots, \psi_{l,1})$, $\psi_{L,<q} \equiv (\psi_{L,q-1}, \psi_{L,q-2}, \ldots, \psi_{L,1})$, and by Assumption A1 applied to other players' strategies.

Now, consider a l-round, say player j, who is $(q+1)$-th in line at some period $t_0 = 0$ (normalized). Player j solves a dynamic decision problem. As before, we extend player j's problem as a (MDP) with an arbitrary initial augmented state in the set $\Theta'_{l,\le q+1} \equiv \bigcup_{q' \le q+1} \Theta'_{l,q'}$. Note that player j's (MDP) is defined by ψ_H, ψ_h, $\psi_{L,\le q}$, $\psi_{l,\le q}$, and by Assumption A1 applied to other players' strategies. As the set of augmented states for player j is still finite, there exists an optimal SD-policy (see Theorem Puterman 3). Moreover, any policy whose values solve the optimality equation is optimal (see Theorem Puterman 2). In particular, it is optimal for player j to follow any optimal SD-policy of his (MDP) until his rank becomes q, after which he switches to any optimal policy of a (MDP) for a l-round who enters as q-th in line. Thus, to find an optimal policy for player j, it is sufficient to find a function $\psi_{l,q+1} : \Theta'_{l,q+1} \to \{H, L\}$ that is consistent with Assumption A1.

Claim 3 *For any* $\theta_j = (s, q+1) \in \Theta'_{l,q+1}$,

$$v_j^*(\theta_j) \le \max\left\{U_l(L), U_l(H) - \frac{(\bar{k}^{fifo} - s_{Hh} + q + 1)c}{p}\right\}.$$

Proof of Claim 3: Take a l-round, say player j, who is $(q+1)$-th in the queue of l-rounds in period $t_0 = 0$ (normalized), and any augmented state $(\mathbf{s}^0, q+1) \in \Theta'_{l,q+1}$. Two observations will be useful:

1. In any augmented state $\theta_j = (\mathbf{s}, q+1) \in \Theta'_{l,q+1}$, if there exists any $q' < q+1$ such that $\psi_{(\mathbf{s},q')} = L$, the maximum expected continuation payoff for player j is at most $U_l(L)$.

2. In any augmented state $\theta_j = (\mathbf{s}, q+1)$ with $s_{Hh} = \bar{k}^{fifo}+1$, the first l-round in the queue matches with an H-square. Thus, the maximum expected continuation payoff for player j (i.e., $v_j^*(\theta_j)$) equals $v_j^*(\mathbf{s}', q)$, where $\mathbf{s}'$ denotes the augmented state after matching the first l-round with an H-square.[44] As $s'_{Hh} = \bar{k}^{fifo}$, by the induction hypothesis holding up to q,
$$v_j^*(\theta_j) = v_j^*(\mathbf{s}', q) \leq \max\left\{U_l(L), U_l(H) - \frac{qc}{p}\right\}.$$

Player j either matches with a L-square and receives $U_l(L)$ while his rank is $q+1$ or has a corresponding augmented state at some period before matching. Moreover, starting from an arbitrary initial augmented state $\theta_j^0 = (\mathbf{s}^0, q+1)$, the second case occurs only after at least $\bar{k}^{fifo} - s_{Hh}^0 + 1$ arrivals of H-squares.

Consider the following optimal stopping problem:

> **[P']** *A boy (l) stands under an apple (A) tree and holds a banana (B). In each period, one apple falls from the tree with probability p. The first $\bar{k}^*(\equiv \bar{k}^{fifo} - s_{Hh}^0)$ falling apples should be handed over to the owner of the apple tree. The boy can consume exactly one piece of fruit, either an apple or the banana. He (weakly) prefers an apple, with payoff $U'_l(H) \equiv \max\left\{U_l(L), U_l(H) - \frac{qc}{p}\right\}$, to the banana, with payoff $U_l(L)$. Thus, while he can consume the banana and walk away in any period, he may want to wait for falling apples. He incurs a cost c for each period of waiting without consuming any fruit.*

Similar to the proof of Claim 1, let $\Theta_{(P')} \equiv \{0, 1, \ldots, \bar{k}^* + 1\} \cup \{\Delta\}$ denote the state space of [P'], where Δ denotes the (absorbing) state after the boy consumes a fruit. The value of [P'] with an arbitrary initial state $\theta \in \Theta_{(P')}$ exists (by similar arguments to those used for the existence of the value of [P]).

The value of [P'] with the initial condition 0, denoted by $v^*_{(P')}(0)$, is an upper bound of the maximal expected payoff for player j. Unlike player j, the boy in [P'] can always consume a banana and walk away. While player j must wait for at least $\bar{k}^* + 1$ arrivals of H-squares to

[44] That is, $(s'_H, s'_L, s'_h, s'_l) = (s_H, s_L, s_h, s_l) - (1, 0, 0, 1)$.

get an expected continuation payoff of $U_l'(H)$, the boy in [P'] is guaranteed to get $U_l'(H)$ after $\bar{k}^*+1$ falling apples. As such, it is sufficient to prove that

$$v^*_{(P')}(0) \leq \max\left\{U_l(L), U_l(H) - \frac{\bar{k}^{fifo} - s^0_{Hh} + q + 1}{p}\right\}.$$

Let

$$\bar{k}^{**} \equiv \left\lfloor \frac{p(U_l'(H) - U_l(L))}{c} \right\rfloor \leq \bar{k}^{fifo}.$$

As in the proof of Claim 1, we consider three cases:

(i) Suppose that $\bar{k}^* < \bar{k}^{**}$. Compared to consuming a banana immediately, it is weakly more profitable to wait until $\bar{k}^*+1 = \bar{k}^{fifo} - s^0_{Hh} + 1 (\leq \bar{k}^{**})$ apples fall. Once the boy waits, he will continue to wait until he obtains an apple. Thus,

$$\begin{aligned} v^*_{(P')}(0) &= U_l'(H) - \frac{(\bar{k}^*+1)c}{p} = U_l'(H) - \frac{(\bar{k}^{fifo} - s^0_{Hh} + 1)c}{p} \\ &\leq \max\left\{U_l(L), U_l(H) - \frac{(\bar{k}^{fifo} - s^0_{Hh} + q + 1)c}{p}\right\}. \end{aligned}$$

(ii) Suppose that $\bar{k}^* = \bar{k}^{**}$. As $v^*_{(P')}(\cdot)$ solves the optimality equation (see Theorem Puterman 1),

$$v^*_{(P')}(0) = \max\left\{U_l(L), -c + p(v^*_{(P')}(1)) + (1-p)(v^*_{(P')}(0))\right\}.$$

Assume, toward a contradiction, that

$$v^*_{(P')}(0) = -c + p(v^*_{(P')}(1)) + (1-p)(v^*_{(P')}(0)) > U_l(L).$$

Then,

$$\begin{aligned} v^*_{(P')}(0) &= v^*_{(P')}(1) - \frac{c}{p} \\ &= \left(U_l'(H) - \frac{\bar{k}^* c}{p}\right) - \frac{c}{p} = U_l'(H) - \frac{(\bar{k}^{**}+1)c}{p} > U_l(L), \end{aligned}$$

where the second equality is from case (i). After the first falling apple, the boy needs to hand over only $\bar{k}^* - 1 (< \bar{k}^{**})$ additional apples to the owner. The last inequality contradicts the definition of $\bar{k}^{**}$. Therefore, $v^*_{(P')}(0) \leq U_l(L)$.

(iii) Suppose that $\bar{k}^* > \bar{k}^{**}$. More apples should be handed over to the owner than in the previous case, so $v^*_{(P')}(0) \leq U_l(L)$.

This concludes the proof of Claim 3.

Claim 4 *There exists* $\psi_{l,q+1} : \Theta'_{l,q+1} \to \{H, L\}$ *with*

$$\psi_{l,q+1}(\theta_j) = L, \quad \text{for all } \theta_j = (\mathbf{s}, q+1) \in \Theta'_{l,q+1} \text{ with } s_{Hh} < q+1,$$

such that $\psi_{l,\leq q+1} = (\psi_{l,q+1}, \psi_{l,q}, \ldots, \psi_{l,1})$ *is an optimal SD-policy of the* (MDP) *for player* j.

Proof of Claim 4: Let $\psi_{l,\leq q+1} : \Theta'_{l,q+1} \to \{H, L\}$ such that

$$\psi_{l,q+1}(\theta_j) = \begin{cases} H & \text{if } v_j^*(\theta_j) > U_l(L) \\ L & \text{if } v_j^*(\theta_j) \leq U_l(L). \end{cases}$$

Then, $v(\cdot; \psi_{l,\leq q+1}) : \Theta'_{l,\leq q+1} \to \mathbb{R} \cup \{-\infty\}$ solves the optimality equation of the (MDP) for player j. It follows from Theorem Puterman 2 that $\psi_{l,\leq q+1}$ is optimal.

By Claim (1), for any $\theta_j = (\mathbf{s}, q+1) \in \Theta'_{l,q+1}$ with $s_{Hh} < q+1$,

$$v_j^*(\theta_j) \leq \max\left\{U_l(L), U_l(H) - \frac{(\bar{k}^{fifo}+1)c}{p}\right\} = U_l(L),$$

so $\psi_{l,q+1}(\theta_j) = L$. This concludes the proof of Claim 4.

To conclude the proof of Lemma A3, let us turn to Assumption A1. Thus far, we have constructed ψ_l, which ascribes a best-response for a l-round for any initial augmented state, given ψ_H, ψ_h, and Assumption A1 applied to strategies of others. To conclude the proof, we need to guarantee that the ψ_l we constructed satisfies Assumption A1. Take any stationary* strategy-profile $\Psi = (\psi_H, \psi_L, \psi_h, \psi_l)$ such that ψ_H and ψ_h satisfy (11), and ψ_l and ψ_L are constructed as described above. Suppose that both L-squares and l-rounds exist in the market in a period t after a new pair arrives. We consider the case of $s_{Hh}^t \geq 0$ (and omit an analogous proof for the case of $s_{Hh}^t < 0$). For any l-round, say player i, with rank $q_i > s_{Hh}^t$, Claims 2 and 4 imply that player i would demand L. The counterpart of Claims 2 and 4 for L-squares implies that every L-square, say player j, demands a l-round as $q_j \geq 1 = 1 + s_{Hh}^-$. Therefore, Assumption A1 describes best-response behavior. This concludes the proof of Lemma A3. ■

Proof of Lemma 1: The proof follows directly from Lemmas A2 and A3 above. ■

Proof of Lemma 2. We show that the part (b) of Lemma A3 guarantees Lemma 2. Take a stationary* equilibrium Ψ^* satisfying conditions (a) and (b) in Lemma A3. Initially, there is no agent waiting in the market. Suppose that both a L-square and a l-round are present in some period t, for the first time ever. Given (11) (and a similar condition for ψ_h^*), it must be that either (i) $s_H^t \geq 0$, $s_h^t = 0$, $s_L^t = 1$, and $s_l^t = s_H^t + s_L^t$, or (ii) $s_H^t = 0$, $s_h^t \geq 0$, $s_L^t = s_h^t + s_l^t$, and $s_l^t = 1$. In both instances, there exists a L-square who finds no available h-round and demands a l-round, and a l-round who finds no available H-square and demands a L-square. As such, one (L, l) pair will be matched, and only incongruent pairs of agents (i.e., either H-squares and l-rounds, or L-squares and h-rounds) wait until period $t+1$. A similar argument shows that in any period in which a L-square and a l-round coexist, for the second time, third time, etc., one (L, l) pair will be formed. ■

Proof of Proposition 3: First, the (signed) length of the H-h queue, denoted by k_{Hh}, constitutes an ergodic Markov chain. Following arguments in the body of the paper, the unique steady state distribution of k_{Hh} is the uniform distribution over $\{-\bar{k}^{dec}, -\bar{k}^{dec}+1, \ldots, \bar{k}^{dec}\}$. At any time t, suppose that $k_{Hh}^t > 0$. Clearly, the queue has no h-rounds. As equal numbers of squares and rounds enter and exit the market, it must be that $k_{Hh} + k_L = k_l$. Lemma 2 guarantees that $k_L = 0$, therefore $k_l = k_{Hh}$. An analogous argument follows for $k_{Hh} \leq 0$. ■

Proof of Corollary 3: Whenever $c > \frac{p(1-p)U}{2}$, the optimal mechanism matches arriving agents immediately, $\bar{k}^{opt} = 0$, and $\bar{k}^{opt} \leq \bar{k}^{dec}$. Suppose, then, that $c < \frac{p(1-p)U}{2}$. We then have

$$\sqrt{\frac{p(1-p)U}{2c}} < \frac{p(1-p)U}{2c} \leq \frac{pU}{2c} \leq \frac{p(U_H(h) - U_H(l))}{c}.$$

and the result follows from the definitions of $\bar{k}^{opt}$ and $\bar{k}^{dec}$. ■

8.3 Proof Regarding Welfare Comparisons

Proof of Proposition 4:

1. As in the proof of Corollary 1, $W^{opt}(c) - W^{fifo}(c)$ is differentiable at any $c < \frac{p(1-p)U}{2}$ such that $\bar{k}^{opt}, \bar{k}^{fifo} \notin \mathbb{Z}_+$. In a small neighborhood around any such c, the thresholds corresponding to both the optimal and discretionary thresholds are constant in c. Therefore,

$$\frac{d(W^{opt}(c) - W^{fifo}(c))}{dc} = -\frac{2\bar{k}^{opt}(\bar{k}^{opt}+1)}{2\bar{k}^{opt}+1} + \frac{2\bar{k}^{fifo}(\bar{k}^{fifo}+1)}{2\bar{k}^{fifo}+1} \geq 0,$$

where the inequality follows from $\bar{k}^{fifo} \geq \bar{k}^{opt}$ (Corollary 3). Furthermore, the proof of Corollary 1 implies that $W^{opt}(c)$ is continuous in c and so $W^{opt}(c) - W^{fifo}(c)$ is increasing in any point c for which $\bar{k}^{fifo} \notin \mathbb{Z}_+$.

Let $\{d_k\}_{k=1}^{\infty}$ denote the decreasing sequence of costs such that $k = \frac{p(U_H(h)-U_H(l))}{d_k}$. That is, cost d_k corresponds to the maximal cost such that the equilibrium threshold is k in the discretionary process under FIFO.

For an arbitrary k, we will show that

$$W^{opt}(d_{k+1}) - W^{fifo}(d_{k+1}) > W^{opt}(d_k) - W^{fifo}(d_k),$$

or equivalently that

$$W^{opt}(d_{k+1}) - W^{opt}(d_k) > W^{fifo}(d_{k+1}) - W^{fifo}(d_k). \tag{13}$$

First, we focus on $W^{opt}(d_{k+1}) - W^{opt}(d_k)$. Note that $W^{opt}(c)$ is piece-wise linear and continuous in c. It follows that:

$$W^{opt}(d_{k+1}) - W^{opt}(d_k) = \int_{d_{k+1}}^{d_k} \frac{2\bar{k}^{opt}(c)(\bar{k}^{opt}(c)+1)}{2\bar{k}^{opt}(c)+1} dc.$$

Let $k^0 \equiv \left\lfloor \sqrt{\frac{p(1-p)U}{2d_k}} \right\rfloor$. For any $c \in [d_{k+1}, d_k]$, $\bar{k}^{opt}(c) \geq k^0$ and

$$\frac{2\bar{k}^{opt}(c)(\bar{k}^{opt}(c)+1)}{2\bar{k}^{opt}(c)+1} = \frac{1}{2}\left((2\bar{k}^{opt}(c)+1) - \frac{1}{2\bar{k}^{opt}(c)+1}\right) \geq \frac{2k^0(k^0+1)}{2k^0+1}.$$

Thus,

$$W^{opt}(d_{k+1}) - W^{opt}(d_k) \geq \int_{d_{k+1}}^{d_k} \frac{2k^0(k^0+1)}{2k^0+1} dc = \frac{2k^0(k^0+1)}{2k^0+1}(d_k - d_{k+1})$$

$$= \frac{2k^0(k^0+1)}{2k^0+1} p(U_H(h) - U_H(l)) \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{2k^0(k^0+1)}{2k^0+1} \frac{p(U_H(h) - U_H(l))}{k(k+1)}.$$

Next, we consider $W^{fifo}(d_{k+1}) - W^{fifo}(d_k)$. Denote by

$$W(m,c) \equiv S_\infty - \frac{p(1-p)U}{2m+1} - \frac{2m(m+1)c}{2m+1}.$$

Note that

$$\begin{aligned} W^{fifo}(d_{k+1}) - W^{fifo}(d_k) &= W(k+1, d_{k+1}) - W(k, d_k) \\ &= W(k+1, d_{k+1}) - W(k+1, d_k) + W(k+1, d_k) - W(k, d_k). \end{aligned}$$

We use the following two observations:

$$W(k+1, d_{k+1}) - W(k+1, d_k) = \frac{2(k+1)(k+2)}{2k+3}(d_k - d_{k+1})$$

$$= \frac{2(k+1)(k+2)}{2k+3}\frac{p(U_H(h)-U_H(l))}{k(k+1)} = \frac{2(k+2)p(U_H(h)-U_H(l))}{k(2k+3)},$$

and

$$W(k+1,d_k) - W(k,d_k) = \frac{p(1-p)U}{2k+1} + \frac{2k(k+1)d_k}{2k+1} - \frac{p(1-p)U}{2k+3} - \frac{2(k+1)(k+2)d_k}{2k+3}$$

$$= \frac{2p(1-p)U - 4(k+1)^2 d_k}{(2k+1)(2k+3)} = \frac{1}{(2k+1)(2k+3)}\left(2p(1-p)U - \frac{4(k+1)^2 p(U_H(h)-U_H(l))}{k}\right).$$

Then,

$$\begin{aligned} & W^{fifo}(d_{k+1}) - W^{fifo}(d_k) \\ = \; & \frac{2(k+2)p(U_H(h)-U_H(l))}{k(2k+3)} + \frac{2p(1-p)U}{(2k+1)(2k+3)} - \frac{4(k+1)^2 p(U_H(h)-U_H(l))}{k(2k+1)(2k+3)}. \end{aligned}$$

To prove (13), it suffices to show that

$$\begin{aligned} \frac{2k^0(k^0+1)}{2k^0+1}\frac{2k+3}{k+1} \; &> \; 2(k+2) + \frac{2k}{2k+1}\frac{(1-p)U}{U_H(h)-U_H(l)} - \frac{4(k+1)^2}{2k+1} \\ &= \; \frac{2k}{2k+1} + \frac{2k}{2k+1}\frac{(1-p)U}{U_H(h)-U_H(l)}. \end{aligned} \tag{14}$$

To prove the above inequality, we consider the following two cases:

(i) $k^0 \geq 2$**.** Note that

$$\frac{U}{U_H(h)-U_H(l)} < \frac{(U_H(h)-U_H(l)) + (U_h(H)-U_h(L))}{U_H(h)-U_H(l)} = 2. \tag{15}$$

- Since the left hand side of (14) is increasing in k^0, for (14) to hold, it suffices that

$$\frac{12}{5}\frac{2k+3}{k+1} > \frac{2k}{2k+1} + \frac{4k}{2k+1} = \frac{6k}{2k+1},$$

  which holds for all k.

(ii) $k^0 = 1$**.** One sufficient condition for (14) using (15) is

$$\frac{4(2k+3)}{3(k+1)} > \frac{6k}{2k+1},$$

which holds for $k = 1, 2$, or 3.

- Since $k^0 = 1$,

$$\frac{p(1-p)U}{2d_k} = \frac{(1-p)Uk}{2(U_H(h)-U_H(l))} < 4.$$

  Thus, another sufficient condition for (14) in this case is

$$\frac{4(2k+3)}{3(k+1)} > \frac{2k+16}{2k+1},$$

  which holds for $k \geq 4$.

To construct the partition in the proposition, let $\overline{k} = \max\{k \mid d_k \geq \underline{c}\}$ and $\underline{k} = \min\{k \mid d_k < \overline{c}\}$. Now define $c_1 = \underline{c}, c_M = \overline{c}$. If $\underline{k} = \overline{k}$, set $M = 2$ and the partition has only one atom. Otherwise, if $\underline{k} < \overline{k}$, set $M = \overline{k} - \underline{k} + 2$ and $c_i = d_{\overline{k}-i+1}$ for $i = 2, ..., M-1$.

2. Notice that

$$\lim_{c\to 0}\left(W^{opt}(c) - W^{fifo}(c)\right) = \lim_{c\to 0}\Psi(c) - \lim_{c\to 0}\Theta(c) = p\left(U_H(h) - U_H(l)\right).$$

In particular, for sufficiently small c, $W^{opt}(c) - W^{fifo}(c)$ is increasing in both p and $U_H(h) - U_H(l)$, as needed. ■

9 References

Akbarpour, Mohammad, Shengwu Li, and Shayan Oveis Gharan. 2018. "Thickness and Information in Dynamic Matching Markets," mimeo.

Anderson, Ross, Itai Ashlagi, David Gamarnik, and Yash Kanoria. 2017. "Efficient Dynamic Barter Exchange," *Operations Research*, **65(6)**, 1446-1459.

Ashlagi, Itai, Maximilien Burq, Patrick Jaillet, and Vahideh H. Manshadi. 2018. "On Matching and Thickness in Heterogeneous Dynamic Markets," mimeo.

Ashlagi, Itai, Patrick Jaillet, and Vahideh H. Manshadi. 2014. "Kidney Exchange in Dynamic Sparse Heterogenous Pools," mimeo.

Baccara, Mariagiovanna, Allan Collard-Wexler, Leonardo Felli, and Leeat Yariv. 2014. "Child-Adoption Matching: Preferences for Gender and Race," *American Economic Journal: Applied Economics*, **6(3)**, 133-158.

Becker, Gary S. 1974. "A Theory of Marriage: Part II," *The Journal of Political Economy*, **82(2)**, S11-S26.

Bloch, Francis and David Cantala. 2017. "Dynamic Assignment of Objects to Queuing Agents," *American Economic Journal: Microeconomics*, **9**, 88-122.

Budish, Eric, Peter Cramton, and John Shim. 2015. "The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response," *The Quarterly Journal of Economics*, **130(4)**, 1547-1621.

Burdett, Ken and Melvyn G. Coles. 1997. "Marriage and Class," *The Quarterly Journal of Economics*, **112(1)**, 141-168.

Choo, Eugene. 2015. "Dynamic Marriage Matching: An Empirical Framework," *Econometrica*., **83(4)**, 1373-1423.

Doval, Laura. 2018. "A Theory of Stability in Dynamic Matching Markets," mimeo.

Doval, Laura and Balász Szentes. 2018. "On the Efficiency of Queueing in Dynamic Matching-Markets," mimeo.

Eeckhout, Jan. 1999. "Bilateral Search and Vertical Heterogeneity," *International Economic Review*, **40(4)**, 869-887.

Feller, William. 1972. *An Introduction to Probability Theory and Its Applications, Volume II* (2nd ed.). New York: John Wiley & Sons.

Gjertson, David W. 2004. "Explainable Variation in Renal Transplant Outcomes: A Comparison of Standard and Expanded Criteria Donors," *Clinical Transplants*, **2004**, 303-314.

Gurvich, Itai and Amy Ward. 2014. "On the Dynamic Control of Matching Queues," *Stochastic Systems*, **4(2)**, 479-523.

Haeringer, Guillaume and Myrna Wooders. 2011. "discretionary Job Matching," *International Journal of Game Theory*, **40**, 1-28.

Hall, Robert E. and Alan B. Krueger. 2012. "Evidence on the Incidence of Wage Posting, Wage Bargaining, and On-the-Job Search." *American Economic Journal: Macroeconomics*, **4(4)**, 56-67.

Hassin, Refael. 1985. "On the Optimality of First Come Last Served Queues," *Econometrica.*, **53(1)**, 201-202.

Herbst, Holger and Benjamin Schickner. 2016. "Dynamic Formation of Teams: When Does Waiting for Good Matches Pay Off," mimeo.

Hitch, Gunter J., Ali Hortacsu, and Dan Ariely. 2010. "Matching and Sorting in Online Dating," *The American Economic Review*, **100(1)**, 130-163.

Hu, Ming and Yun Zhou. 2018. "Dynamic Type Matching," mimeo.

Kemeny, John G. and J. Laurie Snell. 1960. *Finite Markov Chains.* Springer.

Kocer, Yilmaz. 2014. "Dynamic Matching and Learning," working slides.

Leshno, Jacob. 2017. "Dynamic Matching in Overloaded Waiting Lists," mimeo.

Loertscher, Simon, Ellen V. Muir, and Peter G. Taylor. 2018. "Optimal Market Thickness and Clearing," mimeo.

Margaria, Chiara. 2017. "Queueing to Learn," mimeo.

Naor, Pinhas. 1969. "The Regulation of Queue Size by Levying Tolls," *Econometrica.*, **37(1)**, 15-24.

Niederle, Muriel and Leeat Yariv. 2009. "Discretionary Matching with Aligned Preferences," mimeo.

Oien, Cecilia M., Anna V. Reisaeter, Torbjørn Leivestad, Friedo W. Dekker, and Pål-Dag Line. 2007. "Living Donor Kidney Transplantation: The Effects of Donor Age and Gender on Short- and Long-term Outcomes," *Transplantation*, **83(5)**, 600-606.

Pais, Joana V. 2008. "Incentives in discretionary random matching markets," *Games and Economic Behavior*, **64**, 632-649.

Platz, Trine Tornøe and Lars Peter Østerdal. 2017. "The Curse of the First-in-First-Out Queue Discipline," *Games and Economic Behavior*, **104**, 165-176.

Puterman, Martin L. 2005. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, Wiley.

Rogerson, Richard, Robert Shimer, and Randall Wright. 2005. "Search-Theoretic Models of the Labor Market: A Survey," *Journal of Economic Literature*, **XLIII**, 959-988.

Ross, Sheldon M. 2014 *Introduction to Stochastic Dynamic Programming*, Academic Press.

Satterthwaite, Mark and Artyom Shneyerov. 2007. "Dynamic Matching, Two-sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition," *Econometrica*, **75(1)**, 155-200.

Schummer, James. 2017. "Influencing Waiting Lists," mimeo.

Stein, Rob. 2011. "Under Kidney Transplant Proposal, Younger Patients would Get the Best Organs," *The Washington Post*, February 24.

Taylor, Curtis. 1995. "The Long Side of the Market and the Short End of the Stick: Bargaining Power and Price Formation in Buyers', Sellers', and Balanced Markets," *The Quarterly Journal of Economics*, 110(3), 837-855.

Tyurin, Ilya S. 2010. "An Improvement of Upper Estimates of the Constants in the Lyapunov Theorem," *Russian Mathematical Surveys*, **65(3(393))**, 201-202.

Ünver, Utku. 2010. "Dynamic Kidney Exchange," *Review of Economic Studies*, **77**, 372-414.

Zenios, Stefanos A. 1999. "Modeling the Transplant Waiting List: A Queueing Model with Reneging," *Queueing Systems*, **31**, 239-251.