Optimal Markdown Pricing: Implications of Inventory Display Formats in the Presence of Strategic Customers

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Abstract

We consider a retailer who sells a limited inventory of a product over a finite selling season, using one of two inventory display formats: Display All (DA) and Display One (DO). Under the DA format, the retailer displays all available units so that each arriving customer has perfect information about the actual inventory level. Under the DO format, the retailer displays only one unit at a time so that each customer knows about product availability but not the actual inventory level. Recent research suggests that when faced with strategic consumers, the retailer could increase expected profits by making an upfront commitment to a price path. We focus on such pricing strategies in this paper, and address the following questions: When considering the influence of the display formats on the level of inventory information conveyed to customers, which one of the two formats is better for the retailer? Furthermore, can a move from one display format to another be effective in mitigating the adverse impact of strategic consumer behavior? We propose a stylized game-theoretical model to address these questions. We find support to our hypothesis that the DO format could potentially create an increased perception of scarcity among customers, and hence it is better than the DA format. However, while potentially beneficial, the move from a DA to a DO format is very far from eliminating the adverse impact of strategic consumer behavior. We explain the circumstances under which the DO format is better than the DA format. Using our theoretical results, we also argue that by moving from a DA to a DO format, while keeping the price path unchanged, the volatility of the retailer’s profit decreases. This observation may open an interesting window for future research on settings with risk-prone or risk-averse retailers. Finally, we discuss several model extensions and propose ideas for future research.

Key words: Retailing, Dynamic Pricing, Game Theory Applications, Marketing-Operations Interface, Strategic Customers, Revenue Management, Inventory Display.

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1 Introduction

Many retailers use post-season clearance sales as a reactive response to dispose of unsold items at the end of a selling season. However, some retailers proactively pre-announce their price-markdown schedules at the beginning of the selling season. A well-known example of pre-announced markdown pricing strategy has been adopted by Filene’s Basement since 1908. At the Filene’s Basement Boston store, most unsold items after 2, 4 and 6 weeks will be sold at 25%, 50% and 75% off the regular price, respectively. After 2 months, Filene’s Basement donates all unsold items to charity; see Bell and Starr (1998) for more details. Lands’ End Overstocks uses a similar pre-announced markdown pricing strategy to sell their leftover inventory via its “On the Counter” website. Other stores such as Dress for Less\(^1\), Tuesday Morning discount stores, Wanamaker discount department store in Philadelphia, etc., have adopted a similar pre-announced markdown pricing strategy. Also, TKTS ticket booths in New York City and London offer pre-announced discount tickets for the same day Broadway shows.

Pre-announced markdown pricing strategy is intended to segment customers with different product valuations so that high (low) valuation customers will purchase the product at the regular (clearance) price; see Pashigian and Bowen (1991) and Smith and Achabal (1998) for comprehensive discussions. However, one of the drawbacks of pre-announced pricing schemes is that they cannot segment customers completely because they often lead to “strategic waiting”: a phenomenon in which some high valuation customers may postpone their purchases by waiting for the post-season clearance price, even when there is a risk of not getting the product due to stockout at the end of the selling season (c.f., Phillips (2005) and Fisher (2006)).

Given that strategic waiting has detrimental effect on revenues, some retailers have considered various sales mechanisms to discourage high valuation customers to wait for the post-season clearance price. Besides corporate level strategy that calls for no markdown pricing (see discussion of such strategies in Aviv and Pazgal (2007), Cachon and Swinney (2008), and Su and Zhang (2008a)), there are two operational strategies for enticing high valuation customers to purchase the product at the regular price instead of waiting for the clearance price. The first strategy calls for a limited

\(^{1}\)Current web address: www.d4less.net/discount.htm.
supply of the product so that high-valuation customers would face a higher risk of stockouts if they decide to wait for the clearance price; and, therefore, may prefer to purchase the product at the regular price (c.f., Liu and van Ryzin (2008)). The second strategy, which is the key theme of our paper, is the inventory display format. Here, the underlying principle is that a display format can be used as a tool to influence customers’ perceptions about the risk of stockouts if they decide to wait. Therefore, by optimally selecting the display format, a retailer could discourage high valuation customers to wait for the clearance sales.

In this paper we consider two types of display formats that are commonly seen in retailing. The first is called the “Display All” (DA) format under which the retailer displays all available units so that customers have perfect information about the actual inventory level. For example, since 2005, Expedia.com provides their customers with perfect information about the exact number of plane tickets available at a particular price for a particular flight. Similarly, Filene’s Basement and Benetton adopt a similar in-store display format by placing all available items on the sales floor. The second display format is called the “Display One” (DO) format. Here, the retailer displays only one item at a time so that customers have imperfect information about the current inventory level. For example, Lands’ End Overstocks website provides each arriving customer with information about the availability but not the actual inventory level of each product. Similarly, various high-end stores such as the Bally store in Taiwan displays only a single unit of a product at a time.

In addition to influencing customers’ perceptions about product availability, the DA and DO display formats may offer different advantages and impose certain operational consequences. For example, the DA format allows a brick-and-mortar retailer to utilize store space more effectively, and eliminate the need for additional stockrooms. In fact, this is a fundamental reason for stores such as Benetton and Seven-Eleven to adopt the DA format. However, when the display space in a store is limited, the DO format may be the natural choice that enables a retailer to present an assortment of different designs instead of multiple units of the same design. The DO format can also result in a less cluttered shop floor, hence infusing a sense of exclusivity in the product. As a consequence, a retailer may enjoy an increase in store traffic and higher customer valuations for the product. Implementation of the DO format can also be more costly, due to the required handling
involved in the replacement of a new unit after each sale. The store may also incur temporary lost sales if a unit is not replaced immediately after a sale is made. It is conceivable that due to these explicit and implicit costs, the DO format is not highly popular in brick-and-mortar stores. Obviously, when the DO format is implemented in an online retail environment, the aforementioned costs are irrelevant. This is because the difference between the web page display formats boils down to the amount of inventory information communicated to customers.

In this paper, we compare the display formats on the basis of the inventory information conveyed to customers. In particular, we deliberately exclude the aforementioned handling cost, merchandising, and storage-efficiency considerations from our analysis. Consequently, our models can be used as a basis for a more comprehensive examination of the impact of display formats on revenue performance. Our main hypothesis is that the DO display format induces a larger sense of product scarcity than the DA format, since inventory information is only partial. As a consequence, we expect high-valuation customers to feel more pressed to purchase the product at the premium price, rather than wait for a discount. Obviously, our purpose is to scientifically test the validity of this hypothesis, and the extent to which one display format is better or worse than the other. Nonetheless, such scientific comparison happens to be complex. First, the term “sense of scarcity” in our hypothesis is loose. Would a lack of accurate inventory level information always induce a stronger urge to purchase at premium prices? Consider, for instance, a situation in which a large burst of customers arrive to the store very early in the season and purchase the product at the premium price. Suppose that shortly afterwards, only a single unit is left in stock. In this case, the DA format could have been better for the seller, because customers would know that the actual inventory level is very low, rather than speculate. In other words, depending on the realization of customer arrivals to the store, one display format might end up (in retrospect) being better than the other. The second complication is that the two display formats lend themselves to different equilibria, both in terms of the optimal pricing and in terms of the customer purchasing behavior. As a result, it is hard to predict how the two equilibria compare with each other on the basis of intuitive arguments only.

To our knowledge, this paper is the first to examine the impact of display format on revenues under the presence of strategic consumers. We propose a stylized model of a retailer that seeks to
maximize his expected profit by selling a product to a customer base that consists of two classes of customers: one with a high valuation, and the other with a low valuation. The retailer and the customers engage in a competitive situation, as follows. Initially, the retailer determines his level of inventory, and announces a regular price for the main part of the season, as well as a clearance price for the end of the season. Obviously, in order to determine his optimal inventory and pricing strategies, the retailer needs to anticipate the customers’ purchasing behavior. Customers make their purchasing decisions strategically; i.e., they either purchase the product immediately upon their arrival to the store (“buy now”) or postpone their purchase to the end of the season (“wait”). Because customers’ decisions have potential effects on each other, they are engaged in a competitive subgame among themselves.

The remainder of the paper is organized as follows. We first review the related literature in §2. In §3 we describe our model preliminaries. The DA format and the DO format are analyzed in §4 and §5, respectively. In each of these sections, we first analyze the subgame among customers and determine their optimal purchasing strategy for any given announced price path. We then examine the retailer’s best strategy, and present a method for calculating the retailer’s expected profit. In §6 we propose two benchmark models, and a set of metrics for the analysis of the difference between the display formats. We report the results of an extensive numerical study in §7. On the basis of this study, we identify the conditions under which a change from DA to DO could be valuable to the retailer. We discuss possible model extensions in §8 and conclude in §9.

2 Literature Review

Our paper belongs to the stream of management science literature that studies the effects of strategic consumer behavior in the context of revenue management. We provide below a review of a representative group of related research papers. To keep the review concise and effective, we split the presentation into two parts. We first review models that assume complete knowledge of inventory information, and then continue to discuss models that consider partial inventory information. The primary contribution of our research is in providing a bridge between the two types of models. Specifically, our paper compares the impact of two display formats (one that provides customers with complete information, and one that only provides partial information) on consumer purchasing
behavior and retailer’s pricing, inventory levels, and expected profits.

Elmaghraby et al. (2007) belongs to the first category of models. It considers a setting in which a seller uses a pre-announced markdown pricing mechanism, to sell a finite inventory of a product. Specifically, the seller’s objective is to maximize expected revenues by optimally choosing the number of price steps over the season, and the price at each step. All potential buyers are present at the start of the selling period and remain until all the units have been sold or their demand has been satisfied. The buyers, who demand multiple units, may choose to wait and purchase at a lower price, but they must also consider a scarcity in supply. The authors study the potential benefits of segmentation; namely, the difference between the seller’s profit under the optimal markdown mechanism to that under the optimal single price. They also provide a detailed discussion on the design of profitable markdown mechanisms. Su (2007) presents a pricing control model in which consumers are infinitesimally small and arrive continuously according to a deterministic flow of constant rate. The customer population is heterogeneous along two dimensions: valuations, and degree of patience (vis-a-vis waiting). The seller has to decide on pricing and a rationing policy which specifies the fraction of current market demand that is fulfilled. Given these retailer’s choices, customers decide whether or not to purchase the product and whether to stay or leave the market. The paper shows how the seller can determine a revenue-maximizing selling policy in this game. Su demonstrates that the heterogeneity in valuation and degree of patience jointly influence the structure of optimal pricing policies. In particular, when high-valuation customers are proportionately less patient, markdown pricing policies are effective. On the other hand, when the high-valuation customers are more patient than the low-valuation customers, prices should increase over time in order to discourage waiting. Ovchinnikov and Milner (2005) consider a firm that offers last-minute discounts over a series of periods. Their model incorporates both stochastic demand and stochastic customer waiting behavior. Two waiting behaviors are considered in their paper. In the first, called “the smoothing case,” customers interpolate between their previous waiting likelihood and their observation of the firm’s policy. In the second, named “the self-regulating case,” customers anticipate other customers’ behavior and the likelihood that they will receive a unit on sale. They show that, under the self-regulating case, it is generally optimal for the firm to set some units on sale in each period and allow the customer behavior to limit the number of
customers that enjoy the benefit of the reduced price. In contrast, in the smoothing case, the firm can increase its revenues by following a sales policy that regulates the number of customers waiting. The authors conduct numerical simulations to illustrate the value of making decisions optimally, as compared to a set of reasonable benchmark heuristics. They find that the revenues can increase by about 5-15%. The paper also discusses how these benefits are affected if overbooking is allowed, and show that the impact of overbooking is greatly dependent on the proportion of high-valuation customers in the market. Levin et al. (2005) propose a stochastic game-theoretical dynamic pricing model of a monopolistic that sells a product to a population of strategic consumers. They prove the existence of a unique subgame-perfect equilibrium in this game, and explore its structural properties. Using numerical examples, the authors demonstrate that a company that ignores strategic consumer behavior may receive much lower total revenues than one that uses the strategic equilibrium pricing policy. They also show that when the initial capacity is a decision variable, it can be used together with the appropriate pricing policy to effectively reduce the impact of strategic consumer behavior.

The following papers consider models with imperfect information about the prevailing level of inventory. Aviv and Pazgal (2008) examine two types of markdown pricing policies. The first is an announced fixed-discount policy, where prices are predetermined and known in advance of the beginning of the season. The second policy is one that prescribes a clearance price depending on the prevailing level of inventory at the time of discount. In their models, customers only know the initial level of inventory, but need to form a belief about the current inventory level. To do so, they need to take into account their expectations about the behavior of other customers and the retailer in equilibrium. One of their findings is that while intuition may suggest that a flexible, information-dependent policies should be better for the retailer, pre-announced pricing policies can actually generate higher expected profits. Caldentey and Vulcano (2007) analyze a revenue management problem in which a seller operates an online multi-unit auction. Consumers can get the product either from the auction or from an alternative list price channel. The problem is related to the topic of strategic consumer behavior in that customers need to decide whether to buy at the posted list price and get the item immediately and at no risk, or to join the auction and wait until its end, when the winners are revealed and the auction price is disclosed. In one of the model variants of this paper, the seller is a monopolist that owns both the auction and the
list price channel. Because the initial inventory is shared between the two channels, the number of units that end up being allocated for the auction is dependent on the number of customers that buy at the list price. Here, similarly to Aviv and Pazgal (2008), the authors assume that customers only know the initial level of inventory, and need to make statistical inference about the number of units that will be auctioned, on the basis of the equilibrium strategy. Cachon and Swinney (2008) analyze a two-period model of inventory planning and dynamic pricing, to study the value of quick response\(^2\) under strategic consumer behavior. In their model, the initial level of inventory is not announced to the customers. However, by assuming a given framework of rational expectation, they show how this initial inventory level can be revealed through the calculation of a sub-game perfect Nash equilibrium of the game between the retailer and the customers. They argue that when replenishment is feasible, the retailer should avoid committing to a price path over the season, even in the presence of strategic consumers. They suggest that a better approach is to be cautious with the initial quantity, and then markdown optimally. They find that the value of quick response is generally much greater in the presence of strategic consumers than without them. Furthermore, they argue that quick response may have a dramatically larger influence as a mechanism for mitigating strategic consumer behavior, than as a tool for matching of supply with demand. Su and Zhang (2008a) also utilize a rational-expectations model of a seller facing strategic consumers. In their model, the items are sold at a regular price throughout the season and all unsold items are sold at a reduced price at the end of the season. Unlike our models, customers do not form an explicit time-dependent belief about the prevailing level of inventory. Rather, they all assumed to form the same belief about the likelihood of product availability, which is shown to hold under rational expectations. They find that the seller’s stocking level is lower under strategic consumer behavior, because a high inventory level increases the likelihood of leftovers and thus motivates customers to wait for the end-of-season sale. They also argue that the seller’s profit can be improved by making either quantity commitment or price commitment. The paper also offers a detailed discussion on the influence that strategic consumer behavior has on supply chain coordination. Liu and van Ryzin (2008) propose a rational expectation model to investigate whether it is optimal for a firm to create rationing risk by deliberately understocking products. The rationale

\(^2\)A quick response setting is one in which the retailer is able to replenish stock during the season.
behind this is that strategic high-valuation customers will become more inclined to purchase the product at premium prices, rather than wait for a discount. Like the previous two papers, the customers do not form an explicit belief about the prevailing level of inventory. Rather, they are assumed to form the *same* belief about the fill rate (i.e., likelihood of obtaining the product), on the basis of rational expectations. Via its capacity choice, the firm is able to control the fill rate and, hence, the rationing risk faced by customers. Su and Zhang (2008b) study the effect of inventory information on consumer purchase behavior; this is also one of the main themes of our present work. Su and Zhang (2008b) use a rational expectation game-theoretical framework (in the spirit of Su and Zhang (2008a) and Cachon and Swinney (2008)), to study the role of product availability in attracting consumer demand. They analyze two strategies for the seller: A commitment to a particular level of inventory, and providing customers with guarantees on product availability (in the form of ex-post compensation if the product is out of stock). They demonstrate that the seller can improve his profits by making a combination of inventory commitment and availability guarantees.

It is noteworthy that while our paper also focuses on the aspect of inventory-related information, we are handling a fundamentally different problem. In our paper, the initial level of inventory is known, and the key question is whether or not to share the information about the prevailing level of inventory. Finally, Liu and van Ryzin (2007) propose an adaptive-learning model to describe an interaction between a firm and customers over a sequence of seasons. Each of the seasons includes a full-price period and a markdown period. While customers know the firm’s capacity in previous seasons, they do not know its current capacity. A key feature of the model is that customers learn about availability through experience, by using a simple heuristic rule to update their expectations. In this model of *bounded rationality*, customers decide to either buy at the full-price period or the markdown period. The authors analyze the way in which the equilibrium is settled over the seasons.

In this paper, we focus on announced pricing strategies, and address the following questions: When considering the influence of the display formats on the level of inventory information conveyed to customers, which one of the two formats is better for the retailer? Furthermore, can a move from one display format to another be effective in mitigating the adverse impact of strategic consumer behavior? We believe that the contribution of our work also lies in addressing these questions via rigorous scientific analysis. Our analysis is involved with significant challenges due to the stochastic
nature of the consumer arrival process, and the explicit consideration of the customers’ perception of the prevailing and future levels of inventory. In §5, we explain the significance of the difference between this approach and the simplified “rational expectation” approach taken in other research papers.

3 The Base Model

Consider a retailer who orders $Q$ units (a decision) at a unit cost $c$ that is scheduled to be sold over a selling season that spans over $[0,T]$. In our model, we assume that the retailer can place a single order prior to the start of the season; the order will be received and become available for sale at time 0. In addition to deciding on the order quantity $Q$, the retailer needs to decide and pre-announce two prices at time 0: the premium price $p_h$ (i.e., the selling price throughout the entire season), and the post-season clearance price $p_l$. Clearly, $p_h \geq p_l$. All units not sold at either prices can be salvaged at $s$ per unit. The objective of the retailer is to maximize his expected profit by making three inter-related decisions: $Q$, $p_h$ and $p_l$.

Strategic customers\(^3\) arrive at the store according to a Poisson process with rate $\lambda$, where $\lambda$ is independent of the announced prices $p_h$ and $p_l$. Upon arrival, each customer must take his own valuation as well as the announced price path into consideration when making his purchase decision. To capture market heterogeneity, customers are classified into two classes according to their valuations. Specifically, all customers belong to class-0 have valuation of $v_0$ and all customers belong to class-1 have valuation of $v_1$, where $v_0 < v_1$. We assume that the arrival process can be described as a combination of two independent Poisson processes associated with class-0 and class-1 customers. Specifically, we let $\alpha_0$ be the portion of class-0 customers in the market, and $\alpha_1 = 1 - \alpha_0$ be the complementary portion of class-1 customers. Throughout this paper, we assume that the set of parameters $\{\alpha_0, \alpha_1, v_0, v_1, \lambda, T, c, s\}$ is a common knowledge. Admittedly, this assumption is made for mathematical tractability, since an inclusion of parameter uncertainty (or even structural uncertainty) would make our analysis prohibitively complex. It could be possible to extend our work by adding some level of parameterized uncertainty. Such model, however, will

\(^3\)To simplify our analysis, we assume that all arriving customers are strategic. In §8, we discuss the complexity that arise when dealing with a mixture of strategic and “myopic” customers.
need to be based on an even stronger assumption that the seller and customers know the statistical characterization of the game parameters. Another possible approach could be to develop a model of bounded rationality, in which the lack of parameter knowledge (or lack of sophistication) leads the customers and the seller to adopt certain types of heuristics.

To ensure some potential sales at the premium price and to enable the retailer to facilitate effective price discrimination, it suffices to consider the case when the premium price $p_h$ satisfies the inequality: $v_0 \leq p_h \leq v_1$. Because $p_l \leq p_h$, we need to consider two settings: (i) $v_0 < p_l \leq p_h \leq v_1$, and (ii) $p_l \leq v_0 \leq p_h \leq v_1$. The first case corresponds to a setting in which the retailer posts prices that exclude class-0 customers. Such strategy can be desirable especially when the market consists primarily of high valuation customers or when $v_1$ is significantly larger than $v_0$. Obviously, if such exclusive-sales strategy is adopted, it is optimal for the retailer to set $p_l = p_h = v_1$. This way, class-1 customers will purchase the product at $v_1$, since they have no incentive to wait for the clearance price. Hence, regardless of the inventory display format, the retailer’s expected profit can be expressed as:

$$
\Pi_1^1 (Q) \equiv -(c - s)Q + (v_1 - s) \cdot N (Q, \alpha_1 \lambda T)
$$

(1)

where $N (Q, \Lambda) \equiv \sum_x \min (Q, x) \cdot P_x (\Lambda) = Q - \sum_{x=0}^{Q-1} (Q - x) P_x (\Lambda)$, and $P_x (\Lambda)$ is the Poisson probability function (corresponding to a mean parameter of $\Lambda$). The retailer’s optimal expected profit and order quantity can be determined by solving the “newsvendor” problem: $\Pi_1^1 = \max_{Q \geq 0} \{\Pi_1^1 (Q)\}$. The second case (i.e., when $p_l \leq v_0 \leq p_h \leq v_1$) reflects a setting in which the retailer chooses to target both customer classes. This case, which is considerably harder to analyze, is treated extensively in the remainder of this paper.

Let us examine the strategic purchasing behavior in the second setting. Essentially, each arriving customer needs to compare the surpluses associated with two options: “buy now” and “wait.” As $p_l \leq v_0 \leq p_h \leq v_1$, it is clear that all class-0 will always wait for the clearance price, regardless of the display format. However, for a class-1 customer, the decision is less trivial. If he purchases the product immediately at the premium price $p_h$, then he will obtain a surplus $(v_1 - p_h)$. If he waits for the clearance price $p_l$, then his expected surplus could ideally be $(v_1 - p_l)$; nonetheless,
this customer needs to consider the possibility that an item will not be available to him, due to a stockout. To capture this aspect, we need to introduce the customer’s perception of the likelihood of getting the product at the post-season clearance price \( p_l \) if he decides to wait and returns to the store at the end of the season. We refer to this likelihood as the “perceived fill rate”\(^5\); correspondingly, we shall refer to the complimentary probability as the “perceived sense of scarcity.” The customer hence needs to assess the expected surplus (“perceived fill rate”) \( \times (v_1 - p_l) \).

By comparing these two surpluses, it is clear that a class-1 customer will purchase the product at \( p_h \) if the fill rate is less than \( (v_1 - p_h) / (v_1 - p_l) \), and will wait for the clearance price if the fill rate is greater than \( (v_1 - p_h) / (v_1 - p_l) \). Therefore, the optimal purchasing behavior for each class-1 customer hinges upon the individually-assessed fill rates. As we shall see, the assessment of the fill-rate by an individual customer is a challenging task, as it depends on many factors including the inventory information available under the given display format, the customer’s arrival time, and the purchasing behavior of all other customers. In the next two sections, we first determine the fill rates associated with the DA and DO formats, and then we utilize them to prescribe class-1 customers’ purchasing behavior in equilibrium. Anticipating the customers’ purchasing behavior for both classes of customers, we determine the retailer’s expected profit function and analyze the retailer’s optimal order quantity and prices.

4 The “Display All” (DA) Format Model

In this section, we consider the case in which the retailer adopts the “Display All” (DA) format, so that each arriving customer has perfect information about the actual inventory level. We begin by studying the customers’ subgame, and then proceed to the treatment of the retailer’s problem.

4.1 Strategic Purchasing Under the Display All Format

Consider a given order quantity \( Q \) and a pre-announced price path that satisfies \( p_l \leq v_0 \leq p_h \leq v_1 \). Clearly, we anticipate class-0 to always wait for the clearance price \( p_l \), since \( v_0 \leq p_h \). For class-1, consider a customer who arrives at time \( t \) and observes \( k \) units available for sale. Associated with the state \( (k, t) \), let \( H(k, t) \) be the perceived fill rate as defined earlier. Hence, this customer will

\(^5\)The existing literature uses different expressions to denote the perceived fill rates. For example, Aviv and Pazgal (2007) use the term “allocation probability.”
buy immediately at $p_h$ if $H(k, t) \leq \frac{v_1 - p_h}{v_1 - p_l}$; otherwise, the customer will wait for the clearance price $p_l$. To facilitate the analysis of the fill rates $H(k, t)$, we define the following auxiliary function:

$$
\hat{H}_q(\Lambda) = \sum_{x=0}^{q-1} P_x(\Lambda) + \sum_{x=q}^{\infty} \frac{q}{x+1} P_x(\Lambda)
$$

The function $\hat{H}_q(\Lambda)$ represents the probability that a class-1 customer will obtain a unit of the product if he decides to wait for the clearance price after observing $q$ units available at time $t$ and anticipating $X$ additional customers who would also wait for clearance price $p_l$, where $X \sim \text{Poisson}(\Lambda)$. In other words, $\hat{H}_q(\Lambda) = E_X [q/\max(q, X + 1)]$. An implicit assumption behind this interpretation is that in case of shortage, units are allocated to the customers with equal probability.

Indeed, the following proposition demonstrates that the function $\hat{H}_q$ is useful in constructing a lower-bound on the perceived fill rate.

**Proposition 1** Consider any given arbitrary purchasing policy (including randomized strategies). The following bound applies to the perceived fill rate $H(k, t)$:

$$
H(k, t) \geq \sum_{x=0}^{k-1} P_x(\alpha_1 \lambda (T - t)) \cdot \hat{H}_{k-x}((\alpha_0 \lambda T + \alpha_1 \lambda t))
$$

The lower bound in the proposition has an intuitive interpretation, as it corresponds to the “worst case” scenario in which all class-1 customers arriving prior to time $t$ and all class-0 customers will “wait” and all class-1 customers arriving after $t$ will “buy now”. This scenario yields a lower bound on the fill rate because it would generate the highest number of customers who wait and the lowest number of remaining units at the end of the season. Observe from the above proposition that the lower bound is increasing in $k$ (see the Appendix). Hence, there exists a sufficiently large level of inventory (say $k'$) so that the lower bound is large enough to ensure that $H(k, t) > (v_1 - p_h) / (v_1 - p_l)$, for all $k \geq k'$. Consequently, as $k$ is sufficiently large, it is optimal for a class-1 customer to wait regardless of the purchasing behavior of other class-1 customers.

However, as the level of inventory $k$ becomes lower, $H(k, t)$ becomes lower and each class-1 customer needs to take into account the behavior of other class-1 customers. To examine this
formally, we introduce a sequence of values \( \{\tau^*(k)\} \), defined as follows. For all \( k = 1, 2, \ldots \):

\[
\tau^*(k) = \begin{cases} 
0 & \text{if } \frac{v_1 - p_h}{v_1 - p_l} < \hat{H}_k(\lambda T) \\
\text{See Eq. (4) below} & \text{if } \hat{H}_k(\lambda T) \leq \frac{v_1 - p_h}{v_1 - p_l} \leq \hat{H}_k(\alpha_0 \lambda T) \\
T & \text{if } \frac{v_1 - p_h}{v_1 - p_l} > \hat{H}_k(\alpha_0 \lambda T)
\end{cases}
\] (3)

In the second case of (3), the value \( \tau^*(k) \) is the unique solution to the equation

\[
v_1 - p_h = (v_1 - p_l) \hat{H}_k(\lambda (T - \alpha_1 \tau^*(k)))
\] (4)

(For a proof of uniqueness, see Proposition 2 below.) Equation (4) is reminiscent of the equation

\[
v_1 - p_h = (v_1 - p_l) H(k, t)
\]

which holds for times \( t \) (if exist) at which a class-1 customer is indifferent between a “buy now” and a “wait” decision, if \( k \) units are available for sale at time \( t \). Indeed, the values \( \tau^*(k) \) will play an important role in defining the indifference points (or thresholds) for optimal purchasing policies in equilibrium.

**Proposition 2** The sequence \( \{\tau^*(k)\} \) defined in (3) is unique and non-increasing in \( k \). Furthermore, the values \( \tau^*(k) \) are increasing in \( \lambda, v_1 \) and \( p_l \), and decreasing in \( p_h \).

We are now ready to present a key structural property of the optimal purchasing strategy for class-1 customers. To this end, let us consider the following threshold-type policy that is based on the sequence of values \( \{\tau^*(k)\} \) as presented in (3)-(4). This threshold policy can be described as follows: for any class-1 customer who arrives at time \( t \) and observes \( k_t \) units available for sale, he should “buy now” if \( t < \tau^*(k_t) \) and “wait” if \( t > \tau^*(k_t) \). (If \( t = \tau^*(k_t) \in (0, T) \), the policy can prescribe either action at random.)

**Theorem 1** Under the DA format, an “always-wait” strategy for class-0 customers and a threshold-type policy (based on threshold values \( \{\tau^*(k)\} \)) for class-1 customers form the unique Nash equilibrium in the sub-game among customers.

### 4.2 The Retailer’s Problem Under the Display All Format

Anticipating customers’ purchasing behavior in equilibrium as established in Theorem 1, the retailer needs to identify an order quantity \( Q \) and a pair of optimal prices \((p_h, p_l)\) that maximize his expected profit. We first consider the regime \( p_l \leq v_0 \leq p_h \leq v_1 \). Let us define \( f(j, k) \) to be equal to the
retailer’s expected profit gained during the interval \((\tau^*(j), T]\), given that \(k\) units are in stock at time \(\tau^*(j)\); for notational convenience, let \(\tau^*(Q+1) = 0\). Then, the functions \(f\) satisfy the following recursive scheme:

**Proposition 3** Given the customers’ purchasing behavior in equilibrium, the functions \(f\) satisfy the following equations, for all \(1 \leq k \leq j \leq Q + 1\):

\[
f(j, k) = \begin{cases} 
\sum_{x=0}^{k-1} [xp_h + f(k, k - x)] \times P_x(\alpha_1\lambda_1(\tau^*(k) - \tau^*(j))) + \sum_{x=k}^{\infty} P_x(\alpha_1\lambda_1(\tau^*(k) - \tau^*(j))) \times kp_h & \text{if } k < j \\
kp_l - (p_l - s)\sum_{x=0}^{k-1} (k - x) \times P_x(\lambda(T - \alpha_1\tau^*(k))) & \text{if } k = j
\end{cases}
\]

Using the recursive algorithm outlined in Proposition 3, we can compute the retailer’s expected profit for any order quantity \(Q\) and price path \((p_h, p_l)\). Specifically, let us use the notation \(\pi^{DA}(Q, p_h, p_l) \doteq f(Q+1, Q) - cQ\) to denote the retailer’s expected profit. The following proposition characterizes a property associated with the optimal price path associated with \(\pi^{DA}(Q, p_h, p_l)\).

**Proposition 4** For any given order \(Q\), and for any price path \(\{p_l \leq v_0 \leq p_h \leq v_1\}\), there exists an optimal price path that satisfies the conditions: \(p_l = v_0\) and \(p_h \in \Omega^{DA}_Q\), where

\[
\Omega^{DA}_Q \doteq \left\{v_0 \cdot \bar{H}_Q(\alpha_0\lambda T) + v_1 \cdot \left(1 - \bar{H}_Q(\alpha_0\lambda T)\right), v_0 \cdot \bar{H}_Q(\lambda T) + v_1 \cdot \left(1 - \bar{H}_Q(\lambda T)\right)\right\} \subseteq [v_0, v_1]
\]

Under the given price regime, Proposition 4 states that it is optimal to prescribe the clearance price \(p_l = v_0\). For the optimal premium price \(p_h\), the proposition establishes the range \(\Omega^{DA}_Q\). The upper bound of \(\Omega^{DA}_Q\) is based on the price above which class-1 customers always wait, whereas the lower bound is based on the premium price below which all class-1 customers buy immediately.

In order to identify the optimal strategy for the retailer, we conducted a hierarchical search procedure, as follows. We sequentially examined a plausible set of values of the initial inventory level \(Q\). Given each level of \(Q\), we applied a standard line-search procedure to determine the optimal premium price by solving the following problem

\[
\pi^{DA}(Q) \doteq \max_{p_h} \left\{\pi^{DA}(Q, p_h) : p_h \in \Omega^{DA}_Q\right\}
\]

We then searched for the optimal quantity \(Q\) by solving the problem: \(\pi^{DA} \doteq \max_{Q \geq 0} \left\{\pi^{DA}(Q)\right\}\). However, instead of searching all possible values of \(Q\), we reduced our search effort by using Proposition 4 to construct an upper bound on the function \(\pi^{DA}(Q)\) as follows.

\[\text{Clearly, the value of the functions } f \text{ depend on the price path } (p_h, p_l). \text{ However, the price parameters are omitted for brevity of exposition.}\]
Corollary 1 For all $Q \geq 1$,

$$
\pi^D_-(Q) \leq \pi^D (Q) \leq \pi^D_+ (Q),
$$

where

$$
\pi^D_-(Q) = -(c - s) Q + (v_0 - s) \cdot N(Q, \lambda T) + (v_1 - v_0) \cdot \left(1 - \tilde{H}_Q(\alpha_0 \lambda T)\right) \cdot N(Q, \alpha_1 \lambda T)
$$

$$
\pi^D_+(Q) = -(c - s) Q + (v_0 - s) \cdot N(Q, \lambda T) + (v_1 - v_0) \cdot \left(1 - \tilde{H}_Q(\lambda T)\right) \cdot N(Q, \alpha_1 \lambda T)
$$

The verification of the lower bound is straightforward, since it represents the expected profit under the price path that prescribes the lowest value in $\Omega^D_Q$ for the premium price and $p_l = v_0$. The upper bound is constructed by assuming $p_l = v_0$ and by assuming that all class-1 customers purchase the product at the highest possible premium price in the range $\Omega^D_Q$. The bounds in (5) lead to the following result.

Proposition 5 Let $\Delta^D_\pi (Q) = \pi^D (Q + 1) - \pi^D (Q)$. Then,

$$
\Delta^D_\pi (Q) \leq -(c - s) + (v_0 - s) \cdot \left(1 - \sum_{x=0}^{Q} P_x (\lambda T)\right) + (v_1 - v_0) \cdot \left(1 - \tilde{H}_Q(\lambda T)\right) \cdot \alpha_1 \lambda T
$$

Moreover, this bound on $\Delta^D_\pi (Q)$ is decreasing in $Q$.

The value of this proposition is that it allows us to employ a step-by-step (i.e., $Q = 0, 1, \ldots$) evaluation of the right-hand-side of (6), until we identify the first (smallest) $Q$ for which this value is negative; say, a level $\bar{Q}$ such that $\Delta^D_\pi (\bar{Q}) < 0$. This means that adding one more unit to the stock will certainly result in a reduction of the expected profit. This analysis is reminiscent of the “newsvendor model”. Nonetheless, here, we are not using the actual value $\Delta^D_\pi (Q)$, and so $\bar{Q}$ is not necessarily optimal. However, it is obvious that any $Q > \bar{Q}$ should be avoided, in view of the last part of the proposition. Therefore, we can limit our search to the set $Q \in \{0, \ldots, \bar{Q}\}$.

Finally, in order to identify the overall optimal expected profit associated with the DA format denoted hereafter by $\Pi^D$, we need to compare the retailer’s optimal expected profit associated with the second setting (i.e., $\pi^D$) with the retailer’s optimal expected profit associated with the first setting (i.e., $\Pi^1 = \max_{Q \geq 0} \{\Pi^1 (Q)\}$). Thus,

$$
\Pi^D = \max \{\pi^D, \Pi^1\}
$$
In the remainder of the paper, we use the notation \((Q^{DA}, p_h^{DA}, p_l^{DA})\) to express an optimal choice that leads to the performance \(\Pi^{DA}\).

5 The “Display One” (DO) Format Model

Under the “Display One” (DO) format, the retailer displays only a single unit on the sales floor, and keeps the rest in a “storeroom.” Suppose that, upon a sale, the retailer immediately retrieves a new unit from the storeroom and places it on display\(^7\). Let us focus initially on a given order quantity \(Q\), and a price path in the regime \(p_l \leq v_0 \leq p_h \leq v_1\). Clearly, all class-0 will always wait for the clearance price \(p_l\). Class-1 customers still need to consider their perceived fill rates. However, under DO, a class-1 customer only knows his arrival time \((t)\) and whether or not the product is still in stock. We therefore denote the perceived fill rate by \(\tilde{H}(t)\), excluding the inventory state parameter \(k\). The lack of perfect information about inventory turns the estimation of the fill rate into a technically challenging task. To overcome this modeling challenge, let us discuss three possible approaches for establishing the perceived fill rates \(\tilde{H}(t)\).

The first approach has been examined by Yin and Tang (2006). In this approach, it is assumed that all arriving customers have identical beliefs that the current inventory level is equal to \(k\) with probability \(\theta_k\), where \(\theta_k\) is given exogenously in the sense that the belief is independent of the retailer’s decisions \((Q, p_h\) and \(p_l))\) and of other customers’ purchasing behavior. In this case, it is easy to verify that \(\tilde{H}(t) = \sum_k \theta_k H(k,t)\), where \(H(k,t)\) is as defined for the DA format.

In the second approach, one assumes that the perceived fill rate is constant so that \(\tilde{H}(t) = \hat{H}\), where \(\hat{H}\) is determined endogenously. For any given value of \(\hat{H}\), all class-1 customers will purchase the product at \(p_h\) if \(\hat{H} \leq (v_1 - p_h) / (v_1 - p_l)\), and wait for the clearance price \(p_l\) otherwise. Given this customer purchasing behavior, \(\hat{H}\) needs to be the resulting fill rate exactly. Thus, \(\hat{H}\) constitutes a “self fulfilling prophecy” for the fill rate. Since all consumers act in accordance to this belief, \(\hat{H}\) turns out to indeed be the “correct” actual fill rate. It could be argued that the consumers’ strategy and beliefs are “closed” under rational expectations. However, while this approach has an interesting economic interpretation and it allows for reasonable tractability, it neglects the fact

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\(^7\) A situation like this is common in an e-tailing environment. For example, Lands’ End Overstocks’ website provides each arriving customer about product availability but not the actual inventory level.
that the perceived fill rate should be time-dependent.

Because neither the first nor the second approach satisfactorily addresses the time-varying beliefs of the actual inventory level or the fill rate, we take a different tack. In this third approach, we assume that the initial order quantity $Q$ is a common knowledge to all customers. But to capture the fact that the perceived fill rate is time dependent, we let $\tilde{H}_Q(t)$ be the fill rate assessed by a customer who arrives at time $t$. Clearly, this likelihood depends on the purchasing behavior of other customers and is treated rigorously in the next section.

### 5.1 Strategic Purchasing Under the Display One Format

Consider a class-1 customer who arrives at time $t$ and observes a unit on display (i.e., the product is still available for sale). The perceived fill rate $\tilde{H}_Q(t)$ maintains the following properties.

**Proposition 6** For any given (possibly randomized) time-dependent purchasing policy followed by class-1 customers, the perceived fill rate $\tilde{H}_Q(t)$ is continuous and non-decreasing in $t \in [0, T]$. Moreover, $\tilde{H}_Q(t)$ is strictly increasing in $t$ at (and only at) times when the purchasing policy prescribes a “buy now” action with a non-zero probability.

The significance of this proposition is that the expected surplus associated with a “wait” action is non-decreasing as a function of the arrival time. Therefore, if there is a point in time $t$ that has $\tilde{H}_Q(t) > \frac{v_1 - p_h}{v_1 - p_l}$, then it is optimal for all class-1 customers who arrive on or after $t$ to wait for the clearance price $p_l$. This implication is stated formally in the following theorem.

**Theorem 2** For any purchasing policy to be sustained in equilibrium, it must possess the following properties: all class-0 customers must wait for the clearance price $p_l$; and all class-1 customers must follow a threshold policy. Specifically, all class-1 customers arriving prior to a threshold $\tau$ should purchase the product at $p_h$, and all class-1 customers arriving after the threshold $\tau$ should wait for the clearance price $p_l$.

Theorem 2 suggests that it is sufficient to focus on threshold-type policies that can be characterized by a single threshold $\tau$. Hence, in order to explicitly capture the impact of the threshold $\tau$

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8Several online retailers (especially liquidators such as PacificGeek.com) provide consumers with their initial level of inventory for a particular item but do not update it during the selling season.
on the perceived fill rate, we substitute $\bar{H}_Q(t)$ by $L_Q(t|\tau)$. The following proposition provides a closed-form expression for the latter function.

**Proposition 7** The probability function $L_Q(t|\tau)$ is given by

$$L_Q(t|\tau) = \frac{\sum_{x=0}^{Q-1} \bar{H}_{Q-x}(\lambda(T - \alpha_1 \tau)) \times P_x(\alpha_1 \lambda \tau)}{\sum_{i=0}^{Q-1} P_i(\alpha_1 \lambda \cdot \min\{t, \tau\})}$$

(7)

for all $t \in [0, T]$. The function $L_Q(t|\tau)$ satisfies the following properties:

(i) $L_Q(t|\tau)$ is continuous both in $t$ and $\tau$.

(ii) $L_Q(t|\tau)$ is strictly increasing in $t$ on $[0, \tau)$ and constant on $[\tau, T]$.

Equation (7) represents $L_Q(t|\tau)$ as a mix of conditional probabilities, with each one considering a specific number of class-1 customers that buy the product at premium price (anytime during the season). Of course, all of these conditional probabilities assume that the threshold policy (with $\tau$) is followed by all class-1 customers. Part (i) of Proposition 7 implies that the function

$$L_Q(\tau) = L_Q(\tau|\tau)$$

is continuous. Therefore, there always exists an equilibrium (say $\tau_Q$) in the customers’ game. In particular, the following values of $\tau$ represent equilibrium strategies: (i) $\tau_Q = T$, if $L_Q(T) < (v_1 - p_h) / (v_1 - p_l)$; (ii) $\tau_Q = 0$, if $L_Q(0) > (v_1 - p_h) / (v_1 - p_l)$; and (iii) any solution $\tau_Q$ (if exists) that satisfies the equation:

$$L_Q(\tau_Q) = (v_1 - p_h) / (v_1 - p_l).$$

(8)

As follows from our theoretical analysis below, there could be multiple equilibria in the customers’ game; an important issue we will discuss shortly. Part (ii) of the proposition is reminiscent of a similar result for the DA format case: if at any time $t$ it is strictly better for a customer to “wait,” then all customers arriving after time $t$ should “wait” too. We now provide further characterization of the function $L_Q(\tau)$.

---

9 Clearly, under a threshold policy $\tau$, all class-0 customers and only those class-1 customers arriving after $\tau$ will wait for the clearance price. As such, the total number of customers waiting for the clearance price is equal to a Poisson random variable with rate $\alpha_0 \lambda T + \alpha_1 \lambda (T - \tau) = \lambda(T - \alpha_1 \tau)$. Therefore, if $Q - x$ units are available for sale, the fill rate for a focal customer is given by $\bar{H}_{Q-x}(\lambda(T - \alpha_1 \tau))$. 

19
Theorem 3 The function $L_Q(\tau)$ satisfies the following properties:

(i) $L_1(\tau)$ is strictly increasing in $\tau$.

(ii) For all $Q \geq 2$, the function $L_Q(\tau)$ is unimodal (quasi-convex) in $\tau$. Moreover, the function $L_Q(\tau)$ attains its unique minimum in the range $(0, T]$, and it is strictly decreasing (increasing) for all values of $\tau$ below (above) that minimum point.

(iii) For every $Q \geq 1$ and $\tau \in [0, T]$, $L_Q(\tau)$ is decreasing in $\lambda$.

From the perspective of game theory, Theorem 3 is key to the characterization of the equilibrium in the customers’ subgame. It implies that, when $Q = 1$, there is always a unique equilibrium in the game, and that the threshold in equilibrium is increasing in the ratio $(v_1 - p_h) / (v_1 - p_l)$. When $Q > 1$, the unimodality implies that there could be as many as three equilibrium points in the game. Specifically, this happens if $\min_{\tau \in [0, T]} L_Q(\tau) < (v_1 - p_h) / (v_1 - p_l) < \min \{L_Q(0), L_Q(T)\}$, in which case the three equilibrium points are $\tau = 0$ as well as the only two solutions to the equation $L_Q(\tau) = (v_1 - p_h) / (v_1 - p_l)$. The theorem also implies that if $(v_1 - p_h) / (v_1 - p_l) > L_Q(0)$, the game has a unique equilibrium strategy that has a positive threshold (i.e., $0 < \tau \leq T$). Clearly, for $(v_1 - p_h) / (v_1 - p_l) < \min_{\tau \in [0, T]} L_Q(\tau)$, the unique equilibrium is $\tau_Q = 0$ (i.e., everyone waits). The significance of part (iii) of the theorem will become clear shortly.

Given any game, a prediction of players’ behavior which is not a Nash equilibrium cannot be commonly believed. Hence, when we study a game that has only one equilibrium, it must be the only rational prediction of players’ behavior. However, when a game possesses multiple Nash equilibria the assumption that a particular one will be played relies on the existence of a process or mechanism that directs all the players to expect the same outcome. Schelling (1960) argues that when looking at the multiplicity of equilibria one should look at anything that focuses the players’ attention on a particular equilibrium. Specifically, when an equilibrium offers the highest possible payoffs to all players (it is Pareto dominant among the equilibria) it is natural to assume that the players will focus on it. Indeed, as the following proposition shows, the game we analyze possesses such an equilibrium.
Proposition 8 Suppose that \( Q \geq 2 \) and that \( (v_1 - p_h) / (v_1 - p_l) < L_Q(0) \). Then, among all possible existing equilibrium strategies, \( \tau_Q = 0 \) strictly dominates the others (if any) in terms of the expected surplus gained by class-1 customers.

Thus, under the conditions of Proposition 8, the subgame played by class-1 consumers is a coordination game possessing a single Pareto dominant equilibrium strategy profile; namely, when all class-1 consumers do not buy upon arrival but wait for the end-of-season sale, their equilibrium payoff is the highest possible. Since there is an equilibrium whose outcome is strictly preferred by every player to any other equilibrium outcome we can assume that the consumers will focus on it. Specifically, we shall consider the equilibrium that is uniquely defined as follows:

\[
\tau_Q^* = \begin{cases} 
0 & \text{if } \frac{v_1 - p_h}{v_1 - p_l} \leq L_Q(0) \\
\text{The unique positive solution to Eq. (8)} & \text{if } L_Q(0) < \frac{v_1 - p_h}{v_1 - p_l} \leq \max \{L_Q(0), L_Q(T)\} \\
T & \text{if } \frac{v_1 - p_h}{v_1 - p_l} > \max \{L_Q(0), L_Q(T)\}
\end{cases}
\]

(9)

One of the immediate results that follow is similar to Proposition 2 for the DA case.

Proposition 9 Under the DO format, the threshold \( \tau_Q^* \) as defined in (9) is increasing in \( \lambda, v_1 \) and \( p_l \), and decreasing in \( p_h \).

We next present a relationship between the threshold \( \tau_Q^* \), associated with the DO format, and the sequence of values \( \{\tau^*(k) : k = 1, \ldots, Q\} \) associated with the DA format.

Proposition 10 For any given level of inventory \( Q \geq 1 \), and a pair of prices \((p_h, p_l)\) that satisfy the condition \( p_l \leq v_0 \leq p_h \leq v_1 \), we have

\[
\tau^*(Q) \leq \tau_Q^* \leq \tau^*(1)
\]

(1)

Proposition 10 can be interpreted as follows. Consider a situation in which the retailer is currently using a DA format that yields \( \tau^*(Q) > 0 \). Now, suppose that the retailer contemplates moving to a DO format without changing the price path. One of the concerns the retailer may have is that the DO format might lead all class-1 customers to wait. In other words, is it conceivable that \( \tau_Q^* \) will settle at the value 0? Proposition 10 implies that this cannot happen because \( \tau_Q^* \geq \tau^*(Q) > 0 \).

\[\text{For Further discussion, see Section 1.2.4 in Fudenberg and Tirole (1991).}\]
Proposition 10 also suggests that by moving from a DA to DO while keeping the price path unchanged, the volatility of the retailer’s profit under the DO model will be lower than that of the DA model. This is because the DO model appears to be less sensitive to the arrival time of the first class-1 customer than the DA model. To see this, note that if no arrival occurs prior to \( \tau^*(Q) \), then no purchase at premium price will be made during the entire horizon under the DA model. However, purchases could still occur under the DO model until \( \tau^*_Q \geq \tau^*(Q) \). Nonetheless, if sales are made early enough, and so the inventory declines, purchases under the DA model may still continue after \( \tau^*_Q \): the time at which class-1 customers begin to wait under the DO model.

5.2 The Retailer’s Problem Under the Display One Format

Anticipating the threshold policy \( \tau^*_Q \) adopted by all class-1 customers in equilibrium, it is easy to check that the retailer’s expected profit can be expressed as:

\[
\pi^{DO}(Q, p_h, p_l) = - (c - s) Q + (p_l - s) \cdot N(Q, \lambda T) + (p_h - p_l) \cdot N(Q, \alpha_1 \lambda \tau^*_Q (p_h, p_l))
\]

(Note that we explicitly describe \( \tau^*_Q (p_h, p_l) \) to emphasize the dependency of the threshold on the price path) The retailer’s expected profit \( \pi^{DO}(Q, p_h, p_l) \) can be interpreted as follows. The first term represents the loss incurred if no unit was sold during the season. The second term corresponds to the margin of \((p_l - s)\) for each unit that was sold during the season. The last term adds an additional margin of \((p_h - p_l)\) for each unit that was sold at the premium price \(p_h\) instead of the clearance price \(p_l\).

Given the retailer’s profit function \( \pi^{DO}(Q, p_h, p_l) \), the following proposition characterizes a property associated with the optimal price path.

**Proposition 11** Consider a given finite level of initial inventory \( Q > 0 \), and suppose that the price path is limited to the range \( \{p_l \leq v_0 \leq p_h \leq v_1\} \). Then, it is optimal to set \( p_l = v_0 \). Given \( p_l = v_0 \), the value of \( p_h \) can be practically limited to the range

\[
\Omega^{DO}_{Q, \varepsilon} = \begin{cases} 
[L_Q(0) v_0 + (1 - L_Q(0)) v_1 - \varepsilon, L_Q(0) v_0 + (1 - L_Q(0)) v_1] & \text{if } L_Q(0) \geq L_Q(T) \\
[L_Q(T) v_0 + (1 - L_Q(T)) v_1, L_Q(0) v_0 + (1 - L_Q(0)) v_1] & \text{otherwise}
\end{cases}
\]

for any \( \varepsilon > 0 \). Specifically, while it is possible that no optimal value of \( p_h \) exists in this range, there always exists a price \( p_h \in \Omega^{DO}_{Q, \varepsilon} \) that yields a performance that is arbitrarily close to optimal.
Similarly to Proposition 4 (the DA model), under the given price regime, it is optimal for the retailer to set the clearance price \( p_l = v_0 \). The rationale behind this property is that it is ineffective to offer a discounted price that is lower than \( v_0 \). Otherwise, by raising that price to \( v_0 \), the retailer gains in two ways: he reduces the motivation of class-1 customers to wait and obtains higher revenues from those who wait. While, conceptually, this line of proof is not hard to follow, it relies on the fact that in equilibrium, the threshold value \( \tau^*_Q \), associated with the DO format, is non-decreasing in \( p_l \). In other words, it relies heavily on the theoretical results presented in the previous subsection.

How do we choose a good premium price? Essentially, Proposition 11 shows that given the optimal choice of \( p_l = v_0 \), we can limit our attention to premium prices that satisfy the condition \( L_Q(0) < (v_1 - p_h) / (v_1 - p_l) \leq \max \{ L_Q(0), L_Q(T) \} \), virtually without loss of optimality. Note that this is the middle condition of (9), for which the threshold value \( \tau^*_Q \) can be interpreted as the unique indifference point for class-1 customers; i.e., the single time in the season for which a class-1 customer is indifferent between a “buy now” and a “wait” decision. Once again, the bound provides us with computational convenience. For every value of \( p_h \in \Omega^{DO}_{Q,\varepsilon} \), we can simply calculate the ratio \( (v_1 - p_h) / (v_1 - v_0) \), and identify \( \tau^*_Q \) via (8).

As in the DA case, we can determine the optimal premium price \( p_h \) by solving the following problem via a simple line-search.

\[
\pi^{DO}(Q) \doteq \max \left\{ \pi^{DO}(Q, p_h, p_l = v_0) : p_h \in \Omega^{DO}_{Q,\varepsilon} \right\}
\]

for some \( \varepsilon \) value (we used \( \varepsilon = 0.001v_1 \)). Using the results of Proposition 11, and following similar justifications as those explaining Corollary 1, we can use the boundary points of the range \( \Omega^{DO}_{Q,\varepsilon} \) in order to obtain upper and lower bounds on \( \pi^{DO}(Q) \). Specifically\(^{11}\), we have

\[
\pi^{DO}_{-}(Q) \leq \pi^{DO}(Q) < \pi^{DO}_{+}(Q)
\]

\(^{11}\)The lower bound in (5) may need to be corrected to \( \pi^{DO}_{-}(Q) - \tilde{\varepsilon} \), depending on the value of \( \varepsilon \) that defines the set \( \Omega^{DO}_{Q,\varepsilon} \). We avoid this non-crucial detail for the sake of brevity of exposition.
\[\pi_{DO}^{-}(Q) = - (c - s) Q + (v_0 - s) \cdot N(Q, \lambda T) + (v_1 - v_0) \cdot (1 - \max\{L_Q(0), L_Q(T)\}) \cdot N(Q, \alpha_1 \lambda T)\] and \(\pi_{DO}^{+}(Q) = \pi_{DA}^{+}(Q)\). Since the upper bound in (10) is the same as that for the DA model, we can use the results of Proposition 5 to identify a maximal value of \(Q\) that needs to be considered for the purpose of solving the problem \(\pi_{DO} \doteq \max_{\Omega \geq 0} \{\pi_{DO}(Q)\}\). Interestingly, we establish the following result.

**Corollary 2** For all \(Q \geq 1\), \(\pi_{DO}^{-}(Q) \geq \pi_{DA}^{-}(Q)\).

This corollary is justified by the fact that \(L_Q(T)\) is expressed (see definition) as a weighted average of the functions \(\hat{H}_x(\alpha_0 \lambda T)\) where \(x = 1, \ldots, Q\), and that these functions are increasing in \(x\). Thus, \(\hat{H}_Q(\alpha_0 \lambda T) \geq L_Q(T)\). Additionally, since \(\hat{H}_Q(\cdot)\) is a decreasing function of its argument, it is straightforward to argue that \(\hat{H}_Q(\alpha_0 \lambda T) \geq H_Q(\lambda T) = L_Q(0)\). Obviously, since the values in the corollary are just lower bounds, the latter inequality does not imply that the optimal expected profits \(\pi_{DO}(Q)\) and \(\pi_{DA}(Q)\) maintain the same relationship. However, the inequality is at least consistent with our belief that the DO format can serve as a mechanism to place more pressure on class-1 customers (in terms of perceived risk of stockout), so they feel more inclined to purchase at premium prices.

Finally, in order to identify the overall optimal expected profit associated with the DO format denoted by \(\Pi_{DO}\), we need to compare the retailer’s optimal expected profit associated with the second setting (i.e., \(\pi_{DO}\)) with the retailer’s optimal expected profit associated with the first setting (i.e., \(\Pi_1 \doteq \max_{\Omega \geq 0} \{\Pi_1(Q)\}\)). Thus,

\[\Pi_{DO} \doteq \max \{\pi_{DO}, \Pi_1\}\]

In the remainder of the paper, we use the notation \((Q^{DO}, p_h^{DO}, p_l^{DO})\) to express an optimal choice that leads to the performance \(\Pi_{DO}\).
6 Benchmark Models

In this section, we develop two benchmark models. In the first model, all customers are myopic (i.e., non-strategic). In the second model, the price path is constant so that \( p_h = p_l \). First, consider the case when customers are myopic. In this case, regardless of the display format, a customer will purchase the product immediately upon arrival if his valuation is above the premium price \( p_h \); otherwise, he will wait for the clearance price \( p_l \). In the myopic model, it is optimal for the retailer to adopt the price path \( \{ p_h = v_1, p_l = v_0 \} \). Additionally, it is easy to verify that the retailer’s expected profit for a given initial order quantity \( Q \) is given by

\[
\Pi^M (Q) = -(c - s) Q + (v_0 - s) \cdot N(Q, \lambda T) + (v_1 - v_0) \cdot N(Q, \alpha_1 \lambda T)
\]  

The retailer can determine the optimal order quantity \( Q^M \) by solving the following problem:

\[
\Pi^M = \max_Q \Pi^M (Q)
\]

The exact same argument can be made for the DO model.
Because the fixed-price is a feasible price path in the DA and DO models, it is easy to show that

**Proposition 12** For all $Q \geq 1$,

$$\Pi^F (Q) \leq \{\Pi^{DA} (Q), \Pi^{DO} (Q)\} \leq \Pi^M (Q).$$

(12)

Also, $\Pi^F \leq \{\Pi^{DA}, \Pi^{DO}\} \leq \Pi^M$.

The two benchmark models described above allow us to examine the value of markdown pricing, as examined in the next section.

### 7 Numerical Studies

In this section, we present the details of an extensive numerical study we designed for comparing the retailer’s optimal price path $(p_h, p_l)$, optimal initial order quantity $Q$, and optimal expected profit under the DA and the DO formats. The remainder of this section is organized as follows. In §7.1, we present and explain the choice of the parameter space for our numerical study. In §7.2 we consider the case in which the initial order quantity $Q$ is not a decision variable, but rather, it is given. In §7.3, we explore the models of joint ordering and pricing optimization.

#### 7.1 The Parameter Space and Performance Metrics

In our study, we focused on the space of model parameters $\{\alpha_0, \alpha_1, v_0, v_1, \lambda, T, c, s\}$. However, in order to facilitate a clear and focused analysis, we reduced this large parameter space. For example, we set $v_1 = 1$. This is equivalent to a straightforward rescaling of the parameters $\{v_0, c, s\}$, the prices $\{p_h, p_l\}$, and the revenue values. We also set $T = 1$. This time-scaling requires caution in the interpretation of time, which is now expressed as a portion of the sales horizon, and consequently requires an interpretation of $\lambda$ as the average traffic during the entire horizon. As to the value of $s$, note that a simple shift-transformation (of the type $x := x - s$) of all cost, valuation and price parameters, transforms the problem into one with $s = 0$. Finally, in order to study the impact of costs on performance, we considered the scaled value $\bar{c} := c / v_0$ instead of $c$. This transformation lends itself to a clearer interpretation of the numerical results. More importantly, this transformation enables us to reduce the parameter space from 8 parameters to 4; namely, $\{v_0, \alpha_0, \lambda, \bar{c}\}$. 

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The core of our numerical experiments is spanned over $9 \times 9 \times 6 \times 5 = 2,430$ combinations of the following parameter values:

\[ v_0 \in \{0.1, 0.2, \ldots, 0.9\}, \alpha_0 \in \{0.1, 0.2, \ldots, 0.9\} \]
\[ \lambda \in \{1, 2, 5, 10, 15, 20\}, \bar{c} \in \{0.1, 0.3, 0.5, 0.7, 0.9\} \]

Below, we use the term “scenario” to denote each instance of parameter combination.

We now present some of the performance metrics used in our analyses. First, in order to examine the value of markdown pricing relative to fixed-price policies, we can use the benchmark measure $\Pi^F$. For instance, one can define the theoretical value of markdown pricing as the relative increase in expected profits if the seller moves from a fixed-price policy to a markdown policy. For the DA and DO model, this value can be measured by

\[
\eta^{DA} = \frac{\Pi^{DA}}{\Pi^F} - 1, \quad \text{and} \quad \eta^{DO} = \frac{\Pi^{DO}}{\Pi^F} - 1. \tag{13}
\]

Hereafter, we will refer to $\eta^{DA}$ and $\eta^{DO}$ as the benefits of markdown pricing. Another measure of interest is the value of markdown pricing when customers are myopic, which can be defined as

\[
\eta^M = \frac{\Pi^M}{\Pi^F} - 1. \tag{14}
\]

We shall refer to $\eta^M$ as the ideal benefits of markdown pricing. Obviously, in view of (12), we have $0 \leq \{\eta^{DA}, \eta^{DO}\} \leq \eta^M$. For instance, one can interpret the value of $\eta^M - \eta^{DA}$ as the loss in the value of markdown pricing due to strategic consumer behavior (see, e.g., Aviv and Pazgal 2007).

Next, to compare the retailer’s profits under the different display formats, let $\Pi^{DO} - \Pi^{DA}$ denote the marginal benefit of DO. There are a couple of cases of special interest. One case reflects situations in which the retailer has already made the choice of $Q$ (the initial order quantity). In such case, the marginal benefit of DO is given by $\Pi^{DO} (Q) - \Pi^{DA} (Q)$. A second special case is one in which both the order quantities and prices are set, and it is left to choose between the display formats. Here, the marginal benefit of DO is given by $\Pi^{DO} (Q, p_h, p_l) - \Pi^{DA} (Q, p_h, p_l)$. Another related measure of interest, is the relative marginal benefit of DO, defined by

\[
\delta = \frac{\Pi^{DO}}{\Pi^{DA}} - 1.
\]
7.2 The Case of Fixed Inventory: A Revenue Maximization Problem

We initially considered settings in which the level of inventory \( Q \) is given, rather than being a decision variable. Obviously, since under a given choice of inventory, the procurement cost \( cQ = \bar{c}v_0Q \) can be considered as sunk, the retailer’s objective is to maximize expected revenues. In particular, we set \( \bar{c} = 0 \), which allows us to interpret the expected profit measures as expected revenues. Since \( \bar{c} \) is kept fixed, we are left with the 486 combinations of the parameters \( \{v_0, \alpha_0, \lambda\} \).

For each of these parameter combinations, we considered five values of \( Q \in \{1, 2, 5, 10, 15\} \), hence having 2,430 scenarios in total. For each of these scenarios we assessed various performance metrics as discussed above.

Our first step was to explore the impact of the display format alone. Specifically, we considered the gap

\[
\Pi^{DO}(Q, p_{h}^{DA}, p_{l}^{DA}) - \Pi^{DA}(Q),
\]

which measures the gain in expected revenue when the retailer changes from the DA format to the DO format by using the same optimal price path derived in the DA model. Hence, for any given order quantity \( Q \), this gap enables us to measure the impact of observability of inventory on the retailer’s expected revenue when customers are strategic. Clearly, as the supply is limited under both display formats, we speculate that this gap should be non-negative, because there is an additional perceived scarcity under the DO format that is induced by the fact that the inventory level is unobservable. Confirming our intuition, we found that the value of the relative gap \( \Pi^{DO}(Q, p_{h}^{DA}, p_{l}^{DA}) / \Pi^{DA}(Q) - 1 \) was non-negative in all scenarios. However, we found that the average value of this relative gap is small, with a maximum level of 1.7% only.

Naturally, our next step was to explore the potential benefit of a change in display format when optimal price path is used in each format. Specifically, we measure the performance gap: \( \pi^{DO}(Q) - \pi^{DA}(Q) \). Again, in all 2,430 scenarios, we found this gap to be non-negative. Clearly, this outcome is not surprising, in view of the previous observation, and the fact that \( \Pi^{DO}(Q) \geq \Pi^{DO}(Q, p_{h}^{DA}, p_{l}^{DA}) \) by virtue of optimality. Recall that Corollary 2, which establishes the property \( \pi^{DO}(Q) \geq \pi^{DA}(Q) \), supports our observation that \( \Pi^{DO}(Q) \geq \Pi^{DA}(Q) \). Yet, as we explained above, this corollary suggests, but does not rigorously prove that the optimal expected revenues
\( \pi^{DO} (Q) \) and \( \pi^{DA} (Q) \) maintain the same relationship. To prove the latter property, one would have to consider the intrinsic way in which the equilibrium settles under both display formats (in terms of the retailer’s pricing decision). We found that, on average, the value of \( \delta \) is 0.25% only. More than 95% of the scenarios exhibited \( \delta < 2% \), and just about 2% exhibited \( \delta > 3% \). The maximal value of \( \delta \) observed is 6.90%. First, these results suggest that when moving from DA to DO, a price modification (i.e., selecting the optimal price for the DO case) typically plays a larger role than merely changing the display format. As we expected, the optimal regular price \( (p_h) \) under the DO format was higher or equal to that under the DA format across all instances. We attribute this, again, to the retailer’s ability to charge more due to the increased sense of scarcity that the DO format creates. But the results also demonstrate that the relative marginal benefits of DO, even at their highest levels, are not dramatic in magnitude. Nonetheless, one should be cautious not to interpret the value of DO as typically insignificant. Often, even an increase of 1-2% in revenues could have a high impact on expected profits, and in some cases make the difference between a loss and a profit; see, e.g., Marn and Rosiello (1992) and Marn et al. (2003).

In order to put the previous results in perspective, we note that the average benefits of markdown pricing are 1.37% and 1.64% for \( \eta^{DA} \) and \( \eta^{DO} \), respectively (see (13)). The maximal values of \( \eta^{DA} \) and \( \eta^{DO} \) are both 46.5%. Furthermore, when looking at the ideal benefits of markdown pricing (i.e., \( \eta^M \); see (14)), we observe an average of 13.7% and a maximum of 90.0%. The striking difference between \( \eta^M \) and \( \{\eta^{DO}, \eta^{DA}\} \) exhibits the sharp adverse impact that strategic consumer behavior could have on the retailer’s ability to effectively use price segmentation\(^{12}\).

The purpose of our next step is to identify the conditions under which the DO and DA formats are likely to exhibit a noticeable difference in performance. There are three cases to keep in mind. First, when \( \Pi^{DA} (Q) \leq \Pi^{DO} (Q) \leq \Pi^1 (Q) \), it is optimal for the retailer to sell exclusively to class-1 customers. In such case, a fixed-price will be set at \( v_1 \) and hence the display formats will not matter. As can be expected, this situation happens when \( \alpha_0 \) or \( v_0 \) are relatively small. The second case is \( \Pi^{DA} (Q) \leq \Pi^1 (Q) \leq \Pi^{DO} (Q) \). Here, it is beneficial to use a markdown pricing mechanism under DO, but sell exclusively to class-1 customers under DA. Among all 2,430 scenarios, we observed only 17 of this type. The third and most interesting case is that in which \( \Pi^{DA} (Q) > \Pi^1 (Q) \), and

\(^{12}\)For further discussion of this matter, we refer the reader to Aviv and Pazgal (2007).
we focus on it in the remainder of this section.

Consider first the impact of the parameters $Q$ and $\lambda$. Table 1 below shows the distribution of the relative marginal benefit of the DO format over the DA format (i.e., $\delta$) associated with different values of $Q$ and $\lambda$. First, note that when $Q = 1$, the two display formats are identical,

<table>
<thead>
<tr>
<th>$Q$</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>2</td>
<td>0.70%</td>
<td>1.66%</td>
<td>2.10%</td>
<td>1.07%</td>
<td>0.63%</td>
<td>0.42%</td>
<td>1.23%</td>
</tr>
<tr>
<td>5</td>
<td>0.01%</td>
<td>0.11%</td>
<td>1.52%</td>
<td>2.24%</td>
<td>1.61%</td>
<td>1.06%</td>
<td>0.98%</td>
</tr>
<tr>
<td>10</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.05%</td>
<td>1.14%</td>
<td>2.06%</td>
<td>1.86%</td>
<td>0.76%</td>
</tr>
<tr>
<td>15</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.12%</td>
<td>0.95%</td>
<td>1.78%</td>
<td>0.45%</td>
</tr>
<tr>
<td>Total</td>
<td>0.14%</td>
<td>0.36%</td>
<td>0.71%</td>
<td>0.93%</td>
<td>1.26%</td>
<td>1.35%</td>
<td>0.69%</td>
</tr>
</tbody>
</table>

Table 1: The average relative marginal benefits of DO, $\delta$, for different combinations of parameters $Q \in \{1, 2, 5, 10, 15\}$ and $\lambda \in \{1, 2, 5, 10, 15, 20\}$. The table considers only those scenarios under which $\Pi^{DA}(Q) > \Pi^{1}(Q)$.

since knowing the status of inventory (in-stock or sold-out) automatically reveals the exact amount of inventory on hand (1 or 0, respectively). For $Q \geq 2$, the table suggests that the DO format offers noticeable benefits only when the level of $\lambda$ lies within a medium range (relative to $Q$). We explain this phenomenon, as follows. As the store traffic $\lambda$ increases significantly, the thresholds values $\{\tau^*(k)\}$ and $\tau^*_Q$ increase towards $T$ under both display formats; see Propositions 2 and 9. Hence, we expect a larger portion of class-1 customers to buy at premium price (instead of wait) under both display formats. Consequently, the marginal benefit of DO would vanish. If the traffic intensity $\lambda$ is relatively low, then class-1 customers’ perceived fill rates are high. As such, more class-1 customers will wait for the clearance price $p_1$, regardless of the display format.

We also examined the impact of the parameters $v_0$ and $\alpha_0$ on the marginal benefit of DO. We found that for relatively small values of $v_0$ and $\alpha_0$, the retailer is better off targeting class-1 exclusively. Consequently, the display format would not matter. When the values of $v_0$ and $\alpha_0$ are sufficiently high, the DO format is also not beneficial. For example, if $v_0$ is high, then class-1 does not offer much potential for segmentation anyway. Now, suppose that the value of $\alpha_0$ approaches
1. If the traffic rate $\lambda$ is relatively low (i.e., very small $\alpha_1 \lambda T$), a focal class-1 customer need not worry about the behavior of other class-1 customers, since such arrivals are unlikely to happen at all. If $\lambda$ is relatively high such that more than one arrival of class-1 customers is possible, then, again, a focal class-1 customer can ignore the behavior of other class-1 customers. This is because a large number of class-0 customers are likely to storm the store at the end of the season, making it unlikely to obtain a product.

7.3 The Case of Expected Profit Maximization

We now turn our attention to the case in which the retailer selects both the initial order quantity and price path so as to optimize his expected profit. In contrast to the previous section, the retailer can now select the initial order quantity $Q$ as a lever to influence customers’ perception of product scarcity. Recall from the previous section that we speculate that, relative to the DA format, customers’ perception of product scarcity is higher under the DO format because the inventory level is unobservable. This observation seems to suggest that, as the retailer changes from the DA format to the DO format, he may be able to increase his initial order quantity $Q$ without pushing too many class-1 customers to wait for the clearance price. Alternatively, the retailer may order the same amount when changing from the DA format to the DO format, but increase the premium price $p_h$. Furthermore, the retailer may increase the order quantity and premium price when changing from the DA format to the DO format. Unfortunately, due to the complexity of our models, it is generally hard to predict the way in which the optimal ordering and pricing decisions will behave in equilibrium.

To examine the impact of display formats on the joint ordering and pricing decisions, we consider all 2,430 scenarios spanned by the set of combinations of the four parameter $\{v_0, \alpha_0, \lambda, \bar{c}\}$ as described in §7.1. We separate these 2,430 scenarios into two groups. The first group corresponds to scenarios under which it is beneficial to sell exclusively to class-1 customers. In this group of scenarios, the retailer’s profit $\Pi^1$ is the same under both display formats, and hence, the retailer gains nothing from changing the display format. Hence, this group of scenarios is not interesting for our purpose.

For this reason, we shall focus on the second group of scenarios under which $\Pi^{DA} > \Pi^1$. In this
group of scenarios, the optimal clearance price $p_l = v_0$ under both display formats (see Propositions 4 and 11). To illustrate the difference in the optimal ordering and pricing decisions between the DA and DO formats, we map the distribution of the gaps $p^D - p^A$ and $Q^D - Q^A$ in Figure 1 below. Several areas in this figure are of particular interest. First, confirming our intuition, note that no point resides in the lower-left quarter of the chart. This means that a move from a DA to a DO format never leads to a simultaneous decline in inventory and pricing. Interestingly, we do not observe many points in the interior of the upper-right quarter of the figure. The practical meaning of this observation is that the retailer’s strategy, when moving from a DA to a DO format, is generally one of two types. In some cases, the retailer’s optimal action is to increase the order quantity, but maintain the same price or decrease it; this is reflected by the lower-right quarter of the figure. In other cases, the optimal action is to increase the premium price, along with a decrease or no change in the order quantity; this is reflected by the upper-left side of the figure.

Compared to the optimal inventory choice under the case of myopic consumers ($Q^M$), we found that the optimal inventory levels under the DA and DO formats are lower. This is not surprising because when customers are myopic, neither the quantity nor the premium price ($p_h \in [v_0, v_1]$)

Figure 1: The joint distribution of price and inventory changes, when moving from a DA to a DO format. This figure includes all scenarios where $\Pi^A > \Pi^1$. 

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impacts the purchasing behavior of class-1 customers (they always buy the product at $p_h$). Consequently, the seller can bring more units to the market and sell them at a higher price (of $p_h = v_1$) without worrying about driving the high-valuation customers to wait for the clearance price.

8 Model Variants: Extensions and Ideas for Future Research

In this section, we discuss several interesting variants of our models. Some of them can be viewed as relatively simple model extensions, and we explain the way in which they can be analyzed using our existing methodological framework. The other variants call for changes in the assumptions that result in prohibitively hard problems. We explain the source of complexity in each of these challenging problems, and propose these model variants as potential subjects for future research.

8.1 Uncertain Customer Returns

In our paper, we assume that all customers who decide to wait will return to the store at time $T$ and attempt to purchase the product at the post-season clearance price $p_l$. In contrast, suppose that when a customer decides to wait, he is not certain about whether or not he will be able to materialize his plan to return to the store at time $T$. In the Online Addendum of this paper, we describe a possible model extension in which we introduce an exogenous parameter $r_1$ that represents the probability that a “waiting” customer will actually return to the store at time $T$. There are several possible interpretations of this parameter. One way to think of $r_1$, is as representing the possibility that a customer may be busy, and unable to return at time $T$. Another possible interpretation of $r_1$ is as a “proxy” parameter that reflects the inconvenience associated with a wait decision. In other words, think of $r_1$ as a “discount” that customers associate with the expected surplus gained by a wait decision. Or, one can think of a situation in which the customer may lose interest in the product (e.g., if the customer finds an alternative product). Interestingly, the parameter $r_1$ has two conflicting implications on customers’ choices. On one hand, a low value of $r_1$ means that customers are more pressed to purchase the product at the premium price. On the other hand, a low value of $r_1$ means that a focal class-1 customer is less concerned about other waiting class-1 customers, since they are unlikely to get back to the store at the end of the season.
8.2 Costly Customer Returns

Similarly to the discussion above, one could also apply our models to examine the situation of costly returns to the store. Suppose that customers incur a cost (say) $c_1$ if they revisit the store at the time of discount. In other words, a class-1 customer now compares the current surplus $v_1 - p_h$ with the expected surplus $-c_1 + (v_1 - p_l) H(k, t)$. Clearly, the customers’ purchasing policy is exactly the same as it would be in our main models, if the premium price was $p_h - c_1$. Therefore, by Propositions 2 and 9, we anticipate the threshold levels to increase. In particular, class-1 customers will tend to buy at the premium price rather than wait, as $c_1$ increases. In that sense, we speculate that DO can create a larger sense of scarcity (than it would under lower threshold values). However, as $c_1$ increases, class-1 customers are pressed to buy at premium price, regardless of the display format.

8.3 Timing of Discount

In this section we discuss the impact of the time of discount on the retailer’s expected profit and on the marginal contribution of the DO format. Suppose that instead of offering the discounted price only at the end of the season (i.e., time $T$), the retailer could switch to the discount price earlier, at time $T_D \in (0, T]$. As before, customers that arrive prior to the time of discount need to weigh the current surplus (gained by an immediate purchase) against the expected surplus that can be gained if they wait until time $T_D$. Since the discounted price is offered from $T_D$ until the end of the season (time $T$), customers that arrive after $T_D$ make an immediate purchase if a unit is available, and the surplus they can gain is positive. In the Online Addendum of this paper, we discuss the way in which our analysis needs to be modified.

We conducted a numerical study of a representative subgroup of the parameter combinations described in §7.2. Specifically, we measured the expected revenues under the two display formats, and under the two benchmark models – the case of myopic customers, and the case of a fixed-price. First, we found that these expected profits behave monotonically (non-decreasing) with respect to the time of discount, $T_D$. This is not surprising, since an extended period of time for the high-price season exposes more class-1 customers to the premium prices. Furthermore, in the case of strategic customers, a larger value of $T_D$ even increases the sense of product scarcity, since a given customer is likely to compete against more customers for product availability if he decides to
wait for the time of discount. Figure 2 below presents the results for the parameter combination \( \{Q = 5, \lambda = 10, v_0 = \alpha_0 = 0.7\} \). In the horizontal axis, we vary the time of discount \( (T_D) \), from a small fraction of the season (5%) to the end of the season (100%). The ideal value of segmentation is reflected in this figure by the gap between the “myopic” and “fixed-price” benchmark curves. We observe, in this particular figure, that the difference between the expected revenues under DO and DA is non-decreasing in \( T_D \). We found this property to be consistent across all of the parameter combinations we examined. This confirms our hypothesis that as the potential for segmentation increases (i.e., larger \( T_D \)), there is more value to be gained from the sense of scarcity that the DO format creates.

8.4 Heterogeneity in Consumer Behavior

Recall that in this paper we considered two extreme cases of consumer behavior. In our main models, we assumed that all arriving customers are strategic. We compared these models to the “myopic” benchmark case in which all customers are assumed to be myopic. Obviously, it is
conceivable that in most practical settings sellers are encountered with some mixture of strategic and myopic consumers. We believe that such kind of behavior heterogeneity will not have any significant influence on the spirit of our results. In other words, we expect the marginal value of DO to increase when there is a larger percentage of strategic customers among class-1 customers, together with a large potential for segmentation (i.e., a sufficiently large difference between $v_1$ and $v_0$, and a medium value of $\alpha_0$). Nonetheless, it may be interesting to explore this aspect in a rigorous way. For instance, consider the market mixture studied by Cachon and Swinney (2008). There, the market consists of strategic, myopic, as well as last-minute “bargain hunters.” For our DA model, we intuitively expect that the strategic customers will continue to follow an inventory-dependent threshold policy. However, the proof of such property is very challenging; moreover, the calculation of the threshold values is highly complex. The reason for this is as follows. Recall that in our main model, class-0 customers always wait, and all we needed to do is to compute the probability $H(k, t)$ at the threshold point $t = \tau^*(k)$ for class-1 customers. But, this was relatively simple, since in equilibrium, a class-1 customer that arrives at $\tau^*(k)$ and finds $k$ units, knows that no other class-1 customer is currently waiting, and that all class-1 customers who arrive after that time will wait. In contrast, when we have a mix of strategic and myopic customers, the latter property no longer holds\(^{13}\).

8.5 Multi-Class Customers

To capture market heterogeneity, we have considered the case when the customers are classified into two classes according to their valuations. Admittedly, we have utilized the simplest possible form of heterogeneity, in which each class consists of customers with the same valuation, and where valuations do not vary over time. The treatment of multi-class market is known to be challenging, and is most likely to be based on some form of heuristic or a simplifying structural assumption on the customers’ purchasing pattern; see, e.g., Elmaghraby et al. (2007).

A natural extension of our model could be to think of a market that consists of $J$ classes of

\(^{13}\)Consider the following example. Suppose that $\tau^*(3) = 0.4$ and $\tau^*(2) = 0.5$. Now let us consider an arrival path of class-1 customers as follows: {Strategic at time 0.41, Myopic at time 0.44, Strategic at time 0.47}. Furthermore, suppose that the level of inventory at time 0.41 is 3. In this case, the first strategic customer will wait, since $0.41 > \tau^*(3)$. The second (myopic) customer will buy at time 0.44, and inventory will decline to 2 units. But now, the third (strategic) customer will “buy now,” since $0.47 < \tau^*(2)$. 

customers, where \( v_j \) is the valuation of class-\( j \) customers \( (j \in \{1, \ldots, J\}) \), and \( \alpha_j \) is the proportion of this class in the population. Without loss of generality, assume that \( v_1 < v_2 < \ldots < v_J \). Suppose that a price path \((p_h, p_l)\) is announced. Now, consider a “focal” class-\( j \) customer (with \( v_j \geq p_h \)) that arrives at time \( t \) and finds \( k \) units of inventory in stock. This customer needs to compare the current surplus of \( v_j - p_h \), with the expected surplus associated with a return decision; namely, \((v_j - p_l) H(k, t)\). So far, there is nothing conceptually different from the two-class model of our paper. However, when it comes to the evaluation of \( H(k, t) \), the multiplicity of classes adds a significant level of mathematical complexity, for the exact same reason discussed in the previous subsection.

Given that an extension of our model to a multi-class setting is tedious, is there any conjecture that we can make vis-a-vis the marginal benefits of the DO format? We suggest the following framework. First, a retailer should identify the classes of customers he wishes to target. Obviously, classes with relatively low valuations should be excluded, particularly if their presence in the market is small. This is obviously true if deep markdown pricing that aims at such classes, might drive classes with higher valuations to wait. In the numerical study, we found that when the traffic intensity to the store is very high, relative to the number of units available, it makes a greater sense to target high-valuation classes exclusively. Therefore, in a multi-class setting, the retailer needs to identify the classes of customers to target. To this end, he needs to consider the variance in valuations and proportions of the different classes. The marginal benefits of DO appear to be the largest when there is a significant spread in valuations (between the targeted class with the highest valuation and the targeted class with the lowest valuation). Otherwise, high-valuation classes do not offer much potential for segmentation anyway, and hence the display format will hardly matter. The marginal benefits of DO are higher when the proportion of the high-valuation classes among the targeted customer base is at a medium level. If their proportion is very high, it makes sense to target these classes only in the first step mentioned above. If their proportion is too small, these customers’ purchasing behavior will depend on the traffic intensity, but not that much on the display format (see detailed explanation in the numerical study section). Finally, for the DO format to be beneficial, the customers’ arrival rate should be within a medium range, relative to the order quantity.
8.6 Dynamic Pricing Strategies

Our study has focused on settings in which the seller pre-announces the premium and end-of-season prices at the start of the season. Admittedly, this situation reflects the exception, rather than the common practice. In retail settings, sellers typically adopt a dynamic pricing strategy, rather than making a commitment to a markdown price path. Specifically, the seller announces the premium price (only) at the beginning of the season, and postpones the determination and announcement of the end-of-season price to a later point. For example, when the time of discount arrives, the seller may consider his remaining level of inventory, and then set the end-of-season price optimally.

Our main interest in studying pre-announced pricing strategies is driven by recent research (e.g., Aviv and Pazgal (2008)) which shows that under certain circumstances, a price commitment can actually work more effectively than dynamic pricing, in the presence of strategic customers. However, the study of the impact of display formats in dynamic pricing settings is still of significant interest. To perform a study of this kind, a typical approach is to consider a three-phase game between the seller and the strategic customers – the seller begins by announcing a premium price, the customers then determine their purchasing decisions, and finally, the seller announces the end-of-season price. To anticipate the seller’s and customers’ strategies, we need to identify a subgame-perfect Nash equilibrium in this game. Achieving this goal, however, proves to be extremely complex, and is hence beyond the scope of this paper. However, we make the following conjecture on the basis of our research, combined with the lessons from Aviv and Pazgal (2008) mentioned above. Since the adverse impact of strategic consumer behavior is sharper in a dynamic pricing setting, we suspect that the marginal value of DO will be even more positive when sellers adopt dynamic pricing policies.

8.7 Adaptive Learning

Throughout this paper, we have assumed that customers form rational expectations about the purchasing decisions of every other potential customer. Furthermore, we assumed that consumers can perfectly estimate the likelihood of finding a product available if they return to the store at the end of the season. Hence, we consider a market with highly-sophisticated customers that

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14See also Cachon and Swinney (2008) for further discussion.
are able to obtain information, calculate, and follow the equilibrium we predict. To complement
this normative approach, one may choose to examine the aspect of strategic consumer behavior
by exploring alternative descriptive models of the ways in which customers may actually behave.
In particular, it is of interest to study settings in which customers exhibit adaptive expectations
use adaptive learning models to study markets with repeated interactions between a seller and
customers. These recent papers and references therein can provide useful ideas on how to approach
the study of the impact of display formats under adaptive expectations.

8.8 Optimal Dynamic Display Strategies

An interesting extension of our research could be to explore a wider set of display format strategies.
We only consider two models – the DA and DO formats. Obviously, variants of these display formats
could include situations in which the retailer displays only a part of the inventory, or dynamically
decide about the number of items to display on the basis of the time in the season and the actual
inventory left in stock.

9 Concluding Remarks

We consider a retailer who attempts to maximize his expected profit by selling a finite capacity
(inventory) over a sales horizon. Faced with strategic customers, the retailer needs to determine
his order quantity, and announce a price path that consists of a main-season premium price and an
end-of-season clearance price. Customers arrive to the store at random and make the purchasing
choices by comparing current surplus to the expected surplus if they wait for the end-of-season
discounts. Specifically, the retailer and the customers are engaged in a Stackelberg-form game,
with the retailer being a leader (setting the quantity and announcing prices), and the customers
being the followers (making “buy now” or “wait” decisions). The customers are also engaged in a
competitive situation among themselves, since their decisions influence each other. We study two
inventory display formats: Display All (DA) and Display One (DO). Under the DA format, the
retailer displays all available units so that each arriving customer has perfect information about
the actual inventory level. Under the DO format, the retailer displays only one unit at a time so
that each customer knows about product availability but not the actual inventory level.

For each display format and for any given ordering and pricing decisions, we showed that customers follow a threshold-type purchasing rule in equilibrium. Anticipating such behavior, we determined the retailer’s profit function and proved certain properties of the optimal order quantity and optimal prices. We conducted an extensive numerical study to address the following questions: When considering the influence of the display formats on the level of inventory information conveyed to customers, which one of the two formats is better for the retailer? Furthermore, can a move from one display format to another be effective in mitigating the adverse impact of strategic consumer behavior? Our key findings are stated below.

When the level of inventory is exogenously given, it is useful to consider the retailer’s expected revenue. We found that a change from DA to DO, even without a change of the price path, can never worsen the retailer’s expected revenue. This observation provides support to our hypothesis that the DO format increases the perceived level of product scarcity, and hence drives high-valuation customers to purchase the product at the premium price. Percentage-wise, the marginal benefits of DO are relatively small in magnitude. Nonetheless, we suggest caution in interpreting this result, since even small increases in revenues could have a high impact on expected profits. Context matters! We observed that the marginal benefits of DO tend to be the largest when there is a significant spread in valuations, the proportion of the high-valuation class in the population is at a medium level, and the customers’ arrival rate is within a medium range relative to the initial inventory level. Interestingly, we found that a move to a DO format is very far from eliminating the adverse impact of strategic consumer behavior.

We also explored settings in which the retailer selects both inventory and pricing so as to optimize his expected profit. Here, capacity choice is used as a lever to influence the customers’ perception of product scarcity. Confirming our intuition, a move from DA to DO never leads to a simultaneous decline in inventory and pricing. However, the capacity and pricing choices can settle in equilibrium in a couple of ways. In some cases, the retailer’s optimal action is to increase the quantity, but maintain the same premium price or decrease it. In other cases, the retailer’s optimal action is to increase the premium price, along with a decrease or no change in the level of inventory. Interestingly, we did not observe situations in which the retailer increases the inventory
simultaneously with a significant increase in the premium price.

Using our theoretical results, we explained that by moving from a DA to DO while keeping the price path unchanged, we expect that the volatility of the retailer’s profit will decrease. This observation may open an interesting window for future research on settings with risk-prone or risk-averse retailers. Finally, we discussed several model extensions and proposed further ideas for future research.

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**References**


Appendix: Lemmas and Proofs

The following definitions are used in some of the proofs below. Let \( A_i(t_1, t_2) \) be the number of class-\( i \) customers that arrive during the interval \([t_1, t_2)\). Among these arrivals, let \( B_i(t_1, t_2) \) and \( W_i(t_1, t_2) \) denote the number of customers that decide to “buy now” and “wait,” respectively. Without loss of generality, all customers arriving during \([t_1, t_2)\) and find no inventory in stock, are counted as part of \( W_i(t_1, t_2)\).

**Lemma 1** The function \( \tilde{H}_q(\Lambda) \) is strictly decreasing in \( \Lambda \geq 0 \). Additionally, for all \( \Lambda > 0 \), \( \tilde{H}_q(\Lambda) \) is strictly increasing in \( q \), with \( \lim_{q \to \infty} \tilde{H}_q(\Lambda) = 1 \).
Proof. Consider a pair of values 0 ≤ Λ₁ < Λ₂, and let X ∼Poisson(Λ₁) and Y ∼Poisson(Λ₂ − Λ₁) be two independent random variables. Then, note that

\[ \hat{H}_q (Λ₁) = E[q/ \max (q, 1 + X)] > E[q/ \max (q, 1 + X + Y)] = \hat{H}_q (Λ₂) \]

The inequality follows by simple algebra, and it is strict since the event \{Y > 0\} has a non-zero probability. The last equality follows directly from the fact that X + Y ∼Poisson(Λ₂). To complete the proof, it suffices to show that \( q/ \max (q, 1 + X) \) is increasing in \( q \), and that it converges to 1 as \( q \) grows. But this follows by simple algebra. ■

Proof of Proposition 1. We provide below a sketch of the proof. We first argue that given the information \( \{t, k_t = k\} \), and for any possible initial level of inventory \( k_0 \geq k \), the following property applies: \( W_1(0, t) \mid \{k_t = k, k_0 = k' \geq k\} \leq ST \ A_1(0, t) \sim \text{Poisson}(α_1λt) \). In plain words, the conditional distribution of the number of class-1 customers that have arrived during the time interval \([0, t]\) and decided to wait, is stochastically dominated by the unconditional distribution of the total number of class-1 arrivals during that time period. To verify this, consider any possible sample path of the inventory level (i.e., any feasible decreasing step-function from \( k_0 = k' \geq k \) to \( k_t = k \)). It is easy to show now that under the specific realization of \( k_t \) and for any policy, the probability distribution for the associated \( W_1(0, t) \) is stochastically smaller than the distribution of the total number of arrivals during the time interval \([0, t]\); this can be done via simple coupling arguments. The stochastic dominance hence holds for any belief on the initial level of inventory, and not only for a given value \( k_0 \). Clearly, a lower bound on \( H(k, t) \) can now be generated by the worst-case condition under which \( A_1(0, t) \) customers are waiting by time \( t \), and all arrivals after that time are immediate buyers. The lower bound in the proposition is hence given by \( E[(k - A_1(t, T)) \mid \max \{k - A_1(t, T), 1 + W_0(0, T) + Y\}] \), where \( Y \sim \text{Poisson}(α_1λt) \). ■

Proof of Proposition 2. The Proposition follows directly from Lemma 1 and the fact that the ratio \( (v_1 - p_h)/(v_1 - p_l) \) is increasing in \( v_1 \) and \( p_l \), and decreasing in \( p_h \). ■

Lemma 2 Consider any purchasing policy for class-1 customers (including randomized strategies). Also, suppose that all class-0 customers are waiting. Now, consider a “focal” class-1 customer that arrives at time \( t \) and finds \( k \) units in stock. Then, the allocation probability \( H(k, t) \) assessed by this customer satisfies the following inequality:

\[ H(k, t) \leq \hat{H}_k(λ(T - α_1t)) \]

Proof. Recall that \( B_1(t_1, t_2) + W_1(t_1, t_2) \sim \text{Poisson}(α_1λ(t_2 - t_1)) \), and is equal to the number of class-\( i \) customers that arrive during \([t_1, t_2]\). Clearly, \( B_1(t_1, t_2) \) and \( W_1(t_1, t_2) \) could be dependent
of each other, and, individually, they are not necessarily Poisson-distributed. Note that

\[ H(k, t) = E \left[ \frac{k - B_1(t, T)}{\max \{k - B_1(t, T), 1 + W_0(0, T) + W_1(0, t) + W_1(t, T)\}} \right]_{k_t = k} \]

\[ \leq E \left[ \frac{k}{\max \{k, 1 + W_0(0, T) + B_1(t, T) + W_1(t, T)\}} \right]_{k_t = k} \]

\[ = E \left[ \frac{k}{\max \{k, 1 + W_0(0, T) + B_1(t, T) + W_1(t, T)\}} \right] \]

\[ = \tilde{H}_k(\lambda(T - \alpha_1 t)) \]

The first inequality is proven by simple algebra. The second equality follows from the fact that

\[ B_1(t, T) + W_1(t, T) \]

is equal to the total number of arrivals during \([t, T)\), which is independent of the event \(\{k_t = k\}\). The last equality is verified by the definition of \(\tilde{H}\), and by the fact that \(W_0(0, T) + B_1(t, T) + W_1(t, T)\) is Poisson-distributed with a mean of \(\alpha_0 \lambda T + \alpha_1 \lambda (T - t)\).

**Proof of Theorem 1.** The proof for class-0 customers is trivial, following the assumption that \(p_h > v_1\). For class-1, consider a “focal” customer that arrives at time \(t\) and finds \(k\) units in stock. We distinguish between two cases, as follows.

Case 1: Suppose that \(t < \tau^*(k)\). Under this case, the estimated surplus associated with a “wait” decision for this customer satisfies:

\[ (v_1 - p_l) H(k, t) < (v_1 - p_l) \tilde{H}_k(\lambda(T - \alpha_1 \tau^*(k))) \]

The first inequality follows directly from Lemmas 1 and 2. The second inequality follows from the last two cases of Eq. (3). Therefore, it is optimal for any class-1 customer to “buy now” at times \(t < \tau^*(k_t)\). In fact, this is the unique optimal response to any purchasing strategy of the other customers; i.e., not only in equilibrium.

Case 2: Suppose that \(t > \tau^*(k)\) (clearly, we only need to consider the case \(\tau^*(k) < T\)). The treatment of this case is considerably more complex. First, note that since \((v_1 - p_h) / (v_1 - p_l) < 1\), it follows from Lemma 1 and Proposition 1, that there always exists some sufficiently large \(k'\) for which \(\tau^*(k) = 0\) for all \(k > k'\), and such that an optimal policy must prescribe a “wait” action for all class-1 customers that find more than \(k'\) units of inventory at their time of arrival (regardless of when it is during the season). In other words, our proof is complete for all \(k > k'\). We proceed by induction. Suppose now that the theorem holds for \(k + 1, k + 2, \ldots\), and let us show that it must hold for \(k\). Without loss of generality, let \(t' \in [\tau^*(k), T)\) be the first time at which the current policy prescribes a “wait” action to a class-1 customer (note that \(t'\) must exist) with a non-zero probability. Obviously, for the policy to be viable in equilibrium, it must satisfy the condition \(v_1 - p_h \leq (v_1 - p_l) H(k, t')\). Using the induction assumption, note that if a customer arrives at time \(t'\) and finds \(k\) units in stock, this implies that no customer that arrived during the time
period \([0, t')\) has decided to wait. Consequently, at time \(t'\), our focal class-1 customer estimates the availability and allocation probability to be:

\[
H (k, t') = E \left[ \frac{k - B_1(t', T)}{\max \{ k - B_1(t', T), 1 + W_0(0, T) + W_1(t', T) \} } | k_{i'} = k \right]
\]

But let us now take any value \(t \in (t', T)\), and calculate \(H (k, t')\) by conditioning on the number of class-1 customers that buy and wait during the interval \((t', t)\):

\[
H (k, t') = E_{(B_{11}, W_{11})} [G (k, t, B_{11}, W_{11}) | k_{i'} = k]
\]

where \(B_{11} = B_1(t', t), B_{12} = B_1(t, T), W_{11} = W_1(t', t), W_{12} = W_1(t, T)\), and

\[
G (k, t, b, w) = E \left[ \frac{k - b - B_{12}}{\max \{ k - b - B_{12}, 1 + W_0(0, T) + w + W_{12} \} } | k_t = k - b \right]
\]

Note that in the expression for \(G\) we are allowed to condition only on the event \(\{k_t = k - b\}\), since once this event is given, the joint distribution of the random variables \((B_{12}, W_{12})\) is (conditionally) independent of the information \(\{k_{i'} = k, B_{11}, W_{11}\}\). Next, we argue that while \(B_{12} + W_{12}\) is independent of the event \(\{k_t = k - b\}\), the marginal distribution of \(B_{12} + b|k_t = k - b\) is stochastically increasing in \(b\). To show this, simply note that the number of buyers during \((t, T)\) under \(k_t = k\) cannot be larger than that under \(k_t = k - b\) by more than \(b\) units (because if there exists a time \(\tilde{t}\) such that \(k_t\) becomes equal under both starting points, then the subsequent behavior pattern of future customers would become identical). Therefore, with a bit of algebra, one shows that:

\[
G (k, t, b, w) \leq E \left[ \frac{k - B_{12}}{\max \{ k - B_{12}, 1 + W_0(0, T) + (w + b) + W_{12} \} } | k_t = k \right]
\]

where the inequality is strict when \(b > 0\). We hence conclude that:

\[
H (k, t') \leq E_{(B_{11}, W_{11})} \left[ E \left[ \frac{k - B_{12}}{\max \{ k - B_{12}, 1 + W_0(0, T) + (W_{11} + B_{11}) + W_{12} \} } | k_t = k, W_{11}, B_{11} \right] | k_{i'} = k \right]
\]

\[
= E \left[ \frac{k - B_{12}}{\max \{ k - B_{12}, 1 + W_0(0, T) + Z + W_{12} \} } | k_t = k \right]
\]

\[
\leq E \left[ \frac{k - B_{12}}{\max \{ k - B_{12}, 1 + W_0(0, T) + A_1(\tau^*(k), t) + W_{12} \} } | k_t = k \right] = H (k, t)
\]

The random variable \(Z\) represents the distribution of \(\{A_1(t', t) | k_{i'} = k\}\), which is in fact independent of the event \(\{k_{i'} = k\}\). But in the inequality that follows, we substitute the independent variable \(Z\) by the variable \(\{A_1(t', t) | k_t = k\}\), which is stochastically smaller than \(Z\). The verification of the latter property relies on the induction assumption, which implies that the event \(\{k_t = k\}\) can only happen if \(\{k_{i'} = k\}\) and no purchases were made during the time interval \((t', t)\).
Note that at least one of the two inequalities above is strict if there is a non-zero probability that a class-1 customer will “buy now” during \((t', T]\) when observing \(k\) units in stock. We therefore conclude that no “buy now” actions can occur during \((t', T]\) in equilibrium. For otherwise, it would be strictly beneficial to deviate into an “always wait” strategy during that time interval, since \(H(k, t) > (v_1 - p_h)/(v_1 - p_l)\) for some \(t \in (t', T]\). It is noteworthy that thus far we have only shown that a threshold-form policy is optimal, with a unique threshold that is equal to \(t'\). We still need to show that \(t'\) is equals to \(\tau^*(k)\). But this follows directly from the fact that given a threshold policy with \(t'\) (for \(k\) units) and \(\tau^*(k + i)\) for \(k + i\) units \((i \geq 1)\), the following equality holds: \(H(k, t') = \hat{H}_k(\lambda (T - \alpha_1 t'))\); the remainder of the proof follows from the definition of the values \(\tau^*(k)\), and Proposition 2.

**Proof of Proposition 3.** A nice structural property of the threshold purchasing policy in equilibrium is that all class-1 arrivals to the store make a “buy now” decision until the first time that \(t \geq \tau^*(k_t)\). At that time and on, all class-1 arrivals will “wait.” Hence, when at time \(\tau^*(j)\), with inventory level \(k_{\tau^*(j)} = k > j\), all arrivals in the time interval \((\tau^*(j), \tau^*(k)]\) will buy the product at the high price. Recall that the number of class-1 arrivals during that interval is Poisson distributed with a mean of \(\alpha_1 \lambda_1 (\tau^*(k) - \tau^*(j))\). When \(k_{\tau^*(j)} = j\), no high-price revenues are collected, and all arrivals make a decision to wait. Thus, revenues are collected at time \(T\), on the basis of the number of returning customers (Poisson-distributed, with a mean of \(\alpha_0 \lambda T + \alpha_1 \lambda_1 (T - \tau^*(k))\)). Let us call this variable \(X\). It is then easy to see that \(f(j, j) = E[p_l \cdot \min(j, X) + s \cdot \max(j - X, 0)]\), which is equal to \(j p_l + (s - p_l) \cdot E \max(j - X, 0)\).

**Proof of Proposition 4.** For the proof of the first part \((p_l = v_0)\), suppose that there exists an optimal price-path in the range \(\{v_1 \geq p_h \geq v_0 \geq p_l\}\) such that \(p_l < v_0\). Let \(\tau^*(k)\) be the set of threshold values associated with this optimal price path. Now, let us modify the discounted price by increasing it to \(v_0\), without changing \(p_h\). Let \(\tau'(k)\) be the threshold values associated with the new price path \((p_h, v_0)\). Next, consider any sample-path of customers arrivals, and let \(q_t^*\) and \(q_t'\) be the prevailing levels of inventory under the price-paths \((p_h, p_l)\) and \((p_h, v_0)\), respectively. Clearly, from Proposition 2, we have: \(\tau^*(k) \leq \tau'(k)\) for all \(k\), and that both \(\tau^*\) and \(\tau'\) are non-increasing in \(k\). Hence, it is easy to see that starting at \(q_0^* = q_0' = Q\), the inventory level \(q_t^*\) is higher or equal to \(q_t'\) for all \(t \in (0, T]\). In other words, not only that more class-1 customers purchase the product at the premium price under the new price path, but all of the rest of the customers will pay a higher discounted price. Clearly, it is possible that the retailer can identify an even better premium price \(p_h\) in order to obtain a higher expected profit. For the remainder of the proof, let us focus on the case \(p_l = v_0\). From the first case of \((3)\), we can easily see that if the premium price is set such that \((v_1 - p_h)/(v_1 - v_0) < \hat{H}_Q(\lambda T)\), then all customers will decide to wait for the discounted price \(v_0\). However, in such a case it is trivial to see that the retailer
can do no worse by decreasing the value of \( p_h \) to \( v_0 \cdot \tilde{H}_Q(\lambda T) + v_1 \cdot \left(1 - \tilde{H}_Q(\lambda T)\right) \). Similarly, if \( (v_1 - p_h) / (v_1 - v_0) > \tilde{H}_Q(\alpha_0 \lambda T) \geq \ldots \geq \tilde{H}_1(\alpha_0 \lambda T) \), then all class-1 customers will make a “buy now” decision; however, it can be easily seen that the premium price can be further increased towards \( v_0 \cdot \tilde{H}_Q(\alpha_0 \lambda T) + v_1 \cdot \left(1 - \tilde{H}_Q(\alpha_0 \lambda T)\right) \) without driving class-1 customers to wait; hence collecting a higher expected profit.

**Proof of Proposition 5.** Note that

\[
\Delta^D\alpha(Q) = \pi^D\alpha(Q + 1) - \pi^D\alpha(Q) = \pi^D\alpha(Q + 1) - \pi^D\alpha(Q) = -(c - s) + (v_0 - s) \cdot \Delta_N(Q\lambda T) - (v_1 - v_0) \cdot \left(1 - \tilde{H}_Q(\alpha_0 \lambda T)\right) \cdot N(Q, \alpha_1 \lambda T)
\]

where \( \Delta_N(Q|\Lambda) \equiv N(Q + 1, \Lambda) - N(Q, \Lambda) = 1 - \sum_{x=0}^{Q} P_x(\Lambda) \). Obviously, the proposition offers a possible bound that can be obtained by eliminating the expression \((v_1 - v_0) \cdot \left(1 - \tilde{H}_Q(\alpha_0 \lambda T)\right) \cdot N(Q + 1, \alpha_1 \lambda T)\), and by replacing \( N(Q + 1, \alpha_1 \lambda T) \) with the trivial upper bound of \( \alpha_1 \lambda T \). The fact that the bound in the proposition is decreasing in \( Q \) is trivial.

**Proof of Proposition 6.** Note that the allocation probability is formally defined as:

\[
\tilde{H}_Q(t) \equiv E\left[\frac{Q - B_1(0,T)}{\max\{Q - B_1(0,T), 1 + W_0(0,T) + W_1(0,T)\}} | B_1(0,t) < Q \right]
\]

Thus, it is left to show that \( \Pr\{B_1(0,t) < Q\} \) is non-increasing in \( t \), since the expression within the last expected value is independent of \( t \). But this is trivial, since for any starting level of inventory, and any policy adopted by the customers, the unconditional random variable \( B_1(0,t) \) is stochastically non-decreasing in \( t \). Proof of continuity is straightforward, given that the arrival process is Poisson.

**Proof of Theorem 2.** For convenience, let \( \rho \equiv (v_1 - p_h) / (v_1 - p_l) \), and consider a focal customer arriving at time \( t \). Furthermore, let \( \tilde{H}_Q(t) \) be the allocation probability assessed by the focal customer for a given strategy of all other customers. In view of Proposition 6, we consider the following three cases: (i) when \( \tilde{H}_Q(T) < \rho \), it is easy to see that a “buy now” strategy is the only optimal response for the customer at any given time; (ii) when \( \tilde{H}_Q(0) > \rho \), it is easy to see that a “wait” strategy is the only optimal response for the customer at any given time; (iii) when \( \tilde{H}_Q(0) \leq \rho \leq \tilde{H}_Q(T) \), we can argue that there exist \( \tau_1, \tau_2 \in [0, T] \), with \( \tau_2 \geq \tau_1 \), such that \( \tilde{H}_Q(t) < \rho \) for \( t \in [0, \tau_1] \), \( \tilde{H}_Q(t) = \rho \) for \( t \in [\tau_1, \tau_2] \), and \( \tilde{H}_Q(t) > \rho \) for \( t \in (\tau_2, T] \). Under these circumstances, the best response of the focal customer must prescribe a “buy now” action prior to \( \tau_1 \), and a “wait” action after time \( \tau_2 \). For \( t \in [\tau_1, \tau_2] \), a “buy now” or a “wait” action can be made arbitrarily. However, the situation in which \( \tau_1 < \tau_2 \) can only occur when all other customers “wait” if they arrive during \([\tau_1, \tau_2]\) (again, in view of Proposition 6). The theorem now follows.
Proof of Proposition 7. The expression (7) is straightforward; see Proof of Proposition 6, and use the fact that \( B_1 (0,t) \sim \min \{ Q, \text{Poisson} (\alpha_1 \lambda \cdot \min \{ t, \tau \}) \} \). Property (i) follows from the continuity of the Poisson density function, and the continuity of the \( \hat{H} \) function. Property (ii) can be easily verified by using the fact that \( \sum_{x=0}^{Q-1} P(x|a) \) is strictly decreasing in \( a \). ■

Proof of Theorem 3. The proof for the case \( Q = 1 \) follows from the fact that \( L_1 (\tau) = \hat{H}_1 (\lambda (T - \alpha_1 \tau)) \), and Lemma 1 above. For \( Q \geq 2 \), note that \( L_Q (\tau) \) can be written as \( f(\alpha_1 \lambda \tau) \), where \( f(\theta) = \left( \sum_{x=0}^{Q-1} H_{Q-x} (\lambda T - \theta) \cdot P_x (\theta) \right) / \left( \sum_{x=0}^{Q-1} P_x (\theta) \right) \). In the Online Addendum to this paper, we show that \( f(\theta) \) satisfies the properties stated in the theorem with respect to \( \theta \), over the extended range \( \theta \in [0, \lambda T] \). Clearly, this completes our proof, since \( \theta \) is a strictly monotone transformation of \( \tau \). A proof of part (iii) of the theorem appears in the second part of the online addendum. ■

Proof of Proposition 8. Note that because of the quasi-convexity and continuity of \( L_Q (\tau) \), the condition \( (v_1 - p_h) / (v_1 - p_l) < L_Q (0) \) implies that \( \tau = 0 \) is an equilibrium, and so is any solution to the equation \( (v_1 - p_h) / (v_1 - p_l) = L_Q (\tau) \). However, under a strategy characterized by a solution to the latter equation, a class-1 customer gains a surplus of \( v_1 - p_h \) (it is easy to see this, since a customer arriving before \( \tau \) “buys now” and gains \( v_1 - p_h \), and a customer arriving after \( \tau \) “waits” and gains \( (v_1 - p_l) L_Q (t|\tau) = (v_1 - p_l) L_Q (\tau) = v_1 - p_h \)). This is compared to the equilibrium strategy \( \tau = 0 \), in which a class-1 customer is expected to gain \( (v_1 - p_l) L_Q (t|0) = (v_1 - p_l) L_Q (0) > v_1 - p_h \). ■

Proof of Proposition 9. First, note that the ratio \( (v_1 - p_h) / (v_1 - p_l) \) is increasing in \( v_1 \) and decreasing in \( p_h \). Therefore, the monotonicity pattern with respect to these three parameters is immediate in view of parts (i) and (ii) of Theorem 3, and the definition in (9). The monotonicity with respect to \( \lambda \) follows directly from part (iii) of Theorem 3 and the definition in (9). ■

Proof of Proposition 10. The result is straightforward, following directly from the monotonicity of the functions \( \hat{H}_k (\Lambda) \) with respect to \( k \) and \( \Lambda \), and the fact that \( L_Q (\tau) \) can be expressed as a weighted average of the functions \( \hat{H}_k (\lambda T - \alpha_1 \lambda \tau) \). ■

Proof of Proposition 11. In view of our focus on the unique purchasing policy described in (9), and Theorem 3, we can conclude that \( \tau^*_Q \) is increasing in \( p_l \). Hence, using similar arguments as in the proof of Proposition 4, one can verify that it is optimal to set \( p_l = v_0 \) (the argument is even simpler here, since there is only a single threshold that is independent of the prevailing inventory level). Hence, let us fix \( p_l = v_0 \). Suppose that \( p_h \) is set so low such that \( (v_1 - p_h) / (v_1 - v_0) > \max \{ L_Q (0), L_Q (T) \} \). Then, (9) tells us that all class-1 customers make a “buy now” decision. However, observe that it is clearly beneficial to increase \( p_h \) to \( \min \{ \bar{v}(T), \bar{v}(0) \} \) where \( \bar{v}(t) = L_Q (t) v_0 + (1 - L_Q (t)) v_1 \). This is because all class-1 customers still “buy now”, but at a higher price. Now, suppose that \( p_h \) is set so high such that \( (v_1 - p_h) / (v_1 - v_0) < L_Q (0) \). Then, we
conclude from the first case of (9) that all customers wait for the discounted price. Consequently, the retailer can decrease \( p_h \) to \( \bar{v}(0) \) without loss of optimality. But let us see what happens if \( p_h = \bar{v}(0) \). It is useful to distinguish between two cases. First, if \( \tau_Q^* \) is non-continuous with respect to \( p_h \) at \( \bar{v}(0) \), then there exists a value \( \tilde{\tau} > 0 \) such that any decrease in \( p_h \) to \( \bar{v}(0) - \varepsilon \) (\( \varepsilon > 0 \)) will switch class-1’s policy threshold from \( \tau_Q^* = 0 \) to \( \tau_Q^* > \tilde{\tau} \). Thus, there always exists a sufficiently small \( \varepsilon > 0 \) for which the price \( \bar{v}(0) - \varepsilon \) yields a strictly better expected profit performance than \( \bar{v}(0) \). Clearly, it could be that no optimal \( p_h \) exists in this case, since for any value of \( p_h = \bar{v}(0) - \varepsilon \), the price \( \bar{v}(0) - \varepsilon/2 \) performs even better. Second, if \( \tau_Q^* \) is continuous with respect to \( p_h \) at \( \bar{v}(0) \), then the impact of a reduction of \( p_h \) (down from \( v_0 \)) on expected profits can be either negative or positive. Nonetheless, the impact is continuous with respect to \( \varepsilon \geq 0 \). Hence, by continuity, there always exists a value \( \varepsilon > 0 \) such that the expected performance under \( \bar{v}(0) - \varepsilon \) can become arbitrarily close to that under \( \bar{v}(0) \).