

# Dynamic Pricing of Short Life-Cycle Products through Active Learning\*

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## Abstract

Revenue management techniques, practiced for many years in the airline and hotel industries, have gained popularity with retailers. One such technique – dynamic pricing – refers to the sophisticated process of controlling prices over a course of a sales season in a way that maximizes expected revenues. In this paper, we consider a retailer who sells a fashionable good during a short sales season, but is uncertain about how successful the product will be in the market. Will it attract customer attention or not? and to what extent? We propose a continuous-time revenue-maximization model in which the seller continuously learns about the market reaction to his product from sales observations. Customers arrive randomly over the course of the season, and each customer has an individual reservation price, based on which he makes his own purchasing decision. We combine theoretical results with numerical experimentation to study the value of dynamic pricing for a retailer faced with market uncertainty. We compare optimal expected revenues to those obtained under two alternative pricing policies: fixed pricing schemes, and certainty-equivalent policies. The comparative study allows us to explore the value of proactively setting prices to impact the revenue as well as the learning process itself (“active learning”) versus “passive learning,” where learning is performed continuously, but the impact of prices on learning is ignored. We also consider the value of obtaining perfect information about the market conditions.

*Keywords:* Dynamic Pricing, Learning, Revenue Management, Yield Management, Optimal Control, Inventory.

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# 1 Introduction

Retailers who sell fashionable goods during short sales seasons often need to carefully manage the risks associated with the uncertainty surrounding the demand for their products. Any items left at the end of the season are relegated to the discount racks where they generate a narrow profit margin or even a loss. On the other hand, demand shortage due to understocking leads to customer dissatisfaction and loss of potential profits. Retailers are faced with the daunting task of pricing the items so that they are gone by the end of the season but not to discount them so greatly that they sacrifice profit margins unnecessarily. The problem is further exacerbated by the fact that most apparel retailers make a single upfront purchase of the entire desired quantity at the beginning of the sales season without the ability to replenish inventory. Yet according to a recent BearingPoint and National Retail Federation survey only 12% of respondents utilize any kind of price optimization software while the vast majority of retail managers rely mostly on their own intuition and knowledge to gauge what they think is the optimum price<sup>1</sup>.

Revenue management techniques and tools that were first employed by airlines in the early 1970's are currently in wide use in the hospitality industry. Recently, these techniques were re-designed to help retailers determine the timing and depth of markdowns in order to increase profits. In its basic form, revenue management assumes that the retailer possesses exact knowledge of the underlying characteristics of the demand as well as consumer's price elasticity. It attempts to dynamically balance expected current and future revenues by setting and changing the appropriate prices throughout the selling season<sup>2</sup>. High prices carry the potential benefits of large immediate rewards, but bear the risk of selling fewer units. Low prices, on the other hand, offer smaller expected revenues but higher expected number of units sold. In this paper, we examine the value of revenue management in more realistic settings where some parameters of the demand are unknown to the retailer at the start of the sales season. As time progresses, the retailer can improve his knowledge of the demand through sales observations. The optimal pricing strategy in this environment needs to take into account not only the balance between high and low prices discussed above, but the inherent impact of prices on the learning of the demand process itself.

We consider a retail store that sells a certain quantity of a product over a fixed period of time. The arrival rate of potential customers to the store is constant and depends on general market conditions. Each customer has a reservation price for the product, and will buy a single unit of the

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<sup>1</sup>Chain Store Age, "The Price is Right" September 2002.

<sup>2</sup>Revenue management (also known as "Yield Management") encompasses additional techniques such as segmentation, advance purchase requirements, etc. (see, e.g., Weatherford and Bodily, 1992).

product as long as the price is below his or her reservation price. Consumers are assumed to be heterogeneous in their reservation prices<sup>3</sup>. The retailer knows the distribution of the reservation prices in the population, but not their values for each individual customer. Since sales of short life-cycle products are often associated with a high level of demand uncertainty, a key component of an intelligent sales strategy is the proper gathering and integration of demand information.

In this work, we integrate a Bayesian learning mechanism into a revenue-maximization dynamic decision problem. Specifically, we consider the customers arrival rate to be unknown to the seller. The seller has a prior belief about the distribution of this arrival rate, which he updates through observations of sales. The seller's goal is to select an optimal pricing strategy that serves both as a learning tool about demand, as well as a revenue-generating activity. Under such strategies, price experimentation may compromise immediate expected profits, but this loss can possibly be recouped during the remainder of the sales season through use of the improved information about the demand (Trefler, 1993). We show that optimal prices tend to be higher with higher levels of uncertainty about the market condition. In other words, when faced with uncertainty, the seller should "bet" on the better scenarios, particularly when the sales season is long enough. Over the sales season prices should decline continuously as the retailer faces increasing pressure to sell the available units, but should increase after each sale. This increase in price is the result of two factors: there are less units to sell, and hence more time to experiment with higher prices, and in addition, the occurrence of a sale raises the seller's belief that the market condition is good. As expected, the optimal (expected) revenue increases as a function of the length of the sales season, but in a diminishing rate. This result should be considered, for instance, if the length of the sales season can be determined by the retailer. Our study also suggests that expected revenues are concave with respect to the level of available inventory, a property that should be considered when making the initial procurement decision.

A retailer facing the daunting task of setting the optimal price in the described complicated uncertain environment may choose to tackle the problem at several levels of increasing complexity. He may choose to implement a naive pricing policy that completely ignores the uncertainty about the market condition, or he may set an optimally chosen fixed price for the entire season. We show that when the initial uncertainty about market condition is high, there is a heavy price to pay for ignoring the uncertainty or for using best fixed-price policies.

We argue that the benefits of dynamic pricing strategies are critical in settings with high but resolvable initial uncertainty about how successful the product is. But solving the full fledged

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<sup>3</sup>For further discussion of these assumptions, see Bitran and Mondschein (1997), and Gallego and van Ryzin (1994)

complex dynamic pricing decision-problem by taking into account the impact of pricing on the learning process (“active learning”) is not completely necessary. We found that learning must take place, but it is quite reasonable for the seller to adopt a “passive learning” approach, where he updates his belief about the market condition as time progresses, but at each moment the price is set as if the market condition is known and equals to its current estimate. Thus, prices are not used proactively to affect learning, but nevertheless they are used in the learning mechanism to update the belief about the market condition. We show that the sub-optimality due to the use of passive learning is minor particularly when there are many units for sale, or when the sales season is long.

We complete our study by considering the potential benefits that can be gained by exogenously getting perfect information about the actual market condition. Our results suggest that obtaining market information at the beginning of the sales season is particularly beneficial when the number of units available for sale is small (so price experimentation is potentially very costly) and when the sales season is sufficiently long (so the seller can take advantage of this external information.)

The structure of the paper is as follows: In §2 we describe our modeling framework and learning process in greater detail. Section 3 is devoted to the analysis of the optimal policies and the optimal revenue functions. In §4 we describe a so-called certainty-equivalent heuristic policy for dynamic pricing and learning. An advantage of this type of heuristic, compared to optimal policies, is in its simple form. But more importantly, it allows us to examine the value of active versus passive learning. We continue in §5, to consider two additional types of simpler pricing policies: The first is a fixed price policy in which a constant price is set for the entire duration of the season (§5.1). The second is a pricing policy that completely ignores the uncertainty about the parameter of the demand distribution, by replacing it with its initial expected value (§5.2). The value of obtaining perfect information about the unknown parameter of the demand distribution is discussed in §6. We conclude the paper with a summary of our main results in §7.

## 1.1 Relevant Literature

A large body of literature on optimal dynamic pricing strategies exists in several research disciplines, such as marketing, management science, and economics. In the following, we highlight parts of the literature that are most relevant to the problem at hand. Joint inventory and pricing decision models for environments with known but stochastic demand functions, can be mainly found in the management science literature. For example, Federgruen and Heching (1999) study finite and infinite-horizon, periodic-review joint inventory and pricing problems and provide a characterization of the optimal joint pricing and inventory replenishment strategies. We refer the reader to their

extensive literature review, which also contains references to papers dealing with deterministic demand functions. To our knowledge, no quantitative model for joint inventory replenishment<sup>4</sup> and pricing decision problem appears in the literature for the case in which some parameters of the demand process are unknown.

Active learning about demand functions through pricing was studied by Balvers and Cosimano (1990). In their paper, they consider a firm that faces a linear demand curve, with unknown slope and intercept. The model is set as a periodic-review control problem in which price is the sole control variable. Using historical prices and actual demand quantities, the seller is able to update his belief about the actual demand curve parameters. Since no inventories are allowed to be carried over from one period to another, production is always set to be equal to the price-induced demand. This, in turn, makes consecutive decision periods dependent of each other solely through learning (unlike the joint inventory and pricing models mentioned above). Balvers and Cosimano argue that the optimal prices are relatively stable over time (“price stickiness”). They suggest that this phenomenon occurs because price changes add uncertainty to the decision process, as well as hinder the learning process, and therefore they are generally not desirable. The literature review in Balvers and Cosimano (1990) contains references to active learning through pricing in competitive environments, as well as to more theoretical papers relevant to adaptive Bayesian decision problems (see, e.g., Easley and Kieffer 1988).

In the marketing literature several studies come to mind: Kalish (1983) investigates the optimal pricing by a monopolist faced with "learning by doing" (marginal cost of production decreases with output) and a positive "diffusion" effects but no inventory. The inherent dependence of current demand on cumulative sales make the problem dynamic in nature. He shows that prices will decline over time. Dehbar and Oren (1985) analyze the dynamic pricing decision of a monopolist offering a new product or service whose consumption utility depends on the size of the network (e.g. telecommunication services.) They show that higher network growth anticipation by potential customers and lower discount rates lead to lower prices and larger networks. In a more recent paper, Braden and Oren (1994) study a pricing problem for a monopoly that sells products to customers with different tastes that vary according to an index whose value is unknown. They show that it is optimal in their setting to follow a policy that determines prices in a way that does not need to explicitly consider the evolution of information, provided the learning about the unknown demand parameter is done periodically, prior to each decision. Finally, Desiraju and Shugan (1999) analyze the strategic pricing of capacity-constrained services. They find that yield-management

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<sup>4</sup>We refer here to inventory of non-perishable goods, that can be transferred from one period to another.

pricing systems work best when price insensitive consumers prefer to buy later than price sensitive consumers.

Only a few studies have examined finite horizon problems. For instance, Mirman et al. (1993) study a two-period model where a firm needs to optimally determine prices (or quantities) in order to maximize profits. They characterize the optimal strategy for the firm, providing conditions under which prices during the first period are higher or lower than the myopic optimal prices.

Papers that investigate sales of a fixed number of items during a finite sales period include Lazear (1986) and Pashigian (1988). These papers consider consumer populations characterized by fixed reservation prices. The seller does not know the exact values of the reservation prices, but has a prior belief. Lazear treats a single unit case, and shows that prices decline over time. Pashigian extends the analysis to multiple units, and provides empirical support for the model based on data from the fashion industry. Gallego and van Ryzin (1994) develop a continuous-time pricing control model of revenue maximization in which the parameters of the stochastic demand are known. They characterize the optimal policies and expected revenues and develop bounds on their values, and heuristic policies. Finally, Bitran and Mondschein (1997) discuss a similar problem to that of Gallego and van Ryzin (1994). They use the continuous-time model as a benchmark against which they compare periodic-review policies, and pricing schemes with announced discounts. Our paper synthesizes a decision framework similar to that of the latter papers with a Bayesian learning mechanism.

## 2 The Model

Consider a single retail store with  $q$  units of a particular product available for sale over a season of finite length,  $T$ . There is no possibility of replenishment throughout the entire sales season. The only marketing variable at the store's disposal is the sales price  $\{p(s) : s \in [0, T]\}$ , which controls both its revenue, as well as its ability to learn about the characteristics of the demand. Demand for the product occurs in single units and it is modeled as follows: we assume that customers arrive to the store according to a Poisson process with an intensity rate that depends on the general market conditions; we denote this *store visit intensity rate* by  $\Lambda$ . We assume that its value is constant over the sales season, yet it is *unknown* to the seller. Equipped with a prior belief about the value of  $\Lambda$ , the seller updates his belief about its value based on the price and sales history. Each customer's purchasing decision depends on the sales price prevailing at the time of arrival. Hence, customers do not attempt to study the pricing scheme  $p(s)$  to schedule their arrival, nor do they

wait at the store in anticipation of a price reduction. Consumer heterogeneity is modeled through a price-dependent purchasing probability  $\alpha(p) = e^{-\alpha p}$  where  $\alpha$  is a non-negative scalar known to the seller. The purchase probability can be interpreted as a representation of the distribution of customer reservation prices in the population. Specifically,  $\alpha(p)$  is the proportion of the population that has a willingness to pay of at most  $p$  for the good. It is possible without loss of generality to replace the purchase probability parameter  $a$  with 1 (this simply means a rescaling of the currency used for purchase by  $a$ ), and so we assume hereafter that  $\alpha(p) = e^{-p}$ . Probability functions of an exponential form have been widely applied in the literature, and have been found to fit empirical data very well. The reader is referred to Gallego and van Ryzin (1994), Kalyanam (1996), Smith and Achabal (1998), and the references therein. Finally, we assume that the seller observes only completed sales, so that arrival of a customer which may provide information about the market condition, is not recorded if the customer did not purchase an item.

## 2.1 The learning process

Consider a given time period of length  $\delta$  during which a fixed price  $p$  applies. Suppose that the seller's prior belief about the market condition is given by a Gamma distribution with a shape parameter  $m$ , and a scale parameter  $\theta$ . Specifically,

$$f_{\Lambda}(\lambda|m, \theta) = \frac{\theta (\theta\lambda)^{m-1} e^{-\theta\lambda}}{\Gamma(m)}, \quad \lambda \geq 0$$

Weighted with respect to the seller's belief about the market condition  $\Lambda$ , the actual demand<sup>5</sup> during the time period  $\delta$ , denoted by  $N_{\delta}$  has a Gamma-mixed Poisson distribution which yields the following probability distribution

$$\begin{aligned} \Pr(N_{\delta} = n) &= \int_0^{\infty} \frac{(\lambda\delta\alpha(p))^n e^{-\lambda\delta\alpha(p)}}{\Gamma(n+1)} \frac{\theta (\theta\lambda)^{m-1} e^{-\theta\lambda}}{\Gamma(m)} d\lambda \\ &= \frac{\Gamma(n+m)}{\Gamma(n+1)\Gamma(m)} \left( \frac{\delta\alpha(p)}{\delta\alpha(p) + \theta} \right)^n \left( \frac{\theta}{\delta\alpha(p) + \theta} \right)^m \end{aligned}$$

Therefore, if  $n$  units were sold (and there are still units left to be sold), the posterior distribution of the market parameter  $\Lambda$ , is updated to

$$\frac{(\delta\alpha(p) + \theta) [\lambda(\delta\alpha(p) + \theta)]^{m+n-1} e^{-\lambda(\delta\alpha(p) + \theta)}}{\Gamma(n+m)}$$

or, in other words, a Gamma distribution with a shape parameter equal to  $m$  plus the number of units sold during the time period, and a scale parameter that equals to  $\theta$  plus the length of the

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<sup>5</sup>We define actual demand as the number of customers who would purchase units of the product if it was available.

period multiplied by the purchase probability under the prevailing price  $p$ . Hence, we have chosen a model that lends itself to a conjugate pair of statistical distributions (see, e.g., Bernardo and Smith 2003). In other words, throughout the sales season, the seller's belief about the market condition will remain Gamma, but with parameters that vary over time. This simplifies the tractability of the problem, and makes it easier to construct the updating mechanism for the forecasts of future demand.

Particularly, the updating scheme in our problem works as follows: let  $s$  be a certain time during the sales season and denote the belief about  $\Lambda$  at that time, by the pair  $(m_s, \theta_s)$ . Then,  $\theta_s = \theta_0 + \int_0^s \alpha(p(s)) ds$ , and  $m_s = m_0 + N_s$ , where  $N_s$  is the point process counting the number of sales up to time  $s$ , exclusive.

Recall that under the properties of the Gamma distribution, the point-estimate of the intensity rate  $\Lambda$  at time  $s$  is given by its expected value

$$\mu_\Lambda(s) = \frac{m_s}{\theta_s}$$

and the *relative* level of uncertainty about this parameter is given by the coefficient of variation:

$$cv_\Lambda(s) = \frac{\text{Std}[\Lambda|m_s, \theta_s]}{\mu_\Lambda(s)} = \frac{1}{\sqrt{m_s}}$$

Thus,  $cv_\Lambda(s)$  is a function of the current value of the shape parameter only, and not  $\theta_s$ . In other words, while the seller continuously updates the point estimate  $\mu_\Lambda(s)$ , a change (reduction) in the measure  $c_\Lambda$  occurs *only* at sales points. This means, for instance, that increasing the price not only reduces the purchase probability, but also lowers the likelihood of a reduction in the relative level of uncertainty. Thus, the seller has to trade-off an increase in potential revenues against the drawback of slower learning.

## 2.2 A dynamic pricing model

In this section we develop an intensity control framework for solving the store's dynamic optimal pricing problem. Our presentation, occasionally, tends to emphasize intuition rather than delve into precise mathematical details. Let  $\mathcal{P}$  be the set of admissible pricing policies, and consider a specific pricing policy  $p \in \mathcal{P}$ . Suppose that at the beginning of an horizon of length  $t$ , there are  $q$  units in stock ( $q = 1, 2, \dots$ ) and the current belief about the market condition is characterized by  $m$  and  $\theta$ . Define  $J_p(q, t, m, \theta)$  to be the expected store revenue under the given pricing policy  $p$ . We are interested in identifying an optimal pricing scheme  $p^*$  (if exists) and its associated optimal

revenue for the problem; i.e.,

$$J_{p^*}(q, t, m, \theta) = \sup_{p \in \mathcal{P}} J_p(q, t, m, \theta) = J_{q,m}^*(t, \theta)$$

Recall that the objective of the seller is to find at each instantaneous point of time, the optimal balance between setting the price too low risking loss of revenue, versus setting a high price and lowering the purchase probability as well as the likelihood of a reduction in the uncertainty level  $cv_\Lambda$ . Specifically, the problem is described using the following dynamic control formulation: Consider a small time interval  $dt$ , and note that

$$\begin{aligned} J_{q,m}^*(t, \theta) &= \sup_p \left\{ \frac{m}{\theta} \alpha(p) dt \cdot [p + J_{q-1, m+1}^*(t - dt, \theta + \alpha(p) dt)] \right. \\ &\quad \left. + [1 - \frac{m}{\theta} \alpha(p) dt] \cdot J_{q,m}^*(t - dt, \theta + \alpha(p) dt) + o(dt) \right\} \end{aligned} \quad (1)$$

Since the value  $\frac{m}{\theta} \alpha(p) dt$  represents the conditional probability<sup>6</sup> of a purchase of a single unit during the time interval  $dt$  the optimal revenue is comprised of two parts. In case of a sale, the retailer gets the revenue from a single unit at price  $p$ , leaving him with less time and one less unit. In case of no sale (probability of  $1 - \frac{m}{\theta} \alpha(p) dt$ ), the seller is left with  $q$  units, lost some potential selling time but learned something about the arrival rates.

Assuming that  $J^*$  has the required differentiability properties, we obtain<sup>7</sup> the Hamilton-Jacobi (HJ)-type partial differential equation for this problem:

$$\frac{\partial J_{q,m}^*(t, \theta)}{\partial t} = \sup_p \left\{ \frac{m}{\theta} \alpha(p) [p + J_{q-1, m+1}^*(t, \theta) - J_{q,m}^*(t, \theta)] + \alpha(p) \frac{\partial J_{q,m}^*(t, \theta)}{\partial \theta} \right\}. \quad (2)$$

We set  $J_{0,m}^*(t, \theta) = 0$  for all  $m$ , so that (2) is well-defined for all  $q \geq 1$ .

Our first observation, summarized in the following Lemma, is that the problem depends on the length of the sales season ( $t$ ) and the scale parameter of the Gamma distribution ( $\theta$ ) through their ratio  $\frac{t}{\theta}$  only. We define  $R \doteq \frac{t}{\theta}$  to represent the current belief of the seller about the total expected number of store visits during the entire sales horizon.

**Lemma 1** *The optimal pricing policy  $p_{q,m}^*(t, \theta)$  and its associated optimal expected revenue function  $J_{q,m}^*(t, \theta)$  depend on  $(t, q, m, \theta)$  through  $(q, R, m)$  only. In other words, we have*

$$\frac{\partial}{\partial R} J_q^*(R, m) = \sup_p \left\{ \alpha(p) \left[ p - J_q^*(R, m) + J_{q-1}^* \left( \frac{m+1}{m} R, m+1 \right) - \frac{R}{m} \frac{\partial}{\partial R} J_q^*(R, m) \right] \right\} \quad (3)$$

<sup>6</sup>We note that the conditional distribution of the number of purchases in a given interval is negative-binomial, but the fact that  $m_s$  and  $m_s/\theta_s$  are bounded under any sample-path, guarantees that the purchase probabilities in a small interval are as reflected in (1).

<sup>7</sup>This is achieved by subtracting  $J_{q,m}^*(t - dt, \theta)$  from both sides of the equation, dividing both sides by  $dt$  and taking the limit as  $dt$  goes to zero. For a formal justification see Bremaud (1980).

Further, given  $\alpha(p) = e^{-p}$ , it is straightforward to verify that the optimal pricing policy is given by

$$p_q^*(R, m) = 1 + J_q^*(R, m) - J_{q-1}^* \left( \frac{m+1}{m} R, m+1 \right) + \frac{R}{m} \frac{\partial}{\partial R} J_q^*(R, m). \quad (4)$$

Substituting equation (4) into equation (3) yields:

$$\frac{\partial}{\partial R} J_q^*(R, m) = \exp(-p_q^*(R, m)) \quad (5)$$

In Section 3 we study the behavior of the optimal pricing policies  $p_q^*(R, m)$  and their associated expected revenue functions  $J_q^*(R, m)$ . But before we do that, we describe a special case of interest in which there is full information about the market parameter  $\Lambda$ . This is the limit case of our model in which the shape parameter of the Gamma distribution approaches infinity (or  $c_V = 1/\sqrt{m} \rightarrow 0$ ), and its scale parameter  $\theta$  is set so as to reflect a known value  $R$ .

### 2.3 The case of full information

When the store owner knows the exact market condition parameter  $\Lambda$  (or, equivalently, the expected number of store visits  $R = \Lambda t$ ), pricing is used as a tool for revenue maximization only. The seller still has to carefully select the best price at each point of time. A high price may lead to unsold units, while a low price may result in early sales without exploiting the full market potential. In this case, where learning is irrelevant, Equation (5) takes a particularly simple form: Let  $J_q^{FI}(R) = J_q^*(R, \infty)$  be the optimal expected revenue collected over the sales horizon, when  $q$  units are available for sale. Then,

$$\frac{dJ_q^{FI}(R)}{dR} = \exp(-1 - J_q^{FI}(R) + J_{q-1}^{FI}(t, \Lambda))$$

The solution to this differential equation is shown in the following proposition.

**Proposition 2** *The optimal expected revenue functions  $J_q^{FI}(R)$  is given by*

$$J_q^{FI}(R) = \ln \left( \sum_{i=0}^q \frac{1}{i!} \left( \frac{R}{e} \right)^i \right), \quad (6)$$

*and the corresponding optimal pricing policy  $p_q^{FI}(R)$  is given by*

$$p_q^{FI}(R) = 1 + \ln \left( \sum_{i=0}^q \frac{1}{i!} \left( \frac{R}{e} \right)^i \right) - \ln \left( \sum_{i=0}^{q-1} \frac{1}{i!} \left( \frac{R}{e} \right)^i \right) \quad (7)$$

Hence, as expected, the results show that the more customers the seller expect to show up during the season (either due to the length of the season or the high arrival rate), the higher is the initial price. The revenue to the retailer is increasing and concave in the length of the season and is increasing and concave in the initial inventory. We will use this full information model as a benchmark for certain parts of this work. Particularly, the value function  $J_q^{FI}$  will be used to assess the potential increase in revenue that can be achieved when full information about the market condition can be obtained from an external source at the beginning of the sales season (see Section 6).

### 3 Optimal Policies

Finding the optimal pricing policy  $p_q^*(R, m)$  requires the solution of the Hamilton-Jacobi equation (5), from which the price is derived using Equation (4). Alternatively, one can derive an equation that characterizes the optimal price directly as shown in the following proposition.

**Proposition 3** *The optimal pricing scheme  $p_q^*(R, m)$  satisfies the following differential equation:*

$$\frac{\partial p_q^*(R, m)}{\partial R} = \frac{m+1}{m + R e^{-p_q^*(R, m)}} \times \begin{cases} e^{-p_q^*(R, m)} & \text{if } q = 1 \\ e^{-p_q^*(R, m)} - e^{-p_{q-1}^*\left(\frac{m+1}{m}R, m+1\right)} & \text{if } q \geq 2 \end{cases} \quad (8)$$

The differential equation presented in the proposition plays an important role in our analysis of the seller's problem.

Equation (8), is expressed in a recursive fashion, where  $\frac{\partial p_q^*(R, m)}{\partial R}$  depends on  $p_{q-1}^*\left(\frac{m+1}{m}R, m+1\right)$ . Hence, it seems theoretically possible to solve the problem for a single unit in inventory and all possible beliefs about market condition, use this result to solve for the case with an initial inventory of two units, and so forth. Alas, even though the equations are well defined, possess a unique solution, and can be solved numerically, a general analytical solution does not exist<sup>8</sup>. We propose the following qualitative characterization of the optimal prices.

**Theorem 4** *The optimal pricing functions  $p_q^*(R, m)$  satisfy the following properties*

- (i) *For all  $R \geq 0$ ,  $p_q^*(R, m) < p_{q-1}^*\left(\frac{m+1}{m}R, m+1\right)$ .*
- (ii) *For every  $q$  and  $m \geq 1$ ,  $p_q^*(R, m)$  is increasing in  $R$ .*

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<sup>8</sup>For the case of  $q = 1$  there exists an analytic solution of the form:

$$p_{q=1}^*(R, m) = \ln \left( R + e\rho \left( \frac{R}{e} \right) \right) \quad (9)$$

where  $\rho(x) \in [1, \infty)$  is the unique solution to the equation  $\rho^{m+1} - \rho = x$ , in the range  $[1, \infty)$ .

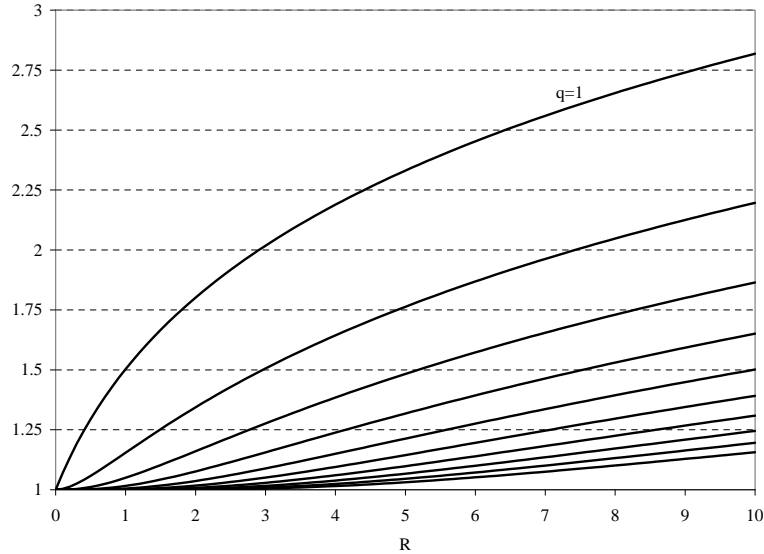


Figure 1: The optimal price as a function of  $R$ , for ten values of  $q = 1, \dots, 10$  and  $cv_\Lambda = 1$ . The curves maintain a decreasing order pattern, with the top curve representing  $q = 1$ .

Part (i) of the Theorem shows that immediately after a sale the optimal price jumps upward. Part (ii) demonstrates that in between sales, the price declines as the seller becomes more and more desperate to sell the units in stock. Additionally, the value of  $\theta$  increases as time progresses; in other words the seller's belief about the market condition worsens. Furthermore, as in the full information case, the higher the number of customers the seller expects to arrive at the store the higher the retail price. In the full information case this was obvious. But when there is uncertainty about the market parameters, lower prices allow the retailer to accelerate his learning (with low prices it is much more likely that there was no sale due to no arrival rather than a low valuation for the good.) Thus, when expecting many arrivals to his store the retailer is faced with two conflicting prices forces. He can afford to increase the price and hope to sell to a high valuation consumer or he can lower the price and learn faster about the arrival rate and price more profitably for the rest of the selling season. In the above theorem we show that the opportunistic force pushing for higher prices has a larger magnitude in this environment.

In Figure 1 we depict the value of the optimal prices as a function of  $R$  for several values of  $q$  and a fixed initial coefficient of variation  $cv_\Lambda = 1/\sqrt{m} = 1$ . We see that for the same expected number of arrival ( $R$ ) the larger the initial inventory the lower the price demanded, this reflects the seller fear of not selling his units and remaining with useless excess inventory. As the ratio between  $q$  and  $R$  increases, the optimal price decreases and converges to the optimal instantaneous price

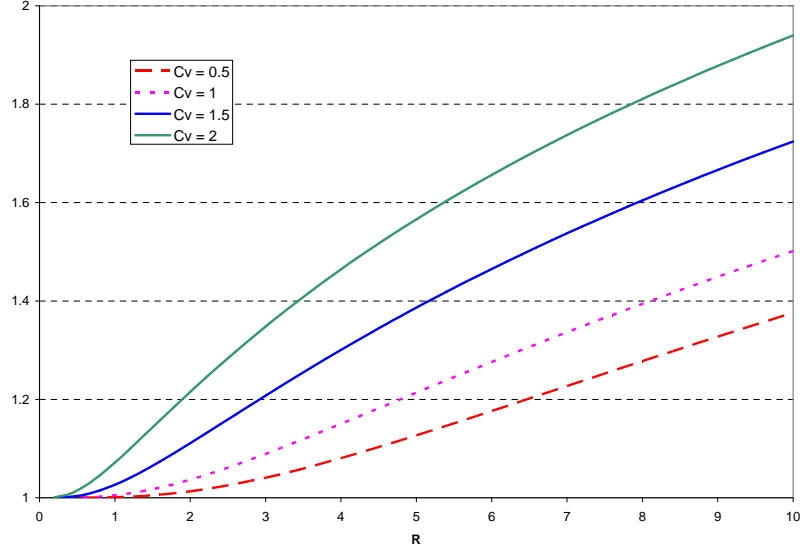


Figure 2: The optimal prices as a function of  $R$ , plotted for four values of uncertainty ( $cv_\Lambda = 0.5, 1, 1.5, 2$ ) and  $q = 5$ .

$p = 1$ . For instance, consider the case  $q = 10$ , and let  $R = 4$ , this refers to a situation at which 4 customers are expected to arrive until the end of the sales season. As a result of the low likelihood that more than 10 customers will arrive, it is reasonable to set the price so as to maximize the expected revenue gained from each actual arrival. Indeed,  $p_{q=10}^*(R = 4, m = 1) = 1.015 \approx 1$ .

In Figure 2 we explore the subject of uncertainty. To do so, we keep the number of units fixed at  $q = 5$ , and consider four different levels of the coefficient of variation  $cv_\Lambda = 0.5, 1, 1.5, 2$ . We observe the clear pattern of the optimal prices increasing in the level of uncertainty.

For the same reasons discussed above, finding the optimal value function  $J_q^*(R, m)$  is as hard as solving  $p_q^*(R, m)$ . Nevertheless, observe from (4) and (5) that  $\frac{d}{dR} J_q^*(R, m) = \exp(-p_q^*(R, m))$ , and so  $J_q^*(R, m)$  can be easily constructed along the numerical procedure that generates  $p_q^*(R, m)$ . In Figure 3 we depicts the expected revenue as a function of  $R$  for several values of  $q$  and a fixed initial coefficient of variation  $cv_\Lambda = 1$ . As we increase the value of  $q$ , the function  $J_q^*(R, m)$  converges to  $\frac{R}{e}$ , the expected profit under infinite number of units in stock. Generally, we can state  $\lim_{q \rightarrow \infty} J_q^*(R, m) = \frac{R}{e}$ . It is interesting to note that the convergence to the infinite-unit curve seems to be greatly dependent on the expected total number of arrivals during the horizon, divided by the number of units available for sale. Figure 3 suggests that an increase of the length of the sales season brings expectations for higher revenues, but in a diminishing rate. The following theorem establishes this result:

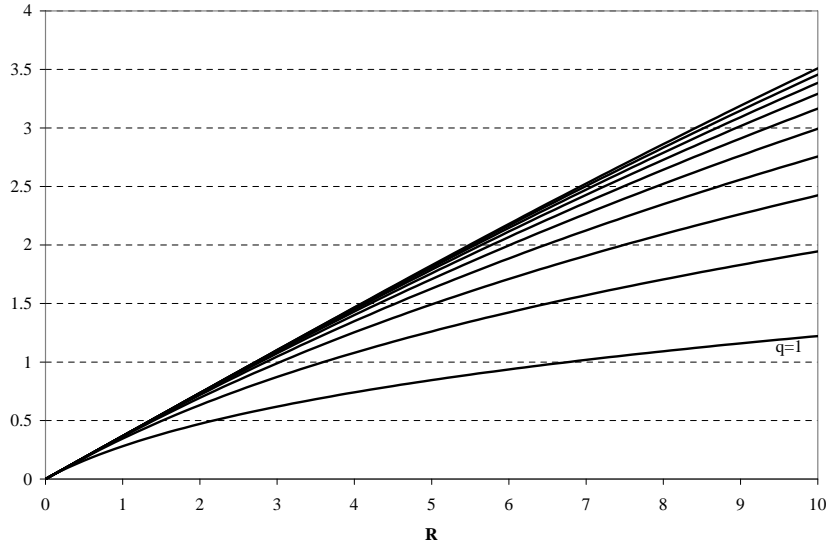


Figure 3: The optimal expected revenue as a function of  $R$ , for ten values of  $q = 1, \dots, 10$  and  $cv_\Lambda = 1$ . The curves maintain an increasing order pattern, with the bottom curve representing  $q = 1$ .

**Theorem 5** *The function  $J_q^*(R, m)$  is strictly increasing and concave in  $r \geq 0$ , for any  $q \geq 1$  and  $m > 0$ .*

Finally, Figure 4 presents the optimal expected revenue for the same parameter-combination considered in Figure 2. As expected, we notice that the uncertainty surrounding the actual market intensity parameter  $\Lambda$  has a negative impact on expected revenues. Additionally, Figure 4 suggests that uncertainty plays a meaningful role only when the length of the horizon is sufficiently high.

## 4 Certainty Equivalent Heuristics

In this section, we consider an alternative to the optimal policy  $p_q^*(R, m)$ , in the form of a *certainty equivalent heuristic*. Specifically, the heuristic we propose below treats the pricing problem at each point of time (say,  $s$ ) as one with a perfectly known market parameter  $\Lambda$ , that equals precisely to the prevailing ratio  $\frac{m(s)}{\theta(s)}$ . It is for this reason that we use the title “certainty equivalent.” While the heuristic ignores uncertainty about  $\Lambda$  when setting the prices, it *does* take into account the random nature of arrivals and individual customer reservation prices. Indeed, at each time  $s$ , the seller solves the full information case discussed in §2.3 above, or more simply, sets the price to

$$p(s) = p_{q(s)}^{FI} \left( \frac{m(s)}{\theta(s)} \cdot (T - s) \right)$$

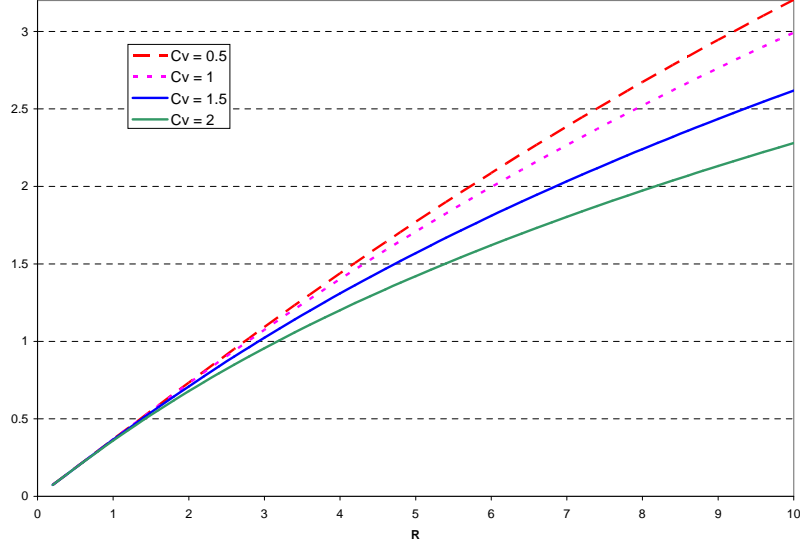


Figure 4: The optimal expected revenue as a function of  $R$ , plotted for four values of uncertainty ( $cv_\Lambda = 0.5, 1, 1.5, 2$ ) and  $q = 5$ .

(See Proposition 2.) As to the learning about  $\Lambda$  – it is done, but “passively.” In other words, prices are not used proactively to affect learning, but nevertheless they are used in the learning mechanism to update the belief about the market condition  $\Lambda$ . Then, the price for the current time interval is set as explained above. The certainty equivalent heuristic serves a dual goal. First, it offers a simpler procedure for calculating the prices than that used to compute the optimal policy, making it more appealing from practical purposes. The second, and more compelling objective, is to allow us to study how important is the consideration of the way by which current prices affect the ability of the seller to learn. In other words, is passive learning sufficient, or should active learning through smart price setting be sought?

To compute the expected revenue under the certainty-equivalent policies, denoted by  $J_q^{CE}(R, m)$ , we numerically solve the following simultaneous set of  $q$  differential equations<sup>9</sup>:

$$\frac{d}{dr} J_{i,m+q-i}^{CE}(r) = \frac{m+q-i}{me^p + R} (p - J_i^{CE}(R, m+q-i) + J_{i-1}^{CE}(R, m+q-i+1)), \quad i = 1, \dots, q \quad (10)$$

with the initial condition  $J_i^{CE}(0, m+q-i) = 0$  for  $i = 1, \dots, q$ , and  $J_0^{CE}(R, m) \equiv 0$ .

Figure 5 displays the percentage-wise sub-optimality gap  $(J_q^*(R, m) - J_q^{CE}(R, m)) / J_q^*(R, m)$ , for  $cv_\Lambda = 1/\sqrt{m} = 1$  and several values of  $q$ . Overall, for  $cv_\Lambda = 1$ , the gap is relatively small (below 1.7%) and we expect it to decrease even further as  $m$  increases. The optimality gap seems to be

<sup>9</sup>Note that  $J_{q,m}^{CE}(r)$  satisfies Equation (3) without the supremum operator.

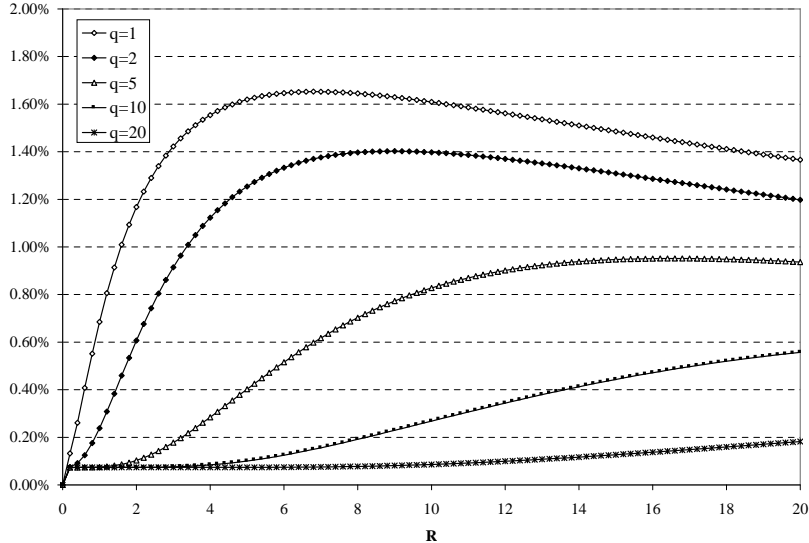


Figure 5: Percentage sub-optimality of the certainty-equivalent policies as a function of  $R$ . ( $cv_\Lambda = 1$ .)

particularly narrow when  $R$  is small relative to the number of units available for sale.

The explanation for this is that the optimal price becomes closer to 1, since only a small number of customers are expected to arrive, hence making the optimal policies and the certainty equivalent heuristic perform equally well (see our discussion of Figure 3 above). Also, as expected, when  $R$  is very large, the percentage-wise gap between the policies diminishes.

Figure 6 below considers the single-unit case ( $q = 1$ ) for four different levels of the coefficient of variance  $cv_\Lambda = 0.5, 1, 1.5, 2$ . Since for the values of  $R$  in the range  $[0, 20]$ , the case  $q = 1$  appears to have the highest gap in Figure 5, we consider the values of Figure 6 as conservative estimates of the sub-optimality gap of the certainty equivalent heuristics. This figure shows, indeed, that as the level of uncertainty about  $\Lambda$  increases, taking into account the impact of prices on learning becomes clearly more important. We note that coefficient of variances larger than 1 are not rare in practical settings, particularly in the apparel fashion industry; see Raman et al. (1994).

## 5 Additional Pricing Schemes

A seller faced with limited inventory, a short selling season and stochastic and unknown demand may find the task of full optimization or even a certainty equivalent heuristic too daunting. The seller may opt for charging a single (optimally chosen) fixed price for the entire season.

Conversely, he may implement a naive policy ignoring market uncertainty, treating his estimate

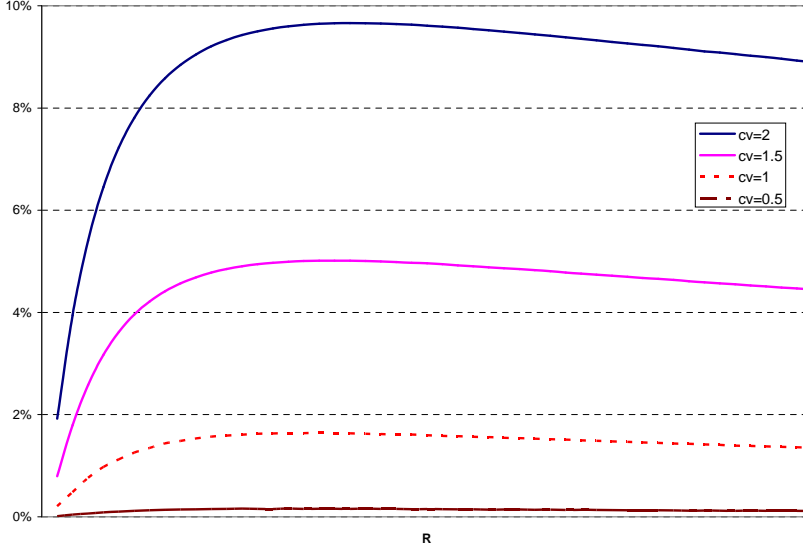


Figure 6: Percentage sub-optimality of the certainty-equivalent policies as a function of  $R$ . ( $q = 1$ .)

of the market condition  $E[\Lambda]$  as completely accurate and never attempt to learn at all. We analyze these pricing schemes below. In §5.1 we consider the case in which a fixed price is set for the entire duration of the season. Then, in §5.2, we discuss a pricing policy that ignores the uncertainty about the market parameter  $\Lambda$ .

## 5.1 Fixed Pricing

As a complementary part for our discussion, we consider the common base case in which a fixed price is set for the entire duration of the season. There are multiple explanations for why fixed prices are so often observed in practice: these range from high cost and complexity of frequent price changes via the lack of dynamic pricing tools and strategy to the need to set a common price for a product in a competitive market. We discuss the analysis of the fixed pricing scheme below, and subsequently provide additional numerical results.

When the store owner sets a constant price<sup>10</sup>  $p$  for the entire duration of the sales season, he gives up the ability to act upon learning about the market condition. Therefore, the parameters  $m$  and  $\theta$  affect the best choice of the price  $p$  not through their evolution in time but only via their representation of the market uncertainty. Let  $J_{q,m}^{FP}(R|p)$  be the expected revenue function for a fixed-price policy with a price  $p$ . One can develop an expression for this function using stochastic

<sup>10</sup>Throughout this section we use the letter  $p$  interchangeably to represent the pricing policy as well as the fixed price it prescribes.

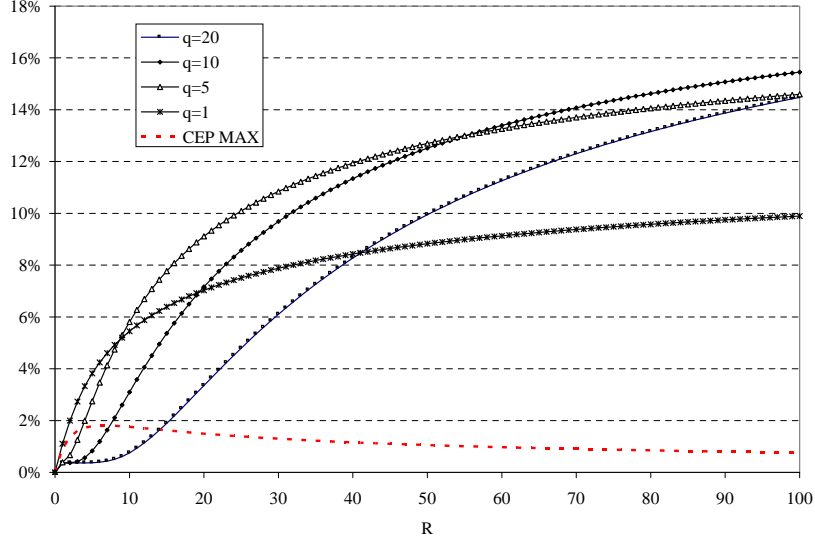


Figure 7: Percentage sub-optimality of best fixed-price policies as a function of  $R$ .

analysis arguments. Assume for a moment that the actual market parameter  $\Lambda$  is equal to a certain value  $\lambda$ . Then, given a constant price  $p$ , the demand during the horizon has a Poisson statistical distribution with the (price-dependent) expected value  $\lambda te^{-p}$ . Consequently, the expected number of units sold is given by the expected value of this Poisson distribution, truncated at  $q$ , the number of units available for sale. This value is given by  $S(\lambda te^{-p}, q)$ , where

$$S(a, q) = \sum_{s=0}^{q-1} s \frac{e^{-a} a^s}{s!} + q \sum_{s=q}^{\infty} \frac{e^{-a} a^s}{s!}$$

The assessment of the expected revenue now boils down to an integration over the prior statistical distribution of  $\Lambda$ :  $J_{q,m}^{FP}(R|p) = pE[S(\Lambda te^{-p}, q)]$ . It can be further shown that the last expression is equal to:

$$J_{q,m}^{FP}(R|p) = p \int_0^{\infty} S(we^{-p}, q) \frac{\frac{m}{R} \left(\frac{wm}{R}\right)^{m-1} e^{-\frac{mw}{R}}}{\Gamma(m)} dw \quad (11)$$

The latter integral can also be presented in an exact analytical form that involves summation over the related negative-binomial and binomial probability distributions.

In our numerical studies we solve the problem  $J_{q,m}^{FP}(R) = \max_p J_{q,m}^{FP}(R|p)$  with little effort using a common solver package. Figure 7 below shows the optimality gap for the best fixed price policy. We varied  $R$  from 0 to 100, and considered four values of  $q = 1, 5, 10, 20$  and  $cv_{\Lambda} = 1$ . The results are quite interesting. Indeed, we observe from this figure that the use of a fixed price policy can be bad in environments with relatively high levels of uncertainty about the market condition.

As explained above, a fixed price policy does not perform poorly when the expected number of arrivals is smaller than the number of units available for sale. But as more time is available for sale (or alternatively, as the expected number of arrivals per unit is stock grows), yield management becomes more important. Again, we expect to see a decline in the curves as  $R$  grows. Nevertheless, this decline from a maximal gap is very slow. Not less importantly, Figure 7 provides an additional insight about active vs. passive learning: The dashed line in this figure represents the maximal gap of the certainty equivalent heuristic across the four values  $q = 1, 5, 10, 20$ . This gap is at most 2%, for the parameter set investigated. This suggests that as long as the uncertainty about the parameter  $\Lambda$  is not significantly high, one need not be concerned about the way by which prices affect learning – it is good enough to set the prices at each point as if the market condition is known and equal to the current estimate of  $\Lambda$ , and then learn from the history of sales and prices through the Bayesian mechanism discussed above.

## 5.2 Policies ignoring market uncertainty

Another benchmark for our study, is the case in which uncertainty and learning about the market condition  $\Lambda$  is completely ignored. In this case, we assume that the seller treats his best estimate of the market condition  $\mathbb{E}[\Lambda]$  as accurate, and uses the pricing policy that is optimal for the full information case. In other words,

$$p_q^N(\mathbb{E}[\Lambda], t) = p_q^{FI}(\mathbb{E}[\Lambda] \cdot t)$$

(the subscript ‘N’ is used to denote a *naive* policy). Unlike the certainty-equivalent heuristic, in this case the seller does not learn about  $\Lambda$ . To make this case comparable to the policies discussed above, we consider the case  $\Lambda \sim \text{Gamma}(m, \theta)$ . We calculate the expected revenue under this benchmark policy using the following procedure: Suppose that the real market condition is  $\lambda$ . Then, a price scheme of the type  $p_q^{FI}(at)$  (for any constant  $a > 0$ ) yields the simultaneous set of differential equations for the conditional expected revenue functions as follows:

$$\begin{aligned} \frac{d}{dt} J_i^N(t|\lambda, a) &= \lambda \exp(-p_i^{FI}(at)) (p_i^{FI}(at) - J_i^N(t|\lambda, a) + J_{i-1}^N(t|\lambda, a)), \quad i = 1, \dots, q \\ J_i^N(0|\lambda, a) &= 0, \quad i = 1, \dots, q \\ J_0^N(t|\lambda, a) &\equiv 0 \end{aligned}$$

Our particular interest is in the expected value  $J_{q,m}^N(t, \theta)$ , which can be assessed by computing:

$$J_{q,m}^N(t, \theta) = \mathbb{E}_\Lambda \left[ J_q^N \left( t | \Lambda, \frac{m}{\theta} \right) \right]$$

**Proposition 6** Define the random variable  $W = \Lambda\theta \sim \text{Gamma}(m, 1)$ . The expected revenue for the pricing policy that ignores uncertainty is given by  $J_{q,m}^N(t, \theta) = J_{q,m}^N(R = \frac{mt}{\theta})$ , where

$$J_{q,m}^N(R) = E_W [J_q^N(R|W, m)]$$

The result of the proposition is helpful for the sake of comparing this policy with the other policies presented above, on the same axis of  $R$ -values. For instance, an intelligent seller (one that is aware of the uncertainty surrounding  $\Lambda$ ) can interpret the difference term  $J_{q,m}^*(r) - J_{q,m}^N(r)$  as the expected loss in revenue, if the uncertainty about  $\Lambda$  is totally ignored. We present some numerical results associated with this case in the next section.

## 6 The Value of Perfect Information

The comparative study above enabled us to shed light on the value of active learning through pricing and sales observations. But at the same time, it also raised an interesting question: how good is a learning done solely through observation of purchases? A variety of alternative learning mechanisms may be available to the seller, such as obtaining advance information about the market condition through a market study, monitoring customer behavior in stores, counting “clicks” on web sites, etc. Looking at this issue from a slightly different angle, one may wonder, for instance, as to whether the increase in expected revenues when switching from a policy that ignores market uncertainty to the optimal pricing scheme  $p^*$  constitutes a large or only a small fraction of the potential increase that could be gained by obtaining perfect information about  $\Lambda$ . To this end, we define the expected revenue under perfect information as follows:

$$\text{Expected revenue under perfect info: } V_{q,m}(t, \theta) = E_\Lambda [J_q^{FI}(\Lambda t)]$$

As stated, this expected revenue is based on the assumption that after acquiring the information about  $\Lambda$ , the optimal pricing scheme  $p_q^{FI}(\Lambda t)$  will be applied. As shown in the following proposition,  $V_{q,m}(t, \theta)$  depends on  $t$  and  $\theta$  only through  $R$ .

**Proposition 7** The expected revenue under perfect information is given by

$$V_q(R, m) = \int_0^\infty \ln \left( \sum_{i=0}^q \frac{1}{i!} \left( \frac{wR}{em} \right)^i \right) \frac{w^{m-1} e^{-w}}{\Gamma(m)} dw$$

We compared this expected value of the revenue under perfect information with the optimal expected revenue function for various combinations of the parameters  $q, m$  and  $R$ . Specifically, we

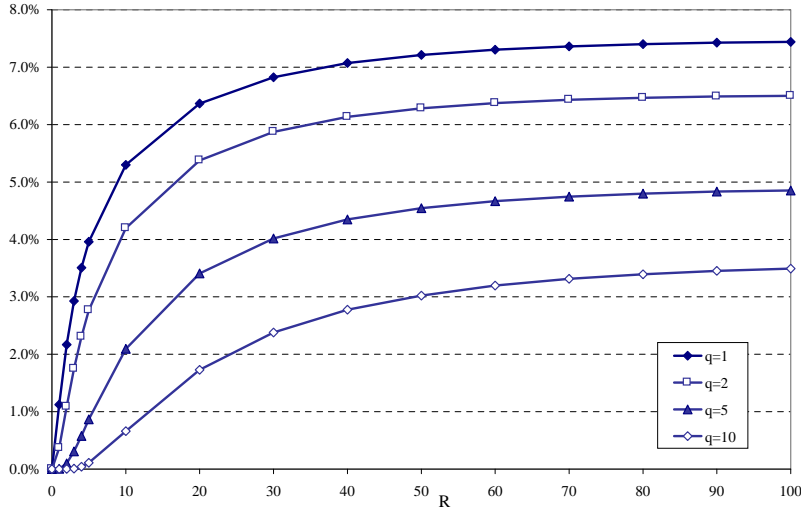


Figure 8: The expected value of perfect information as a function of the scaled time parameter  $R$ , for  $c_v = 1$  and  $q = 1, 2, 5, 10$ .

used the “value of information” metric:

$$\text{Value of perfect info: } \left[ \frac{V_q(R, m) - J_q^*(R, m)}{J_q^*(R, m)} \right] * 100\%$$

Our results are summarized in Figure 8 for the case  $cv_\Lambda = 1$ . We observe that the percentage-wise value of perfect information is increasing in  $R$  up to a relatively constant level which holds for a very long range of values of  $R$  (say  $R = 70$  to several hundreds). The value of perfect information then declines at a very slow rate (for instance, when  $q = 1$ , the value of perfect information declines from its top value of 7.5% to about 6.7% at  $R = 1,000$  and 5.4% at  $R = 10,000$ ).

Figure 8 also suggests that investment in market information brings higher percentage-wise increase in sales revenues when  $q$  is small. This supports our intuition that the larger the number of items, the higher the opportunity to apply the learning from actual sales observations before running out of items – therefore, suppressing the value of getting full market information in some other way. Nevertheless, we suggest some caution in the interpretation of this results: for an assessment of the potential return on investment from market research programs designed to reduce the uncertainty about the market “condition”, it may be more appropriate to consider the magnitude of the value of perfect information rather than its percentage metric discussed above.

Getting back to the question we posed at the beginning of this section, we study in Figure 9 below the value of using a policy other than a naive one for the case of  $q = 5$  and  $cv_\Lambda = 1.5$ .

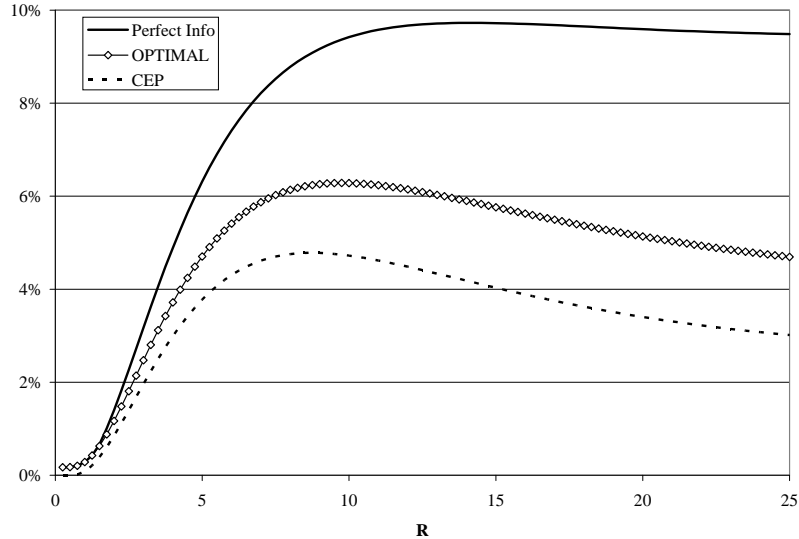


Figure 9: Percentage-wise contribution of various policies with respect to the the policy that ignores the existence of uncertainty ( $p_q^N(R)$ ).

Suppose that a seller currently uses a policy that completely ignores the uncertainty about  $\Lambda$ . How bad can it possibly be? We start by examining the expected value of perfect information about  $\Lambda$ . The figure suggests that the percentage-wise value of perfect information increases with  $R$  first, and then decreases. The figure shows two additional curves: one that relates to the optimal pricing policy, and one that relates to the certainty-equivalent heuristic. We gain from this figure an insight that applies quite exclusively for reasonable values of  $cv_\Lambda$  (say, above 0.5) and mid-level values of  $R/q$  (say, between 1 and 100)<sup>11</sup>: Regardless of whether or not the maximal potential gain (value of perfect information with respect to the naive policy) is significant for the firm, the major increases in expected revenues are obtained by certainty-equivalent policies, and to a larger extent by obtaining perfect information. Trying to set the prices in a way that takes into account their impact on the learning process has a secondary impact only.

## 7 Summary

In this paper, we developed a continuous-time pricing model for a retailer that sells a limited quantity of a short life-cycle good over a finite sales season. The model incorporates three types of uncertainty, prevalent in most realistic settings. First is the uncertainty about the exact number and timing of customer’s arrivals to the store, due to unexplainable, “pure” statistical behavior. The

<sup>11</sup>Based on additional numerical studies not reported here.

second type of uncertainty is about the reservation price of each customer, which affect individual purchasing decision. The third type of uncertainty is about how successful the product is in the market (“market condition”). We propose a Bayesian updating mechanism for learning about the market condition along the sales season, and incorporate it into a revenue maximization problem. Ideally, the seller should set the prices in a way that maximize expected profits, but to do so, the seller has to take into account the intricate way by which prices affect instantaneous and future revenues, as well as the learning process itself.

Our theoretical results together with numerical experimentations yield several important managerial insights. First, optimal prices tend to be higher with higher levels of uncertainty about the market condition. In other words, when faced with uncertainty, the seller should ask for a high price betting on a strong market, particularly when the sales season is long enough to recover if conditions are not as favorable. We show that optimal prices should decline continuously over the sales season, but should increase at the time points of a sale. This increase in price is the result of two factors: there are less units to sell, and hence more time to experiment with higher prices, and in addition, the occurrence of a sale raises the seller’s belief that the market condition is good. The optimal expected revenue increases as a function of the length of the sales season and the number of units available for sale, but in a diminishing rate. These results should be taken into account when the retailer has some influence over the length of the selling season and the initial inventory level.

In the paper, we compared the expected revenues under the optimal pricing policies with several other policies: A fixed price scheme, a certainty-equivalent heuristic, and a naive policy that completely ignores the uncertainty about the market condition. We first show that when the initial uncertainty about market condition is higher, there is a heavy price to pay for ignoring the uncertainty or for using best fixed-price policies. The percentage loss in expected revenues (sub-optimality gap) is the highest when the length of the season is sufficiently large. This results contrasts the important observation made by Gallego and van Ryzin (1994), according to which fixed-price policies tend to perform very well in the case of full information about the market condition. We therefore argue that the benefits of dynamic pricing strategies become critical in settings with high but resolvable initial uncertainty about how successful the product is. But how important is it to solve the complex dynamic pricing decision-problem by taking into account the impact of pricing on the learning process (“active learning”)? We found that unless the level of uncertainty is very high, this is a minor concern. Learning has to take place, but it is quite reasonable to adopt

a “passive learning” approach, as does the certainty-equivalent heuristic – the seller updates his belief about the market condition as time progresses, but at each moment the price is set as if the market condition is known and equals to its current estimate.

We complete our study by taking a look at the potential expected benefits that can be gained from perfect information about the actual market condition. Our results suggest that obtaining market information at the beginning of the sales season is particularly beneficial when the number of units available for sale is small, and when the sales season is sufficiently long (expected number of arrivals per unit of item is about 1 and above). We explain this property by arguing that the larger the number of items, the higher is the opportunity to apply the learning from actual sales observations before running out of items. Additionally, optimal prices are not sensitive to the level of inventory when it is large.

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## Appendix: Proofs of Theorems and Propositions

**Proof of Lemma 1.** We provide below an outline of the proof. It is done by an induction on  $q$  and by construction of the function  $J_{q,m}^*$ . First, we demonstrate that it is sufficient to describe  $p^*$  and  $J^*$  through the triplet  $(q, r = \frac{t}{\theta}, m)$ . The lemma then follows trivially, from the definition of  $R = mr$ . Clearly, the lemma holds for  $q = 0$ . Suppose it holds up to  $q - 1$ , and prove for  $q$ . Let us construct  $J_{q,m}^*(t, \theta)$  through small steps on the  $t$ -axis. For  $t = 0$ , the theorem clearly applies since  $J_{q,m}^*(0, \theta) = 0$ . Suppose that the theorem holds up to  $t$ , and observe that

$$\begin{aligned} \frac{\partial J_{q,m}^*(t, \theta)}{\partial t} &= \sup_p \left\{ \frac{m}{\theta} \alpha(p) [p + J_{q-1,m+1}^*(t, \theta) - J_{q,m}^*(t, \theta)] + \alpha(p) \frac{\partial J_{q,m}^*(t, \theta)}{\partial \theta} \right\} \\ &= \frac{1}{\theta} \sup_p \left\{ m \cdot \alpha(p) \left[ p + J_{q-1,m+1}^*\left(\frac{t}{\theta}\right) - J_{q,m}^*\left(\frac{t}{\theta}\right) \right] - \frac{t}{\theta} \alpha(p) \frac{d}{dr} J_{q,m}^*(r) \right\} \\ &= \frac{1}{\theta} h\left(\frac{t}{\theta}\right) \end{aligned}$$

for some function  $h$ . Clearly, the optimal price that solves the supremum problem depends on  $t$  and  $\theta$  through  $r$ . It now follows by simple calculus arguments, that  $J_{q,m}^*(t, \theta) = J_{q,m}^*(r = \frac{t}{\theta})$  and  $p_{q,m}^*(t, \theta) = p_{q,m}^*(r = \frac{t}{\theta})$ . Next, through a simple transformation of variables we get  $J_q^*(R, m) = J_{q,m}^*(r = R/m)$  (and similarly for  $p^*$ ).

finally substitution leads to

$$\begin{aligned} \frac{\partial}{\partial R} J_q^*(R, m) &= \frac{\theta}{m} \frac{\partial J_q^*(\frac{t}{\theta} m, m)}{\partial t} = \frac{\theta}{m} \frac{\partial J_{q,m}^*(t, \theta)}{\partial t} \Big|_{t=R\theta/m} \\ &= \sup_p \left\{ \alpha(p) \left[ p - J_q^*(R, m) + J_{q-1}^*\left(\frac{m+1}{m} R, m+1\right) - \frac{R}{m} \frac{\partial}{\partial R} J_q^*(R, m) \right] \right\} \end{aligned}$$

■

**Proof of Proposition 2.** Let  $H_q(R) = \exp(J_q^{FI}(R))$  and observe that  $\frac{d}{dR} H_q(R) = \frac{1}{e} H_{q-1}(t)$ . Additionally, since  $J_q^{FI}(0) = 0$ ,  $H_q(0) = 1$ . We complete the proof using induction on  $q$ . For  $q = 1$ , we have  $\frac{d}{dR} H_1(R) = \frac{1}{e}$ , hence  $H_1(R) = \frac{R}{e} + 1$ . Suppose that the proposition holds for all  $q \leq q_0$ . Then,  $\frac{d}{dR} H_{q_0+1}(R) = \frac{1}{e} H_{q_0}(R) = \frac{1}{e} \sum_{i=0}^{q_0} \frac{1}{i!} \left(\frac{R}{e}\right)^i$ . Therefore,  $H_{q_0+1}(R) = 1 + \int_0^R \frac{1}{e} \sum_{i=0}^{q_0} \frac{1}{i!} \left(\frac{s}{e}\right)^i ds = 1 + \frac{1}{e} \sum_{i=0}^{q_0} \frac{1}{(i+1)!} R^{i+1} \left(\frac{1}{e}\right)^i = \sum_{i=0}^{q_0+1} \frac{1}{i!} \left(\frac{R}{e}\right)^i$ . ■

**Proof of Proposition 3.** By taking the derivative of (4), and using (5), we get

$$\begin{aligned} \frac{\partial}{\partial R} p_q^*(R, m) &= \exp(-p_q^*(R, m)) - \frac{m+1}{m} \exp\left(-p_{q-1}^*\left(\frac{m+1}{m} R, m+1\right)\right) \\ &\quad + \frac{1}{m} \exp(-p_q^*(R, m)) - \frac{R}{m} \exp(-p_q^*(R, m)) \cdot \frac{\partial}{\partial R} p_q^*(R, c) \end{aligned}$$

Rearranging terms we get Equation (8), the proposition now follows for all  $q > 1$  (the proof for  $q = 1$  is simpler). ■

**Proof<sup>12</sup> of Theorem 4.** Consider the simultaneous set of  $q$  differential equations:

$$\begin{aligned}\frac{d}{dr}f_1(r) &= (m+q)/(1+r/f_1(r)) \\ \frac{d}{dr}f_i(r) &= (m+q-i+1)\frac{1}{1+r/f_i(r)}(1-f_i(r)/f_{i-1}(r)) \quad \text{for } i=2,\dots,q\end{aligned}\quad (12)$$

Given the initial conditions  $f_1(0) = \dots = f_q(0) = e$ , one can show by induction that for every  $i = 1, \dots, q$ ,  $f_i(r) = e + c_i r^i + O(r^{i+1})$  for some  $c_i > 0$ . In particular, we have  $f_1(r) > f_2(r) > \dots > f_q(r)$  on some interval  $r \in (0, \varepsilon)$ . We propose a proof that  $f_1(r) > f_2(r) > \dots > f_q(r)$  for all  $r > 0$  by contradiction: Suppose that the latter inequalities do not hold for all  $r > 0$ . Then, there is some least  $i = i_0$  for which the inequality fails. In other words, there exists  $r_1 > 0$  with  $f_{i_0}(r_1) = f_{i_0+1}(r_1)$  but  $f_{i_0}(r) > f_{i_0+1}(r)$  for  $r \in (0, r_1)$ . But then,  $f'_{i_0}(r_1) \leq f'_{i_0+1}(r_1) = (m+q-i_0)(1-f_{i_0+1}(r_1)/f_{i_0}(r_1))/(1+r_1/f_{i_0}(r_1)) = 0$ , which requires  $f_{i_0}(r_1) \leq f_{i_0-1}(r_1)$ , contradicting the minimality of  $i_0$ . Note that if we let  $h_{q,m}(r) = \exp(p_q^*(mr, m))$ , we get from (8) that this function satisfies the differential equation

$$\frac{\partial h_{q,m}(r)}{\partial r} = \frac{m+1}{1+m/h_q(R, m)} \times \begin{cases} 1 & \text{if } q=1 \\ 1 - \frac{h_{q,m}(r)}{h_{q-1,m+1}(r)} & \text{if } q \geq 2 \end{cases} \quad (13)$$

with the initial condition  $h_{q,m}(0) = e$ . Thus, a seller starting with  $q$  units and an initial belief characterized by  $cv_\Lambda = \frac{1}{\sqrt{m}}$  can find the optimal price by solving equations 12. This implies that  $h_{q,m}(r) > h_{q-1,m+1}(r)$  for all  $r > 0$  or converting to optimal prices  $p_q^*(R, m) < p_{q-1}^*(\frac{m+1}{m}R, m+1)$  for all  $R > 0$ .

Consequently, part (ii) is easily verified for the exponential of the pricing mechanism from the structure of the differential equations (since  $1 - f_i(r)/f_{i-1}(r) \geq 0$ ). ■

**Proof of Theorem 5.** Observe first that  $\frac{d}{dR}J_q^*(R, m) = \exp(-p_q^*(R, m)) > 0$ . By taking the derivative from both sides of the latter equation, we get the strict concavity result  $\frac{d^2}{dR^2}J_q^*(R, m) = -\frac{d}{dR}J_q^*(R, m) \cdot \frac{d}{dR}p_q^*(R, m) < 0$  where the inequality is due to part (ii) of Theorem 4. ■

**Proof of Proposition 6.** Recall the set of simultaneous differential equations presented in §5.2. It is easily verified that the functions  $J_i^N$  satisfy the property  $J_i^N(\frac{t}{b}|b\lambda, ba) = J_i^N(t|\lambda, a)$  for all  $b > 0$ . Particularly, this property holds for  $a = \frac{m}{\theta}, \lambda = \Lambda$ , and  $b = \theta$ . Hence, we get  $J_q^N(t|\Lambda, \frac{m}{\theta}) = J_q^N(r|W, m)$  and the proposition follows. ■

**Proof of Proposition 7.** The proposition follows directly from the transformation  $W = \theta\Lambda$ :  $E_\Lambda[J_q^{FI}(\Lambda t)] = E_W[J_q^{FI}(Wr)]$ . Also, note that  $W$  has a Gamma distribution with shape parameter  $m$  and scale parameter 1. ■

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<sup>12</sup>The authors wish to thank Prof. Robert Israel for suggesting this short and elegant version of the proof.