Merchants vs. Two-Sided Platforms: Competing Intermediation Models

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PRELIMINARY AND INCOMPLETE - PLEASE DO NOT CITE
February 28, 2014

Abstract

We build and solve a model of competition between two intermediaries that can choose to offer a continuum of product categories in two modes. The "merchant" mode creates higher consumer willingness-to-pay but incurs higher fixed costs relative to the "two-sided platform" (TSP) mode. Consumers are heterogeneous in the willingness-to-pay differential they perceive between a product offered in the higher quality merchant mode and the same product offered in the lower quality TSP mode. We investigate four distinct competition scenarios which differ in two dimensions: i) consumers singlehome or multihome; ii) intermediaries can offer each product in one mode only or in both modes. In all scenarios, one intermediary acts as the Stackelberg leader by choosing its product mix before the Stackelberg follower.

A first key result is that the Stackelberg leader can completely foreclose the follower only when intermediaries are able to offer products in both modes. Second, we show that the follower always makes larger profits under multihoming relative to singlehoming. In contrast, the leader prefers multihoming when intermediaries are restricted to one mode per product, but prefers singlehoming when intermediaries can offer products in both modes. Third and in contrast with classical results in duopoly models with vertical differentiation, we find that in some cases the Stackelberg leader chooses a product mix of lower overall "quality" than the Stackelberg follower.

Keywords: Competing Intermediaries, Merchants, Two-Sided Platforms, Platform Competition.

JEL Classifications: L1, L2, L8

1 Introduction

Many intermediaries (particularly online) can choose their business model along a spectrum ranging from pure "merchant" that buys goods from suppliers and resells them to buyers (e.g. supermarkets, Amazon with respect to products it sells in its own name, Zappos), to pure "two-sided platform"

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that simply enables suppliers to sell directly to buyers (e.g. eBay, Amazon with respect to products sold by third-party sellers). The monopoly choice of the optimal intermediation model along this spectrum is driven by a series of tradeoffs, which are analyzed in Hagiu and Wright (2013, 2014).

In this paper, we investigate the equilibrium intermediation choices of two competing intermediaries by focusing on one specific tradeoff. Specifically, the merchant mode allows an intermediary to create higher willingness-to-pay for buyers (e.g. by providing additional services, a unified buyer interface, more convenient and faster delivery, certification, etc.), but at higher fixed costs, relative to the two-sided platform (TSP) mode. For example, it is significantly cheaper for Amazon to expand into a new product category by allowing third-party sellers to offer it on its site. On the other hand, by incurring the higher fixed cost necessary to build the capability of offering the new product category itself, Amazon could create higher value for buyers.

In our model, all consumers view a product offered in the merchant mode as being of higher quality than the same product offered in the TSP mode, but consumers have different views of the quality difference. Thus, the possibility for intermediaries to choose between two modes (merchant vs. TSP) introduces a new dimension along which they can vertically differentiate. At the same time, however, it also allows the intermediary that moves first to pre-empt (and sometimes fully foreclose) its competitor. Indeed, in our duopoly model one firm (the Stackelberg leader or the incumbent) chooses which products to offer under each one of the two modes before its rival (the Stackelberg follower or the entrant). After the intermediaries have chosen their products line-ups, they simultaneously choose prices for each of their products offered in each mode.\(^1\)

We analyze different variants of this duopoly model, which differ in two important dimensions. First, we distinguish between two alternative scenarios of consumer behavior: multihoming - i.e. consumers can buy products from both intermediaries - or singlehoming - i.e. each consumer must make all their product purchases from the same intermediary. Second, we investigate two possibilities for firms: in one case, firms are restricted to offering any given product in a single mode only; in the other case, firms can offer products in both modes. Consequently, there are four different duopoly competition scenarios. By comparing equilibrium outcomes between them, we derive several interesting results regarding the effect of the nature of competition on foreclosure, incumbency advantage, equilibrium profits and product line-ups.

First, we find that foreclosure is only possible when firms are able to offer products in both modes (M and TSP). In that case, the incumbent can introduce "fighting brands," i.e. products that are offered in the lower quality TSP mode solely in order to pre-empt the entrant from offering

\(^1\)Our modeling approach is related to Champsaur and Rochet (1989), where firms first choose their product lines and then choose prices. There are, however, two important differences. First, in Champsaur and Rochet (1989) consumers only have demand for a single product, whereas in our model consumers are interested in all available product categories. Second, in order to capture the notion of an incumbent, we allow one firm to choose its product mix before the other. In Champsaur and Rochet (1989) firms always move simultaneously.
the same products. The incumbent is then able to extract larger revenues by selling the merchant versions of the same products as a monopolist. A closely related result is that the two intermediary firms have opposite preferences regarding the ability to offer products in both modes: introducing that ability always increases the incumbent’s profits and decreases the entrant’s profits.

Second, we find that the two intermediary firms have different preferences regarding consumer singlehoming and multihoming. The entrant always prefers multihoming to singlehoming. The incumbent prefers multihoming when firms can offer each product in a single mode only, but prefers singlehoming when firms can offer products in both modes. The preference for multihoming is intuitive. Consumer multihoming tends to relax price competition: losing a consumer for one product does not preclude a firm from selling that same consumer a different product. In contrast, consumer singlehoming leads firms to engage in "winner-take-all" competition for each consumer’s business - this creates stronger incentives for firms to try to steal consumers from each other. The reason this intuition fails to apply to the incumbent when firms can offer products in both modes is once again related to the possibility of foreclosure. When foreclosure is possible, more intense price competition (singlehoming) makes it easier for the incumbent to keep the entrant out.

Third, a striking result is that in some cases the "quality" chosen by the incumbent in equilibrium is lower than the entrant’s (although the incumbent always makes higher profits than the entrant). At a high level, this result seems to stand in stark contrast with the classic work of Shaked and Sutton (1982): in a duopoly model with firms offering a single product each, they show that the incumbent always chooses a higher product quality than the entrant. The key difference is that in our model the notion of "quality" is not as straightforward: it is determined by the number of products sold in the merchant mode relative to products sold in the TSP mode (the former are perceived as higher quality by consumers). The scenario with the incumbent offering lower "quality" than the entrant occurs when consumers multihome and firms can offer each product in a single mode only. In that case, it is sometimes profitable for the incumbent to offer all products in the TSP mode and force the entrant to offer a limited number of products in the merchant mode, which overall yields less competition than if the incumbent offered some products in the merchant (higher quality) mode.

The rest of the paper is organized as follows. In the next section, we present the modelling set-up. In section 3, we briefly analyze the monopoly benchmark. In section 4 we analyze the two duopoly scenarios corresponding to consumer multihoming, while section 5 focuses on the two duopoly scenarios corresponding to consumer singlehoming. We compare equilibrium profits and product mixes across the four duopoly settings in section 6, before concluding in section 7.

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2 At a high level, the use of "fighting brands" is similar to the mechanism formalized by Johnson and Myatt (2003), from whom we also borrow the terminology.
2 Model

There is a continuum \([0, N]\) of product varieties, each of which is supplied by an industry of perfectly competitive suppliers, whose cost we normalize to 0. There is a continuum of measure 1 of consumers, each of whom is interested in all product varieties. The only way consumers can have access to products is through one or two intermediary firms. Each product can be offered by each intermediary in one of two modes. One mode is called the "merchant mode" or M mode in short. The other is called the "two-sided platform mode" or TSP mode in short. Each consumer has willingness to pay \(v\) for any of the \(N\) products acquired in the M mode. On the other hand, consumers differ in their willingness to pay for products acquired in the TSP mode. Specifically, consumer \(\lambda\) derives utility \(\lambda v\) from acquiring any of the \(N\) products from an intermediary in the TSP mode, where \(\lambda\) is uniformly distributed on \([\alpha, 1]\) with density \(\frac{1}{1-\alpha}\). We assume \(\alpha \geq 1/2\). An implication of this assumption is that a monopoly intermediary that only offers a product in the TSP mode will always price it at \(v\) so that all consumers buy the product. Also, there will be no scope for price discrimination by offering both versions of the good: a firm always prefers that consumers purchase the good in the M mode.

Our formalization of the difference between the two modes is meant to capture an important aspect of the difference between merchant-type intermediaries and two-sided platform intermediaries (see Hagiu and Wright 2013 for an overview). Specifically, merchants (e.g. supermarkets, Amazon with respect to products it sells in its own name, Zappos) are able to create higher buyer willingness-to-pay by taking control over products from suppliers and providing additional services, e.g. unified and higher-quality buyer interface, more convenient and faster delivery, certification, etc.

In contrast, two-sided platforms (e.g. eBay, Amazon with respect to products sold by third-party suppliers) get less involved in supplier-buyer transactions, which typically leads to a lower-quality buyer experience overall as individual suppliers are unable to match the level of additional services offered by a merchant.

Thus, in our model consumers are heterogenous in how much they care about the superior service implied by the M mode. Higher \(\lambda\) consumers are those who care less about that difference. In turn, we assume that the superior service of the M mode is achieved by incurring higher costs per product variety relative to the TSP mode. Specifically, we assume that an intermediary offering \(N_M\) products in the M mode and \(N_P\) products in the TSP mode incurs total fixed costs equal to \(\frac{\gamma N_M^3}{2} + \varepsilon N_P\), where \(\varepsilon > 0\) is positive but small. Thus, there are diseconomies of scope in offering product varieties in the M mode due to the fact that scarce resources of the firm must be allocated across products categories in order to provide the necessary service. On the other hand, firms can offer products in the TSP mode using a constant returns to scale technology (no additional services are necessary). Marginal costs are normalized to 0 for both modes.
Finally, we assume that regardless of the mode chosen for a given product, the price charged to buyers is always chosen by the intermediary. Thus, in our model the difference between the M mode and the TSP mode is entirely captured by differences in cost structure and consumer willingness-to-pay. In particular, we abstract away from tradeoffs between the two intermediation modes that are driven by differences in the allocation of control rights - these are studied in Hagiu and Wright (2014). Our assumption is not unrealistic: as pointed out by Hagiu and Wright (2014), prices charged to buyers are most likely to be determined through contracts between suppliers and the intermediary. Furthermore, intermediaries hold all the bargaining power: recall we have assumed perfectly competitive suppliers for each product variety.

There are two intermediaries: firm 1 and firm 2. The timing of the competition game we consider throughout the paper is as follows:

1. Firm 1 chooses which products to offer in each mode (M or TSP) and incurs the corresponding fixed costs.

2. Firm 2 chooses which products to offer in each mode (having observed firm 1’s product line-up choices) and incurs the corresponding fixed costs.

3. Firms simultaneously choose prices for their products offered in each mode (possibly different prices for a product offered in both modes). Consumers make their purchasing decisions.

Consistent with this timing, we will sometimes refer to firm 1 as the "Stackelberg leader" or "incumbent" and to firm 2 as the "Stackelberg follower" or "entrant".

There are two reasons for our choice of Stackelberg timing for product line choices instead of simultaneous timing. First, one of the main objectives of our analysis is to determine how the incumbent’s early mover advantage is affected by the mode of competition. In particular, we wish to determine conditions under which firm 1 can foreclose firm 2. Second, from a methodological point of view, if firms make their product line choices simultaneously and before choosing prices (also simultaneously) then in general there may not be a pure strategy equilibrium in the product line selection stage.

Our assumption that firms choose product line-ups (or "quality") first and then prices, is reminiscent of the model in Kreps and Sheinkman (1983), where firms choose capacities and then prices. Indeed, we view product line-ups as a long-run variable: it takes longer for firms to change products offered in each mode, whereas prices can be changed more easily.

We will examine a series of models that differ in two key dimensions. One is whether firms can offer their products in both modes (M and TSP) or in one mode only. The second is whether consumers can make purchases from both firms - i.e. multihome - or must make all of their purchases
from one firm only - i.e. singlehome. We thus examine four different competition scenarios in order to test how the incumbency advantage changes as the firms’ strategy space and consumers’ ability to purchase from multiple firms change.

3 Monopoly

Before turning to the competition settings, we first determine the optimal choices for a monopoly intermediary in terms of prices and product line choices. The monopoly case will serve as a useful benchmark when we turn to analyzing the four competition scenarios.

First, it is never optimal for the monopolist to offer the same product in both modes. Indeed, if both modes are available for a given product, it is always a dominant strategy for the monopolist to only sell the M version of the product at a price equal to \( v \) (slightly below), which extracts the entire consumer surplus. Selling both the M and the TSP version at any combination of prices will necessarily result in revenues less than \( v \). Thus, it would not make sense to offer the TSP mode (which incurs a positive fixed cost) for any product also carried in the M mode.

Second, the monopolist always covers the entire product range. Any product can be offered in the TSP mode only and generate positive net profits \( \alpha v - \varepsilon \) (recall that the fixed cost \( \varepsilon \) of selling products in the TSP mode is small).

Suppose then without loss of generality\(^3\) that the monopolist offers products \([0, N_M]\) in the M mode and products \([N_M, N]\) in the TSP mode. The firm optimally charges \( v \) for M products and \( av \) for TSP products. The resulting monopoly profits are then

\[
\Pi(N_M) = vN_M - \gamma \frac{N_M^2}{2} + \alpha v (N - N_M) - \varepsilon (N - N_M).
\]

The optimal number of products offered in the M mode is

\[
N_M^* = \min \left( \frac{v(1 - \alpha) + \varepsilon}{\gamma}, N \right),
\]

leading to optimal profits:

\[
\Pi_M = \begin{cases} 
  vN - \gamma \frac{N^2}{2} & \text{if } \gamma N \leq v(1 - \alpha) + \varepsilon \\
  \alpha vN + \frac{(v(1 - \alpha) + \varepsilon)^2}{2\gamma} - \varepsilon N & \text{if } \gamma N \geq v(1 - \alpha) + \varepsilon
\end{cases}
\]

Thus, if \( \gamma \) is sufficiently small then the firm is a pure merchant, i.e. offers all products in the M mode. When \( \gamma \) passes a certain threshold, the optimal monopoly intermediation model is a hybrid,

\(^3\)All products are identical so their ordering is immaterial.
with the number of products offered in the M mode decreasing in $\gamma$.

4 Competition with multihoming consumers

Let us now turn to the first set of duopoly scenarios. Throughout this section, we assume that consumers can buy products from both firms (i.e. they multihome) and firms cannot condition their prices on where consumers have made other purchases. Consumer multihoming is likely to prevail in markets in which search costs are low or consumers do not perceive a large inconvenience from browsing multiple intermediaries. Consumer multihoming implies that in Stage 3 (price setting) firms compete for consumers on a product-by-product basis.

4.1 Multihoming with one mode per product

In this subsection we assume that firms are restricted to offer each product in one mode only - either TSP or M (or not at all). This may be because an intermediary having chosen the TSP mode does not want to compete with its suppliers by also offering products in the M mode. Or because the intermediary derives exogenous benefits (not modeled here) from committing to a "pure mode" strategy.\footnote{One such reason might be fending off antitrust investigation.}

The following lemma is helpful in the derivation of the full equilibrium.

**Lemma 1** In any equilibrium with consumers multihoming and firms restricted to one mode per product, every product $n \in [0, N]$ is either offered by firm 1 only in the TSP mode or by both firms, one in the TSP mode and the other in the M mode. Every product that belongs to the first category yields revenues $\alpha v$ for firm 1 in stage 3. Every product that belongs to the second category yields revenues $(1 - \alpha)v/9$ for the firm that offers it in the TSP mode and $4(1 - \alpha)v/9$ for the firm that offers it in the M mode.

**Proof.** If a product $n \in [0, N]$ is not offered by either firm, one of them could deviate (in stage 1 for firm 1 or stage 2 for firm 2) by offering that product in the TSP mode. This would bring in additional profits $\alpha v - \varepsilon > 0$. Furthermore, if both firms offered product $n$ in the same mode, they would derive negative profits from that product (counting fixed costs), which cannot be part of an equilibrium.

Finally, suppose that firm $i$ offers product $n$ in the M mode at price $p_M$, while firm $j$ offers it as a TSP at price $p_P$, where $\{i, j\} = \{1, 2\}$. Consumer $\lambda$ buys the product from firm $i$ if and only if $v - p_M \geq \lambda v - p_P$, which implies that firm $i$’s stage 3 revenues from product $n$ are $\frac{p_M}{1 - \alpha} \left[\frac{v + p_P - p_M}{v} - \alpha\right]$, while firm $j$’s stage 3 revenues from that product are $\frac{p_P}{1 - \alpha} \left[1 - \frac{v + p_P - p_M}{v}\right]$. The stage 3 equilibrium is then $p_M = 2(1 - \alpha)v/3$ and $p_P = (1 - \alpha)v/3$, yielding revenues $4(1 - \alpha)v/9$ for firm $i$ and $(1 - \alpha)v/9$ for firm $j$. 


firm $j$. Thus, if firm 1 offers product $n$ in the M mode in equilibrium, it must be that firm 2 offers the same product in the TSP mode. Note however that the converse is not necessarily true: if firm 1 offers product $n$ in the TSP mode, then in equilibrium firm 2 either offers it in the M mode or not at all (if the fixed cost of the products that firm 2 already offers in the M mode is too high). Finally, note that it is a dominated strategy for firm 1 to not offer all products: it can guarantee positive profits for any product by offering it in the TSP mode.}

Using Lemma 1, we can derive the equilibrium for the entire game (the proof is in the appendix):

**Proposition 1** When consumers multihome and firms are restricted to at most one mode per product, the equilibrium product line-ups and profits are as follows:

- If $\gamma N \leq \frac{(1-\alpha)v}{3} + \varepsilon$ then firm 1 offers all products $[0, N]$ in the M mode, while firm 2 offers all products in the TSP mode. Profits are:

  $$\Pi_1 = \frac{4(1-\alpha)vN}{9} - \frac{\gamma N^2}{2}$$

  $$\Pi_2 = \frac{(1-\alpha)vN}{9} - \varepsilon N$$

- If $\frac{(1-\alpha)v}{3} + \varepsilon \leq \gamma N \leq \frac{(71\alpha+1)(1-\alpha)v}{18(10\alpha-1)} + \frac{\varepsilon(6v(1-\alpha)+9\varepsilon)}{2v(10\alpha-1)}$ then firm 1 offers products $\left[0, \frac{(1-\alpha)v}{3\gamma} + \frac{\varepsilon}{\gamma}\right]$ in the M mode and products $\left[\frac{(1-\alpha)v}{3\gamma} + \frac{\varepsilon}{\gamma}, N\right]$ in the TSP mode, while firm 2 offers products $\left[0, \frac{(1-\alpha)v}{3\gamma} + \frac{\varepsilon}{\gamma}\right]$ in the TSP mode and products $\left[\frac{(1-\alpha)v}{3\gamma} + \frac{\varepsilon}{\gamma}, N\right]$ in the M mode. Profits are:

  $$\Pi_1 = \frac{(1-\alpha)vN}{9} + \frac{(1-\alpha)^2 v^2}{18\gamma} - \varepsilon \left(N - \frac{(1-\alpha)v}{3\gamma} - \frac{\varepsilon}{2\gamma}\right)$$

  $$\Pi_2 = \frac{7(1-\alpha)vN}{9} - \frac{(1-\alpha)^2 v^2}{6\gamma} - \frac{\gamma N^2}{2\gamma} - \varepsilon \left(\frac{(1-\alpha)v}{\gamma} - N + \frac{3\varepsilon}{2\gamma}\right)$$

- If $\gamma N \geq \frac{(71\alpha+1)(1-\alpha)v}{18(10\alpha-1)} + \frac{\varepsilon(6v(1-\alpha)+9\varepsilon)}{2v(10\alpha-1)}$ then firm 1 offers all products $[0, N]$ in the TSP mode, while firm 2 only offers products $\left[0, \frac{4(1-\alpha)v}{9\gamma}\right]$, all in the M mode. Profits are:

  $$\Pi_1 = \alpha vN - \frac{4(10\alpha - 1)(1-\alpha)v^2}{81\gamma} - \varepsilon N$$

  $$\Pi_2 = \frac{8(1-\alpha)^2 v^2}{81\gamma}$$

Several observations are in order. First, consider the third parameter region defined in the proposition above, where $\gamma N \geq \frac{(71\alpha+1)(1-\alpha)v}{18(10\alpha-1)} + \frac{\varepsilon(6v(1-\alpha)+9\varepsilon)}{2v(10\alpha-1)}$. Here, the Stackelberg leader firm 1
does not offer any products in the M mode; instead, it chooses to only offer lower quality products as a pure TSP. This is a counter-intuitive result, especially since the marginal cost for firm 1 to add one product in the M mode is 0. The logic of the result is as follows. If firm 1 chooses to deviate by offering a product in the M mode, then firm 2’s best response will be to offer that same product in the TSP mode, so firm 1 makes profits \( 4(1 - \alpha) v/9 \) from selling that product. If instead firm 1 offers the product as a TSP then firm 2’s best response is not to offer the product at all in this parameter region. This is because firm 2 is already offering too many products in the M mode, so the fixed cost of selling an additional M product is high enough that firm 2 prefers not to compete on it. In this scenario, firm 1 is a monopolist TSP on the product in question, so it makes profits \( \alpha v \) from selling it. For firm 1 to prefer this outcome, \( \alpha v \) must exceed \( 4(1 - \alpha) v/9 \).

Furthermore, firm 2 must prefer not to offer the product at all when firm 1 offers it as a TSP. In other words, the revenue from offering the product in the M mode when firm 1 offers it in the TSP mode, \( 4(1 - \alpha) v/9 \), must be less than or equal to the marginal cost of offering an additional product in the M mode, \( \gamma \times \frac{4(1 - \alpha) v}{9\gamma} = 4(1 - \alpha) v/9 \). The desired inequality clearly holds.

Second, as \( \gamma \) increases, firm 1 offers fewer products in the M mode: it goes from \( N \) to \( \frac{(1 - \alpha) v}{3\gamma} \) and then down to 0. In particular, at \( \gamma = \frac{(7\alpha + 1)(1 - \alpha) v}{18(10\alpha - 1)N} + \frac{\varepsilon(6\alpha(1 - \alpha) + 9\varepsilon)}{2eN(10\alpha - 1)} \), firm 1 goes from offering \( N_1 = \frac{(1 - \alpha) v}{3\gamma} + \frac{\varepsilon}{\gamma} > 0 \) products in the M mode to offering no M products. At that point, the trade-off mentioned above kicks in: firm 1 drops the M mode entirely and offers all products in the TSP mode, knowing that M mode fixed costs are too high for firm 2 to offer all products in the M mode. Thus, there is a positive measure of products that firm 1 prefers to offer as a TSP monopolist, rather than offer them in the M mode and face competition.

Third, firm 1’s profits are decreasing in \( \gamma \) in the first two regions and increasing in \( \gamma \) on the third region. This is simply because in the first two regions, firm 1 offers some positive measure of products in the M mode, hence the higher the fixed cost of M products, the lower firm 1’s profits. On the other hand, in region 3, firm 1 does not offer any products in the M mode, but firm 2 does, so that a higher \( \gamma \) hurts firm 2 by inducing it to lower the number of M products it offers. This in turn benefits firm 1, who becomes a monopolist TSP for more products.

### 4.2 Multihoming with multiple modes per product

Suppose now that firms can offer each product in both the M mode and the TSP mode (they can still offer products in a single mode as well). Again, we start with a lemma.

**Lemma 2** In any equilibrium with consumers multihoming and firms allowed to offer products in both modes, firm 1 always offers all \( N \) products in the TSP mode and firm 2 offers no products in the TSP mode.
Proof. First, suppose firm 1 offers products \([0, N_1]\) in the M mode only. Then firm 2's best response is to offer products \([0, N_1]\) in the TSP mode, which yields revenues \(4v (1 - \alpha) N_1 / 9\) for firm 1 on products \([0, N_1]\). But firm 1 could do strictly better by also offering products \([0, N_1]\) in the TSP mode. Indeed, this would ensure that firm 2 makes negative profits on products \([0, N_1]\) no matter in which mode it offers them, so firm 2's best response is to not offer products \([0, N_1]\) at all. This means that firm 1 would be able to derive revenues \(vN_1\) from products \([0, N_1]\), in exchange for an increase in fixed costs equal to \(\varepsilon N_1\) - clearly a profitable deviation if \(N_1 > 0\).

Second, for any given product, firm 1 makes higher profits by offering it in the TSP mode than by not offering it at all. Indeed, if firm 1 offers a product in the TSP mode then firm 2 will either offer the product in the M mode or not at all (offering the product in the TSP mode is a dominated strategy due to Bertrand competition). This implies that by offering a product in the TSP mode, firm 1 can make profits of at least \((1 - \alpha) v / 9 - \varepsilon > 0\).

Thus, firm 1 always offers all products in the TSP mode. Consequently, firm 2 would lose money on any product offered in the TSP mode. ■

In accordance with the lemma, suppose (without loss of generality) that in equilibrium firm 1 offers products \([0, N_1]\) in both modes and products \([N_1, N]\) in the TSP mode only, while firm 2 only offers products \([N_1, N_2]\) in the M mode, where \(N_2 \leq N\). In a sense, the products \([0, N_1]\) offered by firm 1 in the TSP mode serve as "fighting brands," i.e. a credible commitment which dissuades firm 2 from offering them and thereby allows firm 1 to extract maximum M mode profits from them.

The following proposition (proven in the appendix) completes the characterization of the equilibrium.

Proposition 2 When consumers multihome and firms are allowed to offer products in both modes, firms' product choices and equilibrium profits are as follows:

- If \(\gamma N \leq (1 - \alpha) v \left( 1 + \sqrt{\frac{80\alpha - 8}{81(1 - \alpha)}} \right)\) then firm 1 offers all products \([0, N]\) in both modes and firm 2 is foreclosed (it does not offer any product). Profits are:
  \[
  \Pi_1 = vN - \frac{\gamma N^2}{2} - \varepsilon N \quad \text{and} \quad \Pi_2 = 0
  \]

- If \(\gamma N \geq (1 - \alpha) v \left( 1 + \sqrt{\frac{80\alpha - 8}{81(1 - \alpha)}} \right)\) then firm 1 offers products \([0, \frac{(1 - \alpha) v}{\gamma}]\) in both modes and products \(\left[\frac{(1 - \alpha) v}{\gamma}, N\right]\) in the TSP mode only, while firm 2 offers products \(\left[\frac{(1 - \alpha) v}{\gamma}, \frac{13(1 - \alpha) v}{9\gamma}\right]\) in the M mode only. Profits are:
  \[
  \Pi_1 = \alpha vN + \frac{(1 - \alpha)(89 - 161\alpha)v^2}{162\gamma} - \varepsilon N \quad \text{and} \quad \Pi_2 = \frac{8(1 - \alpha)^2 v^2}{81\gamma}
  \]
Thus, for low $\gamma$, firm 1 finds it profitable to fully foreclose firm 2 by offering all products in both modes. This is simply because a lower $\gamma$ reduces firm 1’s cost of offering products in the M mode (it already offers everything in the TSP mode) and any product offered by firm 1 in both modes effectively forecloses firm 2 from offering it. As $\gamma$ increases, firm 1 no longer finds it profitable to offer all products in the M mode, which opens the door for firm 2 to enter by offering in the M mode some of the products that firm 1 only supplies as a TSP.

5 Competition with singlehoming consumers

In this section we analyze the duopoly scenarios in which consumers can only visit one firm for all of their product purchases, i.e. they singlehome. Consumers are most likely to singlehome when it is costly or inconvenient to browse multiple intermediaries for the products they are looking for. Furthermore, although not captured in our model, consumers may receive benefits from buying more products from the same firm.

Just like in the previous section, we examine two cases: (i) each firm can offer a given product in one mode only (M or TSP); (ii) firms can offer products in both modes.

5.1 Singlehoming with one mode per product

Denote by $N_i^M$ the number of products that firm $i$ offers in the M mode and by $N_i^P$ the number of products that firm $i$ offers in the TSP mode, for $i = 1, 2$. When products can only be offered in a single mode, $N_i^M + N_i^P \leq N$. The next lemma (proven in the appendix) determines the Stage 3 equilibrium as a function of product line choices $(N_i^M, N_i^P), i = 1, 2$.

Lemma 3 Suppose $N_2^P > N_1^P$. Then there exists $y \in [\alpha, 1]$ such that in the Stage 3 equilibrium all consumers $[\alpha, y]$ go to firm 1 and all consumers $[y, 1]$ go to firm 2. There are three cases:

- If $(2\alpha - 1) (N_2^P - N_1^P) \leq N_1^M - N_2^M \leq (2 - \alpha) (N_2^P - N_1^P)$ then Stage 3 equilibrium revenues are

  $\Pi_1 = \frac{v \left[ (N_1^M - N_2^M) - (2\alpha - 1) (N_2^P - N_1^P) \right]^2}{9(1 - \alpha)(N_2^P - N_1^P)}$

  $\Pi_2 = \frac{v \left[ (N_2^M - N_1^M) + (2 - \alpha) (N_2^P - N_1^P) \right]^2}{9(1 - \alpha)(N_2^P - N_1^P)}$

- If $(2\alpha - 1) (N_2^P - N_1^P) \geq N_1^M - N_2^M$ then firm 1 makes 0 revenues in Stage 3.

- If $N_1^M - N_2^M \geq (2 - \alpha) (N_2^P - N_1^P)$ then firm 2 makes 0 revenues in Stage 3.
Results for the case $N_2^P < N_1^P$ are obtained by symmetry.

A key result in the proof of the lemma is that in any Stage 3 equilibrium only the total prices charged by each firm matter. This is intuitive: given that each consumer has the same valuation for all products and she makes all of her product purchases at the same firm (single-homing), consumers make their firm choice based on the total price they expect to pay at each firm.

Note that $N_2^P > N_1^P$ implies $N_1^M > N_2^M$ if both firms are to make positive revenues in Stage 3 (interior equilibrium). Using Lemma 3, we can derive the equilibrium product line choices and profits for the entire game.

**Proposition 3** When consumers singlehome and firms are restricted to at most one mode per product, there exists a unique equilibrium, determined as follows:

- If $\gamma N \leq \frac{4v(1-\alpha)}{9} + \varepsilon$ then firm 1 chooses $N_1^{M*} = N$ and $N_1^{P*} = 0$, while firm 2 chooses $N_2^{M*} = 0$ and $N_2^{P*} = N$. Equilibrium profits are
  \[
  \Pi_1^* = \frac{4v (1 - \alpha) N}{9} - \gamma \frac{N^2}{2} \quad \text{and} \quad \Pi_2^* = \frac{v (1 - \alpha) N}{9} - \varepsilon N
  \]

- If $\gamma N \geq \frac{4v(1-\alpha)}{9} + \varepsilon$ then firm 1 chooses $N_1^{M*} = \frac{4v(1-\alpha)}{9\gamma} + \frac{\varepsilon}{\gamma}$ and $N_1^{P*} = N - N_1^{M*}$, while firm 2 chooses $N_2^{M*} = 0$ and $N_2^{P*} = N$. Equilibrium profits are
  \[
  \Pi_1^* = \frac{8v^2 (1 - \alpha)^2}{81\gamma} - \varepsilon \left[ N - \frac{4v (1 - \alpha)}{9\gamma} - \frac{\varepsilon}{\gamma} \right] \quad \text{and} \quad \Pi_2^* = \frac{4v^2 (1 - \alpha)^2}{81\gamma} - \varepsilon N
  \]

Thus, when the fixed cost of M products $\gamma$ is small, we essentially obtain maximum differentiation between the two firms: firm 1 offers all products in the M mode (higher "quality"), while firm 2 offers all products in the TSP mode (lower "quality"). This is the exact same outcome as with multihoming consumers, single mode per product and low $\gamma$ (first region in Proposition 1). As $\gamma$ increases past $\frac{4v(1-\alpha)}{9N} + \frac{\varepsilon}{N}$, the two firms remain differentiated but the "quality" differential is decreasing because firm 1 starts substituting M products for TSP products in order to lower its fixed costs, whereas firm 2 keeps offering all products in the TSP mode.

Let us now focus on the parameter range $\frac{4v(1-\alpha)}{9} + \varepsilon \leq \gamma N \leq \frac{(7\alpha+1)(1-\alpha)v}{18(10\alpha-1)} + \frac{\varepsilon(6v(1-\alpha)+9\varepsilon)}{2v(10\alpha-1)}$, so that we are in the second region of Proposition 3 (single-homing, single mode) and the second region of Proposition 1 (multihoming, single mode). Firm 1 behaves very similarly in both cases: it offers some products in the M mode and the rest in the TSP mode, with the number of M products declining in $\gamma$. Firm 2 behaves quite differently: it covers the entire product space in both cases,
but under multihoming (Proposition 1) it offers some products in the M mode, whereas under singlehoming (Proposition 3) it only offers TSP products. This difference in behavior by firm 2 can be explained as follows: under multihoming, firm 2 aims to differentiate product by product, whereas under singlehoming, competition is in terms of total utility (cf. comment after Lemma 3), so differentiation is also achieved in terms of total utility. Thus, in order to differentiate under singlehoming, firm 2 chooses the lowest possible total "quality" conditional on offering all products (as the first mover, firm 1 never allows firm 2 to be the higher "quality" firm and thereby derive larger profits).

Furthermore, note that Proposition 3 does not exhibit the discontinuity in the number of M products offered by firm 1, which occurs at \[ \gamma N = \frac{(71\alpha+1)(1-\alpha)v}{18(10\alpha-1)} + \frac{\varepsilon(6\alpha(1-\alpha)+9\varepsilon)}{2\varepsilon(10\alpha-1)} \] in Proposition 1. Indeed, under singlehoming firm 2 never considers offering in the M mode products that firm 1 only offers in the TSP mode. Again, this is because firms compete in total utility only, so firm 2 always achieves maximum differentiation (conditional on being the lower quality firm) by offering all products in the TSP mode.

Finally, note that as \( \gamma \) grows very large, we approach Bertrand competition, with firm 2 offering all products in the TSP mode and firm 1 offering almost all products in the TSP mode.\(^5\)

### 5.2 Singlehoming with multiple modes per product

Suppose now that firms can offer each product in both the M mode and the TSP mode (they can still offer products in a single mode as well). The following proposition (proven in the appendix) establishes that in this scenario the incumbent (firm 1) always forecloses the entrant (firm 2).

**Proposition 4** If consumers singlehome and firms can offer products in both modes then firm 1 always forecloses firm 2 by offering all products in the TSP mode and \( N_{1M^*} = \min \left( \frac{(1-\alpha)v}{\gamma}, N \right) \) products in the M mode. Profits are:

\[
\Pi_1^* = \begin{cases} 
 vN - \frac{\gamma N^2}{2} - \varepsilon N & \text{if } \gamma N \leq v(1-\alpha) \\
 \alpha vN + \frac{v^2(1-\alpha)^2}{2\gamma} - \varepsilon N & \text{if } \gamma N \geq v(1-\alpha)
\end{cases}
\]

\[
\Pi_2^* = 0
\]

Thus, we obtain the rather striking result that firm 1 can foreclose entry by firm 2 simply by adding the TSP mode to all products offered in the M mode in the monopoly product line-up.\(^6\)

---

5. The only reason we do not get all the way to perfect Bertrand competition is \( \varepsilon > 0 \).

6. \( N_{1M^*} \) is very similar to the monopoly choice of products in the M mode. The difference is that here firm 1 always offers all products in the TSP mode.
Note that these TSP products are never purchased in equilibrium: as a de facto monopolist in Stage 3, firm 1 prefers selling the M version only.

Thus, just like in the case when consumers multihome, foreclosure is only possible when firms can offer products in both modes. The difference is that now foreclosure occurs all the time, whereas it only occurred for low enough $\gamma$ when consumers multihomed (cf. Proposition 2). This is understood by noting that singlehoming generally creates more intense price competition than multihoming. Under multihoming, in order to foreclose firm 2, firm 1 had to offer all products in both modes, which becomes too costly for large $\gamma$. In contrast, under singlehoming, firm 1 already makes it very difficult for firm 2 to enter by offering all products in the TSP mode. By adding the monopoly number of M products, firm 1 can then ensure that firm 2 cannot hope to make positive profits by entering, due to intense price competition.

6 Comparison of equilibria

We now turn to the comparison of firm profits and product line choices across the four different competition scenarios that we have studied.

6.1 Equilibrium profits

First, we compare the profits of both firms using Propositions 1, 2, 3, and 4. We obtain the following results (all comparisons involve straightforward calculations in $\alpha$, $\gamma N$ and $v$).

Proposition 5  i) Firm 1 (Stackelberg leader) always makes higher profits than firm 2 (Stackelberg follower).
ii) Firm 1 makes higher profits, whereas Firm 2 makes lower profits when firms can offer products in multiple modes relative to the case when they must offer each product in a single mode.
iii) If firms can offer each product in a single mode only then both firms always makes higher profits when consumers multihome relative to the case when they singlehome.
iv) If firms can offer each product in multiple modes then firm 1 makes higher profits when consumers singlehome, whereas firm 2 makes higher profits when consumers multihome.

Regarding result (i), recall than in a traditional duopoly context, the Stackelberg leader makes larger profits than the Stackelberg follower in a Cournot setting, but lower profits in a Bertrand setting. Our modeling set-up with firms first choosing product line-ups and then prices is similar in spirit to the Kreps and Sheinkman (1983) set-up, in which firms choose capacities and then prices - the outcome in that model is the Cournot equilibrium. In other words, capacity investments
(quantity in Kreps and Sheinkman 1983 and product "quality" in our model) serve as credible commitment devices that provide an advantage to the Stackelberg leader.

Result (ii) is understood by noting that the ability to offer products in multiple modes allows the leader, firm 1, to establish the TSP modes of some products as "fighting brands" that serve solely to reduce competition by pre-empting (and sometime fully foreclosing entry of) firm 2. The follower (firm 2) naturally has the opposite preference. A similar logic applies to result (iv). Conditional on offering products in multiple modes, firm 1’s ability to foreclose firm 2 is enhanced when moving from multihoming to singlehoming consumers, because price competition is more intense under singlehoming. As a result, the two firms have once again opposite preferences: firm 1 prefers singlehoming, whereas firm 2 prefers multihoming.

Result (iii) says that the two firms have the same preference over consumer homing behavior when they are restricted to offer each product in a single mode: both firms make higher profits under multihoming than under singlehoming. This is because multihoming relaxes price competition between the two firms. Indeed, when consumers multihome, competition occurs product by product: losing a consumer for one product does not preclude a firm from selling that same consumer a different product. On the other hand, if consumers singlehome then firms engage in "winner-take-all" competition for each consumer’s business - this creates stronger incentives for firms to try to steal customers from each other. Note the contrast with result (iv): with a single mode per product, foreclosure is not possible so firm 1 also prefers less intense price competition.

6.2 Equilibrium average product "quality"

We now turn to the comparison of equilibrium "quality" choices, i.e. the total consumer utilities created by the equilibrium product mixes of each firm. Specifically, denote by $q_{m1}^i(\lambda)$ the total willingness-to-pay of consumer $\lambda \in [\alpha, 1]$ for firm $i$'s products in the multihoming equilibrium with one mode per product, where $i = L, F$ ($L$ stands for the Stackelberg leader, i.e. firm 1; $F$ stands for the Stackelberg follower, i.e. firm 2). Similarly, define $q_{m2}^i(\lambda)$ for the multihoming equilibrium with multiple modes, $q_{s1}^i(\lambda)$ for the singlehoming equilibrium with one mode, and $q_{s2}^i(\lambda)$ for the singlehoming equilibrium with multiple modes. For simplicity, we refer to these willingnesses-to-pay as "qualities" chosen by the two firms.

Comparing quality choices across the four duopoly equilibria, we obtain:

**Proposition 6** i) All consumers view the equilibrium quality of firm 1 (Stackelberg leader) as higher than that of firm 2 (Stackelberg follower), except when consumers multihome, firms can offer each product in a single mode only and $\frac{4(1-\alpha)\gamma}{9\alpha} \geq \gamma N \geq \frac{(7\alpha + 1)(1-\alpha)\gamma}{18(10\alpha - 1)}$. In that case, consumers $\lambda < \frac{4(1-\alpha)\gamma}{9\gamma N}$ view firm 2’s quality as higher than firm 1’s.

ii) For the Stackelberg leader, $q_{m1}^L(\lambda) \leq q_{s1}^L(\lambda) \leq q_{s2}^L(\lambda) \leq q_{m2}^L(\lambda)$ for all $\lambda$ and all parameter
values.

iii) For the Stackelberg follower, 0 = q^F_2 (\lambda) \leq q^F_1 (\lambda) \leq \min \{q^{m1}_F (\lambda), q^{m2}_F (\lambda)\} for all \lambda and parameter values.

Result (i) indicates that in most cases, firm 1’s product mix if of higher quality than firm 2’s for all consumers. This is in line with the Shaked and Sutton (1982) result that the first mover in a duopoly with vertical differentiation achieves higher profits by always choosing a higher quality. However, our model produces an interesting exception to this broad principle, which occurs when consumers multihome, firms can offer each product in a single mode only and γN ∈ \((\frac{7(1-\alpha)(1-\alpha)^{v2}}{18(1-\alpha-1)}, \frac{4(1-\alpha)^{v2}}{9\alpha})\).

Recall from our discussion of Proposition 1 that in this parameter range firm 1 only offers TSP products because γ is large enough that firm 2 does not react by covering the entire product range with products in the M mode. Thus, firm 1 is actually a monopolist on some products, which yields higher profits than competing as the higher quality producer for the same products. This scenario provides an important difference between our set-up and traditional vertical differentiation models like Shaked and Sutton (1982). Specifically, the scenario in which the Stackelberg leader offers lower "quality" than the Stackelberg follower can only occur when firms compete product by product (consumer multihoming) and firms can choose different qualities for each product. In contrast, the Shaked and Sutton (1982) model is closest to the scenario with singlehoming and single mode per product in our model. A final important difference between our model and the Shaked and Sutton framework is that higher quality is more costly in our model, while in their model it is costless to increase the quality of a firm’s offering.

Result (ii) indicates that the quality chosen by firm 1 is higher when firms are allowed to offer each product in both modes relative to the case when they are restricted to one mode per product - this holds under both singlehoming and multihoming. The reason is as follows. Firm 1 uses multiple modes to engage in pre-emption (and possibly complete foreclosure) of firm 2: specifically, firm 1 can use the TSP mode as a "fighting brand" to pre-empt firm 2 from offering a given product, which then liberates firm 1 to extract maximum profits with the M version of that product, which in turn means firm 1 is able to offer more products in the M mode, leading to higher overall quality. Just like profits in Proposition 5, the respective qualities chosen by firm 2 behave in the opposite way: result (iii) says that firm 2 chooses higher quality when firms are restricted to offer each product in a single mode (both under singlehoming and multihoming). The reason is similar: since firm 1 increases quality by offering more M products (when going from single mode to multiple modes), firm 2 is forced to reduce the number of M products it offers in favor of TSP products, in order to differentiate.

Another interesting part of result (ii) is that when firms can offer each product in one mode only, firm 1 chooses higher quality under singlehoming relative to multihoming, although single-
homing yields lower profits (cf. result (iii) in Proposition 5). This is understood by recalling that foreclosure is not possible when only a single mode is allowed per product. In this case, firm 1 must seek to maximize vertical differentiation from firm 2, which is a more pressing imperative under singlehoming, because this is when price competition is more intense. This result is reversed when firms can offer each product in multiple modes. In that scenario, firm 1 does not have to seek vertical differentiation under singlehoming because it can completely foreclose firm 2. As a result, firm 1’s chosen quality is higher under multihoming.

Finally, note that, with one possible exception, the ranking of qualities chosen by firm 2 is the reverse of that of qualities chosen by firm 1. This plays into the idea that the firms’ qualities are strategic substitutes, just like capacities are strategic substitutes in the Kreps and Scheinkman (1983) model.

7 Conclusion

We have analyzed a duopoly model with competing intermediaries choosing to offer products under one or two modes - merchant or two-sided platform. We have shown that the ability to offer products under multiple modes leads to the possibility of foreclosure. We have also derived several results regarding equilibrium intermediary profits and product mixes that stand in contrast with the traditional literature on vertically differentiated duopolies. Our analysis can be extended in a number of ways. First, one could introduce strategic suppliers that maintain decision rights in the TSP mode. This would enrich the factors that drive the intermediaries to choose one mode or the other. Second, one could introduce demand differences across products, i.e. some products having a larger and/or more elastic demand than others. Again, this would introduce additional factors that drive equilibrium choices of intermediation modes.

References


8 Appendix

8.1 Proof of Proposition 1

From Lemma 1, we know that firm 1 covers the entire range of products categories \([0, N]\). Thus, in any equilibrium firm 1 offers products \([0, N_1]\) in the M mode and products \([N_1, N]\) in the TSP mode (without loss of generality). Lemma 1 then implies that firm 2 necessarily offers products \([0, N_1]\) as a TSP and no other products as a TSP. Thus, assume without loss of generality that firm 2 offers products \([N_1, N_2]\) in the M mode and does not offer products \([N_2, N]\) at all, where \(N_1 \leq N_2 \leq N\).

Firm profits are therefore:

\[
\Pi_1 = \frac{4 (1 - \alpha) v N_1}{9} - \frac{\gamma N_1^2}{2} + \frac{(1 - \alpha) v (N_2 - N_1)}{9} + \alpha v (N - N_2) - \varepsilon (N - N_1)
\]

\[
\Pi_2 = \frac{(1 - \alpha) v N_1}{9} - \varepsilon N_1 + \frac{4 (1 - \alpha) v (N_2 - N_1)}{9} - \frac{\gamma (N_2 - N_1)^2}{2}
\]

Firm 1 chooses \(N_1\), then firm 2 chooses \(N_2\) having observed \(N_1\). We have:

\[
N_2(N_1) = \min \left[ N_1 + \frac{4 (1 - \alpha) v}{9 \gamma}, N \right]
\]
This implies:

\[
\frac{d\Pi_1}{dN_1} = \frac{4(1 - \alpha) v}{9} - \gamma N_1 + \varepsilon - \left\{ \begin{array}{ll}
\alpha v & \text{if } N_1 \leq N - \frac{4(1 - \alpha) v}{9\gamma} \\
\frac{(1 - \alpha) v}{9} & \text{if } N_1 \geq N - \frac{4(1 - \alpha) v}{9\gamma} \end{array} \right.
\]

Thus, \(\frac{d\Pi_1}{dN_1} < 0\) for all \(N_1 \leq N - \frac{4(1 - \alpha) v}{9\gamma}\) (since \(\alpha \geq 1/2\) and \(\varepsilon\) is small).

If \(N \geq \frac{7(1 - \alpha) v}{9\gamma} + \frac{\varepsilon}{\gamma}\), then \(\frac{d\Pi_1}{dN_1} < 0\) for all \(N_1\), so \(N_1^* = 0\) and \(N_2^* = \frac{4(1 - \alpha) v}{9\gamma} < N\). This leads to

\[\Pi_1 = \alpha v N - \frac{4(10\alpha - 1)(1 - \alpha) v^2}{81\gamma} - \varepsilon N \quad \text{and} \quad \Pi_2 = \frac{8(1 - \alpha)^2 v^2}{81\gamma}\]

If \(N \leq \frac{4(1 - \alpha) v}{9\gamma}\) then \(N_2^* = N\) and \(N_1^* = \min\left[\frac{(1 - \alpha) v}{3\gamma} + \frac{\varepsilon}{\gamma}, N\right]\). Thus:

- If \(N \leq \frac{(1 - \alpha) v}{3\gamma} + \frac{\varepsilon}{\gamma}\) then \(\Pi_1 = \frac{4(1 - \alpha) v N}{9} - \frac{2N^2}{2}\) and \(\Pi_2 = \frac{(1 - \alpha) v N}{9} - \varepsilon N\)
- If \(\frac{(1 - \alpha) v}{3\gamma} + \frac{\varepsilon}{\gamma} \leq N \leq \frac{4(1 - \alpha) v}{9\gamma}\) then \(\Pi_1 = \frac{(1 - \alpha) v N}{9} + \frac{(1 - \alpha)^2 v^2}{18\gamma} - \varepsilon \left(N - \frac{(1 - \alpha) v}{3\gamma} - \frac{\varepsilon}{2\gamma}\right)\) and \(\Pi_2 = \frac{7(1 - \alpha) v N}{9} - \frac{(1 - \alpha)^2 v^2}{6\gamma} - \frac{2N^2}{2} - \varepsilon \left(\frac{(1 - \alpha) v}{\gamma} - N + \frac{3\varepsilon}{2\gamma}\right)\)

Suppose now \(\frac{4(1 - \alpha) v}{9\gamma} \leq N \leq \frac{7(1 - \alpha) v}{9\gamma}\). Then \(N_1^* = 0\) or \(N_1^* = \frac{(1 - \alpha) v}{3\gamma} + \frac{\varepsilon}{\gamma}\), depending on which one yields higher profits to firm 1:

- If \(N_1 = 0\) then \(N_2 = \frac{4(1 - \alpha) v}{9\gamma}\) and therefore \(\Pi_1 = \alpha v N - \frac{4(10\alpha - 1)(1 - \alpha) v^2}{81\gamma} - \varepsilon N\) and \(\Pi_2 = \frac{8(1 - \alpha)^2 v^2}{81\gamma}\)
- If \(N_1 = \frac{(1 - \alpha) v}{3\gamma} + \frac{\varepsilon}{\gamma}\) then \(N_2 = N\) and therefore \(\Pi_1 = \frac{(1 - \alpha) v N}{9} + \frac{(1 - \alpha)^2 v^2}{18\gamma} - \varepsilon \left(N - \frac{(1 - \alpha) v}{3\gamma} - \frac{\varepsilon}{2\gamma}\right)\)

and \(\Pi_2 = \frac{7(1 - \alpha) v N}{9} - \frac{(1 - \alpha)^2 v^2}{6\gamma} - \frac{2N^2}{2} - \varepsilon \left(\frac{(1 - \alpha) v}{\gamma} - N + \frac{3\varepsilon}{2\gamma}\right)\)

Comparing the last two profit expressions for firm 1, we have \(N_1^* = 0\) (respectively, \(N_1^* = \frac{(1 - \alpha) v}{3\gamma} + \frac{\varepsilon}{\gamma}\)) if and only if \(\gamma N \geq \frac{(7\alpha + 1)(1 - \alpha) v}{18(10\alpha - 1)} + \frac{\varepsilon(6v(1 - \alpha) + 9\varepsilon)}{2v(10\alpha - 1)}\) (respectively, \(\gamma N \leq \frac{(7\alpha + 1)(1 - \alpha) v}{18(10\alpha - 1)} + \frac{\varepsilon(6v(1 - \alpha) + 9\varepsilon)}{2v(10\alpha - 1)}\)).

### 8.2 Proof of Proposition 2

Given the discussion before Proposition 2 in the text, we know that in equilibrium firms 1 offers products \([0, N_1]\) in both modes and products \([N_1, N]\) in the TSP mode only, while firm 2 offers products \([N_1, N_2]\) in the M mode only, where \(N_1 \leq N_2 \leq N\). Profits are therefore:

\[
\Pi_1 = vN_1 - \frac{\gamma N_1^2}{2} + \frac{(1 - \alpha) v (N_2 - N_1)}{9} + \alpha v (N - N_2) - \varepsilon N
\]

\[
\Pi_2 = \frac{4(1 - \alpha) v (N_2 - N_1)}{9} - \frac{\gamma (N_2 - N_1)^2}{2}
\]

Thus, we must have:

\[
N_2 = \min \left(\frac{4(1 - \alpha) v}{9\gamma} + N_1, N\right)
\]
If $N_1 \geq N - \frac{4(1-\alpha)\nu}{9\gamma}$ then $N_2 = N$ and firm 1’s profits are (taking into account firm 2’s best response):

$$vN_1 - \frac{\gamma N_1^2}{2} + \frac{(1-\alpha)\nu(N-N_1)}{9} - \varepsilon N$$

increasing for $N_1 \leq \frac{(8+\alpha)\nu}{9\gamma}$ and decreasing for $N_1 \geq \frac{(8+\alpha)\nu}{9\gamma}$.

If $N_1 \leq N - \frac{4(1-\alpha)\nu}{9\gamma}$ then $N_2 = \frac{4(1-\alpha)\nu}{9\gamma} + N_1$ and firm 1’s profits are (taking into account firm 2’s best response):

$$vN_1 - \frac{\gamma N_1^2}{2} + \frac{4(1-\alpha)^2\nu^2}{81\gamma} + \alpha\nu \left( N - \frac{4(1-\alpha)\nu}{9\gamma} - N_1 \right) - \varepsilon N$$

increasing for $N_1 \leq \frac{(1-\alpha)\nu}{\gamma}$ and decreasing for $N_1 \geq \frac{(1-\alpha)\nu}{\gamma}$.

Since $\alpha \geq 1/2$, we always have $\frac{(1-\alpha)\nu}{\gamma} \leq \frac{(8+\alpha)\nu}{9\gamma}$.

Suppose first $N - \frac{4(1-\alpha)\nu}{9\gamma} \geq \frac{(8+\alpha)\nu}{9\gamma}$, i.e. $N \geq \frac{(12-3\alpha)\nu}{9\gamma}$. In this case, firm 1’s profits are decreasing for $N_1 \geq N - \frac{4(1-\alpha)\nu}{9\gamma}$, therefore the equilibrium is $N_1^* = \frac{(1-\alpha)\nu}{\gamma} < N$ and $N_2^* = \frac{13(1-\alpha)\nu}{9\gamma} < N$, leading to profits:

$$\Pi_1 = \alpha\nu N + \frac{(1-\alpha)(89-161\alpha)\nu^2}{162\gamma} - \varepsilon N \text{ and } \Pi_2 = \frac{8(1-\alpha)^2\nu^2}{81\gamma}$$

Second, suppose $N - \frac{4(1-\alpha)\nu}{9\gamma} \leq \frac{(1-\alpha)\nu}{\gamma}$, i.e. $N \leq \frac{13(1-\alpha)\nu}{9\gamma}$. Then firm 1’s profits are increasing for $N_1 \leq N - \frac{4(1-\alpha)\nu}{9\gamma}$, therefore the equilibrium is $N_1^* = \min\left(\frac{(8+\alpha)\nu}{9\gamma}, N\right) = N$ and $N_2^* = N$, leading to profits:

$$\Pi_1 = vN - \frac{\gamma N^2}{2} - \varepsilon N \text{ and } \Pi_2 = 0$$

Third, suppose $\frac{13(1-\alpha)\nu}{9\gamma} < \frac{(8+\alpha)\nu}{9\gamma} \leq N \leq \frac{(12-3\alpha)\nu}{9\gamma}$. Then firm 1 chooses between:

- option 1: $N_1 = \frac{(1-\alpha)\nu}{\gamma}$, which results in $N_2 = \frac{13(1-\alpha)\nu}{9\gamma}$ and yields profits $\alpha\nu N + \frac{(1-\alpha)(89-161\alpha)\nu^2}{162\gamma} - \varepsilon N$ for firm 1 (same as the first case above)

- option 2: $N_1 = \frac{(8+\alpha)\nu}{9\gamma}$, which results in $N_2 = N$ and yields profits $\frac{(8+\alpha)^2\nu^2}{162\gamma} + \frac{(1-\alpha)\nu N}{9} - \varepsilon N$ for firm 1

It is straightforward to verify that option 1 dominates for $N \geq \frac{(8+\alpha)\nu}{9\gamma}$. Thus, the equilibrium is the same as in the first case above.

Fourth and finally, suppose $\frac{13(1-\alpha)\nu}{9\gamma} \leq N \leq \frac{(8+\alpha)\nu}{9\gamma} < \frac{(12-3\alpha)\nu}{9\gamma}$. Then firm 1 chooses between:

- option 1: $N_1 = \frac{(1-\alpha)\nu}{\gamma}$, which results in $N_2 = \frac{13(1-\alpha)\nu}{9\gamma}$ and yields profits $\alpha\nu N + \frac{(1-\alpha)(89-161\alpha)\nu^2}{162\gamma} - \varepsilon N$ for firm 1 (same as the first case above)
Let $\Delta \Pi(N) \equiv (1 - \alpha) v N - \frac{\gamma N^2}{2} - \frac{(1-\alpha)(89-161\alpha)v^2}{162\gamma}$ the difference between option 2 and option 1 as a function of $N$. Note that $\Delta \Pi$ is strictly decreasing in $N$ for $N \geq \frac{13(1-\alpha)v}{9\gamma} > \frac{(1-\alpha)v}{\gamma}$. And it is straightforward to verify that $\Delta \Pi \left( \frac{13(1-\alpha)v}{9\gamma}, \frac{(8+\alpha)v}{9\gamma} \right)$ is $N = \frac{(1-\alpha)v}{\gamma} \left(1 + \sqrt{\frac{80\alpha - 8}{81(1-\alpha)}}\right)$. Thus, firm 1 chooses option 2 for $N \leq \bar{N}$ and option 1 for $N \geq \bar{N}$.

8.3 Proof of Lemma 3

For each firm $i$, let us order the products it offers in the TSP mode, $n \in [0, N_i^P]$, such that the price charged, $p_i^P(n)$, is increasing in $n$. Similarly, denote by $p_i^M(n)$ the price charged for products $n \in [N_i^P, N_i^P + N_i^M]$, which are offered in the M mode. Clearly, $p_i^M(n) \leq v$ for all $n \in [N_i^P, N_i^P + N_i^M]$.

Thus, for any consumer $\lambda \in [\alpha, 1]$, the net utility from going to firm $i$ is

$$u_i(\lambda) = \int_0^{N_i^P} \max \left(\lambda v - p_i^P(n), 0\right) dn + \int_{N_i^P}^{N_i^P+N_i^M} (v - p_i^M(n)) dn.$$

Note that $u_i'(\lambda)$ is increasing (possibly discontinuously) for all $\lambda$.

Let then $p_i^P \equiv p_i^P \left( N_i^P \right)$ denote the highest price charged by firm $i$ for a good sold in the TSP mode. We will show that if these prices are part of a stage II equilibrium then only the total price $P_i \equiv \int_0^{N_i^P} p_i^P(n) dn + \int_{N_i^P}^{N_i^P+N_i^M} p_i^M(n) dn$ charged by each firm $i = 1, 2$ matters.

**Case I:** $p_1^P \geq p_2^P$.

Suppose $u_2 \left( p_1^P/v \right) \geq u_1 \left( p_1^P/v \right)$. Since $u_i'(\lambda) = v N_i^P - v N_i^P = u_2'(\lambda)$ for all $\lambda \geq p_1^P/v$, this would then imply that no consumer $\lambda \geq p_1^P/v$ comes to firm 1, so that firm 1 makes 0 sales on its products priced at $p_1^P$. This cannot be an equilibrium: firm 1 could lower its price on all of these products until either it increases its profits or the price becomes equal to the next highest price (and then we can restart the same reasoning). Thus, we must have $u_2 \left( p_1^P/v \right) < u_1 \left( p_1^P/v \right)$.

But this implies that there exists $\theta > 0$ such that all consumers $\lambda \in \left[ p_1^P/v - \theta, p_1^P/v \right]$ come to firm 1 but do not buy the products priced at $p_1^P$. Let then firm 1 decrease $p_1^P$ by $\epsilon$ very small such that $p_1^P - \epsilon$ remains its highest price. The resulting loss is lower than $\frac{\epsilon(1-\alpha)}{1-\alpha}$, while the resulting gain is greater than $\frac{\epsilon p_1^P}{\epsilon(1-\alpha)}$. The net effect is strictly positive if $p_1^P > \alpha v$ (since $\alpha > 1/2$), therefore we must have $\alpha v \geq p_1^P \geq p_2^P$. This implies that any consumer that goes to any of the two firms buys all products available, so that only the total price matters for both firms in equilibrium.

**Case II:** $p_1^P < p_2^P$.  

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If \( u_1(p_2^P/v) < u_2(p_2^P/v) \) then \( u_1(\lambda) < u_2(\lambda) \) for all \( \lambda \geq p_2^P/v \) (again, recall \( u'_1(\lambda) < u'_2(\lambda) \) for all \( \lambda \geq p_2^P/v \)). In this case, if \( p_1^P > \alpha v \) then firm 2 could slightly decrease \( p_2^P \) and increase profits. Thus, we must have \( u_2(p_2^P/v) < u_1(p_2^P/v) \) and \( u_2(\lambda) \) crosses \( u_1(\lambda) \) exactly once, from below, for \( \lambda \geq p_2^P/v \). Denote that point of intersection by \( y \) (i.e. \( u_2(y) = u_1(\lambda) \)).

Suppose \( u_2(p_2^P/v) < u_1(p_2^P/v) \). In this case, if \( p_1^P > \alpha v \) then, by the same reasoning as above, firm 1 could profitably decrease \( p_1^P \). If \( p_1^P \leq \alpha v \) then \( u_1(\lambda) \) is linear and \( u_2(\lambda) \) crosses \( u_1(\lambda) \) for all \( \lambda \in [\alpha, p_2^P/v] \). This implies that for each firm only the total price matters for demand and firm profits.

Suppose then \( u_2(p_2^P/v) \geq u_1(p_2^P/v) \). Since \( u_2(p_2^P/v) < u_1(p_2^P/v) \), \( u_1(\lambda) \) is linear and \( u'_2(\lambda) \) is increasing for all \( \lambda \geq p_1^P/v \), there exists a unique \( x \in [p_1^P/v, p_2^P/v] \) such that \( u_2(x) = u_1(x) \). All consumers with \( \lambda \in [x, y] \) go to firm 1 while all consumers with \( \lambda \in [y, 1] \) go to firm 2. Let then \( \overline{N}_2 \equiv \min \{ n \text{ such that } p^P_2(n) \geq xv \} \). Any given TSP product \( n \geq \overline{N}_2 \) offered by firm 2 is only bought by consumers \( \lambda \in [y, 1] \), who in fact buy all TSP products \( n \geq \overline{N}_2 \) (because \( \lambda v > p^P_2 \)). Thus, we can assume without loss of generality (i.e. without changing anything to demands and profits) that all TSP products \( n \geq \overline{N}_2 \) are priced at \( p^P_2 \), where \( (\overline{N}_2 - \overline{N}_2)p_2^P = \int_{\overline{N}_2}^{\overline{N}_2} p_2^P(n) dn \). Clearly, \( x \) and \( y \) remain unchanged and \( \overline{p}_2^P/v \in [x, p_2^P/v] \), so \( u_2(\overline{p}_2^P/v) < u_1(\overline{p}_2^P/v) \). Thus, we are now in a scenario in which:

\[
u_1(\lambda) = u_1(x) + v(\lambda - x) N_1^P \text{ for all } \lambda \geq x \geq p_1^P/v
\]

\[
u_2(\lambda) =\begin{cases} u_2(x) + v(\lambda - x) \overline{N}_2 & \text{for } \lambda \in [x, \overline{p}_2^P/v] \\ u_2(x) + (\overline{p}_2^P - xv) \overline{N}_2 + v(\lambda - \overline{p}_2^P/v) N_2^P & \text{for } \lambda \in [\overline{p}_2^P/v, 1] \end{cases}
\]

Since \( u_1(\lambda) > u_2(\lambda) \) for \( \lambda \in [x, \overline{p}_2^P/v] \) we must have \( \overline{N}_2 < N_1^P \) (recall \( N_1^P < N_2^P \)). Furthermore, since \( u_1(y) = u_2(y) \), we have:

\[
\frac{dy}{dp_2^P} = \frac{N_2^P - \overline{N}_2}{v(N_2^P - N_1^P)} > \frac{1}{v}
\]

Let now firm 2 deviate by slightly lowering \( \overline{p}_2^P \) by \( \varepsilon > 0 \). The loss due to this deviation is approximately equal to \( \frac{v(\overline{N}_2 - \overline{N}_2)^2(1-y)}{1-\alpha} \), which is smaller than \( \frac{\varepsilon(N_2^P - N_2)^2(1-p_2^P/v)}{1-\alpha} \). The gain is at least \( \frac{N_2^P - \overline{N}_2}{v(1-\alpha)} \), which in turn is greater than \( \frac{\varepsilon(N_2^P - N_2)(1-p_2^P/v)}{1-\alpha} \) whenever \( p_2^P \geq \alpha v > v/2 \). The deviation is therefore profitable whenever \( p_2^P \geq \alpha v > v/2 \), so the initial prices can only be part of an equilibrium if \( p_2^P < \alpha v \). This implies that all firm prices for TSP products are lower than \( \alpha v \), so that only the total prices matter for each firm (recall that all firm prices for M products are lower than \( v \)).

Thus, for any consumer \( \lambda \in [\alpha, 1] \) the respective utilities from going at either firm are:

\[
u_i(\lambda) = vN_i^M + \lambda vN_i^P - P_i
\]
and the indifferent consumer $y$ is given by (we assume that $N_2^P > N_1^P$):

$$y = \frac{(N_1^M - N_2^M)v + P_2 - P_1}{v(N_2^P - N_1^P)}$$

Firms’ profits are:

$$\Pi_1 = P_1\frac{y - \alpha}{1 - \alpha} \quad \text{and} \quad \Pi_2 = P_2\frac{1 - y}{1 - \alpha}$$

Given $N_2^P > N_1^P$, we can solve the stage II pricing equilibrium and obtain firm prices and profits:

$$P_1 = v\left[\frac{(N_1^M - N_2^M) - (2\alpha - 1)(N_2^P - N_1^P)}{3}\right]$$
$$P_2 = v\left[\frac{(N_2^M - N_1^M) + (2 - \alpha)(N_2^P - N_1^P)}{3}\right]$$

$$\Pi_1 = v\left[\frac{(N_1^M - N_2^M) - (2\alpha - 1)(N_2^P - N_1^P)}{9(1 - \alpha)(N_2^P - N_1^P)}\right]^2 - \varepsilon N_1^P - \frac{\gamma(N_1^M)^2}{2}$$
$$\Pi_2 = v\left[\frac{(N_2^M - N_1^M) + (2 - \alpha)(N_2^P - N_1^P)}{9(1 - \alpha)(N_2^P - N_1^P)}\right]^2 - \varepsilon N_2^P - \frac{\gamma(N_2^M)^2}{2}$$

This equilibrium is interior if and only if:

$$(2\alpha - 1)(N_2^P - N_1^P) \leq N_1^M - N_2^M \leq (2 - \alpha)(N_2^P - N_1^P)$$

If the left-hand inequality fails then firm 1 makes 0 sales & profits; if the right-hand inequality fails then firm 2 makes 0 sales and profits.

### 8.4 Proof of Proposition 3

There are two cases to consider, depending on whether $N_1^P < N_2^P$ or $N_1^P > N_2^P$ in the Stackelberg equilibrium.

Suppose first $N_1^P < N_2^P$. The profit expressions from Lemma 3 still apply:

$$\Pi_1 = v\left[\frac{(N_1^M - N_2^M) - (2\alpha - 1)(N_2^P - N_1^P)}{9(1 - \alpha)(N_2^P - N_1^P)}\right]^2 - \varepsilon N_1^P - \frac{\gamma(N_1^M)^2}{2}$$
$$\Pi_2 = v\left[\frac{(2 - \alpha)(N_2^P - N_1^P) - (N_1^M - N_2^M)}{9(1 - \alpha)(N_2^P - N_1^P)}\right]^2 - \varepsilon N_2^P - \frac{\gamma(N_2^M)^2}{2}$$

except that now $(N_2^M, N_2^P)$ are chosen in response to $(N_1^M, N_1^P)$.
Both firms must have non-negative demands and profits in equilibrium so we must have:

\[(2\alpha - 1) (N_2^P - N_1^P) < N_1^M - N_2^M < (2 - \alpha) (N_2^P - N_1^P)\]

which also implies \(N_1^M > N_2^M\).

Given \((N_1^M, N_1^P)\) and keeping \(N_2^M\) constant, the sign of \(\frac{\partial \Pi_2}{\partial N_2^P}\) is equal to the sign of:

\[(2 - \alpha)^2 \left( N_2^P - N_1^P \right)^2 - \left( N_2^M - N_1^M \right)^2\]

which is positive by the inequality above. Therefore, in equilibrium we must have:

\[N_2^P = \min \left\{ N - N_2^M, N_1^P + \frac{N_1^M - N_2^M}{2\alpha - 1} \right\}\]

If \(N_2^P = N_1^P + \frac{N_1^M - N_2^M}{2\alpha - 1}\) then firm 1 makes non-positive profits in equilibrium, which is not possible. Thus, we must have:

\[N_2^P = N - N_2^M\]

Replacing this equality in the expressions of \(\Pi_1\) and \(\Pi_2\), we obtain:

\[\Pi_1 \left( N_1^M, N_1^P, N_2^M \right) = v \frac{\left[ N_1^M + (2\alpha - 1) N_1^P - (2\alpha - 1) N - 2 (1 - \alpha) N_2^M \right]^2}{9 (1 - \alpha) (N - N_2^M - N_1^P)} - \varepsilon N_1^P - \gamma \left( \frac{N_1^M}{2} \right)^2\]

\[\Pi_2 \left( N_1^M, N_1^P, N_2^M \right) = v \frac{\left[ (2 - \alpha) N - N_1^M - (2 - \alpha) N_1^P - (1 - \alpha) N_2^M \right]^2}{9 (1 - \alpha) (N - N_2^M - N_1^P)} - \varepsilon \left( N - N_2^M \right) - \gamma \left( \frac{N_2^M}{2} \right)^2\]

First, note that \(\Pi_1 \left( N_1^M, N_1^P, N_2^M \right)\) is increasing in \(N_1^P\) (easily seen) and decreasing in \(N_2^M\):

\[\text{sign} \left( \frac{\partial \Pi_1 \left( N_1^M, N_1^P, N_2^M \right)}{\partial N_2^M} \right) = \text{sign} \left[ - (3 - 2\alpha) N + N_1^M + (3 - 2\alpha) N_1^P + 2 (1 - \alpha) N_2^M \right]\]

\[= \text{sign} \left[ - N + N_1^M + (3 - 2\alpha) N_1^P - 2 (1 - \alpha) N_2^P \right]\]

\[< 0\]

because \(N_1^P < N_2^P\) and \(N_1^M + N_1^P \leq N\).

Second, we have:

\[\frac{\partial \Pi_2 \left( N_1^M, N_1^P, N_2^M \right)}{\partial N_2^M} = v \frac{\left[ (2 - \alpha) N - N_1^M \right] \left[ \alpha N - N_1^M - \alpha N_1^P \right]}{9 (1 - \alpha) (N - N_2^M - N_1^P)^2} + \varepsilon - \gamma N_2^M\]
Note that \((2 - \alpha)N - N_1^M - (2 - \alpha)N_1^P - (1 - \alpha)N_2^M \geq 0\).

Now suppose \(N_1^P + N_1^M < N\) and let firm 1 deviate to \(N_1^{M'} = N - N_1^M\) (keeping \(N_1^M\) constant). In this case, \(\alpha N - N_1^M - \alpha N_1^{M'} + (1 - \alpha)N_2^M < 0\) for all \(N_2^M < N_1^M\) so firm 2's best response must be \(N_2^M = 0\). Thus, by increasing \(N_1^P\), firm 1 induces firm 2 to decrease \(N_2^M\) or keeps it constant. Since \(\Pi_1 (N_1^M, N_1^P, N_2^M)\) is increasing in \(N_1^P\) and decreasing in \(N_2^M\), the deviation strictly increases firm 1's profits, which is at odds with \((N_1^M, N_1^P, N_2^M)\) being a Stackelberg equilibrium.

We have thus shown that in equilibrium both firms must cover the entire product range:

\[
N_1^M + N_1^P = N_2^M + N_2^P = N
\]

so that we can focus on the choices of \(N_1^M\) and \(N_2^M\).

Equilibrium profits are:

\[
\Pi_1 = \frac{4v(1 - \alpha)(N_1^M - N_2^M)}{9} - \gamma \left(\frac{(N_1^M)^2}{2}\right) - \varepsilon (N - N_1^M)
\]
\[
\Pi_2 = \frac{v(1 - \alpha)(N_1^M - N_2^M)}{9} - \gamma \left(\frac{(N_2^M)^2}{2}\right) - \varepsilon (N - N_2^M)
\]

Clearly, \(\Pi_2\) is decreasing in \(N_2^M\) so in equilibrium it must be that \(N_2^M = 0\) and thus \(N_2^P = N\). The profit expressions become:

\[
\Pi_1 = \frac{4v(1 - \alpha)N_1^M}{9} - \gamma \left(\frac{(N_1^M)^2}{2}\right) - \varepsilon (N - N_1^M)
\]
\[
\Pi_2 = \frac{v(1 - \alpha)N_1^M}{9} - \varepsilon N
\]

Finally, optimizing \(\Pi_1\) over \(N_1^M\) we obtain \(N_1^M = \min \left[\frac{4v(1 - \alpha)}{9\gamma} + \frac{\varepsilon}{\gamma}, N\right]\). It is easily verified that in both cases \(\Pi_1 > \Pi_2\).

Consider now the other potential equilibrium, in which \(N_1^P > N_2^P\) and \(N_1^M < N_2^M\). Suppose such an equilibrium exists. Then profits must be:

\[
\Pi_1 = \frac{v \left[ (2 - \alpha) (N_1^P - N_2^P) - (N_2^M - N_1^M) \right]^2}{9(1 - \alpha)(N_1^P - N_2^P)} - \varepsilon N_1^P - \gamma \left(\frac{(N_1^M)^2}{2}\right)
\]
\[
\Pi_2 = \frac{v \left[ (N_2^M - N_1^M) - (2\alpha - 1) (N_1^P - N_2^P) \right]^2}{9(1 - \alpha)(N_1^P - N_2^P)} - \varepsilon N_2^P - \gamma \left(\frac{(N_2^M)^2}{2}\right)
\]

and:

\[
0 < (2\alpha - 1) (N_1^P - N_2^P) < N_2^M - N_1^M < (2 - \alpha) (N_1^P - N_2^P) \quad (1)
\]
Clearly, $\Pi_2$ is increasing in $N_2^P$ so we must have:

$$N_2^P = \min \left\{ N - N_2^M, N_1^P - \frac{N_2^M - N_1^M}{2 - \alpha} \right\}$$

Again, $N_2^P = N_1^P - \frac{N_2^M - N_1^M}{2 - \alpha}$ would mean that firm 1 makes negative profits in equilibrium, so it must be that:

$$N_2^P = N - N_2^M$$

Replacing this equality in the expressions of $\Pi_1$ and $\Pi_2$, we obtain:

$$\Pi_1 \left( N_1^M, N_1^P, N_2^M \right) = \frac{v \left[ N_1^M + (2 - \alpha) N_1^P - (2 - \alpha) N + (1 - \alpha) N_2^M \right]^2}{9 (1 - \alpha) (N_1^P - N + N_2^M)} - \varepsilon N_1^P - \gamma \frac{(N_1^M)^2}{2}$$

$$\Pi_2 \left( N_1^M, N_1^P, N_2^M \right) = \frac{v \left[ (2\alpha - 1) N - N_1^M - (2\alpha - 1) N_1^P + 2 (1 - \alpha) N_2^M \right]^2}{9 (1 - \alpha) (N_1^P - N + N_2^M)} - \varepsilon (N - N_2^M) - \gamma \frac{(N_2^M)^2}{2}$$

First, $\Pi_1 \left( N_1^M, N_1^P, N_2^M \right)$ is increasing in $N_1^P$ and in $N_2^M$:

$$\text{sign} \left( \frac{\partial \Pi_1 \left( N_1^M, N_1^P, N_2^M \right)}{\partial N_1^P} \right) = \text{sign} \left[ (2 - \alpha)^2 (N_1^P - N_2^P)^2 - (N_2^M - N_1^M)^2 \right] > 0$$

$$\text{sign} \left( \frac{\partial \Pi_1 \left( N_1^M, N_1^P, N_2^M \right)}{\partial N_2^M} \right) = \text{sign} \left[ (1 - \alpha) N_2^M - N_1^M - \alpha N_1^P + \alpha N \right] > 0$$

because $N_1^M + N_1^P \leq N$ and $N_2^M > N_1^M$.

Second, we have:

$$\frac{\partial \Pi_2 \left( N_1^M, N_1^P, N_2^M \right)}{\partial N_2^M} = \frac{v \left[ (2\alpha - 1) N - N_1^M - (2\alpha - 1) N_1^P + 2 (1 - \alpha) N_2^M \right]}{9 (1 - \alpha) (N_1^P - N + N_2^M)^2} \times \left[ 2 (1 - \alpha) N_2^M - (3 - 2\alpha) N + N_1^M + (3 - 2\alpha) N_1^P \right] + \varepsilon - \gamma N_2^M$$

Now suppose $N_1^P + N_1^M < N$ and let firm 1 deviate to $N_1^{P'} = N - N_1^M$ (keeping $N_1^M$ constant). In this case, $2 (1 - \alpha) N_2^M - (3 - 2\alpha) N + N_1^M + (3 - 2\alpha) N_1^{P'} > 0$ for all $N_2^M > N_1^M$, so firm 2’s best response must be $N_2^M = N$ (note indeed that still satisfies 1). Thus, by increasing $N_1^P$, firm 1 induces firm 2 to increase $N_2^M$ or keep it constant. Since $\Pi_1 \left( N_1^M, N_1^P, N_2^M \right)$ is increasing in both $N_1^P$ and $N_2^M$, the deviation strictly increases firm 1’s profits, which is at odds with $\left( N_1^M, N_1^P, N_2^M \right)$ being a Stackelberg equilibrium.

We have thus shown that in equilibrium both firms must cover the entire product range:

$$N_1^M + N_1^P = N_2^M + N_2^P = N$$
Equilibrium profits are therefore:

\[
\Pi_1 = \frac{v (1 - \alpha) (N_2^M - N_1^M)}{9} - \gamma \left(\frac{(N_1^M)^2}{2}\right) - \varepsilon (N - N_1^M)
\]

\[
\Pi_2 = \frac{4v (1 - \alpha) (N_2^M - N_1^M)}{9} - \gamma \left(\frac{(N_2^M)^2}{2}\right) - \varepsilon (N - N_2^M)
\]

From the expressions above, it is apparent that firm 2’s choice of \(N_2^M\) is independent of \(N_1^M\); therefore, since firm 1’s profits are decreasing in \(N_1^M\), it will choose \(N_1^M = 0\). The profit expressions become:

\[
\Pi_1 = \frac{v (1 - \alpha) N_2^M}{9} - \varepsilon N
\]

\[
\Pi_2 = \frac{4v (1 - \alpha) N_2^M}{9} - \gamma \left(\frac{(N_2^M)^2}{2}\right) - \varepsilon (N - N_2^M)
\]

Finally, optimizing \(\Pi_2\) over \(N_2^M\) we obtain \(N_2^M = \min \left[\frac{4v(1-\alpha)}{9\gamma} + \frac{\varepsilon}{\gamma}, N\right]\).

Thus, profits are exactly reversed relative to the case \(N_1^P < N_2^P\). And since the firm that has the higher \(N_2^M\) makes higher profits, firm 1 will choose \(N_1^M = \min \left[\frac{4v(1-\alpha)}{9\gamma} + \frac{\varepsilon}{\gamma}, N\right]\) (the first case above). If \(\frac{4v(1-\alpha)}{9\gamma} + \frac{\varepsilon}{\gamma} > N\) then \(N_1^M = N\) and firm 2 has no choice but to choose \(N_2^M = 0\). If \(\frac{4v(1-\alpha)}{9\gamma} + \frac{\varepsilon}{\gamma} < N\) then \(N_1^M = \frac{4v(1-\alpha)}{9\gamma} + \frac{\varepsilon}{\gamma}\). But in this case, the maximum profits that firm 2 can obtain by choosing \(N_2^M > N_1^M\) are negative, therefore firm 2 will once again have no choice but to choose \(N_2^M = 0\). This means that the unique equilibrium is the one determined under the assumption \(N_1^M > N_2^M\).

### 8.5 Proof of Proposition 4

Assume that firm 1 chooses to foreclose firm 2 - we later show that this is the optimal strategy. For exclusion to be effective, firm 1 must offer all products in the TSP mode (as \(\varepsilon\) goes to 0), otherwise firm 2 could profitably enter by offering all products in the TSP mode and thereby capture a positive measure of consumers with \(\lambda\) close to 1. Let then firm 1 offer \(N_1\) products in the M mode and all products in the TSP mode.

Suppose that firm 2 enters with \(N_2 < N_1\) products in the M mode. Let us show that firm 2 cannot obtain any sales at a positive price. First, note that it is optimal for firm 2 to offer the other \(N - N_2\) products in the TSP mode. Indeed, given positive fixed costs, it does not make sense for firm 2 to offer any product in both modes (firm 1 may do so solely for pre-emption reasons, which do not apply to firm 2). Furthermore, firm 2 wants to maximize the utility offered to consumers by adding as many TSP products as possible (without overlapping with M products).

Suppose that firm 2 attracts a positive measure of consumers in the Stage 3 equilibrium. Suppose there are "gaps" in the customer intervals that go to firm 2, i.e. suppose that there exist \(\lambda_1, \lambda_2, \lambda_3\) and
\( \lambda_4 \) such that \( \alpha \leq \lambda_1 < \lambda_2 < \lambda_3 \leq \lambda_4 \leq 1 \), such that all consumers \( \lambda \in [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4] \) go to firm 2 and all consumers \( \lambda \in [\lambda_2, \lambda_3] \) go to firm 1. Take then any consumer \( \lambda_0 \in [\lambda_2, \lambda_3] \) and suppose that this consumer goes to firm 1 and buys the package of products \( \Omega_1 \). Indeed, since firm 1 offers some products in both modes, different consumers may prefer different packages. On the other hand, firm 2 offers a unique package: \( N_2 \) products in the M mode and \( N - N_2 \) products in the TSP mode. If the number of M products in \( \Omega_1 \) is larger than \( N_2 \) then all consumers \( \lambda \geq \lambda_0 \) should prefer buying \( \Omega_1 \) from firm 1 to going to firm 2, which contradicts the existence of \([\lambda_3, \lambda_4]\). If on the other hand the number of M products in \( \Omega_1 \) is smaller than \( N_2 \) then all consumers \( \lambda \leq \lambda_0 \) should prefer buying \( \Omega_1 \) from firm 1 to going to firm 2, which contradicts the existence of \([\lambda_1, \lambda_2]\). If the number of M products in \( \Omega_1 \) is equal to \( N_2 \) then all consumers should prefer buying \( \Omega_1 \) from firm 1 to going to firm 2. Thus, in all cases, we obtain a contradiction. So it must be that the set of consumers attracted by firm 2 in the Stage 3 equilibrium has no gaps. Let this set be the interval \([\lambda_1, \lambda_2]\), with \( \alpha \leq \lambda_1 < \lambda_2 \leq 1 \).

Denote by \( \Omega_1 \) the firm 1 package preferred by consumers \( \lambda \in [\alpha, \lambda_1] \) and by \( \Omega_2 \) the firm 1 package preferred by consumers \( \lambda \in [\lambda_2, 1] \). Then, by a similar reasoning as above, it must be that the number of M products in \( \Omega_1 \) is strictly greater than \( N_2 \) (the number of M products in the unique package offered by firm 2) and that the number of M products in \( \Omega_2 \) is strictly lower than \( N_2 \). Maximum differentiation and symmetry across (i) firm 1’s TSP products that are also offered in the M mode, and (ii) firm 1’s TSP products that are not offered in the M mode, imply that the number of M products in \( \Omega_1 \) is \( N_1 \), whereas the number of M products in \( \Omega_2 \) is 0. Given this, \( \lambda_1 \) and \( \lambda_2 \) are respectively given by:

\[
\begin{align*}
N_2 (v - p_2^M) + (N - N_2) (\lambda_1 v - p_2^P) &= N_1 (v - p_1^M) + (N - N_1) (\lambda_1 v - p_1^P) \\
N_2 (v - p_2^M) + (N - N_2) (\lambda_2 v - p_2^P) &= N \lambda_2 v - N_1 \tilde{p}_1^P - (N - N_1) p_1^P,
\end{align*}
\]

where \( p_1^M \) and \( p_2^M \) are the prices charged by firm 1 and firm 2 respectively for their M products, \( p_2^P \) is the price charged by firm 2 for its TSP products, \( \tilde{p}_1^P \) is the price charged by firm 1 for its \( N_1 \) TSP products which are also offered in the M mode and \( p_1^P \) is the price charged by firm 1 for its \( (N - N_1) \) TSP products which are not offered in the M mode.

The difference in total price between firm 1’s package \( \Omega_1 \) and firm 2’s package is

\[
N_1 p_1^M + (N - N_1) p_1^P - N_2 p_2^M - (N - N_2) p_2^P = v(N_1 - N_2)(1 - \lambda_1).
\]

This simply says that the indifferent consumer, \( \lambda_1 \), is just willing to pay the additional amount corresponding to the quality difference she perceives between firm 1’s package \( \Omega_1 \) and firm 2’s package.

Similarly, the difference in total price between firm 1’s package \( \Omega_2 \) and firm 2’s package is

\[
N_2 p_2^M + (N - N_2) p_2^P - N_1 \tilde{p}_1^P - (N - N_1) p_1^P = v N_2 (1 - \lambda_2).
\]
Let us now construct a profitable deviation by firm 1 as follows. First, set individual prices on $N_2$ of firm 1’s M products to $p^M_2 = p^P_2 - \varepsilon$ and on $(N - N_2)$ TSP products set a price equal to $p^P_2$. Clearly, this package attracts all the consumers between $\lambda_1$ and $\lambda_2$ that were originally going to firm 2. Second, for firm 1’s remaining $N_1 - N_2$ M products set prices equal to $\tilde{p}^M_1 = \tilde{p}^P_2 + v(1 - \lambda_1)$ and for firm 1’s remaining $N_2$ TSP products set prices equal to $\tilde{p}^P_1 = \frac{N_1\tilde{p}^P_2 + (N - N_1)p^P_1 - (N - N_2)p^P_2}{N_2}$. The total price difference between firm 1’s package $\Omega_1$ and firm 1’s package that has the $N_2$ products in the M mode priced at $p^M_2 - \varepsilon$ and the $(N - N_2)$ TSP products priced at $p^P_2$ is $v(N_1 - N_2)(1 - \lambda_1)$ (up to $\varepsilon$). Thus, consumer $\lambda_1$ is just indifferent between the two packages and all consumers below $\lambda_1$ strictly prefer the package with $N_2$ products in the M mode, just as in the original putative equilibrium. Similarly, the total price difference between firm 1’s package that has the $N_2$ products in the M mode priced at $p^M_2 - \varepsilon$ and the $(N - N_2)$ TSP products priced at $p^P_2$, and firm 1’s package $\Omega_2$ is still $vN_2(1 - \lambda_2)$. Thus, consumer $\lambda_2$ is just indifferent between the two packages and all consumers above $\lambda_2$ strictly prefer the package with $N_1$ products in the TSP mode, just as in the original putative equilibrium.

By construction, the revenue from all the consumers that firm 1 was attracting ($\lambda \leq \lambda_1$ and $\lambda \geq \lambda_2$) is the same as in the putative equilibrium (up to $\varepsilon$) and firm 1 gets a positive revenue from the discrete measure of new consumers that it attracts from firm 2 in the putative equilibrium ($\lambda_1 \leq \lambda \leq \lambda_2$). Thus, the deviation strictly improves firm 1’s profits if $\lambda_1 < \lambda_2$. Thus, firm 2 cannot make positive profits if $N_2 < N_1$.

$N_2 = N_1$ cannot be an equilibrium either because the Bertrand logic would drive equilibrium prices down to 0.

Thus, suppose that firm 2 enters with $N_2 > N_1$ products in the M mode and $N - N_2$ products in the TSP mode. Now, firm 2’s profits are no longer 0 in the pricing equilibrium. Respective firm profits are:

$$
\pi_1 = \frac{P_1}{1 - \alpha} \left( P_2 - P_1 \right) \frac{1}{v(N_2 - N_1)}
$$

$$
\pi_2 = \frac{P_2}{1 - \alpha} \left[ 1 - \alpha - \frac{P_2 - P_1}{v(N_2 - N_1)} \right]
$$

where we have denoted $P_1 \equiv N_1p^M_1 + (N - N_1)\tilde{p}^T_1$ and $P_2 \equiv N_2p^M_2 + (N - N_2)\tilde{p}^T_2$ the total effective prices charged by the two firms - only these prices matter for the equilibrium.\footnote{It is not profitable for firm 2 to offer all of its products as a TSP and try to drive firm 1’s profit to 0 because firm 1 has already sunk its capacity investments and therefore firm 2 will not be able to drive firm 1 out of the market. Thus, firm 2 does not want to add additional products because this induces more aggressive pricing behavior by firm 1 and lowers firm 2’s profits.}
The pricing equilibrium is then:

\[
P_1 = \frac{v (1 - \alpha) (N_2 - N_1)}{3}
\]

\[
P_2 = \frac{2v (1 - \alpha) (N_2 - N_1)}{3}
\]

leading to profits:

\[
\pi_1 = \frac{v (1 - \alpha) (N_2 - N_1)}{9}
\]

\[
\pi_2 = \frac{4v (1 - \alpha) (N_2 - N_1)}{9}
\]

Taking into account the product choice stage, firm 2’s profits net of fixed costs are \( \frac{4v(1-\alpha)(N_2-N_1)}{9} - \frac{\gamma N_2^2}{2} - \varepsilon (N - N_2) \). The optimal choice \( N_2 \) is then \( N_2 = \min \left( \frac{4v(1-\alpha)}{9\gamma}, N \right) \) so that firm 2’s profits net of fixed costs are:

\[
\begin{cases}
\frac{8v^2(1-\alpha)^2}{9\gamma} - \frac{4v(1-\alpha)N_1}{9} & \text{if } \frac{4v(1-\alpha)}{9\gamma} \leq N \\
\frac{4v(1-\alpha)N}{9} - \frac{\gamma N_2^2}{2} - \frac{4v(1-\alpha)N_1}{9} & \text{if } \frac{4v(1-\alpha)}{9\gamma} \geq N
\end{cases}
\]

But recall that a monopolist’s optimal choice of \( N_1 \) (conditional on already offering all products in the TSP mode) is \( \min \left( \frac{v(1-\alpha)}{\gamma}, N \right) \) so that in all cases, if firm 1 sets \( N_1 = \min \left( \frac{v(1-\alpha)}{\gamma}, N \right) \) then firm 2’s best profits with \( N_2 > N_1 \) are negative. Thus, firm 2’s entry is blocked whenever firm 1 offers \( N_1 = \min \left( \frac{v(1-\alpha)}{\gamma}, N \right) \) products in the M mode and all products in the TSP mode. Thus, firm 2 does not enter and firm 1’s profits are:

\[
\begin{align*}
\alpha vN &+ \frac{v^2(1-\alpha)^2}{2\gamma} - \varepsilon N \quad \text{if } \gamma \geq \frac{v(1-\alpha)}{N} \\
vN &- \frac{\gamma N_2^2}{2} - \varepsilon N \quad \text{if } \gamma \leq \frac{v(1-\alpha)}{N}
\end{align*}
\]

Clearly, firm 1 cannot do better than this in any scenario in which firm 2 enters, therefore the equilibrium always involves foreclosure of firm 2.

### 8.6 Proof of Proposition 6

We have:

\[
q_{L}^{p_1}(\lambda) \text{ and } q_{F}^{p_1}(\lambda) = \begin{cases}
\lambda vN + \frac{(1-\lambda)(1-\alpha)v^2}{3\gamma} + \frac{(1-\lambda)\varepsilon}{\gamma} & \text{if } \gamma N \leq \frac{(1-\alpha)v}{3} + \varepsilon \\
\lambda vN &+ \frac{v(1-\lambda)(1-\alpha)v^2}{3\gamma} + \frac{(1-\lambda)\varepsilon}{\gamma} \text{ and } vN &- \frac{(1-\lambda)(1-\alpha)v^2}{3\gamma} - \frac{(1-\lambda)\varepsilon}{\gamma} \geq 0 & \text{if } \gamma N \geq \frac{(71\alpha+1)(1-\alpha)v}{18(10\alpha-1)} + \frac{\varepsilon(6v(1-\alpha)+9\varepsilon)}{2v(10\alpha-1)}
\end{cases}
\]
\[ q_{L}^{m2}(\lambda) \text{ and } q_{F}^{m2}(\lambda) = \begin{cases} 
 N \text{ and } 0 & \text{if } \gamma N \leq v (1 - \alpha) \left( 1 + \frac{80\alpha - 8}{81(1 - \alpha)} \right) 
 \lambda v N + \frac{(1 - \lambda)(1 - \alpha)v^2}{\gamma} \text{ and } \frac{4(1 - \alpha)v^2}{9\gamma} & \text{if } \gamma N \geq v (1 - \alpha) \left( 1 + \frac{80\alpha - 8}{81(1 - \alpha)} \right) 
 \end{cases} \]

\[ q_{L}^{s1}(\lambda) \text{ and } q_{F}^{s1}(\lambda) = \begin{cases} 
 N \text{ and } \lambda v N & \text{if } \gamma N \leq \frac{4v(1 - \alpha)}{9} + \varepsilon 
 \lambda v N + \frac{4(1 - \lambda)(1 - \alpha)v^2}{9\gamma} + \frac{(1 - \lambda)v^2}{\gamma} \text{ and } \lambda v N & \text{if } \gamma N \geq \frac{4v(1 - \alpha)}{9} + \varepsilon 
 \end{cases} \]

\[ q_{L}^{s2}(\lambda) \text{ and } q_{F}^{s2}(\lambda) = \begin{cases} 
 N \text{ and } 0 & \text{if } \gamma N \leq v (1 - \alpha) 
 \lambda v N + \frac{(1 - \lambda)(1 - \alpha)v^2}{\gamma} \text{ and } 0 & \text{if } \gamma N \geq v (1 - \alpha) 
 \end{cases} \]

The results in the text of the proposition are then easily obtained by comparing the corresponding pairs among these expressions.